An ergodic decomposition defined by regular jointly measurable Markov semigroups on Polish spaces

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== MI-2010-02 ==

February 17, 2010

Abstract

For a regular jointly measurable Markov semigroup on the space of finite Borel measures on a Polish space we give a Yosida-type decomposition of the state space, which yields a parametrisation of the ergodic probability measures associated to this semigroup in terms of subsets of the state space. In this way we extend results by Costa and Dufour (J. Appl. Prob. **43**, 767–781). As a consequence we obtain an integral decomposition of every invariant probability measure in terms of the ergodic probability measures. Our approach is completely centered around the reduction to and relationship with the case of a single regular Markov operator associated to the Markov semigroup, the resolvent operator, which enables us to fully exploit results in that situation (Worm and Hille, "Ergodic decompositions associated to regular Markov operators on Polish spaces", accepted by *Ergodic Theory and Dynamical Systems*).

1 Introduction

Regular Markov semigroups appear naturally in the context of continuous-time Markov processes as transition operators. If X_t is the state of the process at time t, i.e. a random variable that takes values in a measurable space S, and μ_0 is the law of X_0, then the law of X_t is given by $P(t)\mu_0$. Here each $P(t)$ is a regular Markov operator: an additive and positively homogeneous map on the convex cone of positive finite measures on S, given by a transition kernel, that leaves the set of probability measures invariant. The family of operators $(P(t))_{t\geq 0}$ forms a one-parameter semigroup. Accordingly, the behaviour of the dynamical system in the set of probability measures defined by a Markov semigroup is of special interest, in particular the question of existence and characterisation of invariant and ergodic probability measures for this semigroup.

More structure on S is needed than solely being a measurable space, in order to obtain a satisfactory theory on this topic. In the literature one may encounter a line of research that focuses on a pure topological setting (e.g. [19] and references found there) and one that takes a metric perspective, in which S has the generality of a Polish space, i.e. a topological space that is metrisable for a metric that makes it a complete separable metric space. We pursuit the latter line, driven by applications in population dynamics in biology, in which the state space S typically carries a natural metric. There is much interest lately in continuous-time Markov processes on non-locally compact Polish spaces, for instance those coming from stochastic differential equations in separable Hilbert spaces [9, 21, 24] or in separable Banach spaces [13]. Other recent research on Markov semigroups on Polish spaces is performed by Szarek and coworkers [18, 20, 26].

In [27] we obtained a Yosida-type ergodic decomposition for regular Markov operators on Polish spaces, extending results by Hernàndez-Lerma, Lasserre and Zaharopol [14, 15, 28, 29] on locally compact separable metric spaces. It yields a parametrisation of ergodic measures for such an operator in terms of classes of subsets of state space and an integral decomposition of invariant measures into ergodic measures using this parametrisation. Moreover, it associates with every ergodic measure a Borel measurable subset of the state space on which it is contained, and is such that the corresponding subsets of distinct ergodic measures are disjoint. Such a subset furthermore contains a Borel set that is invariant under the Markov operator, and such that the corresponding ergodic measure is actually concentrated on this smaller subset. We call this the full Yosida-type ergodic decomposition associated to the Markov operator.

In this paper we will show that analogous results hold for regular Markov semigroups. In view of the Markov operator setting, the main problem here is, of course, to deal with the *uncountable* family of Markov operators $(P(t))_{t\geq 0}$. We show that one can reduce this setting to the operator setting, by considering a single Markov operator that is associated to the Markov semigroup insted, the *resolvent operator* R. When the semigroup $(P(t))_{t\geq 0}$ consists of regular Markov operators and is jointly measurable, i.e. $(t,x)\mapsto P(t)\delta_x(E), \mathbb{R}_+\times S\to S$ is jointly measurable for every Borel set E, then R is a regular Markov operator (Proposition 3.2). It turns out that $(P(t))_{t\geq 0}$ and R have the same invariant and ergodic measures. Moreover, the Cesàro averages for the semigroup and the resolvent,

$$\frac{1}{t}\int_0^t P(s)\mu\,ds, \qquad \text{respectively} \qquad \frac{1}{n}\sum_{k=0}^{n-1} R^k\mu,$$

have the same convergence properties (made precise in Theorem 3.7 and its corollaries). These results imply that the Yosida-type ergodic decomposition associated to R, from [27], actually works for the semigroup $(P(t))_{t\geq 0}$ as well. We also obtain a full Yosida-type ergodic decomposition in Section 4.3, that generalises results by Costa and Dufour [6]. There, the authors consider Markov semigroups on locally compact separable metric spaces, and require more regularity. Our results do not require *continuity* assumptions on the Markov semigroup: the operators $P(t)$ need not be Markov-Feller, only regular, and the orbits $t \mapsto P(t)\mu$ need not be continuous in any sense: jointly measurable is sufficient. These are natural assumptions when considering Markov semigroups associated to Markov processes.

One of the results needed to establish the full Yosida-type ergodic decomposition for Markov semigroups is Theorem 4.5, which also yields the conclusion that the usual equivalent notions for ergodicity of an invariant measure μ of a Markov semigroup $(P(t))_{t\geq 0}$ (see Theorem 4.4) are equivalent to

$$\mu(E) = 0 \text{ or } 1 \text{ for every Borel set } E \text{ that is } (P(t))_{t\geq 0}\text{-invariant},$$

i.e. for every $t \geq 0$, $P(t)\delta_x(E) = 1$ for all $x \in E$. We could not retrieve this natural analogue of the definition of ergodicity for the operator case (see Section 2.3) from the Markov semigroup literature in the generality of a regular jointly measurable Markov semigroup on a Polish space.

In our approach we build on definitions and results from [16, 17] and an ergodic decomposition associated to Markov operators from [27] that we will recall in Section 2 for convenience. In Section 3 we introduce the resolvent operator associated to a Markov semigroup. The main theorem of this section (Theorem 3.7) shows that convergence properties of the Cesàro averages of a Markov semigroup and its resolvent coincide. In Section 4.1 we define ergodicity of invariant measures for Markov operators and Markov semigroups and give several equivalent characterisations. We prove analogues of the ergodic decomposition results from [27] in the setting of Markov semigroups in Section 4.2, and give a full Yosida-type ergodic decomposition in Section 4.3.

Some notational conventions. Unless otherwise mentioned, S will denote a Polish space, viewed as a measurable space with respect to its Borel σ-algebra. We write $\mathcal{M}(S)$ to denote the real vector space of all signed finite Borel measures on S, containing $\mathcal{M}^+(S)$, the cone of positive measures. $\mathcal{P}(S)$ consists of the probability measures in $\mathcal{M}^+(S)$. We denote the total variation norm on $\mathcal{M}(S)$ by $\|\cdot\|_{\mathrm{TV}}$ and write $\mathcal{M}(S)_{\mathrm{TV}}$ for the Banach space consisting of $\mathcal{M}(S)$ endowed with the total variation norm. We write $\mathrm{BM}(S)$ to denote the real vector space of all bounded measurable functions from S to $\mathbb{R}$ and $\mathbb{1}_E$ for the indicator function of $E \subset S$. For $f : S \to \mathbb{R}$ measurable and $\mu \in \mathcal{M}(S)$ we write $\langle \mu, f\rangle$ for $\int_S f\, d\mu$. We write $C_b(S)$ to denote the Banach space of bounded continuous functions from S to $\mathbb{R}$, endowed with the supremum norm $\|\cdot\|_\infty$.

2 Preliminaries

In this section we will recall some definitions and results from [16, 17, 27] which we need later on.

2.1 The space $\mathcal{S}_{\mathrm{BL}}$

Let (S, d) be a complete separable metric space.

BL(S) denotes the Banach space of bounded real-valued Lipschitz functions for the metric d, endowed with the norm $\|f\|_{\mathrm{BL}} := |f|_{\mathrm{Lip}} + \|f\|_\infty$, where $|f|_{\mathrm{Lip}}$ is the global Lipschitz constant of f. The Dirac functionals $\delta_x(f) := f(x)$ for $x \in S$ are in $\mathrm{BL}(S)^*$. We denote the usual dual norm on $\mathrm{BL}(S)^*$ by $\|\cdot\|_{\mathrm{BL}}^*$. BL($S$) is in fact isometrically isomorphic to the dual of a separable Banach space $\mathcal{S}_{\mathrm{BL}}$, which can be defined as the closure of the finite linear span of the δ_x, $x \in S$, in $\mathrm{BL}(S)^*$. Then, as shown in [7, Lemma 6], each $\mu \in \mathcal{M}(S)$ defines a unique element in $\mathrm{BL}(S)^*$, which we will also denote by μ, by sending $f \in \mathrm{BL}(S)$ to $\langle \mu, f \rangle = \int_S f\, d\mu$. A function $f \in \mathrm{BL}(S)$ defines a bounded linear functional on $\mathcal{S}_{\mathrm{BL}}$ by sending ϕ to $\phi(f)$. Using [16, Lemma 3.5] one can show that the map $x \mapsto \delta_x$ is a continuous embedding from S into $\mathcal{S}_{\mathrm{BL}}$.

By [16, Theorem 3.9 and Corollary 3.10], $\mathcal{M}^+(S)$ is a closed convex cone of $\mathcal{S}_{\mathrm{BL}}$, and $\mathcal{M}(S)$ is a $\|\cdot\|_{\mathrm{BL}}^*$-dense subspace of $\mathcal{S}_{\mathrm{BL}}$. The restriction of the weak-star topology on $C_b(S)^*$ to $\mathcal{M}^+(S)$, also called the topology of weak convergence on $\mathcal{M}^+(S)$, *equals* the restriction of the norm topology on $\mathcal{S}_{\mathrm{BL}}$ to $\mathcal{M}^+(S)$ by [7, Theorem 18]. In particular the following lemma holds:

Lemma 2.1. *Let $\mu_n, \mu \in \mathcal{M}^+(S)$. Then $\|\mu_n - \mu\|_{\mathrm{BL}}^* \to 0$ if and only if $\langle \mu_n, f \rangle \to \langle \mu, f \rangle$ for all $f \in C_b(S)$.*

The following results come from [17, Proposition 2.5, Proposition 2.6 and Corollary 2.7]. Let (Ω, Σ) be a measurable space.

Proposition 2.2. *Let $p : \Omega \to \mathcal{S}_{\mathrm{BL}}^+$. Then the following conditions are equivalent:*

(i) p is strongly measurable.

(ii) For each $f \in \mathrm{BM}(S)$, the map $\Omega \to \mathbb{R} : \omega \mapsto \langle p(\omega), f \rangle$ is measurable.

(iii) For each Borel measurable $E \subset S$, the map $\Omega \to \mathbb{R} : \omega \mapsto p(\omega)(E)$ is measurable.

Proposition 2.3. *Let $p : \Omega \to \mathcal{S}_{\mathrm{BL}}^+$ be Bochner integrable with respect to μ in $\mathcal{M}^+(\Omega)$, and define $\nu := \int_\Omega p(\omega) d\mu(\omega)$. Then*

$$\int_S f\, d\nu = \int_\Omega \langle p(\omega), f \rangle d\mu(\omega),$$

for any bounded measurable $f : S \to \mathbb{R}$. Thus for any Borel set $E \subset S$,

$$\left[\int_\Omega p(\omega) d\mu(\omega) \right] (E) = \int_\Omega p(\omega)(E) d\mu(\omega).$$

A collection of measures $M \subset \mathcal{P}(S)$ is *tight* if for every $\epsilon > 0$ there exists a compact $K \subset S$ such that $\mu(K) \geq 1 - \epsilon$ for every $\epsilon > 0$.

By Prokhorov's Theorem and Lemma 2.1 we have the following:

Theorem 2.4. *Let $M \subset \mathcal{P}(S)$. Then M is relatively compact in $\mathcal{S}_{\mathrm{BL}}$ if and only if M is tight.*

2.2 Markov operators, Markov semigroups and regularity

Let S be a Polish space. Let d be a complete metric on S metrising the given topology and $\mathcal{S}_{\mathrm{BL}}$ the Banach space associated with (S, d).

A *Markov operator* is a map $P : \mathcal{M}^+(S) \to \mathcal{M}^+(S)$, such that

(MO1) P is additive and $\mathbb{R}_+$-homogeneous,

(MO2) $\|P\mu\|_{\mathrm{TV}} = \|\mu\|_{\mathrm{TV}}$ for all $\mu \in \mathcal{M}^+(S)$.

Since $(\mathcal{M}(S), \|\cdot\|_{\mathrm{TV}})$ is a Banach lattice, condition *(MO1)* ensures that a Markov operator P extends to a positive *bounded* linear operator on $(\mathcal{M}(S), \|\cdot\|_{\mathrm{TV}})$ given by $P\mu := P(\mu^+) - P(\mu^-)$. The operator norm of this extension is

$$\|P\| = \sup\{\|P\mu\|_{\mathrm{TV}} : \mu \in \mathcal{M}^+(S),\ \|\mu\|_{\mathrm{TV}} \leq 1\} = 1 \tag{1}$$

according to *(MO2)*.

A measure $\mu \in \mathcal{M}(S)$ is *P-invariant* if $P\mu = \mu$. The following result follows from [27, Proposition 2.10, Corollary 2.11 and Proposition 2.12]:

Proposition 2.5. *Let P be a Markov operator. The following are equivalent:*

(i) There exists $U : \mathrm{BM}(S) \to \mathrm{BM}(S)$ such that

$$\langle P\mu, f\rangle = \langle \mu, Uf\rangle \text{ for all } \mu \in \mathcal{M}^+(S), f \in \mathrm{BM}(S).$$

(ii) (a) $x \mapsto P\delta_x, S \to \mathcal{S}_{\mathrm{BL}}$ is strongly measurable, and

(b) $P\mu = \int_S P\delta_x \, d\mu(x)$.

In either case,

$$P\mu(E) = \int_S P\delta_x(E)\, d\mu(x)$$

and

$$P \int_\Omega h(\omega) d\nu(\omega) = \int_\Omega Ph(\omega)\, d\nu(\omega)$$

for any finite measure space (Ω, Σ, ν) and $h : \Omega \to \mathcal{S}_{\mathrm{BL}}^+$ Bochner integrable with respect to ν.

Following [12, 22, 17, 27], we will call a Markov operator P *regular* if it satisfies the conditions of Proposition 2.5. The map $U : \mathrm{BM}(S) \to \mathrm{BM}(S)$ associated to a regular Markov operator P is unique and we call it the *dual* of P.

For a regular Markov operator P we define the *Cesàro averages*

$$P^{(n)} := \frac{1}{n}\sum_{k=0}^{n-1} P^k \qquad \text{and} \qquad U^{(n)} := \frac{1}{n}\sum_{k=0}^{n-1} U^k.$$

A *Markov semigroup* is a semigroup $(P(t))_{t\geq 0}$ of Markov operators. $(P(t))_{t\geq 0}$ is called *regular* if $P(t)$ is regular for all $t \geq 0$. In this case we obtain the *dual semigroup* $(U(t))_{t\geq 0}$ on $\mathrm{BM}(S)$, where $U(t)$ is the dual of $P(t)$ for every $t \in \mathbb{R}_+$. A measure μ is *$(P(t))_{t\geq 0}$-invariant* if $P(t)\mu = \mu$ for every $t \in \mathbb{R}_+$.

We say a Markov semigroup $(P(t))_{t\geq 0}$ is a *jointly measurable Markov semigroup* if $(t,x) \mapsto P(t)\delta_x(E)$ is jointly measurable from $\mathbb{R}_+ \times S$ to $\mathbb{R}$ for every Borel set E in S. This holds, by Proposition 2.2, if and only if $(t,x) \mapsto P(t)\delta_x$ is strongly measurable from $\mathbb{R}_+ \times S$ to $\mathcal{S}_{\mathrm{BL}}$.

Proposition 2.6. *Let $(P(t))_{t\geq 0}$ be a regular jointly measurable Markov semigroup. Then for every $\mu \in \mathcal{M}^+(S)$, $t \mapsto P(t)\mu$ is strongly measurable from $\mathbb{R}_+$ to $\mathcal{S}_{\mathrm{BL}}$. For every $E \subset S$ Borel the map $t \mapsto P(t)\mu(E)$ is measurable from $\mathbb{R}_+$ to $\mathbb{R}$.*

Proof. Let $\mu \in \mathcal{M}^+(S)$ and $E \subset S$ Borel. By Proposition 2.5

$$P(t)\mu(E) = \int_S P(t)\delta_x(E)\, d\mu(x),$$

so it follows from the joint measurability and Tonelli's Theorem that $t \mapsto P(t)\mu(E)$ is measurable from $\mathbb{R}_+$ to $\mathbb{R}$. Hence by Proposition 2.2 $t \mapsto P(t)\mu$ is strongly measurable from $\mathbb{R}_+$ to $\mathcal{S}_{\mathrm{BL}}$. □

We shall write $\mathcal{B}(S)$ to denote the σ-algebra of all Borel sets of S. A map $p : \mathbb{R}_+ \times S \times \mathcal{B}(S) \to \mathbb{R}$ is a *Markov transition function* if:

(TF1) $p(t,x,\cdot)$ is a probability measure for every $t \in \mathbb{R}_+$, $x \in S$ and $p(0,x,\cdot) = \delta_x$ for every $x \in S$.

(TF2) $p(\cdot,\cdot,E)$ is $\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(S)$-measurable for every $E \in \mathcal{B}(S)$.

(TF3) p satisfies the Chapman-Kolmogorov equation

$$p(t+s,x,E) = \int_S p(s,y,E)p(t,x,dy)$$

for every $s,t \in \mathbb{R}_+, x \in S, E \in \mathcal{B}(S)$.

Every Markov transition function p generates a regular jointly measurable Markov semigroup: $P(t)\mu(E) := \int_S p(t,x,E)\, d\mu(x)$. The semigroup property follows from *(TF3)*. Conversely, every regular jointly measurable Markov semigroup $(P(t))_{t\geq 0}$ is generated by a Markov transition function $p(t,x,E) := P(t)\delta_x(E)$. A proof for this can be found in e.g. [19, Proposition 2.4], where a Markov transition function is defined slightly more general. Time-homogeneous Markov processes can be defined using Markov transition functions ([10, Chapter 4, Section 1]), so regular jointly measurable Markov semigroups are a natural object of interest when studying time-homogeneous Markov processes.

We call the Markov semigroup $(P(t))_{t\geq 0}$ *strongly stochastically continuous*, when $t \mapsto \langle P(t)\mu, f\rangle$ is continuous for all $\mu \in \mathcal{M}^+(S)$ and $f \in C_b(S)$, and *strongly stochastically continuous at zero* when $t \mapsto \langle P(t)\mu, f\rangle$ is continuous at zero for all $\mu \in \mathcal{M}^+(S)$ and $f \in C_b(S)$.

Lemma 2.7. *Let $(P(t))_{t\geq 0}$ be a Markov semigroup. Then the following are equivalent:*

(i) $(P(t))_{t\geq 0}$ *is strongly stochastically continuous at zero,*

(ii) $t \mapsto \langle P(t)\mu, f\rangle$ *is continuous at zero for all* $\mu \in \mathcal{M}(S)$ *and* $f \in C_b(S)$.

(iii) $t \mapsto P(t)\mu, \mathbb{R}_+ \to \mathcal{S}_{\mathrm{BL}}$ *is continuous at zero for every* $\mu \in \mathcal{M}(S)$.

The proof is not difficult, and follows from Lemma 2.1. See [17, Lemma 3.5], where an analogous statement is proven (but then continuity at every $t \in \mathbb{R}_+$).

Proposition 2.8. *Let* $(P(t))_{t\geq 0}$ *be a regular Markov semigroup that is strongly stochastically continuous at zero. Then* $(P(t))_{t\geq 0}$ *is a jointly measurable Markov semigroup.*

Proof. By Lemma 2.7 and the semigroup property, $t \mapsto P(t)\delta_x$ is right continuous from $\mathbb{R}_+$ to $\mathcal{S}_{\mathrm{BL}}$ for every $x \in S$. For fixed t, $x \mapsto P(t)\delta_x$ is strongly measurable from S to $\mathcal{S}_{\mathrm{BL}}$ by Proposition 2.5, thus $x \mapsto P(t)\delta_x$ is measurable for the Borel σ-algebra of $\mathcal{S}_{\mathrm{BL}}$. Since $\mathcal{S}_{\mathrm{BL}}$ is a separable Banach space, $(t,x) \mapsto P(t)\delta_x$ is jointly measurable by [5, Lemma 6.4.6]. Thus $(P(t))_{t\geq 0}$ is jointly measurable. □

A class of examples of regular Markov semigroups is induces by semigroups of measurable maps. A *semigroup of measurable maps* on S is a family of maps $(\Phi_t)_{t\geq 0}$, such that $\Phi_t : S \to S$ is measurable, $\Phi_t \circ \Phi_s = \Phi_{t+s}$ and $\Phi_0 = \mathrm{Id}_S$ for all $s, t \in \mathbb{R}_+$. $(\Phi_t)_{t\geq 0}$ is called *strongly continuous* if the map $\mathbb{R}_+ \to S : t \mapsto \Phi_t(x)$ is continuous for all $x \in S$. $(\Phi_t)_{t\geq 0}$ is called *jointly measurable* if $(t,x) \mapsto \Phi_t(x)$ is measurable from $\mathbb{R}_+ \times S$ to S.

Proposition 2.9. *Let* $(\Phi_t)_{t\geq 0}$ *be a semigroup of measurable maps on* S. *Then*

(i) $P(t)\mu := \mu \circ \Phi_t^{-1}$ *defines a regular Markov semigroup* $(P(t))_{t\geq 0}$.

(ii) $(P(t))_{t\geq 0}$ *is strongly stochastically continuous (at zero) if and only if* $(\Phi_t)_{t\geq 0}$ *is strongly continuous (at zero).*

(iii) If $(\Phi_t)_{t\geq 0}$ *is jointly measurable, then* $(P(t))_{t\geq 0}$ *is a jointly measurable Markov semigroup.*

Proof. For *(i)* and *(ii)* see [17, Proposition 3.6]. Assume that $(\Phi_t)_{t\geq 0}$ is jointly measurable. Let $f \in \mathrm{BL}(S) \cong \mathcal{S}_{\mathrm{BL}}^*$. Then $(t,x) \mapsto f(\Phi_t(x))$ is measurable, by measurability of f. Thus $(t,x) \mapsto \langle f, \delta_{\Phi_t(x)}\rangle$ is measurable for every $f \in \mathcal{S}_{\mathrm{BL}}^*$, thus $(t,x) \mapsto \delta_{\Phi_t(x)}$ is weakly measurable from $\mathbb{R}_+ \times S$ to $\mathcal{S}_{\mathrm{BL}}$, hence strongly measurable by separability of $\mathcal{S}_{\mathrm{BL}}$. Therefore, by Proposition 2.2, we can conclude that $(P(t))_{t\geq 0}$ is measurable. □

For the remainder we assume that $\mathbf{P} = (P(t))_{t\geq 0}$ is a regular jointly measurable Markov semigroup with dual $\mathbf{U} = (U(t))_{t\geq 0}$.

For $\mu \in \mathcal{M}^+(S)$ and $t > 0$ we define the Cesàro averages

$$\mathbf{P}^{(t)}\mu := \frac{1}{t}\int_0^t P(s)\mu\, ds, \qquad P^{(0)}\mu := \mu.$$

This integral exists as Bochner integral in $\mathcal{S}_{\mathrm{BL}}$ by Proposition 2.6 and since $\|P(s)\mu\|_{\mathrm{BL}}^* \leq \|\mu\|_{\mathrm{BL}}^*$ for every $s \in \mathbb{R}_+$.

Then $\mathbf{P}^{(t)}\mu \in \mathcal{M}^+(S)$ with

$$\|\mathbf{P}^{(t)}\mu\|_{\mathrm{TV}} = \mathbf{P}^{(t)}\mu(S) = \frac{1}{t}\int_0^t P(s)\mu(S)\, ds = \|\mu\|_{\mathrm{TV}}$$

by *(MO2)* and Proposition 2.3. Thus $\mathbf{P}^{(t)}$ is a Markov operator.

Lemma 2.10. *For every $t > 0$, $\mathbf{P}^{(t)}$ is a regular Markov operator with dual*

$$\mathbf{U}^{(t)} f(x) = \frac{1}{t}\int_0^t U(s)f(x)\,ds.$$

Proof. Let $f \in \mathrm{BM}(S), \mu \in \mathcal{M}^+(S)$ and $t > 0$. Then it follows from Proposition 2.6 and Proposition 2.3 that

$$\begin{aligned} \langle \mathbf{P}^{(t)}\mu, f\rangle &= \frac{1}{t}\int_0^t \langle P(s)\mu, f\rangle\,ds = \frac{1}{t}\int_0^t \langle \mu, U(s)f\rangle\,ds \\ &= \frac{1}{t}\int_0^t \int_S U(s)f(x)\,d\mu(x)\,ds. \end{aligned}$$

Since $(P(t))_{t\geq 0}$ is jointly measurable, Proposition 2.2 implies that $(s,x) \mapsto U(s)f(x) = \langle P(s)\delta_x, f\rangle$ is measurable from $\mathbb{R}_+ \times S$ to $\mathbb{R}$. This implies by Fubini's Theorem that $\mathbf{U}^{(t)} f : x \mapsto \frac{1}{t}\int_0^t U(s)f(x)\,ds$ is measurable. This map is bounded by $\|f\|_\infty$. So we can apply Fubini's Theorem:

$$\begin{aligned} \frac{1}{t}\int_0^t \int_S U(s)f(x)\,d\mu(x)\,ds &= \frac{1}{t}\int_S \int_0^t U(s)f(x)\,ds\,d\mu(x) \\ &= \langle \mu, \mathbf{U}^{(t)} f\rangle \end{aligned}$$

This proves the statement. $\square$

2.3 Ergodic decomposition of regular Markov operators

In this section we summarise results from [27] on an ergodic decomposition associated to regular Markov operators in a Polish space S. Let d be a complete metric on S metrising the given topology and $\mathcal{S}_{\mathrm{BL}}$ the Banach space associated with (S,d). Let P be a regular Markov operator. Define

$$\Gamma_t^P := \{x \in S : \{P^{(n)}\delta_x : n \in \mathbb{N}\} \text{ is tight}\},$$

and

$$\Gamma_{cp}^P := \{x \in S : (P^{(n)})_n \text{ converges in } \mathcal{S}_{\mathrm{BL}}\},$$

$$\epsilon_x := \lim_{n\to\infty} P^{(n)}\delta_x \in \mathcal{S}_{\mathrm{BL}} \text{ for } x \in \Gamma_{cp}^P.$$

Each $\epsilon_x, x \in \Gamma_{cp}^P$ is a probability measure. In general it need not be invariant. Hence we define

$$\Gamma_{cpi}^P := \{x \in \Gamma_{cp}^P : \epsilon_x \text{ is invariant}\}.$$

Following [14, 27, 28, 29], we define an *ergodic measure* to be a P-invariant probability measure μ such that $\mu(E) = 0$ or $\mu(E) = 1$ for every *P-invariant* set E, i.e. a Borel set $E \subset S$ such that $P\delta_x(E) = 1$ for every $x \in E$.

$$\Gamma_{cpie}^P := \{x \in \Gamma_{cpi}^P : \epsilon_x \text{ is ergodic}\}.$$

Clearly $\Gamma_t^P \supset \Gamma_{cp}^P \supset \Gamma_{cpi}^P \supset \Gamma_{cpie}^P$. If P is Markov-Feller, $\Gamma_{cp}^P = \Gamma_{cpi}^P$. In [27] the following is shown:

Theorem 2.11. *$\Gamma_t^P, \Gamma_{cp}^P, \Gamma_{cpi}^P$ and Γ_{cpie}^P are Borel subsets of S and*

$$\mu(\Gamma_t^P) = \mu(\Gamma_{cp}^P) = \mu(\Gamma_{cpi}^P) = \mu(\Gamma_{cpie}^P) = 1$$

for every P-invariant probability measure μ.

The map $x \mapsto \epsilon_x$, from Γ_{cpie}^P to the ergodic measures is not injective generally. So we introduce an equivalence relation $\sim$ on Γ_{cpie}^P as follows: $x \sim y$ whenever $\epsilon_x = \epsilon_y$. We write $[x]$ to denote the equivalence class of $x \in \Gamma_{cpie}^P$. The following result comes from [27, Theorem 4.6]. It implies that we can decompose Γ_{cpie}^P into disjoint Borel measurable subsets, such that each ergodic measure has full measure on exactly one of these subsets.

Theorem 2.12. *(i) For every $x \in \Gamma_{cpie}^P$ the set $[x]$ is Borel measurable and $\epsilon_x([x]) = 1$.*

(ii) Any ergodic measure μ is of the form $\mu = \epsilon_x$ for some $x \in \Gamma_{cpie}^P$.

Using the characterisation we obtain an integral decomposition of invariant probability measures in terms of ergodic measures ([27, Theorem 4.10]).

Theorem 2.13. *Let μ be an invariant probability measure. Then the map*

$$x \mapsto \begin{cases} \epsilon_x & \text{if } x \in \Gamma_{cpie}^P \\ 0 & \text{if } x \notin \Gamma_{cpie}^P. \end{cases}$$

is strongly measurable from S to $\mathcal{S}_{\mathrm{BL}}$ and

$$\mu = \int_{\Gamma_{cpie}^P} \epsilon_x \, d\mu.$$

3 Resolvent operator of a regular jointly measurable Markov semigroup

Let $\mathbf{P} = (P(t))_{t\geq 0}$ be a regular jointly measurable Markov semigroup with dual semigroup $\mathbf{U} = (U(t))_{t\geq 0}$.

Lemma 3.1. *Let $\mu \in \mathcal{M}^+(S)$ and $f \in L^1_+(\mathbb{R}_+)$. Then $\int_{\mathbb{R}_+} f(s)P(s)\mu \, ds$ is well defined as Bochner integral in $\mathcal{S}_{\mathrm{BL}}$ and takes value in $\mathcal{S}_{\mathrm{BL}}^+ = \mathcal{M}^+(S)$. It satisfies:*

(i)

$$\| \int_{\mathbb{R}_+} f(s)P(s)\mu \, ds\|_{\mathrm{TV}} = \| \int_{\mathbb{R}_+} f(s)P(s)\mu \, ds\|_{\mathrm{BL}}^* = \|f\|_1 \|\mu\|_{\mathrm{BL}}^* = \|f\|_1 \|\mu\|_{\mathrm{TV}}.$$

(ii) For every $t > 0$,

$$P(t) \int_{\mathbb{R}_+} f(s)P(s)\mu \, ds = \int_{\mathbb{R}_+} f(s)P(s+t)\mu \, ds.$$

Proof. First note that $s \mapsto f(s)P(s)\mu$ is Bochner integrable from $\mathbb{R}_+$ to $\mathcal{S}_{\mathrm{BL}}$ with respect to the Lebesgue measure, and $f(s)P(s)\mu \in \mathcal{M}^+(S)$ for every $s \in \mathbb{R}_+$, thus $\nu := \int_{\mathbb{R}_+} f(s)P(s)\mu\, ds$ is a well defined element of $\mathcal{M}^+(S)$. By Proposition 2.3 and *(MO2)*,

$$\begin{aligned}\|\nu\|_{\mathrm{TV}} &= \|\nu\|^*_{\mathrm{BL}} = \nu(S) = \int_{\mathbb{R}_+} f(s)P(s)\mu(S)\, ds \\ &= \mu(S) \int_{\mathbb{R}_+} f(s)\, ds = \|\mu\|^*_{\mathrm{BL}}\|f\|_1 = \|\mu\|_{\mathrm{TV}}\|f\|_1.\end{aligned}$$

Let $h \in \mathrm{BM}(S)$. Then by applying Proposition 2.3 several times we obtain

$$\begin{aligned}&\langle P(t) \int_{\mathbb{R}_+} f(s)P(s)\mu\, ds, h\rangle = \langle \int_{\mathbb{R}_+} f(s)P(s)\mu\, ds, U(t)h\rangle \\ &= \int_{\mathbb{R}_+} f(s)\langle P(s)\mu, U(t)h\rangle\, ds = \int_{\mathbb{R}_+} \langle f(s)P(t+s)\mu, h\rangle\, ds \\ &= \langle \int_{\mathbb{R}_+} f(s)P(t+s)\mu\, ds, h\rangle,\end{aligned}$$

which proves *(ii)*. □

We define the *resolvent family* (associated with the Markov semigroup $(P(t))_{t\geq 0}$) to be the collection of operators $\{R_\lambda : \lambda > 0\}$:

$$R_\lambda \mu := \int_{\mathbb{R}_+} e^{-\lambda t} P(t)\mu\, dt \text{ for every } \mu \in \mathcal{M}^+(S).$$

We define the *resolvent operator* (associated with $(P(t))_{t\geq 0}$) to be $R := R_1$.

Proposition 3.2. *For every $\lambda > 0$, λR_λ is a regular Markov operator with dual operator λU_λ given by*

$$\lambda U_\lambda f(x) := \lambda \int_{\mathbb{R}_+} e^{-\lambda t} U(t) f(x)\, dt.$$

Proof. First of all, $g : t \mapsto e^{-\lambda t}$ is in $L^1_+(\mathbb{R}_+)$ and $\|g\|_1 = 1/\lambda$. Thus by Lemma 3.1 R_λ is a positively homogeneous and additive map from $\mathcal{M}^+(S)$ to $\mathcal{M}^+(S)$ and $\|\lambda R_\lambda \mu\|^*_{\mathrm{BL}} = \|\mu\|^*_{\mathrm{BL}}$ for every $\mu \in \mathcal{M}^+(S)$. Thus λR_λ is a Markov operator. Let $f \in \mathrm{BM}(S)$, then by applying Proposition 2.3 we get

$$\begin{aligned}\langle \lambda R_\lambda \mu, f\rangle &= \lambda \int_{\mathbb{R}_+} e^{-\lambda t} \langle P(t)\mu, f\rangle\, dt \\ &= \lambda \int_{\mathbb{R}_+} e^{-\lambda t} \langle \mu, U(t)f\rangle\, dt \\ &= \lambda \int_{\mathbb{R}_+} e^{-\lambda t} \int_S U(t)f(x)\, d\mu(x)\, dt.\end{aligned}$$

Since $(P(t))_{t\geq 0}$ is jointly measurable, Proposition 2.2 implies that $(t, x) \mapsto U(t)f(x) = \langle P(t)\delta_x, f\rangle$ is measurable from $\mathbb{R}_+ \times S$ to $\mathbb{R}$, so we can apply Fubini's Theorem and obtain

$$\langle \lambda R_\lambda \mu, f \rangle = \int_S \int_{\mathbb{R}_+} \lambda e^{-\lambda t} U(t) f(x)\, dt\, d\mu(x).$$

□

The *resolvent identity* holds for $(R_\lambda)_{\lambda>0}$:

Lemma 3.3. *Let $\lambda, \gamma > 0$ and $\mu \in \mathcal{M}^+(S)$. Then*

$$R_\lambda \mu - R_\gamma \mu = (\gamma - \lambda) R_\gamma R_\lambda \mu.$$

Proof. Note that for every $t > 0$

$$e^{-\lambda t} - e^{-\gamma t} = (\gamma - \lambda) e^{-\lambda t} \int_0^t e^{-(\gamma-\lambda)s}\, ds.$$

Now by using this equality and Fubini's Theorem, we obtain

$$\begin{aligned}
R_\lambda \mu - R_\gamma \mu &= \int_{\mathbb{R}_+} (e^{-\lambda t} - e^{-\gamma t}) P(t)\mu\, dt \\
&= \int_{\mathbb{R}_+} P(t)\mu(\gamma - \lambda) e^{-\lambda t} \int_0^t e^{-(\gamma-\lambda)s}\, ds\, dt \\
&= (\gamma - \lambda) \int_{\mathbb{R}_+} \int_0^t e^{-\lambda t} e^{-(\gamma-\lambda)s} P(t)\mu\, ds\, dt \\
&= (\gamma - \lambda) \int_{\mathbb{R}_+} e^{-\gamma s} \int_s^\infty e^{-\lambda(t-s)} P(t)\, \mu\, dt\, ds \\
&= (\gamma - \lambda) \int_{\mathbb{R}_+} e^{-\gamma s} \int_0^\infty e^{-\lambda u} P(u+s)\, \mu\, du\, ds.
\end{aligned}$$

By Lemma 3.1 the last expression equals

$$(\gamma - \lambda) \int_{\mathbb{R}_+} e^{-\gamma s} P(s) \int_{\mathbb{R}_+} e^{-\lambda u} P(u)\, \mu\, du\, ds = (\gamma - \lambda) R_\gamma R_\lambda \mu.$$

□

Lemma 3.4. *Let X be a set and $(\Phi_t)_{t\geq 0}$ a semigroup of maps $\Phi_t : X \to X$. Let $x^* \in X$ be such that $\Phi_t(x^*) = x^*$ for Lebesgue almost all $t \in \mathbb{R}_+$. Then $\Phi_t(x^*) = x^*$ for all $t \in \mathbb{R}_+$.*

Proof. There exists a Lebesgue null set $N \subset \mathbb{R}_+$ such that $\Phi_t(x^*) = x^*$ every $t \in \mathbb{R}_+ \backslash N$. Let $t \in N$. Then there must be an $s \in \mathbb{R}_+ \backslash N$ such that $t + s \in \mathbb{R}_+ \backslash N$, otherwise $t + \mathbb{R}_+ \backslash N$ is contained in N and by translation invariance of Lebesgue measure

$$m(N) \geq m(t + \mathbb{R}_+ \backslash N) = m(\mathbb{R}_+ \backslash N) = \infty,$$

a contradiction. □

Proposition 3.5. *Let $\mu \in \mathcal{M}^+(S)$. Then the following are equivalent:*

(i) μ *is* $(P(t))_{t\geq 0}$*-invariant.*

(ii) $\lambda R_\lambda \mu = \mu$ *for some* $\lambda > 0$*.*

(iii) $\lambda R_\lambda \mu = \mu$ *for every* $\lambda > 0$*.*

Proof. *(i)⇒(ii)*: This is immediate.
(ii)⇒(iii): Suppose that $\lambda > 0$ is such that $\lambda R_\lambda \mu = \mu$. Let $\gamma > 0$, then by the resolvent identity, Lemma 3.3,

$$\frac{1}{\lambda}\mu - R_\gamma \mu = \frac{\gamma - \lambda}{\lambda} R_\gamma \mu.$$

So $\gamma R_\gamma \mu = \mu$.
(iii)⇒(i): By assumption we have for every $\lambda > 0$:

$$\int_{\mathbb{R}_+} e^{-\lambda t} P(t)\mu \, dt = R_\lambda \mu = \frac{1}{\lambda}\mu = \int_{\mathbb{R}_+} e^{-\lambda t}\mu \, dt.$$

Hence by [3, Theorem 1.7.3] $P(t)\mu = \mu$ for Lebesgue almost every $t \in \mathbb{R}_+$. By Lemma 3.4 $P(t)\mu = \mu$ for every $t \in \mathbb{R}_+$. □

Lemma 3.6. *For every* $k \in \mathbb{N}, \lambda > 0$ *and* $\mu \in \mathcal{M}^+(S)$

$$R_\lambda^k \mu = \int_0^\infty \frac{t^{k-1}}{(k-1)!} e^{-\lambda t} P(t)\mu \, dt,$$

a Bochner integral in $\mathcal{S}_{\mathrm{BL}}$*.*

Proof. Let $\lambda > 0$ and $\mu \in \mathcal{M}^+(S)$. We prove the statement by induction. Clearly it holds for $k = 1$. Suppose it holds for some $k \in \mathbb{N}$. By Lemma 3.1 we can bring $P(t)$ in and outside of integrals. Using this result and Fubini's Theorem (because we are considering jointly measurable Markov semigroups) we obtain

$$\begin{aligned}
R_\lambda^{k+1}\mu &= \int_0^\infty e^{-\lambda t} P(t) \int_0^\infty \frac{s^{k-1} e^{-\lambda s}}{(k-1)!} P(s)\mu \, ds \, dt \\
&= \int_0^\infty \frac{s^{k-1}}{(k-1)!} \int_0^\infty e^{-\lambda(t+s)} P(t+s)\mu \, dt \, ds \\
&= \int_0^\infty \frac{s^{k-1}}{(k-1)!} \int_s^\infty e^{-\lambda\sigma} P(\sigma)\mu \, d\sigma \, ds \\
&= \int_0^\infty \left(\int_0^\sigma \frac{s^{k-1}}{(k-1)!} \, ds \right) e^{-\lambda\sigma} P(\sigma)\mu \, d\sigma \\
&= \int_0^\infty \frac{\sigma^k}{k!} e^{-\lambda\sigma} P(\sigma)\mu \, d\sigma.
\end{aligned}$$

So the statement holds for $k + 1$. □

Our main result in this section is the following:

Theorem 3.7. *Let $\mathbf{P} = (P(t))_{t\geq 0}$ be a regular jointly measurable Markov semigroup. For every $\mu \in \mathcal{M}(S)$*

$$\|R^{(n)} - \mathbf{P}^{(n)}\|_{\mathcal{L}(\mathcal{M}(S)_{\mathrm{TV}})} = \mathcal{O}(1/\sqrt{n}) \text{ as } n \to \infty.$$

In particular, for every $\mu \in \mathcal{M}(S)$

$$\lim_{n\to\infty} \|R^{(n)}\mu - \mathbf{P}^{(n)}\mu\|^*_{\mathrm{BL}} = \lim_{n\to\infty} \|R^{(n)}\mu - \mathbf{P}^{(n)}\mu\|_{\mathrm{TV}} = 0.$$

Proof. Note that

$$\|R^{(n)} - \frac{1}{n}\sum_{k=1}^{n} R^k\|_{\mathcal{L}(\mathcal{M}(S)_{\mathrm{TV}})} = \frac{1}{n}\|R^n - I\|_{\mathcal{L}(\mathcal{M}(S)_{\mathrm{TV}})} \leq \frac{2}{n}.$$

So the theorem is proven once we have shown that

$$\|\mathbf{P}^{(n)} - \frac{1}{n}\sum_{k=1}^{n} R^k\|_{\mathcal{L}(\mathcal{M}(S)_{\mathrm{TV}})} = \mathcal{O}(1/\sqrt{n}) \text{ as } n \to \infty.$$

Note that in the following we want to estimate $\|\cdot\|_{\mathrm{TV}}$-norms of Bochner integrals in $\mathcal{S}_{\mathrm{BL}}$ that take values in $\mathcal{M}(S)$. Let $\mu \in \mathcal{P}(S)$. By Lemma 3.6 we obtain

$$\begin{aligned}
\|\mathbf{P}^{(n)}\mu - \frac{1}{n}\sum_{k=1}^{n} R^k\mu\|_{\mathrm{TV}} &= \frac{1}{n}\left\|\int_0^n P(t)\mu\,dt - \sum_{k=0}^{n-1}\int_0^\infty e^{-t}\frac{t^k}{k!}P(t)\mu\,dt\right\|_{\mathrm{TV}} \\
&\leq \frac{1}{n}\sum_{k=0}^{n-1}\left\|\int_n^\infty e^{-t}\frac{t^k}{k!}P(t)\mu\,dt\right\|_{\mathrm{TV}} \\
&\quad + \left\|\frac{1}{n}\int_0^n\left(1 - e^{-t}\sum_{k=0}^{n-1}\frac{t^k}{k!}\right)P(t)\mu\,dt\right\|_{\mathrm{TV}}.
\end{aligned}$$

Since $e^{-t}\sum_{k=0}^{n-1}\frac{t^k}{k!} < e^{-t}e^t = 1$ all Bochner integrals in the last expression take value in $\mathcal{M}^+(S) = \mathcal{S}^+_{\mathrm{BL}}$, so by Lemma 3.1 and *(MO2)* we get

$$\|\mathbf{P}^{(n)}\mu - \frac{1}{n}\sum_{k=1}^{n} R^k\mu\|_{\mathrm{TV}} \leq \left(\frac{1}{n}\sum_{k=0}^{n-1}\int_n^\infty e^{-t}\frac{t^k}{k!}\,dt + \frac{1}{n}\int_0^n 1 - e^{-t}\sum_{k=0}^{n-1}\frac{t^k}{k!}\,dt\right).$$

Now,

$$\begin{aligned}
\frac{1}{n}\int_0^n 1 - e^{-t}\sum_{k=0}^{n-1}\frac{t^k}{k!}\,dt &= \frac{1}{n}\sum_{k=0}^{n-1}[1 - \int_0^n e^{-t}\frac{t^k}{k!}\,dt] \\
&= \frac{1}{n}\sum_{k=0}^{n-1}\int_n^\infty e^{-t}\frac{t^k}{k!}\,dt
\end{aligned}$$

since $\int_0^\infty e^{-t}\frac{t^k}{k!}\,dt = 1$ for every $k \in \mathbb{N}$, thus

$$\|\mathbf{P}^{(n)}\mu - \sum_{k=1}^{n} R^k\mu\|_{\mathrm{TV}} \leq \frac{2}{n}\sum_{k=0}^{n-1}\int_n^\infty e^{-t}\frac{t^k}{k!}\,dt$$

for every $n \in \mathbb{N}$.

By elementary calculation we obtain for every $k, n \in \mathbb{N}$,

$$\int_n^\infty e^{-t} \frac{t^k}{k!}\, dt = e^{-n} \sum_{i=0}^{k} \frac{n^i}{i!},$$

thus

$$\begin{aligned} \sum_{k=0}^{n-1} \int_n^\infty e^{-t} \frac{t^k}{k!}\, dt &= e^{-n} \sum_{k=0}^{n-1} \sum_{i=0}^{k} \frac{n^i}{i!} = e^{-n} \sum_{i=0}^{n-1} \frac{n^i(n-i)}{i!} \\ &= e^{-n} \left(\sum_{i=1}^{n-1} \left(\frac{n^{i+1}}{i!} - \frac{n^i}{(i-1)!} \right) + n \right) = e^{-n} \frac{n^n}{(n-1)!}. \end{aligned}$$

So

$$\|\mathbf{P}^{(n)}\mu - \frac{1}{n} \sum_{k=1}^{n} R^k \mu\|_{\mathrm{TV}} \leq \frac{2n^n}{e^n n!} = \frac{\sqrt{2\pi n} n^n}{e^n n!} \cdot \frac{2}{\sqrt{2\pi n}}.$$

This holds for every $\mu \in \mathcal{P}(S)$. Since $\mathcal{M}(S)_{\mathrm{TV}}$ is an *AL-space* (see e.g. [2, page 194] for a definition), the norm of a bounded linear operator T on $\mathcal{M}(S)_{\mathrm{TV}}$ equals $\sup\{\|T\mu\|_{\mathrm{TV}} : \mu \in \mathbf{P}(S)\}$. So

$$\|\mathbf{P}^{(n)} - \frac{1}{n} \sum_{k=1}^{n} R^k\|_{\mathcal{L}(\mathcal{M}(S)_{\mathrm{TV}})} \leq \frac{\sqrt{2\pi n} n^n}{e^n n!} \cdot \frac{2}{\sqrt{2\pi n}}.$$

By Stirling's Formula, $\lim_{n\to\infty} \frac{\sqrt{2\pi n} n^n}{e^n n!} = 1$. Thus

$$\|\mathbf{P}^{(n)} - \frac{1}{n} \sum_{k=1}^{n} R^k\|_{\mathcal{L}(\mathcal{M}(S)_{\mathrm{TV}})} = \mathcal{O}(1/\sqrt{n}) \text{ as } n \to \infty.$$

The last statement follows from the continuity of the embedding $\mathcal{M}(S)_{\mathrm{TV}} \hookrightarrow \mathcal{S}_{\mathrm{BL}}$. □

We now state some further convergence properties of $(P(t))_{t\geq 0}$ that will be useful later on:

Lemma 3.8. *Let $s, t \in \mathbb{R}_{>0}$. Then*

$$\|\mathbf{P}^{(s)} - \mathbf{P}^{(t)}\|_{\mathcal{L}(\mathcal{M}(S)_{\mathrm{TV}})} \leq \frac{2|t-s|}{\max(s,t)}.$$

Proof. Let $\mu \in \mathcal{P}(S)$. Suppose $0 < s < t$. Then by Lemma 3.1

$$\begin{aligned} \|\mathbf{P}^{(s)}\mu - \mathbf{P}^{(t)}\mu\|_{\mathrm{TV}} &\leq \left\| (1/s - 1/t) \int_0^s P(r)\mu\, dr \right\|_{\mathrm{TV}} + \left\| \frac{1}{t} \int_s^t P(r)\mu\, dr \right\|_{\mathrm{TV}} \\ &\leq \frac{2|t-s|}{t} \end{aligned}$$

Since $\mathcal{M}(S)_{\mathrm{TV}}$ is an AL-space,

$$\|\mathbf{P}^{(s)} - \mathbf{P}^{(t)}\|_{\mathcal{L}(\mathcal{M}(S)_{\mathrm{TV}})} \leq \frac{2|t-s|}{\max(s,t)}.$$

□

This implies the following results:

Corollary 3.9. *The map* $t \mapsto \mathbf{P}^{(t)} : (0,\infty) \to \mathcal{L}(\mathcal{M}(S)_{\mathrm{TV}})$ *is continuous. In particular,* $t \mapsto \mathbf{P}^{(t)}\mu$ *is continuous as map from* $(0,\infty)$ *into* $\mathcal{M}(S)_{\mathrm{TV}}$ *and into* $\mathcal{S}_{\mathrm{BL}}$*, for every* $\mu \in \mathcal{M}(S)$*.*

Corollary 3.10. *Let* $s_n, t_n \in \mathbb{R}_+$*, such that* $M := \sup_n |t_n - s_n| < \infty$ *and* $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = \infty$*. Then*

$$\lim_{n\to\infty} \|\mathbf{P}^{(s_n)} - \mathbf{P}^{(t_n)}\|_{\mathcal{L}(\mathcal{M}(S)_{\mathrm{TV}})} = 0.$$

Thus also

$$\lim_{n\to\infty} \|\mathbf{P}^{(s_n)}\mu - \mathbf{P}^{(t_n)}\mu\|^*_{\mathrm{BL}} = \|\mathbf{P}^{(s_n)}\mu - \mathbf{P}^{(t_n)}\mu\|_{\mathrm{TV}} = 0$$

for every $\mu \in \mathcal{M}^+(S)$*.*

Theorem 3.7 and Corollary 3.10 imply the following:

Corollary 3.11. *Let* $0 < t_n \uparrow \infty$*. Then for every* $\mu \in \mathcal{M}(S)$*,*

$$\lim_{n\to\infty} \|\mathbf{P}^{(t_n)}\mu - R^{(\lfloor t_n \rfloor)}\mu\|^*_{\mathrm{BL}} = \lim_{n\to\infty} \|\mathbf{P}^{(t_n)}\mu - R^{(\lfloor t_n \rfloor)}\mu\|_{\mathrm{TV}} = 0.$$

Lemma 3.12. *Let* $s \in \mathbb{R}_+$ *and* $t > 0$*. Then*

$$\|\mathbf{P}^{(t)}P(s) - \mathbf{P}^{(t)}\|_{\mathcal{L}(\mathcal{M}(S)_{\mathrm{TV}})} \leq \frac{2s}{t}.$$

In particular

$$\lim_{t\to\infty} \|\mathbf{P}^{(t)}P(s) - \mathbf{P}^{(t)}\|_{\mathcal{L}(\mathcal{M}(S)_{\mathrm{TV}})} = 0$$

for every $s \in \mathbb{R}_+$*.*

Proof. Let $\mu \in \mathbf{P}(S)$. Then

$$\begin{aligned}
\|\mathbf{P}^{(t)}P(s)\mu - \mathbf{P}^{(t)}\mu\|_{\mathrm{TV}} &= \frac{1}{t}\left\| \int_s^{s+t} P(r)\mu\,dr - \int_0^t P(r)\mu\,dr \right\|_{\mathrm{TV}} \\
&= \frac{1}{t}\left\| \int_t^{s+t} P(r)\mu\,dr - \int_0^s P(r)\mu\,dr \right\|_{\mathrm{TV}} \\
&\leq \frac{1}{t}\left\| \int_t^{s+t} P(r)\mu\,dr \right\|_{\mathrm{TV}} + \frac{1}{t}\left\| \int_0^s P(r)\mu\,dr \right\|_{\mathrm{TV}}.
\end{aligned}$$

So by Lemma 3.1, $\|\mathbf{P}^{(t)}P(s)\mu - \mathbf{P}^{(t)}\mu\|_{\mathrm{TV}} \leq \frac{2s}{t}$. Since $\mathcal{M}(S)_{\mathrm{TV}}$ is an AL-space,

$$\|\mathbf{P}^{(t)}P(s) - \mathbf{P}^{(t)}\|_{\mathcal{L}(\mathcal{M}(S)_{\mathrm{TV}})} \leq \frac{2s}{t}.$$

□

4 The ergodic decomposition

Let S be a Polish space. Let d be a complete metric on S metrising the given topology and $\mathcal{S}_{\mathrm{BL}}$ the Banach space associated with (S,d).

4.1 Ergodic measures

Let P be a regular Markov operator with dual U. Recall that a Borel set E is a P-invariant set if $P\delta_x(E) = 1$ for every $x \in E$. This holds if and only if $U\mathbb{1}_E \geq \mathbb{1}_E$. Let μ be an invariant probability measure, then $E \subset S$ Borel is called *μ-almost P-invariant* if $U\mathbb{1}_E(x) \geq \mathbb{1}_E(x)$ for μ-a.e. $x \in S$. If E is μ-almost P-invariant, then

$$\begin{aligned} 0 &\leq \int_S |U\mathbb{1}_E(x) - \mathbb{1}_E(x)|\, d\mu(x) = \int_S U\mathbb{1}_E(x)\, d\mu(x) - \int_S \mathbb{1}_E(x)\, d\mu(x) \\ &= \langle P\mu - \mu, \mathbb{1}_E \rangle = 0. \end{aligned}$$

Thus $U\mathbb{1}_E(x) = \mathbb{1}_E(x)$ for μ-a.e. $x \in S$.

The following lemma follows easily from the definitions.

Lemma 4.1. *Let P be a regular Markov operator with invariant probability measure μ, then the following statements are equivalent for a Borel set $E \subset S$:*

(i) *E is μ-almost P-invariant.*

(ii) *$P\delta_x(E) = 1$ for μ-almost every $x \in E$.*

(iii) *$\int_E P\delta_x(E)\, d\mu(x) = \mu(E)$.*

Thus every P-invariant set is μ-almost P-invariant for every invariant probability measure μ, but not necessarily the other way around.

Let $\mathcal{P}^P_{\text{inv}} \subset \mathcal{P}(S)$ denote the convex subset of invariant probability measures. There are several different, but equivalent, definitions for an invariant measure to be ergodic. The definition we stated in Section 2.3 can be found in e.g. [28, 29].

Theorem 4.2. *Let P be a regular Markov operator and μ be a P-invariant probability measure. Then the following are equivalent:*

(i) *μ is ergodic, i.e. $\mu(E) = 0$ or 1 for every P-invariant set E.*

(ii) *$\mu(E) = 0$ or 1 for every μ-almost P-invariant set E.*

(iii) *There exists a Borel subset B of S such that $\mu(B) = 1$ and such that $(U^{(n)}f(x))_n$ converges to $\langle \mu, f \rangle$ for every $x \in B$ and $f \in C_b(S)$.*

(iv) *For every $f \in \mathrm{BM}(S)$, $U^{(n)}f(x)$ converges to $\langle \mu, f \rangle$ for μ-a.e. $x \in S$.*

(v) *μ is an extreme point of $\mathcal{P}^P_{\text{inv}} \subset \mathcal{P}(S)$.*

Proof. *(i)⇔(ii)*: This follows from [27, Lemma 4.1].
(i)⇔(iii): This is shown in [27, Theorem 4.4].
(ii)⇒(iv): This statement is proven in [15, Proposition 2.4.2].
(iv)⇒(iii): This can be shown with similar arguments as in the proof of [27, Theorem 4.4], by using a countable convergence-determining subset of $C_b(S)$.
(ii)⇔(v): This is shown in [1, Theorem 19.25]. □

Remark 4.3. Most of the equivalences in the theorem above are known and hold for more general state spaces. The equivalence of *(iii)* to ergodicity is one of the main results in [27] and its proof uses that the state space is Polish.

Let $\mathbf{P} = (P(t))_{t\geq 0}$ be a regular jointly measurable Markov semigroup on S with resolvent R and dual semigroup $\mathbf{U} = (U(t))_{t\geq 0}$. Ergodicity of an invariant measure for a Markov semigroup is generally defined in the literature through ergodicity of an associated dynamical system. Following [24], we associate with $(P(t))_{t\geq 0}$ a dynamical system on the space of trajectories $\Omega = S^{\mathbb{R}}$, with σ-algebra $\mathcal{F} = \mathcal{B}^{\mathbb{R}}$ and a group of invertible, measurable transformations $(\theta_t)_{t\in\mathbb{R}}$ from Ω to Ω: $(\theta_t\omega)(s) := \omega(t+s)$, $t, s \in \mathbb{R}$. With a $(P(t))_{t\geq 0}$-invariant probability measure μ we associate, as explained in [24, Section 2.1], a probability measure $\mathbb{P}^\mu$ on $(\Omega, \mathcal{F})$, such that $\mathbb{P}^\mu(\theta_t E) = \mathbb{P}^\mu(E)$ for every $t \in \mathbb{R}$ and $E \in \mathcal{F}$. $\mathcal{S}^\mu = (\Omega, \mathcal{F}, \mathbb{P}^\mu, \theta_t)$ is called the *canonical dynamical system* associated to $(P(t))_{t\geq 0}$ and μ. As in [24, Section 2.3] a $(P(t))_{t\geq 0}$-invariant probability measure μ is $(P(t))_{t\geq 0}$-*ergodic* if the dynamical system $\mathcal{S}^\mu$ is ergodic (see [24, Chapter 1] and [23, Chapter 2] for details on ergodic dynamical systems).

It follows from [24, Proposition 3.2.7] that μ is $(P(t))_{t\geq 0}$-ergodic if and only if μ is an extreme point of $\mathcal{P}^{\mathbf{P}}_{\text{inv}}(S)$. In [24] the Markov semigroup is assumed to be regular and strongly stochastically continuous at zero. However, jointly measurability and regularity of the Markov semigroup suffices for the proofs of the relevant results in [24]. Working with ergodic measures either through the canonical dynamical system or the equivalent characterisation as extreme points in $\mathcal{P}^{\mathbf{P}}_{\text{inv}}(S)$ is somewhat inconvenient. Therefore we now want to discuss some equivalent characterisations of ergodicity.

A Borel set $E \subset S$ is μ-*almost* $(P(t))_{t\geq 0}$-*invariant* if it is μ-invariant with respect to $P(t)$ for every $t \in \mathbb{R}_+$. E is called a *Lebesgue-almost* $(P(t))_{t\geq 0}$-*invariant set* if for every $x \in E$, $P(t)\delta_x(E) = 1$ for *almost every* $t \in \mathbb{R}_+$. Observe that E a Lebesgue-almost $(P(t))_{t\geq 0}$-invariant set if and only if E is R-invariant.

We have the following equivalent characterisations for a $(P(t))_{t\geq 0}$-invariant probability measure to be ergodic:

Theorem 4.4. *Let μ be a $(P(t))_{t\geq 0}$-invariant probability measure. Then the following are equivalent:*

(i) *$\mu(E) = 0$ or 1 for every Lebesgue-almost $(P(t))_{t\geq 0}$-invariant set E.*

(ii) *$\mu(E) = 0$ or 1 for every μ-almost $(P(t))_{t\geq 0}$ invariant set E.*

(iii) *There exists a Borel subset B of S such that $\mu(B) = 1$ and such that $(\boldsymbol{U}^{(t)} f(x))_n$ converges to $\langle \mu, f\rangle$ for every $x \in B$ and $f \in C_b(S)$.*

(iv) *For every $f \in \mathrm{BM}(S)$, $\boldsymbol{U}^{(t)} f(x)$ converges to $\langle \mu, f\rangle$ for μ-a.e. $x \in S$.*

(v) *μ is an extreme point of $\mathcal{P}^{\mathbf{P}}_{\text{inv}} \subset \mathcal{P}(S)$.*

(vi) *μ is R-ergodic.*

Proof. Let μ be a $(P(t))_{t\geq 0}$-invariant probability measure. By Proposition 3.5 μ is an R-invariant probability measure.

The equivalence between *(ii)* and *(v)* follows from [24, Theorem 3.2.4 and Proposition 3.2.7].

A Borel set E is R-invariant if and only if it is Lebesgue-almost $(P(t))_{t\geq 0}$-invariant, so μ satisfies *(i)* if and only if μ satisfies Theorem 4.2 *(i)*.

Theorem 3.7 implies that for any $f \in \mathrm{BM}(S)$ and any $x \in S$, $\mathbf{U}^{(t)}f(x) = \langle \mathbf{P}^{(t)}\delta_x, f\rangle$ converges as $t \to \infty$ if and only if $V^{(n)}f(x) = \langle R^{(n)}\delta_x, f\rangle$ converges as $n \to \infty$, where V is the dual of R. Thus *(iii)* and *(iv)* are equivalent to Theorem 4.2 *(iii)* and *(iv)* respectively. And by Proposition 3.5, $\mathcal{P}^{\mathbf{P}}_{\mathrm{inv}} = \mathcal{P}^{R}_{\mathrm{inv}}$ so they have the same extreme points. Thus *(v)* is equivalent to Theorem 4.2 *(v)*.

Now the statement follows from Theorem 4.2. □

Analogous to the definition of P-invariant sets for a regular Markov operator P, we define a Borel set $E \subset S$ to be $(P(t))_{t\geq 0}$*-invariant* if for every $x \in E$, $P(t)\delta_x(E) = 1$ *for all* $t \in \mathbb{R}_+$. Clearly, every $(P(t))_{t\geq 0}$-invariant set E is Lebesgue-almost $(P(t))_{t\geq 0}$-invariant, so $\mu(E) = 0$ or $\mu(E) = 1$ whenever μ is a $(P(t))_{t\geq 0}$-ergodic measure. However, the converse need not hold: one can easily construct Lebesgue-almost $(P(t))_{t\geq 0}$-invariant sets that are not $(P(t))_{t\geq 0}$-invariant.

A natural question arises: is a $(P(t))_{t\geq 0}$-invariant measure μ ergodic whenever $\mu(E) = 0$ or 1 for every $(P(t))_{t\geq 0}$-invariant set E? The following results answer this question affirmatively:

Theorem 4.5. *Let E be a Lebesgue-almost $(P(t))_{t\geq 0}$-invariant set. Then the set*

$$\hat{E} := \{x \in S : \limsup_{t\to\infty} \boldsymbol{P}^{(t)}\delta_x(E) = 1\}$$

contains E, is Borel measurable and satisfies:

(i) $\hat{E}$ *is* $(P(t))_{t\geq 0}$*-invariant,*

(ii) $\mu(E) = \mu(\hat{E})$ *for every* $(P(t))_{t\geq 0}$*-invariant probability measure* μ.

Consequently, any $(P(t))_{t\geq 0}$-invariant probability measure μ is ergodic if and only if $\mu(E) = 0$ or $\mu(E) = 1$ for every $(P(t))_{t\geq 0}$-invariant Borel set E.

Proof. First observe that Corollary 3.10 implies that

$$\hat{E} := \{x \in S : \limsup_{n\to\infty} \mathbf{P}^{(n)}\delta_x(E) = 1\}, \tag{2}$$

where the n ranges over $\mathbb{N}$. Second, if $x \in E$, then $P(t)\delta_x(E) = 1$ for Lebesgue almost every $t \in \mathbb{R}_+$, so $\mathbf{P}^{(t)}\delta_x(E) = 1$ for every $t \in \mathbb{R}_+$. Thus $x \in \hat{E}$.

Let $B = S\backslash\hat{E}$. Then $B = \{x \in S : \limsup_{n\to\infty} \mathbf{P}^{(n)}\delta_z(E) < 1\}$. We can write $B = \cup_{m\in\mathbb{N}} B_m$, where

$$B_m := \{x \in S : \limsup_{n\to\infty} \mathbf{P}^{(n)}\delta_x(E) \leq 1 - \frac{1}{m}\}.$$

Fix $m \in \mathbb{N}$, then $B_m = \cap_{d\in\mathbb{N}} \cup_{N\in\mathbb{N}} \cap_{n\geq N} C_{d,m,n}$, where $C_{d,m,n} = \{x \in S : \mathbf{P}^{(n)}\delta_x(E) \leq 1 - \frac{1}{m} + \frac{1}{d}\}$. Since $\mathbf{P}^{(n)}$ is a regular Markov operator and E is a Borel set, $x \mapsto \mathbf{P}^{(n)}(E)$ is Borel measurable by Proposition 2.5 and Proposition 2.2, thus $C_{d,m,n}$ is a Borel set for every $d, m, n \in \mathbb{N}$.

So B is Borel measurable, and consequently $\hat{E}$ as well.

The crucial part in the proof is the following:

Claim: If $\mu \in \mathcal{P}(S)$ *is such that* $\limsup_{t\to\infty} \mathbf{P}^{(t)}\mu(E) = 1$, *then* $\mu(\hat{E}) = 1$. *Proof of claim:* By assumption there is a sequence $(t_n)_n \subset \mathbb{R}_+$ such that $t_n \uparrow \infty$ and $\mathbf{P}^{(t_n)}\mu(E) \to 1$. Since $0 \le \mathbf{U}^{(t_n)}\mathbb{1}_E(x) = \mathbf{P}^{(t_n)}\delta_x(E) \le 1$ for every $x \in S$, we have $\mathbf{U}^{(t_n)}\mathbb{1}_E \le \mathbb{1}_S$ and thus

$$\int_S |\mathbb{1}_S - \mathbf{U}^{(t_n)}\mathbb{1}_E|\, d\mu = \int_S (\mathbb{1}_S - \mathbf{U}^{(t_n)}\mathbb{1}_E)\, d\mu = 1 - \mathbf{P}^{(t_n)}\mu(E) \to 0$$

as $n \to \infty$. Thus $(\mathbf{U}^{(t_n)}\mathbb{1}_E)_n$ converges to $\mathbb{1}_S$ in $L^1(\mu)$. Then there is a subsequence $(\mathbf{U}^{(t_{n_k})}\mathbb{1}_E)_k$ that converges to $\mathbb{1}_S$ μ-a.e. [11, Corollary 2.32]. So there is a Borel set $D \subset S$ such that $\mu(D) = 1$ and $\mathbf{P}^{(t_{n_k})}\delta_x(E) = \mathbf{U}^{(t_{n_k})}\mathbb{1}_E(x) \to 1$ for every $x \in D$. So $D \subset \hat{E}$ and thus $\mu(\hat{E}) = 1$.

We now prove the remaining two properties:

(i) Let $x \in \hat{E}$ and $t \in \mathbb{R}_+$. There is a sequence $(t_n)_n \subset \mathbb{R}_+$ such that $t_n \uparrow \infty$ and $\mathbf{P}^{(t_n)}\delta_x(E) \to 1$. By Lemma 3.12, $\|\mathbf{P}^{(t_n)}P(t)\delta_x - \mathbf{P}^{(t_n)}\delta_x\|_{\mathrm{TV}} \to 0$ as $n \to \infty$, thus $\mathbf{P}^{(t_n)}P(t)\delta_x(E) \to 1$ as well. So $P(t)\delta_x(\hat{E}) = 1$, according to the claim, thus $\hat{E}$ is $(P(t))_{t\ge0}$-invariant.

(ii) We first show that the statement holds for every R-ergodic measure. Let ν be an R-ergodic measure. E is R-invariant by assumption and by *(i)*, $\hat{E}$ is R-invariant as well. So $\nu(E) \in \{0,1\}$ and $\nu(\hat{E}) \in \{0,1\}$. Assume that $\nu(E) \neq \nu(\hat{E})$. Since $E \subset \hat{E}$, the only possibility is that $\nu(E) = 0$ and $\nu(\hat{E}) = 1$. Then $\mathbf{P}^{(t)}\nu(E) = \nu(E) = 0$ for every $t \in \mathbb{R}_+$. So for every $t \in \mathbb{R}_+$

$$0 = \mathbf{P}^{(t)}\nu(E) = \int_S \mathbf{P}^{(t)}\delta_x(E)\, d\nu(x) = \int_{\hat{E}} \mathbf{P}^{(t)}\delta_x(E)\, d\nu(x).$$

Thus for each $t \in \mathbb{R}_+$ there is a Borel set $F_t \subset \hat{E}$ such that $\nu(F_t) = 1$ and $\mathbf{P}^{(t)}\delta_x(E) = 0$ for every $x \in F_t$. Define $F := \cap_{n\in\mathbb{N}} F_n$. Then F is Borel measurable, $F \subset \hat{E}$ and $\nu(F) = 1$. For every $x \in F$, $\mathbf{P}^{(n)}\delta_x(E) = 0$ for all $n \in \mathbb{N}$. According to (2) $F \cap \hat{E} = \emptyset$, so $F = \emptyset$. This contradicts $\nu(F) = 1$ and we conclude that $\nu(E) = \nu(\hat{E})$.

Now let μ be a $(P(t))_{t\ge0}$-invariant probability measure. Then μ is R-invariant by Proposition 3.5. For every $x \in \Gamma^R_{cpie}$, ϵ_x is ergodic, and thus $\epsilon_x(E) = \epsilon_x(\hat{E})$. Hence

$$\mu(E) = \int_{\Gamma^R_{cpie}} \epsilon_x(E)\, d\mu(x) = \int_{\Gamma^R_{cpie}} \epsilon_x(\hat{E})\, d\mu(x) = \mu(\hat{E})$$

by Theorem 2.13 and Proposition 2.3. The last statement now is a simple consequence of Theorem 4.4. □

4.2 Preliminary Yosida-type decomposition of state space and integral decomposition of invariant measures

In this section we prove results for Markov semigroups that are similar to those obtained in [27] for Markov operators and summarised in Section 2.3.

Recall that we assume $\mathbf{P} = (P(t))_{t\geq 0}$ to be a regular jointly measurable Markov semigroup on S. Let R be its resolvent and $\mathbf{U} = (U(t))_{t\geq 0}$ the associated dual semigroup.

We define

$$\Gamma_t^{\mathbf{P}} := \{x \in S : \{\mathbf{P}^{(t)}\delta_x : t \in \mathbb{R}_{\geq 1}\} \text{ is tight}\},$$
$$\Gamma_{cp}^{\mathbf{P}} := \{x \in S : \mathbf{P}^{(t)}\delta_x \text{ converges in } \mathcal{S}_{\mathrm{BL}} \text{ as } t \to \infty.\}$$

For $x \in \Gamma_{cp}^{\mathbf{P}}$ we define $\varepsilon_x = \lim_{t\to\infty} \mathbf{P}^{(t)}\delta_x$. Then $\varepsilon_x \in \mathcal{P}(S)$. Notice that we distinguish in notation this measure from the Markov operator analogue ϵ_x that we associate to R.

Then we set

$$\Gamma_{cpi}^{\mathbf{P}} = \{x \in S : \varepsilon_x \text{ is } (P(t))_{t\geq 0}\text{-invariant}\}.$$

It need not be true that ε_x is a $(P(t))_{t\geq 0}$-ergodic measure whenever $x \in \Gamma_{cpi}^{\mathbf{P}}$, as the following example shows.

Example 4.6. Let $S = [-1, 1]$ with Euclidean metric d and define for $x \in S$ and $t > 0$,

$$P(t)\delta_x := \begin{cases} \delta_{\min(x-t,-1)} & \text{if } x < 0 \\ \frac{1}{2}\delta_{\min(-t,-1)} + \frac{1}{2}\delta_{\max(t,1)} & \text{if } x = 0 \\ \delta_{\max(x+t,1)} & \text{if } x > 0. \end{cases}$$

It is not difficult to prove that $(P(t))_{t\geq 0}$ defines a Markov semigroup via $P(t)\mu = \int_S P(t)\delta_x \, d\mu(x)$ and that $(P(t))_{t\geq 0}$ is even stochastically continuous and Markov-Feller. Now, $\epsilon_0 = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ is invariant with respect to $(P(t))_{t\geq 0}$, hence $0 \in \Gamma_{cpi}$, but ϵ_0 is not ergodic, since ϵ_0 is not an extreme point of the invariant measures.

Therefore we define

$$\Gamma_{cpie}^{\mathbf{P}} := \{x \in S : \varepsilon_x \text{ is } (P(t))_{t\geq 0}\text{-ergodic}\}.$$

Clearly $\Gamma_t^{\mathbf{P}} \supset \Gamma_{cp}^{\mathbf{P}} \supset \Gamma_{cpi}^{\mathbf{P}} \supset \Gamma_{cpie}^{\mathbf{P}}$.

We define

$$\mathcal{P}_t^{\mathbf{P}} := \{\mu \in \mathcal{P}(S) : (\mathbf{P}^{(t)}\mu)_{t\geq 1} \text{ is tight }\}$$

and

$$\mathcal{P}_{cp}^{\mathbf{P}} := \{\mu \in \mathcal{P}(S) : \mathbf{P}^{(t)}\mu \text{ converges in } \mathcal{S}_{\mathrm{BL}} \text{ as } t \to \infty.\}$$

For $\mu \in \mathcal{P}_{cp}^{\mathbf{P}}$, we define $\varepsilon_\mu := \lim_{t\to\infty} \mathbf{P}^{(t)}\mu$. Then we can define

$$\mathcal{P}_{cpi}^{\mathbf{P}} := \{\mu \in \mathcal{P}_{cp}^{\mathbf{P}} : \varepsilon_\mu \text{ is } (P(t))_{t\geq 0}\text{-invariant}\}$$

and

$$\mathcal{P}_{cpie}^{\mathbf{P}} := \{\mu \in \mathcal{P}_{cpi}^{\mathbf{P}} : \varepsilon_\mu \text{ is } (P(t))_{t\geq 0}\text{-ergodic}\}.$$

Note that $x \in \Gamma_\bullet^{\mathbf{P}}$ if and only if $\delta_x \in \mathcal{P}_\bullet^{\mathbf{P}}$, for $\bullet = t, cp, cpi, cpie$. If $x \in \Gamma_{cp}^{\mathbf{P}}$, then $\varepsilon_{\delta_x} = \varepsilon_x$.

Corollary 3.9 implies that for every $0 < \alpha < \beta$ and $\mu \in \mathbf{P}(S)$, $\{(\mathbf{P}^{(t)}\delta_x : t \in [\alpha, \beta]\}$ is compact in $\mathcal{S}_{\mathrm{BL}}$, thus tight. Consequently, if $\mu \in \mathcal{P}(S)$ is such that $(\mathbf{P}^{(t)}\mu)_{t\geq\alpha}$ is tight for some $\alpha > 0$, then it is tight for every $\alpha > 0$.

Theorem 4.7.

$$\mathcal{P}_\bullet^{\boldsymbol{P}} = \mathcal{P}_\bullet^R \text{ and } \Gamma_\bullet^{\boldsymbol{P}} = \Gamma_\bullet^R,$$

where $\bullet = t, cp, cpi, cpie$. *Moreover,* $\epsilon_\mu = \varepsilon_\mu$ *for every* $\mu \in \mathcal{P}_{cp}^{\boldsymbol{P}}$. *Consequently,* $\Gamma_t^{\mathbf{P}}, \Gamma_{cp}^{\mathbf{P}}, \Gamma_{cpi}^{\mathbf{P}}$ *and* $\Gamma_{cpie}^{\mathbf{P}}$ *are Borel sets and*

$$\mu(\Gamma_t^{\mathbf{P}}) = \mu(\Gamma_{cp}^{\mathbf{P}}) = \mu(\Gamma_{cpi}^{\mathbf{P}}) = \mu(\Gamma_{cpie}^{\mathbf{P}}) = 1$$

for every $(P(t))_{t\geq 0}$*-invariant probability measure* μ.

Proof. Let $\mu \in \mathcal{P}_t^{\mathbf{P}}$. Let $(n_k)_k \subset \mathbb{N}$ such that $n_k \uparrow \infty$. Then there is a subsequence $(n_{k_l})_l$ such that $\mathbf{P}^{(n_{k_l})}\mu$ converges in $\mathcal{S}_{\mathrm{BL}}$, so by Theorem 3.7 $R^{(n_{k_l})}\mu$ converges in $\mathcal{S}_{\mathrm{BL}}$. Thus $\mu \in \mathcal{P}_t^R$.

Let $\mu \in \mathcal{P}_t^R$. Let $(t_k)_k \subset [1,\infty)$. If $(t_k)_k$ is bounded, then it has a converging subsequence, so $\mathbf{P}^{(t_k)}\mu$ has a converging subsequence by Corollary 3.9. Else there is a subsequence of $(t_k)_k$ converging to infinity. Then there is a further subsequence $(t_{k_l})_l$, such that $R^{(\lfloor t_{k_l} \rfloor)}\mu$ converges in $\mathcal{S}_{\mathrm{BL}}$, hence $\mathbf{P}^{(t_{k_l})}\mu$ converges in $\mathcal{S}_{\mathrm{BL}}$ by Corollary 3.11. Thus $\mu \in \mathcal{P}_t^{\mathbf{P}}$. So $\mathcal{P}_t^{\mathbf{P}} = \mathcal{P}_t^R$ and thus $\Gamma_t^{\mathbf{P}} = \Gamma_t^R$.

Corollary 3.11 implies that $\mathbf{P}_{cp}^{\mathbf{P}} = \mathbf{P}_{cp}^R$, $\Gamma_{cp}^{\mathbf{P}} = \Gamma_{cp}^R$ and $\varepsilon_\mu = \epsilon_\mu$ for every $\mu \in \mathbf{P}_{cp}^{\mathbf{P}}$. By Proposition 3.5, $\mathbf{P}_{cpi}^{\mathbf{P}} = \mathbf{P}_{cpi}^R$ and $\Gamma_{cpi}^{\mathbf{P}} = \Gamma_{cpi}^R$. From Theorem 4.4, we obtain that $\mathbf{P}_{cpie}^{\mathbf{P}} = \mathbf{P}_{cpie}^R$ and $\Gamma_{cpie}^{\mathbf{P}} = \Gamma_{cpie}^R$.

The final statement follows from the results obtained in [27] summarised in Section 2.3. □

On $\Gamma_{cpie}^{\mathbf{P}}$ an equivalence relation $\sim$ is defined as follows: $x \sim y$ whenever $\varepsilon_x = \varepsilon_y$. We write $[x]$ to denote the equivalence class of $x \in \Gamma_{cpie}^{\mathbf{P}}$. The following result comes from [27, Theorem 4.6]. It implies that we can decompose $\Gamma_{cpie}^{\mathbf{P}}$ into disjoint Borel measurable subsets, such that each ergodic measure has full measure on exactly one of these subsets.

Theorem 4.8. *(i) For every* $x \in \Gamma_{cpie}^{\mathbf{P}}$ *the set* $[x]$ *is Borel measurable and* $\varepsilon_x([x]) = 1$.

(ii) Any ergodic measure μ *is of the form* $\mu = \varepsilon_x$ *for some* $x \in \Gamma_{cpie}^{\mathbf{P}}$.

Using the characterisation we obtain an integral decomposition of invariant probability measures in terms of ergodic measures ([27, Theorem 4.10]).

Theorem 4.9. *Let* μ *be an invariant probability measure. Then the map*

$$x \mapsto \begin{cases} \varepsilon_x & \text{if } x \in \Gamma_{cpie}^{\mathbf{P}} \\ 0 & \text{if } x \notin \Gamma_{cpie}^{\mathbf{P}}. \end{cases} \tag{3}$$

is strongly measurable from S *to* $\mathcal{S}_{\mathrm{BL}}$ *and*

$$\mu = \int_{\Gamma_{cpie}^{\mathbf{P}}} \varepsilon_x \, d\mu(x).$$

For $f \in C_b(S)$ we define the Borel measurable function

$$f^*(x) = \begin{cases} \langle \varepsilon_x, f \rangle & \text{if } x \in \Gamma_{cp}^{\mathbf{P}} \\ 0 & \text{if } x \notin \Gamma_{cp}^{\mathbf{P}}. \end{cases}$$

By applying Theorem 3.7 and Theorem 4.7 to the results in [27, Section 5], we find analogous results in the Markov semigroup setting.

Proposition 4.10. *Let $\mu \in \mathcal{P}(S)$.*

1. *If $\mu(\Gamma_{cp}^{\boldsymbol{P}}) = 1$, then $\mu \in \mathcal{P}_{cp}^{\boldsymbol{P}}$, $\varepsilon_\mu = \int_{\Gamma_{cp}^{\boldsymbol{P}}} \varepsilon_x \, d\mu(x)$ and for every $f \in C_b(S)$, $\langle \varepsilon_\mu, f\rangle = \langle \mu, f^*\rangle$.*
2. *If $\mu(\Gamma_{cpi}^{\boldsymbol{P}}) = 1$, then $\mu \in \mathcal{P}_{cpi}^{\boldsymbol{P}}$ and $\epsilon_\mu = \int_{\Gamma_{cpi}^{\boldsymbol{P}}} \epsilon_x \, d\mu(x)$.*
3. *If $\mu([z]) = 1$ for some $z \in \Gamma_{cpie}^{\boldsymbol{P}}$, then $\mu \in \mathcal{P}_{cpie}^{\boldsymbol{P}}$ and $\varepsilon_\mu = \varepsilon_z$.*

Proposition 4.11. *Let μ be a finite Borel measure on S such that $|\mu|(S \backslash \Gamma_{cp}^{\boldsymbol{P}}) = 0$. Then there is a finite Borel measure μ^* such that the following statements holds:*

(i) $\|P^{(n)}\mu - \mu^*\|_{\mathrm{BL}}^* \to 0$ *as* $n \to \infty$.

(ii) $\langle \mu^*, f\rangle = \langle \mu, f^*\rangle$ *for every* $f \in C_b(S)$

(iii) *If* $|\mu|(S \backslash \Gamma_{cpi}^{\boldsymbol{P}}) = 0$, *then* μ^* *is invariant.*

Proposition 4.12. *Let ν be an invariant probability measure and $\mu \in \mathcal{M}(S)$ such that $\mu \ll \nu$. Then there is an invariant probability measure μ^* such that $\|P^{(n)}\mu - \mu^*\|_{\mathrm{TV}} \to 0$ and $\langle \mu^*, f\rangle = \langle \mu, f^*\rangle$ for every $f \in C_b(S)$.*

4.3 Full Yosida-type ergodic decomposition

Let P be a regular Markov operator and $\mathbf{P} = (P(t))_{t\geq 0}$ a regular jointly measurable Markov semigroup with resolvent R and dual semigroup $\mathbf{U} = (U(t))_{t\geq 0}$.

Let E be a Borel set in S. There is a natural bijection between $\mathcal{M}(E)$ and $\mathcal{M}_E(S) := \{\mu \in \mathcal{M}(S) : |\mu|(S \backslash E) = 0\}$: we can extend any finite Borel measure μ on E to a finite Borel measure $\overline{\mu}$ on S, by defining $\overline{\mu}(F) := \mu(F \cap E)$ for every Borel set F in S. Then clearly $|\overline{\mu}|(S \backslash E) = 0$. On the other hand, if ν is a finite Borel measure on S such that $|\nu|(S \backslash E) = 0$, then its restriction to E defines a Borel measure μ such that $\overline{\mu} = \nu$.

Let E be a P-invariant Borel set. Then P leaves $\mathcal{M}_E(S)$ invariant: if $\mu \in \mathcal{M}_E(S)$, then by Proposition 2.5

$$P|\mu|(E) = \int_S P\delta_x(E)\, d|\mu|(x) \geq \int_E d|\mu| = |\mu|(E) = |\mu|(S).$$

Thus $|P\mu|(S \backslash E) \leq P|\mu|(S \backslash E) = |\mu|(S \backslash E) = 0$. So we can restrict P to $\mathcal{M}_E(S)$. This gives a 'restriction' of P to a regular Markov operator on $\mathcal{M}(E)$.

Similarly, if E is a $(P(t))$-invariant Borel set, then $(P(t))_{t\geq 0}$ leaves $\mathcal{M}_E(S)$ invariant, which gives a 'restriction' of $(P(t))_{t\geq 0}$ to a regular jointly measurable Markov semigroup on $\mathcal{M}(E)$.

The following theorem is shown in [27, Theorem 4.13], giving a full Yosida-type ergodic decomposition for regular Markov operators on Polish spaces.

Theorem 4.13. *Let S be a Polish space and P a regular Markov operator. If there exist invariant measures or equivalently, if Γ^P_{cpie} is not empty, then for every $x \in \Gamma^P_{cpie}$ the following statements hold:*

(i) There exists a P-invariant Borel set $S_{[x]} \subset [x]$ such that $\epsilon_x(S_{[x]}) = 1$.

(ii) ϵ_x is the unique invariant probability measure of $P_{[x]}$, where $P_{[x]}$ is the restriction of P to $\mathcal{M}(S_{[x]})$.

(iii) $P_{[x]}$ is ergodic in the sense that $S_{[x]}$ cannot be written as the union of two disjoint $P_{[x]}$-invariant sets A and B with $\epsilon_x(A) > 0$ and $\epsilon_x(B) > 0$.

Our aim is to show an analogous result for the Markov semigroup $(P(t))_{t\geq 0}$.

Proposition 4.14. *Let $x \in \Gamma^{\mathbf{P}}_{cpie} = \Gamma^R_{cpie}$ and let $S_{[x]}$ be the R-invariant Borel set in $[x]$ given by Theorem 4.13. Then $S_{[x]}$ is $(P(t))_{t\geq 0}$-invariant.*

Proof. The set $S_{[x]}$ from Theorem 4.13 is defined by [27, Lemma 4.1] as follows: $S_{[x]} = \cap_{n=1}^{\infty} B_n$, where $B_0 = [x]$ and

$$B_n = \{x \in B_{n-1} : R\delta_x(B_{n-1}) = 1\}.$$

Let E be an R-invariant Borel set in $[x]$, then $E \subset B_0$. Assume that $E \subset B_n$ for some $n \in \mathbb{Z}_{\geq 0}$, then $R\delta_x(B_n) \geq R\delta_x(E) = 1$ for every $x \in E$, so $E \subset B_{n+1}$. Thus by induction $E \subset \cap_{n=1}^{\infty} B_n = S_{[x]}$, i.e. $S_{[x]}$ is the largest R-invariant set in $[x]$.

Let $\hat{S}_{[x]}$ be defined as in Theorem 4.5, i.e.

$$\hat{S}_{[x]} := \{z \in S : \limsup_{t\to\infty} \mathbf{P}^{(t)}\delta_z([x]) = 1\}.$$

Then $\hat{S}_{[x]}$ is Borel measurable, $S_{[x]} \subset \hat{S}_{[x]} \subset [x]$ and $\hat{S}_{[x]}$ is $(P(t))_{t\geq 0}$-invariant, by Theorem 4.5. We will show that $\hat{S}_{[x]} \subset [x]$. Since $\hat{S}_{[x]}$ is also R-invariant, this will imply that actually $\hat{S}_{[x]} = S_{[x]}$.

Let $z \in \hat{S}_{[x]}$. Then there is a sequence $(t_n)_n \subset \mathbb{R}_+$, such that $t_n \uparrow \infty$ and $0 < \mathbf{P}^{(t_n)}\delta_z([x]) \to 1$. For $E \subset S$ Borel, we define

$$\nu_n(E) := \frac{\mathbf{P}^{(t_n)}\delta_z(E \cap [x])}{\mathbf{P}^{(t_n)}\delta_z([x])}.$$

Then ν_n defines a probability measure on S, and clearly $\nu_n([x]) = 1$. Proposition 4.10 implies that $\nu_n \in \mathcal{P}^{\mathbf{P}}_{cpie}$ and $\varepsilon_{\nu_n} = \varepsilon_x$.

Since $\mathbf{P}^{(t_n)}\delta_z \geq \mathbf{P}^{(t_n)}\delta_z([x]) \cdot \nu_n$,

$$\begin{aligned} \|\mathbf{P}^{(t_n)}\delta_z - \mathbf{P}^{(t_n)}\delta_z([x]) \cdot \nu_n\|_{\mathrm{TV}} &= \mathbf{P}^{(t_n)}\delta_z(S) - \mathbf{P}^{(t_n)}\delta_z([x])\nu_n(S) \\ &= 1 - \mathbf{P}^{(t_n)}\delta_z([x]) \to 0 \end{aligned}$$

as $n \to \infty$, and

$$\|\mathbf{P}^{(t_n)}\delta_z([x]) \cdot \nu_n - \nu_n\|_{\mathrm{TV}} \leq |\mathbf{P}^{(t_n)}\delta_z([x]) - 1| \to 0,$$

thus $\|\mathbf{P}^{(t_n)}\delta_z - \nu_n\|_{\mathrm{TV}} \to 0$ as $n \to \infty$.

It follows from [18, Lemma 2] that $\lim_{t\to\infty} \|\mathbf{P}^{(t)}\rho - \mathbf{P}^{(t)}\mathbf{P}^{(s)}\rho\|_{\mathrm{TV}} = 0$ for every $\rho \in \mathcal{P}(S)$ and $s \in \mathbb{R}_+$. Thus, because $\mathbf{P}^{(t)}$ is a Markov operator,

$$\begin{aligned}\limsup_{t\to\infty} \|\mathbf{P}^{(t)}\delta_z - \mathbf{P}^{(t)}\nu_n\|_{\mathrm{TV}} &= \limsup_{t\to\infty} \|\mathbf{P}^{(t)}\mathbf{P}^{(t_n)}\delta_z - \mathbf{P}^{(t)}\nu_n\|_{\mathrm{TV}} \\ &\leq \|\mathbf{P}^{(t_n)}\delta_z - \nu_n\|_{\mathrm{TV}}.\end{aligned}$$

Now,

$$\begin{aligned}\limsup_{t\to\infty} \|\mathbf{P}^{(t)}\delta_z - \varepsilon_x\|_{\mathrm{BL}}^* &\leq \limsup_{t\to\infty} \|\mathbf{P}^{(t)}\delta_z - \mathbf{P}^{(t)}\nu_n\|_{\mathrm{TV}} \\ &\leq \|\mathbf{P}^{(t_n)}\delta_z - \nu_n\|_{\mathrm{TV}},\end{aligned}$$

which converges to zero as $n \to \infty$. Thus $z \in [x]$. □

Now we can conclude a full Yosida-type ergodic decomposition result for $(P(t))_{t\geq 0}$.

Theorem 4.15. *Let S be a Polish space and $\boldsymbol{P} = (P(t))_{t\geq 0}$ a regular jointly measurable Markov semigroup. If there exist invariant measures, or equivalently, if $\Gamma^{\mathbf{P}}_{cpie}$ is not empty, then for every $x \in \Gamma^{\mathbf{P}}_{cpie}$ the following statements hold:*

- *(i) There is a $(P(t))_{t\geq 0}$-invariant Borel set $S_{[x]} \subset [x]$ such that $\varepsilon_x(S_{[x]}) = 1$.*
- *(ii) ε_x is the unique invariant probability measure of the restriction $(P(t)_{[x]})_{t\geq 0}$ of $(P(t))_{t\geq 0}$ to $\mathcal{M}(S_{[x]})$.*
- *(iii) $(P(t)_{[x]})_{t\geq 0}$ is ergodic in the sense that $S_{[x]}$ cannot be written as the union of two disjoint $(P(t)_{[x]})_{t\geq 0}$-invariant sets A and B with $\varepsilon_x(A) > 0$ and $\varepsilon_x(B) > 0$.*

Proof. *(i)* We define $S_{[x]}$ as in Proposition 4.14. Then the result follows from that proposition.

(ii) Since $S_{[x]}$ is $(P(t))_{t\geq 0}$-invariant, we can restrict $(P(t))_{t\geq 0}$ to a regular jointly measurable Markov semigroup $(P(t)_{[x]})_{t\geq 0}$ on $\mathcal{M}(S_{[x]})$. Let μ be a $(P(t)_{[x]})_{t\geq 0}$-invariant probability measure on $S_{[x]}$ and $\overline{\mu}$ the extension of μ to S. Then $\overline{\mu}$ is a $(P(t))_{t\geq 0}$-invariant probability measure on S such that $\overline{\mu}(S_{[x]}) = 1$, thus $\overline{\mu}(S_{[x]}) = 1$. Now, by Theorem 4.9 and since $\overline{\mu}(S \backslash S_{[x]}) = 0$

$$\begin{aligned}\overline{\mu} &= \int_{\Gamma^{\mathbf{P}}_{cpie}} \varepsilon_y \, d\overline{\mu}(y) = \int_{S_{[x]}} \varepsilon_y \, d\mu(y) \\ &= \int_{S_{[x]}} \varepsilon_x \, d\mu(y) = \varepsilon_x,\end{aligned}$$

thus μ is the restriction of ε_x to $S_{[x]}$.

(iii) Let A, B be disjoint $(P(t)_{[x]})_{t\geq 0}$-invariant Borel subsets of $S_{[x]}$ such that $\varepsilon_x(A) > 0$ and $\varepsilon_x(B) > 0$. Then A, B are disjoint $(P(t))_{t\geq 0}$-invariant Borel subsets of S, thus by ergodicity of ε_x, $\varepsilon_x(A) = \varepsilon_x(B) = 1$. But then $\varepsilon_x(A \cup B) = 2$, which gives a contradiction with the fact that ε_x is a probability measure. □

Remark 4.16. Theorem 4.15 extends [6, Proposition 5.2 and Theorem 5.1]. There, Markov semigroups associated to Borel right processes on locally compact separable metric spaces are considered (see [25, pp. 104-105] for a definition of Borel-right processes). It follows from [4, Remark 5 in §29] that every locally compact separable metric space is a Polish space, so our setting is more general. Also, we do not need any continuity result on the Markov semigroup: joint measurability and regularity are sufficient. Finally the sets defined in [6, Theorem 5.1] depend on a pre-chosen invariant probability measure, while our sets do not.

Proposition 4.17. *Let* $\mu \in \mathcal{P}(S)$ *be such that*

$$\limsup_{t\to\infty} \mathbf{P}^{(t)}\mu([x]) = 1$$

for some $x \in \Gamma^{\mathbf{P}}_{cpie}$*. Then* $\mu([x]) = 1$*,* $\mu \in \mathcal{P}^{\mathbf{P}}_{cpie}$ *and* $\mathbf{P}^{(t)}\mu \to \varepsilon_x$ *in* $\mathcal{S}_{\mathrm{BL}}$ *as* $t \to \infty$*.*

Proof. Tthere is a sequence $(t_n)_n \in \mathbb{R}_+$ such that $t_n \uparrow \infty$ and $\mathbf{P}^{(t_n)}\mu([x]) \to 1$ as $n \to \infty$. Proceeding as in the proof of Theorem 4.5,

$$\int_S |\mathbb{1}_S - \mathbf{U}^{(t_n)}\mathbb{1}_{[x]}|\, d\mu = 1 - \mathbf{P}^{(t_n)}\mu([x]) \to 0$$

as $n \to \infty$, so by [11, Corollary 2.32] there is a Borel set $D \subset S$ and subsequence $(\mathbf{U}^{(t_{n_k})}\mathbb{1}_{[x]})_k$, such that $\mu(D) = 1$ and $\mathbf{U}^{(t_{n_k})}\mathbb{1}_{[x]}(z) \to 1$ for every $z \in D$.

Re-examination of the proof of Proposition 4.14 shows that

$$\mathbf{P}^{(t_n)}\delta_z([x]) = \mathbf{U}^{(t_n)}\mathbb{1}_{[x]}(z) \to 1 \text{ as } n \to \infty$$

implies that $z \in [x]$. Thus $D \subset [x]$ and consequently $\mu([x]) = 1$. The final statement follows from Proposition 4.10. $\square$

References

[1] Aliprantis, C. and K. Border (2006), *Infinite dimensional analysis. A hitchhiker's guide*, Third edition, Springer, Berlin.

[2] Aliprantis, C. and O. Burkinshaw (1985), *Positive Operators*, Academic Press, New York.

[3] Arendt, W., C. Batty, N. Hieber and F. Neubrander (2001), *Vector-valued Laplace transforms and Cauchy problems*, Birkhäuser Verlag, Basel.

[4] Bauer, H. (2001), *Measure and Integration Theory*, Walter de Gruyter, Berlin.

[5] Bogachev, V.I. (2007), *Measure Theory; Volume II*, Springer.

[6] Costa, O. and F. Dufour (2006), Ergodic properties and ergodic decompositions of continuous-time Markov processes, *J. Appl. Prob.* **43**, 767–781.

[7] Dudley, R.M. (1966), Convergence of Baire measures, *Stud. Math.* **27**, 251-268.

[8] Dudley, R.M. (1974), Correction to: "Convergence of Baire measures", *Stud. Math.* **51**, 275.

[9] Es-Sarhir, A. (2009), Existence and uniqueness of invariant measures for a class of transition semigroups on Hilbert spaces, *J. Math. Anal. Appl.* **353**, 497–507.

[10] Ethier, S.N. and T.G. Kurtz (1986), *Markov Processes; Characterization and Convergence*, Wiley, New York.

[11] G.B. Folland (1984), *Real analysis, modern techniques and their applications*, John Wiley & Sons Inc.,New York.

[12] Gacki, H. (2007), *Applications of the Kantorovich-Rubinstein maximum principle in the theory of Markov semigroups*, Dissertationes Math **448**.

[13] Goldys, B. and J.M.A.M. van Neerven (2003), Transition semigroups of Banach space-valued Ornstein-Uhlenbeck processes, *Acta Appl. Math.* **76**, 283–330.

[14] Hernández-Lerma, O. and J.B. Lasserre (1998), Ergodic theorems and ergodic decomposition for Markov chains, *Acta Appl. Math.* **54**, 99–119.

[15] Hernández-Lerma, O. and J.B. Lasserre (2003), *Markov chains and invariant probabilities*, Birkhäuser, Basel.

[16] Hille, S.C. and D.T.H. Worm (2009), Embedding of semigroups of Lipschitz maps into positive linear semigroups on ordered Banach spaces generated by measures, *Integr. Equ. Oper. Theory* **63**, 351–371.

[17] Hille, S.C. and D.T.H. Worm (2009), Continuity properties of Markov semigroups and their restrictions to invariant L^1-spaces, *Semigroup Forum* **79**, 575–600.

[18] Komorowski, T., S. Peszat and T. Szarek, On ergodicity of some Markov processes, *to be published in: Annals of probability.*

[19] Lant, T. and H.R. Thieme (2007), *Markov transition functions and semigroups of measures*, Semigroup Forum **74**, 337–369.

[20] Lasota, A. and T. Szarek, T. (2006), *Lower bound technique in the theory of a stochastical differential equation*, J. Diff. Eq. **231**, 513–533.

[21] Manca, L. (2008), Kolmogorov equations for measures, *J. Evol. Equ.* **8**, 231–262.

[22] Myjak, J. and T. Szarek (2003), Attractors of iterated function systems and Markov operators, *Abstr. Appl. Anal.* **8**, 479–502.

[23] Petersen, K. (1983), *Ergodic theory*, Cambridge University Press.

[24] Da Prato, G. and J. Zabczyk (1996), *Ergodicity for infinite-dimensional systems*, London Mathematical Society Lecture Note Series, 229, Cambridge University Press, Cambridge.

[25] Sharpe, M. (1988), *General theory of Markov processes*, Academic Press, London.

[26] Szarek, T., M. Ślęczka and M. Urbański, On stability of velocity vectors for some passive tracer models, submitted for publication.

[27] Worm, D.T.H. and S.C. Hille (2009), Ergodic decompositions associated to regular Markov operators on Polish spaces, accepted for publication in *Ergodic Theory and Dynamical Systems.*

[28] Zaharopol, R. (2005), *Invariant probabilities of Markov-Feller operators and their supports*, Birkhäuser, Basel.

[29] Zaharopol, R. (2008), An ergodic decomposition defined by transition probabilities, *Acta. Appl. Math.* **104**, 47–81.