Tensor Products of Archimedean Partially Ordered Vector Spaces

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Abstract

We study the tensor product of two directed Archimedean partially ordered vector spaces X and Y by means of Riesz completions. We give a construction of the vector space tensor product $X \otimes Y$ and of the projective cone. With the aid of the Fremlin tensor product of the Riesz completions of X and Y we show that the projective cone in $X \otimes Y$ is contained in an Archimedean cone. The smallest Archimedean cone containing the projective cone satisfies an appropriate universal mapping property. It is also shown that the Fremlin tensor product of two Archimedean Riesz spaces is the Riesz completion of the vector space tensor product equipped with the order induced by the cone of the Fremlin tensor product.

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Key words: Archimedean partially ordered vector space, Fremlin tensor product, projective cone, Riesz completion, tensor product

1 Introduction

Tensor products of function spaces are common objects in functional analytic considerations. These tensor products often have a natural lattice order, just as their components. D.H. Fremlin has studied a general construction of a tensor product of Riesz spaces (vector lattices) which is compatible with the natural tensor products of function spaces. It turns out that such a construction is quite involved. In his seminal paper [7], Fremlin views the algebraic vector space tensor product of two function spaces as a space of functions on the Cartesian product of the underlying spaces. Then he endows this space with the induced

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order. As this space is in general not a Riesz space, he considers the Riesz subspace generated by the vector space tensor product. Fremlin succeeds with a similar construction to make a Riesz space tensor product for arbitrary Archimedian Riesz spaces. His construction uses representations of Archimedean Riesz spaces with order units as subspace s of spaces of continuous functions.

The main point where the representation theory in Fremlin's construction is used, is the construction of the ambient space of which the vector space tensor product is a subspace. This ambient space yields the induced order on the vector space tensor product and it yields the completion to a Riesz space. The main difficulties in finding an intrinsic construction are the definition of the appropriate order on the vector space tensor product and the completion of this partially ordered vector space to a Riesz space.

An alternative construction of the Fremlin tensor product is provided by the theory of tensor products of ℓ-groups in [5, 11].

Grobler and Labuschagne [8] give an intrinsic construction of the Fremlin tensor product. They begin with the *projective cone* in the vector space tensor product, which is the cone generated by the elementary tensors of positive elements. They show that the relative uniform closure of the projective cone induces the appropriate order on the tensor product. The Fremlin tensor product is then obtained by means of Dedekind completions. On their way they also construct the tensor product of directed Archimedean partially ordered vector spaces with the Riesz decomposition property. In the latter case the order on the tensor product is required to satisfy a universal mapping property (see Definition 5.1). One of the major steps in the construction of Grobler and Labuschagne where the Riesz decomposition property is used is the proof that the relative uniform closure of the projective cone is again a cone and not just a wedge.

It is our intention to study tensor products of partially ordered vector spaces by means of Riesz completions. The notion of pre-Riesz space and Riesz completion have been introduced by van Haandel in [9]. We consider two directed Archimedean partially ordered vector spaces X and Y and show that the relative uniform closure of the projective cone in the vector space tensor product $X \otimes Y$ is a cone with the desired universal mapping property. The proof uses the Riesz completions of X and Y and their Fremlin tensor product. Our result extends [8, Theorem 2.5] since we do not need the Riesz decomposition property.

Section 2 presents a construction of the tensor product of two vector spaces. Section 3 studies the projective cone in the vector space tensor product of two partially ordered vector spaces. In Section 4 we first recall the required terminology on Riesz completions. Then we consider two Archimedean Riesz spaces E and F and the order on the vector space tensor product $E \otimes F$ induced by the order of the Fremlin tensor product. We show that with this order $E \otimes F$ is a pre-Riesz space and the Fremlin tensor product is its Riesz completion. In Section 5 we use Riesz completions to construct the appropriate cone in the tensor product of directed Archimedean partially ordered vector spaces.

2 The tensor product of vector spaces

Definition 2.1. Let X and Y be (real) vector spaces. A pair (T, τ) is called a *tensor product* of X and Y provided
(1) T is a vector space and $\tau\colon X \times Y \to T$ is a bilinear map; and
(2) if S is a vector space and $\sigma\colon X \times Y \to S$ is a bilinear map, then there is a unique linear map $\sigma^*\colon T \to S$ such that $\sigma(x,y) = \sigma^*(\tau(x,y))$ for all $x \in X$ and $y \in Y$.

Remark 2.2. If (T, τ) is a tensor product of X and Y, then $\{\tau(x,y)\colon\ x \in X,\, y \in Y\}$ spans T. Indeed, for

$$S = \operatorname{span}\{\tau(x,y)\colon\ x \in X,\, y \in Y\}$$

and $\sigma = \tau$ there is a linear map $\sigma^*\colon T \to S$ such that $\tau(x,y) = \sigma^*(\tau(x,y))$ for all $x \in X$ and $y \in Y$. Suppose that $T \setminus S \neq \varnothing$. By choosing first a Hamel basis of S and then extending it to a basis of T, one can construct a linear map $\phi\colon T \to \mathbb{R}$ with $\phi \neq 0$ and $\phi = 0$ on S. Choose $s \in S$ with $s \neq 0$ and define $\sigma_2^*\colon T \to S$ by $\sigma_2^*(t) = \sigma^*(t) + \phi(t)s$. Then $\sigma_2^* = \sigma^*$ on S, in particular $\tau(x,y) = \sigma_2^*(\tau(x,y))$ for all $x \in X$ and $y \in Y$, and for $u \in T$ with $\phi(u) \neq 0$ one has $\sigma_2^*(u) = \sigma^*(u) + \phi(u)s \neq \sigma^*(u)$, which contradicts the uniqueness of σ^*.

Given vector spaces X and Y, the following construction leads to a tensor product of X and Y. Put

$$\begin{aligned} Z \;=\; & \{\{(\alpha_1, x_1, y_1), (\alpha_2, x_2, y_2), \ldots, (\alpha_n, x_n, y_n)\} : n \in \mathbb{N},\ \alpha_i \in \mathbb{R},\ x_i \in X,\ y_i \in Y \\ & \text{with } (x_i, y_i) \neq (x_j, y_j) \text{ if } i \neq j\}. \end{aligned} \tag{1}$$

The operation $+\colon Z \times Z \to Z$ defined by

$$\begin{aligned} &\{(\alpha_1, x_1, y_1), \ldots, (\alpha_m, x_m, y_m)\} + \{(\alpha_{m+1}, x_{m+1}, y_{m+1}), \ldots, (\alpha_{m+n}, x_{m+n}, y_{m+n})\} \\ &\quad = \left\{ \left(\sum_{i\colon (x_i,y_i)=(u,v)} \alpha_i, u, v \right) \colon (u,v) \in \{(x_1,y_1), \ldots, (x_{m+n}, y_{m+n})\} \right\} \end{aligned}$$

is associative, i. e. $(Z, +)$ is a semigroup. In addition, define a scalar multiplication[1] $\mathbb{R} \times Z \to Z$ by

$$\lambda \{(\alpha_1, x_1, y_1), \ldots, (\alpha_n, x_n, y_n)\} = \{(\lambda\alpha_1, x_1, y_1), \ldots, (\lambda\alpha_n, x_n, y_n)\}.$$

The properties $\lambda(\mu z) = (\lambda\mu)z$, $\lambda(z_1 + z_2) = \lambda z_1 + \lambda z_2$, $(\lambda + \mu)z = \lambda z + \mu z$ and $1z = z$ for any $\lambda, \mu \in \mathbb{R}$, $z, z_1, z_2 \in Z$ are easily derived. The set

$$\begin{aligned} L_1 \;=\; & \Bigg\{ \bigg\{ (\alpha_{ij}, x_{ij}, y_j)\colon j \in \{1, \ldots, n\},\ y_j \text{ mutually distinct}, \\ & \forall j \text{ one has } i \in \{1, \ldots, n_j\} \text{ and } \sum_{i=1}^{n_j} \alpha_{ij} x_{ij} = 0 \bigg\} \in Z \Bigg\} \end{aligned}$$

[1]Observe that $(Z, +, \cdot)$ is not a vector space, since a zero element does not exist.

is closed under the two operations, as well as the set

$$L_2 = \Bigg\{\Bigg\{(\alpha_{ij}, x_i, y_{ij})\colon i \in \{1,\dots,n\},\ x_i \text{ mutually distinct}, \\ \forall i \text{ one has } j = \{1,\dots,n_i\} \text{ and } \sum_{j=1}^{n_i} \alpha_{ij} y_{ij} = 0\Bigg\} \in Z\Bigg\}.$$

Put $L = \{l_1 + l_2\colon l_1 \in L_1,\ l_2 \in L_2\}$ and observe that for any $z \in Z$ the element $0z$ belongs to L. The relation $\sim \subseteq Z \times Z$ defined by

$$(z_1, z_2) \in \sim \quad \text{if} \ \ z_1 + (-1)z_2 \in L$$

is an equivalence relation, and we put $T = Z/L$. With the operations $[z_1] + [z_2] = [z_1 + z_2]$ and $\lambda[z] = [\lambda z]$ ($z_1, z_2, z \in Z$, $\lambda \in \mathbb{R}$) T is a vector space, where L is its zero element, and for $[z] \in T$ the additive inverse is obtained by $[-z]$. Define

$$\begin{aligned} \tau\colon X \times Y &\to T \\ (x, y) &\mapsto [\{(1, x, y)\}] \end{aligned}$$

and observe that τ is bilinear. So far, we have T and τ of (1) in Definition 2.1.
To establish (2), we need some preliminary statements. Referring to the observation in Remark 2.2, an element $t \in T$ can be written as

$$t = [\{(\alpha_1, x_1, y_1), \dots, (\alpha_n, x_n, y_n)\}] = \alpha_1 \tau(x_1, y_1) + \dots + \alpha_n \tau(x_n, y_n),$$

i.e. the set $\{\tau(x, y)\colon x \in X,\ y \in Y\}$ spans the vector space T. Choose a Hamel basis $\{\varepsilon_i\colon i \in I\}$ of X and $\{\delta_j\colon j \in J\}$ of Y. Since τ is bilinear, the set $\{\tau(\varepsilon_i, \delta_j)\colon i \in I,\ j \in J\}$ spans T. We show that $\{\tau(\varepsilon_i, \delta_j)\colon i \in I,\ j \in J\}$ is linearly independent. Indeed, let

$$\sum_{i,j=1}^{n} \alpha_{ij} \tau(\varepsilon_i, \delta_j) = 0,$$

i.e. $\sum_{i,j=1}^{n} \alpha_{ij} [\{(1, \varepsilon_i, \delta_j)\}] = 0$, which means

$$\{(\alpha_{ij}, \varepsilon_i, \delta_j)\colon\ i, j = 1, \dots, n\} \in L.$$

So, there are $l_1 \in L_1$ and $l_2 \in L_2$ such that $\{(\alpha_{ij}, \varepsilon_i, \delta_j)\colon\ i, j = 1, \dots, n\} = l_1 + l_2$. Put

$$\Lambda_k = \{(i, j)\colon (\alpha_{ij}^{(k)}, \varepsilon_i, \delta_j) \in l_k \ \text{ for some } \ \alpha_{ij}^{(k)} \in \mathbb{R}\}$$

for $k = 1, 2$ and observe that $\Lambda_1 \cup \Lambda_2 = \{(i, j)\colon\ i, j = 1, \dots, n\}$. For $(i, j) \in \Lambda_1 \cap \Lambda_2$ follows $\alpha_{ij} = \alpha_{ij}^{(1)} + \alpha_{ij}^{(2)}$, whereas for $(i, j) \in \Lambda_k \setminus (\Lambda_1 \cap \Lambda_2)$ one gets $\alpha_{ij} = \alpha_{ij}^{(k)}$ for $k = 1, 2$. Since

$$l_1 = \{(\alpha_{ij}^{(1)}, \varepsilon_i, \delta_j)\colon\ (i, j) \in \Lambda_1\},$$

for any j with $(i,j) \in \Lambda_1$ one has

$$\sum_{i\colon (i,j)\in\Lambda_1} \alpha_{ij}^{(1)} \varepsilon_i = 0.$$

As $\{\varepsilon_i\colon (i,j) \in \Lambda_1\}$ is linearly independent, one obtains $\alpha_{ij}^{(1)} = 0$ for all $(i,j) \in \Lambda_1$. Analogously, $\alpha_{ij}^{(2)} = 0$ for all $(i,j) \in \Lambda_2$. So, $\alpha_{ij} = 0$ for all $i,j = 1,\dots,n$. Summing up, the set $\{\tau(\varepsilon_i,\delta_j)\colon i \in I,\, j \in J\}$ is a Hamel basis of T.

Next we show (2) in Definition 2.1. Let S be a vector space and let $\sigma\colon X \times Y \to S$ be a bilinear map. We define

$$\sigma^*(\tau(\varepsilon_i,\delta_j))\colon = \sigma(\varepsilon_i,\delta_j)$$

and extend by linearity to a linear map $\sigma^*\colon T \to S$. The bilinearity of τ and σ and the linearity of σ^* ensure $\sigma^*(\tau(x,y)) = \sigma(x,y)$ for all $x \in X$ and $y \in Y$. Observe that if $\sigma^\sharp\colon T \to S$ is a linear map with $\sigma^\sharp(\tau(x,y)) = \sigma(x,y)$ for all $x \in X$ and $y \in Y$, then $\sigma^\sharp = \sigma^*$. Thus we have shown that (T,τ) is a tensor product of X and Y.

Let $(\hat{T},\hat{\tau})$ be another tensor product of X and Y. We establish that there is a linear bijection $\tau^*\colon \hat{T} \to T$ such that $\tau(x,y) = \tau^*(\hat{\tau}(x,y))$ for all $x \in X$, $y \in Y$. Indeed, due to (2) of Definition 2.1 for the tensor product $(\hat{T},\hat{\tau})$, for the bilinear map τ there is a unique linear map $\tau^*\colon \hat{T} \to T$ such that $\tau(x,y) = \tau^*(\hat{\tau}(x,y))$ for all $x \in X$, $y \in Y$. For $t \in T$ one has $t = \sum_{i=1}^n \alpha_i \tau(x_i,y_i)$, $\hat{t} = \sum_{i=1}^n \alpha_i \hat{\tau}(x_i,y_i) \in \hat{T}$ and $\tau^*(\hat{t}) = t$. So, τ^* is surjective. We show that τ^* is injective. Let $u,v \in \hat{T}$ be such that $\tau^*(u) = \tau^*(v)$. Due to Remark 2.2 we can write

$$u = \sum_{i=1}^n \alpha_i \hat{\tau}(x_i,y_i) \quad \text{and} \quad v = \sum_{j=1}^m \beta_j \hat{\tau}(w_j,z_j)$$

with $\alpha_i,\beta_j \in \mathbb{R}$, $x_i,w_j \in X$, $y_i,z_j \in Y$. We get

$$\begin{aligned} \sum_{i=1}^n \alpha_i \tau^*(\hat{\tau}(x_i,y_i)) &= \sum_{j=1}^m \beta_j \tau^*(\hat{\tau}(w_j,z_j)), \quad \text{which means} \\ \sum_{i=1}^n \alpha_i \tau(x_i,y_i) &= \sum_{j=1}^m \beta_j \tau(w_j,z_j). \end{aligned} \tag{2}$$

Due to (2) of Definition 2.1 for the tensor product (T,τ), for the bilinear map $\hat{\tau}$ there is a unique linear map $\hat{\tau}^*\colon T \to \hat{T}$ such that $\hat{\tau}(x,y) = \hat{\tau}^*(\tau(x,y))$ for all $x \in X$, $y \in Y$. We apply $\hat{\tau}^*$ to the equality (2) and get

$$\begin{aligned} \sum_{i=1}^n \alpha_i \hat{\tau}^*(\tau(x_i,y_i)) &= \sum_{j=1}^m \beta_j \hat{\tau}^*(\tau(w_j,z_j)), \quad \text{i. e.} \\ \sum_{i=1}^n \alpha_i \hat{\tau}(x_i,y_i) &= \sum_{j=1}^m \beta_j \hat{\tau}(w_j,z_j), \quad \text{so} \\ u &= v. \end{aligned}$$

Summing up, we arrive at the following statement.

Theorem 2.3. *Let X and Y be vector spaces.*

(a) *There exists a tensor product (T, τ) of X and Y.*

(b) *If $(\hat{T}, \hat{\tau})$ is another tensor product of X and Y, then there is a linear bijection*

$$\tau^* \colon \hat{T} \to T$$

such that $\tau(x, y) = \tau^*(\hat{\tau}(x, y))$ for all $x \in X$, $y \in Y$.

It follows from Theorem 2.3 that there exists an essentially unique tensor product (T, τ) of X and Y. We denote it as usual by $X \otimes Y$, and for $x \in X$ and $y \in Y$ we use the notation $x \otimes y = \tau(x, y)$.

The next lemma will be needed in the sequel to relate the tensor product of pre-Riesz spaces and the tensor product of their Riesz completions.

Lemma 2.4. *Let X, Y, U and V be vector spaces and let $\rho_X \colon X \to U$ and $\rho_Y \colon Y \to V$ be linear injections. Let*

$$\rho(x, y) := \rho_X(x) \otimes \rho_Y(y), \quad x \in X, \ y \in Y.$$

Then the unique linear map $\rho^* \colon X \otimes Y \to U \otimes V$ satisfying $\rho^*(x \otimes y) = \rho(x, y)$ for all $x \in X$ and $y \in Y$ is injective.

Proof. Let $w, z \in X \otimes Y$ be such that $\rho^*(w) = \rho^*(z)$. Due to Remark 2.2, there are $x_i \in X$, $y_i \in Y$, and $\alpha_i \in \mathbb{R}$, $i \in \{1, \ldots, n\}$ such that

$$z = \sum_{i=1}^{n} \alpha_i x_i \otimes y_i.$$

Choose a Hamel basis $(e_i)_{i \in I}$ of $\rho_X(X)$ and extend it to a Hamel basis $(e_i)_{i \in I'}$ of U and choose a Hamel basis $(f_j)_{j \in J}$ of $\rho_Y(Y)$ and extend it to a Hamel basis $(f_j)_{j \in J'}$ of V. Define $\eta \colon U \times V \to X \otimes Y$ by

$$\eta\left(\sum_{i \in I'} \lambda_i e_i, \sum_{j \in J'} \mu_j f_j\right) := \sum_{i \in I} \sum_{j \in J} \lambda_i \mu_j \rho_X^{-1}(e_i) \otimes \rho_Y^{-1}(f_j).$$

Then η is bilinear and $\eta(\rho_X(x), \rho_Y(y)) = x \otimes y$ for all $x \in X$ and $y \in Y$. Hence there is a unique linear map $\eta^* \colon U \otimes V \to X \otimes Y$ such that $\eta^*(u \otimes v) = \eta(u, v)$ for all $u \in U$, $v \in V$. In particular, $\eta^*(\rho_X(x) \otimes \rho_Y(y)) = x \otimes y$ for all $x \in X$ and $y \in Y$. We obtain that

$$\begin{aligned} \eta^*(\rho^*(z)) &= \sum_{i=1}^{n} \alpha_i \eta^*(\rho^*(x_i \otimes y_i)) = \sum_{i=1}^{n} \alpha_i \eta^*(\rho(x_i, y_i)) \\ &= \sum_{i=1}^{n} \alpha_i \eta^*(\rho_X(x_i) \otimes \rho_Y(y_i)) = \sum_{i=1}^{n} \alpha_i x_i \otimes y_i = z. \end{aligned}$$

Similarly, $\eta^*(\rho^*(w)) = w$. From $\rho^*(z) = \rho^*(w)$ it follows that $z = w$. □

3 An ordering on the tensor product

Before we introduce an ordering on the tensor product, we recall some terminology on partially ordered vector spaces.
By X we denote a real vector space and by K a cone in X, that is, K is a wedge ($x, y \in K$, $\lambda, \mu \geq 0$ imply $\lambda x + \mu y \in K$) and $K \cap (-K) = \{0\}$. In X a partial order is introduced by defining $y \geq x$ if and only if $y - x \in K$. Denote for a subset $M \subseteq X$ the set of all upper bounds by

$$M^u = \{x \in X \colon\ x \geq m \ \text{ for all } \ m \in M\}\,.$$

The space (X, K) is called *Archimedean* if for every $x, y \in X$ with $nx \leq y$ for all $n \in \mathbb{N} \cup \{0\}$ one has $x \leq 0$. A set $M \subseteq X$ is called *directed* if for every $x, y \in M$ there is an element $z \in M$ such that $z \geq x$ and $z \geq y$. X is directed if and only if the cone K is *generating* in X, that is, $X = K - K$. X has the *Riesz decomposition property* if for every $y, x_1, x_2 \in K$ with $y \leq x_1 + x_2$ there exist $y_1, y_2 \in K$ such that $y = y_1 + y_2$ and $y_1 \leq x_1$, $y_2 \leq x_2$. For standard notions in the case that X is a Riesz space, see [1].
Let (X, K_X) and (Y, K_Y) be partially ordered vector spaces. With the notations of the previous section define

$$K_Z = \{\{(\alpha_i, x_i, y_i) \colon i \in \{1, \ldots, n\}\} \in Z \colon \alpha_i \geq 0,\, x_i \in K_X,\, y_i \in K_Y\}.$$

Clearly, from $u, v \in K_Z$ and $\lambda \geq 0$ follows $u + v \in K_Z$ and $\lambda u \in K_Z$. In $T = Z/L$ put

$$K_T = \{[z] \colon z \in K_Z\}.$$

We have to show that K_T is a cone in T. It is straightforward that K_T is a wedge in T, so it remains to establish

$$K_T \cap (-K_T) = \{L\}. \tag{3}$$

Denote $K_L = K_Z \cap L$, $K_{L_1} = K_Z \cap L_1$ and $K_{L_2} = K_Z \cap L_2$. The elements of K_{L_1} can easily be characterized. Namely, let

$$l = \{(\beta_1, u_1, v_1)\,, (\beta_2, u_2, v_2)\,, \ldots, (\beta_r, u_r, v_r)\}\,,\ \text{where } (u_p, v_p) \neq (u_q, v_q) \text{ if } p \neq q\,, \tag{4}$$

be in K_Z, then

$$l \in K_{L_1} \text{ if and only if for all } s \in \{1, \ldots, r\} \text{ one has } \beta_s = 0 \text{ or } u_s = 0\,. \tag{5}$$

An analogous statement holds for K_{L_2}. We intend to determine the elements of K_L similarly.

Lemma 3.1. *For $l \in K_Z$ given by (4) follows*

$$l \in K_L \ \ \textit{if and only if for all} \ \ s \in \{1, \ldots, r\} \ \ \textit{one has} \ \ \beta_s = 0 \ \ \textit{or} \ \ u_s = 0 \ \ \textit{or} \ \ v_s = 0\,.$$

Proof Let $l \in K_L$ be given by (4). If $u_s = 0$ for all $s \in \{1, \ldots, r\}$ or $v_s = 0$ for all $s \in \{1, \ldots, r\}$ then we are done. So, assume that at least one of the u_s is non-zero and that at least one of the v_s is non-zero. There are m pairwise distinct elements $x_1, \ldots, x_m \in K_X \setminus \{0\}$ such that $\{u_1, \ldots, u_r\} \setminus \{0\} = \{x_1, \ldots, x_m\}$. Similarly, there are n pairwise distinct elements $y_1, \ldots, y_n \in K_Y \setminus \{0\}$ such that $\{v_1, \ldots, v_r\} \setminus \{0\} = \{y_1, \ldots, y_n\}$. If $0 \in \{u_1, \ldots, u_r\}$, then put $\overline{m} = m+1$ and $x_{m+1} = 0$, otherwise put $\overline{m} = m$. Analogously, if $0 \in \{v_1, \ldots, v_r\}$, then put $\overline{n} = n+1$ and $y_{n+1} = 0$, otherwise put $\overline{n} = n$. Let

$$\Lambda = \{(i,j) \in \{1, \ldots, \overline{m}\} \times \{1, \ldots, \overline{n}\} \colon \exists s \in \{1, \ldots, r\} \text{ such that } (\beta_s, x_i, y_j) \in l\}\,.$$

As the pairs (u_k, v_k) appearing in l are distinct, for $(i,j) \in \Lambda$ there is a unique $s \in \{1, \ldots, r\}$ such that $(\beta_s, x_i, y_j) \in l$; denote $\alpha_{ij} = \beta_s$. We get the representation

$$l = \{(\alpha_{ij}, x_i, y_j) \colon (i,j) \in \Lambda\}\,.$$

Since $l = l_1 + l_2$ with $l_1 \in L_1$ and $l_2 \in L_2$, there are $\varnothing \neq \Lambda_1 \subseteq \Lambda$, $\varnothing \neq \Lambda_2 \subseteq \Lambda$, $\lambda_{ij} \in \mathbb{R}$ for all $(i,j) \in \Lambda_1$ and $\mu_{ij} \in \mathbb{R}$ for all $(i,j) \in \Lambda_2$ such that

$$l_1 = \{(\lambda_{ij}, x_i, y_j) \colon (i,j) \in \Lambda_1\} \quad \text{and} \quad l_2 = \{(\mu_{ij}, x_i, y_j) \colon (i,j) \in \Lambda_2\}\,.$$

We extend l_1 and l_2 such that all pairs of l appear in them. Put $\lambda_{ij} = 0$ for $(i,j) \in \Lambda \setminus \Lambda_1$, $\mu_{ij} = 0$ for $(i,j) \in \Lambda \setminus \Lambda_2$ and denote

$$\overline{l_1} = \{(\lambda_{ij}, x_i, y_j) \colon (i,j) \in \Lambda\} \quad \text{and} \quad \overline{l_2} = \{(\mu_{ij}, x_i, y_j) \colon (i,j) \in \Lambda\}\,,$$

which yields $\overline{l_1} \in L_1$, $\overline{l_2} \in L_2$ and $l = \overline{l_1} + \overline{l_2}$. Putting $\lambda_{ij} = \mu_{ij} = \alpha_{ij} = 0$ for all $(i,j) \in \{1, \ldots, m\} \times \{1, \ldots, n\} \setminus \Lambda$, we obtain $m \times n$-matrices $\mathcal{A} = (\alpha_{ij})$, $\mathcal{L} = (\lambda_{ij})$ and $\mathcal{M} = (\mu_{ij})$, which satisfy $\mathcal{A} = \mathcal{L} + \mathcal{M}$. As $l \in K_Z$, we have $\mathcal{A} \geq 0$. To prove the statement of the lemma, we show $\mathcal{A} = 0$.

As a next step, we equip the finite-dimensional subspace $X_0 = \operatorname{span}\{x_1, \ldots, x_m\}$ of X with the cone $K_{X_0} = K_X \cap X_0$, which is polyhedral and generating. There is a functional φ on X_0 such that $\varphi(x) > 0$ for all $x \in K_{X_0} \setminus \{0\}$, in particular $\varphi(x_i) > 0$ for all $i \in \{1, \ldots, m\}$. Similarly, on the space $Y_0 = \operatorname{span}\{y_1, \ldots, y_n\}$, equipped with the cone $K_{Y_0} = K_Y \cap Y_0$, there is a functional ψ such that $\psi(y_j) > 0$ for all $j \in \{1, \ldots, n\}$. Define $a = (\varphi(x_1), \ldots, \varphi(x_m))^T$ and $b = (\psi(y_1), \ldots, \psi(y_n))^T$ and observe that a and b are strictly positive. Moreover,

$$a^T \mathcal{L} = 0 \quad \text{and} \quad \mathcal{M} b = 0\,. \tag{6}$$

To show the first equality, fix $j \in \{1, \ldots, n\}$, then due to $\{(\lambda_{pq}, x_p, y_q) \colon (p,q) \in \Lambda\} = \overline{l_1} \in L_1$ we have $\sum_{i \colon (i,j) \in \Lambda} \lambda_{ij} x_i = 0$, and, as $\lambda_{ij} = 0$ if $(i,j) \notin \Lambda$, we find $\sum_{i=1}^{\overline{m}} \lambda_{ij} x_i = 0$. Since $x_{m+1} = 0$ if $\overline{m} = m+1$, we obtain $\sum_{i=1}^{m} \lambda_{ij} x_i = 0$. Then $\sum_{i=1}^{m} \lambda_{ij} \varphi(x_i) = 0$, which means $(a^T \mathcal{L})_j = 0$. Analogously, for a fixed $i \in \{1, \ldots, m\}$ we get $(\mathcal{M} b)_i = 0$. So, (6) is satisfied, and we can conclude

$$a^T \mathcal{A} b = a^T (\mathcal{L} b + \mathcal{M} b) = (a^T \mathcal{L}) b = 0\,. \tag{7}$$

Now we prove the lemma by way of contradiction. Assume that there is $(i,j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$ with $\alpha_{ij} > 0$. Since b is strictly positive, we obtain $(\mathcal{A} b)_i > 0$. Moreover, $a^T \mathcal{A} b > 0$, as a is also strictly positive. This contradicts (7), so we conclude $\mathcal{A} = 0$. $\square$

Lemma 3.2. *If $l \in K_L$, $u \in K_Z$ and $l - u \in K_Z$, then $u \in K_L$.*

Proof Denote

$$\begin{aligned} l &= \{(\alpha_1, a_1, b_1), \ldots, (\alpha_k, a_k, b_k), (\lambda_1, c_1, d_1), \ldots, (\lambda_m, c_m, d_m)\} \quad \text{and} \\ u &= \{(\beta_1, a_1, b_1), \ldots, (\beta_k, a_k, b_k), (\mu_1, e_1, f_1), \ldots, (\mu_n, e_n, f_n)\}, \end{aligned}$$

where all the (a_i, b_i), (c_i, d_i), (e_i, f_i) are distinct pairs. We have $a_i, c_i, e_i \in K_X$, $b_i, d_i, f_i \in K_Y$ and $\alpha_i, \beta_i, \lambda_i, \mu_i \geq 0$. Due to

$$\begin{aligned} l - u = &\{(\alpha_1 - \beta_1, a_1, b_1), \ldots, (\alpha_k - \beta_k, a_k, b_k), (\lambda_1, c_1, d_1), \ldots, (\lambda_m, c_m, d_m), \\ &(-\mu_1, e_1, f_1), \ldots, (-\mu_n, e_n, f_n)\} \in K_Z \end{aligned}$$

follows $\alpha_i - \beta_i \geq 0$ and $-\mu_i \geq 0$. We obtain $\mu_i = 0$ for all $i \in \{1, \ldots, n\}$. Due to Lemma 3.1 and $l \in K_L$, for $i \in \{1, \ldots, k\}$ we have either $a_i = 0$ or $b_i = 0$ or $\alpha_i = 0$. In the latter case follows $0 = \alpha_i \geq \beta_i \geq 0$, so $\beta_i = 0$. Therefore $u \in K_L$. $\square$

Finally, we complete the proof of (3). Let $z \in Z$ be such that $[z] \in K_T \cap (-K_T)$, then there are $u, v \in [z]$ such that $u \in K_Z$ and $-v \in K_Z$. Put $l = u - v$ and observe that $l \in K_L$. Due to Lemma 3.2 we get $u \in K_L$ and, hence, $[z] = L$. We obtain that K_T is a cone in the vector space T.

Theorem 3.3. *(T, K_T) is a partially ordered vector space.*

With the usual tensor notation the definition of K_T becomes

$$K_T = \Big\{ \sum_{i=1}^{n} \alpha_i x_i \otimes y_i \colon\ x_i \in K_X,\ y_i \in K_Y,\ \alpha_i \in \mathbb{R}^+,\ n \in \mathbb{N} \Big\}. \tag{8}$$

K_T is called the *projective cone* in T (see, e.g., [8]).

Proposition 3.4. *If X and Y are directed partially ordered vector spaces, then K_T is generating in T.*

Proof First, let $x \in X$ and $y \in Y$. Choose $u \in X$ with $-u \leq x \leq u$ and $v \in Y$ with $-v \leq y \leq v$. Let $a = \tau(u - x, v - y) + \tau(u, v)$ and $b = \tau(u, v - y) + \tau(u - x, v)$. Then $a, b \in K_T$ and $\tau(x, y) = a - b$.
Next, let $t \in T$. Then there exist $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_n \in Y$ such that $t = \tau(x_1, y_1) + \ldots + \tau(x_n, y_n)$. For suitable $a_1, \ldots, a_n, b_1, \ldots, b_n \in K_T$ constructed as above we obtain

$$t = (a_1 - b_1) + \ldots + (a_n - b_n) = (a_1 + \ldots + a_n) - (b_1 + \ldots + b_n) \in K_T - K_T. \qquad \square$$

In the following simple example (T, K_T) is a Riesz space.

Example 3.5. Let X be the set of all $p \times q$-matrices, Y the set of all $r \times s$-matrices, and U the set of all $pr \times qs$-matrices, equipped with the entrywise ordering, respectively. For matrices $A = (a_{ij}) \in X$ and $B \in Y$ denote by $[a_{ij}B]_{ij}$ the element of U which consists of pq blocks, each of them a corresponding multiple of B. Let Z be as in (1) and define $J \colon Z \to U$ by

$$J\left(\left\{\left(\alpha_k, A^{(k)}, B^{(k)}\right) : k = 1, \ldots, n\right\}\right) = \sum_{k=1}^{n} \alpha_k [a_{ij}^{(k)} B^{(k)}]_{ij} .$$

It is straightforward that for all $u \in L_1$ and $u \in L_2$ one has $J(u) = 0$, respectively. So, for $u, v \in Z$ with $[u] = [v]$ follows $J(u) = J(v)$, and the map $\overline{J} \colon T \to U$ given by

$$\overline{J}([u]) = J(u)$$

is well-defined. Observe that for $u, v \in Z$ and $\nu \in \mathbb{R}$ one has $J(u + \nu v) = J(u) + \nu J(v)$, which implies that $\overline{J}$ is linear. It is obvious that $\overline{J}$ is positive.
Let $E_X^{(kl)} = (c_{ij})$ denote the $p \times q$-matrix with

$$c_{ij} = \begin{cases} 1 & \text{if } (i,j) = (k,l) \\ 0 & \text{otherwise} \end{cases}$$

and $E_Y^{(mn)}$ the analogous $r \times s$-matrix. Denote $E_U^{(klmn)} = [c_{ij} E_Y^{(mn)}]_{ij}$. For every $z \in Z$ there are scalars α_{klmn} such that

$$\begin{aligned} [z] \;=\; & \Big[\Big\{\Big(\alpha_{klmn}, E_X^{(kl)}, E_Y^{(mn)}\Big) : \\ & (k,l,m,n) \in \{1,\ldots,p\} \times \{1,\ldots,q\} \times \{1,\ldots,r\} \times \{1,\ldots,s\}\Big\}\Big] . \end{aligned} \tag{9}$$

We show that $\overline{J}$ is bipositive and, hence, injective. Let $t = [z]$ be such that $\overline{J}(t) \geq 0$ in U. Using the representation (9), we obtain

$$\overline{J}(t) = \sum_{k,l,m,n} \alpha_{klmn} E_U^{(klmn)} .$$

As this matrix is positive, each of its entries is positive, which means that $\alpha_{klmn} \geq 0$ for all k, l, m, n. Hence, $z \in K_Z$ and, therefore, $t \in K_T$.
Finally, it is straightforward that $\overline{J}$ is surjective. Hence, T is identified with the Riesz space U.

4 Riesz completions and the Fremlin tensor product

A partially ordered vector space X is called *pre-Riesz* if for every $x, y, z \in X$ the inclusion $\{x + y, x + z\}^u \subseteq \{y, z\}^u$ implies $x \in K$ [9, Definition 1.1(viii), Theorem 4.15]. Every

pre-Riesz space is directed and every directed Archimedean partially ordered vector space is pre-Riesz [9]. Clearly, each Riesz space is pre-Riesz.
By a subspace of a partially ordered vector space or a Riesz space we mean an arbitrary linear subspace with the inherited order. We do not require it to be a lattice or a sublattice.
We say that a subspace X of a Riesz space Y *generates* Y as a Riesz space if for every $y \in Y$ there exist $a_1, \ldots, a_m, b_1, \ldots, b_n \in X$ such that $y = \bigvee_{i=1}^m a_i - \bigvee_{i=1}^n b_i$.
We call a linear subspace D of a partially ordered vector space X *order dense* in X if for every $x \in X$ we have $x = \inf\{y \in D\colon\ y \geq x\}$, that is, each x is the greatest lower bound of the set $\{y \in D\colon\ y \geq x\}$ in X.
Recall that a linear map $i\colon X \to Y$, where X and Y are partially ordered vector spaces, is called *bipositive* if for every $x \in X$ one has $i(x) \geq 0$ if and only if $x \geq 0$. An embedding map is required to be linear and bipositive, which implies injectivity.
Let X be a partially ordered vector space. According to van Haandel [9, Corollaries 4.9–11 and Theorems 3.5, 3.7, 4.13] the following statements are equivalent:

(i) X is pre-Riesz.

(ii) There exist a Riesz space Y and a bipositive linear map $i\colon\ X \to Y$ such that $i(X)$ is order dense in Y.

(iii) There exist a Riesz space Y and a bipositive linear map $i\colon\ X \to Y$ such that $i(X)$ is order dense in Y and generates Y as a Riesz space.

All spaces Y as in (iii) are isomorphic as Riesz spaces. A pair (Y, i) as in (iii) is called a *Riesz completion* of X. As it is unique up to isomorphism we will speak of *the* Riesz completion of X and denote it by X^ρ. If X is Archimedean, then its Riesz completion coincides with its *enveloping Riesz space*, as defined below Theorem 2.9 in [4].
Van Haandel also introduces a notion of homomorphism adapted to the concept of pre-Riesz spaces. Let X and Y be directed partially ordered vector spaces. A linear map $h\colon X \to Y$ is called a *Riesz* homomorphism* if for any $a, b \in X$ and for every lower bound x of $\{a, b\}^u$ in X one has that $h(x)$ is a lower bound of $\{h(a), h(b)\}^u$ in Y (see [9, Definition 5.1]). If X and Y are Riesz spaces, then $h\colon X \to Y$ is a Riesz* homomorphism if and only if it is a Riesz homomorphism. If X and Y are pre-Riesz spaces, then $h\colon X \to Y$ is a Riesz* homomorphism if and only if it is the restriction of a Riesz homomorphism from X^ρ into Y^ρ. The composition of two Riesz* homomorphisms is a Riesz* homomorphism (see [9, Remarks 5.2(ii), (iii), and Theorem 5.6]).
Next we recall Fremlin's main result on the tensor product of Archimedean Riesz spaces (E, E^+) and (F, F^+).

Theorem 4.1. [7, 4.2] *Let E and F be Archimedean Riesz spaces. Then there is an essentially unique Archimedean Riesz space G and a Riesz bimorphism $\phi\colon E \times F \to G$ such that*

(i) *whenever H is an Archimedean Riesz space and $\psi\colon E \times F \to H$ is a Riesz bimorphism, there is a unique Riesz homomorphism $T\colon G \to H$ such that $T\phi = \psi$;*

(ii) *ϕ induces an embedding of $E \otimes F$ in G;*

(iii) *$E \otimes F$ is dense in G in the sense that for every $w \in G$ there exist $x_0 \in E$ and $y_0 \in F$ such that for every $\delta > 0$ there is a $v \in E \otimes F$ such that $|w - v| \leq \delta x_0 \otimes y_0$;*

(iv) *if $w > 0$ in G, then there exist $x \in E^+$ and $y \in F^+$ such that $0 < x \otimes y \leq w$.*

The essentially unique Archimedean Riesz space G in the above theorem is called the *Fremlin tensor product* of E and F and is denoted by $E\bar{\otimes}F$. The essential uniqueness means that any Archimedean Riesz space G with the properties of the theorem is Riesz isomorphic to $E\bar{\otimes}F$. Usually, one identifies the vector space tensor product $E \otimes F$ with the corresponding linear subspace of $E\bar{\otimes}F$.
The next result says that the Fremlin tensor product is the Riesz completion of the vector space tensor product equipped with the order induced by the Fremlin tensor product. This fact is contained in [5, Theorem 4]. We give a proof here based on the density property in Theorem 4.1(iii) (see also [2]).

Theorem 4.2. *Let E and F be Archimedean Riesz spaces, let $E\bar{\otimes}F$ be the Fremlin tensor product of E and F, and let $E \otimes F$ be the linear subspace generated by all $x \otimes y$, $x \in E$, $y \in F$, endowed with the induced order. Then $E \otimes F$ is a pre-Riesz space and $E\bar{\otimes}F$ is its Riesz completion. Moreover, the inclusion map $\hat{\phi}\colon E \otimes F \to E\bar{\otimes}F$ is a Riesz* homomorphism.*

Proof. Let $\phi\colon E \times F \to E\bar{\otimes}F$ be a Riesz bimorphism and let $\hat{\phi}\colon E \otimes F \to E\bar{\otimes}F$ be its induced embedding. By definition of the order in $E \otimes F$, the map $\hat{\phi}$ is bipositive. In order to show that $E \otimes F$ is a pre-Riesz space, we show that $D = \hat{\phi}(E \otimes F)$ is order dense in $G = E\bar{\otimes}F$. Let $w \in G$. According to Theorem 4.1(iii) there exist $x \in E$ and $y \in F$ such that for every $\delta > 0$ there exists a $v_\delta \in D$ such that $|w - v_\delta| \leq \delta z$, where $z = \hat{\phi}(x \otimes y)$. Then $z \in D$ so $v_\delta + \delta z \in D$ and $w \leq v_\delta + \delta z$. Also $v_\delta - w \leq \delta z$, so $v_\delta \leq w + \delta z$. Therefore, if a is a lower bound of $\{u \in D\colon\ u \geq w\}$, then $a \leq v_\delta + \delta \leq w + \delta z + \delta z$. As G is Archimedean, it follows that $a \leq w$. Hence $w = \inf\{u \in D\colon\ u \geq w\}$ in G. Thus, D is order dense in G. Consequently, D is a pre-Riesz space. Let G_0 be the Riesz subspace of G generated by D. The embedding map $\hat{\phi}\colon E \otimes F \to G_0$ is bipositive and its range is order dense and generates G_0 as a Riesz space, so $(G_0, \hat{\phi})$ is a Riesz completion of $E \otimes F$. In particular, $\hat{\phi}$ is a Riesz* homomorphism.
Finally we show that G_0 and G are isomorphic Riesz spaces. We verify that G_0 satisfies the conditions of Theorem 4.1 just as G does, so that the essential uniqueness yields that G_0 and G are isomorphic Riesz spaces. Since $\phi\colon E \times F \to G$ is a Riesz bimorphism and $\phi(E \times F)$ is contained in G_0, ϕ is a Riesz bimorphism from $E \times F$ into G_0. If H is an Archimedean Riesz space and $\psi\colon E \times F \to H$ is a Riesz bimorphism, then there exists a Riesz homomorphism $T\colon G \to H$ such that $T \circ \phi = \psi$. Then the restriction T_0 of T to G_0 is a Riesz homomorphism from G_0 to H and $T_0 \circ \phi = \psi$. If T_1 is another Riesz homomorphism from G_0 into H with $T_1 \circ \phi = \psi$, then $T_1 = T_0$, as G_0 is the Riesz subspace generated by $\phi(E \times F)$. Hence (ii) of Theorem 4.1 is satisfied by G_0. Clearly also (iii) and

(iv) are satisfied by G_0. Thus G and G_0 are isomorphic Riesz spaces and therefore G is the Riesz completion of D. □

5 Closure of the projective cone

Consider two Archimedean Riesz spaces X and Y, their vector space tensor product $T = X \otimes Y$, and let K_T be the projective cone in $X \otimes Y$ as given by (8). Although the order induced by K_T is natural from a construction point of view, it is often not the natural order in examples. For instance, if $X = Y = C[0,1]$, then the order induced by K_T on $X \otimes Y$ viewed as subspace of $C([0,1] \times [0,1])$ is not the natural pointwise order on $[0,1] \times [0,1]$ (see [7, Counterexample 4.7]). The advantage of Fremlin's construction via representations is that it yields the natural order in examples. The problem of the cone K_T is that it is in general non-Archimedean and smaller than the cone of Fremlin's order. The closure of K_T has been studied in various settings with topologies or order units, see [3, 6, 10, 12, 13].
It turns out that the cone of Fremlin's order can be obtained from K_T by taking a suitable Archimedean closure of K_T. Grobler and Labuschagne have taken this approach in [8] to construct the cone of Fremlin's order intrinsically from K_T. They were able to extend their methods to directed Archimedean partially ordered vector spaces with the Riesz decomposition property. We will go even further and show that the Riesz decomposition property is not needed. Instead of embedding in bidual spaces we will use Riesz completions. Thus, we construct the tensor product of arbitrary directed Archimedean partially ordered vector spaces with the proper order. Our construction is intrinsic. However, for the proof that the constructed wedge in the tensor product is in fact a cone, we need to rely on Fremlin's results.
In the remainder of this section, let (X, K_X) and (Y, K_Y) be two directed Archimedean partially vector spaces, let $T = X \otimes Y$ be their vector space tensor product, and let K_T be the projective cone as given by (8).
The next definition is taken from [8, Section 2].

Definition 5.1. A cone K in $X \otimes Y$ is called an *Archimedean tensor cone* if $K_T \subseteq K$ and the following universal mapping property is satisfied: For every directed Archimedean partially ordered vector space (S, K_S) and every positive bilinear map $\sigma\colon X \times Y \to S$ the induced linear map $\sigma^*\colon (X \otimes Y, K) \to (S, K_S)$ is positive.

If an Archimedean tensor cone in $X \otimes Y$ exists, then it is unique [8, Section 2].
Let (X^ρ, ρ_X) and (Y^ρ, ρ_Y) be the Riesz completions of X and Y, respectively. Then X^ρ and Y^ρ are Archimedean Riesz spaces. Let $X^\rho \bar{\otimes} Y^\rho$ be the Fremlin tensor product. Then there exists a Riesz bimorphism $\phi_F\colon X^\rho \times Y^\rho \to X^\rho \bar{\otimes} Y^\rho$ with the properties as in Theorem 4.1. In particular, there is a linear injection $h_F\colon X^\rho \otimes Y^\rho \to X^\rho \bar{\otimes} Y^\rho$ such that

$$h_F(u \otimes v) = \phi_F(u, v) \text{ for all } u \in X^\rho,\ v \in Y^\rho.$$

Define $\rho\colon X \times Y \to X^\rho \otimes Y^\rho$ by

$$\rho(x, y) := \rho_X(x) \otimes \rho_Y(y),\ x \in X,\ y \in Y.$$

Then ρ is bilinear, so ρ induces a unique linear map $\rho^*\colon X \otimes Y \to X^\rho \otimes Y^\rho$ such that

$$\rho(x, y) = \rho^*(x \otimes y) \text{ for all } x \in X,\ y \in Y.$$

By Lemma 2.4, ρ^* is injective. Hence $X \otimes Y$ is embedded into $X^\rho \bar{\otimes} Y^\rho$ by the injective linear map $h_F \circ \rho^*$. Thus the order of the Fremlin tensor product $X^\rho \bar{\otimes} Y^\rho$ induces an order on $X \otimes Y$. Define

$$K_F := \{w \in X \otimes Y\colon\ h_F(\rho^*(w)) \in (X^\rho \bar{\otimes} Y^\rho)^+\}.$$

Lemma 5.2. *K_F is a cone in $X \otimes Y$, $K_T \subseteq K_F$, and $(X \otimes Y, K_F)$ is Archimedean.*

Proof. Since h_F and ρ^* are linear and injective, K_F is a cone. For $x \in K_X$ and $y \in K_Y$, we have

$$h_F(\rho^*(x \otimes y)) = h_F(\rho(x, y)) = h_F(\rho_X(x) \otimes \rho_Y(y)) = \phi_F(\rho_X(x), \rho_Y(y)) \in (X^\rho \bar{\otimes} Y^\rho)^+,$$

as ϕ_F is a Riesz bimorphism and therefore positive. So $x \otimes y \in K_F$, which implies that $K_T \subseteq K_F$.

If $w_1, w_2 \in X \otimes Y$ are such that $w_1 - nw_2 \in K_F$ for all $n \in \mathbb{N}$, then $h_F(\rho^*(w_1)) - nh_F(\rho^*(w_2)) \in (X^\rho \bar{\otimes} Y^\rho)^+$ for all n. Then $-h_F(\rho^*(w_2)) \in (X^\rho \bar{\otimes} Y^\rho)^+$, as $X^\rho \bar{\otimes} Y^\rho$ is Archimedean. So $-w_2 \in K_F$. Hence $(X \otimes Y, K_F)$ is Archimedean. □

Due to Lemma 5.2, the cone K_T is contained in an Archimedean cone, so there exists a smallest Archimedean cone K in $X \otimes Y$ which contains K_T. K is obtained as the intersection of all Archimedean cones that contain K_T. It turns out that K is the Archimedean tensor cone in $X \otimes Y$. There is an alternative construction of the Archimedean tensor cone as the relatively uniform closure (ru-closure) of K_T, which we need in the proof.

We recall some facts about the relative uniform topology. Let (S, K_S) be a directed partially ordered vector space. A sequence $(s_n)_n$ in S is said to converge relatively uniformly to an $s \in S$, denoted by $s_n \to s$ (ru), if there exist an $a \in K_S$ and a sequence $(\lambda_n)_n$ in $\mathbb{R}^+$ such that $\lambda_n \to 0$ as $n \to \infty$ and $-\lambda_n a \leq s_n - s \leq \lambda_n a$ for all n. A subset C of S is called *ru-closed* if it is closed under ru-convergence of sequences. It can be shown that there exists a vector space topology in S for which the closed sets are exactly the ru-closed subsets of S. This topology is called the ru-topology in S. By the *ru-closure* of a set $C \subseteq S$ we mean the smallest ru-closed set in S which contains C.

Lemma 5.3. *K_F is ru-closed in $(X \otimes Y, K_T)$.*

Proof. Let $(w_n)_n$ in K_F and $w \in X \otimes Y$ be such that $w_n \to w$ (ru). Take $a \in K_T$ and $\lambda_n \to 0$ such that we have in K_T-order for all n that $-\lambda_n a \leq w_n - w \leq \lambda_n a$. Let $\tilde{w} := h_F(\rho^*(w))$, $\tilde{w}_n := h_F(\rho^*(w_n))$, $n \in \mathbb{N}$, and $\tilde{a} := h_F(\rho^*(a))$. Since $h_F \circ \rho^*$ is positive we obtain $-\lambda_n \tilde{a} \leq \tilde{w}_n - \tilde{w} \leq \lambda \tilde{a}$ in $(X^\rho \bar{\otimes} Y^\rho)^+$-order for all n. Then $\tilde{w} \geq \tilde{w}_n - \lambda_n \tilde{a} \geq -\lambda_n \tilde{a}$ for all n. As $(X^\rho \bar{\otimes} Y^\rho)^+$ is Archimedean, we obtain $\tilde{w} \in (X^\rho \bar{\otimes} Y^\rho)^+$, so that $w \in K_F$. □

Theorem 5.4. *For a cone K in $X \otimes Y$ the following four statements are equivalent:*

(a) K is the Archimedean tensor cone.

(b) Let (S, K_S) be a directed Archimedean partially ordered vector space and $\phi\colon X \otimes Y \to S$ a linear map such that $\phi(w) \in K_S$ for all $w \in K_T$. Then $\phi(w) \in K_S$ for all $w \in K$.

(c) K is the smallest Archimedean cone in $X \otimes Y$ with $K_T \subseteq K$.

(d) $K = \overline{K_T}$, where $\overline{K_T}$ is the ru-closure of K_T in $(X \otimes Y, K_T)$.

Proof. (a)⇒(c): Let L be an Archimedean cone in $X \otimes Y$ with $K_T \subseteq L$. Take $S := X \otimes Y$, $K_S := L$, and $\sigma(x, y) := x \otimes y$, $x \in X$, $y \in Y$. Since σ is positive bilinear and K is the Archimedean tensor cone, it follows that $\sigma^*\colon (X \otimes Y, K) \to (X \otimes Y, L)$ is positive. As σ^* is the identity map on $X \otimes Y$, we obtain that $K \subseteq L$.
(c)⇒(d): We first show that K is ru-closed. Let $(w_n)_n$ in K and $w \in X \otimes Y$ be such that $w_n \to w$ (ru). Then there are an $a \in K_T$ and $\lambda_n \to 0$ such that in K_T-order we have

$$-\lambda_n a \leq w_n - w \leq \lambda_n a \quad \text{for all } n \in \mathbb{N}.$$

Then $w + \lambda_n a - w_n \in K_T \subseteq K$, so $w + \lambda_n a \in K$ for all n. As K is Archimedean, we obtain that $w \in K$. Hence K is ru-closed, so that $\overline{K_T} \subseteq K$.
Since $K_T \subseteq K_F$ and K_F is ru-closed according to Lemma 5.3, we have $\overline{K_T} \subseteq K_F$. As K_F is a cone by Lemma 5.2 and $\overline{K_T}$ is a wedge, it follows that $\overline{K_T}$ is a cone.
Next we show that $\overline{K_T}$ is Archimedean. Let $w_1, w_2 \in X \otimes Y$ be such that $w_2 - nw_1 \in \overline{K_T}$ for all $n \in \mathbb{N}$. Since K_T is directed, there is a $w_3 \in K_T$ with $w_3 - w_2 \in K_T$ and $w_3 + w_2 \in K_T$. Then in K_T-order we have for all n that

$$-\tfrac{1}{n} w_3 \leq \tfrac{1}{n} w_2 = \tfrac{1}{n} w_2 - w_1 - (-w_1) = \tfrac{1}{n} w_2 \leq \tfrac{1}{n} w_3,$$

so $\frac{1}{n} w_2 - w_1 \to -w_1$ (ru). Hence $-w_1 \in \overline{K_T}$, so $\overline{K_T}$ is Archimedean. Thus $\overline{K_T} \supseteq K$.
(d)⇒(b): Let (S, K_S) be a directed Archimedean partially ordered vector space and let $\phi\colon X \otimes Y \to S$ be linear and such that $\phi(w) \in K_S$ for all $w \in K_T$. Let

$$C := \{w \in X \otimes Y\colon\ \phi(w) \in K_S\}.$$

Then $C \supseteq K_T$. We show that C is ru-closed. Let $(w_n)_n$ in C and $w \in X \otimes Y$ be such that $w_n \to w$ (ru). Take $a \in K_T$ and $\lambda_n \to 0$ such that in K_T-order we have for all n that $-\lambda_n a \leq w_n - w \leq \lambda_n a$. Then for all n we have

$$-\lambda_n \phi(a) \leq \phi(w_n - w) \leq \lambda_n \phi(a),$$

so $\phi(w) \geq \phi(w_n) - \lambda_n \phi(a) \geq -\lambda_n \phi(a)$. As S is Archimedean, we get $\phi(w) \in K_S$, so $w \in C$. Hence C is ru-closed, so that $K \subseteq C$.
(b)⇒(a): Let (S, K_S) be a directed Archimedean partially ordered vector space and let $\sigma\colon X \times Y \to S$ be a positive bilinear map. Then the induced linear map $\sigma^*\colon X \otimes Y \to S$ is such that $\sigma^*(w) \in K_S$ for all $w \in K_T$. Hence $\sigma^*(w) \in K_S$ for all $w \in K$, which means that $\sigma^*\colon (X \otimes Y, K) \to (S, K_S)$ is positive. □

Remark 5.5. Birnbaum [3] and Peressini and Sherbert [13] study the projective cone K_T and the *biprojective wedge*

$$K_b = \Big\{ \sum_{i=1}^{n} x_i \otimes y_i \colon\ x_i \in X,\ y_i \in Y \text{ such that } \sum_{i=1}^{n} \phi(x_i)\psi(y_i) \geq 0\ \forall \phi \in K'_X,\ \psi \in K'_Y,\ n \in \mathbb{N} \Big\},$$

where K'_X and K'_Y denote the sets of all positive linear functionals on (X, K_X) and (Y, K_Y), respectively. In some cases K_b equals the smallest Archimedean cone containing K_T. However, Birnbaum gives an example [3, p. 1051] where K_b is strictly larger than the smallest Archimedean cone containing K_T, so that K_b is not the Archimedean tensor cone.

It follows from Theorem 5.4 that the Archimedean tensor cone K in $X \otimes Y$ is contained in the cone K_F induced by the order of the Fremlin tensor product of the Riesz completions of X and Y. It is an interesting question whether K and K_F coincide. It may also be interesting to relate K and the pseudo-closure of K_T in the sense of [8].

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