SECOND VARIATION OF ZHANG'S λ-INVARIANT ON THE MODULI SPACE OF CURVES

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Abstract. We study the second variation of the invariant λ, introduced by Zhang, on the complex moduli space $\mathcal{M}_g$ of curves of genus $g \geq 2$, using recent work of Kawazumi. As a result we prove that $(8g+4)\lambda$ is equal, up to a constant, to the invariant β introduced some years ago by Hain and Reed. The λ-invariant measures the difference, at archimedean places, between the height of the canonical Gross-Schoen cycle and the Faltings stable height of a curve over a number field. The β-invariant gives the ratio between the Hodge metric on the determinant of the Hodge bundle and a metric defined by means of the Griffiths normal function on $\mathcal{M}_g$ associated to the Ceresa cycle $X - X^-$.

1. Introduction

In a recent paper [21] S. Zhang studies two real-valued functions λ and φ on the complex moduli space $\mathcal{M}_g$ of curves of genus $g \geq 2$. They are related by the formula

$$\lambda = \frac{g-1}{6(2g+1)}\varphi + \frac{1}{12}\delta \,, \tag{1.1}$$

where δ is Faltings's delta-invariant from [4], suitably normalised. The value of the invariant φ (studied independently by N. Kawazumi [15]) at a curve X of genus $g \geq 2$ is defined as follows: let $(\omega_1, \ldots, \omega_g)$ be an orthonormal basis of $H^0(X, \omega_X)$ with respect to the inner product

$$(\omega, \eta) \mapsto \frac{i}{2} \int_X \omega \, \overline{\eta} \tag{1.2}$$

and put

$$\mu_X = \frac{i}{2g} \sum_{k=1}^{g} \omega_k \, \bar{\omega}_k \,.$$

Note that μ_X is a volume form on X, independent of the choice of orthonormal basis $(\omega_1, \ldots, \omega_g)$. Let Δ_{Ar} be the Laplacian on $L^2(X, \mu_X)$ determined by

$$\frac{\partial\overline{\partial}}{\pi i} f = \Delta_{\mathrm{Ar}}(f) \cdot \mu_X$$

for all $f \in L^2(X, \mu_X)$ and let $(\phi_\ell)_{\ell=0}^{\infty}$ be an orthonormal basis of real eigenforms of Δ_{Ar} with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$. Then we put

$$\varphi(X) = \sum_{\ell > 0} \frac{2}{\lambda_\ell} \sum_{m,n=1}^{g} \left| \int_X \phi_\ell \omega_m \bar{\omega}_n \right|^2 . \tag{1.3}$$

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The value of φ is independent of the choice of $(\omega_1, \ldots, \omega_g)$.

Let k be a number field or a function field of a curve, and let X be a smooth, projective and geometrically connected curve of genus $g \geq 2$ with semistable reduction over k. In [21] Zhang also defines invariants λ and φ associated to the non-archimedean places of k. In this case, the definition of λ and φ is in terms of the semistable reduction graphs of X. Both invariants vanish in the case of good reduction. Let ξ be a k-rational point of $\mathrm{Pic}^1 X$ such that $(2g-2)\xi$ is the class of a canonical divisor and let Δ_ξ in $\mathrm{Ch}^2(X^3)$ be the modified diagonal cycle in X^3 associated to ξ as in B. Gross and C. Schoen [5] (strictly speaking, in [5] it is assumed that ξ is a k-rational point of X, but the construction can be generalised, see [21]). We call Δ_ξ a canonical Gross-Schoen cycle. The interest of the invariants λ and φ lies in the fact that they occur as local contributions in formulas relating the height $\langle \Delta_\xi, \Delta_\xi \rangle$ (defined in [5]) of Δ_ξ to the admissible self-intersection of the relative dualising sheaf $(\omega, \omega)_a$ of X and to the (non-normalised) Faltings height $\deg \det R\pi_*\omega$ of X. More precisely, one has the formulas

$$(\omega, \omega)_a = \frac{2g-2}{2g+1} \left(\langle \Delta_\xi, \Delta_\xi \rangle + \sum_v \varphi(X_v) \log Nv \right) \tag{1.4}$$

and

$$\deg \det R\pi_*\omega = \frac{g-1}{6(2g+1)} \langle \Delta_\xi, \Delta_\xi \rangle + \sum_v \lambda(X_v) \log Nv \,, \tag{1.5}$$

where in both cases the sum is taken over all places v of k, the Nv are certain local factors, and $X_v = X \otimes k_v$. These formulas form the main result of [21]; the equivalence of both formulas follows from (1.1) and the Noether formula.

We mention that formula (1.4) relates to the Bogomolov conjecture for X, i.e., the statement that $(\omega, \omega)_a$ should be strictly positive (in the function field case, one assumes that X is not isotrivial). Indeed, the height of the canonical Gross-Schoen cycle $\langle \Delta_\xi, \Delta_\xi \rangle$ is known to be non-negative in the case of a function field in characteristic zero, and should be non-negative in general by a standard conjecture of Gillet-Soulé ([21], Section 2.4). If v is non-archimedean, the invariant φ_v was conjectured by Zhang (cf. [21], Conjecture 1.4.2) and subsequently proved by Z. Cinkir (cf. [2], Theorem 2.9) to satisfy the lower bound

$$\varphi(X_v) \geq c(g)\delta_0 + \sum_{i=1}^{[g/2]} \frac{2i(g-i)}{g} \delta_i \tag{1.6}$$

where for each $i = 0, \ldots, [g/2]$ the invariant δ_i denotes the number of singular points in the special fiber of X_v such that the local normalisation of that fiber at x is connected if $i = 0$ or a disjoint union of two curves of genera i and $g-i$ if $i > 0$, and where $c(g)$ is a positive constant depending only on g. In fact, one can take $c(g) = \frac{g-1}{6g}$. If v is archimedean, the invariant φ as defined in (1.3) is strictly positive (cf. [21], Remark after Proposition 2.5.3).

In this note we study the invariant λ more closely. To be precise, we compute the second variation of the invariant λ over the complex moduli space $\mathcal{M}_g$. Our result is based on a theorem of N. Kawazumi [15], which gives an expression for the second variation of the invariant φ. Hence, by (1.1), our contribution is essentially to compute the second variation of the Faltings delta-invariant.

The result of Kawazumi connects φ with certain canonical 2-forms over the universal curve $\mathcal{C}_g$ over $\mathcal{M}_g$ associated (following work of S. Morita) to the standard representation H of $\mathrm{Sp}_{2g}(\mathbb{Z})$, its third exterior power $\wedge^3 H$, and the "primitive part" $\wedge^3 H/H$ of the latter. Here H is seen as a subrepresentation of $\wedge^3 H$ by wedging with the standard polarisation form in $\wedge^2 H$. Using Kawazumi's result, the invariant φ gets connected with a real-valued function β on $\mathcal{M}_g$ defined—modulo constants—in the paper [9] of R. Hain and D. Reed.

The β-invariant is given as follows. Let $\mathcal{J}(\wedge^3 H/H)$ be the Griffiths intermediate jacobian fibration over $\mathcal{M}_g$ associated to $\wedge^3 H/H$, and let $\hat{\mathcal{B}}$ be the standard $\mathbb{G}_{\mathrm{m}}$-biextension line bundle on $\mathcal{J}(\wedge^3 H/H)$ (see Section 2 for definitions). The latter bundle comes with a natural hermitian metric $\|\cdot\|_{\hat{\mathcal{B}}}$. Let $\nu\colon \mathcal{M}_g \to \mathcal{J}(\wedge^3 H/H)$ be the normal function that associates to each curve X the point in the intermediate jacobian of $\wedge^3 H_1(X)/H_1(X)$ associated, by the Griffiths Abel-Jacobi map, to the Ceresa cycle $X - X^-$ in the jacobian of X. By a result of Morita we have $\nu^*\hat{\mathcal{B}} \cong \mathcal{L}^{\otimes 8g+4}$, where $\mathcal{L}$ is the determinant of the Hodge bundle on $\mathcal{M}_g$. The isomorphism is determined up to a non-zero scalar. Denote by $\|\cdot\|_{\mathrm{biext}}$ a metric (well-defined up to a non-zero scalar) on $\mathcal{L}^{\otimes 8g+4}$ that one obtains by pulling back $\|\cdot\|_{\hat{\mathcal{B}}}$ along ν, and transporting it to $\mathcal{L}^{\otimes 8g+4}$ using Morita's isomorphism. On $\mathcal{L}^{\otimes 8g+4}$ we have (this time without ambiguity) a second metric $\|\cdot\|_{\mathrm{Hdg}}$, called the Hodge metric, given by (1.2) on $\mathcal{L}$. The invariant β measures their ratio:

$$\beta = \log\left(\frac{\|\cdot\|_{\mathrm{biext}}}{\|\cdot\|_{\mathrm{Hdg}}}\right).$$

Note that the invariant β is defined up to a constant on $\mathcal{M}_g$. Let ω_{HR} be the first Chern form of $(\mathcal{L}, \|\cdot\|_{\mathrm{biext}}^{1/(8g+4)})$ and let ω_{Hdg} be that of $(\mathcal{L}, \|\cdot\|_{\mathrm{Hdg}}^{1/(8g+4)})$. It follows that the second variation of β over $\mathcal{M}_g$ satisfies

$$\frac{\partial\overline{\partial}}{\pi i}\beta = (8g+4)(\omega_{\mathrm{HR}} - \omega_{\mathrm{Hdg}}).$$

Our main result is

Theorem A. *The second variation of Zhang's λ-invariant over $\mathcal{M}_g$ is equal to*

$$\frac{\partial\overline{\partial}}{\pi i}\lambda = \omega_{\mathrm{HR}} - \omega_{\mathrm{Hdg}}.$$

In particular, we have the equality $(8g+4)\lambda = \beta$, up to a constant.

In [9] the asymptotic behavior of β along the boundary components of the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ is computed. For the Faltings delta-invariant δ, this was done by J. Jorgenson [13] and R. Wentworth [18], independently. Combining these results we obtain

Theorem B. *Let $X \to D$ be a proper family of stable curves of genus $g \geq 2$ over the unit disk. Assume that X is smooth and that X_t is smooth for $t \neq 0$.*

- *If X_0 is irreducible with only one node, then*

$$\varphi(X_t) \sim -\frac{g-1}{6g}\log|t|$$

as $t \to 0$.

- *If X_0 is reducible with one node and its components have genera i and $g-i$ then*

$$\varphi(X_t) \sim -\frac{2i(g-i)}{g} \log|t|$$

as $t \to 0$.

Here, if f, g are two functions on the punctured unit disk, the notation $f \sim g$ denotes that $f - g$ is bounded as $t \to 0$. One might view Theorem B as an archimedean analogue of Cinkir's result (1.6) above.

Note that for each curve X of genus $g \geq 2$ with semistable reduction over a number field K there should be a natural (non-normalised) "Ceresa height" $h_{\mathrm{Cer}}(X)$ of X over K associated to the hermitian line bundle $\nu^*\hat{\mathcal{B}}$ over $\mathcal{M}_g$ and the moduli point corresponding to X. Theorem A suggests that this height should satisfy the equality

$$-h_{\mathrm{Cer}}(X) + (8g+4) \deg \det R\pi_*\omega = (8g+4) \sum_v \lambda(X_v) \log Nv \, .$$

A combination with equality (1.5) then would imply that

$$3\, h_{\mathrm{Cer}}(X) = (2g-2)\langle \Delta_\xi, \Delta_\xi \rangle$$

should hold, so that $h_{\mathrm{Cer}}(X)$ allows a simple expression in terms of the height of the canonical Gross-Schoen cycle. A comparison of this formula with [21], Theorem 1.5.6 suggests that there should be a natural interpretation in terms of biextensions of Künnemann's canonical height pairing on homologically trivial cycles, at least in the case when the Ceresa cycle is paired with its Fourier-Mukai dual. In a future paper we plan to return to this connection. Note that to define $h_{\mathrm{Cer}}(X)$ properly, one would need a natural extension of $\nu^*\hat{\mathcal{B}}$ as a line bundle over the moduli stack of stable genus g curves over $\mathbb{Z}$. We refer to [9], Theorem 3 for the statement that at least $\nu^*\hat{\mathcal{B}}$ has a natural extension as a line bundle over the moduli space of stable curves over $\mathbb{C}$.

2. Preliminaries

The basic references for this section are the papers [8] and [9] by Hain and Reed. As is customary, we view the moduli spaces $\mathcal{A}_g$ and $\mathcal{M}_g$ of principally polarised complex abelian varieties and of smooth projective complex curves, respectively, as orbifolds. Let $(V_{\mathbb{Z}}, Q : \wedge^2 V_{\mathbb{Z}} \to \mathbb{Z}(-n))$ be a polarised integral Hodge structure of odd weight $n = -2p+1$ and let $\mathrm{GSp}_{2g} \to \mathrm{GSp}(V_{\mathbb{Z}}, Q)$ be an algebraic representation, together with a lift of the structure morphism $\mathbb{S} \to \mathrm{GSp}(V_{\mathbb{R}}, Q)$, where $\mathbb{S}$ is the Deligne torus, to $\mathrm{GSp}_{2g,\mathbb{R}}$. Let $(\mathcal{V}_{\mathbb{Z}}, \mathcal{Q})$ be the corresponding variation of polarised Hodge structures over $\mathcal{A}_g$. We denote by $\mathcal{J}(V_{\mathbb{Z}})$ the Griffiths intermediate jacobian fibration over $\mathcal{A}_g$ associated to $\mathcal{V}_{\mathbb{Z}}$. Thus, if V_A is the fiber of the local system $\mathcal{V}_{\mathbb{Z}}$ at the point A of $\mathcal{A}_g$, the fiber of $\mathcal{J}(V_{\mathbb{Z}})$ at A is the complex torus $J(V_A) = (V_A \otimes \mathbb{C})/(F^{-p+1}(V_A \otimes \mathbb{C}) + \mathrm{Im}\, V_A)$. The holomorphic tangent bundle of $J(V_A)$ is equipped with a canonical hermitian inner product derived from Q. This hermitian inner product determines a translation-invariant global 2-form on $J(V_A)$.

Proposition 2.1. *There exists a unique 2-form w_V on $\mathcal{J}(V_{\mathbb{Z}})$ such that the restriction of w to each fiber over $\mathcal{A}_g$ is the translation-invariant form associated to Q, and such that the restriction of w along the zero-section is trivial.*

Proof. This is in [9], Section 5. □

We also mention the following result. Suppose that $V_{\mathbb{Z}}$ has weight -1. From [6], Section 3 we recall that the (standard $\mathbb{G}_{\mathrm{m}}$-) biextension line bundle $\mathcal{B}$ associated to $V_{\mathbb{Z}}$ is the set of isomorphism classes of mixed Hodge structures whose weight graded quotients are isomorphic to $\mathbb{Z}, V_{\mathbb{Z}}$ and $\mathbb{Z}(1)$. It has a natural projection to the product $J(V_{\mathbb{Z}}) \times J(\check{V}_{\mathbb{Z}})$ where $J(V_{\mathbb{Z}}) = \mathrm{Ext}_{\mathcal{H}}(\mathbb{Z}, V_{\mathbb{Z}})$ is the Griffiths intermediate jacobian of $V_{\mathbb{Z}}$, given by $M \mapsto (M/W_{-2}M, W_{-1}M)$. This projection equips $\mathcal{B}$ with the structure of a line bundle over $J(V_{\mathbb{Z}}) \times J(\check{V}_{\mathbb{Z}})$. The polarisation of $V_{\mathbb{Z}}$ furnishes a canonical morphism $\lambda\colon J(V_{\mathbb{Z}}) \to J(\check{V}_{\mathbb{Z}})$. By pulling back along (id, λ) one obtains from $\mathcal{B}$ a line bundle $\hat{\mathcal{B}}$ over $J(V_{\mathbb{Z}})$. By abuse of language we refer to $\hat{\mathcal{B}}$ as the biextension line bundle over $J(V_{\mathbb{Z}})$. Proposition 7.3 of [9] then states the following.

Proposition 2.2. *Suppose that $\mathcal{V}_{\mathbb{Z}}$ is a variation of polarised Hodge structures of weight -1 over $\mathcal{A}_g$. Let $\hat{\mathcal{B}}$ be the biextension line bundle over $\mathcal{J}(V_{\mathbb{Z}})$, obtained by applying the above construction to each of the fibers of $\mathcal{J}(V_{\mathbb{Z}})$. Then $\hat{\mathcal{B}}$ has a canonical hermitian metric. The first Chern form of $\hat{\mathcal{B}}$ with this metric is equal to $2\, w_V$.*

We will be mainly concerned with the cases where $V_{\mathbb{Z}}$ is equal to either H, $\wedge^3 H$ or $\wedge^3 H/H$, where $H = H_1(X, \mathbb{Z})$ is the first homology group of a compact Riemann surface X of genus $g \geq 2$, marked with a canonical basis. The polarisation is given by the standard intersection form $Q_H = (,)$ on H. Note that the form Q_H identifies H with its dual. The Hodge structure H is mapped into $\wedge^3 H$ by sending x to $x \wedge \zeta$, where ζ in $\wedge^2 H$ is the dual of Q_H.

The polarisations on the Hodge structures $\wedge^3 H$ and $\wedge^3 H/H$ are given explicitly as follows. The form $Q_{\wedge^3 H}$ on $\wedge^3 H$ sends

$$(x_1 \wedge x_2 \wedge x_3, y_1 \wedge y_2 \wedge y_3) \mapsto \det(x_i, y_j)\,.$$

Next, remark that one has a contraction map $c : \wedge^3 H \to H$, defined by

$$x \wedge y \wedge z \mapsto (x, y)z + (y, z)x + (z, x)y\,. \tag{2.1}$$

One may verify that the composite $H \to \wedge^3 H \to H$ induced by c and $\wedge\zeta$ is equal to $(g-1)$ times the identity. Denote the projection $\wedge^3 H \to \wedge^3 H/H$ by p. The projection p has a canonical splitting j (after tensoring with $\mathbb{Q}$), defined by

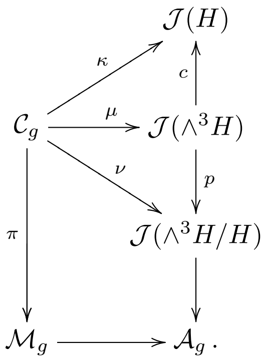
$$p(x \wedge y \wedge z) \mapsto x \wedge y \wedge z - \zeta \wedge c(x \wedge y \wedge z)/(g-1)\,.$$

With these definitions, the form $Q_{\wedge^3 H/H}$ on $\wedge^3 H/H$ is given by

$$(u, v) \mapsto (g-1) Q_{\wedge^3 H}(j(u), j(v))\,.$$

We denote by w_H, $w_{\wedge^3 H}$ and $w_{\wedge^3 H/H}$ the 2-forms on the Griffiths intermediate jacobian fibrations $\mathcal{J}(H)$, $\mathcal{J}(\wedge^3 H)$ and $\mathcal{J}(\wedge^3 H/H)$ over $\mathcal{A}_g$ whose existence is asserted by Proposition 2.1. Note that $\mathcal{J}(H)$ is just the universal abelian variety over $\mathcal{A}_g$.

Let $\pi\colon \mathcal{C}_g \to \mathcal{M}_g$ be the universal curve over $\mathcal{M}_g$, viewed as an orbifold. As is explained in [8], Introduction we have a commutative diagram



Here κ is the map sending a pair (X,x) where X is a curve and x is a point on X to the class of $(2g-2)x - \omega_X$ in the jacobian J of X. The map μ is called the "pointed harmonic volume" (introduced by B. Harris, cf. [10]) and sends a pair (X,x) to the point associated, by the Griffiths Abel-Jacobi map, to the Ceresa cycle at x, i.e. the (homologically trivial) cycle in J given as $X_x - X_x^-$ where X_x is the curve X embedded in J using x and $X_x^- = [-1]_* X_x$. The map ν is called the "harmonic volume" and is just defined as the composite of μ with the map $p\colon \mathcal{J}(\wedge^3 H) \to \mathcal{J}(\wedge^3 H/H)$ induced by the projection $\wedge^3 H \to \wedge^3 H/H$. The map ν factors over $\mathcal{M}_g$, hence defines a Griffiths normal function $\mathcal{M}_g \to \mathcal{J}(\wedge^3 H/H)$ that we shall also denote by ν.

Proposition 2.3. *On $\mathcal{J}(\wedge^3 H)$, the equality of 2-forms*

$$(g-1)w_{\wedge^3 H} = c^* w_H + p^* w_{\wedge^3 H/H}$$

holds.

Proof. According to [8], Proposition 18 we have $(g-1)Q_{\wedge^3 H} = c^* Q_H + p^* Q_{\wedge^3 H/H}$. We obtain the result by taking the associated canonical 2-forms. □

3. Proof of Theorem A

In order to give the proof of Theorem A we pass from $\mathcal{M}_g$ and $\mathcal{C}_g$ to the level-2 moduli orbifolds $\mathcal{M}_g[2]$ and $\mathcal{C}_g[2]$ (see for example [7], Section 7.4 for definitions). These orbifolds can be endowed with a universal theta characteristic α, i.e. a consistent choice of an element $\alpha \in \mathrm{Pic}^{g-1} X$ for each curve X such that 2α is the canonical divisor class. We consider the map

$$j_\alpha\colon \mathcal{C}_g[2] \longrightarrow \mathcal{J}(H)[2]$$

given by sending (X,x) to the class of $(g-1)x - \alpha$ on the jacobian J of X. Note that $\kappa = 2j_\alpha$. The idea of the proof of Theorem A will be to compute the 2-form $j_\alpha^* w_H$ in two ways. This goes by carrying through some of the arguments in [8] on the level of 2-forms, instead of on the level of cohomology classes. The proof is concluded by taking the difference of the two resulting expressions, and by applying Kawazumi's result [15] on the second variation of φ to that difference.

Let e^J be the 2-form

$$e^J = -\frac{1}{2g(2g+1)}(2\kappa^* w_H + 3\mu^* w_{\wedge^3 H}) \tag{3.1}$$

over $\mathcal{C}_g$. By a result of Morita [17] (see also [8], Theorem 6) this 2-form represents the class of $\omega_{\mathcal{C}_g/\mathcal{M}_g}^{-1}$ in $H^2(\mathcal{C}_g,\mathbb{Q})$, where $\omega_{\mathcal{C}_g/\mathcal{M}_g}$ is the relative dualising sheaf of $\mathcal{C}_g$ over $\mathcal{M}_g$. Recall from the Introduction that we have a 2-form ω_{HR} on $\mathcal{M}_g$ by taking the pullback, along ν, of the first Chern form of $(\hat{\mathcal{B}},\|\cdot\|_{\hat{\mathcal{B}}})$, and dividing by $8g+4$.

Lemma 3.1. *Over* $\mathcal{C}_g[2]$*, we have an equality*

$$j_\alpha^* w_H = -\frac{g(g-1)}{2}e^J - \frac{3}{2}\omega_{\mathrm{HR}}$$

of 2*-forms.*

Proof. Upon replacing $H_1(X,\mathbb{Z})$ by $H_1(X,\mathbb{Z}(-1))$ one views the variation of Hodge structures over $\mathcal{A}_g$ determined by $\wedge^3 H/H$ to be one of weight -1 (cf. [9], Section 4). Proposition 2.2 yields that the first Chern form of $(\hat{\mathcal{B}},\|\cdot\|_{\hat{\mathcal{B}}})$ equals $2w_{\wedge^3 H/H}$ so that

$$\nu^* w_{\wedge^3 H/H} = (4g+2)\,\omega_{\mathrm{HR}}\,.$$

Proposition 2.3 then yields

$$(g-1)\mu^* w_{\wedge^3 H} = \kappa^* w_H + (4g+2)\,\omega_{\mathrm{HR}}\,.$$

Combining this equality with the definition of e^J we find

$$\kappa^* w_H = -2g(g-1)e^J - 6\,\omega_{\mathrm{HR}}$$

(cf. [8], Theorem 1). On the other hand we have $[2]^* w_H = 4w_H$ and $\kappa = 2j_\alpha$ which together give

$$\kappa^* w_H = 4j_\alpha^*(w_H)\,.$$

The lemma follows. □

Let X be a compact Riemann surface of genus $g \geq 2$. From [1] we recall that the line bundle $\mathcal{O}(\Delta)$ on $X\times X$, where Δ is the diagonal, comes equipped with a natural hermitian metric given by $\|1\|(x,y) = G(x,y)$, where G is the Arakelov Green's function. By demanding that the adjunction (residue) isomorphism

$$\mathcal{O}(-\Delta)|_\Delta \to \omega_X$$

should be an isometry we obtain a canonical hermitian metric $\|\cdot\|_{\mathrm{Ar}}$ on ω_X, and fiber by fiber we obtain a canonical hermitian metric $\|\cdot\|_{\mathrm{Ar}}$ on $\omega_{\mathcal{C}_g/\mathcal{M}_g}$. Denote by e^A the first Chern form of the dual metric on $\omega_{\mathcal{C}_g/\mathcal{M}_g}^{-1}$. Let δ_F be the Faltings delta-invariant as defined in [4] on p. 402. We remark that δ_F relates to the δ occurring in equation (1.1) via $\delta = \delta_F - 4g\log(2\pi)$.

Lemma 3.2. *Over* $\mathcal{C}_g[2]$*, we have an equality*

$$j_\alpha^* w_H = -\frac{g(g-1)}{2}e^A - \frac{3}{2}\omega_{\mathrm{Hdg}} - \frac{1}{8}\frac{\partial\overline{\partial}}{\pi i}\,\delta_F$$

of 2*-forms.*

Proof. We refer to [4], p. 413 for the first half of this proof. On $\mathcal{J}(H)[2]$ we have a universal theta divisor Θ_α. When restricted to the jacobian J of a curve X, the divisor Θ_α is equal to the image of the canonical theta divisor on $\mathrm{Pic}^{g-1}X$ under the isomorphism $\mathrm{Pic}^{g-1}X \to J$ defined by $x \mapsto x-\alpha$. Further, the orbifold $\mathcal{J}(H)[2]$ can be written as a quotient of the analytic variety $\mathbb{C}^g \times \mathcal{H}_g$ where $\mathcal{H}_g$ is the Siegel upper half space of complex g-by-g matrices with positive definite imaginary part.

When pulled back to $\mathbb{C}^g \times \mathcal{H}_g$, for a suitable choice of universal theta characteristic the divisor Θ_α can be given analytically by Riemann's standard theta function θ. As a result, the line bundle $\mathcal{O}(\Theta_\alpha)$ on $\mathcal{J}(H)[2]$ comes equipped with a natural hermitian metric; the norm of θ in this metric is given by

$$\|\theta\| = (\det \operatorname{Im} \tau)^{1/4} \exp(-\pi \, {}^t y \, (\operatorname{Im} \tau)^{-1} \, y) |\theta(z, \tau)|$$

where $z = x + iy$ is in $\mathbb{C}^g$ and τ is in $\mathcal{H}_g$. With this metric, the first Chern form w_0 of $\mathcal{O}(\Theta_\alpha)$ equals

$$w_0 = w_H + \frac{1}{2}\omega_{\mathrm{Hdg}} \tag{3.2}$$

(cf. [8], Proposition 2). Now as is explained in [8], Section 3 there exists a canonical isomorphism

$$j_\alpha^* \mathcal{O}(\Theta_\alpha) \longrightarrow \omega^{\otimes g(g-1)/2} \otimes \mathcal{L}^{-1}$$

of line bundles over $\mathcal{C}_g[2]$, given by sending $j_\alpha^* \theta$ to a suitable Wronskian differential. By Lemma 3.2 of [11] the norm of this isomorphism is equal to $\exp(\delta_F/8)$, if $\mathcal{L}$ is equipped with the Hodge metric given by (1.2), and ω is equipped with the Arakelov metric $\| \cdot \|_{\mathrm{Ar}}$. By taking first Chern forms we find

$$j_\alpha^* w_0 = -\frac{g(g-1)}{2} e^A - \omega_{\mathrm{Hdg}} - \frac{1}{8} \frac{\partial\overline{\partial}}{\pi i} \, \delta_F \, .$$

We obtain the lemma by inserting (3.2). □

From Lemmas 3.1 and 3.2 we infer that

$$\frac{g(g-1)}{2} \left(e^A - e^J \right) = -\frac{1}{8} \frac{\partial\overline{\partial}}{\pi i} \, \delta_F + \frac{3}{2}\omega_{\mathrm{HR}} - \frac{3}{2}\omega_{\mathrm{Hdg}} \, . \tag{3.3}$$

Now we have

Theorem 3.3. *(Kawazumi [15]) Let φ be Zhang's invariant from (1.3). Then the equality*

$$e^A - e^J = \frac{1}{2g(2g+1)} \frac{\partial\overline{\partial}}{\pi i} \, \varphi$$

holds.

Proof. Theorem 3.1 of [15] reads

$$e^A - e^J = \frac{-2i}{2g(2g+1)} \partial\overline{\partial} \, a_g \, .$$

One verifies easily that the function a_g as defined in the Introduction of [15] is equal to $\frac{1}{2\pi}\varphi$, and that the 2-form e^A on $\mathcal{C}_g$ as defined in [15] is the one defined above. One is left to verify that the 2-form e^J defined in (3.1) is equal to the 2-form called e^J in [15]. The latter is written (cf. Definition (3.10) in [15]) as

$$e^J = -\frac{1}{2g(2g+1)} (M_1 + M_2)(\eta_1^{\otimes 2}) \, ,$$

where the following notation is employed. Let $\mathcal{H}_\mathbb{Z}$ be the local system over $\mathcal{A}_g$ associated to H and consider the derived local systems $\mathcal{H}_\mathbb{R} = \mathcal{H}_\mathbb{Z} \otimes \mathbb{R}$ and $\mathcal{H}_\mathbb{C} = \mathcal{H}_\mathbb{Z} \otimes \mathbb{C}$ over $\mathcal{A}_g$ and $\mathcal{M}_g$. We use the same notation to denote their pullbacks on $\mathcal{C}_g$. Note that when pulled back along $\mathcal{C}_g$, the intermediate jacobian fibration $\mathcal{J}(\wedge^3 H)$ can be seen as a torus bundle over $\mathcal{C}_g$ with fiber $\wedge^3 H \otimes (\mathbb{R}/\mathbb{Z})$. Both M_1, M_2 are real forms in $\operatorname{Hom}(\wedge^2(\wedge^3 \mathcal{H}_\mathbb{C}), \mathbb{C})$, hence global 2-forms on $\mathcal{J}(\wedge^3 H)$, coinciding with the forms C_1, C_2 from [17]. By the discussion in Remark 20 of [8] we can therefore write $M_1 = 2c^* w_H$ and $M_2 = 3w_{\wedge^3 H}$ on $\mathcal{J}(\wedge^3 H)$ where $c\colon \wedge^3 H \to H$ is the contraction map (2.1). The section $\eta_1^{\otimes 2}$ of the local system $\wedge^2(\wedge^3 \mathcal{H}_{\mathbb{C}})$ over $\mathcal{C}_g$ is the one induced by the section $\eta_1 = \eta_1' + \overline{\eta}_1'$ of the local system $\wedge^3 \mathcal{H}_{\mathbb{C}}$ where, as is explained in the introduction to [15], the section η_1' of $\wedge^3 \mathcal{H}_{\mathbb{R}}$ is the first variation of the pointed harmonic volume $\mu\colon \mathcal{C}_g \to \mathcal{J}(\wedge^3 H)$. We obtain $M_1(\eta_1^{\otimes 2}) = 2\mu^* c^* w_H = 2\kappa^* w_H$ and $M_2(\eta_1^{\otimes 2}) = 3\mu^* w_{\wedge^3 H}$ and the equality of Kawazumi's e^J with the one in (3.1) follows. □

The first part of Theorem A follows from a combination of equations (1.1) and (3.3) and Theorem 3.3. The proof is completed by remarking that $\mathcal{M}_g$ allows a surjection from a contractible complex analytic space; for example one could take the Teichmüller space in genus g. This implies that every pluriharmonic function on $\mathcal{M}_g$ is constant.

4. Hyperelliptic curves

As an application of Theorem A we compute the λ-invariant in the hyperelliptic case. Let $\mathcal{H}_g$ be the orbifold moduli space of complex hyperelliptic curves of genus $g \geq 2$. It extends as a moduli stack of hyperelliptic curves over $\mathbb{Z}$. There exists an up to sign unique global trivialising section Λ of the line bundle $\mathcal{L}^{\otimes 8g+4}$ over $\mathcal{H}_g$ that extends as a trivialising section of $\mathcal{L}^{\otimes 8g+4}$ over $\mathbb{Z}$. One has the following formula for $\|\Lambda\|_{\mathrm{Hdg}}$ over $\mathcal{H}_g$ (cf. [12]). Let $n = \binom{2g}{g+1}$ and $r = \binom{2g+1}{g+1}$. Let τ in the Siegel upper half space of complex g-by-g matrices with positive definite imaginary part be the period matrix of a complex hyperelliptic curve of genus g marked with a canonical basis of homology. Let $\varphi_g(\tau)$ be the value at τ of the modular form of weight $4r$ on Siegel upper half space given in [16], Section 3, and put $\Delta_g(\tau) = 2^{-(4g+4)n}\varphi_g(\tau)$. Further put $\|\Delta_g\|(\tau) = (\det \mathrm{Im}\,\tau)^{2r}|\Delta_g(\tau)|$; then for a given hyperelliptic curve X the value of $\|\Delta_g\|(\tau)$ on a period matrix on a canonical basis associated to X does not depend on the choice of such a basis. By the proof of Theorem 8.2 in [12] we have the following formula:

$$\|\Lambda\|_{\mathrm{Hdg}}^n = (2\pi)^{4g^2 r}\|\Delta_g\|^g . \tag{4.1}$$

We derive the following result.

Theorem 4.1. *On the hyperelliptic locus $\mathcal{H}_g$ of genus $g \geq 2$, the λ-invariant is given, up to a constant depending only on g, by*

$$(8g+4)n\lambda = -4g^2 r \log(2\pi) - g\log\|\Delta_g\| .$$

Proof. According to [9], Proposition 6.7 the metric $\|\cdot\|_{\mathrm{biext}}$ restricted to $\mathcal{L}^{\otimes 8g+4}$ over $\mathcal{H}_g$ is a constant metric on a trivial line bundle. It follows that $\beta = -\log\|\Lambda\|_{\mathrm{Hdg}}$, up to a constant. By Theorem A we obtain that $(8g+4)\lambda = -\log\|\Lambda\|_{\mathrm{Hdg}}$ up to a constant. Now apply equation (4.1). □

Using a recent result of K. Yamaki [20] it is possible to compute the constant in Theorem 4.1. Let X be a hyperelliptic curve of genus $g \geq 2$ with semi-stable reduction over a non-archimedean local field k. Let ε be Zhang's epsilon-invariant of X (cf. [21], Section 1.2). Define the invariant ψ as

$$\psi = \varepsilon + \frac{2g-2}{2g+1}\varphi .$$

Let $\mathcal{X}$ be the special fiber of a regular semistable model of X over the ring of integers of k. We say that a double point x of $\mathcal{X}$ is of type 0 if the local normalisation of $\mathcal{X}$ at x is connected. We say that x is of type i, where $i = 1, \ldots, [g/2]$, if the local normalisation of $\mathcal{X}$ at x is the disjoint union of a curve of genus i and a curve of genus $g-i$. Let ι be the involution on $\mathcal{X}$ induced by the hyperelliptic involution on X. Let x be a double point of type 0 on $\mathcal{X}$. If x is fixed by ι, we say that x is of subtype 0. If x is not fixed by ι, the local normalisation of $\mathcal{X}$ at $\{x, \iota(x)\}$ consists of two connected components, of genus j and $g-j-1$, say, where $1 \leq j \leq [(g-1)/2]$. In this case we say that the pair $\{x, \iota(x)\}$ is of subtype j. Let ξ_0 be the number of double points of subtype 0, let ξ_j for $j = 1, \ldots, [(g-1)/2]$ be the number of pairs of double points of subtype j, and let δ_i for $i = 1, \ldots, [g/2]$ be the number of double points of type i. Equality (1.2.5) and Theorem 3.5 of [20] imply that

$$\psi = \frac{g-1}{2g+1}\xi_0 + \sum_{j=1}^{[(g-1)/2]} \frac{6j(g-j-1)+2g-2}{2g+1}\xi_j + \sum_{i=1}^{[g/2]} \left(\frac{12i(g-i)}{2g+1} - 1 \right) \delta_i \, .$$

Hence, by the formulas in [21], Section 1.4 the λ-invariant satisfies

$$(8g+4)\lambda = g\xi_0 + \sum_{j=1}^{[(g-1)/2]} 2(j+1)(g-j)\xi_j + \sum_{i=1}^{[g/2]} 4i(g-i)\delta_i \, .$$

By the local Cornalba-Harris equality [3] [14] [19] we thus obtain

$$(8g+4)\lambda = -\log \|\Lambda\|$$

where now the right hand side denotes the order of vanishing of Λ along the closed point of the spectrum of the ring of integers of k. Now take a hyperelliptic curve X over $\mathbb{Q}$. Then by the relation

$$(8g+4) \deg \det R\pi_*\omega = -\sum_v \log \|\Lambda\|_v \log Nv$$

over a finite field extension of $\mathbb{Q}$ where X acquires semi-stable reduction, equation (1.5) and the known vanishing of $\langle \Delta_\xi, \Delta_\xi \rangle$ in the hyperelliptic case, one obtains that the constant implied by Theorem 4.1 is actually vanishing.

5. Proof of Theorem B

For the proof of Theorem B we just combine the following two results on the asymptotic behavior of β resp. δ, using Theorem A and equation (1.1).

Theorem 5.1. *(Hain-Reed* [9]*) Let $X \to D$ be a proper family of stable curves of genus $g \geq 2$ over the unit disk. Assume that X is smooth and that X_t is smooth for $t \neq 0$.*

- *If X_0 is irreducible with only one node, then*
$$\beta(X_t) \sim -g \log|t| - (4g+2) \log\log(1/|t|)$$
as $t \to 0$.
- *If X_0 is reducible with one node and its components have genera i and $g-i$ then*
$$\beta(X_t) \sim -4i(g-i) \log|t|$$
as $t \to 0$.

Theorem 5.2. *(Jorgenson* [13]*, Wentworth* [18]*) Take the assumptions of the previous theorem.*

- *If X_0 is irreducible with only one node, then*
$$\delta(X_t) \sim -\frac{4g-1}{3g} \log|t| - 6 \log\log(1/|t|)$$
as $t \to 0$.
- *If X_0 is reducible with one node and its components have genera i and $g-i$ then*
$$\delta(X_t) \sim -\frac{4i(g-i)}{g} \log|t|$$
as $t \to 0$.

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