Invariant measures and a stability theorem for locally Lipschitz stochastic delay equations*

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Abstract

We consider a stochastic delay differential equation with exponentially stable drift and diffusion driven by a general Lévy process. The diffusion coefficient is only locally Lipschitz and bounded. Under a mild condition on the large jumps of the Lévy process, we show existence of an invariant measure. Major tools in the proof are a variation-of-constants formula and a stability theorem, which are of independent interest.

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1 Introduction

The main purpose of this paper is to show existence of an invariant measure for a stochastic delay differential equation of the form

$$\mathrm{d}X(t) = \left(\int_{[-\alpha,0]} X(t+s)\mu(\mathrm{d}s)\right)\mathrm{d}t + F(X)(t-)\mathrm{d}L(t), \tag{1}$$

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where L is a Lévy process, α a positive real, and μ is a signed Borel measure on $[-\alpha, 0]$. The diffusion coefficient F may be a function on $\mathbb{R}$ or a functional depending on the segment $(X(t+s)\colon\ -\alpha \le s \le t)$ of the solution X. If the underlying deterministic equation, that is equation (1) with $F = 0$, is exponentially stable, it may be expected that the stochastic equation has an invariant measure under suitable conditions on F. Existence of invariant measures has been shown for an increasingly general class of coefficients F. In 1982 Wolfe [26] dealt with the case where μ is a (negative) point mass at 0 and F is a constant. In 2000, Gushchin and Küchler [11] extended to the case of the general delay measure μ. More recently, Reiß et al. [22] considered nonlinear coefficients F. They assume a global Lipschitz condition on F, boundedness of F, and continuity of F with respect to the Skorohod topology. In each of these works the analysis depends on a variation-of-constants formula for equation (1). In the case of global Lipschitz F, such a formula has been proved in [23].

The theory of stochastic equations in a setting beyond globally Lipschitz conditions has been vastly extended during the last years in the field of stochastic partial differential equations, see, e.g., [5, 6, 13, 19, 20, 10] for some recent developments. Also in the field of stochastic delay differential equations there is interest in results on equations with coefficients that are not globally Lipschitz. In particular, models in financial mathematics can naturally involve a combination of delay, processes with jumps, and locally Lipschitz coefficients [24, (3.6)].

Our main contribution here is extending the results of [22] to equations with diffusion coefficients that are only locally Lipschitz instead of globally Lipschitz. Moreover, the continuity with respect to the Skorohod topology is relaxed to a condition that is considerably better suited for verification in examples. Included in our analysis is the proof of a variation-of-constants formula for (1) for locally Lipschitz F.

If the diffusion coefficient F is only locally Lipschitz with linear growth (see Definition 4.1 for the precise formulation), the eventual Feller property and the variation-of-constants formula, which play a key role in [22, 23], do not follow from the results given there. Establishing these results is the main content of this article. We do so by approximating the locally Lipschitz diffusion coefficient by globally Lipschitz coefficients in a suitable sense. The difficulty is to verify that the solutions of the equations with the approximated coefficients converge to the solution of (1) in an appropriate sense and that the limit inherits the desired properties. It turns out that the reduction steps in the proof of the variation-of-constants formula in [23] have to be changed. The extension to locally Lipschitz coefficients has to be done before increasing the generality of the other components and these steps have to be adapted accordingly. The proof of the eventual Feller property is based on new estimates, which relax the conditions on F even in the globally Lipschitz case. Moreover, we prove a stability theorem.

Two comments on the form of (1) are in order. First, in the spirit of [21, Chapter V] we present our results for the one dimensional equation. At the cost of more complicated notation our arguments can be extended to equations in $\mathbb{R}^n$. Second, (1) is formulated with a linear drift term. However, nonlinear drift terms are covered as well by our theory, due to the generality of the noise processes that we allow. By doubling the dimension and including deterministic components in the process L, locally Lipschitz nonlinearities in the drift are included.

For other approaches to stochastic delay differential equations and invariant measures, see, e.g., [16, 17, 18].

The outline of our arguments is roughly as follows. If the diffusion coefficient F in (1) maps the Skorohod space of real valued càdlàg functions on $[-\alpha, 0]$ into itself and satisfies a suitable locally Lipschitz and growth condition, then it is known that equation (1) has a unique solution X for any initial process on $[-\alpha, 0]$ (see [14]). The solution itself is, however, not a Markov process. Instead, one can consider the segments $X_t = (X(t+a))_{-\alpha\le a\le 0}$ of the solution process. If a solution $X(t)$ of equation (1) is such that all of the segments X_t have the same distribution, then the solution itself is stationary as well. Therefore we want to apply the Krylov-Bogoliubov method to the segment process, and for that we need the state space to be separable. If the driving process L has continuous paths, $(X_t)_t$ takes values in $C[-\alpha, 0]$, which is separable with the supremum norm $\|.\|_\infty$. In general, L may have jumps and then $(X_t)_t$ is a process with as state space the Skorohod space $D[-\alpha, 0]$ of càdlàg functions. This space is not separable under $\|.\|_\infty$, but it is separable when endowed with the Skorohod metric. In order to apply the Krylov-Bogoliubov method and obtain an invariant measure, we need the eventual Feller property (see (13) and (14) in Section 4 below) and tightness of the segment process given by (1). We follow the approach of [22], where the tightness is obtained by means of suitable estimates on the semimartingale characteristics.

In Section 2 we give a brief review of the facts about the Skorohod space and deterministic delay equations that we need in the sequel. In Section 3 we obtain the variation-of-constants formula. In Section 4 we give a precise formulation of the input of equation (1) and introduce the segment process. Section 5 deals with tightness of the segment process. The stability theorem is proved in Section 6 and the Markov and eventual Feller properties and the existence of an invariant measure are established in Section 7.

2 Preliminaries

All the processes we consider are defined on the same filtered probability space $(\Omega, \mathcal{F}_t, \mathcal{F}, \mathbb{P})$. Since we are going to work with a Markov process whose state space is the Skorohod space we recall some facts about it. For $a < b$, let $D[a, b]$ and $D[a, \infty)$ denote the linear spaces of all real-valued càdlàg functions defined on $[a, b]$ and $[a, \infty)$, respectively. Similarly, for $t_0 > 0$, let $\mathbb{D}[0, t_0]$ denote the space of adapted càdlàg processes on $[0, t_0]$ and likewise $\mathbb{D}[0, \infty)$. On $D[a, \infty)$ the Skorohod metric is given by

$$d_S(\varphi, \psi) = \inf_{\lambda\in\Lambda[a,\infty)} \left(\|\varphi\circ\lambda - \psi\|_\infty + |||\lambda|||\right),$$

where $\Lambda[a, \infty)$ is the set of all increasing $[a, \infty) \to [a, \infty)$ homeomorphisms and

$$|||\lambda||| = \sup_{a\le s<t,\, s\neq t} \left|\log \frac{\lambda(t) - \lambda(s)}{t - s}\right|.$$

The space $(D[a,\infty), d_S)$ is complete and separable. Similarly, there is a complete separable metric on $D[a,b]$, which we also denote by d_S.

We will also use a weaker metric on $D[-\alpha,0]$. Consider an arbitrary $\beta > \alpha$. We extend a $\varphi \in D[-\alpha,0]$ to $\overline{\varphi} \in D[-\beta,0]$ by $\overline{\varphi}(t) = \varphi(t)$ for $t \in [-\alpha,0]$ and $\overline{\varphi}(t) = 0$ for $t \in [-\beta,-\alpha)$. Let the metric d_β on $D[-\alpha,0]$ be defined by

$$d_\beta(\varphi,\psi) := d_{D[-\beta,0]}(\overline{\varphi},\overline{\psi}), \quad \varphi,\psi \in D[-\alpha,0],$$

where $d_{D[-\beta,0]}$ denotes the Skorohod metric on $D[-\beta,0]$. It is straightforward to verify that $d_\beta(\varphi,\psi) \le d_S(\varphi,\psi)$ for all $\varphi,\psi \in D[-\alpha,0]$, where d_S still denotes the Skorohod metric on $D[-\alpha,0]$. The metric d_β is actually independent of β and one could even choose $\beta = \infty$.

Lemma 2.1. *1. $(D[-\alpha,0], d_\beta)$ is a Polish space.*

2. d_β and d_S generate the same Borel σ-algebra $\mathcal{B}(D[-\alpha,0])$.

Proof. 1. The set $A := \{\overline{\varphi}\colon\ \varphi \in D[-\alpha,0]\}$ is a closed subset of $D[-\beta,0]$, which is a Polish space. Indeed, Skorokhod convergence implies almost everywhere convergence and we can finish the argument by right continuity of the limit. 2. As $d_\beta \le d_S$, the d_β-Borel σ-algebra is contained in the d_S-Borel σ-algebra. For the opposite inclusion it is enough to show that finite dimensional sets, that is, $\{\varphi \in D[-\alpha,0]\colon\ (\varphi(s_1),\dots,\varphi(s_n)) \in C\}$, where $C \in \mathcal{B}(\mathbb{R}^n)$ and $-\alpha \le s_1 \le \cdots \le s_n \le 0$, are in the d_β-Borel σ-algebra (see [3, (15.2) on p. 157]). This is obvious as $D[-\alpha,0], d_\beta$ is a subspace of $(D[-\beta,0], d_{D[-\beta,0]})$ □

Next we collect some results on the deterministic delay equation

$$\begin{aligned} &x(t) = \varphi(0) + \textstyle\int_0^t \left(\int_{[-\alpha,0]} x(s+a)\mu(\mathrm{d}a) \right) \mathrm{d}s \quad \text{for } t \ge 0, \\ &x(a) = \varphi(a) \quad \text{for } a \in [-\alpha,0]. \end{aligned} \tag{2}$$

Here $\alpha > 0$, μ is a finite signed Borel measure on $[-\alpha,0]$, and the initial condition $\varphi \in D[-\alpha,0]$. The results that we need can be found in a more general framework in [9]. However, we can give these results in a more easily accessible way as follows. According to [8, Theorem (i), p. 972], for each $\varphi \in D[-\alpha,0]$, there exists a unique function $x\colon [-\alpha,\infty) \to \mathbb{R}$ whose restriction to $[0,\infty)$ is continuous and which satisfies (2). Indeed, the map $\psi \mapsto \int_{[-\alpha,0]} \psi(a)\,\mu(\mathrm{d}a)$ is a bounded linear map from $C[-\alpha,0]$ with the uniform norm into $\mathbb{R}$. Therefore it is also bounded on the Sobolev space $W^{1,1}[-\alpha,0]$, as $W^{1,1}$ is continuously embedded in C_b. Since each $\varphi \in D[-\alpha,0]$ is bounded and measurable, we have $(\varphi(0),\varphi) \in M^1 := \mathbb{R} \times L^1[-\alpha,0]$. Hence we may apply [8, Theorem (i)] to obtain a unique solution x to (2.6) of [8]. By Fubini theorem x is the unique solution of (2).

To stress the dependence on the initial condition, we denote the solution of (2) by $x(\cdot,\varphi)$. The solution corresponding to initial condition $\varphi(s) = 0$ for $-\alpha \le s < 0$ and $\varphi(0) = 1$ is called the *fundamental solution* and denoted by r.

The following variation-of-constants formula for $x(\cdot,\varphi)$ holds,

$$x(t,\varphi) = r(t)\varphi(0) + \int_0^t r(t-s) \int_{[-\alpha,-s)} \varphi(s+a)\mu(\mathrm{d}a)\mathrm{d}s, \ t \ge 0, \tag{3}$$

where the inner integral is considered to vanish for $s > \alpha$, hence the outer integral is actually from 0 to $t \wedge \alpha$. Indeed, by Fubini and substitutions one can verify that the right hand side of (3) satisfies (2) and therefore equals the unique solution $x(\cdot, \varphi)$. By similar arguments one can rewrite formula (3) as

$$x(t, \varphi) = \varphi(0)r(t) + \int_{[-\alpha,0]} \int_s^0 r(t + s - a)\varphi(a)\mathrm{d}a\mu(\mathrm{d}s) \quad \text{for } t \geq 0. \tag{4}$$

The delay equation (2) is said to be *stable* if the fundamental solution r converges to zero as $t \to \infty$. The condition

$$v_0(\mu) := \sup\left\{\operatorname{Re}\lambda\colon\ \lambda \in \mathbb{C},\ \lambda - \int_{[-\alpha,0]} \mathrm{e}^{\lambda s}\mu(\mathrm{d}s) = 0\right\} < 0 \tag{5}$$

implies the even stronger property of exponential stability of all solutions , i.e., there exist $\gamma, K > 0$ such that $|x(t, \varphi)| \leq Ke^{-\gamma t}\|\varphi\|_\infty$ for all $t \geq 0$ and for any solution $x(\cdot, \varphi)$ of (2). Indeed, for the stability of the fundamental solution see the text below Corollary 4.1 on p. 182 of [12], and then the exponentrial stability of arbitrary solutions with initial condition $\varphi \in D[-\alpha, 0]$ follows by direct computation from (3).

It is clear from (2) that each of its solutions $x(\cdot, \varphi)$ is absolutely continuous on $[0, T]$ for every $T > 0$ and even continuously differentiable on (α, ∞). If (5) holds, then (2) yields the exponential decay of the derivative $\dot{x}(\cdot, \varphi)$ directly.

3 Variation-of-constants formula

This section establishes a variation-of-constants formula for equation (1). The major point is to show existence and uniqueness for equations of variation-of-constants form.

Recall that for two local martingales M and N their quadratic covariation process is denoted by $[M, N]$. Recall also that for $t \geq 0$, $\int_0^t |dA(s)|$ ($\int_0^\infty |dA(s)|$) denotes the pathwise total variation on $[0, t]$ ($[0, \infty)$, respectively) of a process $A \in D[0, \infty)$. The next two definitions are taken from [21, Section V.2, p. 250–251].

Definition 3.1. Let $1 \leq p \leq \infty$. For a process $X \in \mathbb{D}[0, \infty)$ set

$$\|X\|_{\underline{\underline{S}}^p} := \|\sup_{t\geq 0} |X(t)|\,\|_{L^p}$$

and let $\underline{\underline{S}}^p[0, \infty)$ denote the Banach space of $X \in \mathbb{D}[0, \infty)$ for which $\|X\|_{uuS^p}$ is finite. Further for a local martingale M with $M(0) = 0$ and a càdlàg adapted process A with paths of finite variation on compact intervals a.s. and $A(0) = 0$, set

$$j_p(M, A) := \|[N, N]_\infty^{1/2} + \int_0^\infty |dA(s)|\|_{L^p}.$$

For a semimartingale Z with $Z(0) = 0$ set

$$\|Z\|_{\underline{\underline{H}}^p} := \inf_{Z=M+A} j_p(M, A),$$

where the infimum is taken over all possible decompositions $Z = M + A$ where M is a local martingale with $M(0) = 0$ and a A is a càdlàg adapted proces with paths of finite variation on compact intervals a.s. and $A(0) = 0$. Furthermore, let $\underline{\underline{H}}^p[0,\infty)$ denote the Banach space of all semimartingales Z with $Z(0) = 0$ such that $\|Z\|_{\underline{\underline{H}}^p}$ is finite. For $t_0 > 0$, the analogous Banach spaces of processes only defined on $[0, t_0]$ are denoted by $\underline{\underline{S}}^p[0, t_0]$ and $\underline{\underline{H}}^p[0, t_0]$.

For $1 \leq p < \infty$ there exists a constant c_p such that

$$\|Z\|_{\underline{\underline{S}}^p} \leq c_p \|Z\|_{\underline{\underline{H}}^p} \tag{6}$$

for all semimartingales Z with $Z(0) = 0$ (see [21, Theorem V.2, p.252]. For a process $X \in \mathbb{D}[0,\infty)$ and a stopping time T, the *pre-stopped process* X^{T-} is given by

$$(X^{T-})(t)(\omega) = X(t)(\omega)\mathbf{1}_{\{0\leq t<T(\omega)\}} + X(t\wedge T(\omega)-)(\omega)\mathbf{1}_{\{t\geq T(\omega)>0\}}, \quad \omega\in\Omega,\ t\geq 0.$$

Observe that $X^{T-} \in \mathbb{D}[0,\infty)$.

The following definition from [23] is essentially from [21, p. 256].

Definition 3.2. A map $\Psi\colon \mathbb{D}[0,\infty) \to \mathbb{D}[0,\infty)$ is called *functional Lipschitz* if for any $X, Y \in \mathbb{D}[0, t_0]$

(a) for any stopping time T, $X^{T-} = Y^{T-}$ implies $\Psi(X)^{T-} = \Psi(Y)^{T-}$

(b) there exists a (positive finite) adapted increasing process K such that

$$|\Psi(X)(\omega,t) - \Psi(Y)(\omega,t)| \leq K(\omega,t) \sup_{s\leq t} |X(\omega,s) - Y(\omega,s)|.$$

Definition 3.3. A map $\Psi\colon \mathbb{D}[0,\infty) \to \mathbb{D}[0,\infty)$ is called *locally Lipschitz functional with linear growth* if it satisfies

(a) for all $X, Y \in \mathbb{D}[0,\infty)$ and all stopping times T

$$X^{T-} = Y^{T-} \quad \Longrightarrow \quad \Psi(X)^{T-} = \Psi(Y)^{T-}$$

(b) for each $n \in \mathbb{N}$ there exists an adapted increasing process K_n such that for all $t \geq 0$ and all $\omega \in \Omega$,

$$|\Psi(X)(\omega,t) - \Psi(Y)(\omega,t)| \leq K_n(\omega,t) \sup_{s\leq t} |X(\omega,s) - Y(\omega,s)|$$

whenever $|X(\omega,s)|, |Y(\omega,s)| \leq n$ for all $s \leq t$

(c) there exists a positive increasing adapted process $\gamma(t)$ such that

$$|\Psi(X)(\omega,t)| \leq \gamma(\omega,t)(1 + \sup_{s\leq t} |X(\omega,s)|) \text{ for all } X \in \mathbb{D}[0,\infty).$$

Notice that by (b): $X^{t_0} = Y^{t_0}$ implies $\Psi(X)^{t_0} = \Psi(X)^{t_0}$, for any $X, Y \in \mathbb{D}[0,\infty)$. Therefore, for $U \in \mathbb{D}[0,t_0]$ we can define $\Psi(U) \in \mathbb{D}[0,t_0]$ unambiguously, simply by extending U to $[0,\infty)$. Notice also that any functional Lipschitz map is a locally Lipschitz functional with linear growth.

In the sequel we will need the following condition on a function $g\colon [0,t_0] \to \mathbb{R}$.

Condition 1. For every $Y \in \underline{\underline{S}}^p[0,t_0]$ and $Z \in \underline{\underline{H}}^\infty[0,t_0]$,

$$\int_0^\cdot g(\cdot - s)Y(s-)\mathrm{d}Z(s) \in \underline{\underline{H}}^p[0,t_0] \text{ and}$$
$$\left\| \int_0^\cdot g(\cdot - s)Y(s-)\mathrm{d}Z(s) \right\|_{\underline{\underline{H}}^p[0,t_0]} \le R\|Y\|_{\underline{\underline{S}}^p[0,t_0]}\|Z\|_{\underline{\underline{H}}^\infty[0,t_0]},$$

where $t_0 > 0$, $R > 0$ and $1 < p < \infty$ are given constants.

Lemma 3.4. *Let $t_0 > 0$ be given and let g satisfy Condition 1 for some p and R. Assume that Ψ is locally Lipschitz with linear growth, $\Psi(0) = 0$, and such that the processes $K_n(t)$ and $\gamma(t)$ are constants. Let $Z \in \underline{\underline{H}}^\infty[0,t_0]$ with $\|Z\|_{\underline{\underline{H}}^\infty} < \frac{1}{4c_pR\gamma}$ and let $J \in \underline{\underline{S}}^p[0,t_0]$. Then for any stopping time T the equation*

$$X(t) = J^{T-}(t) + \left(\int_0^\cdot g(\cdot - s)\Psi(X)(s-)\mathrm{d}Z(s) \right)^{T-}(t), \quad 0 \le t \le t_0$$

has a unique solution $X \in \underline{\underline{S}}^p[0,t_0]$. Moreover, X is a semimartingale if J is.

Proof. For each $n \in \mathbb{N}$ and $X \in \mathbb{D}[0,t_0]$ let $X^{(n)}(t) := (X(t) \wedge n) \vee (-n)$, $t \in [0,t_0]$. Now define $\Psi^n\colon \mathbb{D}[0,t_0] \to \mathbb{D}[0,t_0]$ by

$$\Psi^n(X) := \Psi(X^{(n)}).$$

Then Ψ^n is functional Lipschitz: (a) is immediate and for (b) notice that

$$\begin{aligned}|\Psi^n(X)(t) - \Psi^n(Y)(t)| &\le K_n \sup_{s\le t} |X^{(n)}(s) - Y^{(n)}(s)| \\ &\le K_n \sup_{s \le t} |X(s) - Y(s)|.\end{aligned}$$

By [21, Theorem V.5(ii), p. 254], there is a stopping time T^n such that $\mathbb{P}(T^n > t_0) > 1-2^{-n}$ and $Z^{T^n-} \in \mathcal{S}(\frac{1}{2c_pK_nR})$ (notation of [21]), with c_p as in (6). Then by [23, Lemma 5.6] there is a unique $X^n \in \underline{\underline{S}}^p[0,t_0]$ such that

$$X^n(t) = J^{T-}(t) + \left(\int_0^t g(t-s)\Psi^n(X^n)(s-)\mathrm{d}Z^{T^n}(s) \right)^{T-}$$

and we have

$$\begin{aligned}
\|X^n\|_{\underline{\underline{S}}^p} &\le \|J^{T-}\|_{\underline{\underline{S}}^p} + \left\| \left(\int_0^{\cdot} g(\cdot - s)\Psi^n(X^n)(s-) \mathrm{d}Z^{T^n-}(s) \right)^{T-} \right\|_{\underline{\underline{S}}^p} \\
&\le 2\|J\|_{\underline{\underline{S}}^p} + 2c_p \left\| \int_0^{\cdot} g(\cdot - s)\Psi^n(X^n)(s-) \mathrm{d}Z^{T^n-}(s) \right\|_{\underline{\underline{H}}^p} \\
&\le 2\|J\|_{\underline{\underline{S}}^p} + 2c_p R \|\Psi^n(X^n)\|_{\underline{\underline{S}}^p} \|Z^{T^n-}\|_{\underline{\underline{H}}^\infty} \\
&\le 2\|J\|_{\underline{\underline{S}}^p} + 2c_p R\gamma(1 + \|X^n\|_{\underline{\underline{S}}^p}) 2\|Z\|_{\underline{\underline{H}}^\infty},
\end{aligned}$$

where the second inequality follows from [21, Theorem V.2 and proof of Theorem V.5], the third by Condition 1, and the fourth by (c) of Definition 3.3. Since $4c_p R\gamma\|Z\|_{\underline{\underline{H}}^\infty} < 1$ by assumption, we obtain

$$\|X^n\|_{\underline{\underline{S}}^p} \le \frac{2\|J\|_{\underline{\underline{S}}^p} + 4c_p R\gamma\|Z\|_{\underline{\underline{H}}^\infty}}{1 - 4c_p R\gamma\|Z\|_{\underline{\underline{H}}^\infty}} =: C < \infty \text{ for all } n.$$

In particular

$$\mathbb{P}(\sup_{0\le s\le t_0} |X^n(s)| > n) \le \frac{C^p}{n^p} \text{ for all } n \in \mathbb{N}.$$

Let now $A_n := \{\sup_{0\le s\le t_0} |X^n(s)| \le n\}$ and $B_n := \{T^n > t_0\}$. Then$\mathbb{P}(A_n \cap B_n) \ge 1 - 2^{-n} - C^p/n^p$. Set $\widetilde{\Omega}_n := A_n \cap B_n$, $\Omega_n := \bigcap_{k\ge n} \widetilde{\Omega}_k$, $\Omega' := \bigcup_n \Omega_n$, so that $\Omega_n \subset \Omega_{n+1}$, $\mathbb{P}(\Omega_n) \uparrow \mathbb{P}(\Omega') = 1$. On Ω_n we have that

$$\sup_{0\le s\le t_0} |X^k(s)| \le k, \quad Z^{T^k-} = Z,$$

and also $\Psi^{k_1}(X^k) = \Psi^k(X^k) = \Psi(X^k)$ for $k_1 \ge k \ge n$.

If we consider measures $\mathbb{P}^n$ given by $\mathbb{P}^n(A) := \mathbb{P}(A \cap \Omega_n)/\mathbb{P}(\Omega_n)$ for n so large that $\mathbb{P}(\Omega_n) > 0$, we have for $k \ge n$ that

$$\begin{aligned}
X^k(t) &= J^{T-}(t) + \left(\int_0^t g(t-s)\Psi^k(X^k)(s-)\, \mathrm{d}Z(s) \right)^{T-} \\
&= J^{T-}(t) + \left(\int_0^t g(t-s)\Psi(X^k)(s-)\, \mathrm{d}Z(s) \right)^{T-}, \\
X^n(t) &= J^{T-}(t) + \left(\int_0^t g(t-s)\Psi^n(X^n)(s-)\, \mathrm{d}Z(s) \right)^{T-} \\
&= J^{T-}(t) + \left(\int_0^t g(t-s)\Psi(X^n)(s-)\, \mathrm{d}Z(s) \right)^{T-} \\
&= J^{T-}(t) + \left(\int_0^t g(t-s)\Psi^k(X^n)(s-)\, \mathrm{d}Z(s) \right)^{T-}
\end{aligned}$$

$\mathbb{P}_n$-a.s., the stochastic integrals being computed according to the probability $\mathbb{P}_n$. This is easily seen by applying [21, Theorem II.14 and Theorem II.18] and the fact that the stochastic convolutions above are $\mathbb{P}$-a.s. càdlàg. Since Ψ^k is functional Lipschitz in the sense of [23, Definition 5.1]), the uniqueness in [23, Proposition 5.8] yields that we have for $k \geq n$ that $X^k = X^n$ $\mathbb{P}^n$-a.s., thus $\mathbb{P}$-a.s. on Ω_n. Hence there is a process X such that for almost all $\omega \in \Omega'$ and all $t \geq 0$

$$X(t,\omega) = \lim_{n\to\infty} X^n(t,\omega),$$

which is adapted and càdlàg, since the filtration satisfies the usual conditions. Moreover, we have for each n that

$$\begin{aligned} X = X^n &= J^{T-} + \left(\int_0^{\cdot} g(\cdot - s)\Psi^n(X^n)(s-)\,\mathrm{d}Z^{T^n-}(s)\right)^{T-} \\ &= J^{T-} + \left(\int_0^{\cdot} g(\cdot - s)\Psi(X)(s-)\,\mathrm{d}Z(s)\right)^{T-} \end{aligned}$$

on Ω_n a.s. Hence X satisfies the equation. We also have that

$$\sup_{0\leq s\leq t_0} |X(s)|^p = \lim_{n\to\infty} \sup_{0\leq s\leq t_0} |X^n(s)|^p \ \mathbb{P}\text{-a.s.},$$

so that by Fatou's lemma,

$$\|X\|_{\underline{\underline{S}}^p} \leq \sup_n \|X^n\|_{\underline{\underline{S}}^p} < \infty.$$

For uniqueness, suppose X' is another solution in $\mathbb{D}[0,t_0]$. Consider the fundamental sequences $S^n := \inf\{t\colon |X(t)| > n\}$ and $S_1^n := \inf\{t\colon |X'(t)| > n\}$. Then on the set $C_n := \{S^n \wedge S_1^n > t_0\}$ the processes X and X' both solve the equation with Ψ^n, so by uniqueness in [23, Proposition 5.8] $X = X'$ on C_n. Hence $X = X'$ a.s.

Finally, if J is a semimartingale, X^{S^n-} is a semimartingale for each n, by (a) and (c) of Definition 3.3 together with Condition 1, hence X is a semimartingale as well. ☐

Lemma 3.5. *Assume the situation of Lemma 3.4 with $p > 2$, but without the condition $\|Z\|_{\underline{\underline{H}}^\infty[0,t_0]} < \frac{1}{4c_pR\gamma}$. Then for each stopping time T the equation*

$$X(t) = J^{T-}(t) + \left(\int_0^t g(t-s)\Psi(X)(s-)\,\mathrm{d}Z(s)\right)^{T-}, \quad 0 \leq t \leq t_0$$

has a unique solution in $\mathbb{D}[0,t_0]$. If J is a semimartingale, then X is a semimartingale as well.

Proof. Let Ψ^n be the functional Lipschitz maps as in the proof of Lemma 3.4. By [23, Proposition 5.8] for each n there is an $\widetilde{X}^n \in \mathbb{D}[0,t_0]$ such that

$$\widetilde{X}^n(t) = J(t) + \int_0^t g(t-s)\Psi^n(\widetilde{X}^n)(s-)\,\mathrm{d}Z(s),$$

which is a semimartingale if J is a semimartingale. (Indeed, before applying [23, Proposition 5.8], J and Z can be extended constantly after t_0 to $[0,\infty)$ and then the solution can be restricted to $[0,t_0]$.) Stop $\widetilde{X}^n$ at $T-$ to obtain

$$(\widetilde{X}^n)^{T-}(t) = J^{T-}(t) + \left(\int_0^t g(t-s)\Psi^n((\widetilde{X}^n)^{T-})(s-)\,\mathrm{d}Z(s)\right)^{T-}$$

and set $X^n := (\widetilde{X}^n)^{T-}$. As before, Condition 1 and [21, Theorem V.2] yield

$$\|X^n\|_{\underline{\underline{S}}^p[0,t_0]} \le 2\|J\|_{\underline{\underline{S}}^p[0,t_0]} + 2c_pR\|\Psi^n(X^n)\|_{\underline{\underline{S}}^p[0,t_0]}\|Z\|_{\underline{\underline{H}}^\infty[0,t_0]}.$$

By the linear growth of Ψ we have

$$\begin{aligned}|\Psi^n(X^n)(t)| &= |\Psi((X^n\wedge n)\vee(-n))(t)|\\ &\le \gamma(1+\sup_{0\le s\le t}|(X^n(s)\wedge n)\vee(-n)|) \le \gamma(1+n)\end{aligned}$$

and obtain

$$\|X^n\|_{\underline{\underline{S}}^p[0,t_0]} \le 2\|J\|_{\underline{\underline{S}}^p} + 2c_pR\gamma(1+n)\|Z\|_{\underline{\underline{H}}^\infty[0,t_0]}.$$

Let $A_n := \{\sup_{0\le t\le t_0}|X^n(t)| > n\}$, $n\in\mathbb{N}$. Then

$$\mathbb{P}(A_n) \le 2n^{-p}(\|J\|_{\underline{\underline{S}}^p[0,t_0]} + c_pR\gamma(1+n)\|Z\|_{\underline{\underline{H}}^\infty[0,t_0]})$$

by Chebyshev's inequality. Let further $\Omega_n := \bigcap_{k\ge n}(\Omega\setminus A_k)$. Notice that $\Omega_n\subset\Omega_{n+1}$ and $\mathbb{P}(\Omega_n)\uparrow 1$, since $p>2$. Define probability measures $\mathbb{P}^n(A) := \mathbb{P}(\Omega_n\cap A)/\mathbb{P}(\Omega_n)$ for $n\ge N$, where N is such that $\mathbb{P}(\Omega_N)>0$. Arguing as in Lemma 3.4 we obtain for $k\ge n$ that $X^k = X^n$ on Ω_n, so we can define a process $X := \lim X^n$, which is then the unique solution of the equation. If J is a semimartingale then so is X. ☐

Proposition 3.6. *Let Z be a semimartingale and $J\in\mathbb{D}[0,\infty)$. Let g be a function on $[0,\infty)$ which satisfies Condition 1 for each $t_0>0$ with some $R(t_0)>0$ and some fixed $p>2$. Further, suppose that Ψ is a locally Lipschitz functional such that the processes K_n and γ are deterministic. Then the equation*

$$X(t) = J(t) + \int_0^t g(t-s)\Psi(X)(s-)\,\mathrm{d}Z(s),\quad t\ge 0,$$

has a unique solution in $\mathbb{D}[0,\infty)$, which is a semimartingale if J is.

Proof. Fix $t_0>0$. First we show existence and uniqueness on $[0,t_0]$. There is a fundamental sequence T^ℓ of stopping times such that $J^{T^\ell-}\in\underline{\underline{S}}^\infty[0,\infty)$ and $Z^{T^\ell-}\in\underline{\underline{H}}^\infty[0,\infty)$. By the linear growth of Ψ and Condition 1, $\left(\int_0^t g(t-s)\Psi(0)(s-)\,\mathrm{d}Z(s)\right)_{t\ge0}$ is a semimartingale. Hence we may assume that $\Psi(0)=0$.

By the previous lemma, for each ℓ there is an X_ℓ such that

$$X_\ell(t) = J^{T^\ell-}(t) + \left(\int_0^t g(t-s)\Psi(X_\ell)(s-)\,\mathrm{d}Z^{T^\ell-}(s)\right)^{T^\ell-} \quad \text{for all } t \in [0, t_0].$$

Moreover,

$$X_{\ell+k}^{T^\ell-}(t) = J^{T^\ell-} + \left(\int_0^t g(t-s)\Psi(X_{\ell+k}^{T^\ell-})(s-)\,\mathrm{d}Z^{T^\ell-}(s)\right)^{T^\ell-}.$$

Define $X := \lim X_\ell$ and argue as in the two previous lemmas to show that X is the unique solution on $[0, t_0]$. Finally, we can use the same techniques to patch the solutions together obtaining a unique global solution. □

We can now show existence and uniqueness for the general equation of variation-of-constants form.

Theorem 3.7. *Let Ψ be a locally Lipschitz functional with linear growth and let g be a function on $[0, \infty)$ which satisfies Condition 1 for each $t_0 > 0$ with some $R(t_0) > 0$ and for some fixed $p > 2$. Suppose Z is a semimartingale and $J \in \mathbb{D}[0, \infty)$. Then the equation*

$$X(t) = J(t) + \int_0^t g(t-s)\Psi(X)(s-)\,\mathrm{d}Z(s) \tag{7}$$

has a unique solution in $\mathbb{D}[0, \infty)$, which is a semimartingale if J is.

Proof. Fix $t_0 > 0$. For $n, k \in \mathbb{N}$ there are constants $c_{n,k} > 0$, $\gamma_k > 0$ such that

$$\mathbb{P}(\Omega_{n,k}) \geq 1 - 2^{-n-k}, \text{ where } \Omega_{n,k} := \{K_n(t_0, \omega) \leq c_{n,k}\}$$
$$\mathbb{P}(\widetilde{\Omega}_k) \geq 1 - 2^{-k}, \text{ where } \widetilde{\Omega}_k := \{\gamma(t_0, \omega) \leq \gamma_k\}.$$

Set $\Omega_k := \bigcap_n \Omega_{n,k} \cap \widetilde{\Omega}_k$. Then $\mathbb{P}(\Omega_k) \geq 1 - 2^{-k+1}$, and on Ω_k, $K_n(t_0, \omega)$ and $\gamma(t_0, \omega)$ are bounded functions. Now apply the last part of the proof of Proposition 5.8 of [23], using existence and uniqueness from the previous proposition. □

As we mentioned in the introduction, the fundamental solution r of the deterministic delay equation is absolutely continuous on compacts and its derivative is bounded on compacts. Hence due to [23, Lemma 4.2], r satisfies Condition 1 for any $t_0 > 0$ and $p \geq 1$ with $R = 1 + (1 + c_p)t_0 \sup_{0 \leq t \leq t_0} |r'(t)|$. So there is a unique solution of the variation-of-constants formula (7) with $g = r$. This solution also satisfies the stochastic delay differential equation

$$\mathrm{d}X(t) = \left(\int_{[-\alpha,0]} X(t+s)\mu(\mathrm{d}s)\right)\mathrm{d}t + \Psi(X)(t-)\mathrm{d}Z(t). \tag{8}$$

This can be shown by applying the stochastic Fubini theorem. The result is actually Lemma 6.1 in [23]. Although the statement there presupposes Ψ to be functional Lipschitz, the only property of the functional Ψ used in the proof is that it is a map $\mathbb{D}[0, \infty) \to \mathbb{D}[0, \infty)$. As (8) has a unique solution (see [14]), it follows that this solution satisfies the variation-of-constants formula. Stated precisely:

Theorem 3.8. *Let μ be a finite signed Borel measure on $[-\alpha, 0]$ and let r be the fundamental solution of the deterministic delay equation. Let Ψ be a locally Lipschitz functional and let Z and J be semimartingales. The unique solution X of*

$$X(t) = X(0) + J(t) + \int_0^t \int_{[-s,0]} X(s+a)\,\mu(\mathrm{d}a)\mathrm{d}s + \int_0^t \Psi(X)(s-)\,\mathrm{d}Z(s), \quad t \geq 0,$$

satisfies

$$X(t) = r(t)X(0) + \int_0^t r(t-s)\,\mathrm{d}J(s) + \int_0^t r(t-s)\Psi(X)(s-)\,\mathrm{d}Z(s), \quad t \geq 0.$$

4 The equation and the segment process

In the remaining part of the paper we consider (1) and show that it has an invariant measure under suitable conditions.

Our approach is to see the stochastic equation (1) as a perturbation of the deterministic equation (2). If the deterministic part is stable it is plausible to expect existence of an invariant measure under mild conditions on the diffusion part. Therefore we assume that (5) holds. For an analysis of the case where (5) does not hold, see, e.g., [2].

As was shown in [11, Theorem 3.1], even if F is constant, a necessary condition for the existence of an invariant measure on the jumps of the Lévy process L is that $\int_{|x|>1} \log|x|\,\nu(\mathrm{d}x) < \infty$, where ν denotes the Lévy measure of L. As our situation is even more general, we need this condition as well.

As mentioned in the introduction, the main point of this paper is to relax the global Lipschitz condition in [22, Assumption 4.1(c)] to a locally Lipschitz condition. Our locally Lipschitz condition on F is the following.

Definition 4.1. A map $F\colon D[-\alpha,\infty) \to D[-\alpha,\infty)$ is called *a locally Lipschitz functional of deterministic type* (in short *lolidet*) if it satisfies

1 $\forall n \in \mathbb{N}\ \exists K_n > 0$ such that $\forall x, y \in D[-\alpha,\infty)\ \forall t \geq 0$

$$\sup_{s\in[t-\alpha,t]} |x(s)| \vee |y(s)| \leq n \Longrightarrow |F(x)(t) - F(y)(t)| \leq K_n \sup_{s\in[t-\alpha,t]} |x(s)-y(s)|,$$

2 $\exists \gamma > 0$ such that $\forall x \in D[-\alpha,\infty)\ \forall t \geq 0$

$$|F(x)(t)| \leq \gamma \left(1 + \sup_{s\in[t-\alpha,t]} |x(s)|\right).$$

Definition 4.2. Let $(\Phi(s))_{s\in[-\alpha,0]}$ be an initial condition, that is, a process with càdlàg paths and such that $\Phi(s)$ is $\mathcal{F}_0$-measurable for all $-\alpha \le s \le 0$. For $X \in \mathbb{D}[0,\infty)$ or $X \in \mathbb{D}[-\alpha,\infty)$ define $X_{\text{-}\Phi} \in \mathbb{D}[-\alpha,\infty)$ by

$$X_{\text{-}\Phi}(s) := \begin{cases} \Phi(s), & -\alpha \le s < 0, \\ X(s), & s \ge 0. \end{cases}$$

Here we extend the filtration by setting $\mathcal{F}_s := \mathcal{F}_0$ for $s < 0$. Define further $\Psi_\Phi \colon \mathbb{D}[0,\infty) \to \mathbb{D}[0,\infty)$ by

$$\Psi_\Phi(X)(t,\omega) := F(X_{\text{-}\Phi}(\cdot,\omega))(t).$$

By standard arguments, Ψ_Φ indeed maps into $\mathbb{D}[0,\infty)$.

Set $\Omega_0 = \emptyset$ and $\Omega_n := \{\sup_{[-\alpha,0]} |\Phi(s)| > n\}$ for $n \ge 1$. Then $\Omega_n \uparrow \Omega$. Define $C_n \colon \Omega \to \mathbb{R}$ and $\overline{\gamma} \colon \Omega \to \mathbb{R}$ by

$$\begin{aligned} &C_n(\omega) := K_n \text{ on } \Omega_n, \quad C_n(\omega) := K_{n+k} \text{ on } \Omega_{n+k} \setminus \Omega_{n+k-1} \\ &\overline{\gamma}(\omega) := \gamma(1+n) \text{ on } \Omega_n \setminus \Omega_{n-1}. \end{aligned}$$

Notice that C_n and $\overline{\gamma}$ are $\mathcal{F}_0$-measurable for all n. Moreover, Ψ_Φ satisfies

1. for $t \ge 0$ and $X, Y \in \mathbb{D}[0,\infty)$,
$$|\Psi_\Phi(X)(t,\omega) - \Psi_\Phi(Y)(t,\omega)| \le C_n(\omega) \sup_{s\in[(t-\alpha)^+,t]} |X(s,\omega) - Y(s,\omega)|$$
if $\sup_{s\in[(t-\alpha)^+,t]} |X(s,\omega)| \vee |Y(s,\omega)| \le n$, and
2. $$|\Psi_\Phi(X)(t,\omega)| \le \overline{\gamma}(\omega) \left(1 + \sup_{[(t-\alpha)^+,\alpha]} |X(t,\omega)|\right)$$
for all $X \in \mathbb{D}[0,\infty)$.

In other words, Ψ_Φ satisfies (b) and (c) of Definition 3.3, and (a) is obvious.

Define

$$J_\Phi(t) := \int_0^t \int_{[-\alpha,-s)} \Phi(s+a)\mu(\mathrm{d}a)\mathrm{d}s, \quad t \ge 0.$$

Then J_Φ is an adapted (by Fubini arguments) càdlàg process of finite variation. Moreover, $r(t)\Phi(0)$ is an adapted process of finite variation. Hence by [23, Theorem 4.1], $r(t)\Phi(0) + \int_0^t r(t-s)\,\mathrm{d}J(s)$ is a semimartingale.

Let X be the unique solution (see [14, Theorem 4.5]) of

$$\begin{aligned} X(t) &= \Phi(0) + \int_0^t \int_{[-\alpha,0]} X_{\text{-}\Phi}(s+a)\mu(\mathrm{d}a)\mathrm{d}s + \int_0^t F(X_{\text{-}\Phi})(s-)\,\mathrm{d}L(s) \\ &= \Phi(0) + J_\Phi(t) + \int_0^t \int_{[-s,0]} X(s+a)\,\mu(\mathrm{d}a)\mathrm{d}s \\ &\quad + \int_0^t \Psi_\Phi(X)(s-)\,\mathrm{d}L(s). \end{aligned}$$

By Theorem 3.8, X is also a solution of

$$X(t) = r(t)\Phi(0) + \int_0^t r(t-s)\,\mathrm{d}J(s) \\ + \int_0^t r(t-s)\Psi_\Phi(X)(s-)\,\mathrm{d}L(s).$$

Because of (3) Theorem 3.8 takes the following form in the current setting.

Theorem 4.3. *Let $F\colon D[-\alpha,\infty) \to D[-\alpha,\infty)$ be lolidet. Then for $X \in \mathbb{D}[0,\infty)$ and $\Phi \in \mathbb{D}[-\alpha,0]$ the following two statements are equivalent:*

1. *X is the unique solution of*

$$X(t) = \Phi(0) + \int_0^t \int_{[-\alpha,0)} X_{\text{-}\Phi}(s+a)\,\mu(\mathrm{d}a)\mathrm{d}s + \int_0^t F(X_{\text{-}\Phi})(s-)\,\mathrm{d}Z(s),\ t \geq 0,$$

2. *X obeys the variation-of-constants formula*

$$X(t) = x(t,\Phi) + \int_0^t r(t-s)F(X_{\text{-}\Phi})(s-)\,\mathrm{d}Z(s), \quad t \geq 0.$$

Recall that for a process $X \in \mathbb{D}[-\alpha,\infty)$ we denote by $(X_t)_{t\geq 0}$ the segment process, which takes values in $D[-\alpha,0]$ for each t. More precisely, $X_t(s) = X(t+s)$ for $-\alpha \leq s \leq 0$. We wish to show that the segment process is Markov. For this to be true F obviously has to be autonomous in the sense of the following definition.

Definition 4.4. A map $F\colon D[-\alpha,\infty) \to D[-\alpha,\infty)$ is *autonomous* if for all $x \in D[-\alpha,\infty)$ and all $s,t \geq 0$,

$$F(x(s+\cdot))(t) = F(x)(s+t).$$

Assume that F is autonomous. For $u \geq 0$ and $(Y(s))_{-\alpha\leq s\leq 0}$ càdlàg and $\mathcal{F}_u$-measurable, we consider the equation

$$\begin{aligned} X(t) &= Y(0) + \int_0^t \int_{[-\alpha,0]} X(s+a)\mu(\mathrm{d}a)\,\mathrm{d}s \\ &\quad + \int_0^t F(X)(s-)\,\mathrm{d}L^u(s), \quad t \geq 0, \\ X(t) &= Y(t), \quad -\alpha \leq t < 0, \end{aligned}$$

where $L^u(t) = L(t+u) - L(u)$, $t \geq 0$. The underlying filtration $\mathcal{G}_t^u$ is (the right continuous version of) $\sigma(L(s+u)-L(u)\colon\ 0 \leq s \leq t) \vee \mathcal{F}_u$. Denote by $(X_Y^u(t))_{t\geq -\alpha}$ the unique solution of this equation and let $(X_{Y,t}^u)_{t\geq 0}$ denote the corresponding segment process.

For any $\mathcal{F}_0$-adapted initial condition Φ and $t \geq 0$ the process $X_\Phi := X_\Phi^0$ satisfies

$$
\begin{aligned}
X_\Phi(u+\cdot)(t) - X_\Phi(u) &= X_\Phi(t+u) - X_\Phi(u) \\
&= \int_u^{t+u} \int_{[-\alpha,0]} X_\Phi(s+a)\mu(\mathrm{d}a)\,\mathrm{d}s + \int_u^{t+u} F(X_\Phi)(s-)\,\mathrm{d}L(s) \\
&= \int_0^t \int_{[-\alpha,0]} X_\Phi(u+\cdot)(s+a)\mu(\mathrm{d}a)\,\mathrm{d}s + \int_0^t F(X_\Phi(u+\cdot))(s-)\,\mathrm{d}L^u(s),
\end{aligned}
$$

where the latter equality holds by the fact that F is autonomous. The process L^u is a Lévy process relative to the filtration $\mathcal{F}_{u+\cdot}$ and $\mathcal{G}_t^u \subset \mathcal{F}_{t+u}$ for all $t \geq 0$, hence $X^u_{X_{\Phi,u}}$ is also a solution of the equation relative to the filtration $\mathcal{F}_{u+\cdot}$ (see [21, Theorem II.16]). Hence

$$X^u_{X_{\Phi,u}}(t) = X_\Phi(t+u) \tag{9}$$

for all $t \geq 0$, due to the strong uniqueness of the equation.

Under additional conditions we will show below that the segment $(X_{\Phi,t})_t$ is a Markov process, that is,

$$\mathbb{E}[\mathbf{1}_B(X^u_{X_{\Phi,u},t})|\mathcal{F}_u] = \mathcal{A}_B(X_{\Phi,u}), \tag{10}$$

where

$$\mathcal{A}_B = \mathbb{E}\mathbf{1}_B(X^u_{\varphi,t}), \qquad \varphi \in D[-\alpha,0],$$

for every $B \in \mathcal{B}(D[-\alpha,0])$. Notice that $X_{\Phi,t}$ is $\mathcal{F}/\mathcal{B}(D[-\alpha,0])$-measurable, since the Borel σ-algebra generated by the Skorohod topology $\mathcal{B}(D[-\alpha,0])$ equals the σ-algebra generated by the finite dimensional set [3, (15.2) on p. 157]. By $B_b(D[-\alpha,0])$ we denote the space of bounded Borel (relative to the Skorohod topology) functions on $D[-\alpha,0]$. For $0 \leq s \leq t$ we define

$$P_{s,t}f(\varphi) := \mathbb{E}f(X^s_{\varphi,t-s}), \quad \varphi \in D[-\alpha,0],\ f \in B_b(D[-\alpha,0]). \tag{11}$$

We will show that $P_{s,t}$ maps $B_b(D[-\alpha,0])$ into $B_b(D[-\alpha,0])$, that $P_{u,t} = P_{u,s}P_{s,t}$ for $0 \leq u \leq s \leq t$, and that $P_{s,t} = P_{0,t-s}$.

Then $X_{\Phi,t}$ is a homogeneous Markov process and the operators

$$P_t := P_{0,t} \tag{12}$$

form a Markovian semigroup. We will also show that $(P_t)_{t\geq 0}$ is *eventually Feller* in the following sense:

1. for $f \in C_b(D[-\alpha,0])$, $t \geq \alpha$,

$$P_t f \in C_b(D[-\alpha,0]); \tag{13}$$

2. for $t \geq \alpha$, $f \in C_b(D[-\alpha,0])$,

$$\lim_{s\downarrow t} P_s f(\varphi) = P_t f(\varphi) \text{ uniformly in } \varphi. \tag{14}$$

Since $\Delta X_\varphi(t) = F(X_\varphi)(t-)\Delta L(t)$, and L is stochastically continuous, we have that X_φ is stochastically continuous, hence by [22, Lemma 3.2], the segment process $X_{\varphi,t}$ is stochastically continuous as well. So by bounded convergence and subsequence arguments, 2. follows.

The proof of 1. will be more involved and is given in Section 7.

In the sequel we will use several assumptions on the input to our equation (1), so we list them here.

Assumption 1.

1. $v_0(\mu) < 0$ (see (5)).
2. The Lévy measure ν of L satisfies $\int_{|x|>1} \log|x| \, \mathrm{d}\nu(x) < \infty$.
3. F is lolidet.
4. F is autonomous and bounded.

We remark that boundedness of F implies growth condition 2. of Definition 4.1.

5 Tightness of segments

In this section we obtain the tightness of the segment process under Assumption 1. We start by showing that for each fixed $T \geq 0$, and any uniformly bounded sequence Φ_n of initial conditions (i.e. $\sup_n \sup_{-\alpha\leq s\leq 0} \sup_{\omega\in\Omega} |\Phi_n(s,\omega)| < \infty$), that

$$\sup_n \mathbb{P}(\sup_{t\leq T} |X_{\Phi_n}(t)| > K) \to 0 \text{ as } K \to \infty \tag{15}$$

(recall that X_Φ denotes the solution of (1) with initial condition Φ), by showing the following stronger result.

Theorem 5.1. *Let Φ_n be a uniformly bounded sequence of initial conditions. Let further Assumption 1 hold. Then the laws of $(X_{\Phi_n}(t+s)\colon s \in [0,\alpha])_{t\geq 0,\, n\in\mathbb{N}}$ form a tight set. Consequently,*

$$\sup_n \sup_{t\geq 0} \mathbb{P}(\sup_{0\leq u\leq \alpha} |X_{\Phi_n}(t+u)| > K) \to 0 \textit{ as } K \to \infty. \tag{16}$$

Notice that (16) implies (15). In order to prove Theorem 5.1, we need several lemmas.

Lemma 5.2. *Suppose $v_0(\mu) < 0$. The fundamental solution r of the deterministic delay equation is C^1 on $[\alpha,\infty)$ and its total variation $TV_{[-\alpha,\infty)} r$ is finite.*

Proof. Since r is absolutely continuous on $[0,T]$ for every $T > 0$, we have that for $s > t$ and $s_n \to s$ that

$$\int_{[-\alpha,0]} r(s_n + a)\mu(\mathrm{d}a) =: g(s_n) \to g(s) := \int_{[-\alpha,0]} r(s+a)\mu(\mathrm{d}a),$$

by bounded convergence. So $h(t) := \int_\alpha^t g(s)\,\mathrm{d}s$, $t > \alpha$, is an antiderivative of a continuous function, hence C^1, and $h'(t) = g(t)$, which is a continuous function on $[\alpha,\infty)$. Moreover, the estimate $|g(t)| \le C' \exp(-\beta t)$ holds for some $C', \beta > 0$, so that $TV_{[\alpha,\infty)} r \le \int_\alpha^\infty C' \exp(-\beta t)\,\mathrm{d}t < \infty$. Since r is absolutely continuous on $[0,\alpha]$ and $TV_{[-\alpha,0]} r = 1$, we obtain that $TV_{[-\alpha,\infty)} r < \infty$. □

Lemma 5.3. *Suppose $v_0(\mu) < 0$ and let $\dot r(t) := \int_{[-\alpha,0]} r(t+a)\mu(\mathrm{d}a)$, for $t \ge 0$. Then $\dot r$ is almost everywhere on $[0,\infty)$ equal to the derivative of r and the total variation $TV_{[0,\infty)}\dot r$ is finite.*

Proof. The first claim follows directly. For the second claim, let $0 \le t_0 < t_1 < \cdots < t_n < \infty$, then

$$\begin{aligned}\sum_{i=1}^{n-1} |\dot r(t_{i+1}) - \dot r(t_i)| &\le \int_{[-\alpha,0]} \sum_{i=1}^{n-1} |r(t_{i+1}+a) - r(t_i+a)|\,|\mu|(\mathrm{d}a)\\ &\le TV_{[-\alpha,\infty)} r |\mu|.\end{aligned}$$

□

Lemma 5.4. *Suppose $v_0(\mu) < 0$. Let $A \subset D[-\alpha,0]$ such that $\sup_{\varphi\in A}\sup_{-\alpha\le s\le 0}|\varphi(s)| < \infty$. Then the solutions of the deterministic delay equation satisfy*

$$\sup_{\varphi\in A}\sup_{t\in[-\alpha,\infty)} |x(t,\varphi)| < \infty.$$

Proof. By (4), for $\varphi \in A$,

$$\begin{aligned}|x(t,\varphi)| &= \left|\varphi(0)r(t) + \int_{[-\alpha,0]}\int_s^0 r(t+s-a)\varphi(s)\,\mathrm{d}s\mu(\mathrm{d}s)\right|\\ &\le \sup_{\varphi\in A}|\varphi(0)| \sup_{s\in[-\alpha,\infty)}|r(s)| + \sup_{s\in[-\alpha,\infty)}|r(s)|\alpha \sup_{\varphi\in A}\sup_{-\alpha\le s\le 0}|\varphi(s)||\mu| < \infty.\end{aligned}$$

□

Lemma 5.5. *If Assumption 1 holds, $\{\mathcal{L}(X_{\Phi_n}(t))\colon t\ge 0,\, n\in\mathbb{N}\}$ is a tight set of laws on $\mathbb{R}$, provided that $\sup_n \sup_{[-\alpha,0]}\sup_{\omega\in\Omega}|\Phi_n(s,\omega)| < \infty$.*

Proof. Since $X_{\Phi_n}(t) = x(t,\Phi_n) + \int_0^t r(t-s)F(X_{\Phi_n})(s-)\,\mathrm{d}L(s)$ and by Lemma 5.4 $\sup_{t,n}|x(t,\Phi_n)| < \infty$, we can execute the same proof as in [22, Proposition 4.2], as the only property of $F(X_{\Phi_n})$ used there is that it is a bounded process. □

We proceed by showing that the laws of the deducted segments $X_{\Phi_n}(t+\cdot) - X_{\Phi_n}(t)$, $t \ge 0$, are tight as well.

Define processes

$$\begin{aligned} I^n(u) &:= \int_{[-\alpha,0]} X_{\Phi_n}(u+v)\mu(\mathrm{d}v) \\ &= \int_{[-\alpha,0]} \left(x(u+v,\Phi_n) + \int_0^{u+v} r(u+v-s)F(X_{\Phi_n})(s-)\,\mathrm{d}L(s) \right) \mu(\mathrm{d}v) \\ &= \int_{[-\alpha,0]} x(u+v,\Phi_n)\mu(\mathrm{d}v) + \int_0^u \dot{r}(u-s)F(X_{\Phi_n})(s-)\,\mathrm{d}L(s), \end{aligned}$$

where we used the stochastic Fubini theorem and that $r(s) = 0$ for $s < 0$. Hence, since X_{Φ_n} is càdlàg, the processes

$$V^n(u) := \int_0^u \dot{r}(u-s)F(X_{\Phi_n})(s-)\,\mathrm{d}L(s)$$

have càdlàg versions, which we will use in the sequel.

Lemma 5.6. *Processes V^n defined above satisfy*

$$\lim_{K\to\infty} \sup_{n\in\mathbb{N}} \sup_{t\geq 0} \mathbb{P}\big(\sup_{0\leq s\leq\alpha} |V^n(t+s)| > K\big) = 0,$$

in other words, $\{\sup_{0\leq s\leq\alpha} |V^n(t+s)| \colon\, t \geq 0, n \in \mathbb{N}\}$ is a tight set of laws on $\mathbb{R}$.

Proof. First we show that the set of laws $\{\mathcal{L}(V^n(u)) \colon\, u \geq 0,\ n \in \mathbb{N}\}$ is tight. To do so, we examine the proof of [22, Proposition 4.2]. There the authors prove that the family of laws $\{\mathcal{L}(X(t)) \colon\, t \geq 0\}$ is tight, where

$$X(t) = \int_0^t r(t-s)F(X)(s-)\,\mathrm{d}L(s), \quad t \geq 0.$$

In the proof they use the boundedness of $F(X(s-))$, the fact that $r(t)$ decays exponentially for $t \to \infty$, and that the Lévy process L is exactly as in our Assumption 1. Due to our Assumption 1 we have the same bound for $F(X_{\Phi_n})(s-)$ for all n simultaneously. As we also have exponential decay of the function $\dot{r}$, we can execute the same proof for $V^n(u)$ and obtain that $\{\mathcal{L}(V^n(u)) \colon\, u \geq 0,\ n \in \mathbb{N}\}$ is tight.

As

$$\sup_{0\leq s\leq\alpha} |V^n(t+s)| \leq \sup_{0\leq s\leq\alpha} |V^n(t+s) - V^n(t)| + |V^n(t)|,$$

and we have that $\{\mathcal{L}(V^n(t)) \colon\, t \geq 0,\, n \in \mathbb{N}\}$ is tight, it is enough to show tightness of the laws of $\sup_{0\leq s\leq\alpha} |V^n(t+s) - V^n(t)|$, where $n \in \mathbb{N}$ and $t \geq 0$.

Our Lévy process L decomposes into

$$L(t) = bt + M(t) + N(t),$$

where $N(t) = \sum_{s \le t} \Delta L(s) \mathbf{1}_{|\Delta L(s)|>1}$, M is a square integrable Lévy martingale, and $b \in \mathbb{R}$. Then

$$\sup_{0 \le u \le \alpha} \left| \int_0^{t+u} \dot{r}(t+u-s) F(X_{\Phi_n})(s-) \, \mathrm{d}N(s) \right| \\ \le \sum_{s \le t+\alpha} C' m \exp(-\beta(t-s)) |\Delta N(s)|,$$

and the last process is bounded in probability by time reversal for compound Poisson processes and [11, Lemma 4.3] (see also [22, Proof of Proposition 4.2]). Further,

$$\left| \int_0^{t+u} \dot{r}(t+u-s) b \, \mathrm{d}s \right| \le C' b \int_0^{t+u} \exp(-\beta(t+u-s)) \, \mathrm{d}s < C' b.$$

Therefore it is enough to show that the laws of

$$\sup_{0 \le u \le \alpha} \left| \int_0^{t+u} \dot{r}(t+u-s) F(X_{\Phi_n})(s-) \, \mathrm{d}M(s) - \int_0^t \dot{r}(t-s) F(X_{\Phi_n})(s-) \, \mathrm{d}M(s) \right|,$$

$n \in \mathbb{N}$, $t \ge 0$, are a tight family. Now

$$\left| \int_0^{t+u} \dot{r}(t+u-s) F(X_{\Phi_n})(s-) \, \mathrm{d}M(s) - \int_0^t \dot{r}(t-s) F(X_{\Phi_n})(s-) \, \mathrm{d}M(s) \right| \\ \le \left| \int_0^t (\dot{r}(t+u-s) - \dot{r}(t-s)) F(X_{\Phi_n})(s-) \, \mathrm{d}M(s) \right| \\ + \left| \int_t^{t+u} \dot{r}(t+u-s) F(X_{\Phi_n})(s-) \, \mathrm{d}M(s) \right|.$$

For the first term we estimate

$$\left| \int_0^t (\dot{r}(t+u-s) - \dot{r}(t-s)) F(X_{\Phi_n})(s-) \, \mathrm{d}M(s) \right| \\ = \left| \int_0^t \int_0^u \mathrm{d}\dot{r}(t-s+v) F(X_{\Phi_n})(s-) \, \mathrm{d}M(s) \right| \\ = \left| \int_0^t \int_0^{t+u} \mathbf{1}_{t-v \le s \le t-v+u} F(X_{\Phi_n})(s-) \, \mathrm{d}\dot{r}(v) \, \mathrm{d}M(s) \right| \\ = \left| \int_0^{t+u} \left(\int_{t-v}^{t-v+u} F(X_{\Phi_n})(s-) \, \mathrm{d}M(s) \right) \mathrm{d}\dot{r}(v) \right| \\ \le \int_0^{t+\alpha} \sup_{w \le \alpha, \, n \in \mathbb{N}} \left| \int_{t-v}^{t-v+w} F(X_{\Phi_n}(s-) \, \mathrm{d}M(s) \right| \mathrm{d}|\dot{r}|(v),$$

by the stochastic Fubini theorem. Hence

$$\begin{aligned}
&\mathbb{E}\sup_{u\le\alpha}\left|\int_0^t (\dot r(t+u-s)-\dot r(t-s))F(X_{\Phi_n})(s-)\,\mathrm{d}M(s)\right| \\
&\qquad\le \int_0^{t+\alpha}\left(\mathbb{E}\sup_{u\le\alpha}\left|\int_{t-v}^{t-v+u} F(X_{\Phi_n})(s-)\,\mathrm{d}M(s)\right|^2\right)^{1/2}\,\mathrm{d}|\dot r|(v) \\
&\qquad\le (TV_{[0,\infty)}\dot r)4m^2(\mathbb{E}M(\alpha)^2)^{1/2},
\end{aligned}$$

by Doob's inequality, boundedness of F and the fact that M is a Lévy square integrable martingale.

For the second term, we first extend $\dot r$ by $\dot r(s)=\dot r(0)$ for $s<0$ and compute

$$\begin{aligned}
&\left|\int_t^{t+u}\dot r(t+u-s)F(X_{\Phi_n})(s-)\,\mathrm{d}M(s)\right| \\
&\qquad=\left|\int_t^{t+u}\left(\int_0^u \mathrm{d}\dot r(t-s+v)+\dot r(t-s)\right)F(X_{\Phi_n})(s-)\,\mathrm{d}M(s)\right| \\
&\qquad\le\left|\int_t^{t+u}\int_0^u \mathrm{d}\dot r(t-s+v)F(X_{\Phi_n})(s-)\,\mathrm{d}M(s)\right| \\
&\qquad\quad+|\dot r(0)F(X_{\Phi_n})(t-)\Delta M(t)| \\
&\qquad\le\left|\int_t^{t+u}\int_0^u \mathbf{1}_{t-z\le s\le t+u-z}F(X_{\Phi_n}(s-))\,\mathrm{d}\dot r(z)\,\mathrm{d}M(s)\right|+|\dot r(0)|m \\
&\qquad=\left|\int_0^u\left(\int_t^{t+u-z}F(X_{\Phi_n})(s-)\,\mathrm{d}M(s)\right)\mathrm{d}\dot r(z)\right|+|\dot r(0)|m,
\end{aligned}$$

by applying stochastic Fubini theorem and because $|\Delta M|\le 1$. Hence, arguing as above, we obtain

$$\begin{aligned}
&\mathbb{E}\sup_{0\le u\le\alpha}\left|\int_t^{t+u}\dot r(t+u-s)F(X_{\Phi_n})(s-)\,\mathrm{d}M(s)\right| \\
&\qquad\le (TV_{[0,\infty)}\dot r)4m^2(\mathbb{E}M(\alpha)^2)^{1/2}+|\dot r(0)|m.
\end{aligned}$$

Now the proof is complete. □

Proposition 5.7. *Let Assumption 1 hold, and let Φ_n be a uniformly bounded sequence of initial conditions. Then the laws of $(X_{\Phi_n}(t+s)-X_{\Phi_n}(t)\colon s\in[0,\alpha])$, $t\ge 0$, $n\in\mathbb{N}$, are tight in $D[0,\alpha]$.*

Proof. We use the same strategy as in [22, Proposition 4.3] and only need to show that

$$\lim_{K\to\infty}\sup_n\sup_{t\ge0}\mathbb{P}(\sup_{0\le u\le\alpha}|I^n(t+u)|>K)\to 0,$$

where

$$\begin{aligned} I^n(u) &= \int_{[-\alpha,0]} X_{\Phi_n}(u+v)\mu(\mathrm{d}v) \\ &= \int_{[-\alpha,0]} x(u+v,\Phi_n)\mu(\mathrm{d}v) + \int_0^u \dot{r}(u-s)F(X_{\Phi_n})(s-)\,\mathrm{d}L(s). \end{aligned}$$

Since the first term is bounded in n and t by Lemma 5.4, we infer the claim with the aid of Lemma 5.6. □

Now proving Theorem 5.1 is straightforward.

Proof of Theorem 5.1. We have $X_{\Phi_n}(t+\cdot) = (X_{\Phi_n}(t+\cdot) - X_{\Phi^n}(t)\mathbf{1}) + X_{\Phi^n}(t)\mathbf{1}$ for all $t \geq 0$, where $\mathbf{1}(s) = 1$ for all $s \in [0,\alpha]$. Let $\varepsilon > 0$. By Proposition 5.7, there exists a compact set $K \subset D[0,\alpha]$ such that $\mathbb{P}(X_{\Phi_n}(t+\cdot) - X_{\Phi^n}(t)\mathbf{1} \in K) \geq 1 - \varepsilon/2$ for all $t \geq 0$ and $n \in \mathbb{N}$. By Lemma 5.5 there exists a bounded interval $I \subset \mathbb{R}$ such that $\mathbb{P}(X_{\Phi_n}(t) \in I) \geq 1 - \varepsilon/2$. Let $K' := \{\sigma + c\mathbf{1} \colon \sigma \in K,\ c \in I\}$. Then $\mathbb{P}(X_{\Phi_n}(t+\cdot) \in K') \geq 1 - \varepsilon$. The set K' has compact closure in $D[0,\alpha]$, due to [3, Theorem 12.4]. Indeed, as K is compact it satisfies conditions (12.25) and (12.30) of [3]. Then K' satisfies these conditions as well, hence it has compact closure by the same theorem. Thus, $\{\mathcal{L}(X_{\Phi_n}(t+\cdot)) \colon t \geq 0,\ n \in \mathbb{N}\}$ is t ight. □

6 A stability theorem

In this section we prove that $\Phi_n \to \Phi$ in $D[-\alpha, 0]$ w.r.t. d_β in probability implies uniform convergence on compact sets in probability of the corresponding solutions, under Assumption 1 and the following condition:

$$\varphi_n \to \varphi \text{ in } D[-\alpha,0] \text{ w.r.t. } d_\beta \implies \int_0^\alpha \left(F(x_{\text{-}\varphi})(t) - F(x_{\text{-}\varphi_n})(t)\right)^2 \mathrm{d}t \to 0, \tag{17}$$

where $x_{\text{-}\varphi}(t) := x(t)$ for $t \geq 0$ and $x_{\text{-}\varphi}(t) := \varphi(t)$ for $-\alpha \leq t \leq 0$, and likewise for $x_{\text{-}\varphi_n}$.

We need the following approximations of F:

$$F^N(x)(t) := F(x^{(N)})(t), \text{ where } x^N := (x \wedge N) \vee (-N), \text{ for } x \in D[-\alpha,\infty),\ t \geq 0, \text{ and } N > 0.$$

Then F^N is Lipschitz in the sense of [22, (2.5)]. Indeed, for $x, y \in D[-\alpha,\infty)$ and $t \geq 0$ and $N > 0$, we have that $|x^N(t)|, |y^N(t)| \leq N$ for all $t \geq -\alpha$, hence by condition 1 of Definition 4.1 there is K_N such that

$$\begin{aligned} &|F^N(x)(t) - F^N(y)(t)| = |F(x^N)(t) - F(y^N)(t)| \leq \\ &\qquad K_N \sup_{s\in[t-\alpha,t]} |x^N(s) - y^N(s)| \leq K_N \sup_{s\in[t-\alpha,t]} |x(s) - y(s)| \end{aligned}$$

for all $t \geq 0$.

If (17) holds for F then it also holds for the approximations F^N, provided $N > \sup_{n,s}|\varphi_n(s)| \vee |\varphi(s)|$, since then for any $x \in D[0,\infty)$,

$$F^N(x_{\text{-}\varphi}) = F((x_{\text{-}\varphi})^{(N)}) = F((x^{(N)})_{\text{-}\varphi})$$

and the same holds for φ_n.

We need the next lemma.

Lemma 6.1. *Let L be a Lévy process with Lévy measure ν and let $T > 0$. Then for each $K > 0$ there exist constants b and σ such that for each stopping time R with $|\Delta L^{R-}| < K$ we have*

$$\left\| \int_0^{\cdot} H(s-)\,\mathrm{d}L^{R-}(s) \right\|^2_{\underline{\underline{H}}^2[0,T]} \leq 2\left(b^2T + 2\sigma^2 + 2\int_{(-K,K)} u^2\,\nu(du)\right)\int_0^T \mathbb{E}H(t)^2\,\mathrm{d}t,$$

for every predictable process H with $\int_0^T \mathbb{E}H(t)^2\,\mathrm{d}t < \infty$.

Proof. Let $K > 0$ and let R be a stopping time such that $|\Delta L^{R-}| < K$, and let H be a predictable process such that $\int_0^T \mathbb{E}H(t)^2\,\mathrm{d}t < \infty$. Consider the Lévy-Ito decomposition of L (see [1, Theorem 2.4.16]),

$$L(t) = bt + \sigma B(t) + \int_{(-K,K)} u\widetilde{N}(t,\mathrm{d}u) + \int_{|u|\geq K} uN(t,\mathrm{d}u)$$

and set $L_K(t) = L(t) - \int_{|u|\geq K} uN(t,\mathrm{d}u)$. Then $L^{R-} = L_K^{R-}$, so we have

$$\left\| \int_0^{\cdot} H(t)\,\mathrm{d}L^{R-} \right\|_{\underline{\underline{H}}^2[0,T]} = \left\| \int_0^{\cdot} H(t)\,\mathrm{d}L_K^{R-} \right\|_{\underline{\underline{H}}^2[0,T]} \leq 2\left\| \int_0^{\cdot} H(t-)\,\mathrm{d}L_K \right\|_{\underline{\underline{H}}^2[0,T]},$$

argueing as in [21, Proof of Theorem V.5]. Denote

$$I_H(t) = \int_0^t \int_{(-K,K)} uH(t)\,\widetilde{N}(\mathrm{d}t,\mathrm{d}u),$$

where the integral is as defined in [1, Section 4.2]. Notice that $I_H(t)$ equals the usual stochastic integral of H with respect to the Lévy process $\int_{(-K,K)} u\widetilde{N}(t,\mathrm{d}u)$, as one can see from the construction of both integrals. Since $\int_0^T \int_{(-K,K)} \mathbb{E}(uH(t))^2\,\mathrm{d}t\nu < \infty$ by assumption and Fubini, [1, Theorem 4.2.3] yields that I_H is a square integrable martingale and $\mathbb{E}I_H(t)^2 = \int_{(-K,K)} u^2\mathrm{d}\nu \int_0^T \mathbb{E}H(t)^2\,\mathrm{d}t$. Now

$$\int_0^t H(s)\,\mathrm{d}L_K(s) = b\int_0^t H(s)\,\mathrm{d}s + \sigma\int_0^t H(s)\mathrm{d}B(s) + I_H(t),$$

where the first term is a process of bounded variation and the latter two terms are square integrable martingales. Hence by a well known identity for square integrable martingales (see [21, Cor. 3 to Theorem II.27]),

$$\left\|\int_0^\cdot H(s)\,\mathrm{d}L(s)\right\|^2_{\underline{\underline{H}}^2[0,T]} \leq 2\mathbb{E}\left(TV\left(b\int_0^\cdot H(s)\,\mathrm{d}s\right)\right)^2 + 2\mathbb{E}\left(\sigma\int_0^T H(s)\,\mathrm{d}B(s) + I_H(t)\right)^2$$
$$\leq 2b^2T\int_0^T \mathbb{E}H(t)^2\,\mathrm{d}t + 4\sigma^2\int_0^T \mathbb{E}H(t)^2\,\mathrm{d}t + 4\mathbb{E}(I_H(T))^2.$$

□

Theorem 6.2. *Let Assumption 1 and* (17) *hold. If* $\Phi_n \to \Phi$ *in* $D[-\alpha,0]$ *w.r.t.* d_β *in probability, then* $X_{\Phi_n} \to X_\Phi$ *uniformly on compact subintervals of* $[0,\infty)$ *in probability.*

Proof. Write $X = X_\Phi$, $X^n = X_{\Phi_n}$ throughout this proof. Fix $T > 0$ and $\varepsilon > 0$.

Assume first that $\{\Phi_n, \Phi\}$ is a uniformly bounded family. Hence Theorem 5.1 can be applied and (15) holds, so that there exists N_0 such that for $N > N_0$,

$$\sup_n \mathbb{P}(\sup_{t\leq T}|X^n(t)| > N), \mathbb{P}(\sup_{t\leq T}|X(t)| > N) < \varepsilon.$$

Define stopping times $T^n := \inf\{t\colon\ |X^n(t)| > N_0\}$ and $T^\infty := \inf\{t\colon\ |X(t)| > N_0\}$. Then $\mathbb{P}(T^n > T) > 1-\varepsilon$ and $\mathbb{P}(T^\infty > T) > 1-\varepsilon$. Moreover,

$$\begin{aligned}(X^n)^{T^n-}(t) &= \Phi_n(0) + \int_0^{t\wedge T^n-}\int_{[-\alpha,0]}(X^n)^{T^n-}(s+a)\mu(\mathrm{d}a)\,\mathrm{d}s \\ &\quad + \int_0^t F((X^n)^{T^n-})(s-)\,\mathrm{d}L^{T^n-}(s) \\ &= \Phi_n(0) + \int_0^{t\wedge T^n-}\int_{[-\alpha,0]}(X^n)^{T^n-}(s+a)\mu(\mathrm{d}a)\,\mathrm{d}s \\ &\quad + \int_0^t F^{N_0}((X^n)^{T^n-})(s-)\,\mathrm{d}L^{T^n-}(s).\end{aligned}$$

So by uniqueness of solutions, $(X^n)^{T^n-} = (X^{(N_0)}_{\Phi_n})^{T^n-}$, where $X^{(N_0)}_{\Phi_n}$ denotes the solution to the equation (1) with F replaced by F^{N_0}. Likewise, $(X)^{T^\infty-} = (X^{(N_0)}_\Phi)^{T^\infty-}$. Since

$$\begin{aligned}\sup_{t\leq T}|X^n(t) - X(t)| &\leq \sup_{t\leq T}|X^n(t) - X^{(N_0)}_{\Phi_n}(t)| + \sup_{t\leq T}|X^{(N_0)}_{\Phi_n}(t) - X^{(N_0)}_\Phi(t)| \\ &\quad + \sup_{t\leq T}|X^{(N_0)}_\Phi(t) - X(t)|,\end{aligned}$$

we obtain for $\delta > 0$,

$$\begin{aligned}\mathbb{P}(\sup_{t\le T}|X^n(t)-X(t)|>\delta) &\le \mathbb{P}(\sup_{t\le T}|X^n(t)-X^{(N_0)}_{\Phi_n}(t)|>\delta/3)\\ &\quad+\mathbb{P}(\sup_{t\le T}|X^{(N_0)}_{\Phi_n}(t)-X^{(N_0)}_{\Phi}(t)|>\delta/3)+\mathbb{P}(\sup_{t\le T}|X^{(N_0)}_{\Phi}(t)-X(t)|>\delta/3)\\ &\le \mathbb{P}(T^n\le T)+\mathbb{P}(\sup_{t\le T}|X^{(N_0)}_{\Phi_n}(t)-X^{(N_0)}_{\Phi}(t)|>\delta/3)+\mathbb{P}(T^\infty\le T)\\ &\le 2\varepsilon+\mathbb{P}(\sup_{t\le T}|X^{(N_0)}_{\Phi_n}(t)-X^{(N_0)}_{\Phi}(t)|>\delta/3).\end{aligned}$$

Hence in this special case there is no loss of generality by assuming that F is Lipschitz in the sense of [22, (2.5)].

Let R be a stopping time such that L^{R-} has bounded jumps, is α-sliceable for suitably small α, and $\mathbb{P}(R>T)>1-\varepsilon$ (see [21, Theorem V.5]). Denote by Z and Z^n the solutions of equation (1) with L replaced by L^{R-} and initial condition Φ and Φ_n, respectively. By uniqueness of solutions, $(Z^n)^{R-}=(X^n)^{R-}$ and $Z^{R-}=X^{R-}$. Hence for $\delta>0$,

$$\begin{aligned}&\mathbb{P}(\sup_{0\le t\le T}|X^n(t)-X(t)|\ge\delta)\\ &\quad\le \mathbb{P}(\sup_{0\le t\le T}|X^n(t)-X(t)|\ge\delta \text{ and } R\ge T)+\mathbb{P}(R<T)\\ &\quad\le \mathbb{P}(\sup_{0\le t\le T}|Z^n(t)-Z(t)|\ge\delta)+\varepsilon/2,\end{aligned}$$

so it suffices to show that $Z^n\to Z$ uniformly on $[0,T]$ in probability. To show this we introduce some notation:

$$\begin{aligned}Y^n(t)&:=\int_0^t\int_{[-\alpha,0]}Z_{\text{-}\Phi}(s+u)-Z_{\text{-}\Phi_n}(s+u)\,\mu(\mathrm{d}u)\mathrm{d}s+\int_0^t(F(Z_{\text{-}\Phi})-F(Z_{\text{-}\Phi_n}))(s-)\,\mathrm{d}L^{R-}(s),\\ \mathcal{G}^n(U)(t)&:=\int_{[-\alpha,0]}(Z_{\text{-}\Phi_n}(t+u)-(Z-U)_{\text{-}\Phi_n}(t+u))\,\mu(\mathrm{d}u),\\ \mathcal{H}^n(U)(t)&:=F(Z_{\text{-}\Phi_n})(t)-F((Z-U)_{\text{-}\Phi_n})(t),\end{aligned}$$

$t\ge 0$. We obtain for $U^n:=Z-Z^n$ the equation

$$U^n(t)=\Phi(0)-\Phi_n(0)+Y^n(t)+\int_0^t\mathcal{G}^n(U^n)(s)\,\mathrm{d}s+\int_0^t\mathcal{H}^n(U^n)(s-)\,\mathrm{d}L^{R-}(s).$$

Lemma V.3.2 of [21] extended to two driving semimartingales yields $\|U^n\|_{\underline{\underline{S}}^2[0,T]}\le C\|\Phi(0)-\Phi_n(0)+Y^n\|_{\underline{\underline{S}}^2[0,T]}$ with a constant $C>0$ depending on the process L^{R-} and the uniform bound for the Lipschitz constants of mappings $V\mapsto\int_{[-\alpha,0]}V_{\text{-}\Phi}(\cdot+u)\,\mu(\mathrm{d}u)$, $V\mapsto\int_{[-\alpha,0]}V_{\text{-}\Phi_n}(\cdot+u)\,\mu(\mathrm{d}u)$, $V\mapsto F(V_{\text{-}\Phi})$, and $V\mapsto F(V_{\text{-}\Phi_n})$. (Notice that this boun is finite as F is assumed to be Lipschitz and we assumed that Φ_n,Φ is a uniformly bounded family. As $\Phi_n\to\Phi$ w.r.t. d_β in probability, we have $\Phi_n(0)\to\Phi(0)$ in probability, hence $\|U^n\|_{\underline{\underline{S}}^2[0,T]}\to 0$ if $\|Y^n\|_{\underline{\underline{S}}^2[0,T]}\to 0$.

Next we show $\|Y^n\|_{\underline{\underline{S}}^2[0,T]} \to 0$. Due to the continuous embedding of $\underline{\underline{H}}^2[0,T]$ into $\underline{\underline{S}}^2[0,T]$ (see [21, Theorem V.2]) and Lemma 6.1 this follows if

$$\mathbb{E}\int_0^T \left(\left(\int_{[-\alpha,0]} (Z_{\text{-}\Phi}(t+u) - Z_{\text{-}\Phi_n}(t+u))\,\mu(\mathrm{d}u) \right)^2 + (F(Z_{\text{-}\Phi}))(t) - F(Z_{\text{-}\Phi_n})(t))^2 \right) \mathrm{d}t \to 0 \tag{18}$$

as $n \to \infty$. By the boundedness of F and assumption (17) we have $\mathbb{E}\int_0^T (F(Z_{\text{-}\Phi})(t) - F(Z_{\text{-}\Phi_n})(t))^2\,\mathrm{d}t \to 0$ as $n \to \infty$. Moreover,

$$\begin{aligned}
&\int_0^T \left(\int_{[-\alpha,0]} Z(_{\text{-}\Phi}(t+u) - Z_{\text{-}\Phi_n}(t+u))\,\mu(\mathrm{d}u) \right)^2 \mathrm{d}t \\
&\quad = \int_0^T \left(\int_{[-\alpha,0]} (\Phi(t+u) - \Phi_n(t+u))\,\mu(\mathrm{d}u) \right) \left(\int_{[-\alpha,0]} (\Phi(t+v) - \Phi(t+v))\,\mu(\mathrm{d}v) \right) \mathrm{d}t \\
&\quad = \int_{[-\alpha,0]} \int_{[-\alpha,0]} \int_0^T (\Phi(t+u) - \Phi_n(t+u))(\Phi(t+v) - \Phi_n(t+v))\mathbf{1}_{[-\alpha,-t)}(u) \\
&\qquad\qquad\qquad \cdot \mathbf{1}_{[-\alpha,-t)}(v)\,\mathrm{d}t\mu(\mathrm{d}u)\mu(\mathrm{d}v).
\end{aligned}$$

This expression converges almost surely to zero and is bounded in n and ω, as convergence in d_β implies almost everywhere convergence on $[-\alpha, 0]$ and the family Φ_n, Φ is uniformly bounded. Hence (18) holds indeed, and we proved the special case of uniformly bouded initial conditions.

For the general case, notice that since $\Phi_n \to \Phi$ w.r.t. d_β in probability, the laws of Φ_n converge weakly to the law of Φ, and since $(D[-\alpha,0], d_\beta)$ is Polish we have by the Prohorov theorem that the family of laws of Φ_n, Φ is tight. Hence for a $\varepsilon > 0$ there is a set $K \subset D[-\alpha, 0]$ compact w.r.t. d_β such that $\mathbb{P}(\Phi_n \in K,\ \Phi \in K) > 1 - \varepsilon$ for all n. As convergence w.r.t. d_β is implied by Skorokhod convergence, K is also compact w.r.t. d_S. Hence all the functions in K are bounded by some finite constant C. Consider the truncated initial conditions Φ_n^C and Φ^C and let $\overline{X}^n$ and $\overline{X}$ be the solutions of equation (1) with these initial conditions. We have that $\mathbb{P}(\Phi_n = \Phi_n^C,\ \Phi = \Phi^C) > 1 - \varepsilon$, and concentrating $\mathbb{P}$ on the sets $\{\Phi_n = \Phi_n^C\}$ and $\{\Phi = \Phi^C\}$, with the aid of [21, Theorem IV.23] and the uniqueness of solutions we conclude that

$$\begin{aligned}
\overline{X}^n &= X^n \text{ a.s. on } \{\Phi_n = \Phi_n^C\} \\
\overline{X} &= X \text{ a.s. on } \{\Phi = \Phi^C\}
\end{aligned}$$

Moreover, it is easy to check that $\Phi_n^C \to \Phi^C$ w.r.t. d_β in probability, so that be the special case above, $\sup_{0 \le t \le T} |\overline{X}^n(t) - \overline{X}(t)| \to 0$ in probability. Finally, for $\delta > 0$,

$$\begin{aligned}
\mathbb{P}(\sup_{0\le t\le T} |X(t) - X^n(t)| > \delta) &\le \mathbb{P}(\sup_{0\le t\le T} |X(t) - \overline{X}(t)| > \delta/3) + \mathbb{P}(\sup_{0\le t\le T} |\overline{X}(t) - \overline{X}^n(t)| > \delta/3) \\
&\quad + \mathbb{P}(\sup_{0\le t\le T} |\overline{X}^n(t) - X^n(t)| > \delta/3) \\
&\le 2\varepsilon + \mathbb{P}(\sup_{0\le t\le T} |\overline{X}^n(t) - X^n(t)| > \delta/3)
\end{aligned}$$

and the theorem has been proved. □

Remark 6.3. 1. By stopping $X = X_\Phi$ appropriately, we can use similar techniques as before and prove Theorem 6.2 even if F is not bounded, but merely having linear growth.

2. Each of the conditions '$\Phi_n \to \Phi$ w.r.t. d_S in probability', '$\Phi_n \to \Phi$ w.r.t. d_β a.s.', and '$\Phi_n \to \Phi$ w.r.t. d_S a.s.' is stronger than the condition of Theorem 6.2.

In the next Corollary the use of d_β instead of d_S is essential; see [22, Section 3.3] for a counterexample with d_S.

Corollary 6.4. *Let Assumption 1 and* (17) *hold. If $\Phi_n \to \Phi$ with respect to d_β in probability, then $X_{\Phi_n,t} \to X_{\Phi,t}$ with respect to d_β in probability for every $t \geq 0$.*

Proof. If $t \geq \alpha$, the assertion readily follows from Theorem 6.2. Consider $0 < t < \alpha$. Let λ be an increasing homeomorphism from $[-\beta, 0]$ onto itself. Define $\rho\colon [-\beta, 0] \to [-\beta, 0]$ by $\rho(s) = s$ for $s \in [-t, 0]$, $\rho(s) = \lambda(t+s) - t$ for $s \in [-\alpha, -t)$, and affine on $[-\beta, -\alpha]$ with $\rho(-\beta) = -\beta$. Then ρ is an increasing homeomorphism, $\sup_{s\in[-\beta,0]} |\rho(s) - s| \leq \sup_{s\in[-\beta,0]} |\lambda(s) - s|$, and

$$\sup_{s\in[-\alpha,0]} |X_{\Phi_n}(t+s) - X_\Phi(t+\rho(s))| \leq \sup_{s\in[-\alpha,-t]} |\Phi_n(t+s) - \Phi(\lambda(t+s))| \vee \sup_{s\in[-t,0]} |X_{\Phi_n}(t+s) - X_\Phi(t+s)|.$$

Hence $d_\beta(X_{\Phi_n,t}, X_{\Phi,t}) \leq d_\beta(\Phi_n, \Phi) + \sup_{s\in[0,\alpha]} |X_{\Phi_n}(s) - X_\Phi(s)|$, and the proof follows. □

7 Markov and eventual Feller property and existence of invariant measure

Theorem 7.1. *Let Assumption 1 and* (17) *hold. The segment process $(X_{\Phi,t})_{t\geq 0}$ is a Markov process, the transition operators $P_{s,t}$ defined by* (11) *map $B_d(D[-\alpha, 0])$ into $B_d(D[-\alpha, 0])$ and satisfy*

$$P_{u,t} = P_{u,s} P_{s,t} \quad \text{and} \quad P_{s,t} = P_{0,t-s}$$

for all $0 \leq u \leq s \leq t$. Moreover, the Markov semigroup $(P_t)_{t\geq 0}$ defined by (12) *is eventually Feller.*

Proof. In this proof we endow $D[-\alpha, 0]$ with the metric d_β. Recall that by Lemma 2.1 d_β and d_S generate the same Borel σ-algebra $\mathcal{B}(D[-\alpha, 0])$.

We begin by showing (10). Let $0 \leq u \leq t$ and $B \in \mathcal{B}(D[-\alpha, 0])$. Observe that $\mathbf{1}_B(X^u_{X_{\Phi,t-u}})$ is measurable, as $X^u_{X_{\Phi,t-u}} = X_\Phi(t)$, by (9). Let C_b denote the space of bounded functions $f\colon D[-\alpha, 0] \to \mathbb{R}$ that are continuous with respect to d_β. Let $f \in C_b$ and let ξ be an $\mathcal{F}_u$-measurable random variable with values in $D[-\alpha, 0]$. For a $\mathcal{F}_u$-measurable random variable ξ with values in $D[-\alpha,0]$, let $A(\xi,\omega) := f(X^u_{\xi,t}(\omega))$. Let further $\mathcal{A}(\varphi) := \mathbb{E}A(\varphi,\cdot)$ for $\varphi \in D[-\alpha,0]$. Assume first that

$$\xi = \sum_{i=1}^{n} a_i \mathbf{1}_{C_i} \tag{19}$$

with $a_i \in D[-\alpha,0]$ and $C_i \in \mathcal{F}_u$, and C_i mutually disjoint, $\bigcup_i C_i = \Omega$, and $\mathbb{P}(C_i) > 0$ for all i. Then $A(\xi,\omega) = \sum_i A(a_i,\omega)\mathbf{1}_{C_i}(\omega)$ (as before we rescale $\mathbb{P}$ to C_i and use [21, Theorem IV.23] and uniqueness of solutions), so

$$\mathbb{E}[A(\xi(\cdot),\cdot)|\mathcal{F}_u] = \sum_i \mathbf{1}_{C_i}\mathbb{E}A(a_i,\cdot) = \sum_i \mathcal{A}(a_i)\mathbf{1}_{C_i} = \mathcal{A}(\xi(\cdot)).$$

If ξ is an arbitrary $\mathcal{F}_u$-measurable random variable with values in $D[-\alpha,0]$, then there are ξ_m of the form (19) such that $d_\beta(\xi_m(\omega),\xi(\omega)) \to 0$ as $m \to \infty$ for a.e. ω (see [25, Proposition I.1.9]). Due to the continuity of f and Corollary 6.4 we have $A(\xi_m,\cdot) \to A(\xi,\cdot)$ in probability, so that

$$\mathcal{A}(\xi_m) = \mathbb{E}[A(\xi_m,\cdot)|\mathcal{F}_u] \to \mathbb{E}[A(\xi,\cdot)|\mathcal{F}_u],$$

as f is bounded. Again by Corollary 6.4 we have $\mathcal{A}(\varphi_n) \to \mathcal{A}(\varphi)$ whenever $d_\beta(\varphi_n,\varphi) \to 0$, so that $\mathcal{A}(\xi_m) \to \mathcal{A}(\xi)$ a.s. Hence $\mathbb{E}[A(\xi,\cdot)|\mathcal{F}_u] = \mathcal{A}(\xi)$ a.s.

Next let C be a closed subset of $D[-\alpha,0]$ and choose $f_n \in C_b$ such that $f_n \downarrow \mathbf{1}_C$ pointwise. Let $A_n(\omega) := f_n(X^u_{\varphi,t}(\omega))$ and $A(\varphi,\omega) = \mathbf{1}_C(X^u_{\varphi,t}(\omega))$, $\omega \in \Omega$, and $\mathcal{A}_n(\varphi) = \mathbb{E}A_n(\varphi,\cdot)$ and $\mathcal{A}(\varphi) = \mathbb{E}A(\varphi,\cdot)$, $\varphi \in D[-\alpha,0]$. Then $A_n \downarrow A$ and $\mathcal{A}_n \downarrow \mathcal{A}$ pointwise, so

$$\mathbb{E}[A(\xi,\cdot)|\mathcal{F}_u] = \lim_{n\to\infty} \mathbb{E}[A_n(\xi,\cdot)|\mathcal{F}_u] = \lim_{n\to\infty} \mathcal{A}_n(\xi) = \mathcal{A}(\xi) \text{ a.s.}$$

By a monotone class argument we can extend the above identity to any $C \in \mathcal{B}(D[-\alpha,0])$, that is, we have proved (10).

We show that $P_{s,t}$ maps $B_b(D[-\alpha,0])$ into $B_b(D[-\alpha,0])$. Indeed, if $f \in C_b$, then Corollary 6.4 yields that $P_{s,t}f \in C_b$. If C is a closed subset of $D[-\alpha,0]$, then there are $f_n \in C_b$ such that $f_n \downarrow \mathbf{1}_C$ pointwise and then $P_{s,t}f_n \downarrow P_{s,t}\mathbf{1}_C$ pointwise, so $P_{s,t}\mathbf{1}_C \in B_b(D[-\alpha,0])$. By a monotone class argument we obtain $P_{s,t}\mathbf{1}_C \in B_b(D[-\alpha,0])$ for any $F \in \mathbb{B}(D[-\alpha,0])$ and then it follows that $P_{s,t}f \in B_b(D[-\alpha,0])$ for any $f \in B_b(D[-\alpha,0])$.

The Markov property (10) yields for $0 \le u \le s \le t$ that

$$\begin{aligned} P_{u,t}f(\varphi) &= \mathbb{E}f(X^u_{\varphi,t-u}) = \mathbb{E}(\mathbb{E}[f(X^u_{\varphi,t-u})|\mathcal{F}_s]) \\ &= \mathbb{E}(\mathbb{E}[f(X^s_{X^u_{\varphi,s-u},t-s})|\mathcal{F}_s]) = \mathbb{E}(\mathbb{E}f(X^s_{\psi,t-s})|_{X^u_{\varphi,s-u}=\psi}) \\ &= P_{u,s}P_{s,t}f(\varphi). \end{aligned}$$

By uniqueness in law [15, Subsection IX.6c] we have that $(X^u_{\varphi,t})_{t\ge0}$ has the same law as $(X_{\varphi,t})_{t\ge0}$ for each $u \ge 0$, since L^u and L have the same law. Hence $P_{s,t} = P_{0,t-s}$.

Finally, we establish that $(P_t)_t$ is eventually Feller. By Proposition 6.2 we have that for each $t \ge \alpha$, $\varphi_n \to \varphi$ in $D[-\alpha,0]$ implies $X_{\varphi_n,t} \to X_{\varphi,t}$ in $D[-\alpha,0]$ in probability, so $P_tf(\varphi_n) \to P_tf(\varphi)$, hence (13) holds. Property (14) has already been shown. □

Remark 7.2. The condition (17) is rather mild as the following examples show.

1. Let ρ be a finite signed Borel measure on $[-\alpha,0]$ and let f be a locally Lipschitz function on $\mathbb{R}$ with linear growth. Define

$$F(x)(t) := f\left(\int_{[-\alpha,0]} x(t+v)\rho(\mathrm{d}v)\right), \quad t \geq 0,$$
$$F(x)(t) := 0, \quad -\alpha \leq t \leq 0,$$

$x \in D[-\alpha,\infty)$. Then if $\varphi_n \to \varphi$ in $D[-\alpha,0]$ and $x \in D[0,\infty)$,

$$\begin{aligned}
&\int_0^\alpha \left(F(x_{\text{-}\varphi}(t)) - F(x_{\text{-}\varphi_n}(t))\right)^2 \mathrm{d}t \\
&\quad = \int_0^\alpha \left(f\left(\int_{[-\alpha,0]} x_{\text{-}\varphi_n}(t+v)\rho(\mathrm{d}v)\right) - f\left(\int_{[-\alpha,0]} x_{\text{-}\varphi}(t+v)\rho(\mathrm{d}v)\right)\right)^2 \mathrm{d}t \\
&\quad \leq C\int_0^\alpha \left(\int_{[-\alpha,0]} (x_{\text{-}\varphi_n}(t+v) - x_{\text{-}\varphi}(t+v))\rho(\mathrm{d}v)\right)^2 \mathrm{d}t \\
&\quad = C\int_0^\alpha \left(\int_{[-\alpha,0]} (x_{\text{-}\varphi_n}(t+v) - x_{\text{-}\varphi}(t+v))\rho(\mathrm{d}v)\right) \\
&\qquad \cdot \left(\int_{[-\alpha,0]} (x_{\text{-}\varphi_n}(t+u) - x_{\text{-}\varphi}(t+u))\rho(\mathrm{d}u)\right) \mathrm{d}t \\
&\quad = C\int_0^\alpha \int_{[-\alpha,0]}\int_{[-\alpha,0]} \left(\varphi_n(t+v) - \varphi(t+v)\right)\left(\varphi_n(t+u) - \varphi(t+u)\right) \\
&\qquad\qquad \mathbf{1}_{u<-t}\mathbf{1}_{v<-t}\rho(\mathrm{d}u)\rho(\mathrm{d}v)\,\mathrm{d}t,
\end{aligned}$$

for some C depending only on f, x, ρ, and (φ_n), where the latter equality follows by Fubini theorem and the fact that $x_{\text{-}\varphi_n}(t+w) = x_{\text{-}\varphi}(t+w)$ whenever $t+w \geq 0$. Now by the Fubini theorem and dominated convergence F satisfies (17) for each $x \in D[0,\infty)$.

Moreover, F is lolidet: for $t \geq 0$ and $x, y \in D[-\alpha,\infty)$ such that $\sup_{s\in[1-\alpha,t]} |x(s)| \vee |y(s)| \leq n$ we have

$$\left|\int_{[-\alpha,0]} x(t+v)\,\rho(\mathrm{d}v)\right|, \left|\int_{[-\alpha,0]} y(t+v)\,\rho(\mathrm{d}v)\right| \leq n|\rho|.$$

Hence as f is locally Lipschitz there is a $C_n > 0$ such that

$$\begin{aligned}
|F(x)(t) - F(y)(t)| &\leq C_n \left|\int_{[-\alpha,0]} (x(t+v) - y(t+v))\,\rho(\mathrm{d}v)\right| \\
&\leq C_n|\rho| \sup_{v\in[-\alpha,0]} |x(t+v) - y(t+v)|.
\end{aligned}$$

Since f has linear growth, it follows that F is lolidet.

However, F need not be Lipschitz in the sense of [22, (2.5)] if f is not Lipschitz. To see this, take $f(t) = \sin(t^2)$, ρ the Lebesgue measure on $[-\alpha,0]$, and evaluate $F(x_n)(t) = \sin(\alpha^2 n^2)$, where $x_n \equiv n$.

2. Likewise we can take $\rho_1, \dots, \rho_d$ signed Borel measures on $[-\alpha, 0]$ and f a locally Lipschitz on $\mathbb{R}^d$. In particular, we may take for F combinations of finitely many point evaluations.

   As above, F is lolidet but need not be Lipschitz in the sense of [22, (2.5)].

3. Let f be a locally Lipschitz function on $\mathbb{R}$. Let for $x \in D[-\alpha, \infty)$,

$$\begin{aligned} F(x)(t) &:= f\big(\sup_{t-\alpha \le s \le t} |x(s)|\big), \text{ for } t \ge 0, \\ F(x)(t) &:= 0 \text{ for } -\alpha \le t < 0. \end{aligned}$$

   Then if $\varphi_n \to \varphi$ in $D[-\alpha, 0]$, for $x \in D[0, \infty)$,

$$\int_0^\alpha (F(x_{\text{-}\varphi_n})(t) - F(x_{\text{-}\varphi})(t))^2 \, \mathrm{d}t \le C \int_0^\alpha \left(\sup_{t-\alpha \le s \le t} |x_{\text{-}\varphi_n}(s)| - \sup_{t-\alpha \le s \le t} |x_{\text{-}\varphi}(s)| \right)^2 \mathrm{d}t$$

   for some C depending only on f, x, and (φ_n), hence by bounded convergence the last expression above will tend to zero if for a.e. $t \in [0, \alpha]$

$$\sup_{t-\alpha \le s \le t} |x_{\text{-}\varphi_n}(s)| - \sup_{t-\alpha \le s \le t} |x_{\text{-}\varphi}(s)| \to 0. \tag{20}$$

   Let us show that (20) holds for t such that φ is continuous at $t - \alpha$. Let (λ_n) be a sequence of increasing homeomorphisms on $[-\alpha, 0]$ such that $\|\varphi_n - \varphi \circ \lambda_n\|_\infty + \|\lambda_n - I\|_\infty \to 0$. Then

$$\begin{aligned} &\left| \sup_{t-\alpha \le s \le 0} |\varphi_n(s)| - \sup_{t-\alpha \le s \le 0} |\varphi(s)| \right| \\ &\quad \le \left| \sup_{t-\alpha \le s \le 0} |\varphi_n(s)| - \sup_{t-\alpha \le s \le 0} |\varphi \circ \lambda_n(s)| \right| + \left| \sup_{t-\alpha \le s \le 0} |\varphi \circ \lambda_n(s)| - \sup_{t-\alpha \le s \le 0} |\varphi(s)| \right|. \end{aligned}$$

   The first term converges to 0 as $\|\varphi_n - \varphi \circ \lambda_n\|_\infty$. For the second term, let $\varepsilon > 0$ and let $\delta > 0$ be such that $t - \alpha - \delta < s < t - \alpha + \delta$ implies $\varphi(t-\alpha) - \varepsilon < \varphi(s) < \varphi(t-\alpha) + \varepsilon$. Fix N such that $\|\lambda_n - I\|_\infty < \delta$ for all $n \ge N$. Then

$$\sup_{t-\alpha \le s \le 0} |\varphi \circ \lambda_n(s)| = \sup_{\lambda_n(t-\alpha) \le s \le 0} |\varphi(s)| \le \sup_{t-\alpha-\delta \le s \le 0} |\varphi(s)| \le \sup_{t-\alpha \le s \le 0} |\varphi(s)| + \varepsilon$$

   and

$$\sup_{t-\alpha \le s \le 0} |\varphi(s)| - 2\varepsilon \le \sup_{t-\alpha+\delta \le s \le 0} |\varphi(s)| \le \sup_{\lambda_n(t-\alpha) \le s \le 0} |\varphi(s)|,$$

   so that the second term is less than 2ε whenever $n \ge N$. Now since $\sup_{t-\alpha \le s \le t} |x_{\text{-}\varphi}(s)| = \sup_{t-\alpha \le s \le 0} |\varphi(s)| \vee \sup_{0 \le s \le t} |x(s)|$ and the same with φ_n, (20) holds if φ is continuous at $t - \alpha$. Hence F satisfies (17) for each $x \in D[0, \infty)$.

If $x, y \in D[-\alpha, \infty)$ are bounded by n on $[t-\alpha, t]$, then $\left|\sup_{[t-\alpha,t]}|x(s)| - \sup_{[t-\alpha,t]}|y(s)|\right| \leq n$, so by the assumption on f there is a $C_n > 0$ such that

$$|F(X)(t) - F(y)(t)| \leq C_n \left| \sup_{[t-\alpha,t]} |x(s)| - \sup_{[t-\alpha,t]} |y(s)| \right|.$$

As $\sup_{s\in[t-\alpha,t]}|x(s)| \leq \sup_{s\in[t-\alpha,t]}|x(s) - y(s)| + \sup_{s\in[t-\alpha,t]}|y(s)|$, we obtain by symmetry $\left|\sup_{s\in[t-\alpha,t]}|x(s)| - \sup_{s\in[t-\alpha,t]}|y(s)|\right| \leq \sup_{s\in[t-\alpha,t]}|x(s) - y(s)|$. Since f has linear growth, it follows that F is lolidet. Again, F need not be Lipschitz in the sense of [22, (2.5)] if f is not Lipschitz, as we see by taking $f(t) = \sin(t^2)$ and evaluating F on the sequence $x_n \equiv n$.

4. Similar arguments as in 3. can be given for functionals like $f(\sup_{t-\alpha\leq s\leq t} x(s))$, $f(\inf_{t-\alpha\leq s\leq t} x(s))$ and $f(\inf_{t-\alpha\leq s\leq t} |x(s)|)$.

Notice that all functionals F in the previous remark are autonomous in the sense of Definition 4.4. If f is bounded, then F satisfies all conditions of Assumption 1 and (17).

Finally we consider existence of an invariant measure. Denote by $\mathcal{P}$ the set of Borel probability measures on $D[-\alpha, 0]$ endowed with the topology of weak convergence of measures. Let B_b denote the space of all real valued bounded Borel functions on $D[-\alpha, 0]$ and denote $\langle \zeta, f\rangle = \int f \,\mathrm{d}\zeta$, $f \in B_b$, $\zeta \in \mathcal{P}$. The adjoint of the Markov semigroup defined in (12) is given by

$$\langle P_t^* \zeta, f\rangle = \langle \zeta, P_t f\rangle, \quad f \in B_b,\ \zeta \in \mathcal{P}.$$

A measure $\eta \in \mathcal{P}$ is called an *invariant measure* for (1) if

$$P_t^* \eta = \eta \qquad \text{for all } t \geq 0.$$

If η is the distribution of an initial segment Φ, then $P_t^*\eta$ is the distribution of the segment $X_{t,\Phi}$. Therefore if Φ is an $\mathcal{F}_0$-measurable random variable with values in $D[-\alpha, 0]$ whose law is an invariant measure, the segment process corresponding to the solution X of (1) with initial condition Φ is constant in law. In this case the solution X itself is also constant in law.

Theorem 7.3. *Grant Assumption 1 and assume that* (17) *holds for every* $x \in D[0, \infty)$*. Then equation* (1) *has an invariant measure.*

Proof. It follows from Theorem 7.1 that P_t maps $C_b = C_b(D[-\alpha, 0])$ into C_b for $t \geq \alpha$ and that $t \mapsto P_t^*\zeta$ is a continuous map from $[\alpha, \infty)$ to $\mathcal{P}$. Moreover, $P_{s+t}^* = P_s^* P_t^*$ for all $s, t \geq 0$. Theorem 5.1 yields that the set $\{P_t^*\zeta\colon\ t \geq \alpha\}$ is tight, where, for instance, ζ is the distribution of the initial condition $\varphi \equiv 0$.

Next, proceeding as in [22, Section 4.2], the invariant measure η is obtained by means of the Krylov-Bogoliubov method. □

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