Ergodic measures of Markov semigroups with the e–property

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== MI-2010-09 ==

April 12, 2010

Abstract

We study the set of ergodic measures for a Markov semigroup on a Polish state space. The principal assumption on this semigroup is the e–property, an equicontinuity condition. We introduce a weak concentrating condition around a compact set K and show that this condition has several implications on the set of ergodic measures, one of them being the existence of a Borel subset K_0 of K with a bijective map from K_0 to the ergodic measures, by sending a point in K_0 to the weak limit of the Cesàro averages of the Dirac measure on this point. We also give sufficient conditions for the set of ergodic measures to be countable and finite. Finally, we give a quite general condition under which the Cesàro averages of any measure converge to an invariant measure.

1 Introduction

In this paper we are concerned with the study of ergodicity of Markov semigroups. Literature devoted to ergodic properties of Markov semigroups is huge. As a basic introduction may serve the monograph by S. P. Meyn and R. L. Tweedie [14]. Since its publication in 1993 there has been a rapid progress caused by possible applications in stochastic differential equations and fractals. In the theory of stochastic differential equations for instance, it was usually assumed that Markov processes corresponding to solutions of studied equations satisfy the strong Feller property. Since it is a very restrictive assumption, especially

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in the case when noise is degenerate, there has been an urgent need to find out a new tool allowing us to examine degenerate stochastic differential equations. A first attempt in this direction was made by M. Hairer and J. Mattingly who introduced the so-called asymptotic strong Feller property. Its definition is complex, hence omitted here. The reader interested in it is referred to [6]. On the other hand, the main assumption we have made is the e–property. For the first time it was used by A. Lasota and one of the authors in [12], where a sufficient condition for the existence of an invariant measure was formulated and proved. Let $(U(t))_{t\geq 0}$ be a Markov semigroup defined on the class of all bounded Borel measurable functions on some Polish space S. We will say that this semigroup has the e-property if the family $(U(t)\varphi)_{t\geq 0}$ is equicontinuous for any bounded Lipschitz function φ. The criterion mentioned above says that every semigroup with the e-property which is concentrated around some compact set, admits an invariant measure. Further results on semigroups satisfying the e-property are proved in [11]. The authors formulated criteria assuring the existence of an invariant measure and its uniqueness. They also applied them to a stochastic differential equation corresponding to a passive tracer model. Recently the criteria for the existence and uniqueness of an invariant measure have been extended to other stochastic differential equations – equations driven by Lévy noise among others [10]. It is worth mentioning here that the solution to the 2D Navier–Stokes equation with degenerate noise considered by Hairer and Mattingly generates the semigroup with the e-property not only with the asymptotic strong Feller property. It seems that all known examples of Markov processes with the asymptotic strong Feller property satisfy the e-property as well. This gives an additional reason for studying Markov semigroups with the e-property. Recently S.C. Hille and the second author started considering ergodic decompositions of general Markov semigroups on Polish spaces. It appeared that then a quite general Yosida-type decomposition of the state space holds [18]. Similar results were obtained by O. Costa and F. Dufour in [3] in the setting of locally compact separable metric spaces. One of its consequences is a characterisation of ergodic measures in terms of a measurable subset of the state space, and an integral decomposition over this subset of any invariant measure in terms of the ergodic measures. In [19] S.C. Hille and the second author focused on Markov semigroups with the e–property, and showed interesting consequences for the ergodic decompositions, some of which will be used in this paper.

In the present paper we are interested in determining ergodic measures as limits of Cesàro averages starting from some compact set. A concentrating condition related to the one introduced in [12] appears to be perfectly fitted to our task. Namely, we prove that then the number of invariant ergodic measures is closely related to the behaviour of Markov semigroups on this concentrating compact set. This allows us to provide a condition for the existence of finitely many ergodic measures. Similar problems for infinite dimensional systems were studied in [13]. We may also determine whether the Markov semigroup admits countably or uncountably many ergodic invariant measures.

The present paper is organised as follows. We start with some notational conventions and preliminaries. In Section 3 we first show some new consequences of the e–property on the set of ergodic measures, and the remainder is devoted to the study of conclusions we are able to draw from a weak concentrating (around some compact set) condition. The main result, Theorem 3.8, says that the set of all ergodic measures is obtained as Cesàro weak limits starting at some points from the given compact set. In fact, we find in this way a Borel subset of the compact set that maps bijectively to the set of ergodic measures. This allows us to determine how many ergodic measure do exist. The condition assuring the existence of finitely or countably many ergodic measures is provided in Section 4. In Section 5, in turn, we show (Theorem 5.2) that a condition related to our weak concentrating condition ensures on a Markov semigroup with the e–property that for every probability measure the Cesàro weak limit exists and is an invariant measure. This theorem implies corollaries that give necessary and sufficient conditions for a Markov semigroup to be weak* mean ergodic and asymptotically stable.

Some notational conventions. Unless otherwise mentioned, (S, d) will denote a complete separable metric space, viewed as a measurable space with respect to its Borel σ-algebra. We write $\mathcal{M}(S)$ to denote the real vector space of all signed finite Borel measures on S, containing $\mathcal{M}^+(S)$, the cone of positive measures. $\mathcal{P}(S)$ consists of the probability measures in $\mathcal{M}^+(S)$. We denote the total variation norm on $\mathcal{M}(S)$ by $\|\cdot\|_{\mathrm{TV}}$ and write $\mathcal{M}(S)_{\mathrm{TV}}$ for the Banach space consisting of $\mathcal{M}(S)$ endowed with the total variation norm. We write $\mathbb{1}_E$ for the indicator function of $E \subset S$. For $f : S \to \mathbb{R}$ measurable and $\mu \in \mathcal{M}(S)$ we write $\langle \mu, f\rangle$ for $\int_S f\, d\mu$. $C_b(S)$ denotes the Banach space of bounded continuous functions from S to $\mathbb{R}$, endowed with the supremum norm $\|\cdot\|_\infty$, and $\mathrm{BL}(S)$ the Banach space of bounded Lipschitz functions from S to $\mathbb{R}$, with the norm $\|f\|_{\mathrm{BL}} = |f|_{\mathrm{Lip}} + \|f\|_\infty$, where $|f|_{\mathrm{Lip}}$ denotes the Lipschitz constant of f. For $x \in S$ and $r > 0$, $B(x, r)$ denotes the open ball around x with radius r.

2 Preliminaries

Let (S, d) be a complete separable metric space. On $\mathcal{M}(S)$ we can consider the *weak topology* $\sigma(\mathcal{M}(S), C_b(S))$ (not to be confused with the weak topology on $\mathcal{M}(S)$ when viewed as a Banach space with the total variation norm). On $\mathcal{M}^+(S)$ this topology is metrisable by the norm $\|\cdot\|^*_{\mathrm{BL}}$:

$$\|\mu - \nu\|^*_{\mathrm{BL}} = \sup\{|\langle \mu - \nu, f\rangle| : f \in \mathrm{BL}(S) : \|f\|_{\mathrm{BL}} \leq 1\},$$

and $\mathcal{M}^+(S)$ is actually complete with respect to this metric (see e.g. [5, Theorem 9 and Theorem 18]).

We will make use of the well known Alexandrov Theorem several times in this paper. It states that a sequence of probability measures $(\mu_n)_n$ converges weakly to a probability measure μ iff $\liminf_{n\to\infty} \mu_n(U) \geq \mu(U)$ for all open $U \subset S$ The above conditions are equivalent to the condition: $\limsup_{n\to\infty} \mu_n(C) \leq \mu(C)$ for all closed $C \subset S$.

We can define the Banach space $\mathcal{S}_{\mathrm{BL}}$ to be the completion of $\mathcal{M}(S)$ in $\mathrm{BL}(S)^*$, then $\mathcal{M}^+(S)$ is a closed convex cone in $\mathcal{S}_{\mathrm{BL}}$. See [8] for some properties of this Banach space.

We define a Markov semigroup $(P(t))_{t\geq 0}$ on S to be semigroup of maps $P(t) : \mathcal{M}^+(S) \to \mathcal{M}^+(S)$, such that $P(t)$ is a positively homogeneous and additive map and

$$\|P(t)\mu\|_{\mathrm{TV}} = \|\mu\|_{\mathrm{TV}} \text{ for all } \mu \in \mathcal{M}^+(S).$$

Throughout this paper we will assume that $(P(t))_{t\geq 0}$ is *Markov–Feller*, i.e. there is a dual semigroup $(U(t))_{t\geq 0}$ on $C_b(S)$ such that

$$\langle P(t)\mu, f\rangle = \langle \mu, U(t)f\rangle$$

for every $t \in \mathbb{R}_+, \mu \in \mathcal{M}^+(S), f \in C_b(S)$. We also assume that $(P(t))_{t\geq 0}$ is *jointly measurable*, i.e. for every Borel set $E \subset S$ the map $(t, x) \mapsto P(t)\delta_x(E)$ is jointly measurable. The joint measurability implies that for $f \in L^1(\mathbb{R}_+)$, $\int_{\mathbb{R}_+} f(s)P(s)\mu(E)\,ds$ exists for all $E \in \Sigma$ and $\mu \in \mathcal{M}^+(S)$ and this defines a measure. We can also define this integral as Bochner integral in $\mathcal{S}_{\mathrm{BL}}$: as shown in [9, Section 2] we obtain for all $h \in \mathrm{BM}(S)$:

$$\left\langle \int_{\mathbb{R}_+} f(s)P(s)\mu\,ds, h \right\rangle = \int_{\mathbb{R}_+} \langle f(s)P(s)\mu, h\rangle\,ds.$$

We call a measure $\mu \in P(S)$ *invariant* if $P(t)\mu = \mu$ for every $t \in \mathbb{R}_+$ and write $\mathcal{P}_{\mathrm{inv}}(S)$ to denote the convex set of invariant probability measures.

A Borel set E is called μ-invariant if for all $t \in \mathbb{R}_+$, $U(t)\mathbb{1}_E = \mathbb{1}_E$ μ-a.e. Then an invariant probability measure μ is *ergodic* if $\mu(E) = 0$ or $\mu(E) = 1$ for every μ-invariant Borel set E (see e.g. [4]). The ergodic measures are also exactly the extreme points of $\mathcal{P}_{\mathrm{inv}}(S)$. We denote the set of ergodic measures by $\mathcal{P}_{\mathrm{erg}}(S)$.

For $t > 0$ and $\mu \in \mathcal{M}^+(S)$ we can define

$$P^{(t)}\mu = \frac{1}{t}\int_0^t P(s)\mu\,ds,$$

then $P^{(t)}$ defines a Markov–Feller operator.

Write

$$\Gamma_t := \{x \in S : (P^{(t)}\delta_x)_{t\geq 1} \text{ is tight}\}$$

and

$$\Gamma_{cp} := \{x \in S : P^{(t)}\delta_x \text{ converges in } \mathcal{S}_{\mathrm{BL}} \text{ as } t \to +\infty\}.$$

Obviously $\Gamma_{cp} \subset \Gamma_t$. If $x \in \Gamma_{cp}$, then we define $\varepsilon_x = \lim_{t\to+\infty} P^{(t)}\delta_x$. Clearly $\varepsilon_x \in P(S)$. Because $(P(t))_{t\geq 0}$ is Markov–Feller, it is not hard to show that $\varepsilon_x \in P_{\mathrm{inv}}(S)$.

We furthermore define

$$\Gamma_{cpie} := \{x \in \Gamma_{cp} : \varepsilon_x \text{ is ergodic}\}.$$

On Γ_{cpie} we can define an equivalence relation $\sim$: $x \sim y$ iff $\varepsilon_x = \varepsilon_y$. We shall write $[x]$ to denote the equivalence class of x. In [17, 18] it is shown that Γ_t, Γ_{cp} and Γ_{cpie} and $[x]$

are Borel sets and that $\mu(\Gamma_t) = \mu(\Gamma_{cp}) = \mu(\Gamma_{cpie})$ for all invariant probability measures μ. Furthermore, $\mathcal{P}_{\rm erg}(S) = \{\varepsilon_x : x \in \Gamma_{cpie}\}$ and $\varepsilon_x([x]) = 1$ for all $x \in \Gamma_{cpie}$.

A semigroup $(P(t))_{t\geq 0}$ has the *e–property* if for every $f \in \mathrm{BL}(S)$, the family of functions $(U(t)f)_{t\geq 0}$ is equicontinuous.

The e–property has various important consequences: In [19] it was shown that Γ_{cp} and Γ_{cpie} are closed sets. Moreover, the map $\Phi : \Gamma_{cp} \to \mathcal{S}_{\rm BL}$ of the form $\Phi(x) = \varepsilon_x$, is continuous and every invariant probability measure μ satisfies $\mu = \int_{\Gamma_{cpie}} \varepsilon_x \, d\mu(x)$.

From [10, Lemma 1] it follows that $\Gamma_t = \Gamma_{cp}$ (see also [19, Theorem 5.13]).

Since Φ is continuous and Γ_{cpie} is closed, $[x] = \Phi^{-1}(\{\varepsilon_x\})$ is closed in S for every $x \in \Gamma_{cpie}$, thus in particular $\mathrm{supp}(\varepsilon_x) \subset [x]$.

3 Weak concentrating condition

We will always assume that $(P(t))_{t\geq 0}$ is a jointly measurable Markov–Feller semigroup on a complete separable metric space (S, d). We begin this section by giving some properties of the set of ergodic measures when the e–property holds.

Proposition 3.1. *Suppose $(P(t))_{t\geq 0}$ has the e–property. Then $\mathcal{P}_{\rm erg}(S)$ is closed in $\mathcal{S}_{\rm BL}$.*

Proof. An invariant probability measure μ is ergodic if and only if for every $f \in C_b(S)$, $\lim_{t\to+\infty} U^{(t)} f(x) = \langle \mu, f\rangle$ for μ-a.e. $x \in S$.

Let us define for $f \in C_b(S)$

$$f^*(x) := \begin{cases} \langle \epsilon_x, f\rangle & \text{if } x \in \Gamma_{cp}, \\ 0 & \text{if } x \notin \Gamma_{cp}. \end{cases}$$

For every $x \in \Gamma_{cp}$, $U^{(t)} f(x) \to \langle \varepsilon_x, f\rangle = f^*(x)$. Since $\mu(\Gamma_{cp}) = 1$ we obtain that μ is ergodic if and only if $f^* = \langle \mu, f\rangle$ μ-.a.e. for every $f \in C_b(S)$, or equivalently

$$\int_{\Gamma_{cp}} (f^*(x) - \langle \mu, f\rangle)^2 \, d\mu(x) = 0. \tag{1}$$

Now assume that $(\mu_n)_n$ is a sequence of ergodic measures such that $\mu_n \to \mu$ in $\mathcal{S}_{\rm BL}$. Then μ is invariant, since $(P(t))_{t\geq 0}$ is Markov–Feller. Fix $f \in C_b(S)$. We need to show that (1) holds. Since $\|f^*\|_\infty \leq \|f\|_\infty$, we have for every $x \in \Gamma_{cp}$ and $n \in \mathbb{N}$

$$\begin{aligned} |(f^*(x) - \langle \mu, f\rangle)^2 &- (f^*(x) - \langle \mu_n, f\rangle)^2| \\ &\leq 2|f^*(x)||\langle \mu - \mu_n, f\rangle| + |(\langle \mu, f\rangle)^2 - (\langle \mu_n, f\rangle)^2| \\ &\leq 2\|f\|_\infty |\langle \mu - \mu_n, f\rangle| \\ &\quad + |\langle \mu + \mu_n, f\rangle||\langle \mu - \mu_n, f\rangle| \\ &\leq 4\|f\|_\infty |\langle \mu - \mu_n, f\rangle|. \end{aligned}$$

So

$$\left| \int_{\Gamma_{cp}} (f^*(x) - \langle \mu, f \rangle)^2 - (f^*(x) - \langle \mu_n, f \rangle)^2 \, d\mu_n(x) \right| \to 0$$

as $n \to +\infty$.

For every $n \in \mathbb{N}$

$$\begin{aligned} &\left| \int_{\Gamma_{cp}} (f^*(x) - \langle \mu, f \rangle)^2 \, d\mu(x) - \int_{\Gamma_{cp}} (f^*(x) - \langle \mu_n f \rangle)^2 \, d\mu_n(x) \right| \leq \\ &\left| \int_{\Gamma_{cp}} (f^*(x) - \langle \mu, f \rangle)^2 \, d[\mu(x) - \mu_n(x)] \right| \\ &+ \left| \int_{\Gamma_{cp}} (f^*(x) - \langle \mu, f \rangle))^2 - (f^*(y) - \langle \mu_n, f \rangle)^2 \, d\mu_n(x) \right|. \end{aligned} \tag{2}$$

The final term in inequality (2) above goes to zero as $n \to +\infty$.

Since $x \mapsto \varepsilon_x$ is continuous from Γ_{cp} to $\mathcal{S}_{\mathrm{BL}}$, we also know that $x \mapsto (f^*(x) - \langle \mu, f \rangle)^2$ is bounded and continuous from Γ_{cp} to $\mathbb{R}$. Γ_{cp} is closed, thus we can apply the Tietze Extension Theorem, and so there exists a $g \in C_b(S)$, such that $g(x) = (f^*(x) - \langle \mu, f \rangle)^2$ for every $x \in \Gamma_{cp}$. Since $\mu(\Gamma_{cp}) = \mu_n(\Gamma_{cp}) = 1$ for every $n \in \mathbb{N}$, we have

$$\left| \int_{\Gamma_{cp}} (f^*(x) - \langle \mu, f \rangle)^2 \, d[\mu(x) - \mu_n(x)] \right| = |\langle \mu, g \rangle - \langle \mu_n, g \rangle| \to 0$$

as $n \to +\infty$ since $\mu_n \to \mu$ in $\mathcal{S}_{\mathrm{BL}}$ and $g \in C_b(S)$.

Now note that $\int_{\Gamma_{cp}} (f^*(x) - \langle \mu_n, f \rangle)^2 \, d\mu_n(x) = 0$ for every $n \in \mathbb{N}$, since the measures μ_n are ergodic, thus $\int_{\Gamma_{cp}} (f^*(x) - \langle \mu, f \rangle)^2 \, d\mu(x) = 0$ as well. Thus μ is ergodic. □

Proposition 3.2. *If $P_{\mathrm{inv}}(S)$ is non-empty, then there exists an invariant probability measure μ_0 with*

$$\mathrm{supp}(\mu_0) = \overline{\bigcup_{\mu \in P_{\mathrm{inv}}(S)} \mathrm{supp}(\mu)}.$$

Hence $\bigcup_{\mu \in P_{\mathrm{inv}}(S)} \mathrm{supp}(\mu)$ is closed.

Proof. Let $D = \bigcup_{\mu \in P_{\mathrm{inv}}(S)} \mathrm{supp}(\mu)$. Then D is separable, so there exist $(x_n)_n \subset D$ such that $D \subset \overline{\{x_n : n \in \mathbb{N}\}}$. Let $\mu_n \in P_{\mathrm{inv}}(S)$ be such that $x_n \in \mathrm{supp}(\mu_n)$ and define

$$\mu_0 = \sum_{n=1}^{\infty} (1/2^n) \mu_n,$$

then $\mu_0 \in P_{\text{inv}}(S)$ and

$$\bigcup_{n=1}^{\infty} \text{supp}(\mu_n) \subset \text{supp}(\mu_0),$$

thus

$$D = \overline{\bigcup_{n=1}^{\infty} \text{supp}(\mu_n)} = \text{supp}(\mu_0).$$

□

We can also ask ourselves what we can say about the union of the supports of all ergodic measures. Even with the e–property, this set need not be closed, as the following example shows:

Example 3.3. Let $S = [0,1]$. For $x \in S$ and $t \in \mathbb{R}_+$ define

$$P(t)\delta_x = [x + e^{-t}(1-x)]\delta_x + [(1-x) - e^{-t}(1-x)]\delta_{1-x}.$$

Then $P(0)\delta_x = \delta_x$, and easy calculation shows that $P(t)P(s)\delta_x = P(t+s)\delta_x$ for all $x \in S$ and $s, t \in \mathbb{R}_+$. For every $E \subset S$ Borel,

$$P(t)\delta_x(E) = [x + e^{-t}(1-x)]\mathbb{1}_E(x) + [(1-x) - e^{-t}(1-x)]\mathbb{1}_E(1-x)$$

so $(t,x) \mapsto P(t)\delta_x(E)$ is jointly measurable. Thus we can define a jointly measurable Markov semigroup on S as follows:

$$P(t)\mu = \int_S P(t)\delta_x \, d\mu(x) \text{ for all } \mu \in \mathcal{M}^+(S).$$

It can be shown that $(P(t))_{t\geq 0}$ is Markov-Feller and satisfies the e–property. Now, for all $x \in S$, $P(t)\delta_x \to x\delta_x + (1-x)\delta_{1-x} = \varepsilon_x$ as $t \to \infty$. Because these measures cannot be written as the convex combination of different invariant probability measures, these are ergodic measures. So $\Gamma_{cpie} = S$, and thus each ergodic measure equals $x\delta_x + (1-x)\delta_{1-x}$ for some $x \in S$. Now, for all $0 < x < 1$, $\text{supp}(\varepsilon_x) = \{x, 1-x\}$, and $\text{supp}(\varepsilon_0) = \text{supp}(\varepsilon_1) = \{1\}$. Thus

$$\bigcup_{x \in S} \text{supp}(\varepsilon_x) = (0,1],$$

which is open but not closed in S.

We show that in general the e–property implies that union of the supports of ergodic measures is a G_δ subset of S, i.e. a countable intersection of open sets.

Theorem 3.4. *Let $(P(t))_{t\geq 0}$ be a Markov–Feller semigroup with the e-property. Then*

$$D := \bigcup_{\mu \in \mathcal{P}_{\text{erg}}(S)} \text{supp}(\mu)$$

is an G_δ set. In particular, D is a Polish space in its relative topology.

Proof. If $x \in \text{supp}(\mu)$ for an ergodic measure μ, then $x \in \Gamma_{cpie}$, and $\text{supp}(\mu) \subset [x]$, so $\mu = \varepsilon_x$. So we can write

$$D = \{x \in \Gamma_{cpie} : x \in \text{supp}(\varepsilon_x)\} = \bigcap_{k \in \mathbb{N}} D_k,$$

where

$$D_k = \{x \in \Gamma_{cpie} : \varepsilon_x(B(x, 1/k)) > 0\}.$$

Let $E_k := \Gamma_{cpie} \backslash D_k = \{x \in \Gamma_{cpie} : \varepsilon_x(B(x, 1/k)) = 0\}$. We will show that E_k is closed. Let $x_n \in E_k$ such that $x_n \to x$ in S. Then $x \in \Gamma_{cpie}$ and $\varepsilon_{x_n} \to \varepsilon_x$.

For $N \in \mathbb{N}$ define $V_N = B(x, 1/k) \cap (\cap_{n \geq N} B(x_n, 1/k))$. Let $y \in V_N$ and define $r := \sup\{d(y, x_n) : n \geq N\}$. Since $d(y, x) < 1/k$ and $x_n \to x$, $r < 1/k$, which implies that V_N is open in S. Now, for all $N \in \mathbb{N}$,

$$\varepsilon_x(V_N) \leq \liminf_{n \to +\infty} \varepsilon_{x_n}(V_N) \leq \liminf_{n \to +\infty} \varepsilon_{x_n}(B(x_n, 1/k)) = 0.$$

Since $V_N \subset V_{N+1}$ for all N and $\cup_N V_N = B(x, 1/k)$, $\varepsilon_x(B(x, 1/k)) = 0$ and E_k is closed.

We can write:

$$D = \Gamma_{cpie} \cap \bigcap_{k \in \mathbb{N}} (S \backslash E_k).$$

Since Γ_{cpie} is closed, D is a G_δ set.

The final statement follows from [1, Theorem 3.1.2], which states that every G_δ subset of a Polish space is again a Polish space. □

We introduce the *weak concentrating condition*:

($\mathcal{C}$) There exists a compact $K \subset S$ such that for every $\varepsilon > 0$ and every $x \in S$

$$\limsup_{t \to +\infty} P^{(t)} \delta_x(K^\varepsilon) > 0,$$

where $K^\varepsilon = \{x \in S : d(x, K) < \varepsilon\}$.

It turns out that we can obtain every ergodic measure from K:

Lemma 3.5. *Suppose* ($\mathcal{C}$) *is satisfied. For every* $x \in \Gamma_{cpie}, K \cap \text{supp}(\varepsilon_x) \neq \emptyset$.

Proof. Suppose Γ_{cpie} is non-empty and let $x \in \Gamma_{cpie}$ such that $K \cap \text{supp}(\varepsilon_x) = \emptyset$. Since $\text{supp}(\varepsilon_x)$ is closed and K is compact, there exists an $\varepsilon > 0$ such that $K^\varepsilon \cap \text{supp}(\varepsilon_x) = \emptyset$. Thus $\varepsilon_x(K^\varepsilon) = 0$. In particular, $\varepsilon_x(\overline{K^{\varepsilon/2}}) = 0$. Let $y \in \text{supp}(\varepsilon_x) \cap [x]$, which is non-empty since $\varepsilon_x([x]) = 1$, then

$$\limsup_{t \to +\infty} P^{(t)} \delta_y(\overline{K^{\varepsilon/2}}) \leq \varepsilon_x(\overline{K^{\varepsilon/2}}) = 0,$$

which contradicts ($\mathcal{C}$). □

Note that $(\mathcal{C})$ is a stronger condition than condition $(\mathcal{E})$ considered in [12]. Thus [12, Theorem 3.1] implies that there exists an invariant measure when $(\mathcal{C})$ is satisfied and $(P(t))_{t\geq 0}$ satisfies the e–property. Then there must also exist an ergodic measure, and thus Γ_{cpie} and Γ_{cp} are non-empty.

We shall write $\hat{K} := K \cap \Gamma_{cpie}$. Since Γ_{cpie} is closed, $\hat{K}$ is compact. By Lemma 3.5, $\Phi(\hat{K}) = \mathcal{P}_{\rm erg}(S)$, and by continuity of Φ we can conclude:

Corollary 3.6. *If the e–property and $(\mathcal{C})$ hold for $(P(t))_{t\geq 0}$, then $\mathcal{P}_{\rm erg}(S)$ is compact in $\mathcal{S}_{\rm BL}$.*

The following result can be found in [2, Corollary 6.9.18]:

Proposition 3.7. *Let X be a Polish space and R an equivalence relation on X with closed equivalence classes. If $R(Z) \subset X$ is a Borel set for every closed $Z \subset X$, then R admits a Borel section, i.e. there is a Borel set $B \subset X$ such that B contains exactly one element of every equivalent class.*

Theorem 3.8. *If $(P(t))_{t\geq 0}$ satisfies the e–property and $(\mathcal{C})$, then there exists a Borel set $K_0 \subset K$ such that*

(i) $x \in \operatorname{supp}(\varepsilon_x)$ for all $x \in K_0$. In particular $K_0 \subset \Gamma_{cpie}$.

(ii) If $x, y \in K_0$ with $x \neq y$, then $\varepsilon_x \neq \varepsilon_y$.

(iii) For every ergodic measure μ there is an $x \in K_0$ such that $\mu = \varepsilon_x$.

Proof. Let

$$X := \bigcup_{\mu \in \mathcal{P}_{\rm erg}(S)} \operatorname{supp}(\mu) \cap K,$$

then X is a G_δ set by Theorem 3.4, hence a Polish space in its relative topology by [1, Theorem 3.1.2]. Also, $X \subset \Gamma_{cpie}$.

Let us define an equivalence relation R on X as follows: xRy if and only if x and y are in the support of the same ergodic measure, so if and only if $\varepsilon_x = \varepsilon_y$. Note that xRy if and only if $x \in \operatorname{supp}(\varepsilon_y)$ if and only if $y \in \operatorname{supp}(\varepsilon_x)$. Note that R is the restriction to X of the equivalence class $\sim$ on Γ_{cpie} we introduced earlier. For every $x \in X$, $R(x) = \operatorname{supp}(\varepsilon_x) \cap K = \operatorname{supp}(\varepsilon_x) \cap X$, thus $R(x)$ is closed in X. In order to apply Proposition 3.7, we need to show that $R(Z)$ is closed for all closed subsets Z in X.

Let Z be closed in X. Let $\bar{Z}$ be its closure in S, then $Z = \bar{Z} \cap X$. Furthermore, $\bar{Z}$ is a closed subset of $K \cap \Gamma_{cpie}$, hence compact. We claim that

$$R(Z) = \{x \in \Gamma_{cpie} \cap K : \varepsilon_x(\bar{Z}^{1/n}) > 0 \text{ for all } n \in \mathbb{N}\} =: W_Z.$$

Let $x \in R(Z)$, then there is a $z \in Z$ such that $x \in \operatorname{supp}(\varepsilon_z)$, thus also $z \in \operatorname{supp}(\varepsilon_x)$. Hence, for all $n \in \mathbb{N}$, $\varepsilon_x(\bar{Z}^{1/n}) \geq \varepsilon_x(B(z, 1/n)) > 0$, so $x \in W_Z$.

Now let $x \in W_Z$. Suppose that $\text{supp}(\varepsilon_x) \cap \bar{Z} = \emptyset$, then compactness of $\bar{Z}$ implies that there is an $n \in \mathbb{N}$ such that $\varepsilon_x(\bar{Z}^{1/n}) = 0$, which is a contradiction. So there is a $z \in \bar{Z}$ such that $z \in \text{supp}(\varepsilon_x)$. Since $\bar{Z} \subset K$, $z \in \bar{Z} \cap X = Z$. Since $z \in \text{supp}\, \varepsilon_x$, $x \in \text{supp}\, \varepsilon_z$, so $x \in R(Z)$.

Now it remains to show that W_Z is Borel in X. We can write

$$W_Z = \bigcap_{n \in \mathbb{N}} W_Z^n,$$

where $W_Z^n = \{x \in \Gamma_{cpie} \cap K : \varepsilon_x(\bar{Z}^{1/n}) > 0\}$. Let

$$V_Z^n := (\Gamma_{cpie} \cap K) \backslash W_Z^n = \{x \in \Gamma_{cpie} \cap K : \varepsilon_x(\bar{Z}^{1/n}) = 0\},$$

and let $x_k \in V_z^n$ such that $x_k \to x \in S$. Then $x \in \Gamma_{cpie} \cap K$ and

$$\varepsilon_x(\bar{Z}^{1/n}) \leq \liminf_{k \to +\infty} \varepsilon_{x_k}(\bar{Z}^{1/n}) = 0,$$

thus $x \in V_z^n$. So V_z^n is closed, and thus W_z^n is open (in the relative topology on $\Gamma_{cpie} \cap K$). Then $W_z^n \cap X$ is open in X, so

$$R(Z) = W_Z = X \cap W_Z = \bigcap_{n \in \mathbb{N}} W_Z^n \cap X$$

is a G_δ subset of X, thus Borel. Application of Proposition 3.7 yields the existence of a Borel set $K_0 \subset X \subset \Gamma_{cpie} \cap K$ such that for every $x \in X$ there is exactly one $y \in K_0$ such that $y \in R(x)$.

Thus *(i)* and *(ii)* are satisfied. Now let μ be an ergodic measure, then there is an $x \in \Gamma_{cpie}$ with $\mu = \varepsilon_x$. Lemma 3.5 implies that there is a $z \in \text{supp}(\varepsilon_x) \cap K \subset X$ and thus there is exactly one $y \in K_0$ such that $y \in R(z)$. Consequently $\varepsilon_y = \varepsilon_z = \varepsilon_x = \mu$. This concludes the proof. □

Note that the set K_0 from Theorem 3.8 need not be unique. For instance, if we let S be the unit circle and $P(t)\delta_x := \delta_{e^{2\pi i t}x}$, then $(P(t))_{t \geq 0}$ defines a Markov–Feller semigroup with the e–property with a unique ergodic measure given by the Lebesgue measure on S. Obviously we can choose $K_0 = \{z\}$ for any $z \in S$.

Theorem 3.8 raises the interesting question if for Markov–Feller semigroups with the e–property, but without $(\mathcal{C})$, a result analogous to Theorem 3.8 holds.

We say that $(P(t))_{t \geq 0}$ is *sweeping* from some family $\mathcal{A}$ of Borel subsets of S if

$$\lim_{t \to \infty} P(t)\mu(A) = 0,$$

for all $\mu \in P(S)$ and $A \in \mathcal{A}$.

The following result generalises [16, Proposition 3]:

Proposition 3.9. *Let $(P(t))_{t\geq 0}$ be a Markov-Feller semigroup that satisfies the e–property and $(\mathcal{C})$, then $(P(t))_{t\geq 0}$ is sweeping from compact sets disjoint from Γ_t.*

Proof. We define by $K_0 \subset K$ the set of all $x \in K$ such that for any $\varepsilon > 0$

$$\limsup_{t\to+\infty} P^{(t)}\mu(B(x,\varepsilon)) > 0 \tag{3}$$

for some $\mu \in \mathcal{P}(S)$. Observe that K_0 is closed, hence compact. Now we show that $(\mathcal{C})$ is also satisfied with K replaced with K_0. Fix $\varepsilon > 0$. Then for each $x \in K_0$ we define $r_x = \varepsilon/2$. For $x \in K\backslash K_0$ there exists an $r_x > 0$ such that $\lim_{t\to+\infty} P^{(t)}\mu(B(x, 2r_x)) = 0$ for all $\mu \in \mathcal{P}(S)$. By compactness of K there exist $m, n \in \mathbb{N}_0$, $x_1, ..., x_m \in K_0$ and $x_{m+1}, ..., x_n \in K\backslash K_0$ such that $K \subset \cup_{i=1}^n B(x_i, r_{x_i})$. Let $r := \min\{r_{x_i} : 1 \leq i \leq n\}$, then $r > 0$ and

$$K^r \subset \cup_{i=1}^m B(x_i,\varepsilon) \cup \left(\cup_{i=m+1}^n B(x_i, 2r_{x_i})\right) \subset K_0^\varepsilon \cup \left(\cup_{i=m+1}^n B(x_i, 2r_{x_i})\right).$$

Now,

$$\begin{aligned}\limsup_{t\to+\infty} P^{(t)}\delta_x(K_0^\varepsilon) &= \limsup_{t\to+\infty} P^{(t)}\delta_x(K_0^\varepsilon) + \sum_{k=n+1}^{m} \limsup_{t\to+\infty} P^{(t)}\delta_x(B(x_i, 2r_{x_i})) \\ &\geq \limsup_{t\to+\infty} P^{(t)}\delta_x(K^r) > 0.\end{aligned}$$

[19, Proposition 6.1] yields that $K_0 \subset \Gamma_t = \Gamma_{cp}$.

Suppose there is a compact L such that $L \cap \Gamma_t = \emptyset$ and an $\alpha > 0$ and $\mu \in P(S)$ such that $\limsup_{t\to\infty} P(t)\mu(L) > \alpha$. Since Γ_t is closed, there is an $\eta > 0$ such that $L^\eta \cap \Gamma_t = \emptyset$.

We define

$$M := \{\nu \in P(S) : \text{there exists } \gamma < \eta \text{ such that } \liminf_{t\to\infty} P(t)\nu(\Gamma_t^\gamma) > 1 - \alpha/2\}.$$

Note that Γ_t is non-empty and $(P(t))_{t\geq 0}$-invariant (see [16, Lemma 1]), thus $\{\delta_x : x \in \Gamma_t\} \subset M$, and in particular M is non-empty as well. Also, M is convex and $P(t)M \subset M$ for all $t \in \mathbb{R}_+$. From [16, Lemma 3] it follows that M is open in the weak topology, and since $K_0 \subset \Gamma_t$ there is a $\sigma > 0$ such that whenever $\nu \in P(S)$ with $\text{supp}(\nu) \subset K_0^\sigma$, then $\nu \in M$.

Let $x \in L$, then $(\mathcal{C})$ implies that there is a $t_x > 0$ such that

$$\alpha_x := P(t_x)\delta_x(K_0^{\sigma/2}) > 0.$$

Since $P(t_x)$ is Markov-Feller, there is an $r_x > 0$ such that $P(t_x)(\delta_y(K_0^{\sigma/2})) > 0$ for all $y \in B(x, r_x)$. By compactness of L there exist $x_1, ..., x_k \in L$ such that $L \subset \cup_{i=1}^k B(x_i, r_{x_i})$. Define $\Theta = \min_{1\leq i\leq k} \alpha_{x_i}/2$ and

$$\gamma := \sup\{\beta \geq 0 : P(t_0)\mu \geq \beta\nu \text{ for some } \nu \in M, t_0 > 0\}.$$

Now choose $\nu \in M$ and $t_0 > 0$ such that $P(t_0)\mu \geq \beta\nu$ holds with $\beta > \gamma - \Theta\alpha/(2k)$. Then for all $t \geq 0$, $P(t+t_0)\mu \geq \beta P(t)\nu$ and $P(t)\nu \subset M$, thus we can choose $\nu \in M$ and t_0 in such a way that $P(t_0)\mu(L) > \alpha$ and $\nu(L) < \alpha/2$. Then

$$(P(t_0)\mu - \beta\nu)(L) \geq \alpha - \alpha/2 = \alpha/2.$$

So there exist $j \in \{1, ..., k\}$ such that $(P(t_0)\mu - \beta\nu)(B(x_j, r_{x_j})) \geq \alpha/(2k)$. Now

$$\begin{aligned}
\langle P(t_{x_j})(P(t_0)\mu - \beta\nu), \mathbb{1}_{K_0^\sigma}\rangle &= \langle P(t_0)\mu - \beta\nu, U(t_{x_j})\mathbb{1}_{K_0^\sigma}\rangle \\
&= \int_S P(t_{x_j})\delta_x(K_0^\sigma)d[P(t_0)\mu - \beta\nu](x) \\
&\geq \int_{B(x_j, r_{x_j})} P(t_{x_j})\delta_x(K_0^\sigma)d[P(t_0)\mu - \beta\nu](x) \\
&\geq \Theta\alpha/(2k).
\end{aligned}$$

Set

$$\tilde{\nu} = \frac{(P(t_{x_j}+t_0)\mu - \beta P(t_{x_j})\nu)(\cdot \cap K_0^\sigma)}{(P(t_{x_j}+t_0)\mu - \beta P(t_{x_j})\nu)(K_0^\sigma)},$$

then $\tilde{\nu} \in M$, and $\mathrm{supp}(\nu) \subset K_0^\sigma$. Let

$$\hat{\nu} = \beta(\beta + \Theta\alpha/(2k))^{-1}P(t_{x_j})\nu + \Theta\alpha/(2k)(\beta + \Theta\alpha/(2k))^{-1}\tilde{\nu}.$$

Since $P(t_{x_j})\nu$ and $\tilde{\nu}$ are in M, $\hat{\nu}$ is in M as well, since M is convex. Furthermore,

$$P(t_{x_j}+t_0)\mu \geq (\beta + \Theta\alpha/(2k))\hat{\nu},$$

which contradicts the fact that $\gamma < \beta + \Theta\alpha/(2k)$. This completes the proof. □

4 Countably many ergodic measures

Let $(P(t))_{t\geq 0}$ be a jointly measurable Markov–Feller semigroup with dual $(U(t))_{t\geq 0}$. In this section we give some sufficient conditions for the set of ergodic measures to be countable or finite.

First we note that the e–property, even when combined with the weak concentrating condition $(\mathcal{C})$, does not guarantee that the set of ergodic measures is countable. A trivial example is given by $S = [0,1]$ and $(P(t))_{t\geq 0}$ the identity semigroup. Then δ_x is an ergodic measure for all $x \in S$.

Hairer and Mattingly introduced in [6] the asymptotic strong Feller property, which generalises the well-known strong Feller property, and used it in combination with other conditions to show uniqueness of invariant measures. They give a sufficient condition for a Markov semigroup to be asymptotic strong Feller in [6, Proposition 3.12]. However, this condition makes sense only for Hilbert spaces. We give a more general condition that works for Polish spaces as well. We will show that this implies that there are at most countably many ergodic measures, and when combined with $(\mathcal{C})$, the number of ergodic measures is finite.

We fix an $x_0 \in S$. For $f : S \to \mathbb{R}$ and $\theta > 0$ we define the local Lipschitz constant

$$|f|_{\mathrm{Lip},\theta} := \sup\left\{ \frac{|f(x) - f(y)|}{d(x,y)} : x \neq y; x, y \in B(x_0, \theta) \right\}.$$

We assume there are sequences $t_n \geq 0$ and $\delta_n \downarrow 0$ and a non-decreasing function $C : \mathbb{R}_+ \to \mathbb{R}$, such that for all $f \in \mathrm{BL}(S)$ and $\theta > 0$

$$|U(t_n)f|_{\mathrm{Lip},\theta} \leq C(\theta)[\|f\|_\infty + \delta_n |f|_{\mathrm{Lip}}]. \tag{4}$$

Our next result gives lower bounds on distances between points in the supports of different ergodic measures. It generalises [7, Theorem 2.1] and its proof is based on the proof of that theorem. We include it here for completeness.

Proposition 4.1. *Let μ and ν be ergodic measures and $x \in \mathrm{supp}(\mu)$, $y \in \mathrm{supp}(\nu)$. Then* (4) *implies that*

$$d(x,y) \geq \frac{1}{C(d(x,x_0) \vee d(y,x_0))}.$$

Proof. We define for $n \in \mathbb{N}$ the following metric on S:$d_n(x,y) = 1 \wedge (\frac{1}{\sqrt{\delta_n}} d(x,y))$. These metrics induce metrics on $\mathcal{P}(S)$ in the following way:

$$d_n(\mu,\nu) = \sup\{|\langle \mu - \nu, f\rangle| : f \in \mathrm{Lip}^1_{d_n}(S)\},$$

where

$$\mathrm{Lip}^1_{d_n}(S) = \{f : S \to \mathbb{R} : |f(x) - f(y)| \leq d_n(x,y) \text{ for all } x, y \in S\}.$$

Then $d_n(\mu,\nu) \leq 1$ and $\lim_{n\to+\infty} d_n(\mu,\nu) = 1/2\|\mu - \nu\|_{\mathrm{TV}}$, by [6, Lemma 3.4]. Note that it suffices to only consider those $f \in \mathrm{Lip}^1_{d_n}(S)$ for which $f(x_0) = 0$. For such f, $|f(x)| = |f(x) - f(x_0)| \leq d_n(x,x_0) \leq 1$, so $\|f\|_\infty \leq 1$. Moreover

$$|f_n(x) - f_n(y)| \leq \frac{1}{\sqrt{\delta_n}} d(x,y) \text{ for all } x, y \in S,$$

so $|f_n|_{\mathrm{Lip}} \leq \frac{1}{\sqrt{\delta_n}}$. Now we apply (4):

$$\begin{aligned} d_n(P(t_n)\delta_x, P(t_n)\delta_y) &\leq \sup\{|U(t_n)f(x) - U(t_n)f(y)| : f \in \mathrm{Lip}^1_{d_n}(S) \text{ and } f(x_0) = 0\} \\ &\leq d(x,y) C(d(x,x_0) \vee d(y,x_0))(1 + \sqrt{\delta_n}). \end{aligned}$$

Let μ_1 and μ_2 be two distinct ergodic measures, then they are mutually singular, so $\|\mu_1 - \mu_2\|_{\mathrm{TV}} = 2$. Suppose that there are $x \in \mathrm{supp}(\mu_1)$ and $y \in \mathrm{supp}(\mu_2)$ such that

$d(x, y) < (C(d(x, x_0) \vee d(y, x_0)))^{-1}$. Then we will show that $\|\mu_1 - \mu_2\|_{\mathrm{TV}} < 2$, which gives a contradiction. By assumption there is a Borel set E containing x and y such that $\alpha := \min(\mu_1(E), \mu_2(E)) > 0$ and $\beta := \mathrm{diam}(E)C(d(x, x_0) \vee d(y, x_0)) < 1$. We can write $\mu_i = \alpha\nu_i + (1 - \alpha)\rho_i$, with $\nu_i, \rho_i \in P(S)$ and $\nu_i(E) = 1$. Then

$$\begin{aligned} d_n(\mu_1, \mu_2) &= d_n(P(t_n)\mu_1, P(t_n)\mu_2) \leq \alpha d_n(P(t_n)\nu_1, P(t_n)\nu_2) + 1 - \alpha \\ &\leq \alpha \int_S \int_S d_n(P(t_n)\delta_w, P(t_n)\delta_z)\, d\nu_1(w)\, d\nu_2(z) + 1 - \alpha \\ &\leq \alpha\beta(1 + \sqrt{\delta_n}) + 1 - \alpha, \end{aligned}$$

thus

$$1/2\|\mu_1 - \mu_2\|_{\mathrm{TV}} \leq 1 - \alpha(1 - \beta) < 1,$$

which is a contradiction. □

Corollary 4.2. *Assume that* (4) *holds for some non-decreasing* $C : \mathbb{R}_+ \to \mathbb{R}$*. Then there exist at most countably many ergodic measures.*

Proof. We will show that for every bounded set B in S, there exist at most countably many ergodic measures whose support intersects B. Since we can cover S with countably many bounded sets, this proves that there are at most countably many ergodic measures.

Let $B \subset S$ bounded, and define $R := \sup\{d(x, x_0) : x \in B\} < \infty$. Let μ be an ergodic measure with $x \in \mathrm{supp}(\mu) \cap B$. Then Proposition 4.1 implies that for any ergodic measure ν with $\mu \neq \nu$ and $y \in \mathrm{supp}(\nu) \cap B$ we have

$$d(x, y) \geq 1/C(R). \tag{5}$$

Now we choose for every ergodic measure μ with $\mathrm{supp}(\mu) \cap B \neq \emptyset$ an $x \in \mathrm{supp}(\mu) \cap B$ and consider the open ball $B(x, 1/(2C(R)))$. By (5) these balls are mutually disjoint. Separability of S implies that there can be only countably many of such balls, which concludes the proof. □

Corollary 4.3. *Assume that* (4) *holds for some non-decreasing* $C : \mathbb{R}_+ \to \mathbb{R}$ *and that condition* ($\mathcal{C}$) *holds. Then there are only finitely many ergodic measures.*

Proof. Lemma 3.5 implies that the support of every ergodic measure has non-empty intersection with K. Since K is compact, it is bounded, so by the proof of Corollary 4.2 there is an $R > 0$ such that whenever μ and ν are two distinct ergodic measures, then $B(x, 1/(2C(R)) \cap B(y, 1/(2C(R))) = \emptyset$ for every $x \in \mathrm{supp}(\mu) \cap K$ and $y \in \mathrm{supp}(\nu) \cap K$. Since any subset of K is totally bounded, there can be only finitely number of mutually disjoint balls with radius $(2C(R))^{-1}$ and center in K, so the number of ergodic measures is finite as well. □

Notice that the conditions in Corollary 4.3 not necessarily imply the existence of invariant measures. However, when combined with the e–property there do exist invariant measures. There exist examples of Markov–Feller semigroups with the e–property that do not satisfy (4) or even the asymptotic strong Feller property. At the beginning of this section we gave a trivial example of such a semigroup. See also [11, Remark 6]. As for now we do not know any Markov–Feller semigroups that satisfy the asymptotic strong Feller property but not the e–property. We now give a condition to ensure both properties:

Proposition 4.4. *Suppose there exists a non-decreasing* $C : \mathbb{R}_+ \to \mathbb{R}$ *such that for all* $t \in \mathbb{R}_+$ *and* $f \in \mathrm{BL}(S)$

$$|U(t)f|_{\mathrm{Lip},\theta} \leq C(\theta)[\|f\|_\infty + |f|_{\mathrm{Lip}}] = C(\theta)\|f\|_{\mathrm{BL}}.$$

Then $(P(t))_{t\geq 0}$ *satisfies the e–property. If in addition there is a function* $h : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\lim_{t\to+\infty} h(t) = 0$ *and*

$$|U(t)f|_{\mathrm{Lip},\theta} \leq C(\theta)[\|f\|_\infty + h(t)|f|_{\mathrm{Lip}}] \text{ for all } t \in \mathbb{R}_+,$$

then $(P(t))$ *satisfies* (4) *as well.*

Proof. If $x_n \to x \in S$, then for all $f \in \mathrm{BL}(S)$,

$$\sup_{t\geq 0} |U(t)f(x_n) - U(t)f(x)| \leq C(2d(x, x_0))\|f\|_{\mathrm{BL}} d(x_n, x)$$

for n large enough. This proves the e–property. It is clear that under the extra assumption, $(P(t))_{t\geq 0}$ satisfies (4) as well. □

5 Convergence of Cesàro averages

In this section we will formulate a condition on Markov–Feller semigroups with the e–property such that the Cesàro averages of all probability measures will converge weakly to invariant measures.

Note that $(\mathcal{C})$ is not sufficient. See [11, Remark 1] for an example of a Markov–Feller semigroup $(P(t))_{t\geq 0}$ having the e–property that satisfies an even stronger condition than $(\mathcal{C})$, i.e. there is a $z \in S$ such that

$$\liminf_{t\to+\infty} P^{(t)}\delta_x(B(z,\varepsilon)) > 0$$

for all $\varepsilon > 0$ and all $x \in S$. However, as shown in [11, Remark 1], the set Γ_t for this semigroup does not equal the whole space, and since $\Gamma_{cp} = \Gamma_t$, there exist probability measures for which the Cesàro averages do not converge.

It turns out that strengthening $(\mathcal{C})$ by demanding a uniform lower bound depending on ε will give the result. We first prove a preliminary result:

Lemma 5.1. *Let $(P(t))_{t\geq 0}$ be a Markov–Feller semigroup that satisfies the e–property. Let $K \subset S$ be compact. Then for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $y \in K$ and $x \in B(y, \delta)$*

$$\|P^{(t)}\delta_x - P^{(t)}\delta_y\|^*_{\mathrm{BL}} < \varepsilon \text{ for all } t \in \mathbb{R}_+.$$

Proof. Suppose that the statement does not hold. Then there exists an $\varepsilon > 0$, $y_n \in K, x_n \in B(y_n, 1/n)$ and $t_n \in \mathbb{R}_+$ such that

$$\|P^{(t_n)}\delta_{x_n} - P^{(t_n)}\delta_{y_n}\|^*_{\mathrm{BL}} \geq \varepsilon.$$

There is a subsequence y_{n_k} such that $y_{n_k} \to y \in K$ and thus $x_{n_k} \to y$.

The e–property and [19, Theorem 4.2] imply that the family of maps $(P^{(t)}\delta_\cdot)_{t\geq 0}$ from S to $\mathcal{S}_{\mathrm{BL}}$ is equicontinuous. So

$$\|P^{(t_{n_k})}\delta_{y_{n_k}} - P^{(t_{n_k})}\delta_y\|^*_{\mathrm{BL}} \to 0$$

and

$$\|P^{(t_{n_k})}\delta_{x_{n_k}} - P^{(t_{n_k})}\delta_y\|^*_{\mathrm{BL}} \to 0,$$

giving a contradiction. □

Theorem 5.2. *Let $(P(t))_{t\geq 0}$ be a Markov–Feller semigroup that satisfies the e–property. Then the following two statements are equivalent:*

(i) There exists a compact set $K \subset S$ such that for every $\varepsilon > 0$ we may find $\alpha > 0$ such that

$$\limsup_{t\to+\infty} P^{(t)}\delta_x(K^\varepsilon) \geq \alpha \qquad \text{for } x \in S. \tag{6}$$

(ii) The set of ergodic measures is compact and $(P^{(t)}\mu)_{t\geq 0}$ converges to an invariant measure for every $\mu \in \mathcal{P}(S)$.

Proof. (i)⇒(ii): By Corollary 3.6 the set of ergodic measures is compact in $\mathcal{S}_{\mathrm{BL}}$.

It follows from [19, Theorem 5.13] that it is sufficient to show that $(P^{(t)}\mu)_{t\geq 1}$ is tight for any $\mu \in \mathcal{P}(S)$. By $\mathcal{C}_\varepsilon$ we denote the family of all subsets of S who are contained in a finite union of closed ε–balls. Then tightness of $(P^{(t)}\mu)_{t\geq 1}$ is equivalent to the following: for all $\varepsilon > 0$ there is a $C \in \mathcal{C}_\varepsilon$ such that $\liminf_{t\to+\infty} P^{(t)}\mu(C) \geq 1 - \varepsilon$ [15, Lemma 3.2].

We first show that we can replace the lim sup condition by a lim inf condition.
Step 1. There exists a compact set $\hat{K} \subset \Gamma_{cp}$ such that for every $\varepsilon > 0$ we may find $\beta > 0$ such that

$$\liminf_{t\to+\infty} P^{(t)}\delta_x(\hat{K}^\varepsilon) \geq \beta \qquad \text{for } x \in S. \tag{7}$$

Let a compact set K satisfying condition (6) be given. We define by $K_0 \subset K$ the set of all $x \in K$ such that for any $\varepsilon > 0$

$$\limsup_{t\to+\infty} P^{(t)}\mu(B(x,\varepsilon)) > 0 \tag{8}$$

for some $\mu \in \mathcal{P}(S)$. Observe that K_0 is closed, hence compact. Similar as in the proof of Proposition 3.9 it follows that K_0 satisfies (6) with K replaced with K_0. [19, Proposition 6.1] yields that $K_0 \subset \Gamma_t = \Gamma_{cp}$, so $P^{(t)}\delta_x$ converges for all $x \in K_0$.

By compactness the set of ergodic measures is tight. So there exists a compact set K_1 such that $\nu(K_1) \geq 1/2$ for every ergodic measure ν. Now for an arbitrary invariant probability measure μ, we obtain

$$\mu(K_1) = \int_{\Gamma_{cpie}} \varepsilon_x(K_1)\, d\mu(x) \geq \mu(\Gamma_{cpie})/2 = 1/2.$$

Since every invariant measure is concentrated on the closed set Γ_{cp}, we may assume that $K_1 \subset \Gamma_{cp}$. Now we define the compact set $\hat{K} := K_0 \cup K_1 \subset \Gamma_{cp}$.

Fix $0 < \varepsilon < 1/8$. By Lemma 5.1 there exists a $\delta > 0$ such that for all $y \in \hat{K}$ and $x \in B(y,\delta)$, $\|P^{(t)}\delta_x - P^{(t)}\delta_y\|_{\mathrm{BL}}^* < \varepsilon^2$.

Define $g := (1 - \varepsilon d(\cdot,\hat{K})) \vee 0$, then $|g|_{\mathrm{Lip}} \leq 1/\varepsilon$ and $\|g\|_\infty \leq 1$, so $g \in \mathrm{BL}(S)$ with $\|g\|_{\mathrm{BL}} \leq 1/\varepsilon + 1$. Moreover, $1/2\mathbb{1}_{\hat{K}^{\varepsilon/2}} \leq g \leq \mathbb{1}_{\hat{K}^\varepsilon}$. Fix $x \in \hat{K}^\delta$ and let $y \in \hat{K}$ be such that $d(x,y) < \delta$. Then

$$\begin{aligned} P^{(t)}\delta_x(\hat{K}^\varepsilon) &\geq \langle P^{(t)}\delta_x, g\rangle \geq \langle P^{(t)}\delta_y, g\rangle - \|P^{(t)}\delta_x - P^{(t)}\delta_y\|_{\mathrm{BL}}^*(1/\varepsilon + 1) \\ &\geq 1/2P^{(t)}\delta_y(\hat{K}^{\varepsilon/2}) - (\varepsilon + \varepsilon^2). \end{aligned}$$

Since $y \in \Gamma_{cp}$, $P^{(t)}\delta_y$ converges to the invariant probability measure ε_y, so we obtain

$$\begin{aligned} \liminf_{t\to+\infty} P^{(t)}\delta_x(\hat{K}^\varepsilon) &\geq 1/2 \liminf_{t\to+\infty} P^{(t)}\delta_y(\hat{K}^{\varepsilon/2}) - (\varepsilon + \varepsilon^2) \\ &\geq 1/2\varepsilon_y(\hat{K}^{\varepsilon/2}) - (\varepsilon + \varepsilon^2) \\ &\geq 1/4 - \varepsilon - \varepsilon^2 > 1/4 - 2\varepsilon > 0. \end{aligned}$$

Let $\alpha > 0$ be such that (6) is satisfied with ε replaced with δ. Fix $x \in S$. Then there is a $T > 0$ such that $P^{(T)}\delta_x(\hat{K}^\delta) \geq \alpha/2$. Define

$$\rho := \frac{P^{(T)}\delta_x(\hat{K}^\delta \cap \cdot)}{P^{(T)}\delta_x(\hat{K}^\delta)}.$$

Then $\rho \in P(S)$ and $P^{(T)}\delta_x \geq P^{(T)}\delta_x(\hat{K}^\delta)\rho \geq (\alpha/2)\rho$.

By Fatou's lemma

$$\liminf_{t\to+\infty} P^{(t)}\rho(\hat{K}^\varepsilon) \geq \int_S \liminf_{t\to+\infty} P^{(t)}\delta_y(\hat{K}^\varepsilon)\, d\rho(y) \geq 1/4 - 2\varepsilon.$$

Now,

$$\liminf_{t\to+\infty} P^{(t)}\delta_x(\hat{K}^\varepsilon) = \liminf_{t\to+\infty} P^{(t)}P^{(T)}\delta_x(\hat{K}^\varepsilon) \geq \alpha/2(1/4 - 2\varepsilon),$$

where the first equality follows from [11, Lemma 2]. Thus (7) is satisfied with $\beta = \alpha/2(1/4- 2\varepsilon)$.

Step 2. For every $\varepsilon > 0$ there exists an open set U with $\hat{K} \subset U$ such that for all $\mu \in P(S)$ with $\mu(U) = 1$ there is a $C \in \mathcal{C}_\varepsilon$ for which

$$\liminf_{t\to+\infty} P^{(t)}\mu(C) \geq 1 - \varepsilon.$$

Fix $\varepsilon > 0$ and let $x \in \hat{K}$. Since $(P^{(t)}\delta_x)_{t\geq 1}$ is tight, we may find $C_x \in \mathcal{C}_\varepsilon$ and $r_x > 0$ such that

$$\liminf_{t\to+\infty} P^{(t)}\delta_y(C_x) \geq 1 - \varepsilon \qquad \text{for all } y \in B(x, r_x).$$

Indeed, let $\tilde{C}_x \in \mathcal{C}_{\varepsilon/2}$ be such that

$$\liminf_{t\to+\infty} P^{(t)}\delta_x(\tilde{C}_x) \geq 1 - \varepsilon/2.$$

Choose an arbitrary function $f \in \mathrm{BL}(S)$ such that $\mathbb{1}_{\tilde{C}_x} \leq f \leq \mathbb{1}_{C_x}$, where $C_x = \tilde{C}_x^{\varepsilon/2}$. Obviously, $C_x \in \mathcal{C}_\varepsilon$. By the e–property we may find $r_x > 0$ such that $|P(t)f(x)-P(t)f(y)| < \varepsilon/2$ for $t \geq 0$ and $y \in B(x, r_x)$. Then

$$\begin{aligned}\liminf_{t\to+\infty} P^{(t)}\delta_y(C_x) &\geq \liminf_{t\to+\infty} U^{(t)}f(y) \geq \liminf_{t\to+\infty} U^{(t)}f(x) - \varepsilon/2 \\ &\geq \liminf_{t\to+\infty} U^{(t)}\mathbb{1}_{\tilde{C}_x}(x) - \varepsilon/2 \geq 1 - \varepsilon.\end{aligned} \tag{9}$$

Let $\{x_1, \dots x_N\} \subset \hat{K}$ be such that

$$\hat{K} \subset \bigcup_{i=1}^{N} B(x_i, r_{x_i}) := U.$$

Set $C = \bigcup_{i=1}^N C_{x_i}$ and observe that $C \in \mathcal{C}_\varepsilon$. For $\mu \in \mathcal{P}(S)$ with $\mu(U) = 1$ we have by Fatou's lemma

$$\liminf_{t\to+\infty} P^{(t)}\mu(C) \geq \int_S \liminf_{t\to+\infty} P^{(t)}\delta_x(C)\, d\mu(x) \geq 1 - \varepsilon. \tag{10}$$

Step 3. For every $\varepsilon > 0$ and every $\mu \in P(S)$ there is a $C \in \mathcal{C}_\varepsilon$ for which

$$\liminf_{t\to+\infty} P^{(t)}\mu(C) \geq 1 - \varepsilon.$$

Let $\varepsilon > 0$ and $\mu \in P(S)$. Let U and $C \in \mathcal{C}_\varepsilon$ be given by Step 2. Define

$$\gamma = \sup\{\alpha \geq 0 : \exists N, t_1, \ldots, t_N \geq 0,\ P^{(t_1)}P^{(t_2)}\ldots P^{(t_N)}\mu \geq \alpha\nu$$
$$\nu \in \mathcal{P}(S),\ \liminf_{t\to+\infty} P^{(t)}\nu(C) \geq 1-\varepsilon\}.$$

We prove that $\gamma = 1$. Assume, contrary to our claim, that $\gamma < 1$.

Let $\tilde{\varepsilon} > 0$ be such that $\hat{K}^{\tilde{\varepsilon}} \subset U$. Let $\beta \in (0,1)$ be such that condition (7) holds with ε replaced with $\tilde{\varepsilon}$. If $\gamma > 0$, choose $\alpha \in ((\gamma-\beta)(1-\beta)^{-1}, \gamma) \cap [0,1)$ and else choose $\alpha = 0$. Then there exist $N \in \mathbb{N}$, $t_1, \ldots, t_N \geq 0$, and $\nu \in \mathcal{P}(S)$ such that

$$P^{(t_1)}P^{(t_2)}\ldots P^{(t_N)}\mu \geq \alpha\nu$$

and

$$\liminf_{t\to+\infty} P^{(t)}\nu(C) \geq 1-\varepsilon.$$

Set

$$\hat{\mu} = (1-\alpha)^{-1}(P^{(t_1)}P^{(t_2)}\ldots P^{(t_N)}\mu - \alpha\nu)$$

and observe that $\hat{\mu} \in \mathcal{P}(S)$. Further, by (7) and Fatou's lemma,

$$\liminf_{t\to+\infty} P^{(t)}\hat{\mu}(U) \geq \beta,$$

so there is a $t > 0$ such that

$$P^{(t)}\hat{\mu}(U) \geq \beta/2.$$

Define

$$\nu_1 = \frac{P^{(t)}\hat{\mu}(\cdot \cap U)}{P^{(t)}\hat{\mu}(U)}.$$

Then

$$P^{(t)}\hat{\mu} = (1-\alpha)^{-1}(P^{(t)}P^{(t_1)}P^{(t_2)}\ldots P^{(t_N)}\mu - \alpha P^{(t)}\nu) \geq (\beta/2)\nu_1$$

and hence

$$P^{(t)}P^{(t_1)}P^{(t_2)}\ldots P^{(t_N)}\mu \geq \alpha P^{(t)}\nu + \beta/2(1-\alpha)\nu_1$$
$$= (\alpha + \beta/2(1-\alpha))[(\alpha+\beta/2(1-\alpha))^{-1}(\alpha P^{(t)}\nu + \beta/2(1-\alpha)\nu_1)].$$

Set $\nu_2 = (\alpha+\beta/2(1-\alpha))^{-1}(\alpha P^{(t)}\nu + \beta/2(1-\alpha)\nu_1)$. Observe that Step 2 implies that

$$\liminf_{t\to+\infty} P^{(t)}\nu_2(C) \geq (\alpha+\beta(1-\alpha))^{-1}[\alpha \liminf_{t\to+\infty} P^{(t)}\nu(C) + \beta(1-\alpha)\liminf_{t\to+\infty} P^{(t)}\nu_1(C)]$$
$$\geq 1-\varepsilon.$$

Observation that $\alpha + \beta/2(1-\alpha) > \gamma$ leads to contradiction. Hence $\gamma = 1$. Further, for any $\alpha < 1$ we find N and $t_1, \ldots, t_N > 0$ such that

$$\liminf_{t\to+\infty} P^{(t)}\mu(C) = \liminf_{t\to+\infty} P^{(t)}P^{(t_1)}P^{(t_2)}\ldots P^{(t_N)}\mu(C) \geq \alpha \liminf_{t\to+\infty} P^{(t)}\nu(C) \geq \alpha(1-\varepsilon),$$

where the first equality follows from [11, Lemma 2]. Hence

$$\liminf_{t\to+\infty} P^{(t)}\mu(C) \geq 1-\varepsilon,$$

which completes the proof of implication *(i)⇒(ii)*.

(ii)⇒(i): By compactness the set of ergodic measures is tight. So there exists a compact set K such that $\mu(K) \geq 1/2$ for all ergodic μ. Let ν be an invariant probability measure, then

$$\nu(K) = \int_{\Gamma_{cpie}} \varepsilon_x(K)\, d\nu(x) \geq \nu(\Gamma_{cpie})/2 = 1/2.$$

Let $x \in S$. By assumption $P^{(t)}\delta_x$ converges to an invariant probability measure ν, thus for all $\varepsilon > 0$,

$$\liminf_{t\to+\infty} P^{(t)}\delta_x(K^\varepsilon) \geq \nu(K^\varepsilon) \geq 1/2.$$

□

We call a Markov semigroup $(P(t))_{t\geq 0}$ *weak* mean ergodic* if there exists a $\mu_* \in P(S)$ such that

$$\lim_{t\to+\infty} P^{(t)}\mu = \mu_* \text{ for all } \mu \in P(S).$$

In [11, Theorem 2] there are given sufficient conditions for weak* mean ergodicity. We can use Theorem 5.2 to give a necessary and sufficient condition for a Markov semigroup to be weak* mean ergodic.

Corollary 5.3. *Let $(P(t))_{t\geq 0}$ be a Markov–Feller semigroup that satisfies the e–property. Then the following are equivalent:*

(i) $(P(t))_{t\geq 0}$ is weak mean ergodic.*

(ii) There exists a $z \in S$ such that for every $\varepsilon > 0$ we may find $\alpha > 0$ such that

$$\limsup_{t\to+\infty} P^{(t)}\delta_x(B(z,\varepsilon)) \geq \alpha \qquad \text{for } x \in S. \tag{11}$$

Proof. The statement follows from Theorem 5.2 and the observation that if *(ii)* is satisfied, then condition $(\mathcal{C})$ holds with K replaced with $\{z\}$, so Theorem 3.8 implies that there is exactly one ergodic measure. □

We define $(P(t))_{t\geq 0}$ to be *asymptotically stable* if there exists a probability measure μ_* such that $P(t)\mu \to \mu_*$ for all $\mu \in P(S)$. Then μ_* is a unique invariant probability measure.

Corollary 5.4. *Let $(P(t))_{t\geq 0}$ be a Markov–Feller semigroup with the e–property. Then $(P(t))_{t\geq 0}$ is asymptotically stable if and only if there exists a $z \in S$ such that for every $\varepsilon > 0$ we may find $\alpha > 0$ for which*

$$\liminf_{t\to+\infty} P(t)\delta_x(B(z,\varepsilon)) \geq \alpha \text{ for all } x \in S. \tag{12}$$

Proof. If $(P(t))_{t\geq 0}$ is asymptotically stable with invariant probability measure μ_*, then for all $z \in \mathrm{supp}(\mu_*)$ and all $x \in S$

$$\liminf_{t\to+\infty} P(t)\delta_x(B(z,\varepsilon)) \geq \mu_*(B(z,\varepsilon)) > 0.$$

Now suppose there is a $z \in S$ such that (12) is satisfied. Then condition $(\mathcal{C})$ holds with K replaced with $\{z\}$, so Theorem 3.8 implies that there is exactly one ergodic measure, hence one invariant probability measure μ_*. By [16, Theorem 2], $P(t)\mu \to \mu_*$ for all $\mu \in P(S)$ with $\mu(\Gamma_t) = 1$, and by Theorem 5.2 $\Gamma_t = \Gamma_{cp} = S$. This completes the proof. ◻

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