Equicontinuous families of Markov operators on complete separable metric spaces with applications to ergodic decompositions and existence, uniqueness and stability of invariant measures.

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== MI-2010-03 ==

March 4, 2010

Abstract

We consider Markov operators and Markov semigroups with an equicontinuity property on a complete separable metric space. We show that this property has several implications on a Yosida-type ergodic decomposition of state space, given by the authors in (*Ergodic decompositions associated to regular Markov operators on Polish spaces*, to appear in Ergodic Theory and Dynamical Systems) and (*An ergodic decomposition defined by regular jointly measurable Markov semigroups on Polish spaces*, submitted). Using this ergodic decomposition, we obtain several new results on existence, uniqueness and stability of invariant measures.

1 Introduction

Families of Markov operators appear naturally as transition operators on the context of Markov processes. If X_t is the state of the process at time t in a measurable space S, and μ_0 is the law of X_0, then the law of X_t is given by $P(t)\mu_0$, where $P(t)$ is a Markov operator. Depending on time being discrete ($t \in T = \mathbb{N}_0$) of continuous ($t \in T = \mathbb{R}_+$), one obtains a semigroup of Markov operators.

More structure is required to get a satisfactory theory. We will assume that the state space S is a complete separable metric space. This is a convenient setting for applications, e.g. population dynamics in biology, in which S typically carries a natural metric. There is much interest lately in continuous-time Markov processes on non-locally compact complete separable spaces, for instance those coming from stochastic differential equations in separable Hilbert spaces [2, 6, 15] or in separable Banach spaces [8]. There are also recent results by Szarek and coworkers about Markov semigroups on complete separable metric spaces [12, 13, 20]. Furthermore, there exists a line of research in the literature that focuses on a pure topological setting (e.g. [14] and references found there).

We will also require that the Markov operators P are *regular*, i.e. given by a transition kernel

$$P\mu = \int_S p(x)\, d\mu(x), \tag{1}$$

where $p(\cdot)$ is a measure-valued function. Because S is a complete separable metric space, our framework presented in [9, 10], summarised in Section 2.1, enables us to interpret (1) as a *Bochner integral* in a Banach space $\mathcal{S}_{\mathrm{BL}}$ that densely contains the finite Borel measures $\mathcal{M}(S)$ and whose norm $\|\cdot\|_{\mathrm{BL}}^*$ is such that the relative norm topology on the positive measures $\mathcal{M}^+(S)$ coincides with the relative weak-star topology induced by the bounded continuous functions on S, $C_b(S)$.

In this paper we further exploit this viewpoint, which enables us to refine our results on ergodic decompositions previously obtained [21, 22] and summarised in Section 2.3 in the setting of Markov operators and semigroups having the so-called *e-property*, which has also been considered by Szarek *et al.* [12, 13, 19, 20]. If $U(t)$ is the dual of the regular Markov operator $P(t)$, then the family $(P(t))_{t\in T}$ has the e-property if and only if for each bounded Lipschitz function f on S, and any $x \in S$, the family of functions $(U(t)f)_{t\in T}$ is equicontinuous at x. This definition depends on the metric, so the notion of e-property is not well-defined in the generality of a Polish space, i.e. a separable topological space that is metrisable for a complete metric, since no particular metric is designated in this setting. The e-property is weaker than the well-studied strong Feller property [2]. We are able to obtain some stronger results than found in [12, 13, 19, 20], and even go beyond the e-property condition by requiring the weaker Cesàro e-property only. These results entail various characterisations of existence, uniqueness and stability of invariant measures for (discrete or continuous-time) semigroups of Markov operators that have the (Cesàro) e-property.

Our approach is founded on Theorem 3.1, which establishes a relationship between weak convergence and norm convergence in $\mathcal{S}_{\mathrm{BL}}$. It follows from a reinterpretation of a result by Pachl [17, Theorem 3.2] in view of our Banach space $\mathcal{S}_{\mathrm{BL}}$. Pachl's Theorem basically says, among others, that a subset $M \subset \mathcal{M}(S)$ that is bounded on the unit ball in the space of uniformly continuous, bounded functions on S, $(C_{ub}(S), \|\cdot\|_\infty)$, is relatively compact in $\mathcal{S}_{\mathrm{BL}}$ if and only if it is relatively $\sigma(\mathcal{M}(S), C_{ub}(S))$ countably compact. We use this result to obtain equivalent conditions for families of Markov operators to be equicontinuous in Section 4. Together with a new characterisation of tightness of sets of probability measures, Theorem 2.5 [21, Theorem 3.1], we are able to conclude various interesting properties on particular sets of measures associated to such families.

In Section 5, we explore our general results of Section 4 in various settings of semigroups of Markov operators. Similar to our approach in [22] we "reduce" the continuous-time semigroup case to the discrete-time case of a single iterated

Markov operator through the analysis of the *resolvent operator* associated to the continuous-time Markov semigroup. We obtain a stronger version of a Yosida-type ergodic decomposition of the state space under the condition of the Cesàro e-property to hold. Compared to [21, 22], the improvement in this setting consists of closedness and invariance of the sets in the decomposition, and we obtain a *continuous* surjective function from one of these sets to the ergodic invariant probability measures.

Our work on existence, uniqueness and stability of invariant measures was inspired by the interesting work of Szarek and coworkers [12, 13, 19, 20]. Our type of arguments yields various improvements, generalisations and novel results on this topic, all collected in Section 6. Some of these turned out to be obtained independently, with different arguments and under slightly different conditions, in [11].

Some notational conventions. Unless otherwise mentioned, (S,d) will denote a complete separable metric space, viewed as a measurable space with respect to its Borel σ-algebra. We write $\mathcal{M}(S)$ to denote the real vector space of all signed finite Borel measures on S, containing $\mathcal{M}^+(S)$, the cone of positive measures. $\mathcal{P}(S)$ consists of the probability measures in $\mathcal{M}^+(S)$. We denote the total variation norm on $\mathcal{M}(S)$ by $\|\cdot\|_{\mathrm{TV}}$ and write $\mathcal{M}(S)_{\mathrm{TV}}$ for the Banach space consisting of $\mathcal{M}(S)$ endowed with the total variation norm. We write $\mathrm{BM}(S)$ to denote the real vector space of all bounded measurable functions from S to $\mathbb{R}$ and $\mathbb{1}_E$ for the indicator function of $E \subset S$. For $f : S \to \mathbb{R}$ measurable and $\mu \in \mathcal{M}(S)$ we write $\langle \mu, f\rangle$ for $\int_S f\, d\mu$. We write $C_b(S)$ to denote the Banach space of bounded continuous functions from S to $\mathbb{R}$, endowed with the supremum norm $\|\cdot\|_\infty$, and $C_{ub}(S)$ to denote the Banach space of bounded uniformly continuous functions from S to $\mathbb{R}$, also endowed with the supremum norm.

2 Preliminaries

Let (S,d) be a complete separable metric space.

2.1 The space $\mathcal{S}_{\mathrm{BL}}$

In this part we will recall some definitions and results from [4], [9] and [10]. $\mathrm{BL}(S)$ denotes the Banach space of bounded real-valued Lipschitz functions for the metric d, endowed with the norm $\|f\|_{\mathrm{BL}} := |f|_{\mathrm{Lip}} + \|f\|_\infty$, where $|f|_{\mathrm{Lip}}$ is the global Lipschitz constant of f. The Dirac functionals $\delta_x(f) := f(x)$ for $x \in S$ are in $\mathrm{BL}(S)^*$. We denote the usual dual norm on $\mathrm{BL}(S)^*$ by $\|\cdot\|^*_{\mathrm{BL}}$. $\mathrm{BL}(S)$ is in fact isometrically isomorphic to the dual of a separable Banach space $\mathcal{S}_{\mathrm{BL}}$, which can be defined as the closure of the finite linear span of the δ_x, $x \in S$, in $\mathrm{BL}(S)^*$. Then, as shown in [4, Lemma 6], each $\mu \in \mathcal{M}(S)$ defines a unique element in $\mathrm{BL}(S)^*$, which we will also denote by μ, by sending $f \in \mathrm{BL}(S)$ to $\langle \mu, f\rangle = \int_S f\, d\mu$. A function $f \in \mathrm{BL}(S)$ defines a bounded linear functional on $\mathcal{S}_{\mathrm{BL}}$ by sending φ to $\varphi(f)$. Using [9, Lemma 3.5] one can show that the map $x \mapsto \delta_x$ is a continuous embedding from S into $\mathcal{S}_{\mathrm{BL}}$.

By [9, Theorem 3.9 and Corollary 3.10], $\mathcal{M}^+(S)$ is a closed convex cone of $\mathcal{S}_{\mathrm{BL}}$, and $\mathcal{M}(S)$ is a $\|\cdot\|_{\mathrm{BL}}^*$-dense subspace of $\mathcal{S}_{\mathrm{BL}}$. The restriction of the weak-star topology on $C_b(S)^*$ to $\mathcal{M}^+(S)$, also called the topology of weak convergence on $\mathcal{M}^+(S)$, *equals* the restriction of the norm topology on $\mathcal{S}_{\mathrm{BL}}$ to $\mathcal{M}^+(S)$ by [4, Theorem 18]. In particular the following lemma holds:

Lemma 2.1. *Let $\mu_n, \mu \in \mathcal{M}^+(S)$. Then $\|\mu_n - \mu\|_{\mathrm{BL}}^* \to 0$ if and only if $\langle \mu_n, f\rangle \to \langle \mu, f\rangle$ for all $f \in C_b(S)$.*

Let
$$\mathcal{S}_{\mathrm{BL}}^+ := \{\phi \in \mathcal{S}_{\mathrm{BL}} : \phi(f) \geq 0 \text{ for all } f \in \mathrm{BL}(S), f \geq 0\}.$$
Then $\mathcal{S}_{\mathrm{BL}}^+ = \mathcal{M}^+(S)$ by [9, Corollary 4.2].

The following results come from [10, Proposition 2.5, Proposition 2.6 and Corollary 2.7]. Let (Ω, Σ) be a measurable space.

Proposition 2.2. *Let $p : \Omega \to \mathcal{S}_{\mathrm{BL}}^+$. The following conditions are equivalent:*

(i) p is strongly measurable.

(ii) For each $f \in \mathrm{BM}(S)$, the map $\Omega \to \mathbb{R} : \omega \mapsto \langle p(\omega), f\rangle$ is measurable.

(iii) For each Borel measurable $E \subset S$, the map $\Omega \to \mathbb{R} : \omega \mapsto p(\omega)(E)$ is measurable from Ω to $\mathbb{R}$.

Proposition 2.3. *Let $\mu \in \mathcal{M}^+(\Omega)$ and let $p : \Omega \to \mathcal{S}_{\mathrm{BL}}^+$ be Bochner integrable with respect to μ. Define $\nu := \int_\Omega p(\omega) d\mu(\omega)$. Then*
$$\int_S f \, d\nu = \int_\Omega \langle p(\omega), f\rangle d\mu(\omega),$$
for any $f \in \mathrm{BM}(S)$. In particular, for any Borel set $E \subset S$,
$$\left[\int_\Omega p(\omega) d\mu(\omega)\right](E) = \int_\Omega p(\omega)(E) d\mu(\omega).$$

Corollary 2.4. *For any $\mu \in \mathcal{M}^+(S)$, $\int_S \delta_x \, d\mu(x) = \mu$, as a Bochner integral in $\mathcal{S}_{\mathrm{BL}}$.*

A collection of measures $M \subset \mathcal{P}(S)$ is *tight* if for every $\epsilon > 0$ there exists a compact $K \subset S$ such that $\mu(K) \geq 1 - \epsilon$ for every $\epsilon > 0$. For $F \subset S$ and $\delta > 0$ we define $f_F^\delta(x) := \left(1 - \frac{d(x,F)}{\delta}\right)^+$. Then $f_F^\delta \in \mathrm{BL}(S)$ with $\|f_F^\delta\|_{\mathrm{BL}} \leq 1 + \frac{1}{\delta}$.

Theorem 2.5. *Let $D \subset S$ be dense, and let $M \subset \mathcal{P}(S)$. The following statements are equivalent:*

(i) M is tight.

(ii) M is relatively compact in $\mathcal{S}_{\mathrm{BL}}$.

(iii) For every $m \in \mathbb{N}$ there is a finite subset $F \subset D$ such that
$$\langle \mu, f_F^{\frac{1}{m}}\rangle > 1 - \frac{1}{m} \text{ for every } \mu \in M.$$

The equivalence between *(i)* and *(ii)* follows from the well-known Prokhorov Theorem and Lemma 2.1, and the equivalence between *(i)* and *(iii)* is shown in [21, Theorem 3.1].

2.2 Markov operators and Markov semigroups

A *Markov operator* is a a map $P : \mathcal{M}^+(S) \to \mathcal{M}^+(S)$, such that

(MO1) P is additive and $\mathbb{R}_+$-homogeneous,

(MO2) $\|P\mu\|_{\mathrm{TV}} = \|\mu\|_{\mathrm{TV}}$ for all $\mu \in \mathcal{M}^+(S)$.

P extends to a positive *bounded* linear operator on $\mathcal{M}(S)_{\mathrm{TV}}$ given by $P\mu := P\mu^+ - P\mu^-$. The operator norm of this extension is 1.

The following result follows from [21, Proposition 2.10, Corollary 2.11 and Proposition 2.12]:

Proposition 2.6. *Let P be a Markov operator. The following are equivalent:*

(i) There exists $U : \mathrm{BM}(S) \to \mathrm{BM}(S)$ such that

$$\langle P\mu, f\rangle = \langle \mu, Uf\rangle \text{ for all } \mu \in \mathcal{M}^+(S), f \in \mathrm{BM}(S).$$

(ii) (a) $x \mapsto P\delta_x, S \to \mathcal{S}_{\mathrm{BL}}$ is strongly measurable, and

(b) $P\mu = \int_S P\delta_x \, d\mu(x)$.

In either case,

$$P\mu(E) = \int_S P\delta_x(E)\, d\mu(x)$$

and

$$P\int_\Omega h(\omega) d\nu(\omega) = \int_\Omega Ph(\omega)\, d\nu(\omega)$$

for any finite measure space (Ω, Σ, ν) and $h : \Omega \to \mathcal{S}_{\mathrm{BL}}^+$ Bochner integrable with respect to ν.

Following [7, 16, 10, 21], we will call a Markov operator P *regular* if it satisfies the conditions of Proposition 2.6. The map $U : \mathrm{BM}(S) \to \mathrm{BM}(S)$ associated to a regular Markov operator P is unique and we call it the *dual* of P.

For a regular Markov operator P we define the *Cesàro averages*

$$P^{(n)} := \frac{1}{n}\sum_{k=0}^{n-1} P^k \qquad \text{and} \qquad U^{(n)} := \frac{1}{n}\sum_{k=0}^{n-1} U^k.$$

A measure μ is *invariant* (with respect to P) if $P\mu = \mu$. A P-invariant set is a Borel set E such that $P\delta_x(E) = 1$ for every $x \in E$. An invariant probability measure μ is *ergodic* (with respect to P), if for every P-invariant set E, $\mu(E) = 0$ or 1.

A regular Markov operator P is *Markov-Feller* if its dual U satisfies $U(C_b(S)) \subset C_b(S)$.

Proposition 2.7. *A Markov operator P is a Markov-Feller operator if and only if $P : \mathcal{S}_{\mathrm{BL}}^+ \to \mathcal{S}_{\mathrm{BL}}^+$ is continuous.*

Proof. One direction is proven in [10, Lemma 3.3] and the other direction can be shown using Lemma 2.1. □

A *Markov semigroup* is a semigroup $(P(t))_{t\geq 0}$ of Markov operators. $(P(t))_{t\geq 0}$ is called *regular* if $P(t)$ is regular for all $t \geq 0$.

A Markov semigroup $(P(t))_{t\geq 0}$ is a *jointly measurable Markov semigroup* if $(t,x) \mapsto P(t)\delta_x(E)$ is jointly measurable from $\mathbb{R}_+ \times S$ to $\mathbb{R}$ for every Borel set E in S. This holds, by Proposition 2.2, if and only if $(t,x) \mapsto P(t)\delta_x$ is jointly measurable from $\mathbb{R}_+ \times S$ to $\mathcal{S}_{\mathrm{BL}}$. The following result comes from [22, Proposition 2.6].

Proposition 2.8. *Let $(P(t))_{t\geq 0}$ be a regular jointly measurable Markov semigroup and $\mu \in \mathcal{M}^+(S)$. Then $t \mapsto P(t)\mu$ is strongly measurable from $\mathbb{R}_+$ to $\mathcal{S}_{\mathrm{BL}}$. For every $E \subset S$ Borel the map $t \mapsto P(t)\mu(E)$ is measurable from $\mathbb{R}_+$ to $\mathbb{R}$.*

We call the Markov semigroup $(P(t))_{t\geq 0}$ *strongly stochastically continuous*, when $t \mapsto \langle P(t)\mu, f\rangle$ is continuous for all $\mu \in \mathcal{M}^+(S)$ and $f \in C_b(S)$, and *strongly stochastically continuous at zero* when $t \mapsto \langle P(t)\mu, f\rangle$ is continuous at zero for all $\mu \in \mathcal{M}^+(S)$ and $f \in C_b(S)$.

Proposition 2.9. *([22, Proposition 2.8]) Let $(P(t))_{t\geq 0}$ be a regular Markov semigroup that is strongly stochastically continuous at zero. Then $(P(t))_{t\geq 0}$ is a jointly measurable Markov semigroup.*

A measure μ is *invariant* (with respect to $(P(t))_{t\geq 0}$) if $P(t)\mu = \mu$ for every $t \in \mathbb{R}_+$. A $(P(t))_{t\geq 0}$-invariant set is a Borel set E such that E is $P(t)$-invariant for all $t \in \mathbb{R}_+$. Analogous to the Markov operator case, we call an invariant probability measure μ *ergodic* (with respect to $(P(t))_{t\geq 0}$) if for every $(P(t))_{t\geq 0}$-invariant set E, $\mu(E) = 0$ or 1. Note that there are several equivalent definitions of ergodic measures; these are related in [22, Theorem 4.4 and Theorem 4.5].

With $(P(t))_{t\geq 0}$ we can associate a regular Markov operator R:

$$R\mu := \int_{\mathbb{R}_+} e^{-t} P(t)\mu \, dt, \tag{2}$$

the *resolvent operator*.

In [22] it is shown that a probability measure is $(P(t))_{t\geq 0}$-invariant if and only if it is R-invariant and $(P(t))_{t\geq 0}$-ergodic if and only if it is R-ergodic.

2.3 Ergodic decompositions

Here we summarise some definitions and results on the ergodic decompositions defined in [21] and [22].

2.3.1 Ergodic decompositions associated to Markov operators

Let P be a regular Markov operator. We define

$$\Gamma_t^P = \{x \in S : \{P^{(n)}\delta_x : n \in \mathbb{N}\} \text{ is tight}\}$$

and

$$\Gamma_{cp}^P = \{x \in S : (P^{(n)}\delta_x)_n \text{ converges in } \mathcal{S}_{\mathrm{BL}}.\}$$

For $x \in \Gamma^P_{cp}$ we define $\epsilon_x = \lim_{n\to\infty} P^{(n)}\delta_x$, then $\epsilon_x \in \mathcal{P}(S)$. Then we let

$$\Gamma^P_{cpi} = \{x \in S : \epsilon_x \text{ is invariant}\}$$

and

$$\Gamma^P_{cpie} = \{x \in S : \epsilon_x \text{ is ergodic.}\}$$

In a similar manner we can define subsets of $\mathcal{P}(S)$:

$$\mathcal{P}^P_t = \{\mu \in \mathcal{P}(S) : \{P^{(n)}\mu : n \in \mathbb{N}\} \text{ is tight}\}$$

and

$$\mathcal{P}^P_{cp} = \{\mu \in \mathcal{P}(S) : (P^{(n)}\mu)_n \text{ converges in } \mathcal{S}_{\mathrm{BL}}.\}$$

For $\mu \in \mathcal{P}^P_{cp}$ we define $\epsilon_\mu = \lim_{n\to\infty} P^{(n)}\mu$ Then we let

$$\mathcal{P}^P_{cpi} = \{\mu \in \mathcal{P}(S) : \epsilon_\mu \text{ is invariant}\}$$

and

$$\mathcal{P}^P_{cpie} = \{\mu \in \mathcal{P}(S) : \epsilon_\mu \text{ is ergodic.}\}$$

Clearly $\Gamma^P_t \supset \Gamma^P_{cp} \supset \Gamma^P_{cpi} \supset \Gamma^P_{cpie}$ and $\mathcal{P}^P_t \supset \mathcal{P}^P_{cp} \supset \mathcal{P}^P_{cpi} \supset \mathcal{P}^P_{cpie}$. If P is Markov-Feller, $\Gamma^P_{cp} = \Gamma^P_{cpi}$. In [21] the following is shown:

Theorem 2.10. *$\Gamma^P_t, \Gamma^P_{cp}, \Gamma^P_{cpi}$ and Γ^P_{cpie} are Borel sets and*

$$\mu(\Gamma^P_t) = \mu(\Gamma^P_{cp}) = \mu(\Gamma^P_{cpi}) = \mu(\Gamma^P_{cpie}) = 1$$

for every invariant probability measure μ.

On Γ^P_{cpie} an equivalence relation $\sim$ is defined as follows: $x \sim y$ whenever $\epsilon_x = \epsilon_y$. We write $[x]$ to denote the equivalence class of $x \in \Gamma^P_{cpie}$. The following result comes from [21, Theorem 4.6]. It implies that we can decompose Γ^P_{cpie} into disjoint Borel measurable subsets, such that each ergodic measure has full measure on exactly one of these subsets.

Theorem 2.11. *(i) For every $x \in \Gamma^P_{cpie}$ the set $[x]$ is Borel measurable, ϵ_x is an ergodic probability measure and $\epsilon_x([x]) = 1$.*

(ii) Any ergodic measure μ is of the form $\mu = \epsilon_x$ for some $x \in \Gamma^P_{cpie}$.

Using the characterisation we obtain an integral decomposition of invariant probability measures in terms of ergodic measures ([21, Theorem 4.10]).

Theorem 2.12. *Let μ be an invariant probability measure. Then the map*

$$x \mapsto \begin{cases} \epsilon_x & \text{if } x \in \Gamma^P_{cpie} \\ 0 & \text{if } x \notin \Gamma^P_{cpie} \end{cases} \tag{3}$$

is strongly measurable from S to $\mathcal{S}_{\mathrm{BL}}$ and

$$\mu = \int_{\Gamma^P_{cpie}} \epsilon_x \, d\mu.$$

2.3.2 Ergodic decomposition associated to Markov semigroups

For Markov semigroups we can also achieve an ergodic decomposition very similar to those stated in the previous section for Markov operators.

Let $\mathbf{P} = (P(t))_{t\geq 0}$ be a regular jointly measurable Markov semigroup. We define

$$\Gamma_t^{\mathbf{P}} = \{x \in S : \{P^{(t)}\delta_x : t \in \mathbb{R}_{\geq 1}\} \text{ is tight}\}$$

and

$$\Gamma_{cp}^{\mathbf{P}} = \{x \in S : P^{(t)} \text{ converges in } \mathcal{S}_{\mathrm{BL}} \text{ as } t \to \infty.\}$$

For $x \in \Gamma_{cp}^{\mathbf{P}}$ we define $\varepsilon_x = \lim_{t\to\infty} P^{(t)}\delta_x$, then $\varepsilon_x \in \mathcal{P}(S)$. Notice that we distinguish this measure in notation from the Markov operator analogue ϵ_x.

Then we let

$$\Gamma_{cpi}^{\mathbf{P}} = \{x \in S : \varepsilon_x \text{ is invariant}$$

and

$$\Gamma_{cpie}^{\mathbf{P}} = \{x \in S : \varepsilon_x \text{ is ergodic.}$$

Clearly $\Gamma_t^{\mathbf{P}} \supset \Gamma_{cp}^{\mathbf{P}} \supset \Gamma_{cpi}^{\mathbf{P}} \supset \Gamma_{cpie}^{\mathbf{P}}$. If $(P(t))_{t\geq 0}$ is Markov-Feller, $\Gamma_{cp}^{\mathbf{P}} = \Gamma_{cpi}^{\mathbf{P}}$.

Similarly the sets $\mathcal{P}_\bullet^{\mathbf{P}}$ can be defined where $\bullet = t, cp, cpi, cpie$.

In [22] an important connection between the Markov semigroup and its resolvent R is shown:

Proposition 2.13.

$$\Gamma_\bullet^{\mathbf{P}} = \Gamma_\bullet^R \text{ and } \mathcal{P}_\bullet^{\mathbf{P}} = \mathcal{P}_\bullet^R$$

where $\bullet = t, cp, cpi, cpie$*.*

In particular it implies that all the results mentioned about the ergodic decomposition associated to Markov operators holds for Markov semigroups as well.

3 A new result on weak convergence in $\mathcal{S}_{\mathrm{BL}}$

The following theorem is crucial. Its proof is based on results by Pachl [17].

Theorem 3.1. *Let $(\mu_n)_n \subset \mathcal{M}(S)$ and $M \geq 0$ be such that $\langle \mu_n, f\rangle$ converges as $n \to \infty$ for every $f \in \mathrm{BL}(S) = \mathcal{S}_{\mathrm{BL}}^*$ and*

$$\|\mu_n\|_{\mathrm{TV}} \leq M \text{ for every } n \in \mathbb{N}.$$

Then there exists a $\mu \in \mathcal{M}(S)$ such that $\|\mu_n - \mu\|_{\mathrm{BL}}^* \to 0$ as $n \to \infty$.

Proof. In [17] the norm $\|\cdot\|_d$ is defined on $\mathcal{M}(S)$ as follows

$$\|\nu\|_d = \sup\{|\langle \mu, f\rangle| : f \in \mathrm{BL}(S) \text{ such that } \|f\|_\infty \leq 1 \text{ and } |f|_{\mathrm{Lip}} \leq 1.$$

It is easy to see that $\|\nu\|_{BL}^* \leq \|\nu\|_d \leq 2\|\nu\|_{\mathrm{BL}}^*$ for every $\nu \in \mathcal{M}(S)$, so the norms are equivalent. It follows from [17, Theorem 3.2(b)] that the set $\{\mu_n : n \in \mathbb{N}\}$

is $\|\cdot\|_d$-precompact (hence $\|\cdot\|_{\mathrm{BL}}^*$-precompact) whenever $\langle \mu_n, g\rangle$ converges as $n \to \infty$ for every $g \in C_{ub}(S)$.

Fix $g \in C_{ub}(S)$ and $\epsilon > 0$. By [4, Lemma 8] $\mathrm{BL}(S)$ is dense in $C_{ub}(S)$, so there exists an $f \in \mathrm{BL}(S)$ with $\|f-g\|_\infty \le \epsilon$. For every $m, n \in \mathbb{N}$

$$\begin{aligned} |\langle \mu_m, g\rangle - \langle \mu_n, g\rangle| &\le |\langle \mu_m, g-f\rangle| + |\langle \mu_n, g-f\rangle| + |\langle \mu_m, f\rangle - \langle \mu_n, f\rangle| \\ &\le 2M\epsilon + |\langle \mu_m, f\rangle - \langle \mu_n, f\rangle|. \end{aligned}$$

Since $\langle \mu_n, f\rangle$ converges as $n \to \infty$, this implies that $(\langle \mu_n, g\rangle)_n$ is Cauchy and thus converges for every $g \in C_{ub}(S)$. So by [17, Theorem 3.2(a)] there is a $\mu \in \mathcal{M}(S)$ such that

$$\lim_{n\to\infty} \langle \mu_n, g\rangle \to \langle \mu, g\rangle$$

for every $g \in C_{ub}(S)$.

Also, $\{\mu_n : n \in \mathbb{N}\}$ is precompact in $\mathcal{S}_{\mathrm{BL}}$. Suppose that μ_n does not converge to μ in $\mathcal{S}_{\mathrm{BL}}$, then there is a subsequence $(\mu_{n_k})_k$ and an $\epsilon > 0$ such that $\|\mu_{n_k} - \mu\|_{\mathrm{BL}}^* \ge \epsilon$ for every $k \in \mathbb{N}$. By precompactness, there exists a subsequence of $(\mu_{n_k})_k$, that we will also denote by $(\mu_{n_k})_k$, that converges in $\mathcal{S}_{\mathrm{BL}}$, say to $\varphi \in \mathcal{S}_{\mathrm{BL}}$. Then for every $f \in \mathrm{BL}(S)$

$$\langle \mu, f\rangle = \lim_{k\to\infty} \langle \mu_{n_k}, f\rangle = \varphi(f).$$

Since an element in $\mathcal{S}_{\mathrm{BL}}$ is determined uniquely by parings with all elements in $\mathrm{BL}(S) = \mathcal{S}_{\mathrm{BL}}^*$, $\varphi = \mu$. Thus $\|\mu_{n_k} - \mu\|_{\mathrm{BL}}^* \to 0$ which gives a contradiction. □

4 Equicontinuous families of Markov operators

We start by giving a sufficient condition for a regular Markov operator to be Markov-Feller:

Lemma 4.1. *Let P be a regular Markov operator such that $Uf \in C_b(S)$ for every $f \in \mathrm{BL}(S)$, then P is Markov-Feller.*

Proof. By Proposition 2.7, it suffices to show that $P : \mathcal{S}_{\mathrm{BL}}^+ \to \mathcal{S}_{\mathrm{BL}}^+$ is continuous. Let $(\mu_n)_n, \mu$ be in $\mathcal{M}^+(S)$ such that $\|\mu_n - \mu\|_{\mathrm{BL}}^* \to 0$. Let $f \in \mathrm{BL}(S)$. By Lemma 2.1,

$$\langle P\mu_n, f\rangle = \langle \mu_n, Uf\rangle \to \langle \mu, Uf\rangle = \langle P\mu, f\rangle$$

as $n \to \infty$. Note that $\|P\mu_n\|_{\mathrm{TV}} = \|\mu_n\|_{\mathrm{TV}} = \|\mu_n\|_{\mathrm{BL}}^*$ is bounded as n ranges in $\mathbb{N}$, since μ_n converges in $\mathcal{S}_{\mathrm{BL}}$. Theorem 3.1 yields $\|P\mu_n - P\mu\|_{\mathrm{BL}}^* \to 0$. □

Let $H \subset C_b(T, X)$, where T is a topological space and (X, d) a metric space. We say that H is *equicontinuous at* $t \in T$, if for every $\epsilon > 0$ there is an open set $U \subset T$ containing t, such that $d(f(t), f(s)) < \epsilon$ for every $s \in U$. H is an *equicontinuous family* if it is equicontinuous at every $t \in T$. The following result gives several equivalent condition for a family of regular Markov operators to be equicontinuous:

Theorem 4.2. *Let $(P_\lambda)_{\lambda\in\Lambda}$ be a family of regular Markov operators on S. Let U_λ be the dual of P_λ. The following statements are equivalent:*

(i) $(P_\lambda)_{\lambda\in\Lambda}$ *is an equicontinuous family in* $C_b(\mathcal{S}^+_{\mathrm{BL}}, \mathcal{S}^+_{\mathrm{BL}})$.

(ii) *For any topological space* T *and continuous map* $\Phi : T \to \mathcal{S}^+_{\mathrm{BL}}$, $(P_\lambda \circ \Phi)_{\lambda\in\Lambda}$ *is an equicontinuous family in* $C_b(T, \mathcal{S}^+_{\mathrm{BL}})$.

(iii) *For every* $f \in \mathrm{BL}(S)$, $(U_\lambda f)_{\lambda\in\Lambda}$ *is an equicontinuous family in* $C_b(S)$.

Proof. *(i)⇒ (ii)*: This is trivial.
(ii)⇒(iii): Let $T = S$, $\Phi = \delta : x \mapsto \delta_x, S \to \mathcal{S}^+_{\mathrm{BL}}$. If $f = 0$, the statement is trivial. So assume $f \neq 0$ and let $\epsilon > 0$. There exists a $\delta > 0$ such that for all $y \in S$ such that $d(x, y) < \delta$,

$$\|P_\lambda\delta_x - P_\lambda\delta_y\|^*_{\mathrm{BL}} < \frac{\epsilon}{\|f\|_{\mathrm{BL}}}, \text{ for all } \lambda \in \Lambda.$$

Then

$$|U_\lambda f(x) - U_\lambda f(y)| \quad = \quad |\langle P_\lambda\delta_x, f\rangle - \langle P_\lambda\delta_y, f\rangle| \leq \|P_\lambda\delta_x - P_\lambda\delta_y\|^*_{\mathrm{BL}}\|f\|_{\mathrm{BL}} < \epsilon.$$

(iii)⇒(i): Suppose not. Then there exists a $\mu \in \mathcal{S}^+_{\mathrm{BL}} = \mathcal{M}^+(S)$ where $(P_\lambda)_{\lambda\in\Lambda}$ is not equicontinuous. Thus there exists an $\epsilon > 0$, $\mu_k \in \mathcal{M}^+(S)$ and $\lambda_k \in \Lambda$ such that $\|\mu_k - \mu\|^*_{\mathrm{BL}} \leq \frac{1}{k}$ and

$$\|P_{\lambda_k}\mu_k - P_{\lambda_k}\mu\|^*_{\mathrm{BL}} \geq \epsilon$$

for every $k \in \mathbb{N}$. By Lemma 2.1 we know that μ_k converges to μ weak-star. This implies by [4, Theorem 7] that μ_k converges to μ uniformly on any equicontinuous and uniformly bounded class of functions on S.

Take $f \in \mathrm{BL}(S)$. By assumption, $(U_{\lambda_k} f)_k$ is an equicontinuous and uniformly bounded class of functions, thus

$$|\langle P_{\lambda_k}\mu_k - P_{\lambda_k}\mu, f\rangle| = |\langle \mu_k - \mu, U_{\lambda_k} f\rangle| \to 0.$$

So by Theorem 3.1 we obtain that $\|P_{\lambda_k}\mu_k - P_{\lambda_k}\mu\|^*_{\mathrm{BL}} \to 0$ which gives a contradiction. □

Note that condition *(iii)* in the theorem above seems to be far weaker than condition *(i)*, but they are actually equivalent. For a regular Markov operator P we write $P\delta$ to denote the map from S to $\mathcal{S}_{\mathrm{BL}}$ given by $P\delta(x) = P\delta_x$. Note that $P\delta$ is the composition of the continuous map $\delta : x \mapsto \delta_x$ and P, so Theorem 4.2 implies:

Corollary 4.3. *Let* $(P_\lambda)_{\lambda\in\Lambda}$ *be a family of regular Markov operators on* $\mathcal{M}^+(S)$ *with duals* $(U_\lambda)_{\lambda\in\Lambda}$, *such that* $(U_\lambda f)_{\lambda\in\Lambda}$ *is equicontinuous for every* $f \in \mathrm{BL}(S)$. *Then* $(P_\lambda\delta)_{\lambda\in\Lambda}$ *is equicontinuous as well.*

We now prove some interesting properties on certain sets associated to equicontinuous families of Markov operators. These results will turn out to be useful when we discuss ergodic decompositions associated to equicontinuous Markov operators or Markov semigroups in the next section.

Theorem 4.4. *Let $(P_\lambda)_{\lambda\in\Lambda}$ be a family of regular Markov operators on $\mathcal{M}^+(S)$ with duals $(U_\lambda)_{\lambda\in\Lambda}$, such that $(U_\lambda f)_{\lambda\in\Lambda}$ is equicontinuous for every $f \in \mathrm{BL}(S)$. Let*

$$\mathcal{P}_t^\Lambda := \{\mu \in \mathcal{P}(S) : (P_\lambda \mu)_{\lambda\in\Lambda} \text{ is tight}\}$$

and

$$\Gamma_t^\Lambda := \{x \in S : \delta_x \in \mathcal{P}_t^\Lambda\}.$$

Then $\mathcal{P}_t^\Lambda$ is closed in $\mathcal{S}_{\mathrm{BL}}$ and Γ_t^Λ is closed in S.

Proof. The map $\delta : S \to \mathcal{S}_{\mathrm{BL}}, x \mapsto \delta_x$ is continuous. If $\mathcal{P}_t^\Lambda$ is closed in $\mathcal{S}_{\mathrm{BL}}$, then $\Gamma_t^\Lambda = \delta^{-1}(\mathcal{P}_t^\Lambda)$ is closed in S.

If $\mathcal{P}_t^\Lambda = \emptyset$, the statements hold. If $\mathcal{P}_t^\Lambda \neq \emptyset$, let $(\mu_n)_n \in \mathcal{P}_t^\Lambda$ such that $\mu_n \to \mu$ for some $\mu \in \mathcal{S}_{\mathrm{BL}}$, then $\mu \in \mathcal{P}(S)$. Let D be a countable dense subset of S, and $\mathcal{F}$ the collection of finite subsets of D, then $\mathcal{F}$ is countable and by Theorem 2.5 we can write

$$\mathcal{P}_t^\Lambda = \cap_{m\in\mathbb{N}} \cup_{F\in\mathcal{F}} \cap_{\lambda\in\Lambda} \left\{ \nu \in \mathcal{P}(S) : \langle P_\lambda \nu, f_F^{\frac{1}{m}} \rangle \geq 1 - \frac{1}{m} \right\}.$$

Fix $m \in \mathbb{N}$. We will show that there exists an $F \in \mathcal{F}$ such that for every $\lambda \in \Lambda$,

$$\langle P_\lambda \mu, f_F^{\frac{1}{m}} \rangle \geq 1 - \frac{1}{m}.$$

By Theorem 4.2, $(P_\lambda)_{\lambda\in\Lambda}$ is equicontinuous from $\mathcal{S}_{\mathrm{BL}}^+$ to $\mathcal{S}_{\mathrm{BL}}^+$, so there exists an $n_0 \in \mathbb{N}$, such that for all $\lambda \in \Lambda$,

$$\|P_\lambda \mu_{n_0} - P_\lambda \mu\|_{\mathrm{BL}}^* < \frac{1}{2m(2m+1)}.$$

Since $\mu_{n_0} \in \mathcal{P}_t^\Lambda$, there exists an $F_0 \in \mathcal{F}$, such that

$$\langle P_\lambda \mu_{n_0}, f_{F_0}^{\frac{1}{2m}} \rangle \geq 1 - \frac{1}{2m} \text{ for every } \lambda \in \Lambda.$$

Now, $\|f_{F_0}^{\frac{1}{2m}}\|_{\mathrm{BL}}^* \leq 1 + |f_{F_0}^{\frac{1}{2m}}|_{\mathrm{Lip}} \leq 1 + 2m$. So

$$|\langle P_\lambda \mu_{n_0} - P_\lambda \mu, f_{F_0}^{\frac{1}{2m}} \rangle| < \frac{1}{2m} \text{ for every } \lambda \in \Lambda.$$

Thus, for every $\lambda \in \Lambda$ we have

$$\begin{aligned} \langle P_\lambda \mu, f_{F_0}^{\frac{1}{m}} \rangle &\geq \langle P_\lambda \mu, f_{F_0}^{\frac{1}{2m}} \rangle \\ &\geq \langle P_\lambda \mu_{n_0}, f_{F_0}^{\frac{1}{2m}} \rangle - \frac{1}{2m} \geq 1 - \frac{2}{2m} = 1 - \frac{1}{m}. \end{aligned}$$

So $\mu \in \mathcal{P}_t^\Lambda$. $\square$

Theorem 4.5. *Let $(P_\lambda)_{\lambda\in\Lambda}$ be a family of regular Markov operators on $\mathcal{M}^+(S)$ with duals $(U_\lambda)_{\lambda\in\Lambda}$, such that $(U_\lambda f)_{\lambda\in\Lambda}$ is equicontinuous for every $f \in \mathrm{BL}(S)$. Let*

$$\mathcal{P}_t^\Lambda := \{\mu \in \mathcal{P}(S) : (P_\lambda\mu)_{\lambda\in\Lambda} \text{ is tight}\}$$

and

$$\Gamma_t^\Lambda := \{x \in S : \delta_x \in \mathcal{P}_t^\Lambda\}.$$

Then for every $\mu \in \mathcal{P}_t^\Lambda$,

$$\mathrm{supp}(\mu) \subset \Gamma_t^\Lambda.$$

Proof. If $\mathcal{P}_t^\Lambda = \emptyset$, the statement holds. If $\mathcal{P}_t^\Lambda \neq \emptyset$, let $\mu \in \mathcal{P}_t^\Lambda$ and $x \in \mathrm{supp}(\mu)$. Suppose that $x \notin \Gamma_t^\Lambda$. Then $(P_\lambda\delta_x)_{x\in\Lambda}$ is not tight, and there exists a sequence $(\lambda_n)_n \subset \Lambda$ such that $(P_{\lambda_n}\delta_x)_n$ does not have any convergent subsequences. Then $(P_{\lambda_n}\delta_x)_n$ is not tight.

Let $D \subset S$ be countable and dense, and define $\mathcal{F}$ to be the collection of finite subsets of D. Then $\mathcal{F}$ is countable. By Theorem 2.5 there exists an $m \in \mathbb{N}$ such that for every $F \in \mathcal{F}$ there is a strictly increasing sequence $(n_k)_k \subset \mathbb{N}$ for which

$$U_{\lambda_{n_k}} f_F^{\frac{1}{m}}(x) = \langle P_{\lambda_{n_k}}\delta_x, f_F^{\frac{1}{m}}\rangle \le 1 - \frac{1}{m}.$$

By Corollary 4.3 we know that $(P_\lambda\delta)_{\lambda\in\Lambda}$ is equicontinuous in x, so in particular there exists a $\delta > 0$ such that for every $z \in B_x(\delta)$ and $\lambda \in \Lambda$,

$$\|P_\lambda\delta_x - P_\lambda\delta_z\|_{\mathrm{BL}}^* \le \frac{1}{2m(m+1)}.$$

Since $x \in \mathrm{supp}(\mu)$, $\alpha := \mu(B_z(\delta)) > 0$.

By assumption $(P_{\lambda_{n_k}}\mu)_k$ is tight, so it can be shown using Theorem 2.5 that there exists an $F \in \mathcal{F}$ such that

$$\langle P_{\lambda_{n_k}}\mu, f_F^{\frac{1}{m}}\rangle > 1 - \frac{\alpha}{2m} \text{ for every } k \in \mathbb{N}. \tag{4}$$

Now we have for every $z \in B_x(\delta), k \in \mathbb{N}$, that

$$\begin{aligned}
U_{\lambda_{n_k}} f_F^{\frac{1}{m}}(z) &\le U_{\lambda_{n_k}} f_F^{\frac{1}{m}}(x) + |U_{\lambda_{n_k}} f_F^{\frac{1}{m}}(z) - U_{\lambda_{n_k}} f_F^{\frac{1}{m}}(x)| \\
&\le 1 - \frac{1}{m} + \|P_\lambda\delta_z - P_\lambda\delta_x\|_{\mathrm{BL}}^* \|f\|_F^{\frac{1}{m}} \\
&\le 1 - \frac{1}{m} + \frac{1}{2m} = 1 - \frac{1}{2m},
\end{aligned}$$

since $\|f_F^{\frac{1}{m}}\|_{\mathrm{BL}}^* \le m+1$. So for every $k \in \mathbb{N}$ we obtain

$$U_{\lambda_{n_k}} f_F^{\frac{1}{m}} \le \mathbb{1}_{B_x(\delta)}(1 - \frac{1}{2m}) + \mathbb{1}_{S\setminus B_x(\delta)}.$$

Therefore,

$$\langle \mu, U_{\lambda_{n_k}} f_F^{\frac{1}{m}}\rangle \le \alpha(1 - \frac{1}{2m}) + 1 - \alpha = 1 - \frac{\alpha}{2m}$$

for every $k \in \mathbb{N}$, which contradicts (4). $\square$

Let $P : \mathcal{M}^+(S) \to \mathcal{M}^+(S)$ be a regular Markov operator. Following [12], we say that P has the *e-property* if $(U^n f)_{n\in\mathbb{N}_0}$ is equicontinuous for every $f \in \mathrm{BL}(S)$. Every such Markov operator is automatically Markov-Feller by Lemma 4.1.

Remark 4.6. Note that P having the e-property depends on the metric d. There may be many metrics that metrise the topology on S, and under which S becomes a complete separable metric space. It would be interesting to be able to find sufficient conditions on the topology of S, and on the Markov operator P, such that there exists a metric metrising the topology and such that P satisfies the e-property.

P has the *Cesàro e-property* if $(U^{(n)} f)_{n\in\mathbb{N}}$ is equicontinuous for every $f \in \mathrm{BL}(S)$. Note that every Markov operator with the e-property has the Cesàro e-property, though the converse statement need not be true in general. By Theorem 4.2, P has the e-property if and only if $(P^n)_{n\in\mathbb{N}_0}$ is an equicontinuous family in $C_b(\mathcal{S}^+_{\mathrm{BL}}, \mathcal{S}^+_{\mathrm{BL}})$, and the Cesàro e-property if and only if $(P^{(n)})_{n\in\mathbb{N}}$ is an equicontinuous family in $C_b(\mathcal{S}^+_{\mathrm{BL}}, \mathcal{S}^+_{\mathrm{BL}})$.

Let $\mathbf{P} = (P(t))_{t\geq 0}$ be a regular jointly measurable Markov-Feller semigroup. We say $\mathbf{P}$ has the *eventual e-property* if there is a $\tau \geq 0$ such that $(U(t)f)_{t\geq\tau}$ is equicontinuous for every $f \in \mathrm{BL}(S)$ and it has the *e-property* if τ can be chosen to be zero. Note that the corresponding definition of eventual e-property of Markov operators coincides with the definition of e-property, because a finite union of equicontinuous families of functions is equicontinuous.

$\mathbf{P}$ has the *Cesàro e-property* if there is a $\tau > 0$ such that $(\mathbf{U}^{(t)} f)_{t\geq\tau}$ is equicontinuous for every $f \in \mathrm{BL}(S)$. Since $t \mapsto \mathbf{P}^{(t)}$ is continuous from $(0,\infty)$ to $\mathcal{L}(\mathcal{M}(S)_{\mathrm{TV}})$ (endowed with the operator norm) according to [22, Corollary 3.9], the map $t \mapsto \mathbf{U}^{(t)} f$ is continuous from $(0,\infty)$ to $(\mathrm{BM}(S), \|\cdot\|_\infty)$. Using this it can easily be shown that equicontinuity of $(\mathbf{U}^{(t)} f)_{t\geq\tau}$ for some $\tau > 0$ implies equicontinuity of $(\mathbf{U}^{(t)} f)_{t\geq\tau'}$ for any $\tau' > 0$.

Lemma 4.7. *If* $\mathbf{P}$ *has the eventual e-property, then it has the Cesàro e-property.*

Proof. Let $\tau > 0$ be such that $(U(t)f)_{t\geq\tau}$ is equicontinuous for every $f \in \mathrm{BL}(S)$. Let $t \geq \tau$. Then for every $x \in S$,

$$\mathbf{U}^{(t)} f(x) = \frac{1}{t}\int_0^\tau U(s)f(x)\,ds + \frac{1}{t}\int_\tau^t U(s)f(x)\,ds.$$

Note that $x \mapsto \int_0^\tau U(s)f(x)\,ds$ is a bounded continuous function.

Fix $x \in S$ and $\epsilon > 0$. Then there is a $\delta > 0$ such that $d(x,y) < \delta$ implies $|U(s)f(x) - U(s)f(y)| < \epsilon$ for every $s \geq \tau$ and

$$\left|\int_0^\tau U(s)f(x)\,ds - \int_0^\tau U(s)f(y)\,ds\right| < \epsilon.$$

Then for every $t \geq \tau$ and $y \in B_x(\delta)$,

$$|\mathbf{U}^{(t)} f(x) - \mathbf{U}^{(t)} f(y)| \leq \frac{1}{t}\epsilon + \frac{t-\tau}{t}\epsilon < (\frac{1}{\tau} + 1)\epsilon.$$

This implies that $(\mathbf{U}^{(t)} f)_{t\geq\tau}$ is equicontinuous. □

Proposition 4.8. *(i) If $\mathbf{P}$ has the Cesàro e-property, then the resolvent R has the Cesàro e-property.*

(ii) If $\mathbf{P}$ has the eventual e-property, then R has the e-property.

Proof. Let V be the dual operator of the regular Markov operator R (cf.(2)). *(i)* It was shown in [22, Corollary 3.8] that

$$\lim_{n\to\infty} \|V^{(n)} f - \mathbf{U}^{(n)} f\|_\infty = 0.$$

Suppose $\mathbf{P}$ has the Cesàro e-property and let $f \in \mathrm{BL}(S)$ and $x \in S$. Fix $\epsilon > 0$. Then there is a $\delta > 0$ such that $|\mathbf{U}^{(n)} f(x) - \mathbf{U}^{(n)} f(y)| < \epsilon$ for every $y \in B_x(\delta)$ and $n \in \mathbb{N}$. There is also an $N \in \mathbb{N}$ such that $\|V^{(n)} f - \mathbf{U}^{(n)} f\|_\infty < \epsilon$ for every $n \geq N$. This implies that $|V^{(n)} f(x) - V^{(n)} f(y)| < 3\epsilon$ for every $y \in B_x(\delta)$ and $n \geq N$. Since $\{V^{(n)} f : 1 \leq n \leq N-1\}$ is equicontinuous, there is a $0 < \delta' < \delta$ such that $|V^{(n)} f(x) - V^{(n)} f(y)| < 3\epsilon$ for every $y \in B_x(\delta')$. This implies that $(V^{(n)} f)_{n\in\mathbb{N}}$ is equicontinuous at x.

(ii): In [22, Lemma 3.5] we obtained that

$$R^n \mu = \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-t} P(t)\mu \, dt.$$

Consequently, for every $f \in \mathrm{BM}(S)$ and $x \in S$,

$$V^n f(x) = \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-t} U(t) f(x) \, dt.$$

Suppose $\mathbf{P}$ has the eventual e-property, then there is a $\tau > 0$ such that $(U(t)f)_{t\geq\tau}$ is equicontinuous for every $f \in \mathrm{BL}(S)$. Fix $f \in \mathrm{BL}(S)$, $x \in S$ and $\epsilon > 0$. Then there is a $\delta > 0$ such that $|U(t)f(x) - U(t)f(y)| < \epsilon$ for every $y \in B_x(\delta)$ and $t \geq \tau$. Now, for every $y \in B_x(\delta)$ we have

$$\begin{aligned}
|V^n f(x) - V^n f(y)| &\leq \int_0^\tau \frac{t^{n-1}}{(n-1)!} e^{-t} |U(t)f(x) - U(t)f(y)| \, dt \\
&\quad + \int_\tau^\infty \frac{t^{n-1}}{(n-1)!} e^{-t} |U(t)f(x) - U(t)f(y)| \, dt \\
&\leq 2\|f\|_\infty \int_0^\tau \frac{t^{n-1}}{(n-1)!} e^{-t} \, dt + \epsilon \int_\tau^\infty \frac{t^{n-1}}{(n-1)!} e^{-t} \, dt \\
&= 2\|f\|_\infty e^{-\tau} \sum_{k=n}^\infty \frac{\tau^k}{k!} + \epsilon e^{-\tau} \sum_{k=0}^{n-1} \frac{\tau^k}{k!}.
\end{aligned}$$

The first term goes to zero as $n \to \infty$ and the second term is bounded from above by ϵ, so there is an $N \in \mathbb{N}$ such that $|V^n f(x) - V^n f(y)| < 2\epsilon$ for every $n \geq N$ and $y \in B_x(\delta)$. Since $\{V^n f : 0 \leq n \leq N-1\}$ is equicontinuous, there is a $0 < \delta' < \delta$ such that $|V^n f(x) - V^n f(y)| < 2\epsilon$ for every $y \in B_x(\delta')$. This implies that $(V^n f)_{n\in\mathbb{N}}$ is equicontinuous at x. □

The strong Feller property is often used in combination with irreducibility ([2, p. 42]) to show uniqueness of invariant measures for Markov operators and

Markov semigroups, see e.g. [2]. A regular Markov operator P is called *strong Feller* if $U(\mathrm{BM}(S)) \subset C_b(S)$ and eventually strong Feller if there is an $N \in \mathbb{N}$ such that $U^N(\mathrm{BM}(S)) \subset C_b(S)$. Note that $U^N(\mathrm{BM}(S)) \subset C_b(S)$ implies that $U^n(\mathrm{BM}(S)) \subset C_b(S)$ for every $n \geq N$. P is called *ultra Feller* if $x \mapsto P\delta_x$ is continuous from S to $\mathcal{M}(S)_{\mathrm{TV}}$. Obviously, when P is ultra Feller, it is also strong Feller, though not necessary the other way around. However, it is a remarkable fact that if P and Q are both strong Feller operators, then PQ is ultra Feller [3, Théorème IX.18] (see also [18]).

A Markov semigroup $(P(t))_{t\geq 0}$ is *strong Feller* if $P(t)$ is strong Feller for every $t > 0$ and *eventually strong Feller* if there is a $\tau > 0$ such that $P(\tau)$ is strong Feller, which implies that $P(t)$ is strong Feller for every $t \geq \tau$. Since $P(t) = P(t/2)P(t/2)$ $P(t)$ is ultra Feller for every $t > 0$.

Proposition 4.9. *An eventually strong Feller Markov operator P has the e-property. An eventually strong Feller Markov semigroup has the eventual e-property.*

Proof. Let P be an eventually strong Feller Markov operator, then there is an $N \in \mathbb{N}$ such that P^N is strong Feller. Then P^{2N} is ultra Feller. For every $n \in \mathbb{N}$ we have

$$\|P^{n+2N}\delta_x - P^{n+2N}\delta_y\|_{\mathrm{TV}} \leq \|P^{2N}\delta_x - P^{2N}\delta_y\|_{\mathrm{TV}}.$$

This implies that for every $f \in \mathrm{BM}(S)$, $(U^n f)_{n\geq 2N}$ is equicontinuous. Thus $(U^n f)_{n\in\mathbb{N}}$ is equicontinuous for every $n \in \mathbb{N}$.

The proof for the Markov semigroup case proceeds in an analogous manner. $\square$

5 Ergodic decomposition of Markov operators and semigroups with Cesàro e-property

In this section we show that the ergodic decomposition associated to Markov operators and semigroups that we summarised in Section 2.3 has better properties when we assume the Cesàro e-property. We start by considering a regular Markov operator P with the Cesàro e-property.

Theorem 5.1. *Let P be a regular Markov operator that satisfies the Cesàro e-property. Then $\mathcal{P}_t^P = \mathcal{P}_{cp}^P$ and consequently also $\Gamma_t^P = \Gamma_{cp}^P$.*

Proof. It is clear that $\mathcal{P}_{cp}^P \subset \mathcal{P}_t^P$. Let $\mu \in \mathcal{P}_t^P$. Define

$$D_\mu := \{\nu \in \mathcal{P}(S) : \lim_{n\to\infty} \|P^{(n)}\mu - P^{(n)}\nu\|_{\mathrm{BL}}^* \to 0\}.$$

Step 1. D_μ is closed in $\mathcal{S}_{\mathrm{BL}}$.

We can write

$$D_\mu = \bigcap_{m\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \{\rho \in \mathcal{P}(S) : \|P^{(n)}\mu - P^{(n)}\rho\|_{\mathrm{BL}}^* \leq \frac{1}{m} \text{ for all } n \geq N\}.$$

Let $\nu_n \in D_\mu$ such that $\|\nu_n - \nu\|_{\mathrm{BL}}^* \to 0$ for some $\nu \in \mathcal{S}_{\mathrm{BL}}$, then $\nu \in \mathcal{P}(S)$, since $\mathcal{P}(S)$ is closed in $\mathcal{S}_{\mathrm{BL}}$.

Fix $m \in \mathbb{N}$. By Theorem 4.2, $(P^{(n)})_{n\in\mathbb{N}}$ is an equicontinuous family in $C_b(\mathcal{S}_{\mathrm{BL}}^+, \mathcal{S}_{\mathrm{BL}}^+)$. Thus there exists a $k \in \mathbb{N}$ such that

$$\|P^{(n)}\nu_k - P^{(n)}\nu\|_{\mathrm{BL}}^* \leq \frac{1}{2m} \text{ for every } n \in \mathbb{N}.$$

Since $\nu_k \in D_\mu$, there exists $N_0 \in \mathbb{N}$ such that

$$\|P^{(n)}\nu_k - P^{(n)}\mu\|_{\mathrm{BL}}^* \leq \frac{1}{2m} \text{ for every } n \geq N_0.$$

Thus for every $m \in \mathbb{N}$,

$$\nu \in \bigcup_{N\in\mathbb{N}} \{\rho \in \mathcal{P}(S) : \|P^{(n)}\mu - P^{(n)}\rho\|_{\mathrm{BL}}^* \leq \frac{1}{m} \text{ for all } n \geq N\}.$$

So $\nu \in D_\mu$.

Step 2. $P^{(n)}\mu \in \mathcal{D}_\mu$ *for every* $n \in \mathbb{N}$.

Let $\rho = \nu - P\nu$ for some $\nu \in \mathcal{M}^+(S)$. It is easy to see that $P^{(n)}\rho = \frac{1}{n}\rho - \frac{1}{n}P^n\rho$ for every $n \in \mathbb{N}$. Thus $\lim_{n\to\infty} \|P^{(n)}\rho\|_{\mathrm{BL}}^* = 0$.

Fix $n \in \mathbb{N}$. Then we can write $\mu - P^{(n)}\mu = \nu - P\nu$ where

$$\nu = \frac{n-1}{n}\mu + \frac{n-2}{n}P\mu + \ldots + \frac{1}{n}P^{n-2}\mu.$$

Now, $\nu \in \mathcal{M}^+(S)$, thus we know that

$$\lim_{m\to\infty} \|P^{(m)}\mu - P^{(m)}P^{(n)}\mu\|_{\mathrm{BL}}^* = \lim_{m\to\infty} \|P^{(m)}(\nu - P\nu)\|_{\mathrm{BL}}^* = 0$$

thus $P^{(n)}\mu \in D_\mu$.

Step 3. $\mu \in \mathcal{P}_{cp}^P$.

Since $\mu \in \mathcal{P}_t(S)$, there is a subsequence $(P^{(n_k)}\mu)_k$ that converges to a $\mu^* \in \mathcal{P}(S)$. Now, Step 1 and Step 2 imply that $\mu^* \in D_\mu$. Since P is Markov-Feller, μ^* is invariant, so $P^{(n)}\mu^* = \mu^*$ for every $n \in \mathbb{N}$. Thus

$$\|P^{(n)}\mu - \mu^*\|_{\mathrm{BL}}^* = \|P^{(n)}\mu - P^{(n)}\mu^*\|_{\mathrm{BL}}^* \to 0$$

as $n \to \infty$. □

In [11, Lemma 1] the Markov semigroup-analogue of Theorem 5.1 is proven. By exploiting the relationship between the ergodic decomposition of a Markov semigroup and that of its resolvent (Proposition 2.13), we easily obtain the Markov semigroup-analogue from Theorem 5.1.

A direct consequence of Theorem 4.4 and Theorem 5.1 is as follows:

Corollary 5.2. $\mathcal{P}_t^P = \mathcal{P}_{cp}^P$ *is closed in* $\mathcal{S}_{\mathrm{BL}}$ *and* $\Gamma_t^P = \Gamma_{cp}^P$ *is closed in* S.

For $f \in C_b(S)$ we define

$$f^*(x) := \begin{cases} \langle \epsilon_x, f \rangle & \text{if } x \in \Gamma_{cp}^P \\ 0 & \text{if } x \notin \Gamma_{cp}^P. \end{cases}$$

In general, $f^* \in \mathrm{BM}(S)$ (e.g. [21]). However, in our setting, when P is a regular Markov operator with the Cesàro e-property, one has:

Proposition 5.3. *The map $x \mapsto \epsilon_x$ is continuous from Γ_{cp}^P to $\mathcal{S}_{\mathrm{BL}}$ and for every $f \in C_b(S)$, $x \mapsto f^*(x)$ is continuous from Γ_{cp}^P to $\mathbb{R}$.*

Proof. Let $f \in \mathrm{BL}(S)$ and $x \in \Gamma_{cp}^P$. Let $\epsilon > 0$. Since P has the Cesàro e-property, there exists a $\delta > 0$, such that

$$|\langle P^{(n)}\delta_x - P^{(n)}\delta_y, f\rangle| < \epsilon$$

for every $n \in \mathbb{N}$ and $y \in B_x(\delta)$. Let $y \in B_x(\delta) \cap \Gamma_{cp}^P$, then we have

$$|\langle \epsilon_x - \epsilon_y, f\rangle| = \lim_{n\to\infty} |\langle P^{(n)}\delta_x - P^{(n)}\delta_y, f\rangle| \le \epsilon.$$

Thus $x \mapsto \langle \epsilon_x, f\rangle$ is continuous. Now let $x_n \in \Gamma_{cp}^P$ such that $x_n \to x$ in S. Then $x \in \Gamma_{cp}^P$ according to Corollary 5.2, and thus $\langle \epsilon_{x_n}, f\rangle \to \langle \epsilon_x, f\rangle$ for every $f \in \mathrm{BL}(S)$. So $\epsilon_{x_n} \to \epsilon_x$ by Theorem 3.1, hence $x \mapsto \epsilon_x$ is continuous from Γ_{cp}^P to $\mathcal{S}_{\mathrm{BL}}$. The last statement follows from Lemma 2.1 and the fact that $f^*(x) = \langle \epsilon_x, f\rangle$. □

Theorem 5.4. *Γ_{cpie}^P is closed.*

Proof. In [21] we have shown that

$$\Gamma_{cpie}^P = \{x \in \Gamma_{cpi}^P : \int_{\Gamma_{cpi}^P} (f^*(y) - f^*(x))^2 \, d\epsilon_x(y) = 0 \text{ for every } f \in C_b(S)\}.$$

Let $x_n \in \Gamma_{cpie}^P$ such that $x_n \to x$ for some $x \in S$. Then $x \in \Gamma_{cp}^P = \Gamma_{cpi}^P$ by Corollary 5.2 and $\epsilon_{x_n} \to \epsilon_x$ in $\mathcal{S}_{\mathrm{BL}}$ by Proposition 5.3.

Let $f \in C_b(S)$. We need to show that

$$\int_{\Gamma_{cpi}^P} (f^*(y) - f^*(x))^2 \, d\epsilon_x(y) = 0.$$

Since $\|f^*\|_\infty \le \|f\|_\infty$, we have for every $y \in \Gamma_{cpi}^P$ and $n \in \mathbb{N}$

$$\begin{aligned} |(f^*(y) - f^*(x))^2 &- (f^*(y) - f^*(x_n))^2| \\ &\le 2|f^*(y)||f^*(x) - f^*(x_n)| + |f^*(x)^2 - f^*(x_n)^2| \\ &\le 2\|f\|_\infty |f^*(x) - f^*(x_n)| \\ &\quad + |f^*(x) + f^*(x_n)||f^*(x) - f^*(x_n)| \\ &\le 4\|f\|_\infty |f^*(x) - f^*(x_n)|. \end{aligned}$$

So

$$
\begin{aligned}
\left|\int_{\Gamma^P_{cpi}} (f^*(y)-f^*(x))^2\right. &- \left.(f^*(y)-f^*(x_n))^2\, d\epsilon_{x_n}(y)\right| \\
&\le 4\|f\|_\infty \int_{\Gamma^P_{cpi}} |f^*(x)-f^*(x_n)|\, d\epsilon_{x_n}(y) \\
&= 4\|f\|_\infty |f^*(x)-f^*(x_n)| \to 0
\end{aligned}
$$

as $n \to \infty$.

For every $n \in \mathbb{N}$

$$
\begin{aligned}
&\left|\int_{\Gamma^P_{cpi}} (f^*(y)-f^*(x))^2\, d\epsilon_x(y) - \int_{\Gamma^P_{cpi}} (f^*(y)-f^*(x_n))^2\, d\epsilon_{x_n}(y)\right| \le \\
&\left|\int_{\Gamma^P_{cpi}} (f^*(y)-f^*(x))^2\, d[\epsilon_x(y)-\epsilon_{x_n}(y)]\right| \qquad (5)\\
&+\left|\int_{\Gamma^P_{cpi}} (f^*(y)-f^*(x))^2 - (f^*(y)-f^*(x_n))^2\, d\epsilon_{x_n}(y)\right|.
\end{aligned}
$$

The final term in inequality (5) above goes to zero as $n \to \infty$.

By Proposition 5.3 $y \mapsto (f^*(y)-f^*(x))^2$ is bounded and continuous from $\Gamma^P_{cp} = \Gamma^P_{cpi}$ to $\mathbb{R}$. By Corollary 5.2 Γ^P_{cp} is closed, thus we can apply the Tietze Extension Theorem, and so there exists a $g \in C_b(S)$, such that $g(y) = (f^*(y)-f^*(x))^2$ for every $y \in \Gamma^P_{cpi}$. Since $\epsilon_x(\Gamma^P_{cpi}) = \epsilon_{x_n}(\Gamma^P_{cpi}) = 1$ for every $n \in \mathbb{N}$, we have

$$
\left|\int_{\Gamma^P_{cpi}} (f^*(y)-f^*(x))^2\, d[\epsilon_x(y)-\epsilon_{x_n}(y)]\right| = |\langle \epsilon_x, g\rangle - \langle \epsilon_{x_n}, g\rangle| \to 0
$$

as $n \to \infty$ since $\epsilon_{x_n} \to \epsilon_x$ in $\mathcal{S}_{\mathrm{BL}}$ and $g \in C_b(S)$.

Now note that $\int_{\Gamma^P_{cpi}} (f^*(y)-f^*(x_n))^2\, d\epsilon_{x_n}(y) = 0$ for every $n \in \mathbb{N}$, since $x_n \in \Gamma^P_{cpie}$, thus $\int_{\Gamma^P_{cpi}} (f^*(y)-f^*(x))^2\, d\epsilon_x(y) = 0$ as well. Thus $x \in \Gamma^P_{cpie}$. □

Corollary 5.5. *For every $x \in \Gamma^P_{cpie}$, $[x]$ is closed.*

Proof. Let $x \in \Gamma^P_{cpie}$. Then $[x] = \{z : \epsilon_z = \epsilon_x\}$. The map $h : \Gamma^P_{cpie} \to \mathcal{S}_{\mathrm{BL}}, z \mapsto \epsilon_z$ is continuous by Proposition 5.3, thus $[x]$ is closed in Γ^P_{cpie}. Thus $[x]$ is closed in S as well, by Theorem 5.4. □

Now we have shown that all the sets in our ergodic decomposition are closed. Our aim now is to show that the sets are also P-invariant. Theorem 5.1 and Theorem 4.5 imply:

Corollary 5.6. *For every $\mu \in \mathcal{P}^P_{cp} = \mathcal{P}^P_t$, $\mathrm{supp}(\mu) \subset \Gamma^P_{cp} = \Gamma^P_t$, so in particular $\mu(\Gamma^P_{cp}) = 1$.*

The following result follows from [21, Proposition 3.5] and Corollary 5.6:

Proposition 5.7. *Let $\mu \in \mathcal{P}(S)$. Then $\mu \in \mathcal{P}^P_{cp}$ if and only if $\mu(\Gamma^P_{cp}) = 1$.*

Corollary 5.8. *Let $\mu \in \mathcal{P}^P_{cp}$, then*

$$\epsilon_\mu = \int_{\Gamma^P_{cp}} \epsilon_x \, d\mu(x).$$

Proof. By definition, $\epsilon_\mu = \lim_{n\to\infty} P^{(n)}\mu$. Now,

$$P^{(n)}\mu = P^{(n)} \int_S \delta_x \, d\mu(x) = \int_S P^{(n)}\delta_x \, d\mu(x) = \int_{\Gamma^P_{cp}} P^{(n)}\delta_x \, d\mu(x)$$

since $\mu(\Gamma^P_{cp}) = 1$. Now, $P^{(n)}\delta_x \to \epsilon_x$ for every $x \in \Gamma^P_{cp}$, thus by the Dominated Convergence Theorem we have

$$\epsilon_\mu = \lim_{n\to\infty} \int_{\Gamma^P_{cp}} P^{(n)}\delta_x \, d\mu(x) = \int_{\Gamma^P_{cp}} \epsilon_x \, d\mu(x).$$

□

Corollary 5.9. *Γ^P_{cp} is a P-invariant set, i.e.*

$$P\delta_x(\Gamma^P_{cp}) = 1 \textit{ for every } x \in \Gamma^P_{cp}.$$

Proof. Let $x \in \Gamma^P_{cp}$. Since

$$\lim_{n\to\infty} \|P^{(n)}P\delta_x - P^{(n)}\delta_x\|^*_{\mathrm{BL}} \to 0,$$

$P^{(n)}P\delta_x \to \epsilon_x$ in $\mathcal{S}_{\mathrm{BL}}$. Thus $P\delta_x \in \mathcal{P}^P_{cp}$ and by Proposition 5.7 $P\delta_x(\Gamma^P_{cp}) = 1$. □

We can define $\mathcal{P}^P_{cpie} := \{\mu \in \mathcal{P}_{cp} : \epsilon_\mu \text{ is ergodic.}\}$

Lemma 5.10. *Let μ be an invariant probability measure and $z \in \Gamma^P_{cpie}$ such that $\mu([z]) = 1$. Then $\mu = \epsilon_z$.*

Proof. By the integral decomposition of invariant measures into ergodic measures, we have

$$\mu = \int_{\Gamma^P_{cpie}} \epsilon_x \, d\mu(x) = \int_{[z]} \epsilon_x \, d\mu(x),$$

since $\mu([z]) = 1$. But if $x \in [z]$, then $\epsilon_x = \epsilon_z$, so $\mu = \int_{[z]} \epsilon_z \, d\mu(x) = \epsilon_z$. □

Proposition 5.11. *Let $\mu \in \mathcal{P}(S)$. Then $\mu \in \mathcal{P}^P_{cpie}$ if and only if $\mu([z]) = 1$ for some $z \in \Gamma^P_{cpie}$. In that case $\epsilon_\mu = \epsilon_z$.*

Proof. Suppose that $\mu([z]) = 1$ for some $z \in \Gamma^P_{cpie}$, then $\mu(\Gamma_{cp}) = 1$, so $\mu \in \mathcal{P}_{cp}$ by Proposition 5.7, and $\epsilon_\mu = \int_{\Gamma_{cp}} \epsilon_x \, d\mu(x)$ by Corollary 5.8. Since $\mu([z]) = 1$ we have

$$\epsilon_\mu = \int_{\Gamma_{cp}} \epsilon_x \, d\mu(x) = \int_{[z]} \epsilon_x \, d\mu(x) = \int_{[z]} \epsilon_z \, d\mu(x) = \epsilon_z,$$

so ϵ_μ is ergodic, thus $\mu \in \mathcal{P}^P_{cpie}$.

On the other hand, if $\mu \in \mathcal{P}^P_{cpie}$, then there is a $z \in \Gamma^P_{cpie}$ such that $\epsilon_\mu = \epsilon_z$. Since $\mu \in \mathcal{P}_{cp}$, Corollary 5.8 implies

$$1 = \epsilon_z([z]) = \int_{\Gamma_{cp}} \epsilon_x([z])\, d\mu(x),$$

thus $\epsilon_x([z]) = 1$ for μ-a.e. $x \in \Gamma_{cp}$. Since ϵ_x is an invariant probability measure for every $x \in \Gamma_{cp}$, Lemma 5.10 implies that $\epsilon_x = \epsilon_z$ for μ-a.e. $x \in \Gamma_{cp}$. Thus $x \in [z]$ for μ-a.e. $x \in \Gamma_{cp}$. Since $\mu(\Gamma_{cp}) = 1$, this implies that $\mu([z]) = 1$. □

Corollary 5.12. *For every $z \in \Gamma^P_{cpie}$, $[z]$ is a P-invariant set.*

Proof. Let $z \in \Gamma^P_{cpie}$ and $x \in [z]$. Then since

$$\lim_{n\to\infty} \|P^{(n)} P\delta_x - P^{(n)}\delta_x\|^*_{\mathrm{BL}} \to 0,$$

$P^{(n)} P\delta_x \to \epsilon_x = \epsilon_z$, so $\epsilon_{P\delta_x} = \epsilon_z$ and thus Proposition 5.11 implies that $P\delta_x([z]) = 1$. □

Since Γ^P_{cpie} is a union of P-invariant sets, Γ^P_{cpie} is also a P-invariant set.

Now let $(P(t))_{t\geq 0}$ be a jointly measurable Markov-Feller semigroup with the Cesàro e-property. By Proposition 4.8 the resolvent R also has the Cesàro e-property. Proposition 2.13 and the previous results yield:

Theorem 5.13. *Let $\mathbf{P} = (P(t))_{t\geq 0}$ be a jointly measurable Markov-Feller semigroup with the Cesàro e-property. Then the following holds:*

(i) $\mathcal{P}^{\mathbf{P}}_t = \mathcal{P}^{\mathbf{P}}_{cp}$ is closed in $\mathcal{S}_{\mathrm{BL}}$ and $\Gamma^{\mathbf{P}}_t = \Gamma^{\mathbf{P}}_{cp}$ is closed.

(ii) $\Gamma^{\mathbf{P}}_{cpie}$ is closed and $[z]$ is closed for every $z \in \Gamma^{\mathbf{P}}_{cpie}$.

(iii) For $\mu \in \mathcal{P}(S)$, $\mu \in \mathcal{P}^{\mathbf{P}}_{cp}$ if and only if $\mu(\Gamma^{\mathbf{P}}_{cp}) = 1$. In this case $\varepsilon_\mu = \int_{\Gamma^{\mathbf{P}}_{cp}} \varepsilon_x\, d\mu(x)$

(iv) For $\mu \in \mathcal{P}(S)$, $\mu \in \mathcal{P}^{\mathbf{P}}_{cpie}$ with $\varepsilon_\mu = \varepsilon_z$ if and only if $\mu([z]) = 1$.

Lemma 5.14. *For every $t \in \mathbb{R}_+$ and $\mu \in \mathcal{P}(S)$ we have*

$$\lim_{s\to\infty} \|P^{(s)}P(t)\mu - P^{(s)}\mu\|_{\mathrm{TV}} = 0.$$

So in particular

$$\lim_{s\to\infty} \|P^{(s)}P(t)\mu - P^{(s)}\mu\|^*_{\mathrm{BL}} = 0.$$

Proof. Let $\rho \in \mathcal{M}(S)$, then $\|\rho\|_{\mathrm{TV}} = \rho(P) - \rho(N)$, where $S = P \cup N$ is the Hahn decomposition corresponding to μ. So $\|\rho\|_{\mathrm{TV}} \leq 2\sup\{|\rho(E)| : E \subset S \text{ Borel}\}$.

Let E be a Borel set in S, $s > 0$ and $t \in \mathbb{R}_+$. Then by Proposition 2.3 we obtain

$$\begin{aligned} |P^{(s)}P(t)\mu(E) - P^{(s)}\mu(E)| &= |\frac{1}{s}\int_t^{t+s} P(r)\mu(E)\, dr - \frac{1}{s}\int_0^s P(r)\mu(E)\, dr| \\ &\leq |\frac{1}{s}\int_s^{t+s} P(r)\mu(E)\, dr - \frac{1}{s}\int_0^t P(r)\mu(E)\, dr| \\ &\leq \frac{2t}{s}. \end{aligned}$$

Thus $\|P^{(s)}P(t)\mu - P^{(s)}\mu\|_{\mathrm{TV}} \leq \frac{4t}{s} \to 0$ as $s \to \infty$. $\square$

Corollary 5.15. *$\Gamma^{\mathbf{P}}_{cp}$ is a $(P(t))_{t\geq 0}$-invariant set, i.e.*

$$P(t)\delta_x(\Gamma^{\mathbf{P}}_{cp}) = 1 \text{ for every } x \in \Gamma^{\mathbf{P}}_{cp}.$$

Proof. Let $x \in \Gamma^{\mathbf{P}}_{cp}$. Since

$$\lim_{s\to\infty} \|P^{(s)}P(t)\delta_x - P^{(s)}\delta_x\|^*_{\mathrm{BL}} \to 0$$

by Lemma 5.14, $P^{(s)}P(t)\delta_x \to \varepsilon_x$. Thus $P(t)\delta_x \in \mathcal{P}^{\mathbf{P}}_{cp}$ and by Theorem 5.13 $P(t)\delta_x(\Gamma^{\mathbf{P}}_{cp}) = 1$. $\square$

Since every ergodic measure is of the form ε_z for some $z \in \Gamma^{\mathbf{P}}_{cpie}$, we know that for every $\mu \in \mathcal{P}^{\mathbf{P}}_{cpie}$, $\varepsilon_\mu = \varepsilon_z$ for some $z \in \Gamma^{\mathbf{P}}_{cpie}$.

Corollary 5.16. *For every $z \in \Gamma^{\mathbf{P}}_{cpie}$, $[z]$ is a $(P(t))_{t\geq 0}$-invariant set.*

Proof. Let $z \in \Gamma^{\mathbf{P}}_{cpie}$ and $x \in [z]$. Then since

$$\lim_{s\to\infty} \|P^{(s)}P(t)\delta_x - P^{(s)}\delta_x\|^*_{\mathrm{BL}} \to 0$$

by Lemma 5.14, $P^{(s)}P(t)\delta_x \to \varepsilon_x = \varepsilon_z$, so $\varepsilon_{P(t)\delta_x} = \varepsilon_z$ and thus Theorem 5.13 implies that $P(t)\delta_x([z]) = 1$. $\square$

Since $\Gamma^{\mathbf{P}}_{cpie}$ is a union of $(P(t))_{t\geq 0}$-invariant sets, $\Gamma^{\mathbf{P}}_{cpie}$ is also a $(P(t))_{t\geq 0}$-invariant set.

6 Existence, uniqueness and stability of invariant measures

Every Markov semigroup we consider in this section will be Markov-Feller and jointly measurable. We will find conditions on existence, uniqueness and stability of invariant measures of such semigroups with a (Cesàro) e-property. Our proofs can easily be adapted such that similar results hold for regular Markov operators with the (Cesàro) e-property as well.

6.1 Existence of invariant measures

The following result gives some equivalent conditions for the existence of invariant measures. The statement *(v)⇒(i)* is slightly more general than that of [13, Theorem 3.1], since there ν has to be of the form δ_x for some $x \in S$. We also do not require the Markov semigroup to be strongly stochastically continuous at zero; joint measurability is sufficient. Our proof of this statement (or rather of *(iv)⇒(i)*) is based on that of [13, Theorem 3.1], though simplified by exploiting our equicontinuity results.

Theorem 6.1. *Let $(P(t))_{t\geq 0}$ be a Markov semigroup with the eventual e-property. The following results are equivalent:*

(i) $\Gamma_t^{\mathbf{P}}$ *is not empty.*

(ii) There exist invariant measures.

(iii) There exists an $z \in S$ and a $x \in S$ such that for every $\delta > 0$

$$\liminf_{t\to\infty} P^{(t)}\delta_x(B_z(\delta)) > 0.$$

(iv) There exists a $z \in S$ such that for every $\delta > 0$ there is a $\nu \in \mathcal{P}(S)$ such that

$$\limsup_{t\to\infty} P^{(t)}\nu(B_z(\delta)) > 0. \tag{6}$$

(v) There exists a compact $K \subset S$ such that for every open set U with $K \subset U$ there is a $\nu \in \mathcal{P}(S)$ such that

$$\limsup_{t\to\infty} P^{(t)}\nu(U) > 0. \tag{7}$$

Proof. *(i) ⇒ (ii)*: By Theorem 5.13 $\Gamma_t^{\mathbf{P}} = \Gamma_{cp}^{\mathbf{P}}$ and $\Gamma_{cp}^{\mathbf{P}} = \Gamma_{cpi}^{\mathbf{P}}$ since $(P(t))_{t\geq 0}$ is Markov-Feller. So there exist invariant measures.
(ii) ⇒ (iii): If there exist invariant measures, then there exist invariant ergodic probability measures. Let μ be an invariant ergodic probability measure, then there is a $z \in \mathrm{supp}(\mu)$. Now, $\mu = \varepsilon_x$ for some $x \in \Gamma_{cpie}^{\mathbf{P}}$, so $P^{(t)}\delta_x \to \varepsilon_x$ in $\mathcal{S}_{\mathrm{BL}}$ as $t \to \infty$. This implies, by Lemma 2.1 and Alexandrov's Theorem, that for every $\delta > 0$,

$$\liminf_{t\to\infty} P^{(t)}\delta_x(B_z(\delta)) \geq \varepsilon_x(B_z(\delta)) > 0.$$

(iii)⇒ (iv): Take $\nu = \delta_x$.
(iv)⇒ (v): Take $K = \{z\}$.
(v)⇒ (iv): Suppose that for every $x \in K$ there exists a $\delta > 0$ such that $\lim_{t\to\infty} P^{(t)}\nu(B_x(\delta)) = 0$ for every $\nu \in \mathcal{P}(S)$. Then there are $x_1, ..., x_n \in K$, $\delta_1, ..., \delta_n > 0$ such that $\lim_{t\to\infty} P^{(t)}\nu(B_{x_i}(\delta_i)) = 0$, for every $i \in \{1, 2, ..., n\}$ and $\nu \in \mathcal{P}(S)$, and $K \subset \cup_{i=1}^n B_{x_i}(\delta_i)$. By assumption there exists a $\nu \in \mathcal{P}(S)$ such that

$$0 < \limsup_{t\to\infty} P^{(t)}\nu(\cup_{i=1}^n B_{x_i}(\delta_i)) \leq \sum_{i=1}^n \limsup_{t\to\infty} P^{(t)}\nu(B_{x_i}(\delta_i)) = 0,$$

which is a contradiction.
(iv)⇒ (i): Suppose that $\Gamma_t^{\mathbf{P}}$ is empty, the in particular $z \notin \Gamma_t^{\mathbf{P}}$.

We will show that there exist $\epsilon > 0$, compact sets $K_n \subset S$ and $t_n \in \mathbb{R}_+$ for $n \in \mathbb{N}$, such that $t_n \geq n$,

$$P^{(t_n)}\delta_z(K_n) > 2\epsilon \text{ for every } n \in \mathbb{N}$$

and

$$\min\{d(x, y) : x \in K_m, y \in K_n\} \geq \epsilon \text{ whenever } m \neq n.$$

Since $z \notin \Gamma_t^{\mathbf{P}}$, $\{P^{(t)}\delta_z : t \geq N\}$ is not tight for every $N \in \mathbb{N}$. So there exists an $0 < \epsilon\frac{1}{2}$ such that for every $N \in \mathbb{N}$ and every compact $K \subset S$, there is a $t \geq N$, such that

$$P^{(t)}\delta_z(K^\epsilon) \leq P^{(t)}\delta_z(K^{2\epsilon}) < 1 - 2\epsilon.$$

Let $t_1 = 1$, $K_1 \subset S$ be compact such that $P^{(t_1)}\delta_z(K_1) > 2\epsilon$. Then there is a $t_2 \geq 2$ such that $P^{(t_2)}\delta_z(K_1^\epsilon) < 1 - 2\epsilon$. So there is a compact $K_2 \subset S\backslash K_1^\epsilon$ such that $P^{(t_2)}\delta_z(K_2) > 2\epsilon$. And since $K_1 \subset K_2$ is compact, there is a $t_3 \geq 3$ such that $P^{(t_3)}\delta_z((K_1 \cup K_2)^\epsilon) < 1 - 2\epsilon$. Continuing onward we find the required K_n and t_n.

Now we will show that there exist a $(s_n)_n \subset \mathbb{R}_+$ such that $s_n \to \infty$ and $P(s_n)\delta_z(K_n) \geq \epsilon$. Suppose such a sequence does not exist, then there is an $S \in \mathbb{R}_+$ and a $N \in \mathbb{N}$, such that $P(s)\delta_z(K_n) < \epsilon$ for every $s \geq S$ and $n \geq N$. But then, for every $n > \max(N, S)$,

$$\begin{aligned} P^{(t_n)}\delta_z(K_n) &= \frac{1}{t_n}\int_0^S P(t)\delta_z(K_n)\,dt + \frac{1}{t_n}\int_N^{t_n} P(t)\delta_z(K_n)\,dt \\ &\leq \frac{N}{t_n} + \frac{\epsilon}{t_n} \to 0 \end{aligned}$$

as $n \to \infty$.

By the eventual e-property and Theorem 4.2, there is a $\tau > 0$ such that $(P(t)\delta)_{t\geq\tau}$ is equicontinuous in $z \in S$, so it follows that there exists a $\delta > 0$ such that

$$\|P(t)\delta_z - P(t)\delta_y\|_{\mathrm{BL}}^* < \frac{1}{1+\frac{3}{\epsilon}}\frac{\epsilon}{2}$$

for every $t \geq \tau$ and $y \in B_z(\delta)$. Since $s_n \to \infty$, there is an $M \in \mathbb{N}$ such that $s_n \geq \tau$ whenever $n \geq M$. Let $f_n := f_{K_n}^{\frac{\epsilon}{3}}$, then $f_n \in \mathrm{BL}(S)$, with $\|f_n\|_{\mathrm{BL}} \leq 1 + \frac{3}{\epsilon}$, so

$$|\langle P(s_n)\delta_z - P(s_n)\delta_y, f_n\rangle| < \frac{\epsilon}{2} \text{ for every } n \in \mathbb{N}_{\geq M}, y \in B_z(\delta).$$

Since $\mathbb{1}_{K_n} \leq f_n \leq \mathbb{1}_{(K_n)^{\frac{\epsilon}{3}}}$, we have for every $n \in \mathbb{N}_{\geq M}$ and $y \in B_z(\delta)$:

$$\begin{aligned} P(s_n)\delta_y((K_n)^{\frac{\epsilon}{3}}) &\geq \langle P(s_n)\delta_y, f_n\rangle \\ &\geq \langle P(s_n)\delta_z, f_n\rangle - |\langle P(s_n)\delta_z - P(s_n)\delta_y, f_n\rangle| \\ &\geq P(s_n)(K_n) - |\langle P(s_n)\delta_z - P(s_n)\delta_y, f_n\rangle| > \frac{\epsilon}{2}. \end{aligned}$$

Let $\nu \in \mathcal{P}(S)$ be such that (6) holds and let $0 < \alpha < \limsup_{t\to\infty} P^{(t)}\nu(B_z(\delta))$. Then there is a sequence $(r_k)_k \subset \mathbb{R}_+$ such that $r_k \to \infty$ and

$$\lim_{n\to\infty} P^{(r_k)}\nu(B_z(\delta)) > \alpha.$$

Now, for all $t \geq 0$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} P(t+s_n)\nu(K_n^{\frac{\epsilon}{3}}) &= \int_S P(s_n)\delta_y(K_n^{\frac{\epsilon}{3}})\,dP(t)\nu(y) \\ &\geq \int_{B_z(\delta)} P(s_n)\delta_y(K_n^{\frac{\epsilon}{3}})\,dP(t)\nu(y) \geq \frac{\epsilon}{2}P(t)\nu(B_z(\delta)). \end{aligned}$$

Thus for every $n \in \mathbb{N}_{\geq M}$

$$\begin{aligned}
\liminf_{k\to\infty} P^{(r_k)}\nu(K_n^{\frac{\epsilon}{3}}) &= \liminf_{k\to\infty} P^{(r_k)}P(s_n)\nu(K_n^{\frac{\epsilon}{3}}) \\
&= \liminf_{k\to\infty} \frac{1}{r_k}\int_0^{r_k} P(t+s_n)\nu(K_n^{\frac{\epsilon}{3}})\,dt \\
&\geq \frac{\epsilon}{2}\liminf_{k\to\infty} P^{(r_k)}(B_z(\delta)) > \frac{\alpha\epsilon}{2}.
\end{aligned}$$

Since $K_n^{\frac{\epsilon}{3}} \cap K_m^{\frac{\epsilon}{3}} = \emptyset$ whenever $n \neq m$,

$$\liminf_{k\to\infty} P^{(r_k)}(\bigcup_{n=M}^{N} K_n^{\frac{\epsilon}{3}}) > \frac{(N-M+1)\alpha\epsilon}{2}$$

for every $N \geq M$, which is not possible. $\square$

6.2 Uniqueness of invariant measures

The following notion is defined by Szarek in [19]: A Markov operator P *overlaps support* if for every $x, y \in S$ there is an $n_0 \in \mathbb{N}$, such that

$$\operatorname{supp}(P^{n_0}\delta_x) \cap \operatorname{supp}(P^{n_0}\delta_y) \neq \emptyset.$$

The main result in [19] is the following:

Theorem 6.2. *Let P be a regular Markov operator satisfying the e-property and such that P overlaps support. Then P has at most one invariant probability measure.*

The condition of overlapping support is far from necessary for uniqueness of invariant measures. We give a simple example to show this:

Example 6.3. Let $S = \{0, 1\}$, $\Phi(0) := 1$ and $\Phi(1) := 0$. Then $P\mu := \mu \circ \Phi^{-1}$ defines a regular Markov operator satisfying the e-property. P has a unique invariant probability measure $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$, but clearly does not overlap supports, since $\operatorname{supp}(P^n\delta_0) \cap \operatorname{supp}(P^n\delta_1)$ for every $n \in \mathbb{N}$.

We show that a more general condition suffices for uniqueness of invariant probability measures, and the previous example satisfies *this* condition:

Theorem 6.4. *Let P be a regular Markov operator with the Cesàro e-property. Suppose that for every $x, y \in S$ there exists a $z \in S$, such that for every $\delta > 0$ there are $n_1, n_2 \in \mathbb{N}$ such that*

$$P^{n_1}\delta_x(B_z(\delta)) > 0$$

and

$$P^{n_2}\delta_y(B_z(\delta)) > 0.$$

Then there is at most one invariant probability measure.

Proof. Suppose there are at least two invariant probability measures, then it follows from Theorem 2.12 that there are at least two distinct ergodic probability measures ϵ_x and ϵ_y, where $x, y \in \Gamma^P_{cpie}$ such that $x \not\equiv y$. Let $z \in S$ be given by the assumption. Since $[x]$ and $[y]$ are disjoint, z cannot be in both. Say $z \notin [x]$. Since $[x]$ is closed by Corollary 5.5 there is a $\delta > 0$ such that $B_z(\delta) \cap [x] = \emptyset$. By Corollary 5.16 $[x]$ is an invariant set, so $P^n\delta_x([x]) = 1$ for every $n \in \mathbb{N}$, which implies that $P^n\delta_x(B_z(\delta)) = 0$ for every $n \in \mathbb{N}$, which is a contradiction. □

By a similar proof, but now using Theorem 5.13 and Corollary 5.16, we obtain the Markov semigroup-variant of Theorem 6.4:

Theorem 6.5. *Let $(P(t))_{t\geq 0}$ be a Markov semigroup with the Cesàro e-property. Suppose that for every $x, y \in S$ there exists a $z \in S$, such that for every $\delta > 0$ there are $t_1, t_2 > 0$ such that*

$$P(t_1)\delta_x(B_z(\delta)) > 0$$

and

$$P(t_2)\delta_y(B_z(\delta)) > 0.$$

Then there is at most one invariant probability measure.

We recently discovered that, using different techniques, Theorem 6.5 has also been obtained in [11, Theorem 2], though there strong stochastic continuity at zero is required, while joint measurability suffices for our proof .

The following result comes from [12, Theorem 1] (translated into our notation).

Theorem 6.6. *If there is a $z \in S$ such that for every $x \in S$ and every $\delta > 0$*

$$\liminf_{t\to\infty} P^{(t)}\delta_x(B_z(\delta)) > 0,$$

then there exists a unique invariant probability measure μ^* and $P^{(t)}\nu \to \mu^*$ for every $\nu \in \mathcal{P}(S)$ such that $\operatorname{supp}(\nu) \subset \Gamma^{\boldsymbol{P}}_t$.

We generalise this result by showing that we can replace lim inf by lim sup:

Theorem 6.7. *Let $(P(t))_{t\geq 0}$ be a Markov semigroup with the eventual e-property. If there is a $z \in S$ such that for every $x \in S$ and every $\delta > 0$*

$$\limsup_{t\to\infty} P^{(t)}\delta_x(B_z(\delta)) > 0,$$

then there exists a unique invariant probability measure μ^*, and $z \in \operatorname{supp}(\mu^*)$.

Proof. By Theorem 6.1 and Theorem 6.5 there exists a unique invariant measure μ^*.

Suppose $z \notin \operatorname{supp}(\mu^*)$, then there is a $\delta > 0$ such that $\mu^*(B_z(\delta)) = 0$. Let $f := f^{\frac{\delta}{2}}_{B_z(\frac{\delta}{2})}$, then $\mathbb{1}_{B_z(\frac{\delta}{2})} \leq f \leq \mathbb{1}_{B_z(\delta)}$. Let $x \in \Gamma^{\mathbf{P}}_{cp}$, which is non-empty since there exists an invariant measure, then $P^{(t)}\delta_x \to \mu^*$ in $\mathcal{S}_{\mathrm{BL}}$, thus by assumption

$$\begin{aligned} 0 < \limsup_{t\to\infty} P^{(t)}\delta_x(B_z(\frac{\delta}{2})) &\leq \limsup_{t\to\infty} \langle P^{(t)}\delta_x, f\rangle \\ &= \langle \mu^*, f\rangle \leq \mu^*(B_z(\delta)) = 0, \end{aligned}$$

which is a contradiction. □

The second part of Theorem 6.6, that $P^{(t)}\nu \to \mu^*$ for every $\nu \in \mathcal{P}(S)$ such that $\nu(\Gamma_t^{\mathbf{P}}) = 1$, *always* holds whenever the Markov semigroup has the e-property and a unique invariant measure:

Proposition 6.8. *Let $(P(t))_{t\geq 0}$ be a Markov semigroup with the Cesàro e-property, and with a unique invariant probability measure μ^*. Then the following are equivalent for $\nu \in \mathcal{P}(S)$:*

(i) $P^{(t)}\nu \to \mu^*$ *in* $\mathcal{S}_{\mathrm{BL}}$ *as* $t \to \infty$.

(ii) $\nu(\Gamma_t^{\mathbf{P}}) = 1$.

Proof. Let $\nu \in \mathcal{P}(S)$. Since μ^* is the only invariant probability measure, $\lim_{t\to\infty} P^{(t)}\nu = \mu^*$ in $\mathcal{S}_{\mathrm{BL}}$ if and only if $\nu \in \mathcal{P}_{cp}^{\mathbf{P}}$. By Theorem 5.13 this holds if and only if $\nu(\Gamma_t^{\mathbf{P}}) = 1$. □

6.3 Stability of invariant measures

The proof of our next theorem is based on that of [20, Theorem 2]. We will show further on that our statement is actually a generalisation of [20, Theorem 2].

Theorem 6.9. *Let $(P(t))_{t\geq 0}$ be a Markov semigroup with the eventual e-property. Assume that $(P(t))_{t\geq 0}$ has an ergodic probability measure μ^* and let $w \in \Gamma_{cpie}^{\mathbf{P}}$ be such that $\mu^* = \varepsilon_w$. Then the following statements are equivalent:*

(i) There is a $z \in \operatorname{supp}(\mu^*)$ and an $x \in S$, such that

$$\liminf_{t\to\infty} P(t)\delta_x(B_z(\delta)) > 0 \textit{ for every } \delta > 0.$$

(ii) $\lim_{t\to\infty} P(t)\nu = \mu^*$ *in* $\mathcal{S}_{\mathrm{BL}}$ *for every* $\nu \in \mathcal{P}(S)$ *such that* $\nu([w]) = 1$.

(iii) For every $z \in \operatorname{supp}(\mu^*)$ and every $\nu \in \mathcal{P}(S)$ such that $\nu([w]) = 1$

$$\liminf_{t\to\infty} P(t)\nu(B_z(\delta)) > 0 \textit{ for every } \delta > 0.$$

Proof. (i) ⇒ (ii):

Step 1: For every $\delta > 0$ there is an $\alpha > 0$ such that

$$\liminf_{t\to\infty} P(t)\nu(B_z(\delta)) > \alpha$$

for every $\nu \in \mathcal{P}(S)$ such that $\nu([w]) = 1$.

Let $\delta > 0$. Define

$$\gamma := \frac{1}{2} \liminf_{t\to\infty} P(t)\delta_x(B_z(\delta/2)).$$

By assumption $\gamma > 0$. Let $f := f_{B_z(\delta/2)}^{\delta/2}$. Then $\mathbb{1}_{B_z(\delta/2)} \leq f \leq \mathbb{1}_{B_z(\delta)}$ and $f \in \mathrm{BL}(S)$. By the eventual e-property, there exists an $\eta > 0$ and a $T > 0$ such that

$$|\langle P(t)\delta_y - P(t)\delta_x, f\rangle| < \gamma$$

for every $y \in B_x(\eta)$ and $t \geq T$. Thus for all $y \in B_x(\eta)$ and $t \geq T$,

$$\begin{aligned} P(t)\delta_y(B_z(\delta)) &\geq \langle P(t)\delta_y, f\rangle \\ &\geq \langle P(t)\delta_x, f\rangle - |\langle P(t)\delta_y - P(t)\delta_x, f\rangle| \\ &> P(t)\delta_x(B_z(\delta/2)) - \gamma. \end{aligned}$$

Hence

$$\liminf_{t\to\infty} P(t)\delta_y(B_z(\delta)) > \frac{1}{2} \liminf_{s\to\infty} P(s)\delta_x(B_z(\delta/2)) = \gamma$$

for every $y \in B_x(\eta)$. Set $\theta := \frac{\mu^*(B_z(\eta))}{2}$, then $\theta > 0$ since $z \in \operatorname{supp}(\mu^*)$.

Fix $\nu \in \mathcal{P}(S)$ such that $\nu([w]) = 1$. Then Theorem 5.13 implies that $\lim_{t\to\infty} P^{(t)}\nu \to \varepsilon_w = \mu^*$ in $\mathcal{S}_{\mathrm{BL}}$, thus by Lemma 2.1 and Alexandrov's Theorem

$$\liminf_{t\to\infty} P^{(t)}\nu(B_z(\eta)) \geq \mu^*(B_z(\eta)).$$

Then there exists a $t_0 > 0$ such that $P(t_0)\nu(B_z(\eta)) > \theta$. Thus by Fatou's Lemma

$$\begin{aligned} \liminf_{t\to\infty} P(t)\nu(B_z(\delta)) &= \liminf_{t\to\infty} P(t+t_0)\nu(B_z(\delta)) \\ &\geq \int_S \liminf_{t\to\infty} P(t)\delta_y(B_z(\delta))d[P(t_0)\nu](y) \\ &\geq \int_{B_z(\eta)} \liminf_{t\to\infty} P(t)\delta_y(B_z(\delta))d[P(t_0)\nu](y) > \gamma\theta, \end{aligned}$$

so we can take $\alpha = \gamma\theta$.

Step 2: For every $\nu_1, \nu_2 \in \mathcal{P}(S)$ with $\nu_1([w]) = \nu_2([w]) = 1$, $\lim_{t\to\infty} \|P(t)\nu_1 - P(t)\nu_2\|^*_{\mathrm{BL}} = 0$.

Since $\mu^*([w]) = 1$, this will imply that $\lim_{t\to\infty} \|P(t)\nu - \mu^*\|^*_{\mathrm{BL}} = 0$, which proves the theorem.

Fix $\epsilon > 0$. For $\gamma > 0$ we define $\mathcal{P}^\gamma(S) := \{\mu \in \mathcal{P}(S) : \operatorname{supp}(\mu) \subset B_z(\gamma)\}$. Let $\mu_1, \mu_2 \in \mathcal{P}^\gamma(S)$. If $f \in \mathrm{BL}(S)$ with $|f|_{\mathrm{Lip}} \leq \|f\|^*_{\mathrm{BL}} \leq 1$, then $|f_{\sup} - f_{\inf}| \leq 2\delta$, where

$$f_{\inf} := \inf\{f(y) : y \in B_z(\gamma)\} \text{ and } f_{\sup} := \sup\{f(y) : y \in B_z(\gamma)\}.$$

Now, $f_{\inf} \leq \langle \mu_i, f\rangle \leq f_{\sup}$ for $i \in \{1,2\}$, thus $|\langle \mu_1 - \mu_2, f\rangle| \leq 2\delta$. So $\|\mu_1 - \mu_2\|^*_{\mathrm{BL}} \leq 2\delta$. Since $(P(t))_{t\geq 0}$ satisfies the eventual e-property, there exists by Theorem 4.2 a $\gamma > 0$ and a $T > 0$ such that $\|P(t)\mu_1 - P(t)\mu_2\|^*_{\mathrm{BL}} < \epsilon/2$ whenever $\mu_1, \mu_2 \in \mathcal{P}^\gamma(S)$ and $t \geq T$.

By Step 1 there exists an $0 < \alpha < 1$, such that

$$\liminf_{t\to\infty} P(t)\nu(B_z(\gamma)) > \alpha$$

for every $\nu \in \mathcal{P}_t^{\mathbf{P}}$. By induction we will define a sequence of $(t_n)_{n\geq 0} \subset \mathbb{R}_+$ and four sequences of probability measures $(\nu_1^n)_{n\geq 0}$, $(\nu_2^n)_{n\geq 0}$, $(\mu_1^n)_{n\geq 0}$, $(\mu_2^n)_{n\geq 0}$, such that $\nu_i^n \in \mathcal{P}^\gamma(S), \mu_i^n([w]) = 1$ for every $i \in \{1,2\}, n \geq 1$, and

$$P(t_n)\mu_i^{n-1} = \alpha\nu_i^n + (1-\alpha)\mu_i^n \text{ for every } i \in \{1,2\}, n \geq 1. \tag{8}$$

Define $t_0 = 0$, $\nu_1^0 = \mu_1^0 = \nu_1$ and $\nu_2^0 = \mu_2^0 = \nu_2$. If $n \geq 1$ and $t_{n-1}, \nu_1^{n-1}, \nu_2^{n-1}, \mu_1^{n-1}, \mu_2^{n-1}$ are given, then since $\mu_i^{n-1} \in \mathcal{P}_t^{\mathbf{P}}$ for $i \in \{1,2\}$, we may choose $t_n > 0$ such that

$$P(t_n)\mu_i^{n-1}(B_z(\gamma)) > \alpha \text{ for } i \in \{1,2\}.$$

For $E \subset S$ Borel, define

$$\nu_i^n(E) := \frac{P(t_n)\mu_i^{n-1}(E \cap B_z(\gamma))}{P(t_n)\mu_i^{n-1}(B_z(\gamma))} \text{ for } i \in \{1,2\},$$

then $\nu_i^n \in \mathcal{P}^\gamma(S)$ for $i \in \{1,2\}$. Note that

$$\nu_i^n \leq \frac{\mu_i^{n-1}}{P(t_n)\mu_i^{n-1}(B_z(\gamma))} < \frac{\mu_i^{n-1}}{\alpha}.$$

We also define

$$\mu_i^n(E) := \frac{1}{1-\alpha}(P(t_n)\mu_i^{n-1}(E) - \alpha\nu_i^n(E)) \text{ for } i \in \{1,2\}.$$

Then $\mu_i^n \in \mathcal{P}(S)$. Also, for $i \in \{1,2\}$, $\mu_i^n \leq \frac{1}{1-\alpha}P(t_n)\mu_i^{n-1}$. Since $\mu_i^{n-1}([w]) = 1$, $P(t_n)\mu_i^{n-1}([w]) = 1$ as well, since $[w]$ is $(P(t))_{t\geq 0}$-invariant by Corollary 5.16. So $P(t_n)\mu_i^{n-1}(S\backslash[w]) = 0$, and thus $\mu_i^n(S\backslash[w]) = 0$ as well. Hence $\mu_i^n([w]) = 1$ for $i \in \{1,2\}$.

Now choose $N \in \mathbb{N}$ such that $(1-\alpha)^N < \frac{\epsilon}{4}$. From (8) we obtain for every $i \in \{1,2\}$ and $t \in \mathbb{R}_+$:

$$\begin{aligned} P(t_1 + t_2 + ... + t_N)\nu_i &= \alpha P(t_2 + ... + t_N + t)\nu_i^1 + \alpha(1-\alpha)P(t_3 + ... + t_N + t)\nu_i^2 \\ &+ ... + \alpha(1-\alpha)^{N-1}P(t)\nu_i^k + (1-\alpha)^N P(t)\mu_i^N. \end{aligned}$$

Since $\nu_i^n \in \mathcal{P}^\gamma(S)$ for every $i \in \{1,2\}, n \geq 1$, we have $\|P(t)\nu_1^n - P(t)\nu_2^n\|_{\mathrm{BL}}^* \leq \frac{\epsilon}{2}$ for every $t \geq T$. Also,

$$\|(1-\alpha)^N(P(t)\mu_1^N - P(t)\mu_2^N\|_{\mathrm{BL}}^* \leq 2(1-\alpha)^N < \frac{\epsilon}{2}.$$

Thus for every $t \geq t_1 + ... + t_N + T$ we have $\|P(t)\nu_1 - P(t)\nu_2\|_{\mathrm{BL}}^* \leq \epsilon$. Since $\epsilon > 0$ was arbitrary, we have proven Step 2, and thus *(ii)*.

(ii) $\Rightarrow$ *(iii)*: Let $z \in \mathrm{supp}(\mu^*)$ and $\nu([w]) = 1$. $P(t)\nu \to \mu^*$ as $t \to \infty$, so by Alexandrov's Theorem $\liminf_{t\to\infty} P(t)\nu(B_z(\delta)) \geq \mu^*(B_z(\delta)) > 0$.

(iii) $\Rightarrow$ *(i)*: By assumption, $[w]$ is non-empty, so take $x = w$, then $\delta_x([w]) = 1$ and the statement holds. □

If $(P(t))_{t\geq 0}$ has the eventual e-property and a unique invariant probability measure μ^*, then μ^* is ergodic and $\Gamma_t^{\mathbf{P}} = \Gamma_{cpie}^{\mathbf{P}} = [w]$, where $w \in \Gamma_{cpie}^{\mathbf{P}}$. So Theorem 6.9 and Theorem 5.13 imply the following:

Corollary 6.10. *Let $(P(t))_{t\geq 0}$ be a Markov semigroup with the eventual e-property. Assume $(P(t))_{t\geq 0}$ has a unique invariant probability measure μ^*, then the following are equivalent:*

(i) There is a $z \in \operatorname{supp}(\mu^*)$ and an $x \in S$, such that

$$\liminf_{t\to\infty} P(t)\delta_x(B_z(\delta)) > 0 \textit{ for every } \delta > 0.$$

(ii) $\lim_{t\to\infty} P(t)\nu = \mu^*$ in $\mathcal{S}_{\mathrm{BL}}$ for every $\nu \in \mathcal{P}_t^{\mathbf{P}}$.

(iii) For every $z \in \operatorname{supp}(\mu^*)$ and every $\nu \in \mathcal{P}_t^{\mathbf{P}}$

$$\liminf_{t\to\infty} P(t)\nu(B_z(\delta)) > 0 \textit{ for every } \delta > 0.$$

Let $(P(t))_{t\geq 0}$ be a semigroup with the e-property. In [20, Theorem 2] it is shown that if $(P(t))_{t\geq 0}$ satisfies the condition of Theorem 6.6 and if additionally $\liminf_{t\to\infty} P(t)\delta_z(B_z(\delta)) > 0$ with z such as in Theorem 6.6, then $\lim_{t\to\infty} P(t)\nu = \mu^*$ in $\mathcal{S}_{\mathrm{BL}}$ for every $\nu \in \mathcal{P}(S)$ with $\nu(\Gamma_t^{\mathbf{P}}) = 1$, where μ^* is the unique invariant probability measure. As we have shown in our generalisation of Theorem 6.6, Theorem 6.7, the z from that theorem is in the support of the invariant measure. In particular, Corollary 6.10 is a generalisation of [20, Theorem 2], since it implies that $(P(t))_{t\geq 0}$ need not satisfy the condition of Theorem 6.6 (or the condition of its generalisation), as long as it has a unique invariant measure.

We also obtain the following result:

Corollary 6.11. *Let $(P(t))_{t\geq 0}$ be a Markov semigroup with the eventual e-property. Assume that for every ergodic probability measure μ^* there is a $z \in \operatorname{supp}(\mu^*)$ and an $x \in S$ such that*

$$\liminf_{t\to\infty} P(t)\delta_x(B_z(\delta)) > 0 \textit{ for every } \delta > 0.$$

If $\nu \in \mathcal{P}(S)$ is such that $\nu(\Gamma_{cpie}^{\mathbf{P}}) = 1$, then $\lim_{t\to\infty} P(t)\nu = \varepsilon_\nu$ in $\mathcal{S}_{\mathrm{BL}}$.

Proof. Let $\nu \in \mathcal{P}(S)$ be such that $\nu(\Gamma_{cpie}^{\mathbf{P}}) = 1$. Then

$$P(t)\nu = \int_S P(t)\delta_x \, d\nu(x) = \int_{\Gamma_{cpie}^{\mathbf{P}}} P(t)\delta_x \, d\nu(x).$$

By Theorem 6.9, $\lim_{t\to\infty} P(t)\delta_x = \varepsilon_x$ for every $x \in \Gamma_{cpie}^{\mathbf{P}}$, so by the Lebesgue Dominated Convergence Theorem we get

$$\begin{aligned} \lim_{t\to\infty} P(t)\nu &= \int_{\Gamma_{cpie}^{\mathbf{P}}} \lim_{t\to\infty} P(t)\delta_x \, d\nu(x) \\ &= \int_{\Gamma_{cp}^{\mathbf{P}}} \varepsilon_x \, d\nu(x) = \varepsilon_\nu. \end{aligned}$$

$\square$

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