CANONICAL HEIGHT AND LOGARITHMIC EQUIDISTRIBUTION ON SUPERELLIPTIC CURVES

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Abstract. Let X be a smooth projective curve over a number field K given by an affine equation $y^N = f(x)$ for some integer $N > 1$ and for some monic and separable polynomial $f(x)$ over K of degree larger than N and relative prime to N. We prove that the canonical height on the image of X in its jacobian can be written as a sum, over all places of K, of local integrals over X. We also prove that, except for possibly finitely many exceptions, these local integrals can be obtained by averaging over the n-division points of X. Our results extend earlier results in the context of elliptic curves due to Everest, ní Flathúin and Ward, as well as recent results in the context of dynamical systems on the projective line due to Pineiro, Szpiro and Tucker.

1. Introduction

Let K be a number field. Let X be a smooth projective curve over K given by an equation $y^N = f(x)$, where $f(x) \in K[x]$ is a monic separable polynomial of degree $m > N > 1$ and where $\gcd(m, N) = 1$. We call such a curve a superelliptic curve of type (N, m) over K. The curve X has a unique point o at infinity, which is K-rational. The genus g of X equals $g = \frac{1}{2}(N-1)(m-1)$, by Riemann-Hurwitz; in particular g is positive.

Let v be a place of K, and fix a v-adic absolute value $|\cdot|_v$ on K_v. We are interested in local integrals:

$$\lambda_v(p) = \frac{1}{N} \int_{X_v} \log |x - x(p)|_v \, \mu_v$$

for $o \neq p \in X(K_v)$. The measure space (X_v, μ_v) is a suitable "analytic space" associated to X at v; in fact X_v is the Berkovich analytic space associated to $X \otimes \hat{\overline{K_v}}$ if v is non-archimedean, and X_v is the complex analytic space $X(\overline{K_v})$ if v is archimedean. Here $\overline{K_v}$ is an algebraic closure of K_v. The measure μ_v is the canonical Arakelov measure on X_v, as defined in Section 2 below.

Let $o \neq p \in X(K)$. Then $\lambda_v(p)$ vanishes for almost all v and we put:

$$h_{X,x}(p) = \frac{1}{[K:\mathbb{Q}]} \sum_v n_v \, \lambda_v(p) = \frac{1}{[K:\mathbb{Q}]} \frac{1}{N} \sum_v n_v \int_{X_v} \log |x - x(p)|_v \, \mu_v \, ,$$

where the n_v are suitable local factors. We put $h_{X,x}(o) = 0$. Let $\overline{K}$ be an algebraic closure of K; then $h_{X,x}$ extends naturally to a real-valued function on $X(\overline{K})$. Our first result is as follows.

The author is supported by a VENI grant from the Netherlands Organisation for Scientific Research (NWO).

Theorem A. *The function $h_{X,x}$ is a Weil height on $X(\overline{K})$ associated to the line bundle $\mathcal{O}_X(o)$. In fact, let $J = \mathrm{Pic}^0 X$ be the jacobian of X and let h_J be the canonical (Néron-Tate) height on $J(\overline{K})$. Then the formula:*

$$h_{X,x}(p) = \frac{1}{g} h_J([p-o])$$

holds for all $p \in X(\overline{K})$.

Our second result says that, except possibly when p is a Weierstrass point, the local integrals $\lambda_v(p)$ for $p \in X(K)$ can be computed by averaging over the n-division points of X. To introduce these, consider on J the subscheme Θ represented by the classes $[q_1 + \cdots + q_{g-1} - (g-1)o]$ with $q_1, \ldots, q_{g-1}$ running over X. Then Θ is a symmetric theta divisor on J. Now if $n \geq g$ is an integer we define a subscheme H_n of X by putting:

$$H_n = \iota^* [n]^* \Theta \, .$$

Here $[n]$ denotes multiplication by n on J, and $\iota \colon X \to J$ is the natural embedding sending p to $[p-o]$. It can be shown that H_n is an effective divisor on X of degree gn^2. The points in the support of H_n are called the n-division points of X; they naturally generalise the notion of n-torsion points on elliptic curves. We mostly view H_n as a multi-set of $\overline{K}$-points of X of cardinality gn^2.

Theorem B. *Let $o \neq p \in X(K)$, and assume that p is not a Weierstrass point of X. Let v be a place of K. Then:*

$$\frac{1}{gn^2} \sum_{\substack{q \in H_n \\ x(q) \neq x(p), \infty}} \log |x(p) - x(q)|_v \longrightarrow \int_{X_v} \log |x - x(p)|_v \, \mu_v \tag{1.1}$$

for a suitable sequence of natural numbers n tending to infinity. Here the points in H_n are counted with multiplicity.

Note that the left hand side of (1.1) is well-defined. Recall that a $\overline{K}$-point p of X is a Weierstrass point of X if there is a rational function on $X \otimes \overline{K}$ with a pole of multiplicity at most g at p. If $g \geq 2$, each ramification point of $x \colon X \to \mathbb{P}^1_K$ is a Weierstrass point of X. If $x \colon X \to \mathbb{P}^1_K$ is a hyperelliptic curve (so $N = 2$) then conversely each Weierstrass point is a hyperelliptic ramification point. There are at most $g^3 - g$ Weierstrass points in $X(\overline{K})$.

Theorem A and Theorem B are known in the case that (X, o) is an elliptic curve, i.e., when $m = 3, N = 2$. In this case the function $\lambda_v \colon X(K_v) - \{o\} \to \mathbb{R}$ is just the unique Néron function for o on $X(K_v)$ normalised by the condition:

$$\lambda_v(p) - \frac{1}{2} \log |x(p)|_v \to 0 \quad \text{as} \quad p \to o \, .$$

This can be seen for example by integrating the quasi-parallelogram law satisfied by the Néron function:

$$\lambda_v(p+q) + \lambda_v(p-q) = 2\lambda_v(p) + 2\lambda_v(q) - \log |x(p) - x(q)|_v \tag{1.2}$$

against $\mu_v(q)$, using the translation invariance of μ_v. Theorem A is then obtained from the usual formula expressing the canonical (Néron-Tate) height on an elliptic curve as a sum of local Néron functions.

Theorem B and variations of Theorem B in the elliptic curve case were proved in the 1990s by Everest and ní Flathúin [14] and Everest and Ward [15], and more recently by Baker, Ih and Rumely [2] and Szpiro and Tucker [28]. The proofs are based on classical diophantine approximation results (linear forms in logarithms, and Roth's theorem).

We note that the divisors H_n are exactly the divisors of Weierstrass points of the line bundles $\mathcal{O}_X(o)^{\otimes n+g-1}$. A theorem of Neeman [21] therefore implies that, at least if v is archimedean, the multi-sets H_n are equidistributed with respect to μ_v in the weak topology, i.e., the average of each continuous function on X_v over H_n converges to its integral against μ_v, as $n \to \infty$. Theorem B asserts that such a convergence result holds as well for certain functions on X_v with logarithmic singularities, in both the archimedean and the non-archimedean case.

The proof of Theorem A (given in Section 3) is based on expressing the local integrals λ_v within Zhang's theory of admissible local pairing [31], using potential theory on X_v, and by exploiting the connection discovered by Hriljac and Faltings between global intersection pairing on X and the canonical height on J (i.e., the arithmetic Hodge index theorem). The bit of potential theory on X_v that we need is developed in Section 2, based on the thesis of Thuillier [29]. In Section 4 we discuss some additional results in the context of hyperelliptic curves. This features a formula for the admissible self-intersection of the relative dualising sheaf of a hyperelliptic curve as a sum of local double integrals over the places of K.

The proof of Theorem B (given in Section 5) is based on a diophantine approximation result of Faltings [16] on abelian varieties, and a formula relating the "division polynomial" of H_n to λ_v and a suitable Néron function $\lambda_{\Theta,v}$ on J. Incidentally, we show that this formula can be used to prove that the average height of division points remains bounded. This also follows from results of Burnol [8].

We expect that the statement of Theorem B remains true even when $p \neq o$ is a Weierstrass point on X. We prove this in Section 6 when X is a hyperelliptic curve. The proof consists of a direct computation, using a determinantal formula due to D. Cantor [10] for hyperelliptic division polynomials. A corollary of this computation is a generalisation of a result on "one half log discriminant" due to Szpiro and Tucker [27].

As an application of Theorems A and B we prove in Section 7 a finiteness result for divisors H_n which are integral with respect to a given algebraic point. It would be interesting to have function field analogues of the results in this paper, and to extend the results to more general types of curves.

We finish the Introduction by mentioning some recent results similar in spirit to Theorems A and B. First, let $\varphi\colon \mathbb{P}^1_K \to \mathbb{P}^1_K$ be a surjective morphism of degree $d > 1$, i.e., a dynamical system on the projective line over K. Call and Silverman [9] prove that there exists a canonical height $h_\varphi\colon \mathbb{P}^1(\overline{K}) \to \mathbb{R}$ associated to φ, which is non-negative, and satisfies the functional equation $h_\varphi(\varphi(\alpha)) = d h_\varphi(\alpha)$ for all $\alpha \in \mathbb{P}^1(\overline{K})$. An example is the "usual" height h given by $h(\alpha) = \frac{1}{[K:\mathbb{Q}]}\sum_v \log\max\{1, |\alpha|_v\}$ for $\infty \neq \alpha \in \mathbb{P}^1(K)$, which is the canonical height associated to the squaring map $\varphi(x) = x^2$ of degree 2. If ∞ is a fixed point for φ, then Favre and Rivera-Letelier [17] and Pineiro, Szpiro and Tucker [24] prove that for $\infty \neq \alpha \in \mathbb{P}^1(K)$ one has a "Mahler type" formula:

$$h_\varphi(\alpha) = \frac{1}{[K:\mathbb{Q}]}\sum_v n_v \int_{\mathbb{P}^1_v} \log|x-\alpha|_v \, \mu_{\varphi,v} \tag{1.3}$$

for the canonical height associated to φ. In [24] the expressions $\int_{\mathbb{P}^1_v} \log|x-\alpha|_v \mu_{\varphi,v}$ are certain natural local factors defined using intersection theory on suitable models of $\mathbb{P}^1_K$, whereas in [17] it is shown that the expressions $\int_{\mathbb{P}^1_v} \log|x-\alpha|_v \mu_{\varphi,v}$ can be taken to be actual integrals over the Berkovich projective line, for a canonical measure $\mu_{\varphi,v}$ on $\mathbb{P}^1_v$ associated to φ. Formula (1.3) specialises to a well-known formula of Mahler in the case that h_φ is the usual height h associated to $\varphi\colon x \mapsto x^2$.

In [28] Szpiro and Tucker prove two results in the context of dynamical systems analogous to Theorem B, cf. [28], Theorem 4.6 and Theorem 4.7. First, for $\infty \neq \alpha \in \mathbb{P}^1(K)$ and for any place v of K one has:

$$\frac{1}{d^n} \sum_{\substack{w:\varphi^n(w)=w \\ w\neq\alpha,\infty}} \log|w-\alpha|_v \longrightarrow \int_{\mathbb{P}^1_v} \log|x-\alpha|_v \, \mu_{\varphi,v}$$

as $n \to \infty$. Second, if $\alpha_0 \in \mathbb{P}^1(K)$ is not an exceptional point for φ (i.e., the set $\cup_{n=1}^{\infty} (\varphi^n)^{-1}(\alpha_0)$ is infinite) then:

$$\frac{1}{d^n} \sum_{\substack{w:\varphi^n(w)=\alpha_0 \\ w\neq\alpha,\infty}} \log|w-\alpha|_v \longrightarrow \int_{\mathbb{P}^1_v} \log|x-\alpha|_v \, \mu_{\varphi,v}$$

for each $\infty \neq \alpha \in \mathbb{P}^1(K)$, as $n \to \infty$. In both cases, the points w are counted with multiplicity. The proofs in [28] are based upon Roth's theorem.

In [12], Théorème 1.2, Chambert-Loir and Thuillier prove a logarithmic equidistribution result in the general context of polarised projective varieties (X, L) over K. The result asserts the equidistribution, at each place v of K, of any generic sequence of Galois orbits of "small" points, with respect to any function with logarithmic singularities along an effective Cartier divisor on X whose canonical height is equal to the canonical height of X. Here, both the adelic integration measure and the canonical height are determined by the polarisation L, and again one has a "Mahler type" formula for the canonical height, cf. [12], Théorème 1.4. Interestingly, the proof of the result of Chambert-Loir and Thuillier does not rely on diophantine approximation, but instead on a weak equidistribution result of Zhang for polarised projective varieties, combined with a result on arithmetic volumes due to Yuan.

We note finally that our Theorem B does not seem to be a direct consequence of the logarithmic equidistribution results [12] [28] discussed above. Indeed, in both [12] and [28] the logarithmic equidistribution is asserted of an infinite sequence of points whose heights become "small" with respect to the canonical height. Now Theorem A says that the canonical height on X is essentially the restriction to X of the canonical height on the jacobian of X. But the Bogomolov conjecture, first proved by Ullmo [30], implies that X contains no infinite sequence of "small" points, if $g \geq 2$.

2. Potential theory on analytic curves

Let X be a geometrically connected smooth projective curve over a local field $(k, |\cdot|)$, let $\overline{k}$ be an algebraic closure of k, and let $\hat{\overline{k}}$ be the completion of $\overline{k}$. The absolute value $|\cdot|$ extends in a unique way to $\hat{\overline{k}}$. One has associated to X a locally ringed space $\mathfrak{X} = (|\mathfrak{X}|, \mathcal{O}_{\mathfrak{X}})$ where the underlying topological space $|\mathfrak{X}|$ has the following properties: $|\mathfrak{X}|$ is compact, metrisable, path-connected, and contains $X(\overline{k})$ with its topology induced from $|\cdot|$ naturally as a dense subspace. If k is archimedean, we just take $X(\overline{k})$ itself, with its structure of complex analytic space;

if k is non-archimedean we let $\mathfrak{X}$ be the Berkovich analytic space associated to $X \otimes \hat{\overline{k}}$, see [3].

Our purpose in this section is to put a canonical probability measure $\mu_{\mathfrak{X}}$ on $|\mathfrak{X}|$, and to discuss some elements of a potential theory on $\mathfrak{X}$. This is standard for archimedean k; for non-archimedean k we base our discussion on Thuillier's thesis [29], Chapter 3. See also Baker's expository paper [1]. As an application we derive a useful formula for the integral $\int_{\mathfrak{X}} \log|f| \mu_{\mathfrak{X}}$ where f is a non-zero rational function on $X \otimes \overline{k}$. This formula establishes a link with Zhang's theory of local admissible pairing [31].

We start by considering the natural exact sequence:

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{A}^0 \xrightarrow{\mathrm{dd}^c} \mathcal{A}^1 \longrightarrow 0$$

of sheaves of $\mathbb{R}$-vector spaces on $\mathfrak{X}$. Here $\mathcal{H}$ is the sheaf of harmonic functions on $\mathfrak{X}$, $\mathcal{A}^0$ is the sheaf of smooth functions on $\mathfrak{X}$, $\mathcal{A}^1$ is the sheaf of smooth forms on $\mathfrak{X}$, and dd^c is the Laplace operator. The sheaf $\mathcal{A}^0$ is in fact a sheaf of $\mathbb{R}$-algebras and the sheaf $\mathcal{A}^1$ is naturally a sheaf of modules over $\mathcal{A}^0$.

We let $\mathcal{A}^0(\mathfrak{X})$ and $\mathcal{A}^1(\mathfrak{X})$ be the spaces of global sections of $\mathcal{A}^0$ and $\mathcal{A}^1$. Further we put $\mathrm{D}^0(\mathfrak{X}) = \mathcal{A}^1(\mathfrak{X})^*$ and $\mathrm{D}^1(\mathfrak{X}) = \mathcal{A}^0(\mathfrak{X})^*$ for their $\mathbb{R}$-linear duals. We have a natural $\mathbb{R}$-linear integration map $\int_{\mathfrak{X}} \colon \mathcal{A}^1(\mathfrak{X}) \to \mathbb{R}$ and a natural $\mathbb{R}$-bilinear pairing $\mathcal{A}^0(\mathfrak{X}) \times \mathcal{A}^1(\mathfrak{X}) \to \mathbb{R}$ given by $(\varphi, \omega) \mapsto \int_{\mathfrak{X}} \varphi\, \omega$. This pairing yields a natural commutative diagram:

$$\begin{array}{ccc} \mathrm{D}^0(\mathfrak{X}) & \longrightarrow & \mathrm{D}^1(\mathfrak{X}) \\ \uparrow & & \uparrow \\ \mathcal{A}^0(\mathfrak{X}) & \xrightarrow{\mathrm{dd}^c} & \mathcal{A}^1(\mathfrak{X}) \end{array}$$

where the upward arrows are injections. The map $\mathrm{D}^0(\mathfrak{X}) \to \mathrm{D}^1(\mathfrak{X})$ is the dual of the map $\mathcal{A}^0(\mathfrak{X}) \xrightarrow{\mathrm{dd}^c} \mathcal{A}^1(\mathfrak{X})$, and we shall also denote it by dd^c. The kernel of $\mathrm{dd}^c : \mathrm{D}^0(\mathfrak{X}) \to \mathrm{D}^1(\mathfrak{X})$ is a one-dimensional $\mathbb{R}$-vector space, to be identified with the set of constant functions on $\mathfrak{X}$. Elements of $\mathrm{D}^\alpha(\mathfrak{X})$ are called (α, α)-currents; $(1,1)$-currents can be viewed as measures on $|\mathfrak{X}|$. The unit element of $\mathcal{A}^0(\mathfrak{X})$ gives, under the natural map $\mathcal{A}^0(\mathfrak{X}) \to \mathrm{D}^1(\mathfrak{X})^*$, a natural $\mathbb{R}$-linear integration map $\int_{\mathfrak{X}} \colon \mathrm{D}^1(\mathfrak{X}) \to \mathbb{R}$, extending $\int_{\mathfrak{X}}$ on $\mathcal{A}^1(\mathfrak{X})$. Associated to each non-zero rational function f on $X \otimes \overline{k}$ we have a natural $(0,0)$-current $\log|f| \in \mathrm{D}^0(\mathfrak{X})$. For each closed point p on $X \otimes \overline{k}$ we have a Dirac measure $\delta_p \in \mathrm{D}^1(\mathfrak{X})$, and by linearity we obtain a measure $\delta_D \in \mathrm{D}^1(\mathfrak{X})$ for each divisor D on $X \otimes \overline{k}$.

Proposition 2.1. *(i) (Poincaré-Lelong equation) Let f be a non-zero rational function on $X \otimes \overline{k}$. Then the equation:*

$$\mathrm{dd}^c \log|f| - \delta_{\mathrm{div} f} = 0$$

holds in $\mathrm{D}^1(\mathfrak{X})$.
(ii) (The Laplace operator is self-adjoint) We have:

$$\int_{\mathfrak{X}} \varphi\, \mathrm{dd}^c\, \psi = \int_{\mathfrak{X}} \psi\, \mathrm{dd}^c\, \varphi$$

for all $\varphi, \psi \in \mathrm{D}^0(\mathfrak{X})$.
(iii) (Existence and uniqueness of Green's functions) Let $\mu \in \mathcal{A}^1(\mathfrak{X})$ *be a smooth*

measure with $\int_{\mathfrak{X}} \mu = 1$*, and let* $p \in X(\overline{k})$*. Then there exists a unique* $g_{\mu,p} \in \mathrm{D}^0(\mathfrak{X})$ *such that:*

$$\mathrm{dd}^c\, g_{\mu,p} = \mu - \delta_p \, , \quad \int_{\mathfrak{X}} g_{\mu,p}\, \mu = 0 \, .$$

The symmetry relation $g_{\mu,p}(q) = g_{\mu,q}(p)$ *holds for all* $p \neq q \in X(\overline{k})$*.*

Proof. This is well-known for archimedean k. We find (i)–(iii) respectively in [29], Proposition 3.3.15, Proposition 3.2.12 and Proposition 3.3.13, for non-archimedean k. □

We construct a canonical probability measure $\mu_{\mathfrak{X}} \in \mathcal{A}^1(\mathfrak{X})$ as follows. If k is archimedean we let $\mu_{\mathfrak{X}}$ be the canonical Arakelov probability measure on $X(\overline{k})$. One way of giving $\mu_{\mathfrak{X}}$ is as follows: let $\iota \colon X(\overline{k}) \to J(\overline{k})$ be an immersion of $X(\overline{k})$ into the complex torus $J(\overline{k})$, where $J = \mathrm{Pic}^0 X$ is the jacobian of X. Then $\mu_{\mathfrak{X}} = \frac{1}{g}\iota^*\nu$ where ν is the unique translation-invariant $(1,1)$-form representing the first Chern class of the line bundle defining the principal polarisation on $J(\overline{k})$.

Now suppose that k is non-archimedean. We let $\mathcal{R}$ be the reduction graph of Chinburg-Rumely [13] of X. This is a metrised graph, receiving a canonical surjective continuous specialisation map $\mathrm{sp} \colon |\mathfrak{X}| \to \mathcal{R}$. In particular $\mathcal{R}$ is compact and path-connected. The map sp has a canonical section $i \colon \mathcal{R} \to |\mathfrak{X}|$; in fact $i \circ \mathrm{sp} \colon \mathfrak{X} \to i(\mathcal{R})$ is a deformation retraction. The Laplace operator on $\mathfrak{X}$ extends in a natural way the Laplace operator on $\mathcal{R}$.

In [31] a canonical probability measure $\mu_{\mathcal{R}}$ is constructed on the reduction graph $\mathcal{R}$ of X, called admissible measure. It satisfies properties analogous to the Arakelov measure in the archimedean setting; especially, it gives rise to an adjunction formula. We define the measure $\mu_{\mathfrak{X}}$ on $|\mathfrak{X}|$ by putting $\mu_{\mathfrak{X}} = i_*\mu_{\mathcal{R}}$.

Note that we obtain a canonical symmetric pairing $(,)_a$ on $X(\overline{k})$ by putting $(p,q)_a = g_{\mu_{\mathfrak{X}},p}(q)$ for $p \neq q$. This pairing coincides with the admissible pairing $(,)_a$ constructed in [31], Section 4 using Green's functions on $\mathcal{R}$ with respect to $\mu_{\mathcal{R}}$, and the specialisation map.

For example, if X is an elliptic curve with j-invariant j_X in k then $\mathcal{R}$ is a point if $|j_X| \leq 1$, and $\mathcal{R}$ is a circle of circumference $\log|j_X|$ if $\log|j_X| > 1$. A detailed discussion of the associated Berkovich space, including formulas for $(,)_a$ can be found for example in [23].

The following proposition calculates the integrals $\int_{\mathfrak{X}} \log|f|\mu_{\mathfrak{X}}$ in terms of admissible pairing.

Proposition 2.2. *Let* f *be a non-zero rational function on* $X \otimes \overline{k}$*. Then the formula:*

$$\int_{\mathfrak{X}} \log|f|\, \mu_{\mathfrak{X}} = \log|f|(r) + (\mathrm{div} f, r)_a$$

holds. Here r *is an arbitrary point in* $X(\overline{k})$ *outside the support of* f*.*

Proof. By Proposition 2.1(i) we have:

$$\mathrm{dd}^c \, \log|f| = \delta_{\mathrm{div} f} \, .$$

By integrating against $g_{\mu_{\mathfrak{X}},r}$ we obtain:

$$\int_{\mathfrak{X}} g_{\mu_{\mathfrak{X}},r}\, \mathrm{dd}^c \, \log|f| = g_{\mu_{\mathfrak{X}},r}(\mathrm{div} f) = (\mathrm{div} f, r)_a \, .$$

On the other hand, by Proposition 2.1(ii) and (iii) we have:

$$\begin{aligned}\int_{\mathfrak{X}} g_{\mu_{\mathfrak{X}},r}\,\mathrm{dd}^c \log|f| &= \int_{\mathfrak{X}} (\log|f|)\,\mathrm{dd}^c g_{\mu_{\mathfrak{X}},r} \\ &= \int_{\mathfrak{X}} (\log|f|)(\mu_{\mathfrak{X}} - \delta_r) \\ &= -\log|f|(r) + \int_{\mathfrak{X}} \log|f|\,\mu_{\mathfrak{X}}\,.\end{aligned}$$

The proposition follows. □

Compare with [31], Theorem 4.6(iii) which states that $\log|f|(r) + (\mathrm{div} f, r)_a$ is constant outside the support of f. Using potential theory we are thus able to interpret this constant as a suitable integral over $\mathfrak{X}$.

3. Proof of Theorem A

In this section we prove Theorem A. So let X be a superelliptic curve of type (N, m) over K with equation $y^N = f(x)$ and point at infinity o. We recall that this implies that $m = \deg(f)$ with $m > N > 1$ and $\gcd(m, N) = 1$. If v is a place of K we denote by $\overline{K_v}$ an algebraic closure of K_v. We endow each K_v with a (standard) absolute value $|\cdot|_v$ as follows. If v is archimedean, we take the euclidean norm on K_v. If v is non-archimedean, we choose $|\cdot|_v$ such that $|\pi|_v = \mathrm{e}$, where π is a uniformiser of K_v. We let X_v be the analytic space associated to $X \otimes \hat{\overline{K_v}}$, and μ_v be the canonical measure on X_v, as in Section 2. For each $o \neq p \in X(\overline{K_v})$ we thus have a well-defined element:

$$\lambda_v(p) = \frac{1}{N} \int_{X_v} \log|x - x(p)|_v\,\mu_v \tag{3.1}$$

of $\mathbb{R}$. Let $(,)_a$ be the local admissible pairing on $X(\overline{K_v})$ discussed in Section 2 and let $\sigma \in \mathrm{Aut}(X \otimes \overline{K_v})$ be a generator of the automorphism group of $x\colon X \to \mathbb{P}^1_K$ over $\overline{K_v}$. Note that $\mathrm{div}(x - x(p)) = -No + \sum_{i=0}^{N-1} \sigma^i(p)$. From Proposition 2.2 we obtain therefore that:

$$N\lambda_v(p) = \log|x(p) - x(r)|_v + \sum_{i=0}^{N-1} \left(\sigma^i(p) - o, r\right)_a\,. \tag{3.2}$$

Here $r \in X(\overline{K_v})$ is an arbitrary point, not in the support of $x - x(p)$.

Proposition 3.1. *The function $\lambda_v\colon X(\overline{K_v}) - \{o\} \to \mathbb{R}$ extends uniquely as an element of $\mathrm{D}^0(X_v)$. As such it satisfies the dd^c-equation:*

$$\mathrm{dd}^c \lambda_v = \mu_v - \delta_o\,.$$

Furthermore we have:

$$\lambda_v(p) - \frac{1}{N}\log|x(p)|_v \to 0$$

as $p \to o$ on $X(K_v)$. In particular λ_v defines a local Weil function with respect to the divisor o on X.

Proof. We consider equation (3.2) with p as a variable and r fixed. Both $(\sigma^i(p) - o, r)_a$ and $\log|x(p) - x(r)|_v$ extend as $(0,0)$-currents over X_v, hence so does λ_v. The extension is unique, as $X(\overline{K_v})$ is dense in X_v. To prove the first formula, note that $(,)_a$ is canonical, hence invariant under $\mathrm{Aut}(X \otimes \overline{K_v})$. We can thus rewrite (3.2) as:

$$N\lambda_v(p) = \log|x(p) - x(r)|_v + \sum_{i=0}^{N-1} \left(p - o, \sigma^i(r)\right)_a .$$

Taking dd^c we have, by Proposition 2.1:

$$N \,\mathrm{dd}^c \lambda_v = \sum_{i=0}^{N-1} \left(\delta_{\sigma^i(r)} - \delta_o\right) + \sum_{i=0}^{N-1} \left(\mu_v - \delta_{\sigma^i(r)}\right) .$$

It follows that $\mathrm{dd}^c \lambda_v = \mu_v - \delta_o$ as required. To prove the second formula, we let $p \to o$ in (3.2). Then the sum $\sum_{i=0}^{N-1} \left(\sigma^i(p) - o, r\right)_a$ converges to zero. □

Now let v run over all places of K and take a $o \neq p \in X(K)$. From (3.2) it follows that $\lambda_v(p) = 0$ for almost all v, as the right hand side has this property. Hence the function $h_{X,x}$ on $X(K)$ given by:

$$h_{X,x}(p) = \frac{1}{[K:\mathbb{Q}]} \sum_v n_v \lambda_v(p) = \frac{1}{[K:\mathbb{Q}]} \frac{1}{N} \sum_v n_v \int_{X_v} \log|x - x(p)|_v \,\mu_v$$

and $h_{X,x}(o) = 0$ is well-defined. Here n_v is a (standard) local factor defined as follows: if v is real, then $n_v = 1$; if v is complex, then $n_v = 2$; if v is non-archimedean, then n_v is the log of the cardinality of the residue field at v.

Now let $\overline{K}$ be an algebraic closure of K. Using (3.2) with another base field we see that we can extend $h_{X,x}$ compatibly over all finite extensions of K contained in $\overline{K}$, hence we can extend $h_{X,x}$ to $X(\overline{K})$. We call $h_{X,x}$ the canonical height on $X(\overline{K})$ associated to $x \colon X \to \mathbb{P}^1_K$.

Let g be the genus of X. By Riemann-Hurwitz we have $g = \frac{1}{2}(N-1)(m-1)$; in particular g is positive. The following is a restatement of Theorem A.

Theorem 3.2. *The function $h_{X,x}$ on $X(\overline{K})$ is a Weil height with respect to the line bundle $\mathcal{O}_X(o)$ on X. Let $J = \mathrm{Pic}^0 X$ be the jacobian of X and let h_J be the canonical (Néron-Tate) height on $J(\overline{K})$. Then the formula:*

$$h_J([p - o]) = g h_{X,x}(p)$$

holds for all $p \in X(\overline{K})$.

Proof. We can assume without loss of generality that K contains a root of unity of order N, and that $p \in X(K)$. From (3.2) we find by summing over all places v of K that:

$$N h_{X,x}(p) = \frac{1}{[K:\mathbb{Q}]} \sum_{i=0}^{N-1} \left(\sigma^i(p) - o, r\right)_a \tag{3.3}$$

for any $r \in X(K)$ which is not in the support of $x - x(p)$. Here $(,)_a$ now denotes global admissible pairing. Since we can vary r it follows that as an admissible line bundle $\otimes_{i=0}^{N-1} \mathcal{O}_X(\sigma^i(p) - o)$ is a trivial line bundle, with a constant metric at all places v. Hence in (3.3) we can also take $r = o$. By invariance of $(,)_a$ under $\mathrm{Aut}(X)$ we arrive at the simple formula:

$$h_{X,x}(p) = \frac{1}{[K:\mathbb{Q}]} (p - o, o)_a \tag{3.4}$$

for all $p \in X(K)$. This shows that $h_{X,x}$ defines a Weil height with respect to $\mathcal{O}_X(o)$, by [31], Theorem 4.6.(2).

Recall that we have an equation $y^N = f(x)$ for X giving (X,o) the structure of a superelliptic curve of type (N,m) over K. A small computation shows that the differential:

$$\omega_{1,N-1} = -\frac{\mathrm{d}x}{Ny^{N-1}} = -\frac{\mathrm{d}y}{f'(x)}$$

has divisor equal to $2(g-1)o$ on X. Thus, if we let ω be the admissible canonical line bundle on X, then $2(g-1)o - \omega$ is a trivial line bundle, with a constant metric at all places v. In particular we have $(p-o, 2(g-1)o-\omega)_a = 0$ for all $p \in X(K)$. By writing out we obtain:

$$-(p,\omega)_a = -2(g-1)(p,o)_a + 2(g-1)(o,o)_a - (o,\omega)_a \, .$$

Now by the adjunction formula (cf. [31], 5.2) we have $-(o,\omega)_a = (o,o)_a$ so that:

$$-(p,\omega)_a = -2(g-1)(p,o)_a + (2g-1)(o,o)_a \, . \tag{3.5}$$

The arithmetic Hodge index theorem for admissible pairing (cf. [31], 5.4) implies that:

$$-2[K\colon \mathbb{Q}]h_J([p-o]) = (p-o,p-o)_a = (p,p)_a - 2(p,o)_a + (o,o)_a \, .$$

By the adjunction formula we can write $(p,p)_a = -(p,\omega)_a$ and with (3.5) we arrive at $[K\colon\mathbb{Q}]h_J([p-o]) = g(p-o,o)_a$. By (3.4) we find $h_J([p-o]) = g h_{X,x}(p)$. □

4. Complements on hyperelliptic curves

We want to make some additional remarks about the local Weil functions λ_v in the case that $N=2$, i.e., in the case that (X,o) is an elliptic or pointed hyperelliptic curve. As an application we express the admissible self-intersection of the relative dualising sheaf (cf. [31], 5.4) of a hyperelliptic curve as a sum of local *double* integrals over the places of K. We start by giving a formula for the special value of λ_v at a hyperelliptic ramification point.

Let (X,o) be an elliptic or pointed hyperelliptic curve over K given by an equation $y^2 = f(x)$, where $f(x) \in K[x]$ is monic and separable of degree $m = 2g+1$. Fix a place v of K, as well as an algebraic closure $\overline{K_v}$ of K_v. Keep the v-adic absolute value on K_v and $\overline{K_v}$ as defined in Section 3.

Proposition 4.1. *Let $o \neq p \in X(\overline{K_v})$ be a hyperelliptic ramification point of X and let $\alpha = x(p)$ in $\overline{K_v}$. Then the formula:*

$$2\lambda_v(p) = \int_{X_v} \log|x-\alpha|_v \, \mu_v = \frac{1}{2g}\log|f'(\alpha)|_v$$

holds.

Note that this result is especially remarkable if v is an archimedean place: it says that λ_v, which is a certain transcendental function on $X(\overline{K_v}) - \{o\}$, assumes an algebraic special value at each hyperelliptic ramification point of X.

Proof. We distinguish between the case $g=1$ and the case $g \geq 2$. In the case $g=1$, let p_i, p_j, p_k be the non-trivial hyperelliptic ramification points, i.e., the non-trivial 2-torsion points of (X,o), and let $\alpha_i, \alpha_j, \alpha_k$ in $\overline{K_v}$ be the corresponding roots of f. By the quasi-parallelogram law (1.2) we find:

$$2\lambda_v(p_i) = 2\lambda_v(p_j) + 2\lambda_v(p_k) - \log|\alpha_j - \alpha_k|_v \, .$$

By cyclic permutation of (i, j, k) we obtain two other linear relations between $\lambda_v(p_i)$, $\lambda_v(p_j)$ and $\lambda_v(p_k)$. By elimination we find:

$$\begin{aligned} 2\lambda_v(p_i) &= \frac{1}{2} \log \left(|\alpha_i - \alpha_j|_v |\alpha_i - \alpha_k|_v \right) \\ &= \frac{1}{2} \log |f'(\alpha_i)|_v \,, \end{aligned}$$

settling the case $g = 1$. For $g \geq 2$ we use a result on the arithmetic of symmetric roots from [18]. Let $\alpha_1, \ldots, \alpha_{2g+2}$ on $\mathbb{P}^1(\overline{K_v})$ be the branch points of x (this includes ∞). The symmetric root of a triple $(\alpha_i, \alpha_j, \alpha_k)$ of distinct branch points is then defined to be an element:

$$\ell_{ijk} = \frac{\alpha_i - \alpha_k}{\alpha_j - \alpha_k} \sqrt[2g]{-\frac{f'(\alpha_j)}{f'(\alpha_i)}}$$

of $\overline{K_v}^*$. The actual choice of $2g$-th root will be immaterial in what follows. If α_j equals infinity, the formula should be read as follows:

$$\ell_{i\infty k} = (\alpha_i - \alpha_k) \sqrt[2g]{-f'(\alpha_i)}^{-1} \tag{4.1}$$

(recall that f is monic). Now let $w_1, \ldots, w_{2g+2}$ on $X(\overline{K_v})$ be the hyperelliptic ramification points corrresponding to $\alpha_1, \ldots, \alpha_{2g+2}$. Theorem C of [18] then states that if (w_i, w_j, w_k) is a triple of distinct ramification points, the formula:

$$(w_i - w_j, w_k)_a = -\frac{1}{2} \log |\ell_{ijk}|_v \tag{4.2}$$

holds. Here, as before $(,)_a$ denotes Zhang's local admissible pairing on $X(\overline{K_v})$. Applying Proposition 2.2 to the rational function $x - \alpha_i$, with α_i a finite branch point, we find:

$$\int_{X_v} \log |x - \alpha_i|_v \, \mu_v = \log |x(p) - \alpha_i|_v + 2(w_i - o, p)_a$$

for any $p \neq o, w_i$. Taking $p = w_k$ and applying (4.2) we find:

$$\begin{aligned} \int_{X_v} \log |x - \alpha_i|_v \, \mu_v &= \log |\alpha_i - \alpha_k|_v + 2(w_i - o, w_k)_a \\ &= \log |\alpha_i - \alpha_k|_v - \log |\ell_{i\infty k}|_v \,. \end{aligned}$$

Hence by (4.1):

$$2\lambda_v(\alpha_i) = \int_{X_v} \log |x - \alpha_i|_v \, \mu_v = \frac{1}{2g} \log |f'(\alpha_i)|_v \,.$$

The proposition is proven. □

Note that λ_v depends on the choice of monic equation f for the pointed curve (X, o). Let $\Delta = 2^{4g}\Delta(f)$ where $\Delta(f)$ is the discriminant of f. We refer to [20] for properties of Δ. The discriminant Δ generalises the usual definition $\Delta = 2^4\Delta(f)$ in the case where (X, o) is an elliptic curve. We renormalise λ_v by putting:

$$\hat{\lambda}_v(p) = \lambda_v(p) - \frac{1}{4g(2g+1)} \log |\Delta|_v \,.$$

Then $\hat{\lambda}_v$ is independent of the choice of monic equation f for (X, o), as one checks by replacing x by $u^2x + t$ for $u \in K^*$, $t \in K$. We obtain the familiar relation:

$$\hat{\lambda}_v = \lambda_v - \frac{1}{12}\log|\Delta|_v$$

in the case where (X, o) is an elliptic curve. In that case, the $\hat{\lambda}_v$ have the additional property that $\int_{X_v} \hat{\lambda}_v \mu_v = 0$ for each place v. This is no longer true in general when (X, o) is hyperelliptic, though we have the following result. Assume that $g \geq 2$, and let i be an index with $1 \leq i \leq 2g + 2$. Then put:

$$\chi(X_v) = -2g\left(\log|2|_v + \sum_{k \neq i}(w_i, w_k)_a\right) .$$

We proved in [18], Theorem B that $\chi(X_v)$ is independent of the choice of i, hence is an invariant of X_v, and that $\chi(X_v) \geq 0$ if v is non-archimedean. We conjecture that $\chi(X_v) \geq 0$ even if v is archimedean, but this is proved only if $g = 2$. We have $\chi(X_v) = 0$ if v is non-archimedean and X has potentially good reduction at v. The relevance of $\chi(X_v)$ is that it gives a formula in terms of local invariants for the admissible self-intersection of the relative dualising sheaf $(\omega, \omega)_a$ of X, namely:

$$(\omega, \omega)_a = \frac{2g-2}{2g+1}\sum_v n_v \chi(X_v) , \tag{4.3}$$

where v runs over the places of K. The following result relates $\int_{X_v} \hat{\lambda}_v \, \mu_v$ to $\chi(X_v)$. We believe that this formula could be interesting from the point of view of potential theory on graphs.

Proposition 4.2. *Let v be a place of K. Then the formula:*

$$\int_{X_v} \hat{\lambda}_v \, \mu_v = \frac{1}{2g(2g+1)}\chi(X_v)$$

holds.

Proof. From Proposition 3.1 we obtain that λ_v equals $(p, o)_a$ up to an additive constant, more precisely:

$$\lambda_v(p) = (p, o)_a + \int_{X_v} \lambda_v \, \mu_v$$

for each $o \neq p \in X(\overline{K_v})$. By taking $p = \alpha_i$ and using Proposition 4.1 we find:

$$\frac{1}{4g}\log|f'(\alpha_i)|_v = \lambda_v(w_i) = (w_i, o)_a + \int_{X_v} \lambda_v \, \mu_v .$$

Assume that $\alpha_{2g+2} = \infty$. By summing over $i = 1, \ldots, 2g + 1$ we arrive at:

$$\frac{1}{4g}\log|\Delta(f)|_v = -\frac{1}{2g}\chi(X_v) - \log|2|_v + (2g+1)\int_{X_v} \lambda_v \, \mu_v ,$$

using the definition of $\chi(X_v)$ above. Rewriting a bit gives:

$$\log|\Delta|_v = -2\chi(X_v) + 4g(2g+1)\int_{X_v} \lambda_v \, \mu_v$$

and hence:

$$\chi(X_v) = 2g(2g+1)\int_{X_v} \hat{\lambda}_v \, \mu_v$$

as required. ∎

Corollary 4.3. *The formula:*

$$(\omega,\omega)_a = 2g(g-1)\sum_v n_v \int_{X_v}\int_{X_v} \log|x(p)-x(q)|_v \, \mu_v(p)\mu_v(q)$$

holds.

Proof. Using equation (4.3), Proposition 4.2, the product formula, and the definition of λ_v we find:

$$\begin{aligned}(\omega,\omega)_a &= \frac{2g-2}{2g+1}\sum_v n_v \chi(X_v) \\ &= 4g(g-1)\sum_v n_v \int_{X_v} \hat{\lambda}_v \, \mu_v \\ &= 4g(g-1)\sum_v n_v \int_{X_v} \lambda_v \, \mu_v \\ &= 2g(g-1)\sum_v n_v \int_{X_v}\int_{X_v} \log|x(p)-x(q)|_v \, \mu_v(p)\mu_v(q)\end{aligned}$$

which gives the corollary. ∎

It would be interesting to have explicit formulas for $\lambda_v(p)$ à la the ones of Tate (cf. [26], Chapter VI) for elliptic curves, given the type of the reduction graph $\mathcal{R}_v$ of X at v, and the specialisation of p on $\mathcal{R}_v$, if v is non-archimedean. A natural case to start would be the case where X becomes a Mumford curve at v. This occurs if the branch points of X come in pairs of points closer to one another than to the other branch points in a suitable v-adic metric on the projective line [4].

If v is archimedean, an explicit formula can be given using complex uniformisation, and a certain holomorphic function $\sigma_\sharp$ introduced in [22]. Our formula generalises the formula given in [26], Chapter VI, Theorem 3.2. Write:

$$f(x) = a_0 x^{2g+1} + a_1 x^{2g} + \cdots + a_{2g} x + a_{2g+1} \, ,$$

where we view all a_i as complex numbers using the embedding v. Note that $a_0 = 1$. We let $\omega_i = x^{i-1}\mathrm{d}x/(2y)$ for $i = 1,\ldots,g$; then $(\omega_1,\ldots,\omega_g)$ is a basis of holomorphic 1-forms on the compact Riemann surface X_v. Let $(\omega'|\omega'') \in M_{g\times 2g}(\mathbb{C})$ be the period matrix of $(\omega_1,\ldots,\omega_g)$ on a canonical symplectic basis e of $\mathrm{H}_1(X_v,\mathbb{Z})$. Let $\Lambda_v = (\omega'|\omega'')\cdot\mathbb{Z}^{2g}$ be the associated period lattice in $\mathbb{C}^g$ (vectors are considered as column vectors). Let $\kappa\colon \mathbb{C}^g \to \mathbb{C}^g/\Lambda_v$ be the projection and let:

$$\eta_i = \frac{1}{2y}\sum_{j=i}^{2g-i} (j+1-i) a_{2g-i-j} x^j \mathrm{d}x$$

for $i = 1,\ldots,g$. The η_i are standard meromorphic differential forms on X_v with poles only at o, and with vanishing residues. Let $(\eta'|\eta'') \in M_{g\times 2g}(\mathbb{C})$ be the period matrix of $(\eta_1,\ldots,\eta_g)$ on e and let:

$$\delta' = {}^t(\frac{1}{2},\ldots,\frac{1}{2}) \, , \quad \delta'' = {}^t(\frac{g}{2},\frac{g-1}{2},\ldots,\frac{1}{2}) \, , \quad \delta = \begin{bmatrix}\delta' \\ \delta''\end{bmatrix} \, .$$

We put $\tau = \omega'^{-1}\omega''$ and consider the following hyperelliptic sigma-function:

$$\sigma(z) = \gamma \cdot \exp(-\frac{1}{2}{}^t z \eta' \omega'^{-1} z)\theta[\delta](\omega'^{-1}z, \tau)$$

on $\mathbb{C}^g$ with $\theta[\delta]$ the standard theta function with characteristic δ. The constant γ will be fixed in the proof of Proposition 4.4 below. The function $\sigma(z)$ is holomorphic, and satisfies the functional equation:

$$\sigma(z+\ell) = \chi(\ell)\sigma(z)\exp(L(z+\frac{\ell}{2}, \ell)) \tag{4.4}$$

for $\ell \in \Lambda_v$ where:

$$\chi(\ell) = \exp(2\pi i({}^t\ell'\delta'' - {}^t\ell''\delta') - \pi i {}^t\ell'\ell''),$$
$$L(w,z) = {}^t w \cdot (\eta' z' + \eta'' z''),$$

according to [22], Lemma 3.3. Here, the vectors $z', z'' \in \mathbb{R}^g$ and $\ell', \ell'' \in \mathbb{Z}^g$ are uniquely determined by the equations $z = \omega' z' + \omega'' z''$ and $\ell = \omega'\ell' + \omega''\ell''$. It follows that the real-valued function $\|\sigma\|$ given by:

$$\|\sigma\|(z) = |\sigma(z)| \exp(-\frac{1}{2}\mathrm{Re}\, L(z,z))$$

descends to $\mathbb{C}^g/\Lambda_v$. In order to introduce $\sigma_\sharp$ write:

$$\sigma_{ij\cdots k}(z) = \frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j}\cdots\frac{\partial}{\partial z_k}\sigma(z)$$

for any tuple $(ij\ldots k)$ of integers between 1 and g. The $\sigma_\sharp$-function is then defined to be:

$$\sigma_\sharp(z) = \begin{cases} \sigma_{24\cdots g}(z) & \text{if } g \text{ is even}, \\ \sigma_{24\cdots g-1}(z) & \text{if } g \text{ is odd}. \end{cases}$$

In particular the function $\sigma_\sharp$ coincides with σ if $g = 1$.

As is proved in [22], Lemma 6.4 the $\sigma_\sharp$-function satisfies the functional equation (4.4) for z restricted to $\kappa^{-1}(\iota(X_v))$, where $\iota\colon X_v \to \mathbb{C}^g/\Lambda_v$ is the standard immersion given by $p \mapsto \int_o^p {}^t(\omega_1, \ldots, \omega_g)$. Furthermore, by [22], Proposition 6.6 the function $\sigma_\sharp$ is non-vanishing for z in $\kappa^{-1}(\iota(X_v - \{o\}))$ and if $U \subset \kappa^{-1}(\iota(X_v))$ is an open neighbourhood of 0 analytically isomorphic to a small open disc around o on X_v then $\sigma_\sharp$ restricted to U vanishes at 0 with multiplicity equal to g. We put:

$$\|\sigma_\sharp\|(z) = |\sigma_\sharp(z)| \exp(-\frac{1}{2}\mathrm{Re}\, L(z,z))$$

for z in $\kappa^{-1}(\iota(X_v))$. By what we have said above the function $\|\sigma_\sharp\|$ descends to a real-valued continuous function on X_v, vanishing only at o.

The following proposition says that we can express λ_v in terms of $\sigma_\sharp$.

Proposition 4.4. *Assume that (X, o) is an elliptic or pointed hyperelliptic curve of genus g and let v be an archimedean place of K. Let $\iota\colon X_v \to \mathbb{C}^g/\Lambda_v$ be the immersion of X_v in the complex torus $\mathbb{C}^g/\Lambda_v$ as described above, and let $\kappa\colon \mathbb{C}^g \to \mathbb{C}^g/\Lambda_v$ be the projection. Then for all $p \in X(\overline{K_v}) - \{o\}$ the formula:*

$$\lambda_v(p) = -\frac{1}{g}\log \|\sigma_\sharp\|(z)$$

holds, if $\kappa(z) = \iota(p)$.

Proof. Let U be an open disc on the universal abelian cover $\kappa^{-1}(\iota(X_v))$ of X_v, homeomorphic to a disc on X_v not containing o. Then we have:

$$-\,\mathrm{dd}^c \log \|\sigma_\sharp\|(z) = -\frac{\partial\overline{\partial}}{\pi i}\frac{1}{2}\mathrm{Re}\, L(z,z) = \widetilde{\nu}|_U$$

on U where $\widetilde{\nu}$ on $\mathbb{C}^g$ is a lift of the unique translation-invariant $(1,1)$-form ν on $\mathbb{C}^g/\Lambda_v$ representing the first Chern class of the line bundle defining the canonical principal polarisation on $\mathbb{C}^g/\Lambda_v$. As $\iota^*\nu = g\mu_v$ and U is allowed to vary we find:

$$-\,\mathrm{dd}^c \log \|\sigma_\sharp\| = g(\mu_v - \delta_o)$$

as $(1,1)$-currents on X_v. Recall that a germ of $\sigma_\sharp$ around o vanishes at o with multiplicity g. By Proposition 3.1 we conclude that $\lambda_v = -\frac{1}{g}\log\|\sigma_\sharp\| + c$ for some $c \in \mathbb{R}$. We determine c by looking at the Taylor series expansion of $\sigma_\sharp$ around o. It turns out, by [5], Section 2.1.1, that for a suitable choice of γ such that $\gamma^8 = \pi^{-4g}(\det\omega')^4\Delta(f)$ the homogeneous term of least total degree of the Taylor series expansion of $\sigma(z)$ around the origin of $\mathbb{C}^g$ equals the Hankel determinant:

$$H(z_1,\ldots,z_g) = \begin{vmatrix} z_1 & z_2 & \cdots & z_{(g+1)/2} \\ z_2 & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & z_{g-1} \\ z_{(g+1)/2} & \cdots & z_{g-1} & z_g \end{vmatrix}$$

if g is odd, and:

$$H(z_1,\ldots,z_g) = \begin{vmatrix} z_1 & z_2 & \cdots & z_{g/2} \\ z_2 & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & z_{g-2} \\ z_{g/2} & \cdots & z_{g-2} & z_{g-1} \end{vmatrix}$$

if g is even. From [22], Proposition 6.6 it follows then that, up to a sign, the expansion $\sigma_\sharp(t) = t^g(1+O(t))$ holds in the local coordinate $t = \frac{x^g}{y}$ around o on X_v. Since $x = t^{-2}(1+O(t))$ and $\lambda_v(p) - \frac{1}{2}\log|x(p)|_v \to 0$ as $p \to o$ again by Proposition 3.1 we find that $c = 0$. □

We expect that analogues of the results in this section hold true for superelliptic curves in general.

5. Proof of Theorem B

In this section we prove Theorem B. We will make use of the following general diophantine approximation result due to Faltings (cf. [16], Theorem II):

Theorem 5.1. *Let A be an abelian variety over K and let D be an ample divisor on A. Let v be a place of K and let $\lambda_{D,v}$ be a Néron function on $A(K_v)$ with respect to D. Let h be a height on $A(\overline{K})$ associated to an ample line bundle on A, and let $\kappa \in \mathbb{R}_{>0}$. Then there exist only finitely many K-rational points z in $A - D$ such that $\lambda_{D,v}(z) > \kappa \cdot h(z)$.*

Let X be a superelliptic curve of type (N,m) over K of genus g with equation $y^N = f(x)$ and with point at infinity o. We recall the divisors H_n on X from the Introduction: let $J = \mathrm{Pic}^0 X$ be the jacobian of X, and let $\iota\colon X \to J$ be the embedding given by $p \mapsto [p - o]$. We have a theta divisor Θ on J represented by the classes $[q_1+\cdots+q_{g-1}-(g-1)o]$ for $q_1,\ldots,q_{g-1}$ running through X. We note that Θ is symmetric since, as noted above (cf. the proof of Theorem 3), the divisor $2(g-1)o$ is a canonical divisor on X.

If $n \geq g$ is an integer we define H_n to be the subscheme $\iota^*[n]^*\Theta$ of X. This is an effective K-divisor on X of degree gn^2, as can be seen for example by noting that H_n coincides with the scheme of Weierstrass points of the line bundle $\mathcal{O}_X(o)^{\otimes n+g-1}$ [21]. The points in the support of H_n are called n-division points.

For $p \in X(K)$ we put $T(p)=\{n \in \mathbb{Z}_{\geq g} | p \notin H_n\} = \{n \in \mathbb{Z}_{\geq g} | n[p-o] \notin \Theta\}$.

Lemma 5.2. *Let $p \in X(K)$. If $T(p)$ is not empty, then $T(p)$ contains infinitely many elements.*

Proof. For p such that $[p-o]$ is torsion in J the statement is immediate: assume $n_0[p-o] \notin \Theta$, then if k is the order of $[p-o]$ we can take those $n \geq g$ such that $n \equiv n_0 \bmod k$. Assume therefore that $[p-o]$ is not torsion in J. We prove that infinitely many points of the form $n[p-o]$ where n is an integer are not in Θ. This is sufficient by the symmetry of Θ. Let Z be the Zariski closure in J of the subgroup $\{n[p-o] | n \in \mathbb{Z}\}$ of $J(K)$. Then Z is a closed algebraic subgroup of J, by Lemma 5.3 below. Suppose that only finitely many of the $n[p-o]$ are outside Θ. Then Z is the union of a finite positive number of isolated points and a closed subset of Θ. It follows that Z has dimension zero, contradicting the assumption that $[p-o]$ is not torsion. □

Lemma 5.3. *Let G be an algebraic group variety over a field k and let H be a subgroup of G. Then the Zariski closure of H in G is an algebraic subgroup of G.*

Proof. Let Z be the Zariski closure of H in G and for every h in H denote by t_hZ the left translate of Z under h in G. As t_hZ is closed in G and contains H we find that t_hZ contains Z and in fact $t_hZ=Z$. This implies that H is contained in the stabiliser $\mathrm{Stab}(Z)$ of Z, which is a closed algebraic subgroup of G. We conclude that Z is contained in $\mathrm{Stab}(Z)$ and hence Z is itself an algebraic subgroup of G. □

Note that $T(p)$ can be empty for $p \neq o$: for example if $g \geq 2$ and $p \neq o$ is a ramification point of $x\colon X \to \mathbb{P}^1_K$. We have the following theorem.

Theorem 5.4. *Assume that $T(p)$ is not empty, hence infinite. Let v be a place of K. Then one has:*

$$\frac{1}{gn^2}\sum_{\substack{q\in H_n\\ x(q)\neq\infty}} \log|x(p)-x(q)|_v \longrightarrow \int_{X_v} \log|x-x(p)|_v\,\mu_v$$

as $n\to\infty$ over $T(p)$. In the sum, points are counted with multiplicity.

Note that Theorem 5.4 implies Theorem B. Indeed, if $p \neq o$ is not a Weierstrass point we have $g[p-o] \notin \Theta$ so that $T(p)$ is not empty. Moreover if $n \in T(p)$ then automatically $x(q) \neq x(p)$ for all $q \in H_n$ since H_n is acted upon by the automorphism group of $x\colon X \to \mathbb{P}^1_K$ over $\overline{K}$.

The proof of Theorem 5.4 is based on the existence of an identity:

$$\log|a(n)|_v + \frac{1}{N}\sum_{\substack{q\in H_n\\ x(q)\neq\infty}} \log|x(p)-x(q)|_v = -\lambda_{\Theta,v}(n[p-o]) + gn^2\lambda_v(p) \tag{5.1}$$

of functions on $X(K_v)-\{o\}$ where $\lambda_{\Theta,v}$ is a suitable Néron function with respect to Θ on $J(K_v)$ and where $a(n)$ is a function with at most polynomial growth in n.

We obtain Theorem 5.4 by dividing by gn^2 and letting n tend to infinity over $T(p)$, using Faltings's result to see that $\lim_{n\to\infty} \lambda_{\Theta,v}(n[p-o])/gn^2 = 0$. The existence of an identity of the type (5.1) for each v can be seen from general principles; however, we would like to exhibit concrete functions a and $\lambda_{\Theta,v}$ such that (5.1) holds, in order to achieve uniformity in v, cf. Proposition 5.7 below. The constructions are based on a paper [6] by Buchstaber, Enolskii and Leykin.

We start by constructing a certain polynomial $\sigma_{N,m} \in \mathbb{Q}[z_1,\ldots,z_g]$, where $z_1,\ldots,z_g$ are indeterminates, following [6], Sections 1–4. Let $W_{N,m}$ be the set of positive integers of the form $w = -\alpha N + \beta m$, where $\alpha > 0$, and $N > \beta > 0$. Then $W_{N,m}$ is the Weierstrass gap sequence at o on X, consisting of $g = \frac{1}{2}(N-1)(m-1)$ elements. Write $W_{N,m} = \{w_1,\ldots,w_g\}$ where $1 = w_1 < \cdots < w_g = 2g-1$, and put $\pi_k = w_{g-k+1} + k - g$ for $k = 1,\ldots,g$. Then $\pi = (\pi_k)$ is a non-increasing g-tuple of positive integers, the "partition" associated to (N,m). According to [6], Lemma 2.4 the partition π is self-conjugate. Let $e_1,\ldots,e_g$ be the elementary symmetric functions in the variables $x_1,\ldots,x_g$. We call, as is customary:

$$s_\pi = \det(e_{\pi_i - i + j})_{1\le i,j\le g}$$

in $\mathbb{Q}[x_1,\ldots,x_g]$ the Schur polynomial associated to π. Let $p_r = \frac{1}{r}\sum_{i=1}^g x_i^r$ for $r = 1,2,\ldots$ be the Newton polynomials in the variables $x_1,\ldots,x_g$. According to [6], Theorem 4.1 there exists then a unique polynomial σ in $\mathbb{Q}[z_1,\ldots,z_g]$ such that $s_\pi = \sigma(p_{w_1},\ldots,p_{w_g})$. We call this polynomial $\sigma_{N,m}$. If one puts $\mathrm{wt}(z_i) = w_i$ for $i = 1,\ldots,g$ then one sees that $\sigma_{N,m}$ is homogeneous, of total weight:

$$\mathrm{length}(\pi) = \sum_{k=1}^{g} (w_k - k + 1) = w + g\,,$$

where $w = \sum_{k=1}^g (w_k - k)$ is the Weierstrass weight of o on X. One can compute that $w + g = (N^2-1)(m^2-1)/24$.

Let (α_0,β_0) be the unique pair of integers with $\alpha_0 > 0$ and $N > \beta_0 > 0$ such that $-\alpha_0 N + \beta_0 m = w_1 = 1$. We define t to be the element $x^{\alpha_0} y^{-\beta_0}$ in the function field of X. For each pair of integers (α,β) with $\alpha > 0$ and $N > \beta > 0$ such that $-\alpha N + \beta m > 0$ we define $\omega_{\alpha,\beta}$ to be the rational differential form:

$$\omega_{\alpha,\beta} = -\frac{x^{\alpha-1}}{Ny^\beta}\mathrm{d}x = -\frac{x^{\alpha-1}y^{N-1-\beta}}{f'(x)}\mathrm{d}y\,. \tag{5.2}$$

Note that we have seen the special case $\omega_{1,N-1}$ before in Section 3.

Lemma 5.5. *The element t is a local coordinate at o on X. Written in the local coordinate t, the Laurent series expansions:*

$$x = t^{-N}(1+O(t))\,, \quad y = t^{-m}(1+O(t))$$

hold. Let $w_{\alpha,\beta} = -\alpha N + \beta m$. Then $\omega_{\alpha,\beta}$ can be written as:

$$\omega_{\alpha,\beta} = t^{w_{\alpha,\beta}-1}(1+O(t))\,\mathrm{d}t$$

in the local coordinate t. The $\omega_{\alpha,\beta}$ give rise to a basis of regular differential forms on X.

Proof. As x has a pole of order N at o and y has a pole of order m at o, it follows that t is indeed a local coordinate at o. The Laurent series expansions:

$$x = t^{-N}(1+O(t))\,, \quad y = t^{-m}(1+O(t))$$

follow using the relation $y^N = f(x) = x^m(1+O(t))$. The formula:

$$\omega_{\alpha,\beta} = t^{w_{\alpha,\beta}-1}(1+O(t))\,\mathrm{d}t$$

follows then directly from the definition (5.2) of $\omega_{\alpha,\beta}$. Since the $w_{\alpha,\beta}$ are positive, we see that the $\omega_{\alpha,\beta}$ are regular at o. Further it is clear from (5.2) that the $\omega_{\alpha,\beta}$ are also regular away from o. Finally, the $\omega_{\alpha,\beta}$ are linearly independent since the $w_{\alpha,\beta}$ are distinct. As the set of $w_{\alpha,\beta}$ consists of g elements (as we have noted above, they form the Weierstrass gap sequence at o), the $\omega_{\alpha,\beta}$ give rise to a basis of regular differential forms on X. □

In a sense to be explained below, the $\sigma_{N,m}$ are suitable "degenerations" of local equations of theta divisors on superelliptic curves of type (N,m). Consider the g-fold self-product X^g of X with itself. Let $t^{(j)}$ be the local coordinate t around o on the j-th factor of X^g. Let ω_i for $i=1,\ldots,g$ be the set of $\omega_{\alpha,\beta}$ as in Lemma 5.5, ordered such that ω_i vanishes with multiplicity $w_i - 1$ at o. For each i,j with $1 \leq i,j \leq g$ let $\int_o \omega_i^{(j)}$ be the power series in $K[[t^{(j)}]]$ obtained by formally integrating ω_i, seen as an element of $K[[t^{(j)}]]\,\mathrm{d}t^{(j)}$, taking constant term equal to zero. As $\omega_i^{(j)} = (t^{(j)})^{w_i-1}(1+O(t^{(j)}))\,\mathrm{d}t^{(j)}$ by Lemma 5.5 we find $\int_o \omega_i^{(j)} = \frac{1}{w_i}(t^{(j)})^{w_i}(1+O(t^{(j)}))$ in $K[[t^{(j)}]]$. For each $i=1,\ldots,g$ we then let:

$$z_i = \sum_{j=1}^{g} \int_o \omega_i^{(j)}$$

in $K[[t^{(1)},\ldots,t^{(g)}]]$. Note that the latter can naturally be identified with $\widehat{\mathcal{O}}_{X^g,o}$, the completion of the local ring of X^g at $(o,o,\ldots,o)$. Via the Abel-Jacobi map $X^g \to J$ we obtain $\widehat{\mathcal{O}}_{J,0}$, the completion of the local ring of J at 0, naturally as the subring $K[[z_1,\ldots,z_g]]$ of $K[[t^{(1)},\ldots,t^{(g)}]]$. We have the following result.

Proposition 5.6. *(Buchstaber, Enolskii, Leykin [6] [7]) Assign the weight w_i to the variable z_i for $i=1,\ldots,g$. Then up to a scalar in K^*, the lowest weight homogeneous part of a local equation for Θ in $\widehat{\mathcal{O}}_{J,0}$, written in terms of $z_1,\ldots,z_g$, equals $\sigma_{N,m}$.*

This proposition allows us to make two (implicit) definitions. First, it follows that for integers $n \geq g$ the function $\iota^*[n]^*\sigma_{N,m}(z_1,\ldots,z_g)$ is a local equation for $H_n = \iota^*[n]^*\Theta$ in $\widehat{\mathcal{O}}_{X,o}$, the completion of the local ring of X at o. We define $a(u) \in \mathbb{Q}[u]$ to be the unique polynomial such that:

$$\sigma_{N,m}(\ldots,\frac{n}{w_i}t^{w_i}(1+O(t)),\ldots) = a(n)\cdot t^{w+g}(1+O(t)) \tag{5.3}$$

for all $n \in \mathbb{Z}$. As $\sigma_{N,m}$ is homogeneous of total weight $w+g$ in z_i if each z_i is given the weight w_i, cf. our remarks above, the polynomial $a(u)$ is well-defined. As $[n]^*z_i \equiv nz_i \bmod (z_1,\ldots,z_g)^2$ and $\iota^*z_i = \frac{1}{w_i}t^{w_i}(1+O(t))$ for $i=1,\ldots,g$, we have that:

$$\iota^*[n]^*\sigma_{N,m} = a(n)\cdot t^{w+g}(1+O(t)) \tag{5.4}$$

in $\widehat{\mathcal{O}}_{X,o}$ if $\sigma_{N,m} = \sigma_{N,m}(z_1,\ldots,z_g)$ is seen as a function in $\widehat{\mathcal{O}}_{J,0}$. In particular the multiplicity of o in H_n is (a constant) equal to $w+g$ for all $n \geq g$.

Second, let v be a place of K. Then we let $\lambda_{\Theta,v}$ be the unique Néron function with respect to Θ on $J(K_v)$ such that:

$$\lambda_{\Theta,v}(u) + \log|\sigma_{N,m}(z_1(u), \ldots, z_g(u))|_v \to 0 \tag{5.5}$$

as $u \to 0$ in $(J - \Theta)(K_v)$. It follows from Proposition 5.6 that this is well-defined too.

We claim that identity (5.1) holds with the above (implicitly) defined polynomial a and Néron function $\lambda_{\Theta,v}$.

Proposition 5.7. *For all integers n with $n \geq g$, for all places v of K, and for all $p \in X(K_v)$ with $p \notin H_n$, the equality:*

$$\log|a(n)|_v + \frac{1}{N} \sum_{\substack{q \in H_n \\ x(q) \neq \infty}} \log|x(p) - x(q)|_v = -\lambda_{\Theta,v}(n[p - o]) + gn^2 \lambda_v(p)$$

holds, where $\lambda_{\Theta,v}$ is the Néron function defined in (5.5) and where a is the polynomial in $\mathbb{Q}[u]$ defined in (5.3). In the sum, points are counted with their multiplicity.

Proof. Write $\ell_{n,v}(p)$ as a shorthand for $\lambda_{\Theta,v}(n[p - o])$. One can view $L = \mathcal{O}_J(\Theta)$ as an adelic line bundle on J by putting $\|1\|_{L,v}(z) = \exp(-\lambda_{\Theta,v}(z))$ where 1 is the canonical global section of $\mathcal{O}_J(\Theta)$. By pullback one obtains a structure of adelic line bundle on each $L_n = \mathcal{O}_X(H_n) = i^*[n]^*\mathcal{O}_J(\Theta)$ given by $\|1\|_{L_n,v}(p) = \exp(-\ell_{n,v}(p))$ where now 1 is the canonical global section of $\mathcal{O}_X(H_n)$. By [31], (4.7) the resulting adelic metric is admissible; in particular $\ell_{n,v}(p)$ equals the admissible pairing $(p, H_n)_a$ up to an additive constant. As a result $\ell_{n,v}$ extends to $\mathrm{D}^0(X_v)$, the space of $(0,0)$-currents on X_v. As the other terms in the equality to be proven do so as well, we try to prove the equality as an identity in $\mathrm{D}^0(X_v)$. By Proposition 2.1(iii) we are done once we prove that both sides of the claimed equality have the same image under dd^c, and the difference of both sides tends to zero as p tends to o over $X(K_v)$, avoiding H_n. From the observation that $\|1\|_{L_n,v}(p) = \exp(-\ell_{n,v}(p))$ defines an admissible metric on L_n we obtain first of all that:

$$\mathrm{dd}^c \, \ell_{n,v} = (\deg H_n)\mu_v - \delta_{H_n} = gn^2 \mu_v - \delta_{H_n} \, .$$

Next we recall from Proposition 3.1 that $\mathrm{dd}^c \, \lambda_v = \mu_v - \delta_o$. As by the Poincaré-Lelong equation Proposition 2.1(i) we have:

$$\mathrm{dd}^c \, \frac{1}{N} \sum_{\substack{q \in H_n \\ x(q) \neq \infty}} \log|x(p) - x(q)|_v = \delta_{H_n} - gn^2 \delta_o \, ,$$

the first step follows. To see the second step, Proposition 3.1 tells us that:

$$\lambda_v(p) - \frac{1}{N} \log|x(p)|_v \to 0 \quad \text{as} \quad p \to o \, ,$$

that is, by Lemma 5.5:

$$\lambda_v(p) + \log|t(p)|_v \to 0 \quad \text{as} \quad p \to o \, .$$

Next, as by definition:

$$\lambda_{\Theta,v}(u) + \log|\sigma_{N,m}(z_1(u), \ldots, z_g(u))|_v \to 0 \quad \text{as} \quad u \to 0$$

in $(J - \Theta)(K_v)$ we have, upon pulling back along $[n]$ and ι and using (5.3) that:

$$\log|a(n)|_v + \ell_{n,v}(p) + (w + g) \log|t(p)|_v \to 0 \quad \text{as} \quad p \to o$$

outside H_n, by (5.4). Finally by applying once again Lemma 5.5 we have:

$$\frac{1}{N} \sum_{\substack{q \in H_n \\ x(q) \neq \infty}} \log |x(p) - x(q)|_v + (gn^2 - w - g) \log |t(p)|_v \to 0 \quad \text{as} \quad p \to o$$

outside H_n. The second step follows by combining these equalities. □

We can now prove Theorem 5.4.

Proof of Theorem 5.4. Let $p \in X(K)$ be a point such that $T(p)$ is infinite, and let v be a place of K. By Proposition 5.7 we are done once we prove that $\log |a(n)|_v / n^2 \to 0$ as $n \to \infty$ and $\lambda_{\Theta,v}(n[p-o])/n^2 \to 0$ as $n \to \infty$ over $T(p)$. The first statement is immediate since $a(n)$ is a polynomial in n. As to the second statement, note that it follows immediately if $[p-o]$ is torsion since then the set of values $\lambda_{\Theta,v}(n[p-o])$ as n ranges over $T(p)$ is bounded. Assume therefore that $[p-o]$ is not torsion. Then the $n[p-o]$ with n running through $T(p)$ form an infinite set of K-rational points of $J - \Theta$. Since:

$$\frac{\lambda_{\Theta,v}(n[p-o])}{n^2} = h_J([p-o]) \cdot \frac{\lambda_{\Theta,v}(n[p-o])}{h_J(n[p-o])}$$

with $h_J([p-o]) > 0$ Faltings's Theorem 5.1 can be applied to give:

$$\limsup_{\substack{n \to \infty \\ n \in T(p)}} \frac{\lambda_{\Theta,v}(n[p-o])}{n^2} \leq 0\,.$$

On the other hand $\lambda_{\Theta,v}$ is bounded from below so that:

$$\liminf_{\substack{n \to \infty \\ n \in T(p)}} \frac{\lambda_{\Theta,v}(n[p-o])}{n^2} \geq 0\,.$$

Theorem 5.4 follows. □

As we remarked above the divisor H_n equals the divisor of Weierstrass points of the line bundle $\mathcal{O}_X(o)^{n+g-1}$. From [8], 3.2.2 we obtain therefore that the average height of n-division points remains bounded, as $n \to \infty$. We refer to [25], Theorem 1.1 for a similar (but weaker) result in the context of hyperelliptic curves. From Proposition 5.7 we may reobtain this boundedness of the average height, in the following form.

Proposition 5.8. *Let $h_{X,x}$ be the canonical height on $X(\overline{K})$ defined in Section 3. Then the estimates:*

$$\limsup_{n \to \infty} \frac{1}{gn^2} \sum_{q \in H_n} h_{X,x}(q) \leq \frac{1}{[K:\mathbb{Q}]} \sum_v n_v \int_{X_v} \lambda_v \, \mu_v < \infty$$

hold. Here v runs over all places of K, and points in H_n are counted with multiplicity.

Proof. We take the identity from Proposition 5.7, integrate both sides against μ_v, and divide by gn^2. This gives:

$$\frac{1}{gn^2} \log |a(n)|_v + \frac{1}{gn^2} \sum_{\substack{q \in H_n \\ q \neq o}} \lambda_v(q) = -\frac{1}{gn^2} \int_{X_v} \lambda_{\Theta,v}(n[p-o]) \mu_v + \int_{X_v} \lambda_v \, \mu_v$$

for each $n \geq g$ and for each place v of K. As $a(n)$ is a polynomial in n we have $\log|a(n)|_v/n^2 \to 0$ as $n \to \infty$ and as $\lambda_{\Theta,v}$ is bounded from below we have $\liminf_{n\to\infty} \lambda_{\Theta,v}(n[p-o])/n^2 \geq 0$. By Fatou's Lemma we find:

$$\begin{aligned} \limsup_{n\to\infty} -\frac{1}{gn^2}\int_{X_v} \lambda_{\Theta,v}(n[p-o])\mu_v &= -\liminf_{n\to\infty} \frac{1}{gn^2}\int_{X_v} \lambda_{\Theta,v}(n[p-o])\mu_v \\ &\leq -\int_{X_v} \liminf_{n\to\infty} \frac{1}{gn^2}\lambda_{\Theta,v}(n[p-o])\mu_v \leq 0\,. \end{aligned}$$

Hence we have:

$$\limsup_{n\to\infty} \frac{1}{gn^2} \sum_{\substack{q\in H_n \\ q\neq o}} \lambda_v(q) \leq \int_{X_v} \lambda_v\, \mu_v\,.$$

Summing over the places v of K we obtain:

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{gn^2} \sum_{\substack{q\in H_n \\ q\neq o}} \sum_v n_v \lambda_v(q) &\leq \sum_v \limsup_{n\to\infty} \frac{1}{gn^2} n_v \sum_{\substack{q\in H_n \\ q\neq o}} \lambda_v(q) \\ &\leq \sum_v n_v \int_{X_v} \lambda_v\, \mu_v\,, \end{aligned}$$

which is the first estimate. The equality $\lambda_v(p) = (p,o)_a + \int_{X_v} \lambda_v\, \mu_v$ for $p \in X(K_v)$ finally shows that $\int_{X_v} \lambda_v\, \mu_v$ vanishes for almost all v. □

As is explained in [8], 3.2.4 the boundedness of the average height of n-division points implies that the degrees of the fields generated by the H_n are not bounded as $n \to \infty$.

The boundedness of the average height of n-division points also implies that the affine logarithmic height of the "division polynomial" $\prod_{q\in H_n, q\neq o}(x - x(q))$ in $K[x]$ is $O(n^2)$ as $n \to \infty$. This generalises a known result in the context of elliptic curves, cf. [19], Theorem 3.1 and Theorem 3.2.

6. The case of a Weierstrass point

In this section we prove that the asymptotic formula of Theorem B also holds when $p \neq o$ is a Weierstrass point on a hyperelliptic curve. Write $x(p) = \alpha$. By Proposition 4.1 we have:

$$\int_{X_v} \log|x - \alpha|_v\, \mu_v = \frac{1}{2g}\log|f'(\alpha)|_v\,,$$

so we are done once we prove that:

$$\frac{1}{n^2} \sum_{\substack{q\in H_n \\ x(q)\neq \alpha,\infty}} \log|x(q) - \alpha|_v \longrightarrow \frac{1}{2}\log|f'(\alpha)|_v$$

as $n \to \infty$. We will prove in fact a somewhat stronger statement. Let $g \geq 1$ be an integer and let k be a field of characteristic ℓ where either $\ell = 0$ or $\ell \geq 2g+1$. In particular $\ell \neq 2$. Let:

$$T(k,g) = \{n \in \mathbb{Z}_{\geq g} | \ell \nmid (n-g+1)\cdots(n+g-1)\}\,;$$

note that this is an infinite set. Let $f(x) \in k[x]$ be a monic and separable polynomial of degree $2g+1$, and let (X,o) be the elliptic or pointed hyperelliptic curve of genus g over k given by f. For each integer $n \in T(k,g)$ we have effective k-divisors H_n on

X of degree gn^2, as before. They are invariant under the hyperelliptic involution of (X, o) and split over a separable algebraic closure k^s of k. It will be convenient to view the H_n as multi-sets of k^s-points of X of cardinality gn^2. Let $\alpha \in k$ be a root of f.

Theorem 6.1. *Assume that k is endowed with an absolute value $|\cdot|$. Then one has:*

$$\frac{1}{n^2} \sum_{\substack{q \in H_n \\ x(q) \neq \alpha, \infty}} \log|x(q) - \alpha| \longrightarrow \frac{1}{2} \log|f'(\alpha)| \tag{6.1}$$

for n in $T(k, g)$ tending to infinity. The points in H_n are counted with multiplicity.

Note that the left hand side of (6.1) is well-defined. Our proof relies on a rather intricate determinantal formula for the division polynomials related to H_n due to D. Cantor [10], which we introduce first. For each $n \geq g$ we put:

$$H_n^* = \begin{cases} H_n - H_g & n \equiv g \bmod 2\,, \\ H_n - H_{g+1} & n \equiv g+1 \bmod 2\,. \end{cases}$$

It can be shown that these H_n^* are effective k-divisors on X with support away from o. Note that:

$$\deg H_n^* = \begin{cases} g(n^2 - g^2) & n \equiv g \bmod 2\,, \\ g(n^2 - (g+1)^2) & n \equiv g+1 \bmod 2\,. \end{cases}$$

Also note that:

$$H_g = \frac{g(g-1)}{2} R + go\,, \quad H_{g+1} = \frac{g(g+1)}{2} R\,,$$

where R denotes the reduced divisor of degree $2g+2$ on X consisting of the hyperelliptic ramification points of X.

Now let $\mathcal{R}$ be the commutative ring $\mathbb{Z}[a_1, \ldots, a_{2g+1}][1/2]$ where $a_1, \ldots, a_{2g+1}$ are indeterminates. Let $F(x)$ be the polynomial $x^{2g+1} + a_1 x^{2g} + \cdots + a_{2g} x + a_{2g+1}$ in $\mathcal{R}[x]$, and let $\Delta \in \mathcal{R}$ be the discriminant of F. Let y be a variable satisfying $y^2 = F(x)$, and let $E_1(z)$ be the polynomial $E_1(z) = (F(x-z) - y^2)/z$ in $\mathcal{R}[x, z]$. Next put $S(z) = (-1)^{g+1} y \sqrt{1 + zE_1(z)/y^2}$ where $\sqrt{1 + zE_1(z)/y^2}$ is the power series in $\mathcal{R}[x, \frac{1}{y}][[z]]$ obtained by binomial expansion on $1 + zE_1(z)/y^2$. Note that $S(z)^2 = F(x-z)$ and that $S(z) = \sum_{j=0}^{\infty} P_j(x)(2y)^{1-2j} z^j$ for suitable $P_j(x) \in \mathcal{R}[x]$ of degree $2jg$ (cf. [10], Section 8). For each integer $n \geq g$ we define a polynomial ψ_n in $\mathcal{R}[x]$ by:

$$\psi_n = \begin{cases} \begin{vmatrix} P_{g+1} & P_{g+2} & \cdots & P_{(n+g)/2} \\ P_{g+2} & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & P_{n-2} \\ P_{(n+g)/2} & \cdots & P_{n-2} & P_{n-1} \end{vmatrix} & n \equiv g \bmod 2\,, \\ \begin{vmatrix} P_{g+2} & P_{g+3} & \cdots & P_{(n+g+1)/2} \\ P_{g+3} & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & P_{n-2} \\ P_{(n+g+1)/2} & \cdots & P_{n-2} & P_{n-1} \end{vmatrix} & n \equiv g+1 \bmod 2\,. \end{cases} \tag{6.2}$$

Here, for $n = g$ resp. $n = g+1$ we understand that ψ_n is the unit element. We have:

$$\deg \psi_n = \begin{cases} g(n^2-g^2)/2 & n \equiv g \bmod 2\,, \\ g(n^2-(g+1)^2)/2 & n \equiv g+1 \bmod 2\,. \end{cases}$$

The result of D. Cantor is that the H_n^* are given by the ψ_n. Let $b(n)$ in $\mathcal{R}$ be the leading coefficient of ψ_n.

Theorem 6.2. *(D. Cantor [10]) The function $b\colon \mathbb{Z}_{n\geq g} \to \mathcal{R}$ has positive integral values, and is represented by a numerical polynomial. The function b satisfies $\ell \nmid (n-g+1)\cdots(n+g-1) \Rightarrow \ell \nmid b(n)$ for all prime numbers ℓ and all integers $n \geq g$. Furthermore ψ_n is a universal n-division polynomial, in the following sense: if $^-\colon \mathcal{R} \to k$ is a ring homomorphism from $\mathcal{R}$ to a field k such that $\overline{\Delta}$ is non-zero in k, then for the pointed curve (X,o) over k given by $\overline{F}$ one has $H_n^* = Z(\psi_n)$, where $Z(\psi_n)$ is the zero divisor of ψ_n on X, for each $n \in T(k,g)$.*

We deduce Theorem 6.1 from D. Cantor's theorem by evaluating the determinants at the right hand side of identity (6.2) at α.

Proof of Theorem 6.1. Let α be a root of F in an algebraic closure $\overline{Q(\mathcal{R})}$ of the fraction field $Q(\mathcal{R})$ of $\mathcal{R}$. Let $c_m = \frac{1}{2m+1}\binom{2m+1}{m}$ for $m \geq 0$ be the m-th Catalan number. We claim:

Lemma 6.3. *The identity:*

$$P_j(\alpha) = (-1)^g \cdot c_{j-1} \cdot F'(\alpha)^j$$

holds in $\mathcal{R}[\alpha]$ for all integers $j \geq 1$.

Proof. We recall the relations:

$$S(z) = \sum_{j=0}^{\infty} \frac{P_j(x)}{(2y)^{2j-1}} z^j\,, \quad S(z)^2 = F(x-z)\,.$$

We claim that:

$$\frac{1}{j!}\frac{\mathrm{d}^j S(z)}{\mathrm{d}\, z^j} = \frac{R_j(x,z)}{(2S(z))^{2j-1}} \tag{6.3}$$

for some $R_j(x,z) \in Q(\mathcal{R})[x,z]$ with $R_j(\alpha,0) = -c_{j-1}\cdot F'(\alpha)^j$, for all $j \geq 1$. This gives what we want since $S(0) = (-1)^{g+1}y$ and $P_j(x) = R_j(x,0)$.

To prove the claim we argue by induction on j. We have $\frac{\mathrm{d}S}{\mathrm{d}z} = -\frac{F'(x-z)}{2S(z)}$ which settles the case $j=1$ with $R_1(x,z) = -F'(x-z)$. Now assume that (6.3) holds with $R_j(x,z) \in Q(\mathcal{R})[x,z]$, and with $R_j(\alpha,0) = -c_{j-1}\cdot F'(\alpha)^j$ for a certain $j \geq 1$. Then a small calculation yields:

$$\frac{1}{(j+1)!}\frac{\mathrm{d}^{j+1}S}{\mathrm{d}z^{j+1}} = \frac{1}{j+1}\frac{\mathrm{d}}{\mathrm{d}z}\frac{R_j(x,z)}{(2S(z))^{2j-1}} = \frac{R_{j+1}(x,z)}{(2S(z))^{2j+1}}$$

with:

$$R_{j+1}(x,z) = \frac{2}{j+1}\left(2\left(\frac{\mathrm{d}}{\mathrm{d}z}R_j(x,z)\right)F(x-z) + (2j-1)R_j(x,z)F'(x-z)\right)\,.$$

We find $R_{j+1}(x,z) \in Q(\mathcal{R})[x,z]$ and:

$$\begin{aligned} R_{j+1}(\alpha,0) &= \frac{2(2j-1)}{j+1} R_j(\alpha,0) \cdot F'(\alpha) \\ &= -\frac{2(2j-1)}{j+1} c_{j-1} \cdot F'(\alpha)^{j+1} \\ &= -c_j \cdot F'(\alpha)^{j+1} \end{aligned}$$

by the induction hypothesis. This completes the induction step. □

We can now specialise (6.2) at α. This yields the equality:

$$\psi_n(\alpha) = b_0(n) \cdot F'(\alpha)^{d^*(n)} \tag{6.4}$$

in $\mathcal{R}[\alpha]$ where:

$$b_0(n) = \begin{cases} \begin{vmatrix} c_g & c_{g+1} & \cdots & c_{(n+g)/2-1} \\ c_{g+1} & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & c_{n-3} \\ c_{(n+g)/2-1} & \cdots & c_{n-3} & c_{n-2} \end{vmatrix} & n \equiv g \bmod 2\,, \\ \begin{vmatrix} c_{g+1} & c_{g+2} & \cdots & c_{(n+g-1)/2} \\ c_{g+2} & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & c_{n-3} \\ c_{(n+g-1)/2} & \cdots & c_{n-3} & c_{n-2} \end{vmatrix} & n \equiv g+1 \bmod 2\,, \end{cases} \tag{6.5}$$

at least up to a sign, and where $d^*(n) \in \mathbb{Z}$ is given by:

$$2d^*(n) = \begin{cases} (n^2-g^2)/2 & n \equiv g \bmod 2\,, \\ (n^2-(g+1)^2)/2 & n \equiv g+1 \bmod 2\,. \end{cases}$$

To obtain Theorem 6.1 we need that $b_0(n)$ is non-vanishing in k if $n \in T(k,g)$, and that $b_0(n)$ is not "too large" in $\mathbb{Z}$. To achieve this, we can apply a general result due to Desainte-Catherine and Viennot (cf. [11], Section 6), stating that for arbitrary integers $l, m \geq 1$ we have the Hankel determinant:

$$\begin{vmatrix} c_l & c_{l+1} & \cdots & c_{l+m-1} \\ c_{l+1} & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & c_{l+2m-3} \\ c_{l+m-1} & \cdots & c_{l+2m-3} & c_{l+2m-2} \end{vmatrix} = \prod_{1 \leq i \leq j \leq l-1} \frac{i+j+2m}{i+j}\,. \tag{6.6}$$

Applying this to (6.5) we infer that $\ell \nmid (n-g+1)\cdots(n+g-1) \Rightarrow \ell \nmid b_0(n)$ holds for every prime number ℓ and every integer n and that $b_0(n)$ is represented by a numerical polynomial. In particular $b_0(n)$ is non-vanishing in k if $n \in T(k,g)$, and $b_0(n)$ is not "too large" in $\mathbb{Z}$.

Let us now place ourselves in the situation of Theorem 6.1. From Cantor's theorem and equation (6.4) we obtain the identity:

$$b(n)^2 \prod_{q \in H_n^*} (x(q)-\alpha) = \psi_n(\alpha)^2 = b_0(n)^2 \cdot f'(\alpha)^{2d^*(n)} \tag{6.7}$$

in k. Since $f'(\alpha)$ and $b_0(n)$ are both non-zero in k we deduce that $\prod_{q \in H_n^*}(x(q)-\alpha)$ is non-zero in k as well (we knew already that $b(n)$ is non-zero in k). In particular

H_n^* has support disjoint from the hyperelliptic ramification point corresponding to α. By multiplying left and right hand side by:

$$\prod_{\substack{q \in H_n - H_n^* \\ x(q) \neq \alpha, \infty}} (x(q) - \alpha) = \begin{cases} f'(\alpha)^{g(g-1)/2} & n \equiv g \bmod 2\,, \\ f'(\alpha)^{g(g+1)/2} & n \equiv g+1 \bmod 2 \end{cases}$$

we obtain:

$$b(n)^2 \prod_{\substack{q \in H_n \\ x(q) \neq \alpha, \infty}} (x(q) - \alpha) = b_0(n)^2 \cdot f'(\alpha)^{2d(n)}\,,$$

where:

$$2d(n) = \begin{cases} (n^2 - g)/2 & n \equiv g \bmod 2\,, \\ (n^2 - g - 1)/2 & n \equiv g+1 \bmod 2\,. \end{cases}$$

Taking absolute values and then logarithms (which we can do since both sides are non-zero in k) we obtain:

$$\frac{2}{n^2} \log |b(n)| + \frac{1}{n^2} \sum_{\substack{q \in H_n \\ x(q) \neq \alpha, \infty}} \log |x(q) - \alpha| = \frac{2}{n^2} \log |b_0(n)| + \frac{2d(n)}{n^2} \log |f'(\alpha)|\,.$$

As both $b(n)$ and $b_0(n)$ are represented by polynomials in n, the terms $\frac{2}{n^2} \log |b(n)|$ and $\frac{2}{n^2} \log |b_0(n)|$ tend to zero as $n \to \infty$ over $T(k,g)$. Finally we clearly have $\frac{2d(n)}{n^2} \to \frac{1}{2}$ as $n \to \infty$. We end up with:

$$\frac{1}{n^2} \sum_{\substack{q \in H_n \\ x(q) \neq \alpha, \infty}} \log |x(q) - \alpha| \longrightarrow \frac{1}{2} \log |f'(\alpha)|$$

as $n \to \infty$ over $T(k,g)$, as required. □

Remark 6.4. It is possible to give closed expressions for $b(n)$ and $b_0(n)$. First of all for $n \geq g$ put:

$$\sigma_{n,g} = \begin{cases} \binom{n+1}{3}\binom{n+3}{7} \cdots \binom{n+g-1}{2g-1} & \text{if } g \text{ is even}\,, \\ \binom{n}{1}\binom{n+2}{5} \cdots \binom{n+g-1}{2g-1} & \text{if } g \text{ is odd}\,. \end{cases}$$

Then by [10], Theorem 4.1 we have:

$$b(n) = \begin{cases} \frac{\sigma_{n,g}}{\sigma_{g,g}} & n \equiv g \bmod 2\,, \\ \frac{\sigma_{n,g}}{\sigma_{g,g}} 2^{-g} & n \equiv g+1 \bmod 2\,. \end{cases}$$

Next, from (6.6) we obtain, by some rewriting:

$$b_0(n) = \begin{cases} \frac{\sigma_{n,g}}{\sigma_{g,g}} \frac{1 \cdot 3 \cdots (2g-1)}{(n-g+1)(n-g+3)\cdots(n+g-1)} & n \equiv g \bmod 2\,, \\ \frac{\sigma_{n,g}}{\sigma_{g,g}} \cdot 2^{-g} & n \equiv g+1 \bmod 2\,. \end{cases}$$

Note that $b(n) = b_0(n)$ if $n \equiv g+1 \bmod 2$. It would be interesting to investigate if H_n is a non-zero divisor on X for $n \equiv g+1 \bmod 2$ and $n \geq g+1$, even if the characteristic ℓ of k is positive and smaller than $2g+1$.

Remark 6.5. Silverman [25] proves a number of results which are very similar to the ones above; especially compare formula (21) in [25] with our formula (6.7). Formula (6.7) shows that for $n \in T(k,g)$ the divisor H_n^* has support outside the hyperelliptic ramification locus. This generalises Corollary 1.4 of [25]. The main difference with [25] is that we use a slightly more general notion of division point.

The sequence of divisors considered in [25] is basically the sequence of H_n such that $n = (2k-1)(g-1)$ for some $k \geq 2$.

By summing over the finite branch points of (X, o) we obtain the following result.

Corollary 6.6. *Take the assumptions of Theorem 6.1. Then:*

$$\frac{1}{n^2} \sum_{\substack{q \in H_n \\ f(x(q)) \neq 0,\infty}} \log |f(x(q))| \longrightarrow \frac{1}{2} \log |\Delta(f)|$$

as $n \to \infty$ over $T(k, g)$. Here $\Delta(f)$ is the discriminant of f.

This result was shown by Szpiro and Tucker in [27] for the case that (X, o) is an elliptic curve and k is a discrete valuation field such that X has semistable reduction over k. The proof in [27] uses the geometry of the special fiber of the minimal regular model of X over k; such considerations seem to be absent from the arguments above.

7. An application

In this section we deduce a finiteness result from Theorem A and Theorem B. The line of reasoning is inspired upon [2], Introduction, and [12], Section 8.

Let X be any geometrically connected projective curve over the number field K. Let S be a finite set of places of K, including the archimedean ones. Let $\mathcal{X}$ be a proper model of X over the ring of S-integers of K and let D be an effective K-divisor on X. A point $p \in X(\overline{K})$ is called S-integral with respect to D if the Zariski closures of D and of the $\mathrm{Gal}(\overline{K}/K)$-orbit of p in $\mathcal{X}$ are disjoint. If $(D_n)_{n \in \mathbb{N}}$ is a sequence of effective K-divisors on X we are interested in the question whether the number of $n \in \mathbb{N}$ such that p is S-integral with respect to D_n is finite or infinite. Note that the validity of this question does not depend on the choice of model $\mathcal{X}$.

We focus on the special case that X is a superelliptic curve over K, with equation $y^N = f(x)$ and with point at infinity o. We have the sequence $(H_n)_{n \geq g}$ of divisors of n-division points on X. For $p \in X(\overline{K})$ we put $T(p) = \{n \in \mathbb{Z}_{\geq g} | p \notin H_n\}$.

Proposition 7.1. *Let $p \in X(\overline{K})$. Assume that $[p-o]$ is not torsion in the jacobian of X. Then there are only finitely many $n \in T(p)$ such that p is S-integral with respect to H_n.*

Proof. Without loss of generality we may enlarge S until it contains all primes of bad reduction of X. Also we may assume that $T(p)$ is infinite, and that $p \in X(K)$. By Theorem 3.2 and Theorem 5.4 we have, as $[p - o]$ is not torsion:

$$\begin{aligned} 0 < h_J([p-o]) &= g \sum_v n_v \int_{X_v} \log |x - x(p)|_v \, \mu_v \\ &= g \sum_v n_v \lim_{\substack{n\to\infty \\ n \in T(p)}} \frac{1}{n^2} \sum_{\substack{q \in H_n \\ x(q) \neq x(p),\infty}} \log |x(p) - x(q)|_v \, . \end{aligned}$$

Let $T'(p)$ be the set of $n \in T(p)$ such that p is S-integral with respect to H_n. We assume, by contradiction, that $T'(p)$ is infinite. Then we may write:

$$0 < \sum_v n_v \lim_{\substack{n\to\infty \\ n \in T'(p)}} \frac{1}{n^2} \sum_{\substack{q \in H_n \\ x(q) \neq x(p),\infty}} \log |x(p) - x(q)|_v \, . \tag{7.1}$$

However, if p is S-integral with respect to H_n then $x(p)$ and $x(H_n)$ are disjoint mod v for all $v \notin S$, as H_n is invariant for the automorphism group of $x \colon X \to \mathbb{P}^1_K$ over $\overline{K}$. It follows that the contribution in (7.1) to the sum over all v vanishes for $v \notin S$ so that we can conclude:

$$0 < \sum_{v \in S} n_v \lim_{\substack{n \to \infty \\ n \in T'(p)}} \frac{1}{n^2} \sum_{\substack{q \in H_n \\ x(q) \neq x(p), \infty}} \log |x(p) - x(q)|_v \, .$$

We can now interchange the first summation and the limit, yielding:

$$0 < \lim_{\substack{n \to \infty \\ n \in T'(p)}} \frac{1}{n^2} \sum_{\substack{q \in H_n \\ x(q) \neq x(p), \infty}} \sum_v n_v \log |x(p) - x(q)|_v \, ,$$

the latter sum being again over all places of K. We arrive at a contradiction since:

$$\sum_{\substack{q \in H_n \\ x(q) \neq x(p), \infty}} \sum_v n_v \log |x(p) - x(q)|_v = 0$$

for each $n \in T'(p)$ by the product formula. □

Note that by the Manin-Mumford conjecture (proved by Raynaud) the number of $p \in X(\overline{K})$ such that $[p - o]$ is torsion, is finite. It would be interesting to know whether Proposition 7.1 could be strengthened to the following statement, generalizing "Ih's conjecture" (cf. [2], Section 3): assume $[p - o]$ is not torsion, then there are only finitely many division points $\xi_n \in H_n$ such that p is S-integral with respect to the Galois orbit of ξ_n. Such a statement would follow by the arguments above if one had a version of Theorem 5.4 with the H_n replaced by the Galois orbits of a sequence of distinct $\xi_n \in H_n$. Unfortunately proving such a statement seems to be hard; for example one does not seem to know whether H_n (minus the contribution from the branch points of x) is composed of "large enough" Galois orbits as $n \to \infty$ (i.e., whether the division polynomials associated to H_n have "large enough" prime factors). See also [25] for a discussion of this problem in the context of hyperelliptic curves.

In [2], Theorem 0.2 the stronger version of Proposition 7.1 as stated above is proved in the case that (X, o) is an elliptic curve.

An analogue of Proposition 7.1 in the context of dynamical systems on $\mathbb{P}^1_K$ is stated in [28], Proposition 6.3.

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