Invariant measures for stochastic evolution equations with the shift semigroup*

Onno van Gaans
Anna Rusinek

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Abstract

We study infinite dimensional stochastic evolution equations with Lévy noise and the shift semigroup on functional spaces with weight. We show that the solution is a càdlàg process. We present an explicit sufficient condition for the existence of a unique invariant measure. Further, we study growth estimates for the shift semigroup. The results are applied to Heath-Jarrow-Morton-Musiela models of interest rates.

1 Introduction

Heath, Jarrow and Morton [7] introduced a flexible model for the term structure of interest rates assuming that the forward rate process $f(t,\theta)_{t\in[0,\theta]}$ with maturity time θ is an Itô process with finite dimensional Wiener noise. Musiela [10] proposed to define forward rates in term of the remaining time to maturity $x=\theta-t$ and consider forward rate functions $f_t(x)=f(t,t+x)$. This approach leads to the following SPDE on a separable Hilbert H space of real functions defined on $[0,+\infty)$:

$$\begin{aligned} \mathrm{d}f_t &= (\mathbb{D}f_t + (F_Z\circ\sigma)(f_t))\,\mathrm{d}t + \sigma(f_t)\mathrm{d}Z(t), \\ f_0 &= \eta, \end{aligned} \tag{1.1}$$

where $\eta\in H$, the linear operator $\mathbb{D}f=f'$ generates a strongly continuous semigroup of shift operators, Z is a U-valued Lévy process, σ is a mapping from H into the space of linear bounded operators from U into H, denoted by $L(U,H)$, and F_Z is a mapping from $L(U,H)$ into H given by

$$F_Z(B)(\xi) = \frac{\partial}{\partial\xi}\ln\mathbb{E}\exp\left\{-\int_0^{\xi}\left(BZ(1)\right)(x)\mathrm{d}x\right\}, \quad B\in L(U,H), \xi\geq 0.$$

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As appropriate state space for equation (1.1) H has been chosen to be a functional space with weight function w, for instance a weighted L^2 space (see [11], [14] and [15]) or a weighted Sobolev space (see [3], [4], [6], [9], [16] and [17]). Vargiolu [17] proves the existence of an invariant measure for Heath-Jarrow-Morton-Musiela (HJMM) models with constant coefficients for an exponential decreasing weight function $w(x) = \mathrm{e}^{-\alpha x}$. Sufficient conditions for the existence of an invariant measure for the HJMM model are given in Goldys and Musiela [6] for an exponential increasing weight function and in Marinelli [9], Rusinek [15] or Tehranchi [16] for more general increasing weight functions. In this paper we consider general, not necessarily increasing, weight functions.
We study general equations with the shift semigroup on functional spaces with weight. We show that the solution is a càdlàg process. We also derive an explicit sufficient condition for the existence of a unique invariant measure. Similar results on invariant measures for general evolution equations can be found, amongs others, in Chojnowska-Michalik [1], Da Prato and Zabczyk [2], van Gaans [5] or Rusinek [13], [15]. We study estimates for the shift semigroup $S(t)_{t\geq 0}$. We describe the set of (α, K) for which

$$\|S(t)\|_{L(H,H)} \leq K\mathrm{e}^{\alpha t}, \qquad t \geq 0. \tag{1.2}$$

We prove that (1.2) holds for $\gamma = 2\alpha$ and $K^2 \geq C_\gamma$ if and only if

$$C_\gamma = \sup_{x\geq 0} \sup_{y\geq x} \frac{w(x)\mathrm{e}^{\gamma x}}{w(y)\mathrm{e}^{\gamma y}} < +\infty.$$

The sufficient condition for the existence of an invariant measure, where appear both γ and C_γ leads to optimizing the function $-\gamma C_\gamma^{-1}$. For logarithmically convex and logarithmically concave weight function we present an explicit solution to this problem.
In the cited papers [6], [9], [15], [16] sufficient conditions for the existence of an invariant measures for HJMM models were formulated for increasing weights w with

$$\inf_{x\geq 0} \frac{w'(x)}{w(x)} > 0.$$

Our general results make it possible to formulate conditions for weights that are not increasing, as well as for increasing weights w with

$$\inf_{x\geq 0} \frac{w'(x)}{w(x)} = 0.$$

Finally, also for increasing weight functions we present weaker condition ensuring the existence of an invariant measure.

2 Preliminaries

We will consider processes on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $Z(t)$ be a Lévy process (i.e. a process with independent and stationary increments) taking values in a separable Hilbert space $(U, \langle\cdot,\cdot\rangle_U)$. We consider the following stochastic equation on another separable Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$:

$$\begin{aligned} \mathrm{d}X &= (AX + F(X))\mathrm{d}t + G(X)\mathrm{d}Z(t), \\ X(0) &= \eta, \end{aligned} \tag{2.1}$$

where $\eta \in H$, the linear operator A - which in general may be unbounded - generates strongly continuous semigroup $S_A(t)_{t\geq 0}$ of linear operators on H, F is a mapping from H into H and G is a mapping from H into $L(U,H)$. We shall assume that $\overline{m} = \mathbb{E}Z(1) \in U$ exists and the covariance operator of Z, which is given by

$$Qu = \mathbb{E}\left[(Z(1) - \overline{m}) \langle Z(1) - \overline{m}, u\rangle_U\right],$$

exists and is bounded, i.e. $\|Q\|_{L(U)} < +\infty$, where we abbreviate $L(U,U)$ to $L(U)$. This is a less restrictive assumption than square integrability of Z since

$$\mathrm{Var}Z = \mathbb{E}\,\|Z(1) - \overline{m}\|_U^2 = \mathrm{Tr}Q.$$

If the dimension of U is finite then for every $Q \in L(U)$, it holds $\mathrm{Tr}Q < +\infty$. We shall also assume that there exists $L_F > 0$ and $L_G > 0$ such that for every $x, y \in H$

$$\|F(x) - F(y)\|_H \leq L_F \|x - y\|_H, \tag{2.2}$$
$$\|G(x) - G(y)\|_{\mathcal{L}^2(U,H)} \leq L_G \|x - y\|_H, \tag{2.3}$$

where the Hilbert-Schmidt norm of a linear operator $B : U \to H$ is defined as

$$\|B\|^2_{\mathcal{L}^2(U,H)} = \sum_{i=1}^{+\infty} \|Be_i\|_H^2 < +\infty,$$

where $\{e_i\}_i$ is an orthonormal basis in U. The solution to (2.1) with initial condition η shall be denoted by $\{X^\eta(t) : t \geq 0\}$. The following theorem can be found in Rusinek [15] (see Theorem 6.1).

Theorem 2.1. *Suppose $\overline{m} = 0$. Let L_F, L_G be given by (2.2), (2.3) respectively. If*

$$\|S_A(t)\|_{L(H)} \leq \mathrm{e}^{\alpha t},$$

and

$$2\alpha + 2L_F + {L_G}^2 \|Q\|_{L(U)} < 0,$$

then there exists a unique measure μ^* such that for every $\eta \in H$,

$$\mathcal{L}(X^\eta(t)) \to \mu^*.$$

3 SPDEs with the shift semigroup

For a positive continuous $w : \mathbb{R}_+ \to \mathbb{R}_+$, let L^2_w denote the Hilbert space of all functions $f : \mathbb{R}_+ \to \mathbb{R}$, such that

$$\int_0^{+\infty} |f(x)|^2 w(x)\mathrm{d}x < +\infty,$$

with inner product

$$\langle f,g\rangle_{L^2_w} = \int_0^{+\infty} f(x)g(x)w(x)\mathrm{d}x,$$

and let $W^{k,2}_w$ denote the Hilbert space of all functions $f:\mathbb{R}_+\to\mathbb{R}$ such that $f,\mathbb{D}^l f\in L^2_w$ for every $1\le l\le k$ with inner product

$$\langle f,g\rangle_{W^{k,2}_w} = \langle f,g\rangle_{L^2_w} + \sum_{l=1}^{k}\langle \mathbb{D}^l f,\mathbb{D}^l g\rangle_{L^2_w}.$$

We shall consider equation (2.1) on Sobolev spaces $W^{k,2}_w$ with $A=\mathbb{D}$ generating the semigroup of shift operators $S(t)_{t\ge 0}$, i.e.

$$(S(t)f)(x) = f(x+t).$$

Theorem 3.1. *For every $\eta\in W^{k,2}_w$, the solution $\{X^\eta_t : t\ge 0\}$ to (2.1) on $W^{k,2}_w$ with $A=\mathbb{D}$ is càdlàg.*

In the proof of Theorem 3.1 we shall use the following proposition.

Proposition 3.2. *The following conditions are equivalent:*

$$w(x)\mathrm{e}^{2\alpha x} \le K^2 w(y)\mathrm{e}^{2\alpha y}, \quad y\ge x\ge 0, \tag{3.1}$$

$$\|S(t)\|_{L(W^{k,2}_w)} \le K\mathrm{e}^{\alpha t}, \quad t\ge 0. \tag{3.2}$$

Proof of Proposition 3.2. To avoid technicalities we shall present the proof for $k=1$. Assume that (3.1) does not hold, so for some $b>t>0$, we have

$$w(b-t) - K^2\mathrm{e}^{2\alpha t}w(b) > 0.$$

Let $g(x) = \mathbf{1}_{[t,+\infty]}(x)\left(w(x-t) - K^2\mathrm{e}^{2\alpha t}w(x)\right)$. Since $g(b)>0$ and w is continuous, there exist $\varepsilon,\delta>0$ such that $g(x)>\varepsilon$ on $[b,b+2\delta]$. Let

$$h(x) = (x-b)\mathbf{1}_{[b,b+\delta]}(x) - (x-(b+2\delta))\mathbf{1}_{(b+\delta,b+2\delta]}(x).$$

Then $h'(x) = \mathbf{1}_{[b,b+\delta]}(x) - \mathbf{1}_{(b+\delta,b+2\delta]}(x)$. Since $b>t$, it follows that

$$\begin{aligned}\|S(t)h\|^2_{W^{k,2}_w} - K^2\mathrm{e}^{2\alpha t}\|h\|^2_{W^{k,2}_w} &= \int_t^{+\infty}\left(|h(x)|^2 + |h'(x)|^2\right)g(x)\mathrm{d}x \\ &\ge \int_t^{+\infty}|h'(x)|^2 g(x)\mathrm{d}x > 2\varepsilon\delta > 0.\end{aligned}$$

Hence (3.2) does not hold.
Now assume that (3.1) holds. We have

$$\begin{aligned}\|S(t)f\|^2_{W^{k,2}_w} &\le \int_0^{+\infty}\left(|f(x+t)|^2 + |f'(x+t)|^2\right)K^2 w(x+t)\mathrm{e}^{2\alpha t}\mathrm{d}x \\ &= K^2\mathrm{e}^{2\alpha t}\int_t^{+\infty}\left(|f(\xi)|^2 + |f'(\xi)|^2\right)w(\xi)\mathrm{d}\xi \le K^2\mathrm{e}^{2\alpha t}\|f\|^2_{W^{k,2}_w}.\end{aligned}$$

□

Proof of Theorem 3.1. Since S is strongly continuous it follows that for some $\alpha, K \in \mathbb{R}$ we have (see, for instance, [12])

$$\|S(t)\|_{L(W_w^{k,2})} \leq K\mathrm{e}^{\alpha t}.$$

Let

$$\widetilde{w}(x) = \mathrm{e}^{-2\alpha x} \sup_{\xi \in [0,x]} \left[w(\xi)\mathrm{e}^{2\alpha\xi} \right]. \tag{3.3}$$

Then clearly

$$\widetilde{w}(x)\mathrm{e}^{2\alpha x} \leq \widetilde{w}(y)\mathrm{e}^{2\alpha y}, \quad y \geq x \geq 0,$$

hence by Proposition 3.2, we get

$$\|S(t)\|_{L(W_{\widetilde{w}}^{k,2})} \leq \mathrm{e}^{\alpha t}. \tag{3.4}$$

The Kotelenez theorem [8] ensures that under assumption (3.4) solution to (2.1) on $W_{\widetilde{w}}^{k,2}$ is càdlàg. The proof is finished once we show that norms $\|\cdot\|_{W_{\widetilde{w}}^{k,2}}$, $\|\cdot\|_{W_w^{k,2}}$ are equivalent. Since $\|S(t)\|_{L(W_w^{k,2})} \leq K\mathrm{e}^{\alpha t}$, by Proposition 3.2 for every $x \geq 0$, we have

$$\sup_{\xi \in [0,x]} \left[w(\xi)\mathrm{e}^{2\alpha\xi} \right] \leq K^2 w(x)\mathrm{e}^{2\alpha x}.$$

And

$$\begin{aligned} w(x) &= \mathrm{e}^{-2\alpha x} w(x)\mathrm{e}^{2\alpha x} \\ &\leq \mathrm{e}^{-2\alpha x} \sup_{\xi \in [0,x]} \left[w(\xi)\mathrm{e}^{2\alpha\xi} \right], \end{aligned}$$

thus

$$w(x) \leq \widetilde{w}(x) \leq K^2 w(x), \quad x \geq 0.$$

□

For equations on $W_w^{k,2}$ with the shift semigroup analogous result to Theorem 2.1 holds under less restrictive condition for the semigroup.

Theorem 3.3. *Consider equation* (2.1) *on* $W_w^{k,2}$ *with* $A = \mathbb{D}$. *Suppose* $\overline{m} = 0$. *Let* L_F, L_G *be given by* (2.2), (2.3) *respectively. If*

$$\|S(t)\|_{L(W_w^{k,2})} \leq K\mathrm{e}^{\alpha t}, \tag{3.5}$$

and

$$2\alpha + 2KL_F + K^2 {L_G}^2 \|Q\|_{L(U)} < 0, \tag{3.6}$$

then there exists a unique measure μ^* *such that for every* $\eta \in W_w^{k,2}$,

$$\mathcal{L}(X^\eta(t)) \to \mu^*.$$

Proof of Theorem 3.3. Let $\widetilde{w} : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by (3.3). Easy computation shows that

$$\|F(x) - F(y)\|_{W^{k,2}_{\widetilde{w}}} \leq KL_F \|x - y\|_{W^{k,2}_{\widetilde{w}}},$$
$$\|G(x) - G(y)\|_{\mathcal{L}^2(U, W^{k,2}_{\widetilde{w}})} \leq KL_G \|x - y\|_{W^{k,2}_{\widetilde{w}}}.$$

Therefore condition (3.6) implies existence of a stationary distribution on space $W^{k,2}_{\widetilde{w}}$ from Theorem 2.1. □

Note that by Proposition 3.2 thesis of Theorem 3.3 holds if for some $\gamma \in \mathbb{R}$

$$2L_F + {L_G}^2 \|Q\|_{L(U)} < \frac{-\gamma}{C_\gamma},$$

where for every $\gamma \in \mathbb{R}$, we define C_γ as

$$C_\gamma = \sup_{x \geq 0} \sup_{y \geq x} \frac{w(x) \mathrm{e}^{\gamma x}}{w(y) \mathrm{e}^{\gamma y}}.$$

Let

$$\beta_w = \sup_{\gamma \in \mathbb{R}} \frac{-\gamma}{C_\gamma}. \tag{3.7}$$

The following result presents an explicit formula for β_w in two important cases when function $(\ln w)'$ is monotonic. The proof if left to Section 5.

Theorem 3.4. *If $\ln w$ is a convex function and $w(+\infty) = +\infty$, then*

$$\beta_w = \frac{w(x_w) \mathrm{e}^{-1}}{w(0) x_w},$$

where x_w is the solution to

$$\frac{w'(x_w)}{w(x_w)} = \frac{1}{x_w}.$$

If $\ln w$ is a concave function, then

$$\beta_w = \lim_{x \to +\infty} \frac{w'(x)}{w(x)}.$$

Remark 3.5. *If $w(x) = \mathrm{e}^{\alpha x}$, $\alpha > 0$, then both formulas correspond, since $x_w = \alpha^{-1}$.*

Example 3.6. *Let $w(x) = \mathrm{e}^{x^2}$. We have $2x_w = \frac{1}{x_w}$ for $x_w = \frac{1}{\sqrt{2}}$. Thus*

$$\beta_w = \frac{\mathrm{e}^{\frac{1}{2}} \sqrt{2}}{\mathrm{e}} = \sqrt{\frac{2}{\mathrm{e}}}.$$

Example 3.7. *Let $w(x) = x^x$. We have $1 + \ln x_w = \frac{1}{x_w}$ for $x_w = 1$. Thus*

$$\beta_w = \frac{1}{\mathrm{e}}.$$

4 Growth estimates for the shift semigroup

In this section we shall investigate set Γ_w defined as

$$\Gamma_w = \{\gamma \in \mathbb{R} : C_\gamma < +\infty\}.$$

Note that $\Gamma_w = \left\{\gamma \in \mathbb{R} : \exists C \geq 1 : \|S(t)\|^2_{L(L^2_w)} \leq C\mathrm{e}^{\gamma t}\right\}$, thus set Γ_w is not empty if and only if S is strongly continuous on L^2_w. Indeed, assume that there exists γ in Γ_w, fix $f \in L^2_w$ and $\varepsilon > 0$. There exists $r > 0$ and $g \in L^2_w$ of class C^∞ such that $g(x) = 0$ for every $x > r$ and $\|f - g\|_{L^2_w} < \varepsilon$. Since g is uniformly continuous, there exists $\delta \in [0,1]$ such that

$$\forall t \in [0,\delta] \qquad \sup_{x\geq 0} |g(x+t) - g(x)| < \varepsilon.$$

It follows that for $t \in [0,\delta]$ we have

$$\begin{aligned}\|S(t)f - f\|_{L^2_w} &\leq \|S(t)f - S(t)g\|_{L^2_w} + \|g - f\|_{L^2_w} + \|S(t)g - g\|_{L^2_w} \\ &\leq (\sqrt{C_\gamma}\mathrm{e}^{\frac{1}{2}|\gamma|} + 1)\varepsilon + \varepsilon \left(\int_0^r w(x)\mathrm{d}x\right)^{1/2}.\end{aligned}$$

The same way one can prove that Γ_w is not empty if and only if S is strongly continuous on $W^{k,2}_w$.
We are interested in finding estimates and formulas for

$$\gamma^* := \inf \Gamma_w = \{\gamma \in \mathbb{R} : C_\gamma < +\infty\}.$$

Proposition 4.1. *Let* $\tilde{\delta}_w = \liminf_{x\to+\infty} \frac{w'(x)}{w(x)}$, $\tilde{d}_w = \liminf_{x\to+\infty} \frac{\ln w(x)}{x}$. *We have*

$$-\tilde{d}_w \leq \gamma^* \leq -\tilde{\delta}_w.$$

Lemma 4.2. $\gamma \in \Gamma_w$ *if and only if there exist* $R, D \geq 0$ *such that*

$$\ln w(x) - \ln w(y) + \gamma(x-y) \leq D, \quad \forall y \geq x \geq R. \tag{4.1}$$

Proof of Lemma 4.2. Clearly if $C_\gamma < +\infty$, then (4.1) holds with $R = 0$, $D = \ln C_\gamma$. Now assume that (4.1) holds, which means that for $C = \mathrm{e}^D \geq 1$

$$\sup_{y\in[R,+\infty)} \sup_{x\in[R,y]} \frac{w(x)\mathrm{e}^{\gamma x}}{w(y)\mathrm{e}^{\gamma y}} \leq C.$$

Since

$$\widetilde{C} = \sup_{y\in[0,R]} \sup_{x\in[0,y]} \frac{w(x)\mathrm{e}^{\gamma x}}{w(y)\mathrm{e}^{\gamma y}} < +\infty,$$

it is enough to observe that if $0 \leq x \leq R \leq y$, then

$$\frac{w(x)\mathrm{e}^{\gamma x}}{w(y)\mathrm{e}^{\gamma y}} = \frac{w(x)\mathrm{e}^{\gamma x}}{w(R)\mathrm{e}^{\gamma R}} \cdot \frac{w(R)\mathrm{e}^{\gamma R}}{w(y)\mathrm{e}^{\gamma y}} \leq \widetilde{C}C.$$

□

Proof of Proposition 4.1. For every $\gamma > -\tilde{\delta}_w$ there exists $R > 0$ such that $\forall \xi \geq R$, we have $\phi'(\xi) + \gamma > 0$, where $\phi(\xi) = \ln w(\xi)$. Thus for every $y \geq x \geq R$

$$\phi(x) - \phi(y) + \gamma(x-y) = -\int_x^y (\phi'(\xi) + \gamma)\mathrm{d}\xi < 0.$$

Now assume that $\gamma \in \Gamma_w$. Then

$$w(0)C_\gamma^{-1}\mathrm{e}^{-\gamma y} \leq w(y), \quad y \geq 0.$$

Therefore

$$\frac{\phi(0) - \ln C_\gamma}{y} - \gamma \leq \frac{\phi(y)}{y},$$

so

$$-\gamma \leq \tilde{d}_w.$$

□

Example 4.3. *Let* $w(x) = \exp\{\sin(x^2)\}$, $\xi_n = \sqrt{2\pi n + \pi}$. *We have* $(\ln w)'(\xi_n) = -2\xi_n$, *so* $\tilde{\delta}_w = -\infty$. *The set* Γ_w *is not empty:* (4.1) *holds for every positive* γ *and* $D = 2$, *since* $\ln w(x) \in [-1, 1]$.

Example 4.4. *Let* $w(x) = \exp\{x \cos x\}$, $y_n = 2\pi n + \frac{\pi}{2}$, $x_n = 2\pi n$. *Clearly* $\tilde{d}_w = -1$. *If* $\Gamma_w \neq \emptyset$, *then for some* γ, *we have*

$$\ln w(x_n) - \ln w(y_n) \leq \gamma\frac{\pi}{2} + \ln C_\gamma.$$

But $\ln w(x_n) - \ln w(y_n) = 2\pi n$, *so it must be that* $\Gamma_w = \emptyset$.

Proposition 4.1 and Examples 4.3, 4.4 show that $\tilde{\delta}_w > -\infty$ is a sufficient but not a necessary condition for S to be strongly continuous and $\tilde{d}_w > -\infty$ is necessary but not sufficient.
Since the existence of $\lim\limits_{x\to+\infty} \phi'(x)$ implies existence of $\lim\limits_{x\to+\infty} \frac{\phi(x)}{x} = \lim\limits_{x\to+\infty} \phi'(x)$ as a consequence of Lemma 4.1 we get the following result.

Lemma 4.5. *If the limit* $\lim\limits_{x\to+\infty} \frac{w'(x)}{w(x)} = \delta_w$ *exists, then* $\gamma^* = \delta_w$.

5 Explicit formulas for C_γ

Let

$$\alpha_w = \inf_{x\geq 0} \frac{w'(x)}{w(x)}.$$

The following estimate can be found in Tehranchi [16]:

$$\|S(t)\|_{L(W_w^{k,2})} \leq \mathrm{e}^{-\frac{1}{2}\alpha_w t}.$$

The next result shows that $-\alpha_w$ is also the smallest γ for which

$$\|S(t)\|_{L(W_w^{k,2})} \leq \mathrm{e}^{\frac{1}{2}\gamma t}.$$

Lemma 5.1. $\gamma \in [-\alpha_w, +\infty)$ *if and only if* $C_\gamma = 1$.

Proof of Lemma 5.1. For every $\gamma \in \mathbb{R}$, the function $\widehat{w} : \mathbb{R}_+ \to \mathbb{R}$, given by $\widehat{w}(x) = w(x)\mathrm{e}^{\gamma x}$ is nondecreasing if and only if

$$\gamma \geq -\frac{w'(x)}{w(x)}, \quad \forall x \geq 0,$$

since

$$\widehat{w}'(x) = w(x)\mathrm{e}^{\gamma x}\left(\frac{w'(x)}{w(x)} + \gamma\right).$$

□

If $\ln w$ is concave, then $\alpha_w = \delta_w$, $\Gamma_w = [-\delta_w, +\infty]$ and for every $\gamma \in \Gamma_w$ we have $C_\gamma = 1$. If $\ln w$ is convex, then $\alpha_w = (\ln w)'(0)$,

$$\sup_{x\geq 0}(\ln w)'(x) = (\ln w)'(+\infty) = \delta_w.$$

Therefore $(\ln w)' : [0, +\infty) \to [\alpha_w, \delta_w)$, thus the inverse of $(\ln w)'$ acts from $[\alpha_w, \delta_w)$ into $[0, +\infty)$. The next result gives an explicit formula for C_γ in case of a logarithmically convex weight function.

Lemma 5.2. *If* $\ln w$ *is a convex function, then for every* $\gamma \in (-\delta_w, -\alpha_w)$, *we have*

$$C_\gamma = \frac{w(0)}{w(\psi(-\gamma))}\mathrm{e}^{-\gamma\psi(-\gamma)},$$

where ψ *denotes the inverse of* $(\ln w)'$.

Example 5.3. *Let* $w(x) = \mathrm{e}^{\delta x} + \mathrm{e}^{-\delta x}$, *where* $\delta > 0$. *Since*

$$\frac{w'(x)}{w(x)} = \frac{\delta \mathrm{e}^{\delta x} - \delta \mathrm{e}^{-\delta x}}{\mathrm{e}^{\delta x} + \mathrm{e}^{-\delta x}} = \frac{\delta - \delta \mathrm{e}^{-2\delta x}}{1 + \mathrm{e}^{-2\delta x}},$$

we have $\gamma^* = -\delta_w = -\delta$. *And*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w'(x)}{w(x)}\right) = \frac{4\delta^2}{(w(x))^2} \geq 0,$$

so $\alpha_w = \frac{w'(0)}{w(0)} = 0$ *and* $\ln w$ *is a convex function. For every* $\gamma \geq 0$, *we have* $C_\gamma = 1$. *To find* C_γ *for* $\gamma \in [-\delta, 0)$, *we can apply the above lemma. We have* $\psi(-\gamma) = \ln\left(\frac{\delta-\gamma}{\delta+\gamma}\right)^{\frac{1}{2\delta}}$. *Thus*

$$\begin{aligned} w(\psi(-\gamma))\mathrm{e}^{\gamma\psi(-\gamma)} &= (\mathrm{e}^{\psi(-\gamma)})^{(\delta+\gamma)} + (\mathrm{e}^{\psi(-\gamma)})^{(-\delta+\gamma)} \\ &= \left(\frac{\delta-\gamma}{\delta+\gamma}\right)^{\frac{\delta+\gamma}{2\delta}} + \left(\frac{\delta-\gamma}{\delta+\gamma}\right)^{\frac{-\delta+\gamma}{2\delta}}. \end{aligned}$$

With the notation $q = \frac{\delta-\gamma}{2\delta}$*, we obtain*

$$
\begin{aligned}
w(\psi(-\gamma))\mathrm{e}^{\gamma\psi(-\gamma)} &= \left(\frac{q}{1-q}\right)^{1-q} + \left(\frac{q}{1-q}\right)^{-q} \\
&= \left(\frac{q}{1-q}\right)^{-q}\left(\frac{q}{1-q}+1\right) \\
&= \left(\frac{q}{1-q}\right)^{-q}\frac{1}{1-q} = \left(q^q(1-q)^{1-q}\right)^{-1}.
\end{aligned}
$$

Hence

$$C_\gamma = 2\left(\frac{\delta-\gamma}{2\delta}\right)^{\frac{\delta-\gamma}{2\delta}}\left(\frac{\delta+\gamma}{2\delta}\right)^{\frac{\delta+\gamma}{2\delta}}.$$

Proof of Lemma 5.2. Let $G(x,t) = \phi(x) - \phi(x+t) - \gamma t$, where $\phi = \ln w$. Then

$$\ln C_\gamma = \sup_{x,t\geq 0} G(x,t).$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}x}G(x,t) = \phi'(x) - \phi'(x+t) \leq 0,$$

we get

$$\ln C_\gamma = \sup_{t\geq 0}\left[\phi(0) - \phi(t) - \gamma t\right].$$

To find the supremum of g_γ given by

$$g_\gamma(t) = \phi(0) - \phi(t) - \gamma t,$$

note that g'_γ is concave and $g'_\gamma(\psi(-\gamma)) = 0$, thus

$$\ln C_\gamma = \phi(0) - \phi(\psi(-\gamma)) - \gamma\psi(-\gamma).$$

□

Proof of Theorem 3.4. Assume that $\phi = \ln w$ is convex and let

$$f(\gamma) = \ln(-\gamma) - \ln C_\gamma.$$

By Lemma 5.2 we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\gamma}\ln C_\gamma &= -(\phi'(\psi(-\gamma)) + \gamma)\frac{\mathrm{d}}{\mathrm{d}\gamma}\psi(-\gamma) - \psi(-\gamma) \\
&= -\psi(-\gamma),
\end{aligned}
$$

thus $f'(\gamma) = \frac{1}{\gamma} + \psi(-\gamma)$. If $\delta_w \leq 0$, then $\ln w$ and in turn w is bounded from above. Therefore $w(+\infty) = +\infty$ implies $\delta_w > 0$, so x_w is well-defined. Since for $-\gamma \in [\alpha_w, x_w^{-1}]$, we have $f'(\gamma) < 0$, for $-\gamma \in [x_w^{-1}, \delta_w)$, we have $f'(\gamma) > 0$ and $f'(-x_w^{-1}) = 0$, it follows that

$$\sup_{\gamma\in[-\delta_w,-\alpha_w]} f(\gamma) = f(-x_w^{-1}).$$

Hence

$$\sup_{\gamma\in[-\delta_w,-\alpha_w]} \exp f(\gamma) = \exp f(-x_w^{-1}) = \frac{x_w^{-1}}{C_{-x_w^{-1}}}.$$

To finish the proof of the first part it is enough to observe that $\psi(x_w^{-1}) = x_w$, hence by Lemma 5.2, $C_{-x_w^{-1}} = \frac{w(0)}{w(x_w)}\mathrm{e}^1$. To prove the second part it is enough to observe that if $\ln w$ is concave, then $\delta_w = \alpha_w$, hence $\Gamma_w = [-\alpha_w, +\infty)$ and for every $\gamma \geq -\alpha_w$ we have $C_\gamma = 1$. □

As a corollary to the above proof we get the following result.

Corollary 5.4. *Let*

$$\gamma_w = \inf\left\{\gamma : \frac{-\gamma}{C_\gamma} = \beta_w\right\}. \tag{5.1}$$

If $\ln w$ *is convex and* $w(+\infty) = +\infty$*, then* $-\gamma_w = x_w^{-1}$ *and if* $\ln w$ *is concave, then*

$$-\gamma_w = \alpha_w = \delta_w = \beta_w.$$

Remark 5.5. *We claim that*

$$\frac{-\gamma_w}{C_{\gamma_w}} = \beta_w.$$

Indeed, first note that for $f(x) = \frac{-x}{C_x}$*, we have* $f(x) \leq 0$*,* $x > 0$*, and* $f(0) = 0$*, so*

$$\beta_w = \sup_{\gamma\leq 0} f(\gamma) \geq 0.$$

Moreover $f(-\infty) = 0$*, since*

$$\lim_{x\to-\infty} \frac{-x}{C_x} \leq \lim_{x\to-\infty} \frac{w(1)}{w(0)}\frac{-x}{\mathrm{e}^{-x}} = 0.$$

It follows that $\beta_w = 0$ *if and only if* $f(x) = 0$ *for every* $x < 0$ *and in turn if and only if* $\gamma_w = -\infty$*. If* $\beta_w > 0$*, then* $0 > \gamma_w > -\infty$*. Note that function* $g(\gamma) = C_\gamma$ *is lower semi-continuous as a supremum of continuous functions, hence there exists a sequence* $\gamma^{(n)} \to \gamma_w$ *such that*

$$\frac{-\gamma^{(n)}}{C_{\gamma^{(n)}}} = \beta_w,$$

and

$$C_{\gamma_w} \leq \lim_{n\to+\infty} C_{\gamma^{(n)}}.$$

We conclude that

$$\frac{-\gamma_w}{C_{\gamma_w}} \geq \lim_{n\to+\infty} \frac{-\gamma^{(n)}}{C_{\gamma^{(n)}}} = \beta_w.$$

We end this section with a technical result being a generalization of Lemma 5.2. In general, the function $(\ln w)'$ might be not invertible, so instead of $\psi(-\gamma)$ from Lemma 5.2, we shall consider the inverse images $f^{-1}[Y]$ for $f = (\ln w)'$ and $Y = \{-\gamma\}$.

Lemma 5.6. *For every $\gamma > -\tilde{\delta}_w$, we have*

$$C_\gamma = \sup_{z\in\mathcal{O}_\gamma} \sup_{\xi\in[z,+\infty)\cap\mathcal{O}_\gamma} \frac{w(z)}{w(\xi)} \mathrm{e}^{\gamma(z-\xi)}, \tag{5.2}$$

where

$$\mathcal{O}_\gamma = \{0\} \cup \{x \geq 0 : (\ln w)'(x) = -\gamma\}.$$

Proof of Lemma 5.6. Fix $\gamma \in \Gamma_w$ and let $G(x,t) = \phi(x) - \phi(x+t) - \gamma t$. Then clearly

$$\begin{aligned} \ln C_\gamma &= \sup_{x,t\geq 0} G(x,t) \\ &= \max\left\{ \sup_{t\geq 0} G(0,t), \sup_{x\geq 0} G(x,0), \limsup_{(x,t)\to(+\infty,+\infty)} G(x,t), P \right\} \end{aligned}$$

where

$$P = \sup\left\{ G(x,t) : \frac{\mathrm{d}}{\mathrm{d}x} G(x,t) = \frac{\mathrm{d}}{\mathrm{d}t} G(x,t) = 0 \right\}$$

We have

$$\sup_{t\geq 0} G(0,t) = \max\left\{ G(0,0), \limsup_{t\to+\infty} G(0,t), \sup\{G(0,t) : \phi'(t) = -\gamma\} \right\}.$$

Since $G(0,t) = \phi(0) - t\left(\frac{\phi(t)}{t} + \gamma\right)$ and $\liminf_{t\to+\infty} \frac{\phi(t)}{t} + \gamma = \tilde{d}_w + \gamma > 0$, we conclude that $G(0,+\infty) = -\infty$. Moreover $G(x,0) = G(0,0) = 0$ for every $x \geq 0$, thus

$$\sup_{x,t\geq 0} G(x,t) = \max\left\{ \sup\{G(0,t) : t \in \mathcal{O}_\gamma\}, \limsup_{(x,t)\to(+\infty,+\infty)} G(x,t), P \right\}.$$

Since an easy computation shows that $\frac{\mathrm{d}}{\mathrm{d}x} G(x,t) = \frac{\mathrm{d}}{\mathrm{d}t} G(x,t) = 0$ if and only if

$$\phi'(x) = \phi'(x+t) = -\gamma,$$

the proof is finished once we show that if $x \to +\infty$, $t \to +\infty$, then

$$\phi(x) - \phi(x+t) - \gamma t \to -\infty.$$

Since $\gamma > -\tilde{\delta}_w$, there exists $R, \varepsilon > 0$ such that $\phi'(\xi) + \gamma \geq \varepsilon$ for every $\xi \geq R$. It follows that

$$\begin{aligned} \phi(x) - \phi(x+t) - \gamma t &= -\int_x^{x+t} (\phi'(\xi) + \gamma)\, \mathrm{d}\xi \\ &\leq -\varepsilon t. \end{aligned}$$

□

Lemma 5.7. *If $\phi = \ln w$ is bounded, then formula* (5.2) *holds for every $\gamma > \gamma^* = 0$.*

Remark 5.8. *If* $\widehat{w}(x) = w(x)e^{\alpha x}$*, then* $\widehat{\gamma}^* = \gamma^* - \alpha$ *and for* $\gamma > \widehat{\gamma}^*$*, we have* $\widehat{C}_\gamma = C_{\gamma+\alpha}$*.*

Proof of Lemma 5.7. It is clear that $\tilde{d}_w = 0$. Thus $\gamma^* \geq 0$ from Proposition 4.1. To show that $0 \in \Gamma_w$, note that

$$\phi(x) - \phi(y) \leq 2 \sup_{\xi \geq 0} |\phi(\xi)| < +\infty.$$

The proof of the formula follows the proof of Lemma 5.6 except for the last step, where we now prove that $\lim_{x,t\to+\infty} [\phi(x) - \phi(x+t) - \gamma t] = -\infty$ observing that

$$\phi(x) - \phi(x+t) - \gamma t \leq 2 \sup_{\xi \geq 0} |\phi(\xi)| - \gamma t.$$

□

Example 5.9. *For* $w(x) = e^{\sin x}$*, we have* $\alpha_w = -1$*. We claim that for every* $\gamma \in [0, 1)$*, we have*

$$C_\gamma = \exp\left\{2\left(\sqrt{1-\gamma^2} - \gamma \arccos \gamma\right)\right\}.$$

Indeed, for every $\gamma \in [0, 1)$ *let* $z_\gamma = \arccos(-\gamma) \in \left[\frac{\pi}{2}, \pi\right)$*. Then*

$$\mathcal{O}_\gamma = \{0\} \cup \{x_k = z_\gamma + 2k\pi, k \in \mathbb{N}\} \cup \{y_k = 2\pi - z_\gamma + 2k\pi, k \in \mathbb{N}\}.$$

Let $F(x, y) = \sin x - \sin y - \gamma(y - x)$*,* $x \leq y$*. Note that* $\sin x_k = \sin z_\gamma$*,* $\sin y_k = -\sin z_\gamma$ *and* $\sin z_\gamma > 0$*,* $\gamma > 0$*, hence for* $j \geq k$*, we get*

$$\begin{aligned}
F(0, x_k) &= -\sin z_\gamma - \gamma z_\gamma < 0, \\
F(0, y_k) &= F(0, x_k) + F(x_k, y_k) < F(x_k, y_k), \\
F(y_k, y_j) &= -\gamma(y_j - y_k) \leq 0, \\
F(x_k, x_j) &= -\gamma(x_j - x_k) \leq 0, \\
F(y_k, x_{j+1}) &= -2\sin z_\gamma - \gamma(x_{j+1} - y_k) < 0, \\
F(x_k, y_j) &= F(x_k, y_k) + F(y_k, y_j) \leq F(x_k, y_k)
\end{aligned}$$

and

$$\begin{aligned}
F(x_k, y_k) &= 2\sin z_\gamma - \gamma(2\pi - 2z_\gamma) \\
&= 2f(z_\gamma),
\end{aligned}$$

where $f(z) = \sin z + \cos z(\pi - z)$*. Since* $f(\pi) = 0$ *and* $f'(z) = -(\pi - z)\sin z < 0$ *for* $z \in \left[\frac{\pi}{2}, \pi\right)$*, it follows that for* $z \in \left[\frac{\pi}{2}, \pi\right)$ *we have* $f(z) > 0$*.*

6 HJMM models

In this section as an illustration of the results from the previous sections we consider equation (1.1) on H_w^0 as defined in [16] as the space of all functions $f : \mathbb{R}_+ \to \mathbb{R}$ for which $f' \in L_w^2$ and $f(+\infty) = 0$, with inner product

$$\langle f, g\rangle_{H_w^0} = \langle f', g'\rangle_{L_w^2}.$$

For every $f \in H_w^0$, $(S(t)f)' = S(t)(f')$, hence Theorems 3.1, 3.3 hold for space H_w^0 as well.

We shall assume the driving process Z in (1.1) is the Wiener process W with covariance operator $Q = Id$, i.e. $Qu = u$. Tehranchi [16] presents the following condition ensuring that the solution to (1.1) on H_w^0 has a law-limit:

$$8 \left\| w^{-\frac{1}{3}} \right\|_{L^1}^{\frac{3}{2}} ML + L^2 < \alpha_w, \tag{6.1}$$

provided that

$$\|\sigma(f) - \sigma(g)\|_{\mathcal{L}^2(U,H_w^0)} \leq L \|f - g\|_{H_w^0}, \tag{6.2}$$

$$\|\sigma(f)\|_{\mathcal{L}^2(U,H_w^0)} \leq M. \tag{6.3}$$

We shall illustrate with three examples how the theory presented above leads to an improvement of this result. Proceeding as in the proof of Theorem 3.3 we obtain the following condition ensuring a law-limit:

$$8 \left\| w^{-\frac{1}{3}} \right\|_{L^1}^{\frac{3}{2}} ML + L^2 < \beta_w, \tag{6.4}$$

where β_w was defined by (3.7). Since $C_{-\alpha_w} = 1$, it is clear that

$$\alpha_w = \frac{\alpha_w}{C_{-\alpha_w}} \leq \sup_{\gamma \in \mathbb{R}} \frac{-\gamma}{C_\gamma} = \beta_w.$$

Our condition is strictly weaker whenever $\alpha_w < \beta_w$.

Example 6.1. *Let $w(x) = \mathrm{e}^{\alpha x} + \mathrm{e}^{\delta x}$, where $\delta > -\alpha > 0$. It is easy to check that $\alpha_w = \frac{1}{2}(\alpha + \delta) > 0$, $\delta_w = \delta$ and $C_{-\delta_w} = 2$. We have*

$$0 < \alpha_w < \frac{\delta_w}{C_{-\delta_w}} \leq \beta_w.$$

In the next example the weight w is increasing, but $\alpha_w = 0$, hence condition (6.1) does not hold no matter how small the noise is.

Example 6.2. *Let $w(x) = \mathrm{e}^{x^2}$. Condition* (6.4) *becomes*

$$8 \left(\frac{3\pi}{2} \right)^{\frac{3}{4}} ML + L^2 < \sqrt{\frac{2}{\mathrm{e}}}.$$

Indeed, $\beta_w = \sqrt{\frac{2}{\mathrm{e}}}$ (see Example 3.6*) and*

$$\left\| w^{-\frac{1}{3}} \right\|_{L^1} = \sqrt{6\pi} \int_0^{+\infty} \frac{1}{\sqrt{6\pi}} \mathrm{e}^{-\frac{x^2}{3}} \mathrm{d}x = \sqrt{\frac{3\pi}{2}}.$$

Finally in case w is not increasing, neither condition (6.1) nor (6.4) ensure the existence of a law-limit. In the next theorem we present a condition ensuring the law-limit for general weights.

Theorem 6.3. *Let $L, M \geq 0$ be given by* (6.2) *and* (6.3)*. If*

$$\frac{4\sqrt{2}+4}{\sqrt{w(0)(-\gamma_w)^3}}ML + L^2 < \beta_w, \tag{6.5}$$

then there exists a unique measure μ^* such that for every $\eta \in L^2_w$ we have $\lim_{t\to+\infty} \mathcal{L}(f^\eta_t) = \mu^*$.

Example 6.4. *Let $w(x) = x^x$. Then $\gamma_w = -1$, $\beta_w = \mathrm{e}^{-1}$, hence* (6.5) *becomes*

$$4\left(\sqrt{2}+1\right)ML + L^2 < \mathrm{e}^{-1}.$$

Example 6.5. *Let $w(x) = \mathrm{e}^{\alpha x}$, $\alpha > 0$. Then $-\gamma_w = \beta_w = \alpha$, hence* (6.5) *becomes*

$$4\left(\sqrt{2}+1\right)\alpha^{-\frac{3}{2}}ML + L^2 < \alpha.$$

Note that for such a weight the above condition is weaker than the one derived from (6.1)*:*

$$8 \cdot 3^{\frac{3}{2}}\alpha^{-\frac{3}{2}}ML + L^2 < \alpha.$$

6.1 Proof of Theorem 6.3

We start with introducing constants $\bar{K}_w$ and $\hat{K}_w$ connected to the weight function w. Let

$$\bar{K}_w = \left[\int_0^{+\infty} \frac{x^2}{w(x)}\mathrm{d}x\right]^{\frac{1}{2}},$$

$$\hat{K}_w = \left[\int_0^{+\infty} \left(\int_\xi^{+\infty} \frac{1}{w(x)}\mathrm{d}x\right)^2 w(\xi)\mathrm{d}\xi\right]^{\frac{1}{2}}.$$

The following inequality can be found in Filipovic [3] (see the proof of inequality (3.8) in [3]):

$$\left\|f^4 w\right\|_{L^1} \leq \hat{K}^2_w \left\|f\right\|^4_{H^0_w}. \tag{6.6}$$

Note that if $f \in H^0_w$, then $f(x) = \int_x^{+\infty} -f'(\xi)\mathrm{d}\xi$, so

$$\begin{aligned}\|f\|_{L^1} &\leq \int_0^{+\infty}\int_x^{+\infty} |f'(\xi)|\,\mathrm{d}\xi\mathrm{d}x = \int_0^{+\infty}\int_0^{\xi} |f'(\xi)|\,\mathrm{d}x\mathrm{d}\xi \\ &= \int_0^{+\infty} \xi\,|f'(\xi)|\,\mathrm{d}\xi.\end{aligned}$$

Hence from the Hölder inequality, we obtain

$$\|f\|_{L^1} \leq \bar{K}_w \|f\|_{H^0_w}. \tag{6.7}$$

In the proof of Theorem 6.3 we shall also need the following lemma.

Lemma 6.6. *If $\gamma \in \Gamma_w \cap \mathbb{R}_-$, then*

$$\bar{K}_w \leq \frac{\sqrt{2C_\gamma}}{\sqrt{w(0)(-\gamma)^3}},$$
$$\hat{K}_w \leq \frac{C_\gamma}{\sqrt{w(0)(-\gamma)^3}}.$$

Proof of Lemma 6.6. We have $(w(x))^{-1} \leq \mathrm{e}^{\gamma x} C_\gamma (w(0))^{-1}$, so

$$(\bar{K}_w)^2 \leq \frac{C_\gamma}{(-\gamma)w(0)} \int_0^{+\infty} x^2(-\gamma)\mathrm{e}^{\gamma x}\mathrm{d}x = \frac{C_\gamma}{w(0)} \frac{2}{(-\gamma)^3}.$$

Since

$$\frac{1}{\sqrt{w(x)}} \leq \sqrt{\frac{C_\gamma}{w(0)}}\mathrm{e}^{\frac{\gamma}{2}x},$$
$$\sqrt{\frac{w(\xi)}{w(x)}} \leq \sqrt{C_\gamma}\mathrm{e}^{\frac{\gamma}{2}(x-\xi)},$$

we get

$$\left(\int_\xi^{+\infty} \frac{\sqrt{w(\xi)}}{w(x)}\mathrm{d}x \right)^2 \leq \frac{C_\gamma^2}{w(0)} \mathrm{e}^{-\gamma\xi} \frac{\mathrm{e}^{2\gamma\xi}}{(-\gamma)^2}.$$

Thus

$$\int_0^{+\infty} \left(\int_\xi^{+\infty} \frac{\sqrt{w(\xi)}}{w(x)}\mathrm{d}x \right)^2 \mathrm{d}\xi \leq \frac{C_\gamma^2}{w(0)(-\gamma)^3}.$$

□

The next result follows from (6.7), (6.6) and the proof of Corollary 3.7 in [3] (see also the proof of Lemma 4.1 in [15])

Proposition 6.7. *If $\|A\|_{\mathcal{L}^2(U,H_w^0)}, \|B\|_{\mathcal{L}^2(U,H_w^0)} \leq M$, then*

$$\|F_W(A) - F_W(B)\|_{H_w^0} \leq L_{F_W} \|A - B\|_{\mathcal{L}^2(U,H_w^0)},$$

with

$$L_{F_W} = 2\left(\bar{K}_w + \hat{K}_w\right) M.$$

Proof of Theorem 6.3. Let $\widetilde{w}$ be given by

$$\widetilde{w}(x) = \mathrm{e}^{-\gamma_w x} \sup_{\xi \in [0,x]} [\mathrm{e}^{\gamma_w \xi} w(\xi)].$$

Then

$$\begin{aligned}\|S(t)\|_{L(L^2_{\tilde{w}})} &\le \mathrm{e}^{\gamma_w t},\\ \|\sigma(f)-\sigma(g)\|_{\mathcal{L}^2(U,L^2_{\tilde{w}})} &\le \tilde{L}\,\|f-g\|_{L^2_{\tilde{w}}},\\ \|\sigma(f)\|_{\mathcal{L}^2(U,L^2_{\tilde{w}})} &\le \tilde{M}.\end{aligned}$$

with $\tilde{L} \le \sqrt{C_{\gamma_w}}L$, $\tilde{M} \le \sqrt{C_{\gamma_w}}M$. By Proposition 6.7

$$\|(F_W\circ\sigma)(f)-(F_W\circ\sigma)(g)\|_{L^2_{\tilde{w}}} \le 2(\bar{K}_{\tilde{w}}+\hat{K}_{\tilde{w}})\tilde{M}\tilde{L}\,\|f-g\|_{L^2_{\tilde{w}}}.$$

From Lemma 6.6, we have

$$\begin{aligned}\bar{K}_{\tilde{w}} &\le \frac{\sqrt{2}}{\sqrt{w(0)(-\gamma_w)^3}},\\ \hat{K}_{\tilde{w}} &\le \frac{1}{\sqrt{w(0)(-\gamma_w)^3}}.\end{aligned}$$

We conclude that

$$\begin{aligned}4\left(\bar{K}_{\tilde{w}}+\hat{K}_{\tilde{w}}\right)\tilde{M}\tilde{L}+\tilde{L}^2 &\le \frac{4\sqrt{2}+4}{\sqrt{w(0)(-\gamma_w)^3}}C_{\gamma_w}ML+C_{\gamma_w}L^2\\ &< \beta_w C_{\gamma_w}\\ &= -\gamma_w,\end{aligned}$$

which implies the existence of a law-limit for equation (1.1) on $L^2_{\tilde{w}}$. □

Remark 6.8. *The presented theory can be applied to Lévy processes with jumps. For example, for the standard real valued Poisson process $N(t)_{t\ge0}$ a condition similar to* (6.5) *can be formulated under the assumption that $\sigma(f)(x)\ge 0$ for every $x\ge 0$ and $f\in H^0_w$:*

$$\frac{4\sqrt{2}+4}{\sqrt{w(0)(-\gamma_w)^3}}ML+\frac{2\sqrt{2}}{w(0)\sqrt{(-\gamma_w)^5\beta_w}}M^2L+L^2<\beta_w.$$

The derivation follows the proof of Theorem 6.3*, only now the Lipschitz constant of the drift coeffiecient $F_N\circ\sigma$ is given by (see the proof of Lemma* 4.2 *in* [15]*)*

$$L_{F_N\circ\sigma}=2\left(\bar{K}_w+\hat{K}_w\right)ML+\bar{K}_w\hat{K}_wM^2L.$$

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