Functional representations and Riesz-completions of partially ordered vector spaces

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Abstract

We give a condition under which a partially ordered vector space X can be represented as an order dense subspace of a space of continuous functions. This yields a convenient representation of the Riesz completion of X. To illustrate the method we determine the Riesz completion for Lorentz cones and polyhedral cones. We use the representation to characterise the bands in spaces with polyhedral cones. Moreover, we analyse disjointness of positive semidefinite matrices.

1 Introduction

A useful tool in the analysis of partially ordered vector spaces is the "functional representation method", which, roughly speaking, represents each Archimedean partially ordered vector space with order unit as a subspace of $C(\Omega)$, where $C(\Omega)$ is the space of continuous functions on a compact Hausdorff space Ω. There are various natural ways to do this, e.g. Krein [12] and Nachbin [13]. For our purpose we need to follow Kadison [8].

Let X be a real vector space containing a cone[1] K and consider its induced partial order. Recall that K is *Archimedean* if for any $x, y \in X$ with $nx \leq y$ for all $n \in \mathbb{N}$ we have $x \leq 0$. In that case (X, K) is called an *Archimedean partially ordered vector space*. Throughout the exposition we assume that (X, K) has an *order unit* $u \in K$, i.e., for each $x \in X$ there exists $\alpha \geq 0$ such that

$$-\alpha u \leq x \leq \alpha u.$$

As usual, an order unit $u \in K$ gives rise to a norm $\|\cdot\|_u$ on X by

$$\|x\|_u = \inf\{\alpha > 0 \colon -\alpha u \leq x \leq \alpha u\}.$$

[1] K equals its positive-linear hull, and if $x \in K$ and $-x \in K$, then $x = 0$.

This norm is called an *order unit norm.* Note that with respect to this norm each positive[2] linear functional $\varphi\colon X \to \mathbb{R}$ is continuous, as $|\varphi(x)| \leq \varphi(u)$ for all $x \in X$ with $\|x\|_u \leq 1$.

The set $K^* = \{\varphi\colon X \to \mathbb{R}\colon \varphi$ is a positive linear functional$\}$ is a cone in the dual space X^* of $(X, \|\cdot\|_u)$. Denote

$$\Sigma = \{\varphi \in K^*\colon \varphi(u) = 1\}.$$

By the Banach-Alaoglu theorem the closed unit ball B^* of X^* is weak-$\star$ compact. Σ is a weak-$\star$ closed subset of B^*, and hence weak-$\star$ compact. Let

$$\Lambda = \{\varphi \in \Sigma\colon \varphi \text{ is an extreme point of } \Sigma\}.$$

Recall that $\varphi \in \Sigma$ is *extreme* if φ is not in the (relative) interior of a line-segment in Σ. In general, Λ need not be weak-$\star$ closed, not even if X is finite dimensional. We let $\overline{\Lambda}$ denote the weak-$\star$ closure of Λ in Σ.

In the sequel we write $\Phi\colon X \to C(\overline{\Lambda})$ to denote the linear map

$$(\Phi(x))(\varphi) = \varphi(x) \quad \text{for } \varphi \in \overline{\Lambda}. \tag{1}$$

Recall that a linear map $\Psi\colon X \to Y$, where X and Y are partially ordered vector spaces, is called *bipositive* if

$$x \geq 0 \quad \Longleftrightarrow \quad \Psi(x) \geq 0$$

for any $x \in X$. The following theorem is due to Kadison [8, Theorem 2.1]. For completeness we provide the proof here.

Theorem 1 (Kadison)**.** *If (X, K) is an Archimedean partially ordered vector space with order unit u, then $\Phi\colon X \to C(\overline{\Lambda})$ is a bipositive linear map, which maps u to the constant 1 function in $C(\overline{\Lambda})$.*

Proof. It is clear that Φ is a positive linear map from X into $C(\overline{\Lambda})$ which maps u to the constant 1 function. In order to prove bipositivity, we begin by showing that K is $\|\cdot\|_u$-closed. If $(x_n)_{n\in\mathbb{N}}$ in K and $x \in X$ are such that $\|x_n - x\|_u \to 0$, then $\|x_{n_k} - x\|_u < 1/k$ for a suitable subsequence, so $x \geq x_{n_k} - (1/k)u \geq -(1/k)u$ for all k. Hence $x \in K$, as X is Archimedean. Consequently, if $x \in X \setminus K$, then there exists a continuous linear functional $\varphi\colon X \to \mathbb{R}$ such that $\varphi \geq 0$ on K and $\varphi(x) < 0$. Thus

$$x \in K \iff \varphi(x) \geq 0 \text{ for all } \varphi \in K^*.$$

By the Krein-Milman theorem, Σ equals the weak-$\star$ closed convex hull of Λ. Now suppose that $\Phi(x) \geq 0$. This implies $\varphi(x) \geq 0$ for all $\varphi \in \Lambda$. As x acts as a weak-$\star$ continuous functional on X^*, we get $\varphi(x) \geq 0$ for all $\varphi \in \Sigma$, hence for all $\varphi \in K^*$. Thus, $x \in K$, which shows that Φ is bipositive. $\square$

[2] $\varphi(K) \subseteq [0, \infty)$

As Kadison [8, Section 4] pointed out, the map Φ can be used to prove Kakutani's representation theorem for vector lattices, which says that each Banach lattice with order unit is lattice isomorphic to $C(\Omega)$, where Ω is a compact Hausdorff space. In fact, he showed that $\Phi(X) = C(\overline{\Lambda})$ and $\Lambda = \overline{\Lambda}$.

In view of Kadison's Theorem 1 it is interesting to further analyse the properties of $\Phi(X)$ inside $C(\overline{\Lambda})$ when (X, K) is merely an Archimedean partially ordered vector space with order unit. Of course, in that generality we cannot expect $\Phi(X)$ to be norm dense in $C(\overline{\Lambda})$, but the following is true. Denote by $L(X, K)$ the *vector lattice generated by* $\Phi(X)$ inside $C(\overline{\Lambda})$; so,

$$L(X,K) = \Big\{ \bigvee_{j=1}^{m} f_j + \bigwedge_{j=1}^{n} g_j \colon f_1, \ldots, f_m, g_1, \ldots, g_n \in \Phi(X) \Big\}.$$

Proposition 2. *If (X, K) is an Archimedean partially ordered vector space with order unit, then $L(X, K)$ is norm dense in $C(\overline{\Lambda})$.*

Proof. The assertion is an immediate consequence of the Stone-Weierstrass theorem. Indeed, note that $L(X, K)$ contains the constant 1 function in $C(\overline{\Lambda})$, and $f + g$ and $f \vee g$ are in $L(X, K)$ for all $f, g \in L(X, K)$. Clearly, for $\lambda \neq \mu$ in $\overline{\Lambda}$ there exists an $x \in X$ such that $\lambda(x) \neq \mu(x)$, so $L(X, K)$ separates the points in $\overline{\Lambda}$. □

Instead of norm denseness we consider the much weaker notion of order denseness. Recall that a linear subspace D of a partially ordered vector space Y is *order dense* if for each $h \in Y$ we have that h is the greatest lower bound of the set $\{g \in D \colon g \geq h\}$ in Y, that is,

$$h = \inf\{g \in D \colon g \geq h\}.$$

The notion of order denseness considered here is taken from [1, p. 360]. It is closely related to the Riesz-completion, as studied in [3, 6]. A *Riesz-completion* of a partially ordered vector space (X, K) is a vector lattice Y for which there exists a bipositive linear map $i \colon X \to Y$ such that $i(X)$ is order dense in Y and no proper vector sublattice of Y contains $i(X)$. It was shown in [6, Remark 3.2] that Y is unique up to isomorphism, and hence we may speak of *the Riesz-completion* of (X, K). Riesz completions can be used to extend and analyse vector lattice notions as disjointness, ideals, and bands in partially ordered vector spaces [3, 4]. It was shown in [6, Theorems 1.7, 3.5, 3.7, 4.13 and Corollaries 4.9–11] that, among other spaces, every Archimedean partially ordered vector space (X, K) that is directed, i.e. $X = K - K$, has a Riesz completion, but the general construction given in [6] is rather indirect.

In the setting of Theorem 1, the vector sublattice $L(X, K)$ in $C(\overline{\Lambda})$ is the Riesz completion of (X, K) if $\Phi(X)$ is order dense in $C(\overline{\Lambda})$. It is therefore of interest to ask:

Question 3. *For which Archimedean partially ordered vector space (X, K) with order unit is $\Phi(X)$ order dense in $C(\overline{\Lambda})$?*

Question 4. *Assuming $\Phi(X)$ is order dense, when is $L(X, K) = C(\overline{\Lambda})$?*

In this note we present some partial answers to these questions and illustrate the usefulness of this representation by characterising the bands in spaces with polyhedral cones.

2 Order dense embeddings

To analyse Question 3 we introduce some additional notions. A point $\varphi \in \Sigma$ is called *exposed* if there exists $x \in K$ with $\varphi(x) = 0$ and $\psi(x) > 0$ for all $\psi \in \Sigma \setminus \{\varphi\}$. Observe that the exposed points in Σ are contained in Λ. We say that $\varphi \in \overline{\Lambda}$ is *approximately exposed* if for every open neighborhood U of φ in $\overline{\Lambda}$ there exists $x \in K$ such that

$$\varphi(x) = 0 \text{ and } \psi(x) > 0 \text{ for all } \psi \in \overline{\Lambda} \setminus U.$$

Clearly, every exposed point of Σ is approximately exposed.

Theorem 5. *Let (X, K) be an Archimedean partially ordered vector space with order unit u. If the approximately exposed points are dense in $\overline{\Lambda}$, then $\Phi(X)$ is order dense in $C(\overline{\Lambda})$.*

Proof. Let Λ' denote the set of all approximately exposed points in $\overline{\Lambda}$.

First, we show that for every $f \in C(\overline{\Lambda})$, $\varphi_0 \in \Lambda'$ and $\varepsilon > 0$ there exists $g \in \Phi(X)$ such that $g(\varphi_0) = f(\varphi_0) + \varepsilon$ and $g \geq f$ on $\overline{\Lambda}$.

Indeed, choose an open set $U \subseteq \overline{\Lambda}$ with $\varphi_0 \in U$ such that $f(\varphi) < f(\varphi_0) + \varepsilon$ for all $\varphi \in U$. By definition of Λ' there exists $x \in K$ such that $\varphi_0(x) = 0$ and $\psi(x) > 0$ for all $\psi \in \overline{\Lambda} \setminus U$. As $\overline{\Lambda} \setminus U$ is weak-$\star$ compact and x is weak-$\star$ continuous, $\alpha := \min\{\psi(x) \colon \ \psi \in \overline{\Lambda} \setminus U\} > 0$. Define

$$g(\varphi) = \frac{\|f\|_\infty - f(\varphi_0)}{\alpha}\varphi(x) + f(\varphi_0) + \varepsilon \quad \text{for } \varphi \in \overline{\Lambda},$$

that is, $g = \Phi(\alpha^{-1}(\|f\|_\infty - f(\varphi_0))x + (f(\varphi_0) + \varepsilon)u)$. Hence $g \in \Phi(X)$ and $g(\varphi_0) = f(\varphi_0) + \varepsilon$. For $\varphi \in U$ one has $g(\varphi) \geq f(\varphi_0) + \varepsilon > f(\varphi)$. For $\varphi \in \overline{\Lambda} \setminus U$ one obtains

$$g(\varphi) \geq \frac{\|f\|_\infty - f(\varphi_0)}{\alpha}\alpha + f(\varphi_0) + \varepsilon > \|f\|_\infty \geq f(\varphi).$$

Consequently, $g \geq f$.

As a second step, let $h \in C(\overline{\Lambda})$; we show that $h = \inf A$, where $A = \{h' \in \Phi(X) \colon \ h' \geq h\}$. Suppose that h is not the greatest lower bound of A. Then there exists $f' \in C(\overline{\Lambda})$ with $h' \geq f'$ for all $h' \in A$, but $f' \not\leq h$. Since $C(\overline{\Lambda})$ is a vector lattice, we can consider the element $f = f' \vee h$, which satisfies $f \geq h$, $f \neq h$ and $h' \geq f$ for all $h' \in A$. Since Λ' is dense in $\overline{\Lambda}$, we can choose $\varphi_0 \in \Lambda'$ such that $f(\varphi_0) > h(\varphi_0)$. Choose $0 < \varepsilon < f(\varphi_0) - h(\varphi_0)$. By the first step of the proof, there exists $g \in \Phi(X)$ such that $g(\varphi_0) = f(\varphi_0) + \varepsilon$ and $g \geq f$ on $\overline{\Lambda}$. Then $g \geq h$, so $g \in A$. On the other hand, $g(\varphi_0) < h(\varphi_0) + \varepsilon < f(\varphi_0)$, which is a contradiction.

We conclude that $\Phi(X)$ is order dense in $C(\overline{\Lambda})$. □

Clearly, if (X, K) is an Archimedean partially ordered vector space with order unit and the exposed points of Σ are weak-$\star$ dense in Λ, then it follows from Theorem 5 that $\Phi(X)$ is order dense in $C(\overline{\Lambda})$. According to Straszewicz's theorem, see [11], the exposed points of a compact convex set S in $\mathbb{R}^n$ are dense in the set of extreme points of S. Thus we obtain the following consequence of Theorem 5.

Corollary 6. *Let K be a closed cone in $\mathbb{R}^n$ with order unit u. Then $\Phi(\mathbb{R}^n)$ is order dense in $C(\overline{\Lambda})$.*

Remark 7. If X is infinite dimensional, finding conditions under which the exposed points are dense in the extreme points is much more delicate. This problem has been addressed by Klee [11]. Unfortunately, his results are not amenable to our setting.

We also like to point out that at present we do not know if approximately exposed is equivalent to exposed. However, it is easy to show that the notions coincide when $\overline{\Lambda}$ is first countable. Indeed, in that case, each $\varphi \in \overline{\Lambda}$ has a countable neighbourhood base $\{U_i \colon i = 1, 2, \ldots\}$. For each i there exist $x_i \in K$ with $\|x_i\|_u = 1$ such that $\varphi(x_i) = 0$ and $\psi(x_i) > 0$ for all $\psi \in \overline{\Lambda} \setminus U_i$. Now let $x = \sum_i 2^{-i} x_i$ and note that $\varphi(x) = 0$ and $\psi(x) > 0$ for all $\psi \neq \varphi$, as $\overline{\Lambda}$ is Hausdorff. In particular, if $(X, \|\cdot\|_u)$ is separable, then $\overline{\Lambda} \subseteq B^*$ is metrizable, and hence first countable.

Example 8. (Lorentz cones) Let H be a (possibly infinite dimensional) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and define $X = \mathbb{R} \times H$. The *Lorentz cone* in X is given by

$$L_H = \{(r, u) \in \mathbb{R} \times H \colon r^2 - \langle u, u \rangle \geq 0 \text{ and } r \geq 0\}.$$

In case $H = \mathbb{R}^2$ we get the familiar ice-cream cone. On X we consider the inner product $\langle x \mid y \rangle = rs + \langle u, v \rangle$, where $x = (r, u)$ and $y = (s, v)$, which turns X into a Hilbert space. With respect to this inner product L_H is self-dual. To show this consider $y = (s, v) \in L_H$. For each $x = (r, u) \in L_H$ we have, by Cauchy-Schwartz, that

$$\langle x \mid y \rangle = rs + \langle u, v \rangle \geq rs - \|u\| \|v\| \geq 0. \tag{2}$$

Thus, $y \in L_H^*$. (Here L_H^* denotes the dual of L_H with respect to the Hilbert norm.) Conversely, if $y = (s, v) \in L_H^*$, then $s \geq 0$. If $v = 0$, then clearly

$y \in L_H$. If $v \neq 0$, then consider $x = (r, u)$, where $r = \|v\|$ and $u = -v$. As $r^2 - \langle u, u \rangle = 0$, $x \in L_H$, and hence $\langle x \mid y \rangle \geq 0$. This implies that

$$\langle x \mid y \rangle = rs - \langle v, v \rangle = \|v\|s - \|v\|^2 \geq 0,$$

so that $s - \|v\| \geq 0$, which shows that $y \in L_H$.

Note that L_H has non-empty interior. In fact, it contains the point $e = (1, 0)$. To see this remark that $\langle e - (r, v) \mid e - (r, v) \rangle < 1/4$ implies $(1 - r)^2 + \|v\|^2 < 1/4$, which gives $1/2 < r < 3/2$ and $\|v\| < 1/2$. Therefore, $r^2 - \|v\|^2 > 0$ and hence $(r, v) \in L_H$. The point e is an order unit for L_H.

As L_H has non-empty interior, it follows from [14, Corollary 2.17] that every positive functional on L_H is continuous under the Hilbert norm on X. This implies that $K^* = L_H^* = L_H$, so that $\Sigma = \{(r, u) \in L_H \colon r = 1\}$. Clearly the extreme points of Σ are contained in

$$\{(1, u) \in L_H \colon \|u\| = 1\}.$$

We claim that each point $z = (1, w)$ with $\|w\| = 1$ is extreme. To prove this let $x = (1, u)$ and $y = (1, v)$ be such that $u \neq v$, $\|u\| \leq 1$ and $\|v\| \leq 1$. For the sake of contradiction suppose that $z = \lambda x + (1 - \lambda)y$ with $0 < \lambda < 1$. As

$$1 - \|\lambda u + (1 - \lambda)v\|^2 > 1 - (\lambda\|u\| + (1 - \lambda)\|v\|)^2 \geq 1 - (\lambda + (1 - \lambda))^2 = 0,$$

we find that $\|w\| < 1$, which is a contradiction. Thus, the set of extreme points is $\Lambda = \{(1, u) \in L_H \colon \|u\| = 1\}$. Now note that each extreme point is exposed. Indeed, if $x = (1, u) \in \Lambda$ and $y = (s, v) \in L_H$, then $\langle x \mid y \rangle = 0$ if and only if $v = -su$ by (2). By applying Theorem 5 we conclude that $\Phi(X)$ is order dense in $C(\overline{\Lambda})$ and the Riesz completion of (X, L_H) is $L(X, L_H)$.

In case $H = \mathbb{R}^2$, the set $\overline{\Lambda} = \Lambda$ can be identified with the unit circle S^1. Note that each $f \in \Phi(X)$ has a unique extension to an affine function on the unit disc D. From this observation it is easy to deduce that $L(\mathbb{R}^3, L_{\mathbb{R}^2}) \neq C(S^1)$. Simply consider $h \colon S^1 \to \mathbb{R}$ with

$$h(x, y) = x^2.$$

Remark that each $g \in L(\mathbb{R}^3, L_{\mathbb{R}^2})$ is the restriction of a piece-wise affine function on D to S^1. Such a function is smooth if and only if it is the restriction of an affine function on D, and hence $h \notin L(\mathbb{R}^3, L_{\mathbb{R}^2})$.

3 An extension of a theorem of Kakutani

Kakutani's famous theorem on representation of abstract M-spaces states that every Archimedean vector lattice with an order unit is isometric and as a vector lattice isomorphic to a norm dense vector sublattice of a space of continuous functions $C(\Omega)$, where Ω is a compact Hausdorff space [10, Theorem 2]. The set $\overline{\Lambda}$ that we use in Theorem 5 coincides with the set Ω in Kakutani's construction. If (X, K) is an Archimedean partially ordered vector space with order unit one can apply Kakutani's theorem to the Riesz completion of X to obtain that the Riesz completion of X is isometric and as a vector lattice isomorphic to a norm dense vector sublattice of a space $C(\Omega)$. In this way one does not know whether the underlying set Ω coincides with the set $\overline{\Lambda}$ corresponding to X. If we can apply Theorem 5 it follows that the Riesz completion of X can be embedded in $C(\overline{\Lambda})$. In fact we obtain the following extension of Kakutani's theorem.

Theorem 9. *Let (X, K) be an Archimedean partially ordered vector space with order unit u. If the approximately exposed points are dense in $\overline{\Lambda}$, then the Riesz completion of (X, K) is isometric and as a vector lattice isomorphic to a norm dense subspace of $C(\overline{\Lambda})$.*

Proof. According to Theorem 5, $\Phi(X)$ is order dense in $C(\overline{\Lambda})$ and according to Proposition 2 the vector lattice $L(X, K)$ inside $C(\overline{\Lambda})$ generated by $\Phi(X)$ is norm dense in $C(\overline{\Lambda})$. Moreover, $\Phi(X)$ is order dense in $L(X, K)$, so $L(X, K)$ is the Riesz completion of X. □

That Kakutani's theorem is indeed a consequence of Theorem 9 follows from Proposition 11 below. For its proof we need Riesz homomorphisms and Hayes's theorem.

Recall that a map $\varphi\colon (X, K) \to (Y, L)$ between two directed partially ordered vector spaces is a *Riesz homomorphism* if

$$[\varphi(\{a, b\}^u)]^l = \{\varphi(a), \varphi(b)\}^{ul} \text{ for all } a, b \in X.$$

Here we denote

$$A^u = \{u \in X\colon\ u \geq a \text{ for all } a \in A\}, \quad A^l = \{u \in X\colon\ u \leq a \text{ for all } a \in A\},$$

where A is a subset of X.

Theorem 10. (Hayes [7, Theorem 1.8.1]) *Let (X, K) be a directed partially ordered vector space and $\varphi\colon X \to \mathbb{R}$ linear and nonzero. Then φ is an extreme ray of K^* if and only if φ is a Riesz homomorphism.*

Proposition 11. *If (X, K) is an Archimedean vector lattice with an order unit, then all points of $\overline{\Lambda}$ are approximately exposed.*

Proof. Let $\varphi_0 \in \Lambda$ and let $U \subseteq \overline{\Lambda}$ be open with $\varphi_0 \in U$. We have to find $x \in K$ with $\varphi_0(x) = 0$ and $\varphi(x) > 0$ for all $\varphi \in \overline{\Lambda} \setminus U$. For any $\varphi \in \overline{\Lambda} \setminus \{\varphi_0\}$ there exists $x_\varphi \in X$ such that $\varphi_0(x_\varphi) = 0$ and $\varphi(x_\varphi) \neq 0$. Without loss of generality, $\varphi(x_\varphi) > 0$. Then $x_\varphi^+ \in K$ and

$$\varphi_0(x_\varphi^+) = \varphi_0(x_\varphi)^+ = 0,$$

as φ_0 is a Riesz homomorphism due to Hayes's theorem. Also, $\varphi(x_\varphi^+) \geq \varphi(x_\varphi) > 0$. By weak* continuity of the map $\psi \mapsto \psi(x_\varphi^+)$, there exists an open subset $U_\varphi \subseteq \overline{\Lambda}$ with $\varphi \in U_\varphi$ and $\psi(x_\varphi^+) > 0$ for all $\psi \in U_\varphi$.

As $\overline{\Lambda}\backslash U$ is compact and $\{U_\varphi \colon \varphi \in \overline{\Lambda}\backslash U\}$ is an open cover of this set, there are $\varphi_1, \ldots, \varphi_n \in \overline{\Lambda}\backslash U$ such that $\bigcup_{i=1}^n U_{\varphi_i}$ covers $\overline{\Lambda}\backslash U$. Set $x = x_{\varphi_1}^+ + \cdots + x_{\varphi_n}^+$. Thus $x \in K$, $\varphi_0(x) = 0$ and $\varphi(x) > 0$ for all $\varphi \in \overline{\Lambda} \setminus U$. □

4 The Riesz completion for polyhedral cones

In this section we show that Question 4 has a positive answer if K is a polyhedral cone with non-empty interior. We start by introducing the necessary notations. Let K be a polyhedral cone in $\mathbb{R}^n$ with non-empty interior, so, there are $y_1, \ldots, y_m \in K$ such that

$$K = \left\{ \sum_{i=1}^m \alpha_i y_i \colon \ \alpha_i \geq 0 \text{ for } i = 1, \ldots, m \right\}.$$

Rather than thinking of a polyhedral cone K as the convex hull of finitely many rays, we can think of it as the intersection of finitely many closed half-spaces. In fact, a basic result in polyhedral combinatorics says that there exist $k \geq n$ linear functionals $f_1, \ldots, f_k$ such that

$$K = \{x \in \mathbb{R}^n \colon \ f_i(x) \geq 0 \text{ for } i = 1, \ldots, k\}, \tag{3}$$

and each f_i defines a facet of K meaning, $\dim(\{x \in K \colon f_i(x) = 0\}) = n - 1$.

As K is closed, $(\mathbb{R}^n, K)$ is Archimedean. We fix an order unit by taking an interior point u of K. The dual cone K^* is polyhedral in $\mathbb{R}^n$ with extremal elements $f_1, \ldots, f_k$, which can be scaled such that $\Lambda = \{f_1, \ldots, f_k\} (= \overline{\Lambda})$. As we are working in finite dimensions, we can apply Corollary 6. So, $\Phi \colon (\mathbb{R}^n, K) \to C(\Lambda)$, where

$$\Phi(x) = (f_1(x), \ldots, f_k(x)) \quad \text{for } x \in \mathbb{R}^n, \tag{4}$$

is bipositive and $\Phi(\mathbb{R}^n)$ is order dense in $(\mathbb{R}^k, \mathbb{R}^k_+)$. It may be somewhat surprising that $L(\mathbb{R}^n, K) = (\mathbb{R}^k, \mathbb{R}^k_+)$.

Corollary 12. *If K is a polyhedral cone with k facets and non-empty interior, then $\Phi(\mathbb{R}^n)$ is order dense in $(\mathbb{R}^k, \mathbb{R}^k_+)$, and $(\mathbb{R}^k, \mathbb{R}^k_+)$ is the Riesz completion of $(\mathbb{R}^n, K)$.*

Proof. Corollary 6 yields that $\Phi(\mathbb{R}^n)$ is order dense in $(\mathbb{R}^k, \mathbb{R}^k_+)$. It remains to show that $L(\mathbb{R}^n, K) = \mathbb{R}^k$. Fix $1 \leq \ell \leq k$ and consider the unit vector $y \in \mathbb{R}^k$ with $y_i = 1$ if $i = \ell$ and $y_i = 0$ otherwise. As f_ℓ is exposed, there exists $x \in K$ such that $f_\ell(x) = 0$ and $f_i(x) > 0$ whenever $i \neq \ell$. Choose $\alpha = \min_{i \neq \ell} f_i(x)$ and $a = u - \alpha^{-1} x$. Then $\Phi(a) \vee 0 = y$. Hence the smallest Riesz subspace of $\mathbb{R}^k$ containing $\Phi(\mathbb{R}^n)$ equals $\mathbb{R}^k$. □

5 Bands in spaces with polyhedral cones

Bands are a well established notion in the theory of vector lattices and were introduced as a notion in Archimedean partially ordered vector spaces in [3]. A question which arises naturally is to characterize the bands in spaces ordered by polyhedral cones. On the one hand, such a space X can be embedded under Φ given in (4) into the standard vector lattice $\mathbb{R}^k$. In $\mathbb{R}^k$ the bands can easily be described and the question arises how these bands relate to those of X. On the other hand, if X is embedded in its Riesz completion Y, an extension property for bands was established in [4]. This is of use if the bands in Y can easily be described. The link between the embedding of X under Φ and the Riesz completion of X is now available due to Corollary 12. In this section we apply this result to obtain a characterization of bands in spaces with polyhedral cones.

We start by recalling some notions in an Archimedean partially ordered vector space (X, K). The elements $x, y \in X$ are called *disjoint*, in symbols $x \perp y$, if the set of all upper bounds of $\{x + y, x - y\}$ equals the set of all upper bounds of $\{x - y, -x + y\}$ [3]. The *disjoint complement* of a subset $B \subseteq X$ is the set $B^d = \{y \in X : x \perp y \text{ for all } x \in B\}$. A linear subspace B of X is called a *band* if $(B^d)^d = B$.

If there is a vector lattice Y and a bipositive linear map $i\colon X \to Y$ such that $i(X)$ is order dense in Y, then $x \perp y$ if and only if $i(x) \perp i(y)$ [3, Theorem 4.1]. In this setting, the extension property for bands is shown in [4, Proposition 5.12], i.e., if B is a band in X, then there is a band $\widehat{B}$ in Y such that $i(B) = \widehat{B} \cap i(X)$.

Recall that $\widehat{B}$ is a band in the vector lattice $(\mathbb{R}^k, \mathbb{R}^k_+)$ if and only if there is an $N \subseteq \{1, \ldots, k\}$ such that

$$\widehat{B} = \left\{(x_1, \ldots, x_k)^T : x_i = 0 \text{ for all } i \in N\right\}. \tag{5}$$

In the following we consider $X = \mathbb{R}^n$ equipped with a generating polyhedral cone K, where K is represented by k functionals as in (3). We will characterize the bands in X in terms of special subsets of $\{1, \ldots, k\}$. We deal with subspaces of the form

$$D = \{x \in \mathbb{R}^n \colon\ f_i(x) = 0 \text{ for all } i \in M\}, \tag{6}$$

where $M \subseteq \{1, \ldots, k\}$. The set M is then called an *index set* for D. For $M \subseteq \{1, \ldots, k\}$ denote

$$\operatorname{sat}(M) = \{j \in \{1, \ldots, k\} : \ f_j \in \operatorname{span}\{f_i \colon\ i \in M\}\}.$$

We say that the index set M is *saturated* if $M = \operatorname{sat}(M)$. If D is a subspace given by (6), then

$$D = \{x \in \mathbb{R}^n \colon\ f_i(x) = 0 \text{ for all } i \in \operatorname{sat}(M)\}.$$

An index set M is *bisaturated* if both M and $M^c = \{1, \ldots, k\} \setminus M$ are saturated. The set M is called *weakly bisaturated* if

$$M = \text{sat}\left((\text{sat}\,(M^c))^c\right).$$

Clearly, if M is bisaturated then M is weakly bisaturated.

Lemma 13. *Let D be a subspace of $\mathbb{R}^n$ of form* (6) *given by a saturated index set M. If $j \notin M$, then there exists an $x \in D$ such that $f_j(x) = 1$.*

Proof. We have that $f_j \notin \text{span}\,\{f_i \colon i \in M\}$. Choose a basis $f_{\ell_1}, \ldots, f_{\ell_p}$ of $\text{span}\,\{f_i \colon i \in M\}$. We view the functionals f_i as row vectors of length n. Consider the matrix A with the rows 1 through p equal to f_{ℓ_1} through f_{ℓ_p}, respectively, and the $(p+1)$th row equal to f_j. The matrix A has full rank $p+1$. So there exists a vector $x \in \mathbb{R}^n$ such that $(Ax)_i = 0$ for $1 \leq i \leq p$ and $(Ax)_{p+1} = 1$. Then $f_i(x) = 0$ for all $i \in M$, so $x \in D$ and $f_j(x) = 1$. ◻

Lemma 14. *If B is a band in $(\mathbb{R}^n, K)$, then there exists a unique saturated index set $M \subseteq \{1, \ldots, k\}$ such that*

$$B = \{x \in \mathbb{R}^n \colon\ f_i(x) = 0 \textit{ for all } i \in M\}. \tag{7}$$

Proof. Let Φ be the embedding given in (4). Due to Corollary 12 $\Phi(\mathbb{R}^n)$ is order dense in $\mathbb{R}^k$, so the extension property for bands holds. This means that for the band B there is a band $\widehat{B}$ in $\mathbb{R}^k$ such that $B = \{x \in X \colon\ \Phi(x) \in \widehat{B}\}$. Using (5), there exists a set $N \subseteq \{1, \ldots, k\}$ which is an index set for B. By defining $M = \text{sat}\,(N)$ we obtain the representation (7).

In order to show uniqueness of the representation, suppose there exists another saturated index set L for B such that (without loss of generality) $M \setminus L$ contains an element j. Then by Lemma 13 there exists

$$y \in \{x \in \mathbb{R}^n \colon\ f_i(x) = 0 \text{ for all } i \in L\} = B$$

such that $f_j(y) = 1$, which contradicts that M is an index set for B. ◻

Lemma 15. *Let D be a subspace of $\mathbb{R}^n$ given by a saturated index set M. Then the disjoint complement of D equals*

$$D^d = \{x \in \mathbb{R}^n \colon\ f_i(x) = 0 \textit{ for all } i \in \text{sat}\,(M^c)\}.$$

Proof. First let $x \in D^d$. We show that $f_i(x) = 0$ for all $i \in M^c$. Indeed, for any $i \in M^c$ by Lemma 13 there exists a $y \in D$ such that $f_i(y) = 1$. Then x and y are disjoint in X. Using the embedding Φ given by (4) and the fact that $\Phi(\mathbb{R}^n)$ is order dense in $\mathbb{R}^k$ due to Corollary 12, we obtain that $\Phi(x)$ and $\Phi(y)$ are disjoint in $\mathbb{R}^k$. Hence $f_i(x) = 0$.

Conversely, let $x \in X$ be such that $f_i(x) = 0$ for all $i \in M^c$. For $y \in D$ we have $f_i(y) = 0$ for all $i \in M$, so $\Phi(x)$ and $\Phi(y)$ are disjoint in $\mathbb{R}^k$, which implies that x and y are disjoint in X. Hence $x \in D^d$.

We conclude that M^c is an index set for D^d, so $\text{sat}\,(M^c)$ is an index set for D^d as well. ◻

Theorem 16. *A subset B of X is a band if and only if there exists a weakly bisaturated index set $M \subseteq \{1, \dots, k\}$ such that*

$$B = \{x \in \mathbb{R}^n \colon\ f_i(x) = 0 \textit{ for all } i \in M\}. \tag{8}$$

Proof. Let M be a weakly bisaturated index set and let B be given by (8). Then Lemma 15 yields that B^d has index set $\operatorname{sat}(M^c)$. Applying the lemma once more we obtain that B^{dd} has the index set $\operatorname{sat}((\operatorname{sat}(M^c))^c)$, which equals M. Thus $B = B^{dd}$ and B is a band.

Let B be a band in X. According to Lemma 14 there exists a saturated index set M such that (8) holds. We have to show that M is weakly bisaturated. Applying Lemma 15 twice, we obtain that $\operatorname{sat}((\operatorname{sat}(M^c))^c)$ is a saturated index set for $B^{dd} = B$. Due to the uniqueness of the representation in Lemma 14, we obtain $M = \operatorname{sat}((\operatorname{sat}(M^c))^c)$. □

Example 17. Let $X = \mathbb{R}^4$, define

$$f_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix},\ f_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix},\ f_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix},\ f_4 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix},\ f_5 = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and consider $K = \{x \in \mathbb{R}^4 \colon\ f_i(x) \geq 0 \text{ for all } i = 1, \dots, 5\}$. K^* is contained in $\mathbb{R}^4_+$, so it is a cone. Since f_1, f_2, f_3, f_5 are linearly independent, K^* is generating in $\mathbb{R}^4$, hence $K = K^{**}$ is a cone in $\mathbb{R}^4$. For $u = (1, 1, 1, 1)^T$ we have $f_i(u) = 1$ for all $i = 1, \dots, 5$, so u is an interior point of K, and we obtain

$$\Sigma = \{f \in K^* \colon\ f(u) = 1\} = \left\{ \sum_{i=1}^{5} \alpha_i f_i \colon\ \alpha_i \geq 0,\ \sum_{i=1}^{5} \alpha_i = 1 \right\}.$$

Observe that $\operatorname{span}\{f_1, f_2, f_3, f_4\}$ has dimension 3. Indeed, $f_1 + f_2 - f_3 - f_4 = 0$, and f_1, f_2, f_3 are linearly independent. A straightforward calculation shows that f_i is an extreme point of Σ for all i. We embed $(\mathbb{R}^4, K)$ into $(\mathbb{R}^5, \mathbb{R}^5_+)$ under Φ. To determine the bands in $(\mathbb{R}^4, K)$, we list the weakly bisaturated index sets: $\emptyset$, $\{5\}$, $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$, $\{1, 2, 5\}$, $\{1, 3, 5\}$, $\{1, 4, 5\}$, $\{2, 3, 5\}$, $\{2, 4, 5\}$, $\{3, 4, 5\}$, $\{1, 2, 3, 4\}$, $\{1, 2, 3, 4, 5\}$. By the formula in Theorem 16 we obtain 16 bands in $(\mathbb{R}^4, K)$.

Observe that in Example 17 an index set is weakly bisaturated if and only if it is bisaturated. We pose the following question: In which spaces with a generating polyhedral cone is every weakly bisaturated index set bisaturated?

As illustrated by Example 17, Theorem 16 provides a method to list all bands in a space with a generating polyhedral cone. The question arises whether other types of subspaces, such as directed bands, solvex ideals (as defined in [4]), etc., could be determined by a similar method. Moreover, one could think of adapting the above approach to more general partially ordered vector spaces covered by Theorem 5.

6 Positive semi-definite matrices

Let V be the vector space of $n \times n$ symmetric matrices equipped with the inner-product $\langle A, B\rangle = \mathrm{tr}\,(AB)$. Let Pos_n be the cone of positive semi-definite matrices in V. It is well-known [2] that Pos_n is self-dual in V and that the identity matrix I is an order unit. Thus,

$$\Sigma = \{A \in \mathrm{Pos}_n : \mathrm{tr}\,(A) = 1\}$$

in this case.

The following lemma characterises the extreme points of Λ.

Lemma 18. *The set Λ consisting of the extreme points of Σ is equal to the set Δ consisting of those $A \in \Sigma$ that have exactly 1 non-zero eigenvalue, which is equal to 1.*

Proof. First note that if $A \in \Sigma$, then by the spectral theorem there exists an orthogonal matrix S such that

$$A = S^T \mathrm{Diag}(\lambda_1, \ldots, \lambda_n) S = \sum_i \lambda_i S^T \mathrm{Diag}(e_i) S,$$

where $\lambda_n \geq \lambda_{n-1} \geq \ldots \geq \lambda_1 \geq 0$ are the eigenvalues of A, $\sum_i \lambda_i = 1$, and e_i is the i-th unit vector. Thus, Λ is contained in Δ. To see that each $A \in \Delta$ is an extreme point, we remark that the group of orthogonal matrices, $O(n)$, acts transitively on Δ by conjugation, i.e., for $A, B \in \Delta$ there exists $Q \in O(n)$ such that $Q^T A Q = B$. For each $Q \in O(n)$ the map $A \mapsto Q^T A Q$ is an invertible linear map on V which maps Σ onto itself. So, it maps extreme points of Σ to extreme points of Σ. As Λ is a non-empty subset of Δ and $O(n)$ acts transitively on Δ, it follows that each $A \in \Delta$ is an extreme point of Σ. Thus, $\Lambda = \Delta$. □

Let S^{n-1} be the $(n-1)$-dimensional sphere and consider the equivalence relation $\sim$ in S^{n-1} given by $q \sim p$ if $q = p$ of $q = -p$.

Lemma 19. *The map $[q] \in S^{n-1}/\sim\ \mapsto qq^T \in \Lambda$ is a bijection.*

Proof. If $p, q \in S^{n-1}$ and $pp^T = qq^T$, then $p = qq^T p$, so p is a multiple of q, consequently $p = q$ or $p = -q$. Hence, the map is injective.

To show that it is surjective, let $A \in \Lambda$ and $E = \mathrm{Diag}(1, 0, \ldots, 0)$. By the spectral theorem there exists $Q = (q_1 q_2 \cdots q_n) \in O(n)$, where $q_1, \ldots, q_n$ denote the columns of Q, such that $Q^T A Q = E$, so $A = QEQ^T = q_1 q_1^T$, while $q_1 \in S^{n-1}$. □

Remark that for $q \in S^{n-1}$ the matrix $A = qq^T$ is an orthogonal projection matrix with as range the span of q.

It is known that there is very little lattice structure present in (V, Pos_n). In fact, Kadison [9] proved that (V, Pos_n) is an anti-lattice, meaning that A

and B have a greatest lower bound if and only if A and B are comparable. By means of the Riesz completion we now show that there are no non-trivial disjoint elements in (V, Pos_n). The functional representation serves as a tool for the calculation.

Proposition 20. *There are no non-trivial disjoint elements in* (V, Pos_n).

Proof. Let $A, B \in V \setminus \{0\}$ and note that

$$\begin{aligned} A \perp B &\iff \Phi(A) \perp \Phi(B) \\ &\iff |\mathrm{tr}\,(AC)| \wedge |\mathrm{tr}\,(BC)| = 0 \quad \text{for all } C \in \Lambda \\ &\iff |\mathrm{tr}\,((Aq)q^\top)| \wedge |\mathrm{tr}\,((Bq)q^\top)| = 0 \quad \text{for all } q \in S^{n-1} \\ &\iff |\langle Aq, q\rangle| \wedge |\langle Bq, q\rangle| = 0 \quad \text{for all } q \in S^{n-1}. \end{aligned}$$

As $A \neq 0$ and symmetric, it follows from the spectral theorem that A has an eigenvalue $\lambda \neq 0$ with a normalised eigenvector s. Likewise B has an eigenvalue $\mu \neq 0$ with normalised eigenvector t. Write $q = \alpha s + \beta t$ with $\alpha, \beta \in \mathbb{R}$. Then

$$\langle Aq, q\rangle = \langle \lambda\alpha s + \beta At, \alpha s + \beta t\rangle = \lambda\alpha^2 + \alpha\beta\langle At, s\rangle + \lambda\alpha\beta\langle s, t\rangle + \beta^2\langle At, t\rangle$$

and

$$\langle Bq, q\rangle = \mu\beta^2 + \alpha\beta\langle Bs, t\rangle + \mu\alpha\beta\langle s, t\rangle + \alpha^2\langle Bs, s\rangle.$$

If $\langle Bs, s\rangle \neq 0$ then $|\langle As, s\rangle| \wedge |\langle Bs, s\rangle| \neq 0$. If $\langle Bs, s\rangle = 0$, we can take $\alpha = 1/\beta > 0$. Then

$$|\langle Aq, q\rangle| = |\lambda\alpha^2 + \langle At, s\rangle + \lambda\langle s, t\rangle + \frac{1}{\alpha^2}\langle At, t\rangle| \to \infty \quad \text{as } \alpha \to \infty,$$

and

$$\langle Bq, q\rangle = \frac{\mu}{\alpha^2} + \langle Bs, t\rangle + \mu\langle s, t\rangle \neq 0 \quad \text{for } \alpha \text{ large}.$$

Therefore, $|\langle Aq, q\rangle| \wedge |\langle Bq, q\rangle| \neq 0$ for some $q \in \mathbb{R}^n$. Thus A and B are not disjoint. □

7 Further Examples

The purpose of the following (artificial) example is to illustrate that, in general, for $f \in C\left(\overline{\Lambda}\right)$ the greatest lower bound $\inf\{g \in \Phi(X) \colon g \geq f\}$ need not be the pointwise infimum.

Example 21. Let $D \subset \mathbb{R}^3$ be the convex hull of $\{(\cos\vartheta, \sin\vartheta, 0) \colon 0 \leq \vartheta \leq 2\pi\}$ and $\{(1,0,1),(1,0,-1)\}$. Note that the set of extreme points of D is not closed, as it does not contain $(1,0,0)$. Define $D' = \{(x_1,\ldots,x_4) \in \mathbb{R}^4 \colon (x_1,x_2,x_3) \in D \text{ and } x_4 = 1\}$ and let C be the cone generated by D', i.e.,

$$C = \{\lambda x \in \mathbb{R}^4 \colon \lambda \geq 0 \text{ and } x \in D'\}.$$

Denote the dual of C by K, and remark that K is a closed cone in $\mathbb{R}^4$ with non-empty interior such that $K^* = C^{**} = C$. The point $u = (0,0,0,1)$ satisfies

$$\langle x, u\rangle > 0 \text{ for all } x \in C \setminus \{0\}.$$

This implies that u is in the interior of K and hence u is an order unit for K. Moreover,

$$\Sigma = \{y \in C \colon \langle y, u\rangle = 1\} = D'.$$

For $0 \leq \vartheta \leq 2\pi$, let $\varphi_\vartheta = (\cos\vartheta, \sin\vartheta, 0, 1)$. Remark that Λ consists of $\eta_1 = (1,0,1,1)$, $\eta_{-1} = (1,0,-1,1)$, and φ_ϑ for $0 < \vartheta < 2\pi$.

Now let $f \in C\left(\overline{\Lambda}\right)$ be given by $f(\varphi_\vartheta) = -1$ for $0 \leq \vartheta \leq 2\pi$, and $f(\eta_{\pm 1}) = 0$. Clearly, for each $x \in \mathbb{R}^4$ we have that

$$\Phi(x)(\varphi_0) = \langle x, \varphi_0\rangle = \tfrac{1}{2}(\langle x, \eta_1\rangle + \langle x, \eta_{-1}\rangle) \geq \tfrac{1}{2}(f(\eta_1) + f(\eta_{-1})) \geq 0.$$

Thus, the point-wise infimum of $\{g \in \Phi(X) \colon g \geq f\}$ is not equal to f.

The next example presents an infinite dimensional space X where the answer to Question 4 is positive. In [5, Example 2.2] the Riesz completion of the space has been computed. Here we determine the functional embedding.

Example 22. Consider the vector space

$$c = \{(x_n) = (x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}} \colon \lim_{n\to\infty} x_n \text{ exists}\}$$

ordered by its natural cone c^+. Denote by e the constant 1 sequence and by e_n the unit vectors with $(e_n)_i = 1$ if $i = n$ and 0 otherwise. Let

$$\varphi(x) = \sum_{n=1}^{\infty} 2^{-n} x_n - \lim_{n\to\infty} x_n, \quad x = (x_n) \in c,$$

and let $K = \{x \in c^+ \colon \varphi(x) \geq 0\}$. Then (X, K) is an Archimedean partially ordered vector space. The element $u = e + 2e_1$ is an order unit in (X, K), as $-\lambda u \leq x \leq \lambda u$ in (X, K) whenever $x \in X$ and $\lambda = \sup_n |x_n| + |\varphi(x)|$. We will show that $L(X, K)$ equals $C(\overline{\Lambda})$.

The norm on X induced by u is equivalent to the usual supremum norm and therefore the dual space is as Banach space isomorphic to $\mathbb{R} \times l^1$, where $\psi = (\alpha_0, \alpha_1, \ldots) \in \mathbb{R} \times l^1$ acts on $x = (x_1, x_2, \ldots) \in c$ by

$$\psi(x) = \sum_{n=1}^{\infty} \alpha_n x_n + \alpha_0 \lim_{n\to\infty} x_n.$$

First we show that $\psi = (\alpha_0, \alpha_1, \ldots)$ is in K^* if and only if

$$\alpha_n \geq 0 \text{ and } 2^n \alpha_n + \alpha_0 \geq 0 \text{ for all } n \geq 1. \tag{9}$$

If $\psi = (\alpha_0, \alpha_1, \dots)$ satisfies (9) and $x \in K$, then $\psi(x) \geq 0$ if $\alpha_0 \geq 0$, and if $\alpha_0 < 0$ then

$$\psi(x) = \sum_{n=1}^{\infty} \alpha_n x_n + \alpha_0 \lim_{n\to\infty} x_n \geq -\sum_{n=1}^{\infty} 2^{-n} \alpha_0 x_n + \alpha_0 \lim_{n\to\infty} x_n \geq 0,$$

so that $\psi \in K^*$. If $\psi = (\alpha_0, \alpha_1, \dots) \in K^*$, then $\psi(e_n) \geq 0$, so $\alpha_n \geq 0$ for $n \geq 1$ and $\psi((u - \sum_{k=1}^{m} e_k) + 2^n e_n) \geq 0$ for all m, so $2^n \alpha_n + \alpha_0 \geq 0$ for each $n \geq 1$.

Next we show that

$$\overline{\Lambda} = \{\psi_n \colon\, n \geq 0\} \cup \{\varphi\},$$

where $\psi_n \in K^*$ are given for $x = (x_n) \in X$ by $\psi_0(x) = \lim_n x_n$, $\psi_1(x) = \frac{x_1}{3}$, $\psi_n(x) = x_n$ for $n \geq 2$. Note that $\psi_n \in \Sigma$ for all $n \geq 0$ and $\varphi \in \Sigma$. For each $n \geq 1$ the point ψ_n is exposed in Σ and hence in Λ. Indeed, take $x = (x_k) \in X$ such that $x_n = 0$, $x_k > 0$ for $k \neq n$, and $0 < \lim_{k\to\infty} x_k < \sum_{k=1}^{\infty} 2^{-k} x_k$. Then $\psi_n(x) = 0$ and if $\psi = (\alpha_0, \alpha_1, \dots) \in \Sigma$ is such that $\psi \neq \psi_n$, then $\alpha_k > 0$ for some $k \neq n$, so $\psi(x) = \sum_{k=1}^{\infty} \alpha_k x_k + \alpha_0 \lim_k x_k > 0$. Also φ is exposed in Σ, since $\varphi(e) = 0$ and if $\psi = (\alpha_0, \alpha_1, \dots) \in \Sigma$ then $\psi(e) = \sum_{k=0}^{\infty} \alpha_k > \alpha_0 - \sum_{k=1}^{\infty} 2^{-k} \alpha_0 = 0$ or $2^k \alpha_k = -\alpha_0$ for all k, which yields that $\psi = -\alpha_0 \varphi$ hence $\psi = \varphi$ as $\psi(u) = \varphi(u) = 1$.

If $\psi = (\alpha_0, \alpha_1, \dots) \in K^*$, then if $\alpha_0 \geq 0$, we consider

$$\psi(x) = \sum_{k=1}^{\infty} \alpha_k \psi_k(x) + \alpha_0 \lim_k \psi_k(x) \text{ for all } x \in X$$

and if $\alpha_0 < 0$,

$$\psi(x) = \sum_{k=1}^{\infty} (\alpha_k + 2^{-k} \alpha_0) \psi_k(x) - \alpha_0 \varphi(x) \text{ for all } x \in X,$$

so ψ is contained in the weak-$\star$ closure of the positive linear combinations of ψ_n, $n \geq 0$, and φ. So we have determined $\overline{\Lambda}$.

The topology of $\overline{\Lambda}$ is such that ψ_0 is the only limit point and it is the limit of ψ_n for $n \to \infty$. Indeed, if $n \geq 1$ then $\psi_n(e_n) = 1$ and $\psi_m(e_n) = 0$ for $m \neq n$ and e_n is a weak-$\star$ continuous functional on X^*, so none of the ψ_n is a limit point of the others for $n \geq 1$. Further, $\psi_0(x) = \lim_n \psi_n(x)$ for all $x \in X$, so ψ_0 is the limit of ψ_n as $n \to \infty$, and $\varphi(e) = 0$ whereas $\psi_n(e) \geq 1/3$ for $n \geq 1$, so φ is not a limit point of $\{\psi_n \colon\, n \geq 1\}$.

Therefore $\overline{\Lambda}$ is homeomorpic to $T = \{1/n \colon\, n \geq 1\} \cup \{0\} \cup \{2\}$, if we let ψ_n correspond to $1/n$, ψ_0 to 0 and φ to 2. Then $C(\overline{\Lambda})$ is naturally isomorphic to $c \times \mathbb{R}$ and

$$\Phi(x) = ((x_n), \varphi(x)), \quad x \in X.$$

Due to Theorem 5, $\Phi(X)$ is order dense in $c \times \mathbb{R}$. As

$$\{f \in C(\overline{\Lambda})\colon\ (f, r) \in \Phi(X) \text{ for some } r \in \mathbb{R}\} = c$$

and $L(X, K)$ should be a strictly larger subspace of $c \times \mathbb{R}$ than $\Phi(X)$, we obtain that $L(X, K) = c \times \mathbb{R}$. So the Riesz completion of (X, K) equals $c \times \mathbb{R}$.

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