Large deviation principle for one-dimensional random walk in dynamic random environment: attractive spin-flips and simple symmetric exclusion

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Abstract

Consider a one-dimensional shift-invariant attractive spin-flip system in equilibrium, constituting a dynamic random environment, together with a nearest-neighbor random walk that on occupied sites has a local drift to the right but on vacant sites has a local drift to the left. In [2] we proved a law of large numbers for dynamic random environments satisfying a space-time mixing property called cone-mixing. If an attractive spin-flip system has a finite average coupling time at the origin for two copies starting from the all-occupied and the all-vacant configuration, respectively, then it is cone-mixing.

In the present paper we prove a large deviation principle for the empirical speed of the random walk, both quenched and annealed, and exhibit some properties of the associated rate functions. Under an exponential space-time mixing condition for the spin-flip system, which is stronger than cone-mixing, the two rate functions have a unique zero, i.e., the slow-down phenomenon known to be possible in a static random environment does not survive in a fast mixing dynamic random environment. In contrast, we show that for the simple symmetric exclusion dynamics, which is not cone-mixing (and which is not a spin-flip system either), slow-down does occur.

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1 Introduction and main results

1.1 Random walk in dynamic random environment: attractive spin-flips

Let

$$\xi = (\xi_t)_{t \geq 0} \quad \text{with} \quad \xi_t = \{\xi_t(x) \colon\ x \in \mathbb{Z}\} \tag{1.1}$$

denote a one-dimensional *spin-flip system*, i.e., a Markov process on state space $\Omega = \{0,1\}^{\mathbb{Z}}$ with generator L given by

$$(Lf)(\eta) = \sum_{x \in \mathbb{Z}} c(x,\eta)[f(\eta^x) - f(\eta)], \qquad \eta \in \Omega, \tag{1.2}$$

where f is any cylinder function on Ω, $c(x,\eta)$ is the local rate to flip the spin at site x in the configuration η, and η^x is the configuration obtained from η by flipping the spin at site x. We think of $\xi_t(x) = 1$ ($\xi_t(x) = 0$) as meaning that site x is occupied (vacant) at time t. We assume that ξ is *shift-invariant*, i.e., for all $x \in \mathbb{Z}$ and $\eta \in \Omega$,

$$c(x,\eta) = c(x+y, \tau_y \eta), \qquad y \in \mathbb{Z}, \tag{1.3}$$

where $(\tau_y \eta)(z) = \eta(z-y)$, $z \in \mathbb{Z}$, and also that ξ is *attractive*, i.e., if $\eta \leq \zeta$, then, for all $x \in \mathbb{Z}$,

$$\begin{aligned} c(x,\eta) &\leq c(x,\zeta) \quad \text{if} \quad \eta(x) = \zeta(x) = 0, \\ c(x,\eta) &\geq c(x,\zeta) \quad \text{if} \quad \eta(x) = \zeta(x) = 1. \end{aligned} \tag{1.4}$$

For more on shift-invariant attractive spin-flip systems, see [23], Chapter III. Examples are the (ferromagnetic) Stochastic Ising Model, the Voter Model, the Majority Vote Process and the Contact Process.

We assume that

$$\xi \text{ has an equilibrium } \mu \text{ that is shift-invariant and shift-ergodic.} \tag{1.5}$$

For $\eta \in \Omega$, we write P^η to denote the law of ξ starting from $\xi(0) = \eta$, which is a probability measure on path space $D_\Omega[0,\infty)$, the space of càdlàg paths in Ω. We further write

$$P^\mu(\cdot) = \int_\Omega P^\eta(\cdot)\, \mu(\mathrm{d}\eta) \tag{1.6}$$

to denote the law of ξ when $\xi(0)$ is drawn from μ. We further assume that

$$P^\mu \text{ is tail trivial.} \tag{1.7}$$

Conditional on ξ, let

$$X = (X_t)_{t \geq 0} \tag{1.8}$$

be the random walk with local transition rates

$$\begin{aligned} x \to x+1 \quad &\text{at rate} \quad \alpha\, \xi_t(x) + \beta\, [1 - \xi_t(x)], \\ x \to x-1 \quad &\text{at rate} \quad \beta\, \xi_t(x) + \alpha\, [1 - \xi_t(x)], \end{aligned} \tag{1.9}$$

where w.l.o.g.

$$0 < \beta < \alpha < \infty. \tag{1.10}$$

In words, on occupied sites the random walk jumps to the right at rate α and to the left at rate β, while at vacant sites it does the opposite. Note that, by (1.10), on occupied sites the drift is positive, while on vacant sites it is negative. Also note that the sum of the jump rates is $\alpha+\beta$ and is independent of ξ. For $x \in \mathbb{Z}$, we write P_0^ξ to denote the law of X starting from $X(0)=0$ conditional on ξ, and

$$\mathbb{P}_{\mu,0}(\cdot) = \int_{D_\Omega[0,\infty)} P_0^\xi(\cdot)\, P^\mu(\mathrm{d}\xi) \tag{1.11}$$

to denote the law of X averaged over ξ. We refer to P_0^ξ as the *quenched* law and to $\mathbb{P}_{\mu,0}$ as the *annealed* law.

1.2 Large deviation principles

In [2] we proved that if ξ is cone-mixing, then X satisfies a law of large numbers (LLN), i.e., there exists a $v \in \mathbb{R}$ such that

$$\lim_{t\to\infty} t^{-1} X_t = v \qquad \mathbb{P}_{\mu,0} - a.s. \tag{1.12}$$

All attractive spin-flip systems for which the coupling time at the origin, starting from the configurations $\eta \equiv 1$ and $\eta \equiv 0$, has finite mean are cone-mixing. Theorems 1.1–1.2 below state that X satisfies both an annealed and a quenched large deviation principle (LDP); the interval K in (1.15) and (1.18) can be either open, closed or half open and half closed.

Define

$$\begin{aligned} M &= \sum_{x\neq 0} \sup_{\eta\in\Omega} |c(0,\eta) - c(0,\eta^x)|, \\ \epsilon &= \inf_{\eta\in\Omega} |c(0,\eta) + c(0,\eta^0)|. \end{aligned} \tag{1.13}$$

The interpretation of (1.13) is that M is a measure for the *maximal* dependence of the transition rates on the states of single sites, while ϵ is a measure for the *minimal* rate at which the states of single sites change. See [23], Section I.4, for examples. In [2] we showed that if $M < \epsilon$ then ξ is cone-mixing.

Theorem 1.1. **[Annealed LDP]** *Assume* (1.3–1.5).
(a) There exists a convex rate function $I^{\mathrm{ann}}\colon \mathbb{R} \to [0,\infty)$*, satisfying*

$$I^{\mathrm{ann}}(\theta) \begin{cases} = 0, & \text{if } \theta \in [v_-^{\mathrm{ann}}, v_+^{\mathrm{ann}}], \\ > 0, & \text{if } \theta \in \mathbb{R}\backslash[v_-^{\mathrm{ann}}, v_+^{\mathrm{ann}}], \end{cases} \tag{1.14}$$

for some $-(\alpha-\beta) \le v_-^{\mathrm{ann}} \le v \le v_+^{\mathrm{ann}} \le \alpha-\beta$*, such that*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\mu,0}\big(t^{-1}X_t \in K\big) = -\inf_{\theta\in K} I^{\mathrm{ann}}(\theta) \tag{1.15}$$

for all intervals K *such that either* $K \subsetneq [v_-^{\mathrm{ann}}, v_+^{\mathrm{ann}}]$ *or* $\mathrm{int}(K) \ni v$.
(b) $\lim_{|\theta|\to\infty} I^{\mathrm{ann}}(\theta)/|\theta| = \infty$.
(c) If $M < \epsilon$ *and* $\alpha - \beta < \frac{1}{2}(\epsilon - M)$*, then*

$$v_-^{\mathrm{ann}} = v = v_+^{\mathrm{ann}}. \tag{1.16}$$

Theorem 1.2. [Quenched LDP] *Assume* (1.3–1.5) *and* (1.7).
(a) There exists a convex rate function $I^{\rm que}\colon\,\mathbb{R}\to[0,\infty)$, *satisfying*

$$I^{\rm que}(\theta)\ \begin{cases} =0, & \text{if } \theta\in[v_-^{\rm que},v_+^{\rm que}],\\ >0, & \text{if } \theta\in\mathbb{R}\backslash[v_-^{\rm que},v_+^{\rm que}],\end{cases} \tag{1.17}$$

for some $-(\alpha-\beta)\leq v_-^{\rm que}\leq v\leq v_+^{\rm que}\leq\alpha-\beta$, *such that*

$$\lim_{t\to\infty}\frac{1}{t}\log P^\xi\big(t^{-1}X_t\in K\big)=-\inf_{\theta\in K}I^{\rm que}(\theta)\qquad \xi-a.s. \tag{1.18}$$

for all intervals K.
(b) $\lim_{|\theta|\to\infty}I^{\rm que}(\theta)/|\theta|=\infty$ *and*

$$I^{\rm que}(-\theta)=I^{\rm que}(\theta)+\theta(2\rho-1)\log(\alpha/\beta),\qquad \theta\geq 0. \tag{1.19}$$

(c) If $M<\epsilon$ *and* $\alpha-\beta<\frac{1}{2}(\epsilon-M)$, *then*

$$v_-^{\rm que}=v=v_+^{\rm que}. \tag{1.20}$$

Theorems 1.1 and 1.2 are proved in Sections 2 and 3, respectively. We are not able to show that (1.15) holds for all closed intervals K, although we expect this to be true in general.

Because

$$I^{\rm que}\geq I^{\rm ann}, \tag{1.21}$$

Theorems 1.2(b–c) follow from Theorems 1.1(b–c), with the exception of the symmetry relation (1.19). There is no symmetry relation analogous to (1.19) for $I^{\rm ann}$. It follows from (1.21) that

$$v_-^{\rm ann}\leq v_-^{\rm que}\leq v\leq v_+^{\rm que}\leq v_+^{\rm ann}. \tag{1.22}$$

1.3 Random walk in dynamic random environment: simple symmetric exclusion

It is natural to ask whether in a dynamic random environment the rate functions always have a unique zero. The answer is no. In this section we show that when ξ is the *simple symmetric exclusion process* in equilibrium with an arbitrary density of occupied sites $\rho\in(0,1)$, then for any $0<\beta<\alpha<\infty$ the probability that X_t is near the origin decays slower than exponential in t. Thus, slow-down is possible not only in a static random environment (see Section 1.4), but also in a dynamic random environment, provided it is not fast mixing. Indeed, the simple symmetric exclusion process is not even cone-mixing.

The one-dimensional simple symmetric exclusion process

$$\xi=\{\xi_t(x)\colon\ x\in\mathbb{Z},\,t\geq 0\} \tag{1.23}$$

is the Markov process on state space $\Omega=\{0,1\}^{\mathbb{Z}}$ with generator L given by

$$(Lf)(\eta)=\sum_{\substack{x,y\in\mathbb{Z}\\ x\sim y}}[f(\eta^{xy})-f(\eta)],\qquad \eta\in\Omega, \tag{1.24}$$

where f is any cylinder function on $\mathbb{R}$, the sum runs over unordered neighboring pairs of sites in $\mathbb{Z}$, and η^{xy} is the configuration obtained from η by interchanging the states at sites x and

y. We will asume that ξ starts from the Bernoulli product measure with density $\rho \in (0,1)$, i.e., at time $t = 0$ each site is occupied with probability ρ and vacant with probability $1-\rho$. This measure, which we denote by ν_ρ, is an equilibrium for the dynamics (see [23], Theorem VIII.1.44).

Conditional on ξ, the random walk

$$X = (X_t)_{t\geq 0} \tag{1.25}$$

has the same local transition rates as in (1.9–1.10). We also retain the definition of the quenched law P_0^ξ and the annealed law $\mathbb{P}_{\nu_\rho,0}$, as in (1.11) with $\mu = \nu_\rho$.

Since the simple symmetric exclusion process is *not* cone-mixing (the space-time mixing property assumed in [2]), we do not have the LLN. Since it is *not* an attractive spin-flip system either, we also do not have the LDP. We plan to address these issues in future work. Our main result here is the following.

Theorem 1.3. *For all $\rho \in (0,1)$,*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\nu_\rho,0}\big(|X_t| \leq 2\sqrt{t\log t}\big) = 0. \tag{1.26}$$

Theorem 1.3 is proved in Section 4.

1.4 Discussion

Literature. Random walk in *static* random environment has been an intensive research area since the early 1970's. One-dimensional models are well understood. In particular, recurrence vs. transience criteria, laws of large numbers and central theorems have been derived, as well as quenched and annealed large deviation principles. In higher dimensions a lot is known as well, but some important questions still remain open. For an overview of these results, we refer the reader to [33, 34] and [30]. See the homepage of Firas Rassoul-Agha [WWW.MATH.UTAH.EDU/~FIRAS/RESEARCH] for an up-to-date list of references.

For random walk in *dynamic* random environment the state of the art is rather more modest, even in one dimension. Early work was done in [24], which considers a one-dimensional environment consisting of spins flipping independently between -1 and $+1$, and a walk that at integer times jumps left or right according to the spin it sees at that time. A necessary and sufficient criterion for recurrence is derived, as well as a law of large numbers.

Three classes of models have been studied in the literature so far:

(1) *Space-time random environment*: globally updated at each unit of time [12, 13, 14, 4, 25, 32];

(2) *Markovian random environment*: independent in space and locally updated according to a single-site Markov chain [13, 21, 3];

(3) *Weak random environment*: small perturbation of homogeneous random walk (possibly with a feedback of the walk on the environment) [9, 10, 11, 7, 8, 22, 15].

The focus of these references is: transience vs. recurrence [24, 21], central limit theorem [9, 7, 12, 8, 13, 14, 4, 25, 15], law of large numbers and central limit theorem [3], decay of correlations in space and time [10], convergence of the law of the environment as seen from the walk [11], large deviations [22, 32]. In classes (1) and (2) the random environment is uncorrelated in time, respectively, in space. In [2] we moved away from this restriction by proving a law of large numbers for a class of dynamic random environments correlated in space *and* time, satisfying a space-time mixing condition called *cone-mixing*. We showed that a large class of uniquely ergodic attractive spin-flip systems falls into this class.

Consider a *static* random environment η with law ν_ρ, the Bernoulli product measure with density $\rho \in (0,1)$, and a random walk $X = (X_t)_{t\geq 0}$ with transition rates (compare with (1.9))

$$\begin{aligned} x &\to x+1 \quad \text{at rate} \quad \alpha\eta(x) + \beta[1-\eta(x)], \\ x &\to x-1 \quad \text{at rate} \quad \beta\eta(x) + \alpha[1-\eta(x)], \end{aligned} \tag{1.27}$$

where $0 < \beta < \alpha < \infty$. In [27] it is shown that X is recurrent when $\rho = \frac{1}{2}$ and transient to the right when $\rho > \frac{1}{2}$. In the transient case both ballistic and non-ballistic behavior occur, i.e., $\lim_{t\to\infty} X_t/t = v$ for $\mathbb{P}_{\nu_\rho}$-a.e. ξ, and

$$v \begin{cases} = 0 & \text{if } \rho \in [\frac{1}{2}, \rho_c], \\ > 0 & \text{if } \rho \in (\rho_c, 1], \end{cases} \tag{1.28}$$

where

$$\rho_c = \frac{\alpha}{\alpha+\beta} \in (\tfrac{1}{2}, 1), \tag{1.29}$$

and, for $\rho \in (\rho_c, 1]$,

$$v = v(\rho,\alpha,\beta) = (\alpha+\beta)\,\frac{\alpha\beta + \rho(\alpha^2-\beta^2) - \alpha^2}{\alpha\beta - \rho(\alpha^2-\beta^2) + \alpha^2} = (\alpha-\beta)\,\frac{\rho-\rho_c}{\rho(1-\rho_c)+\rho_c(1-\rho)}. \tag{1.30}$$

Attractive spin flips. The analogues of (1.15) and (1.18) in the *static* random environment (with no restriction on the interval K in the annealed case) were proved in [18] (quenched) and [16] (quenched and annealed). Both I^{ann} and I^{que} are zero on the interval $[0, v]$ and are strictly positive outside ("slow-down phenomenon"). For I^{que} the same symmetry property as in (1.19) holds. Moreover, an explicit formula for I^{que} is known in terms of random continued fractions.

We do not have explicit expressions for I^{ann} and I^{que} in the *dynamic* random environment. Even the characterization of their zero sets remains open, although under the stronger assumptions that $M < \epsilon$ and $\alpha - \beta < \frac{1}{2}(\epsilon - M)$ we know that both have a unique zero at v.

Theorems 1.1–1.2 can be generalized beyond spin-flip systems, i.e., systems where more than one site can flip state at a time. We will see in Sections 2–3 that what really matters is that the system has positive correlations in space and time. As shown in [19], this holds for monotone systems (see [23], Definition II.2.3) if and only if all transitions are such that they make the configuration either larger or smaller in the partial order induced by inclusion.

Simple symmetric exclusion. What Theorem 1.3 says is that, for all choices of the parameters, the annealed rate function (if it exists) is zero at 0, and so there is a slow-down phenomenon similar to what happens in the static random environment. We will see in Section 4 that this slow-down comes from the fact that the simple symmetric exclusion process suffers "traffic jams", i.e., long strings of occupied and vacant sites have an appreciable probability to survive for a long time.

To test the validity of the LLN for the simple symmetric exclusion process, we performed a simulation the outcome of which is drawn in Figs. 1–2. For each point in these figures, we drew 10^3 initial configurations according to the Bernoulli product measure with density ρ, and from each of these configurations ran a discrete-time exclusion process with parallel updating for 10^4 steps. Given the latter, we ran a discrete-time random walk for 10^4 steps, both in the static environment (ignoring the updating) and in the dynamic environment (respecting the updating), and afterwards averaged the displacement of the walk over the 10^3 initial configurations. The probability to jump to the right was taken to be p on an occupied site and $q = 1 - p$ on a vacant site, where p replaces $\alpha/(\alpha+\beta)$ in the continuous-time model. In Figs. 1–2, the speeds resulting from these simulations are plotted as a function of p for $\rho = 0.8$, respectively, as a function of ρ for $p = 0.7$. In each figure we plot four curves: (1) the theoretical speed in the static case (as described by (1.30)); (2) the simulated speed in the static case; (3) the simulated speed in the dynamic case; (4) the speed for the average environment, i.e., $(2\rho-1)(2p-1)$. The order in which these curves appear in the figures is from bottom to top.

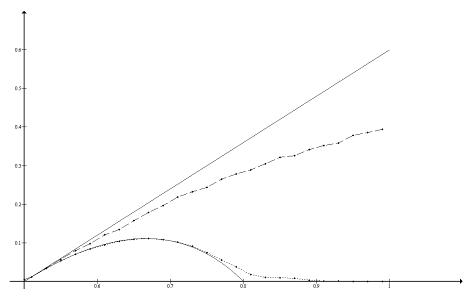

Figure 1: Speeds as a function of p for $\rho = 0.8$.

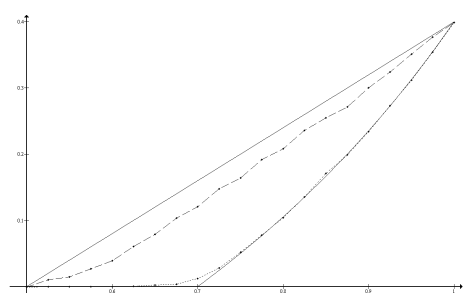

Figure 2: Speeds as a function of ρ for $p = 0.7$.

Fig. 1 shows that, in the static case with ρ fixed, as p increases the speed first goes up (because there are more occupied than vacant sites), and then goes down (because the vancant sites become more efficient to act as a barrier). In the dynamic case, however, the speed is an increasing function of p: the vacant site are not frozen but move around and make way for the walk. It is clear from Fig. 2 that the only value of ρ for which there is a zero speed in the dynamic case is $\rho = \frac{1}{2}$, for which the random walk is recurrent. Thus, the simulation suggests that there is no (!) non-ballistic behavior in the transient case. In view of Theorem 1.3, this in turn suggests that the annealed rate function (if it exists) has zero set $[0, v]$.

In both pictures the two curves at the bottom should coincide. Indeed, they almost coincide, except for values of the parameters that are close to the transition between ballistic and non-ballistic behavior, for which fluctuations are to be expected. Note that the simulated speed in the dynamic environment lies inbetween the speed for the static environment and the speed for the average environment. We may think of the latter as corresponding to a simple symmetric exclusion process running at rate 0, respectively, ∞ rather than at rate 1 as in (1.24).

2 Proof of Theorem 1.1

In Section 2.1 we prove three lemmas for the probability that the empirical speed is above a given threshold. These lemmas will be used in Section 2.2 to prove Theorems 1.1(a–b). In Section 2.3 we prove Theorems 1.1(c).

2.1 Three lemmas

Lemma 2.1. *For all $\theta \in \mathbb{R}$,*

$$J^+(\theta) = -\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\mu,0}(X_t \geq \theta t) \quad \textit{exist and is finite.} \tag{2.1}$$

Proof. For $z \in \mathbb{Z}$ and $u \geq 0$, let $\sigma_{z,u}$ denote the operator acting on ξ as

$$(\sigma_{z,u}\xi)(x,t) = \xi(z+x, u+t), \qquad x \in \mathbb{Z},\ t \geq 0. \tag{2.2}$$

Fix $\theta \neq 0$, and let $G_\theta = \{t \geq 0\colon\ \theta t \in \mathbb{Z}\}$ be the non-negative grid of width $1/|\theta|$. For any $s, t \in G_\theta$, we have

$$\begin{aligned}
&\mathbb{P}_{\mu,0}\big(X_{s+t} \geq \theta(s+t)\big) = E^\mu\Big[P_0^\xi\big(X_{s+t} \geq \theta(s+t)\big)\Big] \\
&= \sum_{y\in\mathbb{Z}} E^\mu\Big[P_0^\xi(X_s = y)\, P_y^{\sigma_{0,s}\xi}\big(X_t \geq \theta(s+t)\big)\Big] \geq \sum_{y\geq\theta s} E^\mu\Big[P_0^\xi(X_s = y)\, P_{\theta s}^{\sigma_{0,s}\xi}\big(X_t \geq \theta(s+t)\big)\Big] \\
&= E^\mu\Big[P_0^\xi(X_s \geq \theta s)\, P_0^{\sigma_{\theta s,s}\xi}\big(X_t \geq \theta t\big)\Big] \geq E^\mu\Big[P_0^\xi(X_s \geq \theta s)\Big]\, E^\mu\Big[P_0^{\sigma_{\theta s,s}\xi}\big(X_t \geq \theta t\big)\Big] \\
&= \mathbb{P}_{\mu,0}(X_s \geq \theta s)\, \mathbb{P}_{\mu,0}(X_t \geq \theta t).
\end{aligned} \tag{2.3}$$

The first inequality holds because two copies of the random walk running on the same realization of the random environment can be coupled so that they remain ordered. The second inequality uses that

$$\xi \mapsto P_0^\xi(X_s \geq \theta s) \quad \text{and} \quad \xi \mapsto P_0^{\sigma_{\theta s,s}\xi}\big(X_t \geq \theta t\big) \tag{2.4}$$

are non-decreasing and that the law P^μ of an attractive spin-flip system has the FKG-property in space-time (see [23], Corollary II.2.12). Let

$$g(t) = -\log \mathbb{P}_{\mu,0}(X_t \geq \theta t). \tag{2.5}$$

Then it follows from (2.3) that $(g(t))_{t\geq 0}$ is subadditive along G_θ, i.e., $g(s+t)\leq g(s)+g(t)$ for all $s,t\in G_\theta$. Since $\mathbb{P}_{\mu,0}(X_t\geq\theta t)>0$ for all $t\geq 0$, it thefore follows that

$$J^+(\theta)=-\lim_{\substack{t\to\infty\\ t\in G_\theta}}\frac{1}{t}\log\mathbb{P}_{\mu,0}(X_t\geq\theta t)\quad\text{exist and is finite.}\tag{2.6}$$

Because X takes values in $\mathbb{Z}$, the restriction $t\in G_\theta$ can be removed. This proves the claim for $\theta\neq 0$. The claim easily extends to $\theta=0$, because the transition rates of the random walk are bounded away from 0 and ∞ uniformly in ξ (recall (1.9)). ∎

Lemma 2.2. *$\theta\mapsto J^+(\theta)$ is non-decreasing and convex on $\mathbb{R}$.*

Proof. We follow an argument similar to that in the proof of Proposition 2.1. Fix $\theta,\gamma\in\mathbb{R}$ and $p\in[0,1]$ such that $p\gamma,(1-p)\theta\in\mathbb{Z}$. Estimate

$$\begin{aligned}
&\mathbb{P}_{\mu,0}\big(X_t\geq[p\gamma+(1-p)\theta]t\big)=E^\mu\Big[P_0^\xi\big(X_t\geq[p\gamma+(1-p)\theta]t\big)\Big]\\
&=\sum_{y\in\mathbb{Z}}E^\mu\Big[P_0^\xi(X_{pt}=y)\,P_y^{\sigma_{0,pt}\xi}\big(X_{t(1-p)}\geq[p\gamma+(1-p)\theta]t\big)\Big]\\
&\geq\sum_{y\geq p\gamma t}E^\mu\Big[P_0^\xi(X_{pt}=y)\,P_{p\gamma t}^{\sigma_{0,pt}\xi}\big(X_{t(1-p)}\geq[p\gamma+(1-p)\theta]t\big)\Big]\\
&=E^\mu\Big[P_0^\xi(X_{pt}\geq p\gamma t)\,P_0^{\sigma_{p\gamma t,pt}\xi}\big(X_{t(1-p)}\geq(1-p)\theta t\big)\Big]\\
&\geq E^\mu\Big[P_0^\xi(X_{pt}\geq p\gamma t)\Big]\,E^\mu\Big[P_{p\gamma t}^{\sigma_{p\gamma t,pt}\xi}\big(X_{t(1-p)}\geq(1-p)\theta t\big)\Big]\\
&=\mathbb{P}_{\mu,0}(X_{pt}\geq p\gamma t)\,\mathbb{P}_{\mu,0}\big(X_{t(1-p)}\geq(1-p)\theta t\big).
\end{aligned}\tag{2.7}$$

It follows from (2.7) and the remark below (2.6) that

$$-J^+\big(p\gamma+(1-p)\theta\big)\geq -pJ^+(\gamma)-(1-p)J^+(\theta),\tag{2.8}$$

which settles the convexity. ∎

Lemma 2.3. *$J^+(\theta)>0$ for $\theta>\alpha-\beta$ and $\lim_{\theta\to\infty}J^+(\theta)/\theta=\infty$.*

Proof. Let $(Y_t)_{t\geq 0}$ be the nearest-neighbor random walk on $\mathbb{Z}$ that jumps to the right at rate α and to the left at rate β. Write $\mathbb{P}_0^{\mathrm{RW}}$ to denote its law starting from $Y(0)=0$. Clearly,

$$\mathbb{P}_{\mu,0}(X_t\geq\theta t)\leq\mathbb{P}_0^{\mathrm{RW}}(Y_t\geq\theta t)\qquad\forall\,\theta\in\mathbb{R}.\tag{2.9}$$

Moreover,

$$J^{\mathrm{RW}}(\theta)=-\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}_0^{\mathrm{RW}}(Y_t\geq\theta t)\tag{2.10}$$

exists, is finite and satisfies

$$J^{\mathrm{RW}}(\alpha-\beta)=0,\quad J^{\mathrm{RW}}(\theta)>0\text{ for }\theta>\alpha-\beta,\qquad\lim_{\theta\to\infty}J^{\mathrm{RW}}(\theta)/\theta=\infty.\tag{2.11}$$

Combining (2.9–2.11), we get the claim. ∎

Lemmas 2.1–2.3 imply that an *upward* annealed LPD holds with a rate function J^+ whose qualitative shape is given in Fig. 3.

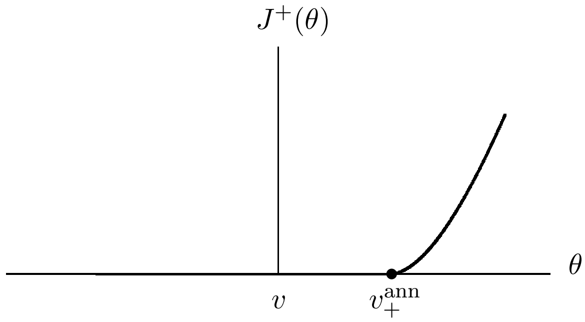

Figure 3: Shape of $\theta \mapsto J^+(\theta)$.

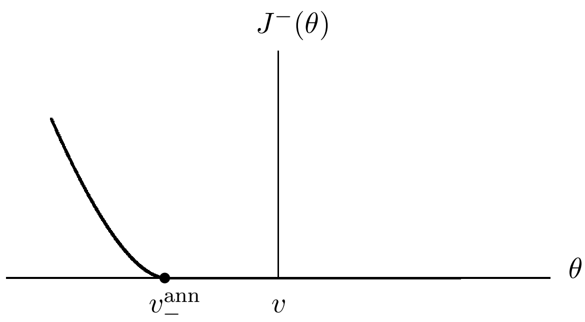

Figure 4: Shape of $\theta \mapsto J^-(\theta)$.

2.2 Annealed LDP

Clearly, J^+ depends on P^μ, α and β. Write

$$J^+ = J_{P^\mu,\alpha,\beta} \tag{2.12}$$

to exhibit this dependence. So far we have not used the restriction $\alpha > \beta$ in (1.10). By noting that $-X_t$ is equal in distribution to X_t when α and β are swapped and P^μ is replaced by $\bar{P}^\mu$, the image of P^μ under reflection in the origin (recall (1.9)), we see that the *upward* annealed LDP proved in Section 2.1 also yields a *downward* annealed LDP

$$J^-(\theta) = -\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\mu,0}(X_t \le \theta t), \qquad \theta \in \mathbb{R}, \tag{2.13}$$

with

$$J^- = J_{\bar{P}^\mu,\beta,\alpha}, \tag{2.14}$$

whose qualitative shape is given in Fig. 4. Note that

$$v_-^{\mathrm{ann}} \le v \le v_+^{\mathrm{ann}}, \tag{2.15}$$

because v, the speed in the LLN proved in [2], must lie in the zero set of both J^+ and J^-.

Our task is to turn the upward and downward annealed LDP's into the annealed LDP of Theorem 1.1.

Proposition 2.4. *Let*

$$I^{\mathrm{ann}}(\theta) = \begin{cases} J_{P^\mu,\alpha,\beta}(\theta) & \text{if } \theta \geq v, \\ J_{\bar{P}^\mu,\beta,\alpha}(-\theta) & \text{if } \theta \leq v. \end{cases} \tag{2.16}$$

Then

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\mu,0}(t^{-1}X_t \in K) = -\inf_{\theta\in K} I^{\mathrm{ann}}(\theta) \tag{2.17}$$

for all closed intervals such that either $K \subsetneq [v_-^{\mathrm{ann}}, v_+^{\mathrm{ann}}]$ *or* $\mathrm{int}(K) \ni v$.

Proof. We distinguish three cases.

(1) $K \subset [v,\infty)$, $K \subsetneq [v, v_+^{\mathrm{ann}}]$: Let $\mathrm{cl}(K) = [a,b]$. Then, because J^+ is continuous,

$$\frac{1}{t} \log \mathbb{P}_{\mu,0}\big(t^{-1}X_t \in K\big) = \frac{1}{t} \log \Big[e^{-tJ^+(a)+o(t)} - e^{-tJ^+(b)+o(t)} \Big]. \tag{2.18}$$

By Lemma 2.2, J^+ is strictly increasing on $[v_+^{\mathrm{ann}},\infty)$, and so $J^+(b) > J^+(a)$. Letting $t\to\infty$ in (2.18), we therefore see that

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\mu,0}\big(t^{-1}X_t \in K\big) = -J^+(a) = -\inf_{\theta\in K} I^{\mathrm{ann}}(\theta). \tag{2.19}$$

(2) $K \subset (-\infty, v]$, $K \subsetneq [v_-^{\mathrm{ann}}, v]$: Same as for (1) with J^- replacing J^+.

(3) $\mathrm{int}(K) \ni v$: In this case (2.17) is an immediate consequence of the LLN in (1.12). ∎

Proposition 2.4 completes the proof of Theorems 1.1(a–b). Recall (2.12) and (2.14). The restriction on K comes from the fact that the difference of two terms that are both $\exp[o(t)]$ may itself not be $\exp[o(t)]$.

2.3 Unique zero of I^{ann} when $M < \epsilon$

In [2] we showed that if $M < \epsilon$ and $\alpha - \beta < \frac{1}{2}(\epsilon - M)$, then a proof of the LLN can be given that is based on a perturbation argument for the generator of the *environment process*

$$\zeta = (\zeta_t)_{t\geq 0}, \qquad \zeta_t = \tau_{X_t}\xi_t, \tag{2.20}$$

i.e., the random environment as seen relative to the random walk. In particular, it is shown that ζ is uniquely ergodic with equilibrium μ_e. This leads to a series expansion for v in powers of $\alpha - \beta$, with coefficients that are functions of P^μ and $\alpha + \beta$ and that are computable via a recursive scheme. The speed in the LLN is given by

$$v = (2\widetilde{\rho} - 1)(\alpha - \beta) \tag{2.21}$$

with $\widetilde{\rho} = \langle \eta(0) \rangle_{\mu_e}$, where $\langle\cdot\rangle_{\mu_e}$ denotes expectation over μ_e ($\widetilde{\rho}$ is the fraction of time X spends on occupied sites).

Proposition 2.5. *Let ξ be an attractive spin-flip system with $M < \epsilon$. If $\alpha - \beta < \frac{1}{2}(\epsilon - M)$, then the rate function I^{ann} in* (2.18) *has a unique zero at v.*

Proof. It suffices to show that

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\mu,0}\big(|t^{-1}X_t - v| \geq 2\delta\big) < 0 \qquad \forall\, \delta > 0. \tag{2.22}$$

To that end, put $\gamma = \delta/2(\alpha-\beta) > 0$. Then, by (2.21), $v \pm \delta = [2(\widetilde{\rho} \pm \gamma) - 1](\alpha - \beta)$. Let

$$A_t = \int_0^t \xi_s(X_s)\,\mathrm{d}s \tag{2.23}$$

be the time X spends on occupied sites up to time t, and define

$$E_t = \big\{|t^{-1}A_t - \widetilde{\rho}| \geq \gamma\big\}. \tag{2.24}$$

Estimate

$$\mathbb{P}_{\mu,0}(|t^{-1}X_t - v| \geq 2\delta\big) \leq \mathbb{P}_{\mu,0}(E_t) + \mathbb{P}_{\mu,0}\big(|t^{-1}X_t - v| \geq 2\delta \mid E_t^c\big). \tag{2.25}$$

Conditional on E_t^c, X behaves like a homogeneous random walk with speed in $[v-\delta, v+\delta]$. Therefore the second term in the r.h.s. of (2.25) vanishes exponentially fast in t. In [2], Lemma 3.4, Eq. (3.26) and Eq. (3.36), we proved that

$$|||S(t)f||| \leq \mathrm{e}^{-c_1 t}\, |||f||| \quad \text{ and } \quad \Big\|S(t)f - \langle f\rangle_{\mu_e}\Big\|_\infty \leq c_2\, \mathrm{e}^{-(\epsilon - M)t}\, |||f||| \tag{2.26}$$

for some $c_1, c_2 \in (0,\infty)$, where $S = (S(t))_{t\geq 0}$ denotes the semigroup associated with the environment process ζ, and $|||f|||$ denotes the triple norm of f. As shown in [26], (2.26) implies a Gaussian concentration bound for additive functionals, namely,

$$\mathbb{P}_{\mu,0}\left(\left|t^{-1}\int_0^t f(\zeta_s) - \langle f\rangle_\mu\right| \geq \gamma\right) \leq c_3\, \mathrm{e}^{-\gamma^2 t/c_4\, |||f|||^2} \tag{2.27}$$

for some $c_3, c_4 \in (0,\infty)$, uniformly in $t > 0$, f with $|||f||| < \infty$ and $\gamma > 0$. By picking $f(\eta) = \eta(0)$, $\eta \in \Omega$, we get

$$\mathbb{P}_{\mu,0}\big(E_t\big) \leq c_5\, \mathrm{e}^{-c_6 t} \tag{2.28}$$

for some $c_5, c_6 \in (0,\infty)$. Therefore also the first term in the r.h.s. of (2.25) vanishes exponentially fast in t. ∎

Proposition 2.5 completes the proof of Theorems 1.1(c).

3 Proof of Theorem 1.2

In Section 3.1 we prove three lemmas for the probability that the empirical speed equals a given value. These lemmas will be used in Section 3.2 to prove Theorems 1.2(a–b). In Section 3.3 we prove Theorem 1.2(c). Theorem 1.2(d) follows from Theorem 1.1(c) because $I^{\mathrm{que}} \geq I^{\mathrm{ann}}$.

3.1 Three lemmas

In this section we state three lemmas that are the analogues of Lemmas 2.1–2.3.

Lemma 3.1. *For all $\theta \in \mathbb{R}$,*

$$I^{\text{que}}(\theta) = -\lim_{t\to\infty} \frac{1}{t} \log P_0^{\xi}(X_t = \lfloor \theta t \rfloor) \quad \textit{exists, is finite and is constant } \xi\textit{-a.s.} \tag{3.1}$$

Proof. Fix $\theta \neq 0$, and recall that $G_\theta = \{t \geq 0 \colon \theta t \in \mathbb{Z}\}$ is the non-negative grid of width $1/|\theta|$. For any $s, t \in G_\theta$, we have

$$\begin{aligned} P_0^{\xi}\big(X_{s+t} = \theta(s+t)\big) &\geq P_0^{\xi}\big(X_s = \theta s\big)\, P_0^{\xi}\big(X_{s+t} = \theta(s+t) \mid X_s = \theta s\big) \\ &= P_0^{\xi}\big(X_s = \theta s\big)\, P_0^{T_s \xi}(X_t = \theta t), \end{aligned} \tag{3.2}$$

where $T_s = \sigma_{\theta s, s}$. Let

$$g_t(\xi) = -\log P_0^{\xi}(X_t = \theta t). \tag{3.3}$$

Then it follows from (3.2) that $(g_t(\xi))_{t\geq 0}$ is a subadditive random process along G_θ, i.e., $g_{s+t}(\xi) \leq g_s(\xi) + g_t(T_s \xi)$ for all $s, t \in G_\theta$. From Kingman's subadditive ergodic theorem (see e.g. [29]) it therefore follows that

$$\lim_{\substack{t\to\infty \\ t \in G_\theta}} \frac{1}{t} \log P_0^{\xi}(X_t = \theta t) = -I^{\text{que}}(\theta) \tag{3.4}$$

exists, is finite ξ-a.s, and is T_s-invariant for every $s \in G_\theta$. Moreover, since ξ is ergodic under space-time shifts (recall (1.5) and (1.7)), this limit is constant ξ-a.s. Because the transition rates of the random walk are bounded away from 0 and ∞ uniformly in ξ (recall (1.9)), the restriction $t \in G_\theta$ may be removed after $X_t = \theta t$ is replaced by $X_t = \lfloor \theta t \rfloor$ in (3.4). This proves the claim for $\theta \neq 0$. By the boundedness of the transition rates, the claim easily extends to $\theta = 0$. ∎

Lemma 3.2. *$\theta \mapsto I^{\text{que}}(\theta)$ is convex on $\mathbb{R}$.*

Proof. The proof is similar to that of Proposition 2.1. Fix $\theta, \zeta \in \mathbb{R}$ and $p \in [0,1]$. For any $t \geq 0$ such that $p\zeta t, (1-p)\theta t \in \mathbb{Z}$, we have

$$\begin{aligned} P_0^{\xi}\big(X_t \geq [p\zeta + (1-p)\theta] t\big) &\geq P_0^{\xi}\big(X_{pt} = p\zeta t\big)\, P_0^{\xi}\big(X_t = [p\zeta + (1-p)\theta]t \mid X_{pt} = p\zeta t\big) \\ &= P_0^{\xi}\big(X_{pt} = p\zeta t\big)\, P_0^{\sigma_{p\zeta t, pt}\xi}\big(X_{(1-p)t} = (1-p)\theta t\big). \end{aligned} \tag{3.5}$$

It follows from (3.5) and the remark below (2.6) that

$$-I^{\text{que}}\big(p\zeta + (1-p)\theta\big) \geq -pI^{\text{que}}(\zeta) - (1-p) I^{\text{que}}(\theta), \tag{3.6}$$

which settles the convexity. ∎

Lemma 3.3. *$I^{\text{que}}(\theta) > 0$ for $|\theta| > \alpha - \beta$ and $\lim_{\theta\to\infty} I^{\text{que}}(\theta)/|\theta| = \infty$.*

Proof. Same as Lemma 2.3. ∎

3.2 Quenched LDP

We are now ready to prove the quenched LDP.

Proposition 3.4. *For P^μ-a.e. ξ, the family of probability measures $P_0^\xi(X_t/t \in \cdot)$, $t > 0$, satisfies the LDP with rate t and with deterministic rate function I^{que}.*

Proof. Use Lemmas 3.1–3.3. ∎

Proposition 3.4 completes the proof of Theorems 1.2, except for the symmetry relation in (1.19), which will be proved in Section 3.3. Recall (1.21) and the remark below it.

3.3 A quenched symmetry relation

Proposition 3.5. *For all $\theta \in \mathbb{R}$, the rate function in Theorem 3.4 satisfies the symmetry relation*

$$I^{\mathrm{que}}(-\theta) = I^{\mathrm{que}}(\theta) + \theta(2\rho - 1)\log(\alpha/\beta). \tag{3.7}$$

Proof. We first consider a discrete-time random walk, i.e., a random walk that observes the random environment and jumps at integer times. Afterwards we will extend the argument to the continuous-time random walk defined in (1.8–1.10).

1. Path probabilities. Let

$$X = (X_n)_{n\in\mathbb{N}_0} \tag{3.8}$$

be the random walk with transition probablities

$$\begin{aligned} x \to x+1 \quad &\text{with probability} \quad p\,\xi_n(x) + q\,[1-\xi_n(x)],\\ x \to x-1 \quad &\text{with probability} \quad q\,\xi_n(x) + p\,[1-\xi_n(x)], \end{aligned} \tag{3.9}$$

where w.l.o.g. $p > q$. For an oriented edge $e = (i, i\pm 1)$, $i \in \mathbb{Z}$, write $\bar{e} = (i\pm 1, i)$ to denote the reverse edge. Let $p_n(e)$ denote the probability for the walk to jump along the edge e at time n. Note that in the *static* random environment these probabilities are time-independent, i.e., $p_n(e) = p_0(e)$ for all $n \in \mathbb{N}$.

We will be interested in n-step paths $\omega = (\omega_0, \dots, \omega_n) \in \mathbb{Z}^n$ with $\omega_0 = 0$ and $\omega_n = \lfloor \theta n \rfloor$ for a given $\theta \neq 0$. Write $\Theta\omega$ to denote the time-reversed path, i.e., $\Theta\omega = (\omega_n, \dots, \omega_0)$. Let $N_e(\omega)$ denote the number of times the edge e is crossed by ω, and write $t_e^j(\omega)$, $j = 1, \dots, N_e(\omega)$, to denote the successive times at which the edge e is crossed. Let $E(\omega)$ denote the set of edges in the path ω, and $E^+(\omega)$ the subset of forward edges, i.e., edges of the form $(i, i+1)$. Then we have

$$N_e(\Theta\omega) = N_{\bar{e}}(\omega) \tag{3.10}$$

and

$$t_e^j(\Theta\omega) = n + 1 - t_{\bar{e}}^{N_{\bar{e}}(\omega)+1-j}(\omega), \qquad j = 1, \dots, N_e(\Theta\omega) = N_{\bar{e}}(\omega). \tag{3.11}$$

Given a realization of ξ, the probability that the walk follows the path ω equals

$$P^\xi(\omega) = \prod_{e\in E(\omega)} \prod_{j=1}^{N_e(\omega)} p_{t_e^j(\omega)}(e) = \prod_{e\in E^+(\omega)} \prod_{j=1}^{N_e(\omega)} p_{t_e^j(\omega)}(e) \prod_{j=1}^{N_{\bar{e}}(\omega)} p_{t_{\bar{e}}^j(\omega)}(\bar{e}). \tag{3.12}$$

The probability of the reversed path is, by (3.10–3.11),

$$\begin{aligned} P^{\xi}(\Theta\omega) &= \prod_{e\in E(\omega)} \prod_{j=1}^{N_e(\Theta\omega)} p_{t_e^j(\Theta\omega)}(e) = \prod_{e\in E(\omega)} \prod_{j=1}^{N_{\bar e}(\omega)} p_{n+1-t_{\bar e}^{N_{\bar e}(\omega)+1-j}(\omega)}(e) \\ &= \prod_{e\in E(\omega)} \prod_{j=1}^{N_{\bar e}(\omega)} p_{n+1-t_{\bar e}^j(\omega)}(e) = \prod_{e\in E^+(\omega)} \prod_{j=1}^{N_e(\omega)} p_{n+1-t_e^j(\omega)}(\bar e) \prod_{j=1}^{N_{\bar e}(\omega)} p_{n+1-t_{\bar e}^j(\omega)}(e). \end{aligned} \tag{3.13}$$

Given a path going from ω_0 to ω_n, all the edges e in between ω_0 and ω_n pointing in the direction of ω_n, which we denote by $\mathcal{E}(\omega_0,\omega_n)$, are traversed one time more than their reverse edges, while all other edges are traversed as often as their reverse edges. Therefore we obtain, assuming w.l.o.g. that $\omega_n > \omega_0$ (or $\theta > 0$),

$$\begin{aligned} \log \frac{P^{\xi}(\Theta\omega)}{P^{\xi}(\omega)} &= \sum_{e\in\mathcal{E}(\omega_0,\omega_n)} \log \frac{p_{n+1-t_e^{N_e(\omega)}(\omega)}(\bar e)}{p_{t_e^{N_e(\omega)}(\omega)}(e)} \\ &\quad + \sum_{e\in E^+(\omega)} \sum_{j=1}^{N_{\bar e}(\omega)} \log\left(\frac{p_{n+1-t_{\bar e}^j(\omega)}(e) p_{n+1-t_e^j(\omega)}(\bar e)}{p_{t_e^j(\omega)}(e) p_{t_{\bar e}^j(\omega)}(\bar e)} \right). \end{aligned} \tag{3.14}$$

In the *static* random environment we have $p_n(e) = p_0(e)$ for all $n \in \mathbb{N}$ and $e \in E(\omega)$, and hence the second sum in (3.14) is identically zero, while by the ergodic theorem the first sum equals

$$(\omega_n - \omega_0)\langle \log[p_0(1,0)/p_0(0,1)]\rangle_{\nu_\rho} + o(n) = (\omega_n-\omega_0)(2\rho-1)\log(p/q) + o(n), \qquad n\to\infty, \tag{3.15}$$

where ν_ρ is the Bernoulli product measure on Ω with density ρ (which is the law that is typically chosen for the static random environment). In the *dynamic* random environment, both sums in (3.14) still look like ergodic sums, but since in general

$$p_{t_e^j(\omega)}(e) \neq p_{t_e^i(\omega)}(e), \qquad i \neq j, \tag{3.16}$$

we have to use space-time ergodicity.

2. Space-time ergodicity. Rewrite (3.14) as

$$\begin{aligned} \log \frac{P^{\xi}(\Theta\omega)}{P^{\xi}(\omega)} &= \sum_{e\in\mathcal{E}(\omega_0,\omega_n)} \log p_{n+1-t_e^{N_e(\omega)}(\omega)}(\bar e) - \sum_{e\in\mathcal{E}(\omega_0,\omega_n)} \log p_{t_e^{N_e(\omega)}(\omega)}(e) \\ &\quad + \sum_{e\in E^+(\omega)} \log p_{n+1-t_{\bar e}^1(\omega)}(e) + \sum_{e\in E^+(\omega)} \log p_{n+1-t_e^1(\omega)}(\bar e) \\ &\quad - \sum_{e\in E^+(\omega)} \log p_{t_e^1(\omega)}(e) - \sum_{e\in E^+(\omega)} \log p_{t_{\bar e}^1(\omega)}(\bar e) \\ &\quad + \sum_{e\in E^+(\omega)} \sum_{j=2}^{N_{\bar e}(\omega)} \log\left(\frac{p_{n+1-t_{\bar e}^j(\omega)}(e) p_{n+1-t_e^j(\omega)}(\bar e)}{p_{t_e^j(\omega)}(e) p_{t_{\bar e}^j(\omega)}(\bar e)} \right), \end{aligned} \tag{3.17}$$

and note that all the sums in (3.17) are of the form

$$\sum_{i=1}^{N} \log p_{t(i)}(\omega_0+i) = \begin{cases} (\log p) \sum_{i=1}^{N} \xi_{t_i}(\omega_0+i) + (\log q) \sum_{i=1}^{N} [1-\xi_{t_i}(\omega_0+i)], \\ (\log q) \sum_{i=1}^{N} \xi_{t_i}(\omega_0+i) + (\log p) \sum_{i=1}^{N} [1-\xi_{t_i}(\omega_0+i)], \end{cases} \tag{3.18}$$

where $t_i = t((i,i+1))$, with $t = t(\omega)\colon \{0,1,\ldots,N\} \to \{0,1,\ldots,n\}$ either strictly increasing or strictly decreasing with image set $I_n(t) \subset \{0,1,\ldots,n\}$ such that $|I_n(t)|$ is of order n. Note that $N = N(\omega) = |\mathcal{E}(\omega_0,\omega_n)| = \omega_n - \omega_0 = \lfloor \theta n \rfloor$ in the first two sums in (3.17), $N = N(\omega) = |E^+(\omega)| \geq \omega_n - \omega_0 = \lfloor \theta n \rfloor$ in the remaining sums, and

$$|t_j - t_i| \geq j - i, \qquad j > i. \tag{3.19}$$

The aim is to show that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \log p_{t_i}(i) = \langle \log p_0(0) \rangle_\mu = \rho \log p + (1-\rho) \log q \qquad \xi - a.s. \text{ for all } \omega \tag{3.20}$$

or, equivalently,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \xi_{t_i}(i) = \langle \xi_0(0) \rangle_\mu = \rho \qquad \xi - a.s. \text{ for all } \omega, \tag{3.21}$$

where, since we take the limit $N \to \infty$, we think of ω as an infinite path in which the n-step path $(\omega_0,\ldots,\omega_n)$ with $\omega_0 = 0$ and $\omega_n = \lfloor \theta n \rfloor$ is embedded. Because P^μ is tail trivial (recall (1.7)) and $\lim_{i\to\infty} t_i = \infty$ for all ω by (3.19), the limit exists ξ-a.s. for all ω. To prove that the limit equals ρ we argue as follows. Write

$$\mathrm{Var}^{P^\mu}\left(\frac{1}{N} \sum_{i=1}^{N} \xi_{t_i}(i) \right) = \frac{\rho(1-\rho)}{N} + \frac{2}{N^2} \sum_{i=1}^{N} \sum_{j>i} \mathrm{Cov}^{P^\mu}\big(\xi_{t_i}(i), \xi_{t_j}(j)\big). \tag{3.22}$$

By (1.5), we have

$$\mathrm{Cov}^{P^\mu}\big(\xi_{t_i}(i), \xi_{t_j}(j)\big) = \mathrm{Cov}^{P^\mu}\big(\xi_0(0), \xi_{|t_j-t_i|}(j-i)\big). \tag{3.23}$$

In view of (3.19), it therefore follows that

$$\lim_{k\to\infty} \sup_{l\geq k} \mathrm{Cov}^{P^\mu}\big(\xi_0(0), \xi_l(k)\big) = 0 \quad \Longrightarrow \quad \lim_{N\to\infty} \mathrm{Var}^{P^\mu}\left(\frac{1}{N} \sum_{i=1}^{N} \xi_{t_i}(i) \right) = 0. \tag{3.24}$$

But the l.h.s. of (3.24) is true by the tail triviality of P^μ.

3. Implication for the rate function. Having proved (3.20) holds, we can now use (3.17–3.18) and (3.20–3.21) to obtain

$$\frac{P^\xi(\Theta\omega)}{P^\xi(\omega)} = \mathrm{e}^{A(\omega_n-\omega_0)+o(n)} \quad \text{with} \quad A = (2\rho-1)\log(p/q). \tag{3.25}$$

Thus, the probability that the walk moves from 0 to $\lfloor \theta n \rfloor$ in n steps is given by

$$
\begin{aligned}
P^\xi(\omega_n = \lfloor \theta n \rfloor \mid \omega_0 = 0) &= \sum_{\substack{\omega\colon\, |\omega|=n \\ \omega_0=0,\omega_n=\lfloor \theta n \rfloor}} P^\xi(\omega) = \sum_{\substack{\omega\colon\, |\omega|=n \\ \omega_0=0,\omega_n=\lfloor \theta n \rfloor}} P^\xi(\Theta\omega)\,\mathrm{e}^{-A\lfloor \theta n \rfloor + o(n)} \\
&= \mathrm{e}^{-A\lfloor \theta n \rfloor + o(n)} \sum_{\substack{\omega\colon\, |\omega|=n \\ \omega_0=\lfloor \theta n \rfloor,\omega_n=0}} P^\xi(\omega) \quad = \mathrm{e}^{-A\lfloor \theta n \rfloor + o(n)}\, P^\xi(\omega_n = 0 \mid \omega_0 = \lfloor \theta n \rfloor).
\end{aligned}
\tag{3.26}
$$

Since the quenched rate function is ξ-a.s. constant, we have

$$
\begin{aligned}
P^\xi(\omega_n = \lfloor \theta n \rfloor \mid \omega_0 = 0) &= P_0^\xi(X_n = \lfloor \theta n \rfloor) = \mathrm{e}^{-nI^{\mathrm{que}}(\theta) + o(n)}, \\
P^\xi(\omega_n = 0 \mid \omega_0 = \lfloor \theta n \rfloor) &= P_0^{\tau_{\lfloor \theta n \rfloor}\xi}(X_n = -\lfloor \theta n \rfloor) = \mathrm{e}^{-nI^{\mathrm{que}}(-\theta) + o(n)},
\end{aligned}
\tag{3.27}
$$

and hence

$$
\frac{1}{n} \log\left(\frac{P^\xi(\omega_n = \lfloor \theta n \rfloor \mid \omega_0 = 0)}{P^\xi(\omega_n = 0 \mid \omega_0 = \lfloor \theta n \rfloor)} \right) \to -I^{\mathrm{que}}(\theta) + I^{\mathrm{que}}(-\theta). \tag{3.28}
$$

Together with (3.26), this leads to the symmetry relation

$$
-I^{\mathrm{que}}(\theta) + I^{\mathrm{que}}(-\theta) = -A\theta. \tag{3.29}
$$

4. From discrete to continuous time. Let $\chi = (\chi_n)_{n\in\mathbb{N}_0}$ denote the jump times of the continuous-time random walk $X = (X_t)_{t\geq 0}$ (with $\chi_0 = 0$). Let Q denote the law of χ. The increments of χ are i.i.d. random variables, independent of ξ, whose distribution is exponential with mean $1/(\alpha+\beta)$. Define

$$
\begin{array}{rclcrcl}
\xi^* &=& (\xi^*_n)_{n\in\mathbb{N}_0} & \text{with} & \xi^*_n &=& \xi_{\chi_n}, \\
X^* &=& (X^*_n)_{n\in\mathbb{N}_0} & \text{with} & X^*_n &=& X_{\chi_n}.
\end{array}
\tag{3.30}
$$

Then X^* is a discrete-time random walk in a random environment ξ^* of the type considered in Steps 1–3, with $p = \alpha/(\alpha+\beta)$ and $q = \beta/(\alpha+\beta)$. The analogue of (3.21) reads

$$
\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \xi_{\chi_{t_i}}(i) = \rho \qquad \xi,\chi - a.s. \text{ for all } \omega, \tag{3.31}
$$

where we use that the law of χ is invariant under permutations of its increments. All we have to do is to show that

$$
\lim_{N\to\infty} E^Q\left(\mathrm{Var}^{P^\mu}\left(\frac{1}{N} \sum_{i=1}^{N} \xi_{\chi_{t_i}}(i) \right) \right) = 0. \tag{3.32}
$$

But

$$
E^Q\left(\mathrm{Cov}^{P^\mu}\left(\xi_{\chi_{t_i}}(i), \xi_{\chi_{t_j}}(j) \right) \right) = E^Q\left(\mathrm{Cov}^{P^\mu}\left(\xi_0(0), \xi_{|\chi_{t_j} - \chi_{t_i}|}(j-i) \right) \right), \tag{3.33}
$$

while (3.19) ensures that $\lim_{j\to\infty} |\chi_{t_j} - \chi_{t_i}| \to \infty$ χ-a.s. for all ω as $j - i \to \infty$. Together with the tail triviality of P^μ assumed in (1.7), this proves (3.32). ∎

4 Proof of Theorem 1.3

In Section 4.1 we show that the simple symmetric exclusion process suffers traffic jams. In Section 4.2 we prove that these traffic jams cause the slow-down of the random walk.

4.1 Traffic jams

In this section we derive two lemmas stating that long strings of occupied and vacant sites have an appreciable probability to survive for a long time under the simple symmetric exclusion dynamics, both when they are alone (Lemma 4.1) and when they are together but sufficiently separated from each other (Lemma 4.2). These lemmas, which are proved with the help of the *graphical representation*, are in the spirit of [1].

In the *graphical representation* of the simple symmetric exclusion process, space is drawn sidewards, time is drawn upwards, and for each pair of nearest-neighbor sites $x, y \in \mathbb{Z}$ links are drawn between x and y at Poisson rate 1. The configuration at time t is obtained from the one at time 0 by transporting the local states along paths that move upwards with time and sidewards along links (see Fig. 5).

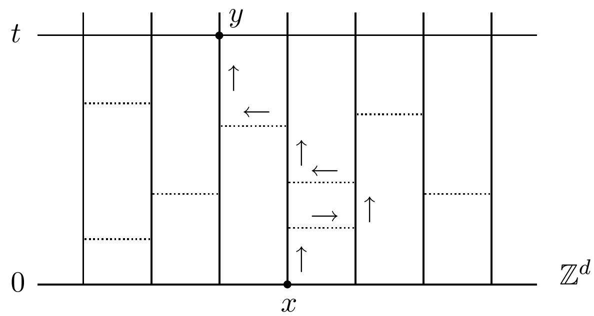

Figure 5: Graphical representation. The dashed lines are links. The arrows represent a path from $(x, 0)$ to (y, t).

Lemma 4.1. *There exists a $C = C(\rho) > 0$ such that, for all $Q \subset \mathbb{Z}$ and all $t \geq 1$,*

$$P^{\nu_\rho}\Big(\xi_s(x) = 0\ \forall\, x \in Q\ \forall\, s \in [0, t]\Big) \geq \mathrm{e}^{-C|Q|\sqrt{t}}. \tag{4.1}$$

Proof. Let

$$H_t^Q = \Big\{x \in \mathbb{Z}\colon\ \exists \text{ path in } \mathcal{G} \text{ from } (x, 0) \text{ to } Q \times [0, t]\Big\}. \tag{4.2}$$

Note that $H_0^Q = Q$ and that $t \mapsto H_t^Q$ is non-decreasing. Denote by $\mathcal{P}$ and $\mathcal{E}$, respectively, probability and expectation w.r.t. $\mathcal{G}$. Let $V_0 = \{x \in \mathbb{Z}\colon\ \xi(x, 0) = 0\}$ be the set of initial locations of the vacancies. Then

$$P^{\nu_\rho}\Big(\xi_s(x) = 0\ \forall\, x \in Q\ \forall\, s \in [0, t]\Big) = (\mathcal{P} \otimes \nu_\rho)\left(H_t^Q \subset V_0\right). \tag{4.3}$$

Indeed, if $\xi(x, 0) = 1$ for some $x \in H_t^Q$, then this 1 will propagate into Q prior to time t (see Fig. 6).

By Jensen's inequality,

$$(\mathcal{P} \otimes \nu_\rho)\left(H_t^Q \subset V_0\right) = \mathcal{E}\left((1-\rho)^{|H_t^Q|}\right) \geq (1-\rho)^{\mathcal{E}(|H_t^Q|)}. \tag{4.4}$$

Moreover, since $H_t^Q = \cup_{x \in Q} H_t^x$ and $\mathcal{E}(|H_t^x|)$ does not depend on x, we have

$$\mathcal{E}(|H_t^Q|) \leq |Q|\, \mathcal{E}(|H_t^0|), \tag{4.5}$$

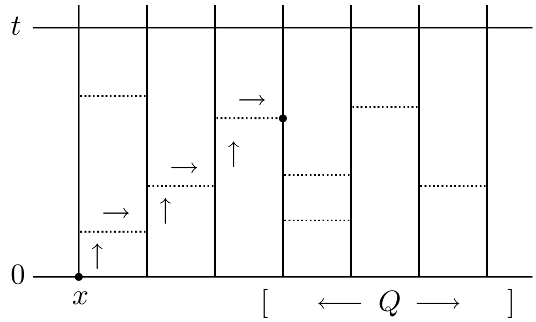

Figure 6: A path from $(x,0)$ to $Q\times[0,t]$.

and, by time reversal, we see that

$$\begin{aligned}\mathcal{E}(|H_t^0|) &= \sum_{x\in\mathbb{Z}} \mathcal{P}\Big(\exists \text{ path in } \mathcal{G} \text{ from } (x,0) \text{ to } \{0\}\times[0,t]\Big)\\ &= \sum_{x\in\mathbb{Z}} \mathbb{P}_0^{\mathrm{SRW}}(\tau_x \le t) = \mathbb{E}_0^{\mathrm{SRW}}(|R_t|),\end{aligned} \tag{4.6}$$

where $\mathbb{P}_0^{\mathrm{SRW}}$ is the law of simple symmetric random walk jumping at rate 1 starting from 0, R_t is the range (= number of distinct sites visited) at time t and τ_x is the first hitting time of x. Combining (4.3–4.6), we get

$$P^{\nu_\rho}\Big(\xi_s(x)=0\ \forall\, x\in Q\ \forall\, s\in[0,t]\Big) \ge (1-\rho)^{|Q|\,\mathbb{E}_0^{\mathrm{SRW}}(|R_t|)}. \tag{4.7}$$

The claim now follows from the fact that $R_0=1$ and $\mathbb{E}_0^{\mathrm{SRW}}(|R_t|)\sim C'\sqrt{t}$ as $t\to\infty$ for some $C'>0$ (see [28], Section 1). ∎

Lemma 4.2. *There exist $C=C(\rho)>0$ and $\delta>0$ such that, for all intervals $Q,Q'\subset\mathbb{Z}$ separated by a distance at least $2\sqrt{t\log t}$ and all $t\ge 1$,*

$$P^{\nu_\rho}\Big\{\xi_s(x)=1,\ \xi_s(y)=0\ \forall\, x\in Q\ \forall\, y\in Q'\ \forall\, s\in[0,t]\Big\} \ge \delta\,\mathrm{e}^{-C(|Q|+|Q'|)\sqrt{t}}. \tag{4.8}$$

Proof. Recall (4.2) and abbreviate $A_t=\{H_t^Q\cap H_t^{Q'}=\emptyset\}$. Similarly as in (4.3–4.4), we have

$$\text{l.h.s.}(4.8) = (\mathcal{P}\otimes\nu_\rho)(A_t) = \mathcal{E}\left(1_{A_t}\,\rho^{|H_t^Q|}\,(1-\rho)^{|H_t^{Q'}|}\right). \tag{4.9}$$

Both $|H_t^Q|$ and $|H_t^{Q'}|$ are non-decreasing in the number of arrows in $\mathcal{G}$, while 1_{A_t} is non-increasing in the number of arrows in $\mathcal{G}$. Therefore, by the FKG-inequality ([23], Chapter II), we have

$$\mathcal{E}\left(1_{A_t}\,\rho^{|H_t^Q|}(1-\rho)^{|H_t^{Q'}|}\right) \ge \mathcal{P}(A_t)\,\mathcal{E}\Big(\rho^{|H_t^Q|}\Big)\,\mathcal{E}\Big((1-\rho)^{|H_t^{Q'}|}\Big). \tag{4.10}$$

We saw in the proof of Lemma 4.1 that, for $t\ge 1$ and some $C>0$,

$$\mathcal{E}\Big(\rho^{|H_t^Q|}\Big)\,\mathcal{E}\Big((1-\rho)^{|H_t^{Q'}|}\Big) \ge \mathrm{e}^{-C(|Q|+|Q'|)\sqrt{t}}. \tag{4.11}$$

Thus, to complete the proof it suffices to show that there exists a $\delta > 0$ such that

$$\mathcal{P}(A_t) \geq \delta \text{ for } t \geq 1. \tag{4.12}$$

To that end, let $q = \max\{x \in Q\}$, $q' = \min\{x' \in Q'\}$ (where without loss of generality we assume that Q lies to the left of Q'). Then, using that Q, Q' are intervals, we may estimate (see Fig. 6)

$$\begin{aligned}
\mathcal{P}([A_t]^c) &= \mathcal{P}\Big(\exists\, z \in \mathbb{Z}\colon\ (z,0) \to \partial Q \times [0,t],\ (z,0) \to \partial Q' \times [0,t]\Big) \\
&\leq \sum_{\substack{x \in \partial Q \\ x' \in \partial Q'}} \int_0^t \Big[\mathcal{P}\Big(\exists\, z \in \mathbb{Z}\colon\ (z,0) \to x \times [s,s+\mathrm{d}s],\ (x,s) \to x' \times [s,t]\Big) \\
&\qquad\qquad + \mathcal{P}\Big(\exists\, z \in \mathbb{Z}\colon\ (z,0) \to x' \times [s,s+ds],\ (x',s) \to x \times [s,t]\Big)\Big] \\
&= \sum_{\substack{x \in \partial Q \\ x' \in \partial Q'}} \int_0^t \Big[\mathcal{P}\Big(\exists\, z \in \mathbb{Z}\colon\ (z,0) \to x \times [s,s+\mathrm{d}s]\Big)\, \mathcal{P}\Big((x,s) \to x' \times [s,t]\Big) \\
&\qquad\qquad + \mathcal{P}\Big(\exists\, z \in \mathbb{Z}\colon\ (z,0) \to x' \times [s,s+\mathrm{d}s]\Big)\, \mathcal{P}\Big((x',s) \to x \times [s,t]\Big)\Big] \\
&\leq 4 \int_0^t \mathcal{P}\Big(\exists\, z \in \mathbb{Z}\colon\ (z,0) \to 0 \times [s,s+\mathrm{d}s]\Big)\, \mathcal{P}\Big((0,0) \to q'-q \times [0,t-s]\Big) \\
&\leq 4\, \mathbb{E}_0^{\mathrm{SRW}}(|R_t|)\, \mathbb{P}_0^{\mathrm{SRW}}(\tau_{q'-q} \leq t),
\end{aligned} \tag{4.13}$$

where the last inequality uses (4.6). We already saw that $\mathbb{E}_0^{\mathrm{SRW}}(|R_t|) \sim C'\sqrt{t}$ as $t \to \infty$. By using, respectively, the reflection principle, the fact that $q' - q \geq 2\sqrt{t \log t}$, and the Azuma-Hoeffding inequality (see [31], (E14.2)), we get

$$\mathbb{P}_0^{\mathrm{SRW}}(\tau_{q'-q} \leq t) = 2\mathbb{P}_0^{\mathrm{SRW}}(S_t \geq q'-q) \leq 2\mathbb{P}_0^{\mathrm{SRW}}(S_t \geq 2\sqrt{t \log t}) \leq 2e^{-\frac{4t \log t}{2t}} = \frac{2}{t^2}. \tag{4.14}$$

Combining (4.13–4.14), we get $\mathcal{P}([A_t]^c) \leq 2C'/t^{3/2}$, which tends to zero as $t \to \infty$. This proves the claim in (4.12), because $\mathcal{P}(A_t) > 0$ for all $t \geq 0$. ∎

4.2 Slow-down

We are now ready to prove Theorem 1.3. The proof comes in two lemmas.

Lemma 4.3. *For all $\rho \in (0,1)$ and $C > 1/\log(\alpha/\beta)$,*

$$\begin{aligned}
\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\nu_\rho,0}(X_t \leq C \log t) = 0, \\
\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\nu_\rho,0}(X_t \geq -C \log t) = 0.
\end{aligned} \tag{4.15}$$

Proof. To prove the first half of (4.15), the idea is to force ξ to vacate an interval of length $C \log t$ to the right of 0 up to time t and to show that, with probability tending to 1 as $t \to \infty$, X does not manage to cross this interval up to time t when C is large enough.

For $t > 0$, let $L_t = C \log t$ and

$$E_t = \big\{\xi_s(x) = 0\ \forall\, x \in [0, L_t] \cap \mathbb{Z}\ \forall\, s \in [0,t]\big\}. \tag{4.16}$$

By Lemma 4.1 we have, for some $C' > 0$ and t large enough,

$$P^{\nu_\rho}(E_t) \geq \mathrm{e}^{-C'\sqrt{t}\log t}. \tag{4.17}$$

Hence

$$\mathbb{P}_{\nu_\rho,0}(X_t \leq L_t) \geq \mathbb{P}_{\nu_\rho,0}(X_t \leq L_t \mid E_t)\, P^{\nu_\rho}(E_t) \geq \mathbb{P}_{\nu_\rho,0}(X_t \leq L_t \mid E_t)\, \mathrm{e}^{-C'\sqrt{t}\log t}. \tag{4.18}$$

To complete the proof it therefore suffices to show that

$$\lim_{t\to\infty} \mathbb{P}_{\nu_\rho,0}(X_t \leq L_t \mid E_t) = 1. \tag{4.19}$$

Let $\tau_{L_t} = \inf\{t \geq 0\colon\ X_t > L_t\}$. Then $\{X_t \leq L_t \mid E_t\} \supset \{\tau_{L_t} > t \mid E_t\}$, and so it suffices to show that

$$\lim_{t\to\infty} \mathbb{P}_{\nu_\rho,0}(\tau_{L_t} > t \mid E_t) = 1. \tag{4.20}$$

We say that X starts a *trial* when it enters the interval $[0, L_t] \cap \mathbb{Z}$ from the left prior. We say that the trial is successful when X hits L_t before returning to 0. Let $M(t)$ be the number of trials prior to time t, and let A_n be the event that the n-th trial is successful. Since

$$\{\tau_{L_t} \leq t\} \subset \bigcup_{n=1}^{M(t)} A_n, \tag{4.21}$$

we have

$$\begin{aligned}
\mathbb{P}_{\nu_\rho,0}\big(\tau_{L_t} \leq t \mid E_t\big) &\leq \mathbb{P}_{\nu_\rho,0}\left(\bigcup_{n=1}^{M(t)} A_n \,\middle|\, E_t\right) \\
&\leq \mathbb{P}_{\nu_\rho,0}\left(\bigcup_{n=1}^{2(\alpha+\beta)t} A_n,\ M(t) \leq 2(\alpha+\beta)t \,\middle|\, E_t\right) \\
&\quad + \mathbb{P}_{\nu_\rho,0}\Big(M(t) > 2(\alpha+\beta)t \mid E_t\Big).
\end{aligned} \tag{4.22}$$

We will show that both terms in the r.h.s. tend to zero as $t \to \infty$.

To estimate the second term in (4.22), let $N(t)$ be the number of jumps by X prior to time t, which is Poisson distributed with mean $(\alpha+\beta)t$ and is independent of ξ. Since $N(t) \geq M(t)$, it follows that

$$\mathbb{P}_{\nu_\rho,0}\Big(M(t) > 2(\alpha+\beta)t \mid E_t\Big) \leq \mathrm{Poi}\big(N(t) > 2(\alpha+\beta)t\big), \tag{4.23}$$

which tends to zero as $t \to \infty$. To estimate the first term in (4.22), note that, since $\mathbb{P}_{\nu_\rho,0}(A_n)$ is independent of n, we have

$$\begin{aligned}
&\mathbb{P}_{\nu_\rho,0}\left(\bigcup_{n=1}^{2(\alpha+\beta)t} A_n,\ M(t) \leq 2(\alpha+\beta)t \,\middle|\, E_t\right) \\
&\quad \leq \mathbb{P}_{\nu_\rho,0}\left(\bigcup_{n=1}^{2(\alpha+\beta)t} A_n \,\middle|\, E_t\right) \leq 2(\alpha+\beta)t\ \mathbb{P}_{\nu_\rho,0}\big(A_1 \mid E_t\big).
\end{aligned} \tag{4.24}$$

But $\mathbb{P}_{\nu_\rho,0}(A_1 \mid E_t)$ is the probability that the random walk on $\mathbb{Z}$ that jumps to the right with probability $\beta/(\alpha+\beta)$ and to the left with probability $\alpha/(\alpha+\beta)$ hits L_t before 0 when it starts from 1. Consequently,

$$2(\alpha+\beta)t\ \mathbb{P}_{\nu_\rho,0}(A_1 \mid E_t) = 2(\alpha+\beta)t\ \frac{(\alpha/\beta)-1}{(\alpha/\beta)^{L_t}-1}, \tag{4.25}$$

which tends to zero as $t \to \infty$ when $L_t > C\log t$ with $C > 1/\log(\alpha/\beta)$. This completes the proof of the first half of (4.15).

To get the second half of (4.15), note that $-X_t$ is equal in distribution to X_t when ρ is replaced by $1-\rho$. ∎

Lemma 4.4. *For all* $\rho \in (0,1)$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\nu_\rho,0}(|X_t| \leq 2\sqrt{t\log t}) = 0. \tag{4.26}$$

Proof. The idea is to create a trap around 0 by forcing ξ up to time t to vacate an interval to the right of 0 and occupy an interval to the left of 0, separated by a suitable distance.

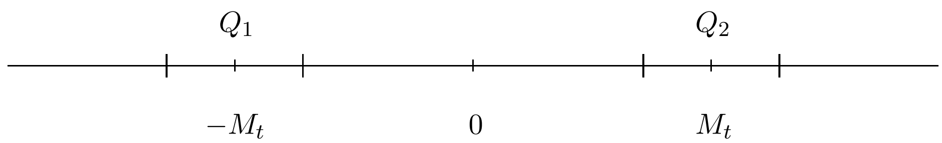

Figure 7: Location of the intervals Q_1 and Q_2. The width of Q_1, Q_2 is $2L_t$. The interval spanning Q_1, Q_2 and the space in between is I_t.

For $t > 0$, let $L_t = C\log t$ with $C > \log(\alpha/\beta)$, $M_t = \sqrt{t\log t}$,

$$Q_1 = \big(-M_t + [-L_t, L_t]\big) \cap \mathbb{Z}, \qquad Q_2 = \big(M_t + [-L_t, L_t]\big) \cap \mathbb{Z}, \tag{4.27}$$

and $I_t = [-M_t - L_t, M_t + L_t] \cap \mathbb{Z}$ (see Fig. 7). For $i = 1,2$ and $j = 0,1$, define the event

$$E_i^j = \Big\{\xi_s(x) = j\ \forall\, x \in Q_i,\ \forall\, s \in [0,t]\Big\}. \tag{4.28}$$

Estimate, noting that $L_t \leq M_t$ for t large enough,

$$\begin{aligned} &\mathbb{P}_{\nu_\rho,0}\Big(|X_t| \leq 2M_t\Big) \geq \mathbb{P}_{\nu_\rho,0}\Big(X_t \in I_t\Big) \\ &\quad \geq \mathbb{P}_{\nu_\rho,0}\Big(X_t \in I_t,\, E_1^1, E_2^0\Big) = \mathbb{P}_{\nu_\rho,0}\Big(X_t \in I_t \mid E_1^1, E_2^0\Big)\, \mathbb{P}_{\nu_\rho,0}\big(E_1^1, E_2^0\big). \end{aligned} \tag{4.29}$$

Since $\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\nu_\rho,0}\big(E_1^1, E_2^0\big) = 0$ by Lemma 4.2, it suffices to show that

$$\lim_{t\to\infty} \mathbb{P}_{\nu_\rho,0}\Big(X_t \in I_t \mid E_1^1, E_2^0\Big) = 1. \tag{4.30}$$

To that end, estimate

$$\begin{aligned} \mathbb{P}_{\nu_\rho,0}\Big(X_t \in I_t \mid E_1^1, E_2^0\Big) &\geq \mathbb{P}_{\nu_\rho,0}\Big(X_t \leq M_t + L_t \mid E_1^1, E_2^0\Big) \\ &\quad + \mathbb{P}_{\nu_\rho,0}\Big(X_t \geq -M_t - L_t \mid E_1^1, E_2^0\Big) - 1. \end{aligned} \tag{4.31}$$

Now, irrespective of what ξ does in between Q_1 and Q_2 up to time t, the same argument as in the proof of Lemma 4.3 shows that

$$\begin{aligned} &\lim_{t\to\infty} \mathbb{P}_{\nu_\rho,0}\Big(X_t \leq M_t + L_t \mid E_1^1, E_2^0\Big) = 1, \\ &\lim_{t\to\infty} \mathbb{P}_{\nu_\rho,0}\Big(X_t \geq -M_t - L_t \mid E_1^1, E_2^0\Big) = 1. \end{aligned} \tag{4.32}$$

Combine this with (4.31) to obtain (4.30). ∎

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