# ON METRIZATION OF UNIONS OF FUNCTION SPACES ON DIFFERENT INTERVALS

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Abstract. Let $(S, d_s)$ be a metric space. We define a metric on the space $C_s^\tau$ which consists of the disjoint union of continuous functions defined on different intervals $[0, \tau]$, $\tau > 0$, with values in the metric space $S$. From this, we define another metric space, $\left(C_s^{\mathbb{N}}\right)$, where elements are sequences of functions from the space $C_s^\tau$. We study the traditional properties of these metric spaces like separability and completeness.

The abstract mathematical problem that we discuss here is motivated by the study of a stochastic model in population dynamics. That is, we consider a deterministic system, that is subject to interventions at discrete points in time randomly, in which the deterministic system jumps to a new position, also randomly. However, we feel that the results discussed here are of separate, mathematical interest.

Modeling population dynamics is of great interest for biologists, ecologists, entomologists and epidemiologists (e.g. Berryman [1], Metz and Diekmann [5] and Murray [6]) . The first definitive theoretical treatment of population dynamics was Thomas Malthus (1798). Forty years later, Verhulst (1838) formed Malthus' "principle of population" into a mathematical model. The next major advance in population dynamics theory was in 1925 and the year later, Lotka and Volterra, in separate works, formulated the predator-prey model. Biologically more realistic models were subsequently introduced, that include, for example, predator satiation (Rosenzweig-MacArthur model).

The mathematical language in which these models are being formulated is that of ordinary differential equations (ODEs). We would like to add stochastic interventions to these type of models, as mentioned above. The trajectory would then be composed of continuous parts between the interventions and the jumps. It is our idea to view the continuous parts as elements of a space of continuous functions with values in a metric space $S$, $C_s([0, \tau])$, where $\tau$ varies according to the time in-between interventions. In addition, the random interventions in between two continuous parts will be modeled by a map from one space $C_s([0, \tau_1])$ to another $C_s([0, \tau_2])$. An entire trajectory will thus be of the form $(f_1, f_2, f_3, ....)$ with $f_i \in C_s([0, \tau_i])$ . We want to view each trajectory as an element of one fixed state

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space. In view of the randomness in the model it will be important to have the state space a complete separable metric space. This article is therefore concerned with defining and studying metrics and their properties on the space of sequences in the disjoint union of $C_s([0,\tau_i])$ over a family of $\tau_i > 0$. This approach contrasts with the common approach in stochastic processes of viewing the trajectories directly in the space of Cadlag functions $D_s([0,\infty))$, (c.f. Skorohod [7], Ethier and Kurtz [4] and Billingsley [2]).

Our approach is inspired by the definition of the Skorohod distance (c.f. Skorohod [7], Ethier and Kurtz [4] and Billingsley [2]) between two functions $f$ and $g$, which can be seen as the uniform distance between $f$ and a deformation of $g$ plus a penalty depending on the amount of deformation. We extend this idea to functions defined on different intervals and analyze how properties of the metric defined accordingly are related to properties of the chosen penalty function.

## 1. Metrics on the Space $C_s^\tau$

Let $(S, d_s)$ be a metric space and let $\tau = (\tau_\alpha) \subset [0,\infty)$ be all different; i.e $\tau_\alpha \neq \tau_\beta$ when $\alpha \neq \beta$. Put $C_s^\tau := \bigcup_\alpha C_s([0,\tau_\alpha])$ where

$$C_s([0,\tau_\alpha]) := \{f : [0,\tau_\alpha] \longrightarrow S,\ f \text{ continuous}\}.$$

We will define a metric on $C_s^\tau$. Let for each pair $(\alpha,\beta)$, $\Delta_{\alpha,\beta} \neq \emptyset$ where

$$\Delta_{\alpha,\beta} \subseteq \{\lambda : [0,\tau_\alpha] \longrightarrow [0,\tau_\beta], \lambda \text{ strictly increasing continuous; } \lambda 0 = 0, \lambda\tau_\alpha = \tau_\beta\},$$

(sometimes we use $\lambda\tau$ for $\lambda(\tau)$ ) such that:

**(D1):** $\Delta_{\beta,\alpha} = \Delta_{\alpha,\beta}^{-1}$.
**(D2):** $\Delta_{\beta,\gamma} \circ \Delta_{\alpha,\beta} \subset \Delta_{\alpha,\gamma}$.

Note that (D1) and (D2) imply $I \in \Delta_{\alpha,\alpha}$, where $I$ is the identity map. Moreover, any $\lambda \in \Delta_{\alpha,\beta}$ must be a homeomorphism.

Let $\Delta = \bigcup_{\alpha,\beta} \Delta_{\alpha,\beta}$. Moreover, assume that we have a *penalty function* $P : \Delta \longrightarrow [0,\infty)$ that satisfies the following properties:

**(P1):** $P(I) = 0$.
**(P2):** $P(\lambda^{-1}) = P(\lambda)$.
**(P3):** $P(\lambda_1\lambda_2) \leq P(\lambda_1) + P(\lambda_2)$.
**(P4):** If $\lambda_n \in \Delta_{\alpha,\beta}$ and $P(\lambda_n) \longrightarrow 0$ as $n \longrightarrow \infty$, then $\alpha = \beta$ and $\lambda_n \longrightarrow I$ uniformly on $[0,\tau_\alpha]$.

Condition (P2) will yield the symmetry of the metric (that we will define), (P3) the triangle inequality and (P4) is involved in the positive definiteness of the metric.

Recall that we call a function *lipeomorphism* if it is Lipschitz, bijective and has a Lipschitz inverse. Also, an increasing lipeomorphism should be strictly increasing.

Good candidates for such $\Delta$ and $P$ are:

(1) $\Delta_{\alpha,\beta}$ consists of all strictly increasing, continuous mappings of $[0,\tau_\alpha]$ onto $[0,\tau_\beta]$ (i.e $\lambda : [0,\tau_\alpha] \longrightarrow [0,\tau_\beta]$ such that $\lambda(0)=0$ and $\lambda(\tau_\alpha)=\tau_\beta$),

$$P(\lambda) = \| \lambda - I \| = \sup_{t\in[0,\tau_\alpha]} | \lambda(t) - t |,$$

$\lambda \in \Delta_{\alpha,\beta}$.

(2) $\Delta_{\alpha,\beta}$ consists of all increasing lipeomorphisms, which are also strictly increasing,

$$P(\lambda) = \sup_{0\le t<s\le \tau_\alpha} | \log \frac{\lambda(s)-\lambda(t)}{s-t} |,$$

$\lambda \in \Delta_{\alpha,\beta}$.

Let us call the following condition (P4′):

**(P4′):** If $\lambda_n \in \Delta_{\alpha,\beta_n}$ and $P(\lambda_n) \longrightarrow 0$ as $n \longrightarrow \infty$, then $\tau_{\beta_n} \longrightarrow \tau_\alpha$ as $n \longrightarrow \infty$ and $\lambda_n \longrightarrow I$ uniformly on $[0,\tau_\alpha]$.

Note that (P4′) implies (P4).

We will establish more properties of the penalty function

$$P(\lambda) = \sup_{0\le t<s\le \tau_\alpha} | \log \frac{\lambda(s)-\lambda(t)}{s-t} |:$$

**Proposition 1.** *$P(\lambda)$ satisfies properties (P1)-(P3) mentioned above and if $\lambda_n \in \Delta_{\alpha_n,\beta_n}$ and $P(\lambda_n) \longrightarrow 0$ as $n \longrightarrow \infty$, then $\frac{\tau_{\beta_n}}{\tau_{\alpha_n}} \longrightarrow 1$ as $n \longrightarrow \infty$ and*

$$\frac{1}{\tau_{\alpha_n}} \sup_{s\in[0,\tau_{\alpha_n}]} | \lambda_n(s) - s | \longrightarrow 0.$$

*As a special case, if $(\tau_{\alpha_n})$ is bounded, then $\tau_{\beta_n} - \tau_{\alpha_n} \longrightarrow 0$ as $n \longrightarrow \infty$. Consequently, $P(\lambda)$ satisfies (P4′) and, therefore, (P4).*

*Proof.* For (P1) and (P2) the proof is obvious. For (P3), let $\lambda_2 : [0,\tau_\alpha] \longrightarrow [0,\tau_\beta]$ and $\lambda_1 : [0,\tau_\beta] \longrightarrow [0,\tau_\gamma]$ then we have

$$\begin{aligned}
| \log \frac{\lambda_1\lambda_2(s)-\lambda_1\lambda_2(t)}{s-t} | &= | \log \frac{\lambda_1\lambda_2(s)-\lambda_1\lambda_2(t)}{\lambda_2(s)-\lambda_2(t)} . \frac{\lambda_2(s)-\lambda_2(t)}{s-t} | \\
&= | \log \left( \frac{\lambda_1\lambda_2(s)-\lambda_1\lambda_2(t)}{\lambda_2(s)-\lambda_2(t)} \right) + \log \left( \frac{\lambda_2(s)-\lambda_2(t)}{s-t} \right) | \\
&\le | \log \frac{\lambda_1\lambda_2(s)-\lambda_1\lambda_2(t)}{\lambda_2(s)-\lambda_2(t)} | + | \log \frac{\lambda_2(s)-\lambda_2(t)}{s-t} | \\
&= | \log \frac{\lambda_1(w)-\lambda_1(v)}{w-v} | + | \log \frac{\lambda_2(s)-\lambda_2(t)}{s-t} | .
\end{aligned}$$

Taking the supremum on the interval $[0,\tau_\alpha]$ for both sides of the inequality we get

$$\begin{aligned}
P(\lambda_1\lambda_2) &= \sup_{0\le t<s\le\tau_\alpha} | \log \frac{\lambda_1\lambda_2(s)-\lambda_1\lambda_2(t)}{s-t} | \\
&\le \sup_{0\le v<w\le\tau_\beta} | \log \frac{\lambda_1(w)-\lambda_1(v)}{w-v} | + \sup_{0\le t<s\le\tau_\alpha} | \log \frac{\lambda_2(s)-\lambda_2(t)}{s-t} | \\
&= P(\lambda_1) + P(\lambda_2).
\end{aligned}$$

Furthermore, let $\lambda_n \in \Delta_{\alpha_n,\beta_n}$ be such that $P(\lambda_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Since $P(\lambda_n) \longrightarrow 0$, $\sup_{0\leq t<s\leq \tau_{\alpha_n}} \mid \log \frac{\lambda_n(s)-\lambda_n(t)}{s-t} \mid \longrightarrow 0$. So, with $t=0$ and $s=\tau_{\alpha_n}$ we obtain $\mid \log \frac{\tau_{\beta_n}}{\tau_{\alpha_n}} \mid \longrightarrow 0$. Hence, $\frac{\tau_{\beta_n}}{\tau_{\alpha_n}} \longrightarrow 1$ as $n \longrightarrow \infty$. Given $\epsilon > 0$, choose $\epsilon_1 > 0$ such that

$$\left(\max\left\{(\mathrm{e}^{\epsilon_1}-1),(1-\mathrm{e}^{-\epsilon_1})\right\}\right) < \epsilon.$$

There exists $N$ such that $\mid \log \frac{\lambda_n(s)-\lambda_n(t)}{s-t} \mid < \epsilon_1$ for all $\tau_{\alpha_n} \geq s > t \geq 0$ and for all $n \geq N$. So, $-\epsilon_1 < \log \frac{\lambda_n(s)-\lambda_n(t)}{s-t} < \epsilon_1$, which implies, $\mathrm{e}^{-\epsilon_1} < \frac{\lambda_n(s)-\lambda_n(t)}{s-t} < \mathrm{e}^{\epsilon_1}$, equivalently, $\mathrm{e}^{-\epsilon_1}(s-t) < \lambda_n(s)-\lambda_n(t) < \mathrm{e}^{\epsilon_1}(s-t)$ for all $\tau_{\alpha_n} \geq s > t \geq 0$ and for all $n \geq N$. Thus at $t=0$, $\mathrm{e}^{-\epsilon_1}s < \lambda_n(s) < \mathrm{e}^{\epsilon_1}s$ for all $s>0$ and for all $n \geq N$. Hence, $-(1-\mathrm{e}^{-\epsilon_1})s < \lambda_n(s)-s < (\mathrm{e}^{\epsilon_1}-1)s$ for all $s>0$ and for all $n \geq N$. Hence,

$$\frac{1}{\tau_{\alpha_n}} \sup_{s\in[0,\tau_{\alpha_n}]} \mid \lambda_n(s)-s \mid \leq \left(\max\left\{(\mathrm{e}^{\epsilon_1}-1),(1-\mathrm{e}^{-\epsilon_1})\right\}\right) < \epsilon$$

for all $n \geq N$. Thus, if $(\tau_{\alpha_n})$ is bounded, then $\tau_{\beta_n}-\tau_{\alpha_n} = \left(\frac{\tau_{\beta_n}}{\tau_{\alpha_n}}-1\right)\tau_{\alpha_n} \longrightarrow 0$ as $n \longrightarrow \infty$ . The proof that $P(\lambda)$ satisfies (P4$'$) follows immediately from the above argument by taking $\lambda_n \in \Delta_{\alpha,\beta_n}$ and choosing $\epsilon_1 > 0$ such that

$$\left(\max\left\{(\mathrm{e}^{\epsilon_1}-1),(1-\mathrm{e}^{-\epsilon_1})\right\}\right)\tau_\alpha < \epsilon.$$

Consequently, we get

$$\sup_{s\in[0,\tau_\alpha]} \mid \lambda_n(s)-s \mid \leq \left(\max\left\{(\mathrm{e}^{\epsilon_1}-1),(1-\mathrm{e}^{-\epsilon_1})\right\}\right)\tau_\alpha < \epsilon$$

for all $n \geq N$. Thus, $\lambda_n \longrightarrow I$ uniformly on $[0,\tau_\alpha]$ . It will then follow that $P(\lambda)$ satisfies (P4). Indeed, (P4) is a special case of (P4$'$). To see this take $\tau_{\beta_n} = \tau_\beta$, i.e $\lambda_n \in \Delta_{\alpha,\beta}$. Then we get, $\lambda_n \longrightarrow I$ uniformly on $[0,\tau_\alpha]$ . Moreover, for $s=\tau_\alpha$, we have $\lambda_n(s) = \tau_\beta$ and from the above supremum we get $\mid \tau_\beta - \tau_\alpha \mid < \epsilon$ for all $n \geq N$ which implies $\tau_\alpha = \tau_\beta$ . □

The following remarks (2,3) concern $P(\lambda) = \sup_{0\leq t<s\leq\tau_\alpha} \mid \log \frac{\lambda(s)-\lambda(t)}{s-t} \mid$:

*Remark* 2. The definitions (1) and (2) for a penalty function may be used for nondecreasing functions $\lambda$, not necessarily continuous, when allowing $+\infty$ as value of the penalty function. In case (2), let $\lambda$ be a nondecreasing function that maps $[0,\tau_\alpha]$ to $[0,\tau_\beta]$ satisfying $\lambda(0)=0$ and $\lambda(\tau_\alpha)=\tau_\beta$. If $P(\lambda)<\infty$, then $\lambda$ is strictly increasing and Lipschitz continuous. The inverse satisfies $P(\lambda^{-1}) = P(\lambda) < \infty$. Thus, also $\lambda^{-1}$ is Lipschitz. Moreover, $P(\lambda)<\infty$ if and only if $\lambda$ is a lipeomorphism. On the other way around, there exist strictly increasing continuous functions $\lambda$ for which $P(\lambda)=\infty$.

*Proof.* If $P(\lambda)<\infty$, then the slopes $\frac{\lambda(s)-\lambda(t)}{s-t}$ should satisfy $0 < \frac{\lambda(s)-\lambda(t)}{s-t} < \infty$ for $s>t$. Therefore, $\lambda$ is strictly increasing. Now to show that $\lambda$ is a lipeomorphism when $P(\lambda)$ is finite, let $P(\lambda)=c<\infty$, then $\mid \log \frac{\lambda(s)-\lambda(t)}{s-t} \mid \leq c$ for all $\tau_\alpha \geq s > t \geq 0$, so, $-c \leq \log \frac{\lambda(s)-\lambda(t)}{s-t} \leq c$. Hence, $\mathrm{e}^{-c} \leq \frac{\lambda(s)-\lambda(t)}{s-t} \leq \mathrm{e}^{c}$, or equivalently,

$\mathrm{e}^{-\mathrm{c}}(s-t) \leq \lambda(s)-\lambda(t) \leq \mathrm{e}^{\mathrm{c}}(s-t)$. Thus,

$$| \lambda(s)-\lambda(t) | \leq \max \left\{\mathrm{e}^{-\mathrm{c}}, \mathrm{e}^{\mathrm{c}}\right\} | s-t |= L | s-t |,$$

for all $\tau_\alpha \geq s > t \geq 0$, where $L = \max\{\mathrm{e}^{-\mathrm{c}}, \mathrm{e}^{\mathrm{c}}\}$. So, $\lambda$ is Lipschitz. By (P2) and using the same argument as above, $\lambda^{-1}$ is Lipschitz. Therefore, $\lambda$ is a lipeomorphism and hence is a member of $\Delta_{\alpha,\beta}$. Conversely, if $\lambda$ is a lipeomorphism, then $| \lambda(s) - \lambda(t) |\leq L | s-t |$, and $| \lambda^{-1}(s) - \lambda^{-1}(t) |\leq \ell | s-t |$ with $\ell, L \geq 1$. Consequently, $0 < \frac{1}{\ell} \leq| \frac{\lambda(s)-\lambda(t)}{s-t} |\leq L$ for $s \neq t$. Hence, $| \log \frac{\lambda(s)-\lambda(t)}{s-t} |\leq c$ for all $\tau_\alpha \geq s > t \geq 0$, where $c = \max(\log \ell, \log L)$. Thus, $P(\lambda) \leq c < \infty$. □

*Remark* 3. For $\lambda \in \Delta_{\alpha,\beta}$, we have

$$P(\lambda) = \sup_{0\leq t<s\leq\tau_\alpha} | \log \frac{\lambda(s)-\lambda(t)}{s-t} |=\operatorname{ess}\sup_{t\in[0,\tau_\alpha]} | \log \lambda'(t) |=:\| \log \lambda' \|_{\infty,[0,\tau_\alpha]} .$$

*Proof.* See [4], page 117. □

Now we proceed to define a metric. Given a family $\Delta_{\alpha,\beta}$ as above and $P$ satisfying (P1)-(P4) with $P(\lambda) < \infty$ we define a function $\hat{d}$ on $C_s^\tau \times C_s^\tau$ as follows:

For $f, g \in C_s^\tau$ with $f \in C_s([0,\tau_\alpha]), g \in C_s([0,\tau_\beta])$, let

$$d_\lambda(f,g) := \sup_{t\in[0,\tau_\alpha]} d_s(f(t), g\circ\lambda(t)),$$
$$\hat{d}(f,g) := \inf_{\lambda\in\Delta_{\alpha,\beta}} \{d_\lambda(f,g) + P(\lambda)\}.$$

*Remark* 4. If $f, g \in C_s([0,\tau_\alpha])$ then $\hat{d}(f,g) \leq \sup_{t\in[0,\tau_\alpha]} d_s(f(t), g(t))$.

*Proof.* According to assumptions $I \in \Delta_{\alpha,\alpha}$ and (P1) we have

$$\hat{d}(f,g) = \inf_{\lambda\in\Delta_{\alpha,\beta}} \{d_\lambda(f,g) + P(\lambda)\} \leq \{d_{\lambda=I}(f,g) + P(I)\} = \sup_{t\in[0,\tau_\alpha]} d_s(f(t), g(t)).$$

□

**Proposition 5.** *If $\Delta_{\alpha,\beta}$ and $P : \Delta \longrightarrow [0,\infty)$ satisfy (D1),(D2) and (P1)-(P4) respectively, then $\hat{d}$ is a metric on $C_s^\tau$.*

*Proof.* For $f, g \in C_s^\tau$ with $f \in C_s([0,\tau_\alpha]), g \in C_s([0,\tau_\beta])$ let $\hat{d}(f,g) = 0$. Then there exist $\lambda_n : [0,\tau_\alpha] \longrightarrow [0,\tau_\beta] \in \Delta_{\alpha,\beta}$ such that $\lim_{n\longrightarrow\infty}\{d_{\lambda_n}(f,g)+P(\lambda_n)\} = 0$. Consequently, $P(\lambda_n) \longrightarrow 0$, $d_{\lambda_n}(f,g) \longrightarrow 0$. Now, $P(\lambda_n) \longrightarrow 0$ gives $\tau_\alpha = \tau_\beta$ and $\lambda_n \longrightarrow I$ uniformly on $[0,\tau_\alpha]$ according to (P4). Therefore, $\lim_{n\longrightarrow\infty} d_{\lambda_n}(f,g) = 0$ gives $\lim_{n\longrightarrow\infty} \sup_{t\in[0,\tau_\alpha]} d_s(f(t), g\circ\lambda_n(t)) = 0$ and this yields $\sup_{t\in[0,\tau_\alpha]} d_s(f(t), g\circ I(t)) = 0$. Thus, $f = g$. For the other way around let $f = g$ and $f, g : [0,\tau_\alpha] \longrightarrow [0,\tau_\alpha]$. Then, as mentioned above, $\hat{d}(f,g) \leq \sup_{t\in[0,\tau_\alpha]} d_s(f(t), g(t))$ and having $d_s$ a metric gives $\hat{d}(f,g) = 0$. For the symmetry we have: $d_\lambda(f,g) = \sup_{t\in[0,\tau_\alpha]} d_s(f(t), g\circ\lambda(t))$, now take $t = \lambda^{-1}(s)$ then $\lambda(t) = s$ and having $\lambda$ :

$[0,\tau_\alpha] \longrightarrow [0,\tau_\beta]$ gives $\lambda^{-1} : [0,\tau_\beta] \longrightarrow [0,\tau_\alpha]$. So,

$$\begin{aligned} d_\lambda(f,g) &= \sup_{t\in[0,\tau_\alpha]} d_s(f(t), g\circ\lambda(t)) = \sup_{t\in[0,\tau_\beta]} d_s(f(\lambda^{-1}(t)), g(t)) \\ &= \sup_{t\in[0,\tau_\beta]} d_s(g(t), f(\lambda^{-1}(t))) = d_{\lambda^{-1}}(g,f). \end{aligned}$$

Note that taking the infimum over $\lambda^{-1} \in \Delta_{\beta,\alpha} = \Delta^{-1}_{\alpha,\beta}$ is equivalent to taking the infimum over $\lambda \in \Delta_{\alpha,\beta}$. So, by assumption (P2),

$$\begin{aligned} \hat{d}(f,g) :&= \inf_{\lambda\in\Delta_{\alpha,\beta}} \{d_\lambda(f,g) + P(\lambda)\} \\ &= \inf_{\lambda^{-1}\in\Delta^{-1}_{\alpha,\beta}=\Delta_{\beta,\alpha}} \{d_{\lambda^{-1}}(g,f) + P(\lambda^{-1})\} \\ &= \hat{d}(g,f). \end{aligned}$$

For the triangle inequality, let $f, g$ be as stated above and let $h \in C_s([0,\tau_\gamma])$ then we have:

$$\hat{d}(f,g) + \hat{d}(g,h) \quad = \quad \inf_{\lambda_1\in\Delta_{\alpha,\beta}} (d_{\lambda_1}(f,g) + P(\lambda_1)) + \inf_{\lambda_2\in\Delta_{\beta,\gamma}} (d_{\lambda_2}(g,h) + P(\lambda_2)).$$

But,

$$d_{\lambda_1}(f,g) + d_{\lambda_2}(g,h) \quad = \quad \sup_{t\in[0,\tau_\alpha]} d_s(f(t), g(\lambda_1(t))) + \sup_{t\in[0,\tau_\beta]} d_s(g(t), h(\lambda_2(t)))$$

and in the latter supremum if we take $t = \lambda_1(s)$ and since $t \in [0,\tau_\beta]$ then $s \in [0,\tau_\alpha]$ thus we get $\sup_{s\in[0,\tau_\alpha]} d_s(g(\lambda_1(s)), h(\lambda_2\lambda_1(s)))$ . So,

$$\begin{aligned} d_{\lambda_1}(f,g) + d_{\lambda_2}(g,h) &= \sup_{t\in[0,\tau_\alpha]} d_s\left(f(t), g(\lambda_1(t))\right) + \sup_{t\in[0,\tau_\beta]} d_s\left(g(t), h(\lambda_2(t))\right) \\ &\geq \sup_{t\in[0,\tau_\alpha]} d_s\left(f(t), g(\lambda_1(t))\right) + \sup_{s\in[0,\tau_\alpha]} d_s\left(g(\lambda_1(t)), h(\lambda_2\lambda_1(t))\right) \\ &\geq \sup_{t\in[0,\tau_\alpha]} d_s\left(f(t), h(\lambda_2\lambda_1(t))\right) = d_{\lambda_2\lambda_1}(f,h). \end{aligned}$$

Note that $\lambda_2\lambda_1 \in \Delta_{\beta,\gamma}\Delta_{\alpha,\beta}$ and by assumption (D2), $\lambda_2\lambda_1 \in \Delta_{\alpha,\gamma}$. Therefore,

$$\begin{aligned} \hat{d}(f,g) + \hat{d}(g,h) &= \inf_{\lambda_1\in\Delta_{\alpha,\beta}} \{d_{\lambda_1}(f,g) + P(\lambda_1))\} + \inf_{\lambda_2\in\Delta_{\beta,\gamma}} \{d_{\lambda_2}(g,h) + P(\lambda_2)\} \\ &= \inf_{\{\lambda_1\in\Delta_{\alpha,\beta},\lambda_2\in\Delta_{\beta,\gamma}\}} \{d_{\lambda_1}(f,g) + d_{\lambda_2}(g,h) + P(\lambda_1) + P(\lambda_2)\} \\ &\geq \inf_{\{\lambda_1\in\Delta_{\alpha,\beta},\lambda_2\in\Delta_{\beta,\gamma}\}} \{d_{\lambda_2\lambda_1}(f,h) + P(\lambda_2\lambda_1)\} \\ &\geq \inf_{\lambda_2\lambda_1\in\Delta_{\alpha,\gamma}} \{d_{\lambda_2\lambda_1}(f,h) + P(\lambda_2\lambda_1)\} \\ &\geq \inf_{\lambda\in\Delta_{\alpha,\gamma}} \{d_\lambda(f,h) + P(\lambda)\} = \hat{d}(f,h). \end{aligned}$$

□

It is interesting to observe the following. If we have $(f_n) \in C_s([0,\tau_{\alpha_n}])$, $f \in C_s([0,\tau_\alpha])$ and we would evaluate $\hat{d}(f_n, f)$, then, to be able to evaluate $d_{\lambda_n}(f_n, f) = \sup_{t\in[0,\tau_{\alpha_n}]} d_s(f_n(t), f(\lambda_n(t)))$, we should only consider $\lambda_n : [0,\tau_{\alpha_n}] \longrightarrow [0,\tau_\alpha]$, or $\lambda_n^{-1} : [0, \tau_\alpha] \longrightarrow [0, \tau_{\alpha_n}]$. i.e either the domain or the range of $\lambda_n$ is independent of $n$. It is also worthy to clarify the concept of convergence under the metric $\hat{d}$. So, if it is given that $(f_n) \in C_s([0, \tau_{\alpha_n}])$ and $f \in C_s([0, \tau_\alpha])$ such that $(f_n) \longrightarrow f$ under the metric $\hat{d}$ then we conclude that there exist $\lambda_n \in \Delta_{\alpha_n, \alpha}$ such that $d_{\lambda_n}(f_n, f) = \sup_{t \in [0, \tau_\alpha]} d_s\left(f_n\left(\lambda_n^{-1}(t)\right), f(t)\right) \longrightarrow 0$, i.e $f_n \circ \lambda_n^{-1} \longrightarrow f$ uniformly on the interval on which $f$ is defined, $[0, \tau_\alpha]$, under the metric $d_s$. Moreover, $P(\lambda_n) \longrightarrow 0$ which, by (P4$'$), implies that $\frac{\tau_\alpha}{\tau_{\alpha_n}} \longrightarrow 1$, or equivalently,

$$\tau_{\alpha_n} \longrightarrow \tau_\alpha. \tag{1.1}$$

Conversely, if we have $(f_n) \in C_s([0, \tau_n])$ and we would prove $\hat{d}(f_n, f) \longrightarrow 0$ then we should find $f \in C_s([0, \tau_\alpha])$ and functions $\lambda_n \in \Delta_{n,\alpha}$ such that $P(\lambda_n) \longrightarrow 0$ and $\sup_{t \in [0, \tau_n]} d_s(f_n(t), f \circ \lambda_n(t)) = \sup_{t \in [0, \tau_\alpha]} d_s\left(f_n \circ \lambda_n^{-1}(t), f(t)\right) \longrightarrow 0$.

For $\Delta_{\alpha,\beta} := \{\lambda{:}[0, \tau_\alpha] \longrightarrow [0, \tau_\beta] : \lambda$ increasing lipeomorphism$\}$ let $\mid \lambda \mid_\ell$ stands for the Lipschitz constant of the function $\lambda$. An example of a function $P$ such that $\hat{d}$ is not a metric, where the definiteness property fails, but it is a semimetric is

**Example 6.** $P(\lambda) = \log(\mid \lambda \mid_\ell \mid \lambda^{-1} \mid_\ell)$, with $\Delta_{\alpha,\beta}$ as described above. Taking $f(t) = t$, $t \in [0, 1]$, $g(t) = 2t$, $t \in \left[0, \frac{1}{2}\right]$ gives that $\hat{d}(f, g) = 0$ but $f \neq g$. To see this, pick $\lambda(t) = \frac{1}{2}t$ so, $P(\lambda) = \log(\frac{1}{2} \times 2) = 0$ and $d_\lambda(f, g) = \sup_{t \in [0,1]}(f(t), g \circ \lambda(t)) = \sup_{t \in [0,1]}(t, t) = 0$ and $\inf_{\lambda \in \Delta_{\alpha,\beta}}\{d_\lambda(f, g) + P(\lambda)\} \leq d_\lambda(f, g) + P(\lambda)$ when $\lambda(t) = \frac{1}{2}t$. Note that $P$ satisfies properties (P1), (P2) and (P3) above but it does not satisfy property (P4).

In a special case we obtain a metric space with this $P$:

**Example 7.** For a fixed $\tau$ the space $\left(C_s([0, \tau]), \hat{d}\right)$ with $P(\lambda) = \log(\mid \lambda \mid_\ell \mid \lambda^{-1} \mid_\ell)$ and $\Delta := \{\lambda{:}[0, \tau] \longrightarrow [0, \tau]$ increasing lipeomorphism$\}$ is a metric space. Here we need to show that $P$ satisfies property (P4). For this purpose we need the following lemma.

Remark first that one could not find two sequences $(x_n)$, $(y_n)$ such that $(x_n) \geq 1$ and $(y_n) \geq 1$ and $(x_n y_n) \longrightarrow 1$ but the statement $\{(x_n) \longrightarrow 1, (y_n) \longrightarrow 1\}$ is not true.

**Lemma 8.** *Let $\lambda_n : [0, \tau] \longrightarrow [0, \tau]$ be strictly increasing lipeomorphisms such that $|\lambda_n|_\ell |\lambda_n^{-1}|_\ell \longrightarrow 1$. Then we have:*

1. $|\lambda_n|_\ell \longrightarrow 1$ and $|\lambda_n^{-1}|_\ell \longrightarrow 1$.
2. $\lambda_n(t) \longrightarrow t$, for all $t$ (pointwise convergence).
3. $\lambda_n \longrightarrow I$ uniformly ($\|\lambda_n - I\| \longrightarrow 0$ ).

*Proof.* We have $|\lambda_n(\tau) - \lambda_n(0)| \leq |\lambda_n|_\ell |\tau - 0|$ but having $\lambda_n(0) = 0, \lambda_n(\tau) = \tau$ gives $|\lambda_n(\tau) - \lambda_n(0)| = |\tau - 0|$. So, $|\lambda_n|_\ell \geq 1$. $\lambda_n$ is increasing, therefore, $\lambda_n^{-1}$ is increasing and hence $|\lambda_n^{-1}|_\ell \geq 1$ as well. Together with $|\lambda_n|_\ell|\lambda_n^{-1}|_\ell \longrightarrow 1$ , we get $|\lambda_n|_\ell \longrightarrow 1$ and $|\lambda_n^{-1}|_\ell \longrightarrow 1$. In fact,

$$|\lambda_n|_\ell - 1 \leq |\lambda_n^{-1}|_\ell \left(|\lambda_n|_\ell - 1\right) = |\lambda_n^{-1}|_\ell|\lambda_n|_\ell - 1 - \left(|\lambda_n^{-1}|_\ell - 1\right)$$

Thus,

$$0 \leq \left(|\lambda_n|_\ell - 1\right) + \left(|\lambda_n^{-1}|_\ell - 1\right) \leq |\lambda_n|_\ell|\lambda_n^{-1}|_\ell - 1.$$

Since $|\lambda_n|_\ell|\lambda_n^{-1}|_\ell \longrightarrow 1$ and $|\lambda_n|_\ell \geq 1$, $|\lambda_n^{-1}|_\ell \geq 1$, we find $|\lambda_n|_\ell \longrightarrow 1$ and $|\lambda_n^{-1}|_\ell \longrightarrow 1$. For the second point of the lemma we need to show that $\lim_{n\to\infty} \lambda_n(t) = t$ for all $t$. We will do this by contradiction. So, assume there exists $t_0$ such that $\lambda_n(t_0)$ does not converge to $t_0$. Then $t_0 \neq 0$ and $t_0 \neq \tau$. Moreover, there exist $\epsilon > 0$ and a subsequence $(n_k)$ such that either $\lambda_{n_k}(t_0) \leq t_0 - \epsilon$ or $\lambda_{n_k}(t_0) \leq t_0 + \epsilon$ for all $k$. In the first case, $|\lambda_n(t_0) - \lambda_n(\tau)| \leq |\lambda_n|_\ell|t_0 - \tau|$ or $|\lambda_n|_\ell \geq \frac{|\lambda_n(t_0) - \lambda_n(\tau)|}{|t_0 - \tau|}$ :

$$|\lambda_{n_k}|_\ell \geq \frac{|\lambda_{n_k}(t_0) - \lambda_{n_k}(\tau)|}{|t_0 - \tau|} = \frac{\tau - \lambda_{n_k}(t_0)}{\tau - t_0} \geq \frac{\tau - t_0 + \epsilon}{\tau - t_0} = 1 + \frac{\epsilon}{\tau - t_0} > 1.$$

In the second case,

$$|\lambda_{n_k}|_\ell \geq \frac{|\lambda_{n_k}(t_0) - \lambda_{n_k}(0)|}{|t_0 - 0|} = \frac{\lambda_{n_k}(t_0)}{t_0} \geq \frac{t_0 + \epsilon}{t_0} = 1 + \frac{\epsilon}{t_0} > 1.$$

This contradicts $|\lambda_n|_\ell \longrightarrow 1$ as $n \longrightarrow \infty$ . For the last part of the lemma, let $\epsilon > 0$. From the pointwise convergence we have: there exist $N_x$ such that: $|\lambda_n(x) - x| < \frac{\epsilon}{3}$ for all $n \geq N_x$. Let $M = \sup_n |\lambda_n|$. Then, $M \geq 1$. Now, if $|x - y| < \frac{\epsilon}{3M}$ then we have:

$$\begin{aligned}
|\lambda_n(y) - y| \leq& |\lambda_n(y) - \lambda_n(x)| + |\lambda_n(x) - x| + |x - y| \\
\leq& |\lambda_n|_\ell|y - x| + \frac{\epsilon}{3} + \frac{\epsilon}{3M} \\
<& \frac{\epsilon|\lambda_n|_\ell}{3M} + \frac{\epsilon}{3} + \frac{\epsilon}{3M} \text{ (note } \frac{|\lambda_n|_\ell}{M} \leq 1, \text{also } \frac{1}{M} \leq 1) \\
\leq& \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
=& \epsilon.
\end{aligned}$$

Since $[0, \tau]$ is compact, there exist finitely many $x_i \in [0, \tau]$ such that $[0, \tau]$ is covered by open intervals $\left(x_i - \frac{\epsilon}{3M}, x_i + \frac{\epsilon}{3M}\right)$. Put $N = \max_i \left(N_{x_i}\right)$ then

$$\sup_{y \in [0,\tau]} |\lambda_n(y) - y| < \epsilon$$

for all $n \geq N$. i.e $\|\lambda_n - I\| \longrightarrow 0$. □

**Lemma 9.** *Let $X$ be a Banach space. Let $S$ be a convex subset of $X$ which is separable in the relative topology, then the space $C_s([0, \tau_\alpha])$ equipped with the uniform metric is separable. In particular, $\left(C_s([0, \tau_\alpha]), \hat{d}\right)$ is separable for all $\alpha$.*

*Proof.* Let $E$ be a countable dense subset of $S$ and let

$$\xi = \{f : [0, \tau_\alpha] \longrightarrow S : \exists\, 0 = t_1 < t_2 < .... < t_n = \tau_\alpha, t_i \in \mathbb{Q} : f(t_1), f(t_2), ....., f(t_n) \in E; f(\lambda t_i + (1-\lambda)t_{i+1}) = \lambda f(t_i) + (1-\lambda) f(t_{i+1}) \,\forall\, 1 \le i < n \, and \, \lambda \in [0,1]\}.$$

$\xi$ is countable because it could be rewritten as

$$\bigcup_n \bigcup_{(t_1,t_2,...,t_n)\in\mathbb{Q}^n} \bigcup_{(e_1,e_2,...,e_n)\in E^n} \{f : [0,\tau_\alpha] \longrightarrow S : \ f(t_1) = e_1, ...., f(t_n) = e_n; \ f \text{ convex on each } [t_i, t_{i+1}]\}$$

and the set described inside the unions contains only one element. We need to show that $\xi$ is dense in $C_s([0,\tau_\alpha])$. For this purpose let $f \in C_s([0,\tau_\alpha])$. Given $\epsilon > 0$, choose $\delta > 0$ such that $d_s(f(s), f(t)) < \epsilon$ whenever $\mid s - t \mid < \delta$. Choose $0 = t_1 < t_2 < ..... < t_n = \tau_\alpha$ in $[0,\tau_\alpha] \cap \mathbb{Q}$ such that $t_{i+1} - t_i < \delta$. Choose $e_1, e_2, ....., e_n \in E$ such that $d_s(f(t_i), e_i) < \epsilon$. Construct $g \in \xi$ such that $g(t_i) = e_i$. If we prove that $\sup_{t\in[0,\tau_\alpha]} d_s(f(t), g(t)) \le 5\epsilon$, then Remark 4 yields $\hat{d}(f,g) < 5\epsilon$, and this ends the proof. Now to prove that $\sup_{t\in[0,\tau_\alpha]} d_s(f(t), g(t)) \le 5\epsilon$, let $t \in [0,\tau_\alpha]$. Then there exists $i$ such that $t \in [t_i, t_{i+1}]$. Thus, $\mid t - t_i \mid < \delta$. Therefore, $d_s(f(t), f(t_i)) < \epsilon$. Hence,

$$\begin{aligned} d_s(f(t), g(t)) \le & d_s(f(t), f(t_i)) + d_s(f(t_i), e_i) + d_s(e_i, g(t_i)) + d_s(g(t_i), g(t)) \\ < & \epsilon + \epsilon + 0 + d_s(g(t_i), g(t)). \end{aligned}$$

Since

$$t = \frac{t_{i+1} - t}{t_{i+1} - t_i} t_i + \frac{t - t_i}{t_{i+1} - t_i} t_{i+1}$$

and $\frac{t_{i+1}-t}{t_{i+1}-t_i} + \frac{t-t_i}{t_{i+1}-t_i} = 1$, it follows by the convexity of $g$ on $[t_i, t_{i+1}]$ that

$$g(t) = \frac{t_{i+1} - t}{t_{i+1} - t_i} g(t_i) + \frac{t - t_i}{t_{i+1} - t_i} g(t_{i+1}).$$

Consequently,

$$\begin{aligned} d_s(g(t_i), g(t)) &= \parallel g(t_i) - g(t) \parallel \\ &= \parallel \frac{t_{i+1} - t}{t_{i+1} - t_i} g(t_i) + \frac{t - t_i}{t_{i+1} - t_i} g(t_i) - \left( \frac{t_{i+1} - t}{t_{i+1} - t_i} g(t_i) + \frac{t - t_i}{t_{i+1} - t_i} g(t_{i+1}) \right) \parallel \\ &= \frac{t - t_i}{t_{i+1} - t_i} \parallel g(t_i) - g(t_{i+1}) \parallel \\ &\le 1 \parallel e_i - e_{i+1} \parallel = d_s(e_i, e_{i+1}). \end{aligned}$$

And

$$\begin{aligned} d_s(e_i, e_{i+1}) \le d_s(e_i, f(t_i)) + & d_s(f(t_i), f(t_{i+1})) + d_s(f(t_{i+1}), e_{i+1}) \\ < & \epsilon + \epsilon + \epsilon = 3\epsilon. \end{aligned}$$

Thus, $d_s(g(t_i), g(t)) \le 3\epsilon$, and therefore, $d_s(f(t), g(t)) < 5\epsilon$. As a result of this $\sup_{t\in[0,\tau_\alpha]} d_s(f(t), g(t)) \le 5\epsilon$. □

**Lemma 10.** *Suppose that there exists a countable subset* $Q \subseteq \{\tau_\alpha : \tau_\alpha \in \mathbb{R}_+, \alpha \in \Lambda\}$ *such that for all* $\tau_\alpha$ *there exist* $\tau_n$ *in* $Q$ *and there exist* $\lambda_n \in \Delta_{\alpha,n}$*, such that* $P(\lambda_n) \longrightarrow 0$*. If* $C_s\left([0,\tau_\alpha]\right)$ *is separable for all* $\alpha$*, then the space* $C_s^\tau := \bigcup_\alpha C_s\left([0,\tau_\alpha]\right)$ *equipped with the metric* $\hat{d}$ *is separable.*

*Proof.* For each $\alpha$, let $M\left[0,\tau_\alpha\right]$ be a countable dense subset of $C_s\left([0,\tau_\alpha]\right)$. We claim that the set $M = \bigcup_{\tau_\alpha \in Q} M\left[0,\tau_\alpha\right]$, is a countable dense subset in $C_s^\tau$. Obviously, taking the disjoint union over $\tau_\alpha \in Q$ makes $M$ countable. Now, to show that the defined set $M$ is dense we only need to show that if $f \in C_s\left[0,\tau_\alpha\right]; \tau_\alpha \notin Q$ then $f$ can be approximated by functions from $M$ (the other case, $f \in C_s\left[0,\tau_\alpha\right]; \tau_\alpha \in Q$ is obvious). So, let $f \in C_s\left([0,\tau_\alpha]\right); \tau_\alpha \notin Q$ and take $\tau_n \in Q$ and $\lambda_n : [0,\tau_\alpha] \longrightarrow [0,\tau_n]$ such that $P(\lambda_n) \longrightarrow 0$. Define $f_n \in C_s\left([0,\tau_n]\right)$ as $f_n = f \circ \lambda_n^{-1}$. Then we have: $f_n\left(\lambda_n(t)\right) = f\left(\lambda_n^{-1}(\lambda_n(t))\right) = f(t)$. Therefore,

$$d_{\lambda_n^{-1}}(f_n, f) = \sup_{t\in[0,\tau_n]} d_s\left(f_n(t), f\circ\lambda_n^{-1}(t)\right) = \sup_{t\in[0,\tau_n]} d_s\left(f_n(t), f_n(t)\right) = 0.$$

Thus, $\hat{d}\left(f_n, f\right) \leq d_{\lambda_n^{-1}}(f_n, f) + P(\lambda_n) = P(\lambda_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Let $g_n \in M\left[0,\tau_n\right]$ such that $\hat{d}(g_n, f_n) < \frac{1}{n}$. Then,

$$\hat{d}(g_n, f) \leq \hat{d}(g_n, f_n) + \hat{d}(f_n, f) \leq \frac{1}{n} + P(\lambda_n) \longrightarrow 0.$$

□

*Remark* 11. Both functions $P(\lambda) = \parallel \lambda - I \parallel$and $P(\lambda) = \sup_{\tau_\alpha \geq s > t \geq 0} \mid \log \frac{\lambda(s)-\lambda(t)}{s-t} \mid$ satisfy the required condition in the previous lemma when $\{\tau_\alpha, \alpha \in \Lambda\} = (0,\infty)$.

*Proof.* Choose $Q = \mathbb{Q} \cap (0,\infty)$. For $\tau_\alpha \in (0,\infty)$ choose $\tau_n$ in $Q$ such that $\tau_n \longrightarrow \tau_\alpha$. Define $\lambda_n(t) := \frac{\tau_n}{\tau_\alpha} t$, clearly, $\lambda_n \in \Delta_{\alpha,n}$. Moreover, in the first case,

$$\begin{aligned} P(\lambda_n) = \parallel \lambda_n - I \parallel &= \sup_{t\in[0,\tau_\alpha]} \mid \lambda_n(t) - t \mid = \sup_{t\in[0,\tau_\alpha]} \mid \frac{\tau_n}{\tau_\alpha} t - t \mid \\ &= \sup_{t\in[0,\tau_\alpha]} \mid \frac{\tau_n}{\tau_\alpha} - 1 \mid t \leq \mid \frac{\tau_n}{\tau_\alpha} - 1 \mid \tau_\alpha \longrightarrow 0. \end{aligned}$$

In the second case,

$$P(\lambda_n) = \sup_{\tau_\alpha \geq s > t \geq 0} \mid \log \frac{\lambda_n(s) - \lambda_n(t)}{s-t} \mid = \sup_{\tau_\alpha \geq s > t \geq 0} \mid \log \frac{\tau_n}{\tau_\alpha} \mid \longrightarrow 0.$$

□

Next we investigate completeness of $\left(C_s^\tau, \hat{d}\right)$.

*Remark* 12. In a metric space, a Cauchy sequence converges if and only if it has a convergent subsequence.

*Proof.* See [3], page 20. □

*Remark* 13. In a metric space $(X,d)$, if $(x_m)$ is a Cauchy sequence, then it contains a subsequence $(y_n)=(x_{m_n})$ such that $d(y_n,y_{n+1})<\frac{1}{2^n}$. Moreover, if a sequence $(x_n)$ satisfies $\sum_{n=1}^{\infty} d(x_n,x_{n+1})<\infty$, then $(x_n)$ is Cauchy. As a special case, if a sequence $(x_n)$ satisfies $d(x_n,x_{n+1})<\frac{1}{2^n}$, then $(x_n)$ is Cauchy.

*Proof.* Let $(x_m)$ be a Cauchy sequence. For $k=1$, choose $N_1$ such that $d(y_n,y_m)<\frac{1}{2}$ for all $n,m\geq N_1$. For $k=2$, choose $N_2>N_1$ such that $d(y_n,y_m)<\frac{1}{2^2}$ for all $n,m\geq N_2$. Proceed like this .... Choose $N_k>N_{k-1}$ such that $d(y_n,y_m)<\frac{1}{2^k}$ for all $n,m\geq N_k$. Then the sequence $(y_{N_k})$ is as required since $d\left(y_{N_k},y_{N_{k+1}}\right)<\frac{1}{2^k}$. Furthermore, let $(x_n)$ be such that $\sum_{n=1}^{\infty} d(x_n,x_{n+1})<\infty$. Need to show $(x_n)$ is Cauchy. We have

$$\begin{aligned} d(x_n,x_{n+k}) \leq & d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+...+d(x_{n+k-1},x_{n+k}) \\ \leq & \sum_{i=n}^{\infty} d(x_i,x_{i+1}) \longrightarrow 0. \end{aligned}$$

Thus, $d(x_n,x_m)<\epsilon$ for all $n,m\geq N$. For the special case, let $(x_n)$ be such that $d(x_n,x_{n+1})<\frac{1}{2^n}$. Need to show $(x_n)$ is Cauchy. We have

$$\begin{aligned} d(x_n,x_{n+k}) \leq & d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+...+d(x_{n+k-1},x_{n+k}) \\ \leq & \frac{1}{2^n}+\frac{1}{2^{n+1}}+...+\frac{1}{2^{n+k-1}} \\ = & \frac{1}{2^n}\left\{1+\frac{1}{2}+...+\frac{1}{2^{k-1}}\right\} \leq \frac{1}{2^n}\{2\} = \frac{1}{2^{n-1}} \longrightarrow 0. \end{aligned}$$

Thus, $d(x_n,x_m)<\epsilon$ for all $n,m\geq N$. ☐

**Lemma 14.** *Let $P(\lambda)$ be the penalty function $P(\lambda)=\sup_{\tau_\alpha\geq s>t\geq 0}\mid \log\frac{\lambda(s)-\lambda(t)}{s-t}\mid$.* If $\lambda\in\Delta$, then

$$\sup_{t\in[0,\tau_\alpha]}\mid \lambda(t)-t\mid \leq \tau_\alpha\left(\mathrm{e}^{\mathrm{P}(\lambda)}-1\right).$$

*Proof.* Since $\mid x-1\mid\leq \mathrm{e}^{|\log x|}-1$ for $x>0$,

$$\begin{aligned} \mid \lambda t-t\mid = t\mid \frac{\lambda t-\lambda 0}{t-0}-1\mid \leq & t\left(\mathrm{e}^{|\log\frac{\lambda t-\lambda 0}{t-0}|}-1\right) \\ \leq & t\left(\mathrm{e}^{\sup_{0\leq s<r\leq\tau_\alpha}|\log\frac{\lambda r-\lambda s}{r-s}|}-1\right) \\ = & t\left(\mathrm{e}^{\mathrm{P}(\lambda)}-1\right). \end{aligned}$$

Taking the supremum over the interval $[0,\tau_\alpha]$ we get

$$\begin{aligned} \sup_{0\leq t\leq\tau_\alpha}\mid\lambda(t)-t\mid = & \sup_{0<t\leq\tau_\alpha} t\mid\frac{\lambda(t)-\lambda(0)}{t-0}-1\mid \\ \leq & \sup_{0\leq t\leq\tau_\alpha} t\left(\mathrm{e}^{\mathrm{P}(\lambda)}-1\right)\leq\tau_\alpha\left(\mathrm{e}^{\mathrm{P}(\lambda)}-1\right). \end{aligned}$$

☐

**Lemma 15.** *If $(S, d_s)$ is a complete metric space and $\hat{d}$ is considered with the penalty function $P(\lambda) = \sup_{\tau_\alpha \geq s > t \geq 0} \mid \log \frac{\lambda(s)-\lambda(t)}{s-t} \mid$, then the space $\left(C_s\left([0,\tau_\alpha]\right), \hat{d}\right)$ is complete for all $\alpha$.*

*Proof.* According to Remark 12 above, it suffices to show that each Cauchy sequence contains a convergent subsequence. Now, Let $(f_k)$ be a Cauchy sequence in $\left(C_s\left([0,\tau_\alpha]\right), \hat{d}\right)$. Then, according to Remark 13, it contains a subsequence $(g_n) = (f_{k_n})$ such that $\hat{d}(g_n, g_{n+1}) < \frac{1}{2^n}$. We shall prove that $(g_n)$ is convergent. Now, by $\hat{d}(g_n, g_{n+1}) < \frac{1}{2^n}$, $\Delta_{\alpha,\alpha}$ contains a $\mu_n : [0,\tau_\alpha] \longrightarrow [0,\tau_\alpha]$ such that $d_{\mu_n}(g_n, g_{n+1}) + P(\mu_n) < \frac{1}{2^n}$. This implies

$$(1.2) \qquad \begin{aligned} d_{\mu_n}(g_n, g_{n+1}) &= \sup_{t\in[0,\tau_\alpha]} d_s\left(g_n(t), g_{n+1}(\mu_n t)\right) \\ &= \sup_{t\in[0,\tau_\alpha]} d_s\left(g_n(\mu_n^{-1} t), g_{n+1}(t)\right) < \frac{1}{2^n} \end{aligned}$$

and

$$(1.3) \qquad P(\mu_n) < \frac{1}{2^n}.$$

The idea of the proof is to find a function $g$ in $C_s\left([0,\tau_\alpha]\right)$ and functions $\lambda_n$ in $\Delta_{\alpha,\alpha}$ for which $P(\lambda_n) \longrightarrow 0$ and $g_n(\lambda_n^{-1} t) \longrightarrow g(t)$ uniformly in $t$ with respect to the metric $d_s$. Remark 16 presents the idea of how to construct such functions $\lambda_n$. Since $\mathrm{e}^x - 1 \leq 2x$ for $0 \leq x \leq \frac{1}{2}$, it follows by (1.3) and Lemma 14 that

$$\begin{aligned} &\sup_{t\in[0,\tau_\alpha]} \mid \mu_{n+m+1} \circ \mu_{n+m} \circ ...\mu_{n+1} \circ \mu_n(t) - \mu_{n+m} \circ ...\mu_{n+1} \circ \mu_n(t) \mid \\ &\qquad = \sup_{s\in[0,\tau_\alpha]} \mid \mu_{n+m+1}(s) - s \mid \\ &\qquad \leq \tau_\alpha \left(\mathrm{e}^{\mathrm{P}(\mu_{n+m+1})} - 1\right) \leq 2\tau_\alpha P(\mu_{n+m+1}) < \frac{\tau_\alpha}{2^{n+m}}. \end{aligned}$$

According to Remark 13 with $d$ the uniform metric, for fixed $n$, the functions $\mu_{n+m}...\mu_{n+1}\mu_n$ (we will use this notation instead of $\mu_{n+m} \circ ...\mu_{n+1} \circ \mu_n$) are uniformly Cauchy as $m \longrightarrow \infty$ (Cauchy in the uniform metric on $\mathbb{R}$). Since the uniform metric is complete, the sequence converges uniformly to a limit

$$\lambda_n t = \lim_{m\longrightarrow\infty} \mu_{n+m}...\mu_{n+1}\mu_n t.$$

Or

$$\lambda_n = \lim_{m\longrightarrow\infty} \mu_{n+m}...\mu_{n+1}\mu_n.$$

Consequently,

$$(1.4) \qquad \begin{aligned} \lambda_{n+1} \circ \mu_n &= \left(\lim_{m\longrightarrow\infty} \mu_{n+m}...\mu_{n+1}\right) \circ \mu_n \\ &= \lim_{m\longrightarrow\infty} \mu_{n+m}...\mu_{n+1}\mu_n = \lambda_n. \end{aligned}$$

The function $\lambda_n$ is a uniform limit of continuous functions and thus continuous. Also, it is nondecreasing and fixes 0, and $\tau_\alpha$. If we prove that $P(\lambda_n)$ is finite, it will then follow by Remark 2 that $\lambda_n$ is a strictly increasing lipeomorphism and hence is a member of $\Delta_{\alpha,\alpha}$. By (1.3) and (P3),

$$\begin{aligned}
|\log \frac{\mu_{n+m}...\mu_{n+1}\mu_n(t)-\mu_{n+m}...\mu_{n+1}\mu_n(s)}{t-s}| &\leq P(\mu_{n+m}...\mu_{n+1}\mu_n)\\
\leq P(\mu_{n+m})+...+P(\mu_{n+1})+P(\mu_n) &< \frac{1}{2^n}+\frac{1}{2^{n+1}}+...+\frac{1}{2^{n+m}}\\
=\frac{1}{2^n}\left\{1+\frac{1}{2}+...+\frac{1}{2^m}\right\} &\leq \frac{1}{2^n}\{2\}=\frac{1}{2^{n-1}}.
\end{aligned}$$

Taking the limit as $m\longrightarrow\infty$ gives

$$\lim_{m\longrightarrow\infty}|\log \frac{\mu_{n+m}...\mu_{n+1}\mu_n(t)-\mu_{n+m}...\mu_{n+1}\mu_n(s)}{t-s}|=|\log\frac{\lambda_n(t)-\lambda_n(s)}{t-s}|\leq\frac{1}{2^{n-1}}.$$

And this shows that $P(\lambda_n)\leq\frac{1}{2^{n-1}}$. In particular, $P(\lambda_n)$ is finite and $\lambda_n\in\Delta_{\alpha,\alpha}$. Now (1.4) is equivalent to $\lambda_{n+1}^{-1}=\mu_n\circ\lambda_n^{-1}$. Therefore, by (1.2),

$$\begin{aligned}
\sup_{t\in[0,\tau_\alpha]} d_s\left(g_n(\lambda_n^{-1}t),g_{n+1}(\lambda_{n+1}^{-1}t)\right) &= \sup_{t\in[0,\tau_\alpha]} d_s\left(g_n(\lambda_n^{-1}t),g_{n+1}(\mu_n\lambda_n^{-1}t)\right)\\
&=\sup_{\omega\in[0,\tau_\alpha]} d_s\left(g_n(\omega),g_{n+1}(\mu_n\omega)\right)<\frac{1}{2^n}.
\end{aligned}$$

It follows that the sequence of functions $g_n(\lambda_n^{-1}t)$, which are elements of $C_s([0,\tau_\alpha])$, is uniformly Cauchy under the metric $d_s$, which is complete, and hence converges uniformly to a limit function $g(t)$. Also, it is clear that $g$ is continuous and therefore an element of $C_s([0,\tau_\alpha])$. Finally, since

$$\sup_{t\in[0,\tau_\alpha]} d_s\left(g_n(t),g(\lambda_n t)\right)=\sup_{t\in[0,\tau_\alpha]} d_s\left(g_n(\lambda_n^{-1}t),g(t)\right)\longrightarrow 0$$

and $P(\lambda_n)\longrightarrow 0$, we have $\hat{d}(g_n,g)\longrightarrow 0$. □

*Remark* 16. Suppose that $g_n(\lambda_n^{-1}t)$ does go uniformly to a limit $g(t)$. By (1.2), $d_s\left(g_n(\mu_n^{-1}\lambda_{n+1}^{-1}t),g_{n+1}(\lambda_{n+1}^{-1}t)\right)<\frac{1}{2^n}$, which is the same thing as $g_n(\mu_n^{-1}\lambda_{n+1}^{-1}t)$ is within $\frac{1}{2^n}$ of $g_{n+1}(\lambda_{n+1}^{-1}t)$, and therefore it, like $g_n(\lambda_n^{-1}t)$, must go uniformly to $g(t)$. This suggests trying to choose $\lambda_n$ in such a way that $g_n(\mu_n^{-1}\lambda_{n+1}^{-1}t)$ is in fact identically equal to $g_n(\lambda_n^{-1}t)$, or $\mu_n^{-1}\lambda_{n+1}^{-1}=\lambda_n^{-1}$, or $\lambda_n=\lambda_{n+1}\mu_n=\lambda_{n+2}\mu_{n+1}\mu_n=$ ..., and this in turn suggests trying to define $\lambda_n$ as an infinitely iterated composition: $\lambda_n=...\mu_{n+1}\mu_n$. According to this idea, $\lambda_n$ should be near the identity for large $n$, just as the tail of a convergent infinite product is near 1.

Note that the proof of Lemma 15 will not work for the penalty function $P(\lambda)=\|\lambda-I\|$. This is because $\sup_t|\lambda_n(t)-t|\leq\frac{1}{2^{n-1}}$ does not imply that $\lambda_n$ is strictly increasing and therefore we cannot assume that the limit function $\lambda_n=\lim_{m\longrightarrow\infty}\mu_{n+m}...\mu_{n+1}\mu_n$ will still be invertible.

For the completeness of the general metric space $\left(C_s^\tau,\hat{d}\right)$ we will need the following condition (P3′) that will replace (P3).

- **(P3′):** If $\mu_m\in\Delta_{m,m+1}$ for $m=1,2,....$ are such that $\sum_{m=1}^\infty P(\mu_m)<\infty$, then the sequence $(\mu^{(m)})_{m=1}^\infty$ defined by $\mu^{(m)}:=\mu_m\circ\mu_{m-1}\circ.....\circ\mu_1$ converges uniformly to some $\lambda\in\Delta$ and $P(\lambda)\leq\sum_{m=1}^\infty P(\mu_m)$.

Note that (P3′) implies (P3).

*Remark* 17. Suppose $(\Delta, P)$ satisfy (P3′) and $(\mu_n)$ is a sequence satisfying the conditions of (P3′). Define $\lambda_n := \lim_{m\longrightarrow\infty} \mu_{n+m}...\mu_{n+1}\mu_n$. Then $P(\lambda_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Moreover, there exists $\tau_\beta; 0 < \tau_\beta < \infty$ such that $\lambda_n \in \Delta_{n,\beta}$ and $\tau_n \longrightarrow \tau_\beta$. Hence, $\lambda_n : [0, \tau_n] \longrightarrow [0, \lim_{m\longrightarrow\infty} \tau_{n+m+1}] = [0, \tau_\beta]$ and $\tau_{n+m+1} \leq M$ for some $M \in \mathbb{R}$.

*Proof.* By (P3′), $P(\lambda_n) \leq \sum_{m=n}^{\infty} P(\mu_m)$. Since $\sum_{m=1}^{\infty} P(\mu_m) < \infty$,

$$\sum_{m=n}^{\infty} P(\mu_m) \longrightarrow 0$$

as $n \longrightarrow \infty$. Thus, $P(\lambda_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Also the other part follows from $(\mu_n)$ satisfy (P3′), which in turn yields, $\lambda_n \in \Delta$. i.e there exist $\tau_{\beta_n}; 0 < \tau_{\beta_n} < \infty$ such that $\lambda_n \in \Delta_{n,\beta_n}$. Then, Since $\tau_n = \mu_{n-1}...\mu_1(\tau_1)$,

$$\lambda_n(\tau_n) = \lim_{m\longrightarrow\infty} \mu_{n+m}...\mu_{n+1}\mu_n(\tau_n) = \lim_{m\longrightarrow\infty} \mu_{n+m}...\mu_1(\tau_1) = \lambda_1(\tau_1),$$

for each $n$, so $\tau_{\beta_n} = \tau_\beta := \lambda_1(\tau_1)$ for all $n$. Furthermore, $\tau_n = \mu_{n-1}...\mu_1(\tau_1)$, so $\tau_n \longrightarrow \tau_\beta$. Also,

$$\lambda_n(\tau_n) = \lim_{m\longrightarrow\infty} \mu_{n+m}...\mu_{n+1}\mu_n(\tau_n) = \lim_{m\longrightarrow\infty} \tau_{n+m+1} = \lambda_1(\tau_1) = \tau_\beta.$$

Clearly, for fixed $n$, $(\tau_{n+m+1})_m$ converges, so it is bounded. i.e $| \tau_{n+m+1} | \leq M$, for some $M \in \mathbb{R}$. □

**Proposition 18.** *Let* $P : \Delta \longrightarrow [0, \infty)$ *be the function* $P(\lambda) = \sup_{0\leq t<s\leq\tau_\alpha} | \log \frac{\lambda(s)-\lambda(t)}{s-t} |$. *Let* $\mu_m : [0, \tau_m] \longrightarrow [0, \tau_{m+1}] \in \Delta_{m,m+1}$ *be such that* $\sum_{m=1}^{\infty} P(\mu_m) < \infty$. *Then the following holds:*

(1) There exist $m, M$; $0 < m < M$ such that $m \leq \tau_n \leq M$ for all $n$.
(2) For each $n$, the sequence $\left(\mu_n^{(m)}\right)_{m=1}^{\infty}$ in $\Delta$ defined by

$$\mu_n^{(m)} := \mu_{n+m} \circ ... \circ \mu_{n+1} \circ \mu_n$$

is Cauchy in the uniform metric.
(3) The uniform limit $\lambda_n := \lim_{m\longrightarrow\infty} \mu_n^{(m)}$ is in $\Delta$, $P(\lambda_n) \leq \sum_{m=0}^{\infty} P(\mu_{n+m})$ and consequently $P(\lambda_n) \longrightarrow 0$ as $n \longrightarrow \infty$.

In particular, $(\Delta, P)$ satisfies (P3′).

*Proof.* In order to prove that there exist $m, M$; $0 < m < M$ such that $0 < m \leq \tau_n \leq M$, we consider the following: $P(\mu_n) = \sup_{0\leq t<s\leq\tau_n} | \log \frac{\mu_n(s)-\mu_n(t)}{s-t} |$, and with $t = 0$, $s = \tau_n$, we have

$$| \log \frac{\mu_n(\tau_n) - \mu_n(0)}{\tau_n - 0} | = | \log(\tau_{n+1}) - \log(\tau_n) | \leq P(\mu_n).$$

Since $P(\mu_n)$ is summable, $(\log(\tau_n))$ is Cauchy. Hence, $(\log(\tau_n))$ converges and it is bounded. i.e $| \log \tau_n | \leq L$, for some $L \in \mathbb{R}$. Thus, $e^{-L} \leq \tau_n \leq e^L$. Or,

$0 < m \leq \tau_n \leq M$, where $m = \mathrm{e}^{-L}$ and $M = \mathrm{e}^{L}$. Also, $\log(\tau_n) \longrightarrow \zeta$ implies $\tau_n = \mathrm{e}^{\log \tau_n} \longrightarrow \mathrm{e}^{\zeta} := \tau_\beta$. i.e $\lim_{n\longrightarrow\infty} \tau_n = \tau_\beta$ where $0 < \tau_\beta < \infty$. Now to prove $\mu_n^{(m)}$ is Cauchy in the uniform metric we consider

$$\begin{aligned}\sup_{t\in[0,\tau_n]} \mid \mu_n^{(m+1)}(t) - \mu_n^{(m)}(t) \mid &= \sup_{t\in[0,\tau_n]} \mid \mu_{n+m+1}...\mu_{n+1}\mu_n(t) - \mu_{n+m}...\mu_{n+1}\mu_n(t) \mid \\ &= \sup_{s\in[0,\tau_{n+m+1}]} \mid \mu_{n+m+1}(s) - s \mid \\ &\leq \tau_{n+m+1}\left(\mathrm{e}^{P(\mu_{n+m+1})} - 1\right).\end{aligned}$$

Since $\sum_{m=1}^{\infty} P(\mu_m) < \infty$, $\lim_{m\longrightarrow\infty} P(\mu_m) = 0$. Let $N$ be such that $P(\mu_m) \leq \frac{1}{2}$ for all $m \geq N$. Since $\mathrm{e}^x - 1 \leq 2x$ for $0 \leq x \leq \frac{1}{2}$, then for all $m \geq N$ we have

$$\begin{aligned}\sup_{t\in[0,\tau_n]} \mid \mu_n^{(m+1)}(t) - \mu_n^{(m)}(t) \mid &\leq \tau_{n+m+1} 2P(\mu_{n+m+1}) \\ &\leq 2MP(\mu_{n+m+1}).\end{aligned}$$

Since $\sum_{m=1}^{\infty} P(\mu_m) < \infty$, $\sum_{m=1}^{\infty} 2MP(\mu_{n+m+1}) < \infty$. ( i.e $\sum_{m=1}^{\infty} d(x_m, x_{m+1}) < \infty$) . Thus, by Remark 13, $\left(\mu_n^{(m)}\right)_{m=1}^{\infty}$ is uniformly Cauchy and there exists $\lambda_n = \lim_{m\longrightarrow\infty} \mu_n^{(m)}$. Moreover, for the third part, for fixed $n$, the function $\lambda_n$ is a uniform limit of continuous functions and thus continuous. Also, it is nondecreasing and satisfies $\lambda_n(0) = 0$, $\lambda_n(\tau_n) = \lim_{m\to\infty} \mu_{n+m} \circ ... \circ \mu_{n+1} \circ \mu_n(\tau_n) = \lim_{m\to\infty} \tau_{n+m+1} = \tau_\beta$. Since

$$\begin{aligned}\mid \log \frac{\mu_{n+m}...\mu_{n+1}\mu_n(t) - \mu_{n+m}...\mu_{n+1}\mu_n(s)}{t-s} \mid &\leq P(\mu_{n+m}...\mu_{n+1}\mu_n) \\ \leq P(\mu_{n+m}) + ... + P(\mu_{n+1}) + P(\mu_n) &= \sum_{i=n}^{n+m} P(\mu_i),\end{aligned}$$

taking the limit as $m \longrightarrow \infty$ gives

$$\begin{aligned}\lim_{m\longrightarrow\infty} \mid \log \frac{\mu_{n+m}...\mu_{n+1}\mu_n(t) - \mu_{n+m}...\mu_{n+1}\mu_n(s)}{t-s} \mid &= \mid \log \frac{\lambda_n(t) - \lambda_n(s)}{t-s} \mid \\ &\leq \lim_{m\longrightarrow\infty} \sum_{i=n}^{n+m} P(\mu_i) = \sum_{i=n}^{\infty} P(\mu_i).\end{aligned}$$

Thus, $P(\lambda_n) \leq \sum_{i=n}^{\infty} P(\mu_i) < \infty$. Hence, by Remark 2, $\lambda_n \in \Delta$. Since

$$\sum_{i=n}^{\infty} P(\mu_i) \longrightarrow 0$$

as $n \longrightarrow \infty$, $P(\lambda_n) \longrightarrow 0$ as $n \longrightarrow \infty$. □

**Theorem 19.** *Let $(\Delta, P)$ satisfy conditions (D1), (D2), (P1), (P2),(P3′) and (P4). If $(S, d_s)$ is a complete metric space, then the space $\left(C_s^\tau, \hat{d}\right)$ is complete.*

*Proof.* The procedure of the proof is similar to the one for a single $\alpha$ case, Lemma 15, with some differences. Let $(f_k)$ be a Cauchy sequence in $C_s^\tau = \bigcup_\alpha C_s([0, \tau_\alpha])$ under the metric $\hat{d}$ associated to $(\Delta, P)$. It contains a subsequence $(g_n) = (f_{k_n})$ such that $g_n \in C_s([0, \tau_n])$ and $\hat{d}(g_n, g_{n+1}) < \frac{1}{2^n}$. We shall prove that $(g_n)$ is convergent. Now, by $\hat{d}(g_n, g_{n+1}) < \frac{1}{2^n}$, $\Delta$ contains $\mu_n : [0, \tau_n] \longrightarrow [0, \tau_{n+1}]$ such that $d_{\mu_n}(g_n, g_{n+1}) + P(\mu_n) < \frac{1}{2^n}$. This implies

$$d_{\mu_n}(g_n, g_{n+1}) = \sup_{t \in [0,\tau_n]} d_s\left(g_n(t), g_{n+1}(\mu_n t)\right) < \frac{1}{2^n} \tag{1.5}$$

and

$$P(\mu_n) < \frac{1}{2^n}. \tag{1.6}$$

Here we need to find a function $g$ in $C_s([0, \tau_\beta])$, $(\tau_\beta \neq \infty)$, and functions $\lambda_n$ in $\Delta_{n,\beta}$ for which $P(\lambda_n) \longrightarrow 0$ and $g_n(\lambda_n^{-1}(t)) \longrightarrow g(t)$ uniformly in $t$ with respect to the metric $d_s$. Since $P(\mu_n) < \frac{1}{2^n}$, $\sum_{n=1}^{\infty} P(\mu_n) < \infty$. Thus, by (P3′), $\lambda_n = \lim_{m \longrightarrow \infty} \mu_n^{(m)}$ exists for each $n$, $\lambda_n \in \Delta$ and $P(\lambda_n) \leq \sum_{m=n}^{\infty} P(\mu_m) \longrightarrow 0$ as $n \longrightarrow \infty$. Since

$$\lambda_n = \lim_{m \longrightarrow \infty} \mu_{n+m} \cdots \mu_{n+1}\mu_n,$$

$$\lambda_n = \lambda_{n+1} \circ \mu_n. \tag{1.7}$$

Therefore, $\lambda_n$ and $\lambda_{n+1}$ must have the same range. Since $\lambda_n, \lambda_{n+1} \in \Delta$, $\lambda_n \in \Delta_{n,\beta}$ for some fixed $\tau_\beta$. This means that the range of $\lambda_n : [0, \tau_n] \longrightarrow [0, \tau_\beta]$ is independent of $n$, and this will make it possible to evaluate the supremum below. Now (1.7) is equivalent to $\lambda_{n+1}^{-1} = \mu_n \lambda_n^{-1}$. Therefore, by (1.5)

$$\sup_{t \in [0,\tau_\beta]} d_s\left(g_n(\lambda_n^{-1}t), g_{n+1}(\lambda_{n+1}^{-1}t)\right) = \sup_{\omega \in [0,\tau_n]} d_s\left(g_n(\omega), g_{n+1}(\mu_n \omega)\right) < \frac{1}{2^n}.$$

It follows that the functions $g_n(\lambda_n^{-1}t)$, which are elements of $C_s([0, \tau_\beta])$, are uniformly Cauchy and hence converge uniformly to a limit function $g(t)$. Also, it is clear that $g$ is continuous and therefore an element of $C_s([0, \tau_\beta])$. Finally, since

$$\sup_{t \in [0,\tau_n]} d_s\left(g_n(t), g(\lambda_n t)\right) \longrightarrow 0$$

and $P(\lambda_n) \longrightarrow 0$, we have $\hat{d}(g_n, g) \longrightarrow 0$. □

Part of the nice properties established in proposition 18 may already be derived from the following general property which we call (P5):

**(P5):** There exists a continuous function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$\lim_{x \longrightarrow 0^+} \frac{f(x)}{x} = L$$

(exists) and such that for $\lambda \in \Delta_{\alpha,\beta}$

$$\sup_{t \in [0,\tau_\alpha]} \mid \lambda(t) - t \mid \leq \tau_\alpha f(P(\lambda)).$$

*Remark* 20. The function $P(\lambda) = \sup_{0 \leq t < s \leq \tau_\alpha} \mid \log \frac{\lambda(s) - \lambda(t)}{s - t} \mid$ satisfies (P5).

*Proof.* Take $f(x) = e^x - 1$, the result follows immediately from Lemma 14. □

Note that (P5) implies (P4). Moreover, we have the following result towards (P3′).

**Lemma 21.** *Assume* $(\Delta, P)$ *satisfies the condition (P5). Let* $\mu_m : [0, \tau_m] \longrightarrow [0, \tau_{m+1}] \in \Delta_{m,m+1}$ *be such that* $\sum_{m=1}^{\infty} P(\mu_m) < \infty$. *Then the following holds:*

(1) There exist $m, M$; $0 < m < M$ such that $m \leq \tau_n \leq M$ for all $n$.
(2) For each $n$, the sequence $\left(\mu_n^{(m)}\right)_{m=1}^{\infty}$ in $\Delta$ defined by
$$\mu_n^{(m)} := \mu_{n+m} \circ ... \circ \mu_{n+1} \circ \mu_n$$
is Cauchy in the uniform metric.
(3) If $\lambda_n := \lim_{m \longrightarrow \infty} \mu_n^{(m)}$ is in $\Delta$ for some $n = n_0 \in \mathbb{N}$, then $\lambda_n \in \Delta$ for all $n \geq n_0$.

*Proof.* Since $\lim_{x \longrightarrow 0^+} \frac{f(x)}{x} = L$, $f(x) \leq (L + \epsilon) x$ for sufficiently small $x$ and $f(x) \longrightarrow 0$ as $x \longrightarrow 0$. Since $\sum_{m=1}^{\infty} P(\mu_m) < \infty$, $P(\mu_m) \longrightarrow 0$ as $m \longrightarrow \infty$. Thus, $f(P(\mu_m)) \longrightarrow 0$ as $m \longrightarrow \infty$. Moreover, since $f(P(\mu_m)) \leq (L + \epsilon) P(\mu_m)$ and $\sum_{m=1}^{\infty} P(\mu_m) < \infty$, $\sum_{m=1}^{\infty} f(P(\mu_m)) < \infty$. By (P5),

$$\begin{aligned} \mid \tau_{m+1} - \tau_m \mid &= \mid \mu_m(\tau_m) - \tau_m \mid \\ &\leq \sup_{0 \leq t \leq \tau_m} \mid \mu_m(t) - t \mid \leq \tau_m f(P(\mu_m)). \end{aligned}$$

Thus,

$$\mid \frac{\tau_{m+1}}{\tau_m} - 1 \mid \leq f(P(\mu_m)) \longrightarrow 0$$

as $m \longrightarrow \infty$. Or, $\frac{\tau_{m+1}}{\tau_m} \longrightarrow 1$ as $m \longrightarrow \infty$. Since $\log x \leq x - 1$ for all $x > 0$,

$$\log(\tau_{m+1}) - \log(\tau_m) = \log\left(\frac{\tau_{m+1}}{\tau_m}\right) \leq \frac{\tau_{m+1}}{\tau_m} - 1.$$

Also,

$$\begin{aligned} -(\log(\tau_{m+1}) - \log(\tau_m)) &= \log\left(\frac{\tau_m}{\tau_{m+1}}\right) \leq \frac{\tau_m}{\tau_{m+1}} - 1 \\ &= \frac{\tau_m}{\tau_{m+1}}\left(1 - \frac{\tau_{m+1}}{\tau_m}\right). \end{aligned}$$

Thus,

$$\begin{aligned} \mid \log(\tau_{m+1}) - \log(\tau_m) \mid &\leq \max\left(\frac{\tau_{m+1}}{\tau_m} - 1, \frac{\tau_m}{\tau_{m+1}}\left(1 - \frac{\tau_{m+1}}{\tau_m}\right)\right) \\ &\leq \left(1 + \frac{\tau_m}{\tau_{m+1}}\right) \mid \frac{\tau_{m+1}}{\tau_m} - 1 \mid. \end{aligned}$$

Since $\frac{\tau_m}{\tau_{m+1}} \longrightarrow 1$ as $m \longrightarrow \infty$, $\left(1 + \frac{\tau_m}{\tau_{m+1}}\right) \leq 3$. Hence,

$$\mid \log(\tau_{m+1}) - \log(\tau_m) \mid \leq 3 \mid \frac{\tau_{m+1}}{\tau_m} - 1 \mid \leq 3 f(P(\mu_m)).$$

Since $\sum_{m=1}^{\infty} f(P(\mu_m)) < \infty$, by Remark 13, $(\log \tau_m)_{m=1}^{\infty}$ is Cauchy. Hence, it is convergent and bounded. That is to say, there exist $m, M$; $0 < m < M$ such that

$m \leq \tau_n \leq M$ for all $n$. For the second part,

$$\begin{aligned}\sup_{t\in[0,\tau_n]} \mid \mu_n^{(m+1)}(t) - \mu_n^{(m)}(t) \mid &= \sup_{t\in[0,\tau_n]} \mid \mu_{n+m+1}...\mu_{n+1}\mu_n(t) - \mu_{n+m}...\mu_{n+1}\mu_n(t) \mid \\ &= \sup_{s\in[0,\tau_{n+m+1}]} \mid \mu_{n+m+1}(s) - s \mid \\ &\leq \tau_{n+m+1} f\left(P\left(\mu_{n+m+1}\right)\right).\end{aligned}$$

Since $\sum_{m=1}^{\infty} P(\mu_m) < \infty$, $\lim_{m\longrightarrow\infty} P(\mu_m) = 0$. Let $N$ be such that $P(\mu_m) \leq \delta$ for all $m \geq N$. Then for all $m \geq N$ we have

$$\begin{aligned}\sup_{t\in[0,\tau_n]} \mid \mu_n^{(m+1)}(t) - \mu_n^{(m)}(t) \mid &\leq \tau_{n+m+1}(L+\epsilon) P\left(\mu_{n+m+1}\right) \\ &\leq M(L+\epsilon) P\left(\mu_{n+m+1}\right).\end{aligned}$$

Since $\sum_{m=1}^{\infty} P(\mu_m) < \infty$, $\sum_{m=1}^{\infty} M(L+\epsilon) P\left(\mu_{n+m+1}\right) < \infty$. i.e

$$\sum_{m=1}^{\infty} d(x_m, x_{m+1}) < \infty.$$

Thus, by Remark 13, $\left(\mu_n^{(m)}\right)_{m=1}^{\infty}$ is uniformly Cauchy. Moreover, there exists $\lambda_n = \lim_{m\longrightarrow\infty} \mu_n^{(m)}$. For the third part, let $\lambda_n = \lim_{m\longrightarrow\infty} \mu_n^{(m)}$ be in $\Delta$ for some $n \geq n_0$. Then $\lambda_n^{-1} \in \Delta$. By (1.7), $\lambda_{n+1}^{-1} = \mu_n \lambda_n^{-1}$. So by induction $\lambda_n^{-1} \in \Delta$ for all $n \geq n_0$. Hence, $\lambda_n \in \Delta$ for all $n \geq n_0$. □

*Remark* 22. In the general case, (P5) does not seem to allow to draw any conclusion on the domination of $P(\lambda_n)$ by $\sum_{m=n}^{\infty} P(\mu_m)$ as required in (P3′) unfortunately. Additional properties of $(\Delta, P)$ are required to establish that part.

## 2. Spaces of Sequences in $C_s^\tau$

We denote the space of all sequences of continuous functions from $C_s^\tau$ by $(C_s^\tau)^{\mathbb{N}}$or briefly $\left(C_s^{\mathbb{N}}\right)$ i.e, $\left(C_s^{\mathbb{N}}\right) := \{(f_n) = (f_1, f_2, ....)\,;\, f_i \in C_s([0,\tau_i])\}$. Now, we would like to define a metric on $\left(C_s^{\mathbb{N}}\right)$, for this purpose we will make use of the metric $\hat{d}$. Let $\hat{d}_0(f,g) = \frac{\hat{d}(f,g)}{1+\hat{d}(f,g)}$ then $\hat{d}_0$ is a metric on $C_s^\tau$ equivalent to $\hat{d}$. Moreover, completeness of $C_s^\tau$ under $\hat{d}$ yields that $C_s^\tau$ is complete under $\hat{d}_0$. So, if $(f_n), (g_n) \in \left(C_s^{\mathbb{N}}\right)$ then we define a metric $d$ as follows:

$$d\left((f_n),(g_n)\right) = \sum_{i=1}^{\infty} 2^{-i} \hat{d}_0(f_i, g_i).$$

The subset $\left(C_s^{\mathbb{N}}\right)_\infty$ consists of all sequences $(f_n) \in \left(C_s^{\mathbb{N}}\right)$ such that $\sum_{i=1}^{\infty} \tau_i = \infty$. In fact, $\left(C_s^{\mathbb{N}}\right)_\infty := \left\{(f_n) \in \left(C_s^{\mathbb{N}}\right); \sum_{i=1}^{\infty} \tau_i = \infty\right\}$ is a proper subspace of $\left(C_s^{\mathbb{N}}\right)$. Let $\xi_k \in \left(\left(C_s^{\mathbb{N}}\right), d\right)$ be such that $\xi_k \longrightarrow \xi$. Then $\xi_k$ is a sequence of sequences of functions which could be denoted by $\left(f_k^{(n)}\right)_{k=1}^{\infty}$ with the sequence $f_k^{(n)} \in \left(C_s^\tau, \hat{d}\right)$ and, therefore, by $\xi$ we mean the sequence $(f_1, f_2, ....) = (f_k)$.

The following explanation will be used to avoid any possible confusing by notations. Let $\left(f_k^{(n)}\right)_{k=1}^{\infty} \in \left(C_s^{\mathbb{N}}\right)_\infty$ be such that $\left(f_k^{(n)}\right)_{k=1}^{\infty} \longrightarrow (f_k)$ where $(f_k) \in \left(C_s^{\mathbb{N}}\right)$. We have

$\left(f_k^{(n)}\right) = \left(\left(f_1^{(n)}\right), \left(f_2^{(n)}\right), \left(f_3^{(n)}\right), ..., \left(f_k^{(n)}\right), ...\right) \longrightarrow (f_1, f_2, f_3, ..., f_k, ...) = (f_k)$.

For $k = 1$, we have

$$\left(f_1^{(n)}\right) = f_1^{(n)} = \left(f_1^1, f_1^2, f_1^3, ..., f_1^n, ....\right) \longrightarrow f_1,$$

where $f_1^n \in C_s([0, \tau_1^n])$ for all $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \tau_1^n = \infty$, and $f_1 \in C_s([0, \tau_1])$. By (1.1); the convergence of $\left(f_1^{(n)}\right)$ to $f_1$, $\tau_1^n \longrightarrow \tau_1$. For $k = 2$, we have

$$\left(f_2^{(n)}\right) = \left(f_2^1, f_2^2, f_2^3, ..., f_2^n, ....\right) \longrightarrow f_2,$$

where $f_2^n \in C_s([0, \tau_2^n])$ for all $n \in \mathbb{N}$, $\sum_{n=1}^{\infty} \tau_2^n = \infty$, and $f_2 \in C_s([0, \tau_2])$. Also, $\tau_2^n \longrightarrow \tau_2$. So we have $\tau_1^n \longrightarrow \tau_1$, $\tau_2^n \longrightarrow \tau_2$,..... .Thus, the sequence

$$\left(\tau_1^n, \tau_2^n, \tau_3^n, ...., \tau_k^n, ...\right) \longrightarrow (\tau_1, \tau_2, \tau_3, ...., \tau_k, ....).$$

Since $\sum_{n=1}^{\infty} \tau_k^n = \infty$ for all $k$, $\sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \tau_k^n\right) = \infty$. This also could be obtained from $\left(f_k^{(n)}\right)_{k=1}^{\infty} \in \left(C_s^{\mathbb{N}}\right)_{\infty}$. Hence, $\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \tau_k^n = \infty$ and $(\tau_k^n) \longrightarrow (\tau_k)$.

**Lemma 23.** *The space* $\left(\left(C_s^{\mathbb{N}}\right), d\right)$ *is complete, whenever* $\left(C_s^{\tau}, \hat{d}\right)$ *is complete.*

*Proof.* The space $\left(\left(C_s^{\mathbb{N}}\right), d\right)$ is the cartesian product of the complete metric spaces $\left(C_s^{\tau}, \hat{d}\right)$ and hence it is complete. □

**Lemma 24.** *The space* $\left(C_s^{\mathbb{N}}\right)_{\infty}$ *is not a closed subspace of* $\left(C_s^{\mathbb{N}}\right)$.

*Proof.* We give an example of a sequence $\left(f_k^{(n)}\right)_{k=1}^{\infty} \in \left(C_s^{\mathbb{N}}\right)_{\infty}$ such that $\left(f_k^{(n)}\right) \longrightarrow (f_k)$ and $(f_k) \notin \left(C_s^{\mathbb{N}}\right)_{\infty}$. Let $\left(f_k^{(n)}\right) \in C_s\left(\left[0, \frac{1}{n^2} + \frac{1}{2^k}\right]\right)$ be such that $\left(f_k^{(n)}\right) \longrightarrow (f_k)$ for some $(f_k) \in \left(C_s^{\mathbb{N}}\right)$. Then $\left(f_k^{(n)}\right) \in \left(C_s^{\mathbb{N}}\right)_{\infty}$ since $\sum_k \sum_n \frac{1}{n^2} + \frac{1}{2^k} = \infty$. But $(f_k) \notin \left(C_s^{\mathbb{N}}\right)_{\infty}$ since $\sum_k \tau_k = \sum_k \frac{1}{2^k} \neq \infty$. □

Recall that a closed subspace of a complete metric space is complete. Conversely, a complete subspace of a metric space is closed. [Lemma 7, Dunford and Schwartz, page 20].

**Lemma 25.** *The space* $\left(\left(C_s^{\mathbb{N}}\right)_{\infty}, d\right)$ *is not complete.*

*Proof.* The subset $\left(C_s^{\mathbb{N}}\right)_{\infty}$ is not a closed subset of $\left(\left(C_s^{\mathbb{N}}\right), d\right)$ and hence it is not complete. □

**Proposition 26.** *If* $\left(C_s^{\tau}, \hat{d}\right)$ *is separable, then the space* $\left(\left(C_s^{\mathbb{N}}\right), d\right)$ *is separable.*

*Proof.* The space $\left(\left(C_s^{\mathbb{N}}\right), d\right)$ is the cartesian product of the separable metric spaces $\left(C_s^{\tau}, \hat{d}\right)$ and hence it is separable. Indeed,

$$\left(C_s^{\mathbb{N}}\right) = \prod_{n=1}^{\infty} C_s^{\tau}.$$

To verify this, let $M$ be a countable dense subset of $C_s^\tau$. Fix $f_0 \in M$ and define the set $\mathfrak{M} \in \left(C_s^{\mathbb{N}}\right)$ as follows:

$$\mathfrak{M} = \left\{(f_1, f_2, f_3, ..., f_m, f_0, f_0, f_0, ...) : f_i \in M\, \forall i, m \in \mathbb{N}\right\}.$$

We claim that the set is a countable dense subset of the the space $\left(\left(C_s^{\mathbb{N}}\right), d\right)$. Countability follows from

$$\mathfrak{M} = \bigcup_{m\in\mathbb{N}} \left\{(f_1, f_2, f_3, ..., f_m, f_0, f_0, f_0, ...) : f_i \in M\, \forall i\right\},$$

which has the same cardinality as $\bigcup_{m\in\mathbb{N}} M^m$. For the density, let $(g_k)_{k=1}^\infty \in \left(C_s^{\mathbb{N}}\right)$. Need to show that $(g_k)_{k=1}^\infty$ could be approximated by elements of $\mathfrak{M}$.

$$(g_k) = (g_1, g_2, g_3, ...).$$

For $k = 1$, we have for $g_1$: there exists $g_1^{(n)} \in M$ such that $d_0\left(g_1^{(n)}, g_1\right) \longrightarrow 0$ as $n \longrightarrow \infty$. This is because $C_s^\tau$ is separable. For $k = 2$, we have for $g_2$: there exists $g_2^{(n)} \in M$ such that $d_0\left(g_2^{(n)}, g_2\right) \longrightarrow 0$ as $n \longrightarrow \infty$. In general, for $g_k$ there exists $g_k^{(n)} \in M$ such that $d_0\left(g_k^{(n)}, g_k\right) \longrightarrow 0$ as $n \longrightarrow \infty$. Thus, for any sequence $(g_k)_{k=1}^\infty \in \left(C_s^{\mathbb{N}}\right)$ there exists a sequence of sequences of functions $\left(g_k^{(n)}\right)_{k=1}^\infty \in M$ (note that $\left(g_k^{(n)}\right)_{k=1}^\infty \in \left(C_s^{\mathbb{N}}\right)$) that satisfies

$$d\left(\left(g_k^{(n)}\right), (g_k)\right) = \sum_{k=1}^\infty 2^{-k}\hat{d}_0\left(g_k^{(n)}, g_k\right) \longrightarrow 0$$

as $n \longrightarrow \infty$. If we could choose $\left(f_k^{(n)}\right) \in \mathfrak{M}$ in such a way that

$$d\left(\left(f_k^{(n)}\right), \left(g_k^{(n)}\right)\right) \longrightarrow 0$$

as $n \longrightarrow \infty$, then we are done since, then,

$$d\left(\left(f_k^{(n)}\right), (g_k)\right) \leq d\left(\left(f_k^{(n)}\right), \left(g_k^{(n)}\right)\right) + d\left(\left(g_k^{(n)}\right), (g_k)\right) \longrightarrow 0$$

as $n \longrightarrow \infty$. The idea is that we approximate the whole sequence $\left(g_k^{(n)}\right)_{k=1}^\infty$ as a "block" by $\left(f_k^{(n)}\right)_{k=1}^\infty$. To construct the sequence $\left(f_k^{(n)}\right)_{k=1}^\infty$, define each element $f_k^n$ in $\left(f_k^{(n)}\right)_{k=1}^\infty$ as follows:

$$f_k^n = \begin{cases} g_k^n, & k \leq n \\ f_0, & k > n \end{cases},$$

then consider the sequences $f_{(k)}^{n}$ in $\left(f_{(k)}^{n}\right)_{n=1}^{\infty}$ as follows:

$$
\begin{aligned}
f_{(k)}^{1} &= \left(f_{k}^{1}\right)_{k=1}^{\infty} = \left(g_{1}^{1}, f_0, f_0, f_0, ....\right) \in \mathfrak{M} \\
f_{(k)}^{2} &= \left(f_{k}^{2}\right)_{k=1}^{\infty} = \left(g_{1}^{2}, g_{2}^{2}, f_0, f_0, f_0, ...\right) \in \mathfrak{M} \\
f_{(k)}^{3} &= \left(f_{k}^{3}\right)_{k=1}^{\infty} = \left(g_{1}^{3}, g_{2}^{3}, g_{3}^{3}, f_0, f_0, f_0, ...\right) \in \mathfrak{M} \\
f_{(k)}^{n} &= \left(f_{k}^{n}\right)_{k=1}^{\infty} = \left(g_{1}^{n}, g_{2}^{n}, g_{3}^{n}, g_{4}^{n}, ..., g_{n}^{n}, f_0, f_0, f_0, ...\right) \in \mathfrak{M}
\end{aligned}
$$

Then, clearly, $\left(f_{(k)}^{n}\right)_{n=1}^{\infty} \in \mathfrak{M}$. Note that each of the above sequences do not approximate the corresponding sequences in $\left(g_{k}^{(n)}\right)_{k=1}^{\infty}$, i.e $f_{(k)}^{1}$ do not approximate $g_{1}^{(n)}$ and so on, but $\left(f_{(k)}^{n}\right)_{n=1}^{\infty}$ the whole sequence of sequences do approximate the entire sequence $\left(g_{k}^{(n)}\right)_{k=1}^{\infty}$. This is to say, $d\left(\left(f_{k}^{(n)}\right), \left(g_{k}^{(n)}\right)\right) \longrightarrow 0$ as $n \longrightarrow \infty$. To see this

$$
\begin{aligned}
d\left(\left(f_{k}^{(n)}\right), \left(g_{k}^{(n)}\right)\right) &= \sum_{k=1}^{\infty} 2^{-k} \hat{d}_0 \left(f_{k}^{(n)}, g_{k}^{(n)}\right) \\
&= \left(\sum_{k=1}^{\infty} 2^{-k} \hat{d}_0 \left(f_{k}^{n}, g_{k}^{n}\right)\right)_{n=1}^{\infty} \\
&= \sum_{k=1}^{n} 2^{-k} \hat{d}_0 \left(f_{k}^{n}, g_{k}^{n}\right) + \sum_{k=n+1}^{\infty} 2^{-k} \hat{d}_0 \left(f_{k}^{n}, g_{k}^{n}\right) \\
&\leq \sum_{k=n+1}^{\infty} 2^{-k} \longrightarrow 0
\end{aligned}
$$

as $n \longrightarrow \infty$. □

**Lemma 27.** *If $\left(C_s^{\tau}, \hat{d}\right)$ is separable, then the space $\left(\left(C_s^{\mathbb{N}}\right)_{\infty}, d\right)$ is separable.*

*Proof.* The proof is similar to the one for the space $\left(\left(C_s^{\mathbb{N}}\right), d\right)$, Proposition 26, with slight differences. We proceed as above. The interval on which $f_0$ is defined, however, is chosen to be constant and thus $\sum_{i=1}^{\infty} \tau_i = \infty$. □

*Remark* 28. Every incomplete metric space $(X, d)$ can be isometrically embedded as a dense complete metric space $\left(\tilde{X}, \tilde{d}\right)$. In other words; there exists a one to one function $f : X \longrightarrow \tilde{X}$ such that $d(x, y) = \tilde{d}(f(x), f(y))$ for all $x, y \in X$.

**Lemma 29.** *$\left(C_s^{\mathbb{N}}\right)_{\infty}$ is dense in $\left(\left(C_s^{\mathbb{N}}\right), d\right)$. In particular, when $\left(C_s^{\tau}, \hat{d}\right)$ is complete, then $\left(\left(C_s^{\mathbb{N}}\right), d\right)$ is a completion of the space $\left(\left(C_s^{\mathbb{N}}\right)_{\infty}, d\right)$.*

*Proof.* From the proof of Proposition 26, $\mathfrak{M}$ is dense in $\left(\left(C_s^{\mathbb{N}}\right), d\right)$ and

$$\mathfrak{M} \subseteq \left(C_s^{\mathbb{N}}\right)_{\infty} \subseteq \left(C_s^{\mathbb{N}}\right),$$

so $\left(C_s^{\mathbb{N}}\right)_{\infty}$ is dense in $\left(C_s^{\mathbb{N}}\right)$. Thus, when $\left(C_s^{\tau}, \hat{d}\right)$ is complete, it follows from Lemma 23 that $\left(\left(C_s^{\mathbb{N}}\right), d\right)$ is a completion of $\left(C_s^{\mathbb{N}}\right)_{\infty}$. □

## 3. Relation to Cadlag functions

**Definition 30.** We define a concatenation map $\gamma$, from $\left(C_s^{\mathbb{N}}\right)_\infty$ into the space of all cadlag functions $D_s\left([0,\infty)\right)$ as follows:

$$\gamma(f_1, f_2, .....)(t) := f_k(t), \tau_1 + ... + \tau_{k-1} \leq t < \tau_1 + .... + \tau_k,$$

i.e

$$\gamma : \left(C_s^{\mathbb{N}}\right)_\infty \longrightarrow D_s\left([0,\infty)\right).$$

It is interesting to observe that the concatenation map is not injective. This could be easily seen from the following example:

**Example 31.** Put $f_n\left(t\right) = 7$, for $t \in [0, 2^n]$, $g_n\left(t\right) = 7$, for $t \in [0,1]$. Then clearly $f_n \in C_s\left([0,2^n]\right)$, $g_n \in C_s\left([0,1]\right)$, $(f_n) \neq (g_n)$ and $\gamma\left((f_n)\right) = \gamma\left((g_n)\right)$, namely the constant function 7 in $D_s\left([0,\infty)\right)$.

Continuity or measurability properties of the concatenation map are not yet completely clear. It will be the topic of further investigation in subsequent work.

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