A copolymer near a selective interface: variational characterization of the free energy

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Abstract

In this paper we consider a random copolymer near a selective interface separating two solvents. The configurations of the copolymer are directed paths that can make i.i.d. excursions of finite length above and below the interface. The excursion length distribution is assumed to have a tail that is logarithmically equivalent to a power law with exponent $\alpha \geq 1$. The monomers carry i.i.d. real-valued types whose distribution is assumed to have zero mean, unit variance, and a finite moment generating function. The interaction Hamiltonian rewards matches and penalizes mismatches of the monomer types and the solvents, and depends on two parameters: the interaction strength $\beta \geq 0$ and the interaction bias $h \geq 0$. We are interested in the behavior of the copolymer in the limit as its length tends to infinity.

The quenched free energy per monomer $(\beta, h) \mapsto g^{\text{que}}(\beta, h)$ has a phase transition along a quenched critical curve $\beta \mapsto h_c^{\text{que}}(\beta)$ separating a localized phase, where the copolymer stays close to the interface, from a delocalized phase, where the copolymer wanders away from the interface. We derive *variational formulas* for both these quantities. We compare these variational formulas with their analogues for the annealed free energy per monomer $(\beta, h) \mapsto g^{\text{ann}}(\beta, h)$ and the annealed critical curve $\beta \mapsto h_c^{\text{ann}}(\beta)$, both of which are explicitly computable. This comparison leads to:

(1) A proof that $h_c^{\text{ann}}(\beta/\alpha) < h_c^{\text{que}}(\beta) < h_c^{\text{ann}}(\beta)$ for all $\alpha > 1$ and $\beta > 0$.

(2) A proof that $g^{\text{que}}(\beta, h) < g^{\text{ann}}(\beta, h)$ for all $\alpha \geq 1$ and (β, h) in the annealed localized phase.

(3) An estimate of the total number of times the copolymer visits the interface in the interior of the quenched delocalized phase.

(4) An identification of the asymptotic frequency at which the copolymer visits the interface in the quenched localized phase.

The copolymer model has been studied extensively in the literature. The goal of the present paper is to open up a window with a variational view and to settle a number of open problems.

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1 Introduction and main results

In Section 1.1 we define the model. In Sections 1.2 and 1.3 we define the quenched and the annealed free energy and critical curve. In Section 1.4 we state our main results, while in Section 1.5 we place these results in the context of earlier work. For more background and key results in the literature, we refer the reader to Giacomin [19], Chapters 6–8, and den Hollander [20], Chapter 9.

1.1 A copolymer near a selective interface

Let $\omega = (\omega_k)_{k\in\mathbb{N}}$ be i.i.d. random variables with a probability distribution ν on $\mathbb{R}$ having zero mean and unit variance:

$$\int_{\mathbb{R}} x\,\nu(dx) = 0, \qquad \int_{\mathbb{R}} x^2\,\nu(dx) = 1, \tag{1.1}$$

and a finite cumulant generating function:

$$M(\lambda) = \log \int_{\mathbb{R}} e^{-\lambda x}\,\nu(dx) < \infty \qquad \forall\,\lambda\in\mathbb{R}. \tag{1.2}$$

Let

$$\Pi = \big\{\pi = (k,\pi_k)_{k\in\mathbb{N}_0}\colon\ \pi_0 = 0,\ \pi_k\in\mathbb{Z}\ \forall\,k\in\mathbb{N}\big\}. \tag{1.3}$$

denote the set of infinite directed paths on $\mathbb{N}_0\times\mathbb{Z}$ (with $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$). Fix $n\in\mathbb{N}_0$ and $\beta,h\geq 0$. For given ω, let

$$H_n^{\beta,h,\omega}(\pi) = -\beta\sum_{k=1}^{n}(\omega_k + h)\,\mathrm{sign}(\pi_{k-1},\pi_k), \qquad \pi\in\Pi, \tag{1.4}$$

be the n-step Hamiltonian on Π, and let

$$P_n^{\beta,h,\omega}(\pi) = \frac{1}{Z_n^{\beta,h,\omega}}\,e^{-H_n^{\beta,h,\omega}(\pi)}\,P(\pi), \qquad \pi\in\Pi, \tag{1.5}$$

be the n-step path measure on Π, where P is any probability distribution on Π under which the excursions away from the interface are i.i.d., lie with equal probability above and below the interface, and have a length whose probability distribution ρ on $\mathbb{N}$ has infinite support and a *polynomial tail*

$$\lim_{\substack{n\to\infty\\ \rho(n)>0}} \frac{\log\rho(n)}{\log n} = -\alpha \text{ for some } \alpha\geq 1. \tag{1.6}$$

Note that the Hamiltonian in (1.4) only depends on the signs of the excursions and on their starting and ending points in ω, not on their shape.

Example. For the special case where ν is the binary distribution $\nu(-1) = \nu(+1) = \frac{1}{2}$ and P is simple random walk on $\mathbb{Z}$, the above definitions have the following interpretation (see Fig. 1). Think of $\pi\in\Pi$ in (1.3) as the path of a directed copolymer on $\mathbb{N}_0\times\mathbb{Z}$, consisting of monomers represented by the edges (π_{k-1},π_k), $k\in\mathbb{N}$, pointing either north-east of south-east. Think of the lower half-plane as water and the upper half-plane as oil. The monomers are labeled by ω, with $\omega_k = -1$ indicating that monomer k is hydrophilic and $\omega_k = +1$ that it is hydrophobic. Both types occur with density $\frac{1}{2}$. The factor $\mathrm{sign}(\pi_{k-1},\pi_k)$ in (1.4) equals -1 or $+1$ depending on whether monomer k lies in the water or in the oil. The interaction Hamiltonian in (1.4) therefore rewards matches and penalizes mismatches of the monomer types and the solvents. The parameter β is the *interaction strength* (or inverse temperature), the parameter h plays the role of

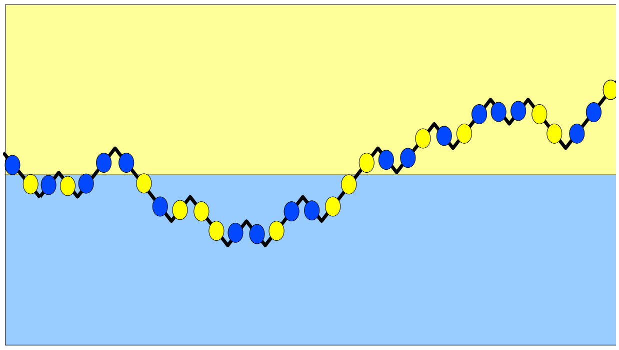


Figure 1: A directed copolymer near a linear interface. Oil in the upper half plane and hydrophobic monomers in the polymer chain are shaded light, water in the lower half plane and hydrophilic monomers in the polymer chain are shaded dark. (Courtesy of N. Pétrélis.)

the *interaction bias*: $h = 0$ corresponds to the hydrophobic and hydrophilic monomers interacting equally strongly, while $h = 1$ corresponds to the hydrophilic monomers not interacting at all. The probability distribution of the copolymer given ω is the quenched Gibbs distribution in (1.5). For simple random walk the support of ρ is $2\mathbb{N}$ and the exponent is $\alpha = \frac{3}{2}$: $\rho(2n) \sim 1/\pi^{1/2}n^{3/2}$ as $n \to \infty$ (Spitzer [23], Section 1).

1.2 Quenched free energy and critical curve

The model in Section 1.1 was introduced in Garel, Huse, Leibler and Orland [14]. It was shown in Bolthausen and den Hollander [7] that for every $\beta, h \geq 0$ the *quenched free energy* per monomer

$$f^{\text{que}}(\beta,h) = \lim_{n\to\infty} \frac{1}{n} \log Z_n^{\beta,h,\omega} \qquad \text{exists, is finite and is constant } \omega\text{-a.s.} \tag{1.7}$$

It was further noted that

$$f^{\text{que}}(\beta,h) \geq \beta h. \tag{1.8}$$

This lower bound comes from the strategy where the path spends all of its time above the interface, i.e., $\pi_k > 0$ for $1 \leq k \leq n$. Indeed, in that case $\text{sign}(\pi_{k-1},\pi_k) = +1$ for $1 \leq k \leq n$, resulting in $H_n^{\beta,h,\omega}(\pi) = -\beta h n[1+o(1)]$ ω-a.s. as $n \to \infty$ by the strong law of large numbers for ω (recall (1.1)). Since $P(\{\pi \in \Pi\colon\ \pi_k > 0 \text{ for } 1 \leq k \leq n\}) = \sum_{k>n} \rho(n) = n^{1-\alpha+o(1)}$ as $n \to \infty$ by (1.6, the cost of this strategy under P is negligible on an exponential scale.

In view of (1.8), it is natural to introduce the *quenched excess free energy*

$$g^{\text{que}}(\beta,h) = f^{\text{que}}(\beta,h) - \beta h, \tag{1.9}$$

to define the two phases

$$\begin{aligned} \mathcal{D}^{\text{que}} &= \{(\beta,h)\colon\ g^{\text{que}}(\beta,h) = 0\},\\ \mathcal{L}^{\text{que}} &= \{(\beta,h)\colon\ g^{\text{que}}(\beta,h) > 0\}, \end{aligned} \tag{1.10}$$

and to refer to $\mathcal{D}^{\text{que}}$ as the *quenched delocalized phase*, where the strategy of staying above the interface is optimal, and to $\mathcal{L}^{\text{que}}$ as the *quenched localized phase*, where this strategy is not optimal. The presence of these two phases is the result of a competition between entropy and energy: by staying close to the interface the copolymer looses entropy, but it gains energy because it can more easily switch between the two sides of the interface in an attempt to place as many monomers as possible in their preferred solvent.

General convexity arguments show that $\mathcal{D}^{\rm que}$ and $\mathcal{L}^{\rm que}$ are separated by a *quenched critical curve* $\beta \mapsto h_c^{\rm que}(\beta)$ given by

$$h_c^{\rm que}(\beta) = \sup\{h \geq 0\colon\, g^{\rm que}(\beta,h) > 0\} = \inf\{h \geq 0\colon\, g^{\rm que}(\beta,h) = 0\}, \qquad \beta \geq 0, \tag{1.11}$$

with the property that $h_c^{\rm que}(0) = 0$, $\beta \mapsto h_c^{\rm que}(\beta)$ is strictly increasing and finite on $[0,\infty)$, and $\beta \mapsto \beta h_c^{\rm que}(\beta)$ is strictly convex on $[0,\infty)$. Moreover, it is easy to check that $\lim_{\beta\to\infty} h_c^{\rm que}(\beta) = \sup[\mathrm{supp}(\nu)]$, the supremum of the support of ν (see Fig. 2).

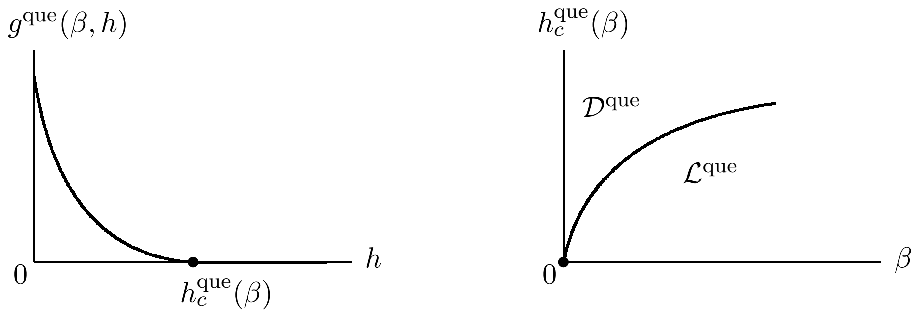


Figure 2: Qualitative pictures of $h \mapsto g^{\rm que}(\beta,h)$ for fixed $\beta > 0$, respectively, $\beta \mapsto h_c^{\rm que}(\beta)$. The quenched critical curve is part of $\mathcal{D}^{\rm que}$.

The following bounds are known for the quenched critical curve:

$$\left(\tfrac{2\beta}{\alpha}\right)^{-1} M\left(\tfrac{2\beta}{\alpha}\right) \leq h_c^{\rm que}(\beta) \leq (2\beta)^{-1} M(2\beta) \qquad \forall\, \beta > 0. \tag{1.12}$$

The upper bound was proved in Bolthausen and den Hollander [7], and comes from an annealed estimate on ω. The lower bound was proved in Bodineau and Giacomin [5], and comes from strategies where the copolymer dips below the interface during rare stretches in ω where the empirical density is sufficiently biased downwards.

Remark: In the literature ρ is typically assumed to be *regularly varying at infinity*, i.e.,

$$\rho(n) = n^{-\alpha} L(n) \text{ for some } \alpha \geq 1 \text{ with } L \text{ slowly varying at infinity.} \tag{1.13}$$

However, the proof of (1.12) in [7] and [5] is easily extended to ρ satisfying the weaker assumption in (1.6). Sometimes results in the literature are derived under assumptions on ν that are stronger than (1.2), e.g. Gaussian or sub-Gaussian tails. Also this is not necessary for (1.12).

1.3 Annealed free energy and critical curve

Recalling (1.3–1.5), (1.7) and (1.9), and using that $\beta \sum_{k=1}^n (\omega_k + h) = \beta h n[1+o(1)]$ ω-a.s. as $n \to \infty$, we see that the quenched excess free energy is given by

$$g^{\rm que}(\beta,h) = \lim_{n\to\infty} \frac{1}{n} \log \widetilde{Z}_n^{\beta,h,\omega} \qquad \omega\text{-a.s.} \tag{1.14}$$

with

$$\widetilde{Z}_n^{\beta,h,\omega} = \sum_{\pi\in\Pi} P(\pi)\, \exp\left[\beta \sum_{k=1}^n (\omega_k + h)\,[\mathrm{sign}(\pi_{k-1},\pi_k) - 1]\right]. \tag{1.15}$$

In this partition sum only the excursions of the copolymer below the interface contribute. The annealed version of the model has partition sum

$$\mathbb{E}(\widetilde{Z}_n^{\beta,h,\omega}) = \sum_{\pi\in\Pi} P(\pi) \prod_{k=1}^{n} \left[1_{\{\mathrm{sign}(\pi_{k-1},\pi_k)=1\}} + e^{M(2\beta)-2\beta h}\, 1_{\{\mathrm{sign}(\pi_{k-1},\pi_k)=-1\}}\right], \tag{1.16}$$

where $\mathbb{E}$ is expectation w.r.t. $\mathbb{P} = \nu^{\otimes\mathbb{N}}$, the probability distribution of ω. The *annealed excess free energy* is therefore given by

$$g^{\mathrm{ann}}(\beta,h) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}(\widetilde{Z}_n^{\beta,h,\omega}). \tag{1.17}$$

(*Note:* In the annealed model the average w.r.t. $\mathbb{P}$ is taken on the partition sum $\widetilde{Z}_n^{\beta,h,\omega}$ in (1.15) rather than on the original partition sum $Z_n^{\beta,h,\omega}$ in (1.5).) The two corresponding phases are

$$\begin{aligned} \mathcal{D}^{\mathrm{ann}} &= \{(\beta,h)\colon\ g^{\mathrm{ann}}(\beta,h) = 0\},\\ \mathcal{L}^{\mathrm{ann}} &= \{(\beta,h)\colon\ g^{\mathrm{ann}}(\beta,h) > 0\}, \end{aligned} \tag{1.18}$$

which are referred to as the *annealed delocalized phase*, respectively, the *annealed localized phase*, and are separated by an *annealed critical curve* $\beta \mapsto h_c^{\mathrm{ann}}(\beta)$ given by

$$h_c^{\mathrm{ann}}(\beta) = \sup\{h \geq 0\colon\ g^{\mathrm{ann}}(\beta,h) > 0\} = \inf\{h \geq 0\colon\ g^{\mathrm{ann}}(\beta,h) = 0\}, \qquad \beta \geq 0. \tag{1.19}$$

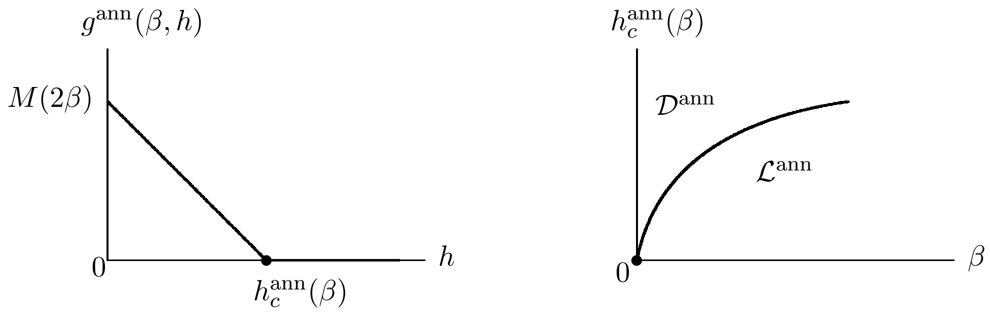


Figure 3: Qualitative picture of $h \mapsto g^{\mathrm{ann}}(\beta,h)$ for fixed $\beta > 0$, respectively, $\beta \mapsto h_c^{\mathrm{ann}}(\beta)$. The annealed critical curve is part of $\mathcal{D}^{\mathrm{ann}}$.

An easy computation based on (1.16) gives that (see Fig. 3)

$$g^{\mathrm{ann}}(\beta,h) = 0 \vee [M(2\beta) - 2\beta h], \qquad \beta, h \geq 0, \tag{1.20}$$

and

$$h_c^{\mathrm{ann}}(\beta) = (2\beta)^{-1} M(2\beta), \qquad \beta > 0. \tag{1.21}$$

Thus, the upper bound in (1.12) equals $h_c^{\mathrm{ann}}(\beta)$, while the lower bound equals $h_c^{\mathrm{ann}}(\beta/\alpha)$.

1.4 Main results

Our variational characterization of the excess free energies and the critical curves are contained in the following theorem.

Theorem 1.1 *Assume* (1.2) *and* (1.6).
(i) For every $\beta, h > 0$*, there are lower semi-continuous, convex and non-increasing functions*

$$\begin{aligned} g &\mapsto S^{\mathrm{que}}(\beta, h; g), \\ g &\mapsto S^{\mathrm{ann}}(\beta, h; g), \end{aligned} \tag{1.22}$$

given by explicit variational formulas, such that

$$\begin{aligned} g^{\mathrm{que}}(\beta, h) &= \inf\{g \in \mathbb{R} \colon\ S^{\mathrm{que}}(\beta, h; g) < 0\}, \\ g^{\mathrm{ann}}(\beta, h) &= \inf\{g \in \mathbb{R} \colon\ S^{\mathrm{ann}}(\beta, h; g) < 0\}. \end{aligned} \tag{1.23}$$

(ii) For every $\beta > 0$*,* $g^{\mathrm{que}}(\beta, h)$ *and* $g^{\mathrm{ann}}(\beta, h)$ *are the unique solutions of the equations*

$$\begin{aligned} S^{\mathrm{que}}(\beta, h; g) &= 0 \quad \text{for } 0 < h \leq h_c^{\mathrm{que}}(\beta), \\ S^{\mathrm{ann}}(\beta, h; g) &= 0 \quad \text{for } h = h_c^{\mathrm{ann}}(\beta). \end{aligned} \tag{1.24}$$

(iii) For every $\beta > 0$*,* $h_c^{\mathrm{que}}(\beta)$ *and* $h_c^{\mathrm{ann}}(\beta)$ *are the unique solutions of the equations*

$$\begin{aligned} S^{\mathrm{que}}(\beta, h; 0) &= 0, \\ S^{\mathrm{ann}}(\beta, h; 0) &= 0. \end{aligned} \tag{1.25}$$

The variational formulas for $S^{\mathrm{que}}(\beta, h; g)$ and $S^{\mathrm{ann}}(\beta, h; g)$ are given in Theorem 3.1, respectively, Theorem 3.2 in Section 3. Figs. 5–8 in Section 3 show how these functions depend on β, h and g, which is crucial for our analysis.

Next we state five corollaries that are consequences of the variational formulas. The first three corollaries are strict inequalities for the excess free energies and the critical curves.

Corollary 1.2 $g^{\mathrm{que}}(\beta, h) < g^{\mathrm{ann}}(\beta, h)$ *for all* $(\beta, h) \in \mathcal{L}^{\mathrm{ann}}$.

Corollary 1.3 *If* $\alpha > 1$*, then* $h_c^{\mathrm{que}}(\beta) < h_c^{\mathrm{ann}}(\beta)$ *for all* $\beta > 0$.

Corollary 1.4 *If* $\alpha > 1$*, then* $h_c^{\mathrm{que}}(\beta) > h_c^{\mathrm{ann}}(\beta/\alpha)$ *for all* $\beta > 0$.

The last two corollaries concern the typical path behavior. Let $\widetilde{\mathcal{P}}_n^{\beta,h,\omega}$ denote the path measure associated with the constrained partition sum $\widetilde{Z}_n^{\beta,h,\omega}$ defined in (1.15). Write $\mathcal{M}_n = |\{1 \leq i \leq n \colon\ \pi_i = 0\}|$ to denote the number of times π returns to the interface up to time n.

Corollary 1.5 *For every* $(\beta, h) \in \mathrm{int}(\mathcal{D}^{\mathrm{que}})$ *and* $c > \alpha/[-S^{\mathrm{que}}(\beta, h; 0)] \in (0, \infty)$*,*

$$\lim_{n\to\infty} \widetilde{\mathcal{P}}_n^{\beta,h,\omega}\left(\mathcal{M}_n \geq c \log n\right) = 0 \qquad \omega - a.s. \tag{1.26}$$

Corollary 1.6 *For every* $(\beta, h) \in \mathcal{L}^{\mathrm{que}}$*,*

$$\lim_{n\to\infty} \widetilde{\mathcal{P}}_n^{\beta,h,\omega}\left(|\tfrac{1}{n}\mathcal{M}_n - C| \leq \varepsilon\right) = 1 \qquad \omega - a.s. \quad \forall \varepsilon > 0, \tag{1.27}$$

where

$$-\frac{1}{C} = \frac{\partial}{\partial g} S^{\mathrm{que}}\big(\beta, h; g^{\mathrm{que}}(\beta, h)\big) \in (-\infty, 0), \tag{1.28}$$

provided this derivative exists. (By convexity, at least the left-derivative and the right-derivative exist.)

1.5 Discussion

1. The main importance of our results in Section 1.4 is that they open up a window on the copolymer model with a variational view. Whereas the results in the literature were obtained with the help of a variety of *estimation techniques*, Theorem 1.1 provides *variational formulas* that are new and explicit. As we will see in Section 3, these variational formulas are not easy to manipulate. However, they provide a natural setting, and are robust in the sense that they can be applied to other polymer models as well, e.g. the pinning model with disorder (Cheliotis and den Hollander [12]). Still other applications involve certain classes of interacting stochastic systems (Birkner, Greven and den Hollander [3]). For an overview, see den Hollander [21].

2. The gap between the excess free energies stated in Corollary 1.2 has never been claimed in the literature, but follows from known results. Fix $\beta > 0$. We know that $h \mapsto g^{\text{ann}}(\beta, h)$ is strictly positive, strictly decreasing and linear on $(0, h_c^{\text{ann}}(\beta)]$, and zero on $[h_c^{\text{ann}}(\beta), \infty)$ (see Fig. 3). We also know that $h \mapsto g^{\text{que}}(\beta, h)$ is strictly positive, strictly decreasing and convex on $(0, h_c^{\text{que}}(\beta)]$, and zero on $[h_c^{\text{que}}(\beta), \infty)$. It was shown in Giacomin and Toninelli [16, 17] that $h \mapsto g^{\text{que}}(\beta, h)$ drops below a quadratic as $h \uparrow h_c^{\text{que}}(\beta)$, i.e., the phase transition is "at least of second order" (see Fig. 2). Hence, the gap is present in a left-neighborhood of $h_c^{\text{que}}(\beta)$. Combining this observation with the fact that $g^{\text{que}}(\beta, h) \le g^{\text{ann}}(\beta, h)$ and $h_c^{\text{que}}(\beta) \le h_c^{\text{ann}}(\beta)$, it follows that the gap is present for all $h \in (0, h_c^{\text{ann}}(\beta))$. *Note:* The above argument crucially relies on the linearity of $h \mapsto g^{\text{ann}}(\beta, h)$ on $(0, h_c^{\text{ann}}(\beta)]$. However, we will see in Section 3 that our proof of Corollary 1.2 is robust and does not depend on this linearity.

3. For a number of years, all attempts in the literature to improve (1.12) had failed. As explained in Orlandini, Rechnitzer and Whittington [22] and Caravenna and Giacomin [8], the reason behind this failure is that any improvement of (1.12) necessarily requires a deep understanding of the global behavior of the copolymer when the parameters are close to the quenched critical curve. Toninelli [24] proved the strict upper bound in Corollary 1.3 with the help of *fractional moment estimates* for unbounded disorder and large β subject to (1.2) and (1.13), and this result was later extended by Bodineau, Giacomin, Lacoin and Toninelli [6] to arbitrary disorder and arbitrary β, again subject to (1.2) and (1.13). The latter paper also proved the strict lower bound in Corollary 1.4 with the help of *appropriate localization strategies* for small β and $\alpha \ge \alpha_0$, where $\alpha_0 \approx 1.801$ (theoretical bound) and $\alpha_0 \approx 1.65$ (numerical bound), which unfortunately excludes the simple random walk example in Section 1.1 for which $\alpha = \frac{3}{2}$. Corollaries 1.3 and 1.4 settle the strict inequalities in full generality subject to (1.2) and (1.6).

4. A point of heated debate has been the slope of the quenched critical curve at $\beta = 0$,

$$\lim_{\beta \downarrow 0} \frac{1}{\beta}\, h_c^{\text{que}}(\beta) = K_c, \tag{1.29}$$

which is believed to be *universal*, i.e., to depend on α alone and to be robust under changes of the fine details of the interaction Hamiltonian. The existence of the limit was proved in Bolthausen and den Hollander [7] for simple random walk, via a Brownian approximation of the copolymer model. This result was extended in Caravenna and Giacomin [10] to ρ satisfying (1.13) with $\alpha \in (1, 2)$ for disorder with a moment generating function that is finite in a neighborhood of the origin. The proof uses a Lévy approximation of the copolymer model. The Lévy copolymer serves as the attractor of a universality class, indexed by the exponent $\alpha \in (1, 2)$. The bounds in (1.12) imply that $K_c \in [\alpha^{-1}, 1]$, and various claims were made in the literature arguing in favor of $K_c = \alpha^{-1}$, respectively, $K_c = 1$. However, in Bodineau, Giacomin, Lacoin and Toninelli [6] it was shown that

$K_c < 1$ for $\alpha > 2$ and $K_c > \alpha^{-1}$ for $\alpha \geq \alpha_0$. For an overview, see Caravenna, Giacomin and Toninelli [11].

5. A numerical analysis carried out in Caravenna, Giacomin and Gubinelli [9] (see also Giacomin [19], Chapter 9) showed that for simple random walk and binary disorder

$$h_c^{\mathrm{que}}(\beta) \approx (2K_c\beta)^{-1} \log\cosh(2K_c\beta) \text{ for moderate } \beta \text{ with } K_c \in [0.82, 0.84]. \tag{1.30}$$

Thus, for this case the quenched critical curve lies "somewhere halfway" between the two bounds in (1.12), and so it remains a challenge to quantify the strict inequalities in Corollaries 1.3 and 1.4. For the upper bound some quantification is offered in Bodineau, Giacomin, Lacoin and Toninelli [6].

6. Because of (1.12), it was suggested that the quenched critical curve possibly depends on the exponent α of ρ alone and not on the fine details of ρ. However, it was shown in Bodineau, Giacomin, Lacoin and Toninelli [6] that for every $\alpha > 1$, $\beta > 0$ and $\epsilon > 0$ there exists a ρ satisfying (1.13) such that $h_c^{\mathrm{que}}(\beta)$ is ϵ-close to the upper bound, which rules out such a scenario. Our variational characterization in Section 3 confirms this observation, and makes it quite evident that the fine details of ρ do indeed matter.

7. Special cases of Corollaries 1.5 and 1.6 were proved in Biskup and den Hollander [4] (for simple random walk and binary disorder) and Giacomin and Toninelli [15, 18] (subject to (1.13), for disorder with a finite moment generating function in a neighborhood of the origin satisfying a Gaussian concentration of measure bound, and under the average quenched measure, i.e., $\mathbb{E}(P_n^{\beta,h,\omega})$). However, no formulas were obtained for the relevant constants.

1.6 Outline

In Section 2 we recall two large deviation principles (LDP's) derived in Birkner [1] and Birkner, Greven and den Hollander [2], which describe the large deviation behavior of the empirical process of words cut out from a random letter sequence according to a random renewal process with exponentially bounded, respectively, polynomial tails. In Section 3 we use these LDP's to prove Theorem 1.1. In Sections 4, 5 and 6 we prove Corollaries 1.2, 1.3 and 1.4, respectively. The proofs of Corollaries 1.5 and 1.6 are given in Section 7. Appendices A–C contain a number of technical estimates that are needed in Section 3.

In Cheliotis and den Hollander [12], the LDP's in [2] were applied to the pinning model with disorder, and variational formulas were derived for the critical curves (not the free energies). The Hamiltonian is similar in spirit to (1.4), except that the disorder is felt only *at* the interface, which makes the pinning model easier than the copolymer model. The present paper borrows ideas from [12]. However, the new challenges that come up are considerable.

2 Large deviation principles

In this section we recall the LDP's from Birkner [1] and Birkner, Greven and den Hollander [2], which are the key tools in the present paper. Section 2.1 introduces the relevant notation, while Sections 2.2 and 2.3 state the annealed, respectively, quenched version of the LDP. Apart from minor modifications, this section is copied from [2]. We repeat it here in order to set the notation and to keep the paper self-contained.

2.1 Notation

Let E be a Polish space, playing the role of an alphabet, i.e., a set of *letters*. Let $\widetilde{E} = \cup_{k\in\mathbb{N}} E^k$ be the set of *finite words* drawn from E, which can be metrized to become a Polish space. Write $\mathcal{P}(E)$ and $\mathcal{P}(\widetilde{E})$ to denote the set of probability measures on E and $\widetilde{E}$.

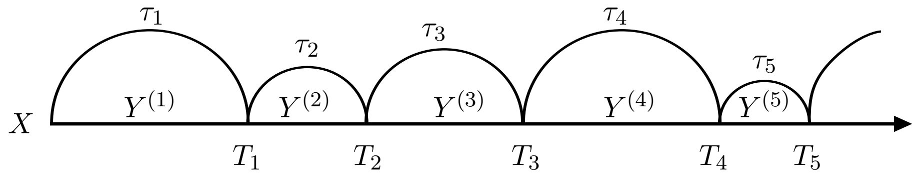


Figure 4: Cutting words out from a sequence of letters according to renewal times.

Fix $\nu \in \mathcal{P}(E)$, and $\rho \in \mathcal{P}(\mathbb{N})$ satisfying (1.6). Let $X = (X_k)_{k\in\mathbb{N}}$ be i.i.d. E-valued random variables with marginal law ν, and $\tau = (\tau_i)_{i\in\mathbb{N}}$ i.i.d. $\mathbb{N}$-valued random variables with marginal law ρ. Assume that X and τ are independent, and write P^* to denote their joint law. Cut words out of the letter sequence X according to τ (see Fig. 4), i.e., put

$$T_0 = 0 \quad \text{and} \quad T_i = T_{i-1} + \tau_i, \quad i \in \mathbb{N}, \tag{2.1}$$

and let

$$Y^{(i)} = \big(X_{T_{i-1}+1}, X_{T_{i-1}+2}, \dots, X_{T_i}\big), \quad i \in \mathbb{N}. \tag{2.2}$$

Under the law P^*, $Y = (Y^{(i)})_{i\in\mathbb{N}}$ is an i.i.d. sequence of words with marginal law $q_{\rho,\nu}$ on $\widetilde{E}$ given by

$$\begin{aligned} q_{\rho,\nu}\big(dx_1,\dots,dx_n\big) &= P^*\big(Y^{(1)} \in (dx_1,\dots,dx_n)\big) \\ &= \rho(n)\,\nu(dx_1)\times\cdots\times\nu(dx_n), \qquad n\in\mathbb{N},\, x_1,\dots,x_n \in E. \end{aligned} \tag{2.3}$$

We define ρ_g as the tilted version of ρ given by

$$\rho_g(n) = \frac{e^{-gn}\rho(n)}{\mathcal{N}(g)}, \quad n \in \mathbb{N}, \qquad \mathcal{N}(g) = \sum_{n\in\mathbb{N}} e^{-gn}\rho(n), \quad g \in [0,\infty). \tag{2.4}$$

Note that if $g > 0$, then ρ_g has an *exponentially bounded tail*. For $g = 0$ we write ρ instead of ρ_0. We write P_g^* and $q_{\rho_g,\nu}$ for the analogues of P^* and $q_{\rho,\nu}$ when ρ is replaced by ρ_g defined in (2.4).

The reverse operation of *cutting* words out of a sequence of letters is *glueing* words together into a sequence of letters. Formally, this is done by defining a *concatenation* map κ from $\widetilde{E}^{\mathbb{N}}$ to $E^{\mathbb{N}}$. This map induces in a natural way a map fsrom $\mathcal{P}(\widetilde{E}^{\mathbb{N}})$ to $\mathcal{P}(E^{\mathbb{N}})$, the sets of probability measures on $\widetilde{E}^{\mathbb{N}}$ and $E^{\mathbb{N}}$ (endowed with the topology of weak convergence). The concatenation $q_{\rho,\nu}^{\otimes\mathbb{N}} \circ \kappa^{-1}$ of $q_{\rho,\nu}^{\otimes\mathbb{N}}$ equals $\nu^{\mathbb{N}}$, as is evident from (2.3).

Let $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ be the set of probability measures on $\widetilde{E}^{\mathbb{N}}$ that are invariant under the left-shift $\widetilde{\theta}$ acting on $\widetilde{E}^{\mathbb{N}}$. For $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$, let $H(Q \mid q_{\rho,\nu}^{\otimes\mathbb{N}})$ be the *specific relative entropy of* Q *w.r.t.* $q_{\rho,\nu}^{\otimes\mathbb{N}}$ defined by

$$H(Q \mid q_{\rho,\nu}^{\otimes\mathbb{N}}) = \lim_{N\to\infty} \frac{1}{N}\, h(\pi_N Q \mid q_{\rho,\nu}^N), \tag{2.5}$$

where $\pi_N Q \in \mathcal{P}(\widetilde{E}^N)$ denotes the projection of Q onto the first N words, $h(\,\cdot \mid \cdot\,)$ denotes relative entropy, and the limit is non-increasing. The following lemma relates the specific relative entropies of Q w.r.t. $q_{\rho,\nu}^{\otimes\mathbb{N}}$ and $q_{\rho_g,\nu}^{\otimes\mathbb{N}}$.

Lemma 2.1 *For $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ and $g \in [0,\infty)$,*

$$H(Q \mid q_{\rho_g,\nu}^{\otimes\mathbb{N}}) = H(Q \mid q_{\rho,\nu}^{\otimes\mathbb{N}}) \ + \log\mathcal{N}(g) + g m_Q \tag{2.6}$$

with $\mathcal{N}(g) \in (0,1]$ defined in (2.4) and $m_Q = E_Q(\tau_1) \in [1,\infty]$ the average word length under Q (E_Q denotes expectation under the law Q and τ_1 is the length of the first word).

Proof. Observe from (2.4) that

$$\begin{aligned} h(\pi_N Q \mid q_{\rho_g,\nu}^N) &= \int_{\widetilde{E}^N} (\pi_N Q)(dy)\, \log\left(\frac{d\pi_N Q}{dq_{\rho_g,\nu}^N}(y)\right) \\ &= \int_{\widetilde{E}^N} (\pi_N Q)(dy)\, \log\left(\frac{\mathcal{N}(g)^N}{e^{-g\sum_{i=1}^N |y^{(i)}|}}\,\frac{d\pi_N Q}{dq_{\rho,\nu}^N}(y)\right) \\ &= h(\pi_N Q \mid q_{\rho,\nu}^N) + N\log\mathcal{N}(g) + Ngm_Q, \end{aligned} \tag{2.7}$$

where $|y^{(i)}|$ is the length of the i-th word and the second equality uses that $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$. Let $N \to \infty$ and use (2.5), to get the claim. ∎

Lemma 2.1 implies that if $g > 0$, then $m_Q < \infty$ whenever $H(Q \mid q_{\rho_g,\nu}^{\otimes\mathbb{N}}) < \infty$. This is a special case of [1], Lemma 7.

2.2 Annealed LDP

For $N \in \mathbb{N}$, let $(Y^{(1)},\dots,Y^{(N)})^{\mathrm{per}}$ be the periodic extension of the N-tuple $(Y^{(1)},\dots,Y^{(N)}) \in \widetilde{E}^N$ to an element of $\widetilde{E}^{\mathbb{N}}$, and define

$$R_N = \frac{1}{N}\sum_{i=0}^{N-1} \delta_{\widetilde{\theta}^i (Y^{(1)},\dots,Y^{(N)})^{\mathrm{per}}} \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}). \tag{2.8}$$

This is the *empirical process of N-tuples of words.* The following *annealed LDP* is standard (see e.g. Dembo and Zeitouni [13], Section 6.5).

Theorem 2.2 *For every $g \in [0,\infty)$, the family $P_g^*(R_N \in \cdot)$, $N \in \mathbb{N}$, satisfies the LDP on $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ with rate N and with rate function I_g^{ann} given by*

$$I_g^{\mathrm{ann}}(Q) = H\big(Q \mid q_{\rho_g,\nu}^{\otimes\mathbb{N}}\big), \qquad Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}). \tag{2.9}$$

This rate function is lower semi-continuous, has compact level sets, has a unique zero at $q_{\rho_g,\nu}^{\otimes\mathbb{N}}$, and is affine.

It follows from Lemma 2.1 that

$$I_g^{\mathrm{ann}}(Q) = I^{\mathrm{ann}}(Q) + \log\mathcal{N}(g) + gm_Q, \tag{2.10}$$

where $I^{\mathrm{ann}}(Q) = H(Q \mid q_{\rho,\nu}^{\otimes\mathbb{N}})$, the annealed rate function for $g = 0$.

2.3 Quenched LDP

To formulate the quenched analogue of Theorem 2.2, we need some more notation. Let $\mathcal{P}^{\mathrm{inv}}(E^{\mathbb{N}})$ be the set of probability measures on $E^{\mathbb{N}}$ that are invariant under the left-shift θ acting on $E^{\mathbb{N}}$. For $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ such that $m_Q < \infty$, define

$$\Psi_Q = \frac{1}{m_Q} E_Q\left(\sum_{k=0}^{\tau_1 - 1} \delta_{\theta^k \kappa(Y)}\right) \in \mathcal{P}^{\mathrm{inv}}(E^{\mathbb{N}}). \tag{2.11}$$

Think of Ψ_Q as the shift-invariant version of $Q \circ \kappa^{-1}$ obtained after *randomizing* the location of the origin. This randomization is necessary because a shift-invariant Q in general does not give rise to a shift-invariant $Q \circ \kappa^{-1}$.

For $\mathrm{tr} \in \mathbb{N}$, let $[\cdot]_{\mathrm{tr}} \colon \widetilde{E} \to [\widetilde{E}]_{\mathrm{tr}} = \cup_{n=1}^{\mathrm{tr}} E^n$ denote the *truncation map* on words defined by

$$y = (x_1, \dots, x_n) \mapsto [y]_{\mathrm{tr}} = (x_1, \dots, x_{n \wedge \mathrm{tr}}), \qquad n \in \mathbb{N},\ x_1, \dots, x_n \in E, \tag{2.12}$$

i.e., $[y]_{\mathrm{tr}}$ is the word of length $\leq \mathrm{tr}$ obtained from the word y by dropping all the letters with label $> \mathrm{tr}$. This map induces in a natural way a map from $\widetilde{E}^{\mathbb{N}}$ to $[\widetilde{E}]_{\mathrm{tr}}^{\mathbb{N}}$, and from $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ to $\mathcal{P}^{\mathrm{inv}}([\widetilde{E}]_{\mathrm{tr}}^{\mathbb{N}})$. Note that if $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$, then $[Q]_{\mathrm{tr}}$ is an element of the set

$$\mathcal{P}^{\mathrm{inv,fin}}(\widetilde{E}^{\mathbb{N}}) = \{Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}) \colon\ m_Q < \infty\}. \tag{2.13}$$

Define (w-lim means weak limit)

$$\mathcal{R} = \left\{ Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}) \colon\ \mathrm{w} - \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \delta_{\theta^k \kappa(Y)} = \nu^{\otimes \mathbb{N}} \quad Q - a.s. \right\}, \tag{2.14}$$

i.e., the set of probability measures in $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ under which the concatenation of words almost surely has the same asymptotic statistics as a typical realization of X.

Theorem 2.3 (Birkner [1]; Birkner, Greven and den Hollander [2]) *Assume* (1.2) *and* (1.6)*. Then, for $\nu^{\otimes \mathbb{N}}$–a.s. all X and all $g \in [0, \infty)$, the family of (regular) conditional probability distributions $P_g^*(R_N \in \cdot \mid X)$, $N \in \mathbb{N}$, satisfies the LDP on $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ with rate N and with deterministic rate function I_g^{que} given by*

$$I_g^{\mathrm{que}}(Q) = \begin{cases} I_g^{\mathrm{ann}}(Q), & \text{if } Q \in \mathcal{R}, \\ \infty, & \text{otherwise}, \end{cases} \quad \text{when } g > 0, \tag{2.15}$$

and

$$I^{\mathrm{que}}(Q) = \begin{cases} I^{\mathrm{fin}}(Q), & \text{if } Q \in \mathcal{P}^{\mathrm{inv,fin}}(\widetilde{E}^{\mathbb{N}}), \\ \lim_{\mathrm{tr} \to \infty} I^{\mathrm{fin}}\big([Q]_{\mathrm{tr}}\big), & \text{otherwise}, \end{cases} \quad \text{when } g = 0, \tag{2.16}$$

where

$$I^{\mathrm{fin}}(Q) = H(Q \mid q_{\rho,\nu}^{\otimes \mathbb{N}}) + (\alpha - 1)\, m_Q\, H\big(\Psi_Q \mid \nu^{\otimes \mathbb{N}}\big). \tag{2.17}$$

This rate function is lower semi-continuous, has compact level sets, has a unique zero at $q_{\rho_g,\nu}^{\otimes \mathbb{N}}$, and is affine.

The difference between (2.15) for $g > 0$ and (2.16–2.17) for $g = 0$ can be explained as follows. For $g = 0$, the word length distribution ρ has a polynomial tail. It therefore is only exponentially costly to cut out a few words of an exponentially large length in order to move to stretches in X that are suitable to build a large deviation $\{R_N \approx Q\}$ with words whose length is of order 1. This is precisely where the second term in (2.17) comes from: this term is the extra cost to find these stretches under the quenched law rather than to create them "on the spot" under the annealed law. For $g > 0$, on the other hand, the word length distribution ρ_g has an exponentially bounded tail, and hence exponentially long words are too costly, so that suitable stretches far away cannot be reached. Phrased differently, $g > 0$ and $\alpha \in [1, \infty)$ is qualitatively similar to $g = 0$ and $\alpha = \infty$, for which we see that the expression in (2.17) is finite if and only $\Psi_Q = \nu^{\otimes\mathbb{N}}$. It was shown in [1], Lemma 2, that

$$\Psi_Q = \nu^{\otimes\mathbb{N}} \iff Q \in \mathcal{R} \qquad \text{on } \mathcal{P}^{\mathrm{inv,fin}}(\widetilde{E}^{\mathbb{N}}), \tag{2.18}$$

and so this explains why the restriction $Q \in \mathcal{R}$ appears in (2.15). For more background, see [2].

Note that $I^{\mathrm{que}}(Q)$ requires a truncation approximation when $m_Q = \infty$, for which case there is no closed form expression like in (2.17). As we will see later on, the cases $m_Q < \infty$ and $m_Q = \infty$ need to be separated. For later reference we remark that, for all $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$,

$$\begin{aligned} I^{\mathrm{ann}}(Q) &= \lim_{\mathrm{tr}\to\infty} I^{\mathrm{ann}}([Q]_{\mathrm{tr}}) = \sup_{\mathrm{tr}\in\mathbb{N}} I^{\mathrm{ann}}([Q]_{\mathrm{tr}}),\\ I^{\mathrm{que}}(Q) &= \lim_{\mathrm{tr}\to\infty} I^{\mathrm{que}}([Q]_{\mathrm{tr}}) = \sup_{\mathrm{tr}\in\mathbb{N}} I^{\mathrm{que}}([Q]_{\mathrm{tr}}), \end{aligned} \tag{2.19}$$

as shown in [2], Lemma A.1.

3 Proof of Theorem 1.1

We are now ready to return to the copolymer and start our variational analysis.

In Sections 3.1 and 3.2 we derive the variational formulas for the quenched and the annealed excess free energies and critical curves that were announced in Theorem 1.1. These variational formulas are stated in Theorems 3.1 and 3.2 below and imply part (i) of Theorem 1.1. In Section 3.3 we state additional properties that imply parts (ii) and (iii).

3.1 Quenched excess free energy and critical curve

Let

$$\widetilde{Z}_{n,0}^{\beta,h,\omega} = E\left(\exp\left[\beta\sum_{k=1}^{n}(\omega_k + h)\left[\mathrm{sign}(\pi_{k-1},\pi_k) - 1\right]\right] 1_{\{\pi_n=0\}}\right), \tag{3.1}$$

which differs from $\widetilde{Z}_n^{\beta,h,\omega}$ in (1.15) because of the extra indicator $1_{\{\pi_n=0\}}$. This indicator is harmless in the limit as $n \to \infty$ (see Bolthausen and den Hollander [7], Lemma 2) and is added for convenience. To derive a variational expression for $g^{\mathrm{que}}(\beta, h) = \lim_{n\to\infty} \frac{1}{n} \log \widetilde{Z}_{n,0}^{\beta,h,\omega}$ $\omega - a.s.$, we use Theorem 2.3 with

$$X = \omega, \quad E = \mathbb{R}, \quad \widetilde{E} = \cup_{n\in\mathbb{N}} \mathbb{R}^n, \quad \nu \in \mathcal{P}(\mathbb{R}), \quad \rho \in \mathcal{P}(\mathbb{N}), \tag{3.2}$$

where ν satisfies (1.2) and ρ satisfies (1.6), with $\rho(n) = P(\{\pi \in \Pi\colon\ \pi_k \neq 0\ \forall\, 1 \leq k < n, \pi_n = 0\})$, $n \in \mathbb{N}$, the excursion length distribution.

Abbreviate

$$\mathcal{C} = \{Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})\colon\ I^{\mathrm{ann}}(Q) < \infty\}, \qquad \mathcal{C}^{\mathrm{fin}} = \{Q \in \mathcal{C}\colon\ m_Q < \infty\}. \tag{3.3}$$

Theorem 3.1 *Assume* (1.2) *and* (1.6). *Fix* $\beta, h > 0$.
(i) The quenched excess free energy is given by

$$g^{\text{que}}(\beta,h) = \inf\{g \in \mathbb{R}\colon\ S^{\text{que}}(\beta,h;g) < 0\}, \tag{3.4}$$

where

$$S^{\text{que}}(\beta,h;g) = \sup_{Q \in \mathcal{C}^{\text{fin}} \cap \mathcal{R}} [\Phi_{\beta,h}(Q) - g m_Q - I^{\text{ann}}(Q)] \tag{3.5}$$

with

$$\Phi_{\beta,h}(Q) = \int_{\widetilde{E}} (\pi_1 Q)(dy) \log \phi_{\beta,h}(y), \tag{3.6}$$

$$\phi_{\beta,h}(y) = \tfrac{1}{2}\left(1 + e^{-2\beta h\, \tau(y) - 2\beta\, \sigma(y)}\right), \tag{3.7}$$

where $\pi_1\colon \widetilde{E}^{\mathbb{N}} \to \widetilde{E}$ *is the projection onto the first word, i.e.,* $\pi_1 Q = Q \circ \pi_1^{-1}$, *and* $\tau(y), \sigma(y)$ *are the length, respectively, the sum of the letters in the word* y.
(ii) An alternative variational formula at $g = 0$ *is* $S^{\text{que}}(\beta,h;0) = S_*^{\text{que}}(\beta,h)$ *with*

$$S_*^{\text{que}}(\beta,h) = \sup_{Q \in \mathcal{C}^{\text{fin}}} [\Phi_{\beta,h}(Q) - I^{\text{que}}(Q)]\,. \tag{3.8}$$

(iii) The function $g \mapsto S^{\text{que}}(\beta,h;g)$ *is lower semi-continuous, convex and non-increasing on* $\mathbb{R}$, *is infinite on* $(-\infty,0)$, *and is finite, continuous and strictly decreasing on* $(0,\infty)$.

Proof. The proof comes in 5 steps. Throughout the proof $\beta, h > 0$ are fixed.

1. Let $t_n = t_n(\pi)$ denote the number of excursions in π away from the interface (recall that $\pi_n = 0$ in (3.1)). For $i = 1, \ldots, t_n$, let $I_i = I_i(\pi)$ denote the i-th excursion interval in π. Then

$$\beta \sum_{k=1}^{n} (\omega_k + h)[\text{sign}(\pi_{k-1},\pi_k) - 1] = \beta \sum_{i=1}^{t_n} \sum_{k \in I_i} (\omega_k + h)[\text{sign}(\pi_{k-1},\pi_k) - 1]. \tag{3.9}$$

During the i-th excursion, π cuts out the word $\omega_{I_i} = (\omega_k)_{k \in I_i}$ from ω. Each excursion can be either above or below the interface, with probability $\frac{1}{2}$ each, and so the contribution to $\widetilde{Z}_{n,0}^{\beta,h,\omega}$ in (3.1) coming from the i-th excursion is

$$\psi_{\beta,h}^{\omega}(I_i) = \tfrac{1}{2}\left(1 + \exp\left[-2\beta \sum_{k \in I_i} (\omega_k + h)\right]\right). \tag{3.10}$$

Hence, putting $I_i = (k_{i-1}, k_i] \cap \mathbb{N}$, we have

$$\widetilde{Z}_{n,0}^{\beta,h,\omega} = \sum_{N \in \mathbb{N}} \sum_{0 = k_0 < k_1 < \cdots < k_N = n} \prod_{i=1}^{N} \rho(k_i - k_{i-1})\ \psi_{\beta,h}^{\omega}\big((k_{i-1}, k_i]\big). \tag{3.11}$$

Summing on n, we get

$$\sum_{n \in \mathbb{N}} e^{-gn}\, \widetilde{Z}_{n,0}^{\beta,h,\omega} = \sum_{N \in \mathbb{N}} F_N^{\beta,h,\omega}(g), \qquad g \in [0,\infty), \tag{3.12}$$

with (recall (2.4))

$$F_N^{\beta,h,\omega}(g) = \mathcal{N}(g)^N \sum_{0=k_0<k_1<\cdots<k_N<\infty} \left(\prod_{i=1}^N \rho_g(k_i - k_{i-1})\right) \exp\left[\sum_{i=1}^N \log \psi^\omega_{\beta,h}\big((k_{i-1},k_i]\big)\right]. \quad (3.13)$$

2. Let

$$R_N^\omega = \frac{1}{N}\sum_{i=1}^N \delta_{\widetilde{\theta}^i(\omega_{I_1},\dots,\omega_{I_N})^{\rm per}} \quad (3.14)$$

denote the *empirical process of N-tuples of words* in ω cut out by the successive excursions. Then (3.13) gives

$$\begin{aligned} F_N^{\beta,h,\omega}(g) &= \mathcal{N}(g)^N\, E_g^*\left(\exp\left[N\int_{\widetilde{E}} (\pi_1 R_N^\omega)(dy)\, \log\phi_{\beta,h}(y)\right]\right) \\ &= \mathcal{N}(g)^N\, E_g^*\big(\exp\left[N\Phi_{\beta,h}(R_N^\omega)\right]\big) \end{aligned} \quad (3.15)$$

with $\Phi_{\beta,h}$ and $\phi_{\beta,h}$ defined in (3.6–3.7). Next, let

$$\bar{S}^{\rm que}(\beta,h;g) = \limsup_{N\to\infty} \frac{1}{N}\log F_N^{\beta,h,\omega}(g), \qquad g\in[0,\infty), \quad (3.16)$$

and note that the limsup exists and is constant (possibly infinity) ω-a.s. because it is measurable w.r.t. the tail sigma-algebra of ω (which is trivial). By (1.14), the left-hand side of (3.12) is a power series that converges for $g > g^{\rm que}(\beta,h)$ and diverges for $g < g^{\rm que}(\beta,h)$. Hence we have

$$g^{\rm que}(\beta,h) = \inf\{g\in\mathbb{R}\colon\ \bar{S}^{\rm que}(\beta,h;g) < 0\}. \quad (3.17)$$

Below we will see that $g\mapsto \bar{S}^{\rm que}(\beta,h;g)$ is strictly decreasing when finite, so that $\bar{S}^{\rm que}(\beta,h;g)$ changes sign precisely at $g = g^{\rm que}(\beta,h)$.

3. A *naive* application of Varadhan's lemma to (3.15–3.16) based on the quenched LDP in Theorem 2.3 yields that

$$\bar{S}^{\rm que}(\beta,h;g) = \log\mathcal{N}(g) + \sup_{Q\in\mathcal{P}^{\rm inv}(\widetilde{E}^{\mathbb{N}})} \left[\Phi_{\beta,h}(Q) - I_g^{\rm que}(Q)\right]. \quad (3.18)$$

This variational formula brings us close to where we want, because Lemma 2.1 and the formulas for $I_g^{\rm que}(Q)$ given in Theorem 2.3 tell us that

$$\text{r.h.s. (3.18)} = \begin{cases} \sup_{Q\in\mathcal{R}} \left[\Phi_{\beta,h}(Q) - g m_Q - I^{\rm ann}(Q)\right], & \text{if } g\in(0,\infty), \\ \sup_{Q\in\mathcal{P}^{\rm inv}(\widetilde{E}^{\mathbb{N}})} \left[\Phi_{\beta,h}(Q) - I^{\rm que}(Q)\right], & \text{if } g = 0, \end{cases} \quad (3.19)$$

which is the same as the variational formulas in (3.5) and (3.8), except that the suprema in (3.19) are not restricted to $\mathcal{C}^{\rm fin}$. Unfortunately, the application of Varadhan's lemma is *problematic*, because $Q\mapsto m_Q$ and $Q\mapsto\Phi_{\beta,h}(Q)$ are neither bounded nor continuous in the weak topology. The proof of (3.18–3.19) therefore requires an *approximation argument*, which is written out in Appendix B. This approximation argument also shows how the restriction to $\mathcal{C}^{\rm fin}$ comes in. This restriction is needed to make the variational formulas proper, namely, it is shown in Appendix A that if $I^{\rm ann}$ is finite, then also $\Phi_{\beta,h}$ is finite. In Appendix B we further show that the variational formula in (3.5) at $g = 0$ equals the variational formula in (3.8), i.e., $S^{\rm que}(\beta, h; 0) = S_*^{\rm que}(\beta, h)$. Thus, we have

$$\bar{S}^{\rm que}(\beta, h; g) = S^{\rm que}(\beta, h; g), \qquad g \in [0, \infty). \tag{3.20}$$

4. To include $g \in (-\infty, 0)$ in (3.20) we argue as follows. We see from (3.6–3.7) and (3.15) that $F_N^{\beta,h,\omega}(g) \geq \left[\frac{1}{2}\mathcal{N}(g)\right]^N$. Since $\mathcal{N}(g) = \infty$ for $g \in (-\infty, 0)$, it follows from (3.16) that $\bar{S}^{\rm que}(\beta, h; g) = \infty$ for $g \in (-\infty, 0)$. Moreover, we have

$$S^{\rm que}(\beta, h; g) \geq \log(\tfrac{1}{2}) + \sup_{\rho' \in \mathcal{P}(\mathbb{N})} \left[-g m_{\rho'} - h(\rho' \mid \rho)\right], \tag{3.21}$$

which is obtained from (3.5–3.7) by picking $Q = q'^{\otimes \mathbb{N}}$ with $q'(dx_1, \dots, dx_n) = \rho'(n)\nu(dx_1) \times \cdots \times \nu(dx_n)$, $n \in \mathbb{N}$, $x_1, \dots, x_n \in \mathbb{R}$ (compare with (2.3)). By picking $\rho'(n) = \delta_{nL}$, $n \in \mathbb{N}$, with $L \in \mathbb{N}$ arbitrary, we get from (3.21) that $S^{\rm que}(\beta, h; g) \geq \log(\frac{1}{2}) - gL + \log \rho(L)$. Letting $L \to \infty$ and using (1.6), we obtain that $S^{\rm que}(\beta, h; g) = \infty$ for $g \in (-\infty, 0)$. Thus, (3.20) extends to

$$\bar{S}^{\rm que}(\beta, h; g) = S^{\rm que}(\beta, h; g), \qquad g \in \mathbb{R}. \tag{3.22}$$

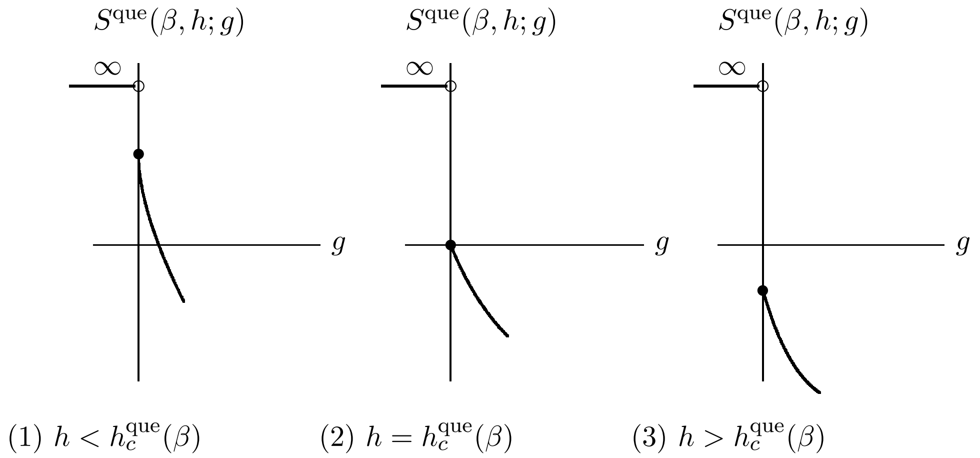


Figure 5: Qualitative picture of $g \mapsto S^{\rm que}(\beta, h; g)$ for $\beta, h > 0$.

5. In Section 6 we will show, with the help of a fractional moment estimate, that $\bar{S}^{\rm que}(\beta, h; g) < \infty$ for $g \in (0, \infty)$. By (3.5), $g \mapsto S^{\rm que}(\beta, h; g)$ is a supremum of functions that are finite and linear on $\mathbb{R}$. Hence, $g \mapsto S^{\rm que}(\beta, h; g)$ is lower semi-continuous and convex on $\mathbb{R}$ and, being finite on $(0, \infty)$, is continuous on $(0, \infty)$. Moreover, since $m_Q \geq 1$, it is strictly decreasing on $(0, \infty)$ as well. This completes the proof of part (iii). ∎

Fig. 5 provides a sketch of $g \mapsto S^{\rm que}(\beta, h; g)$ for (β, h) drawn from $\mathcal{L}^{\rm que}$, $\partial\mathcal{D}^{\rm que}$ and $\mathrm{int}(\mathcal{D}^{\rm que})$, respectively, and completes the variational characterization in Theorem 3.1. In Section 3.3 we look at $h \mapsto S^{\rm que}(\beta, h; 0)$ and obtain the picture drawn in Fig. 6, which is crucial for our analysis.

Remark: A major advantage of the variational formula in (3.8) over the one in (3.5) at $g = 0$ is that the supremum runs over $\mathcal{C}^{\rm fin}$ rather than $\mathcal{C}^{\rm fin} \cap \mathcal{R}$. This will be crucial for the proof of Corollaries 1.3 and 1.4 in Sections 5 and 6, respectively. In Section 6 we will show that

$$S^{\rm que}(\beta, h_c^{\rm ann}(\tfrac{\beta}{\alpha}); 0) > 0. \tag{3.23}$$

It will turn out that $S^{\rm que}(\beta, h_c^{\rm ann}(\frac{\beta}{\alpha}); 0) < \infty$ for some choices of ρ, but we do not know whether it is finite in general.

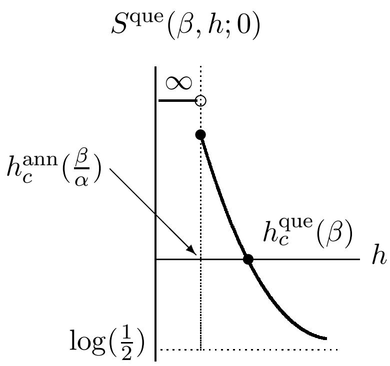


Figure 6: Qualitative picture of $h \mapsto S^{\text{que}}(\beta,h;0)$ for $\beta > 0$.

3.2 Annealed excess free energy and critical curve

In order to exploit Theorem 3.1, we need an analogous variational expression for the annealed excess free energy defined in (1.16–1.17). This variational expression will serve as a *comparison object* and will be crucial for the proof of Corollaries 1.2–1.4.

Theorem 3.2 *Assume* (1.2) *and* (1.6)*. Fix* $\beta, h > 0$*.*
(i) The annealed excess free energy is given by

$$g^{\text{ann}}(\beta,h) = \inf\{g \in \mathbb{R}\colon\ S^{\text{ann}}(\beta,h;g) < 0\}, \tag{3.24}$$

where

$$S^{\text{ann}}(\beta,h;g) = \sup_{Q\in\mathcal{C}^{\text{fin}}} \left[\Phi_{\beta,h}(Q) - g m_Q - I^{\text{ann}}(Q)\right]. \tag{3.25}$$

(ii) The function $g \mapsto S^{\text{ann}}(\beta,h;g)$ *is lower semi-continuous, convex and non-increasing on* $\mathbb{R}$*, infinite on* $(-\infty, g^{\text{ann}}(\beta,h))$*, and finite, continuous and strictly decreasing on* $[g^{\text{ann}}(\beta,h),\infty)$*.*

Proof. Throughout the proof $\beta, h > 0$ are fixed.

(i) Replacing $\widetilde{Z}_n^{\beta,h,\omega}$ by $\mathbb{E}(\widetilde{Z}_n^{\beta,h,\omega})$ in (3.12–3.13), we obtain from (3.16) that

$$\bar{S}^{\text{ann}}(\beta,h;g) = \limsup_{N\to\infty} \frac{1}{N} \log \mathbb{E}\left(F_N^{\beta,h,\omega}(g)\right). \tag{3.26}$$

Using (2.3–2.4), (3.10) and (3.13), and abbreviating

$$\phi_{\beta,h}(k,l) = \tfrac12\Big(1 + e^{-2\beta h k - 2\beta l}\Big), \qquad k \in \mathbb{N},\ l \in \mathbb{R}, \tag{3.27}$$

we compute

$$\bar{S}^{\text{ann}}(\beta,h;g) = \log \mathcal{N}(\beta,h;g) \tag{3.28}$$

with

$$\begin{aligned} \mathcal{N}(\beta,h;g) &= \sum_{k\in\mathbb{N}} \int_{l\in\mathbb{R}} q_{\rho,\nu}(k,dl)\, e^{-gk}\, \phi_{\beta,h}(k,l) \\ &= \sum_{k\in\mathbb{N}} \int_{l\in\mathbb{R}} \rho(k)\, \nu^{\circledast k}(dl)\ e^{-gk}\ \tfrac12\Big(1 + e^{-2\beta h k - 2\beta l}\Big) \\ &= \tfrac12 \sum_{k\in\mathbb{N}} \rho(k)\, e^{-gk} \ + \tfrac12 \sum_{k\in\mathbb{N}} \rho(k)\, e^{-gk}\ \Big[e^{-2\beta h + M(2\beta)}\Big]^k \\ &= \tfrac12\, \mathcal{N}(g) + \tfrac12\, \mathcal{N}\big(g - [M(2\beta) - 2\beta h]\big), \end{aligned} \tag{3.29}$$

where $\mathcal{N}(g)$ is the normalization constant in (2.4), and $\nu^{\circledast k}$ is the k-fold convolution of ν. The right-hand side of (3.28) has the behavior as sketched in Fig. 7. It is therefore immediate that (3.24–3.25) is consistent with (1.20), provided we have

$$S^{\text{ann}}(\beta, h; g) = \bar{S}^{\text{ann}}(\beta, h; g). \tag{3.30}$$

To prove this equality we must distinguish three cases.

(I) $g(\beta, h) \geq g^{\text{ann}}(\beta, h) = 0 \vee [M(2\beta) - 2\beta h]$. The proof comes in 2 steps. Note that the right-hand side of (3.29) is finite.

1. Note that $\Phi_{\beta,h}(Q)$ defined in (3.6) is a functional of $\pi_1 Q$. Moreover, by (2.5),

$$\inf_{\substack{Q \in \mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}}) \\ \pi_1 Q = q}} H(Q \mid q_{\rho,\nu}^{\otimes \mathbb{N}}) = h(q \mid q_{\rho,\nu}) \qquad \forall\, q \in \mathcal{P}(\widetilde{E}) \tag{3.31}$$

with the infimum *uniquely* attained at $Q = q^{\otimes \mathbb{N}}$, where the right-hand side denotes the relative entropy of q w.r.t. $q_{\rho,\nu}$. (The uniqueness of the minimum is easily deduced from the strict convexity of relative entropy on finite cylinders.) Consequently, the variational formula in (3.25) reduces to

$$S^{\text{ann}}(\beta, h; g) = \sup_{\substack{q \in \mathcal{P}(\widetilde{E}) \\ m_q < \infty,\, h(q|q_{\rho,\nu}) < \infty}} \left\{ \int_{\widetilde{E}} q(dy)\, [-g\tau(y) + \log \phi_{\beta,h}(y)] - h(q \mid q_{\rho,\nu}) \right\} \tag{3.32}$$

with $\phi_{\beta,h}(y)$ defined in (3.7) and $m_q = \int_{\widetilde{E}} q(dy)\tau(y)$. A further reduction is possible by noting that, in view of (3.7), the integral is a functional of the law of $(\tau(y), \sigma(y))$ under q. Hence, projecting further from $\widetilde{E}$ to $\mathbb{N} \times \mathbb{R}$ and using the analogue of (3.31) for this projection, we have

$$\begin{aligned} S^{\text{ann}}(\beta, h; g) = \sup_{\substack{q \in \mathcal{P}(\mathbb{N} \times \mathbb{R}) \\ m_q < \infty,\, h(q|q_{\rho,\nu}) < \infty}} \Big\{ & \sum_{k \in \mathbb{N}} \int_{l \in \mathbb{R}} q(k, dl)\, [-gk + \log \phi_{\beta,h}(k, l)] \\ & - \sum_{k \in \mathbb{N}} \int_{l \in \mathbb{R}} q(k, dl) \log \left(\frac{q(k, dl)}{q_{\rho,\nu}(k, dl)} \right) \Big\} \end{aligned} \tag{3.33}$$

with $m_q = \sum_{k \in \mathbb{N}} \int_{l \in \mathbb{R}} k q(k, dl)$.

2. Define

$$q_{\beta,h;g}(k, dl) = \frac{1}{\mathcal{N}(\beta, h; g)}\, q_{\rho,\nu}(k, dl)\, e^{-gk}\, \phi_{\beta,h}(k, l), \qquad (k, l) \in \mathbb{N} \times \mathbb{R}, \tag{3.34}$$

with $\mathcal{N}(\beta, h; g)$ the normalizing constant in (3.29) (which is finite because $g \geq [M(2\beta) - 2\beta h]$). Then the term between braces in (3.33) can be rewritten as

$$\log \mathcal{N}(\beta, h; g) - h(q \mid q_{\beta,h;g}), \tag{3.35}$$

and so we have two cases:

(1) if both $m_{q_{\beta,h;g}} < \infty$ and $h(q_{\beta,h;g} \mid q_{\rho,\nu}) < \infty$, then the supremum in (3.33) has a unique maximizer at $q = q_{\beta,h;g}$;

(2) if $m_{q_{\beta,h;g}} = \infty$ and/or $h(q_{\beta,h;g} \mid q_{\rho,\nu}) = \infty$, then any maximizing sequence $(q_n)_{n \in \mathbb{N}}$ with $m_{q_n} < \infty$ and $h(q_n \mid q_{\rho,\nu}) < \infty$ for all $n \in \mathbb{N}$ satisfies $\text{w} - \lim_{n \to \infty} q_n = q_{\beta,h;g}$ (weak limit).

In both cases

$$S^{\mathrm{ann}}(\beta, h; g) = \log \mathcal{N}(\beta, h; g), \tag{3.36}$$

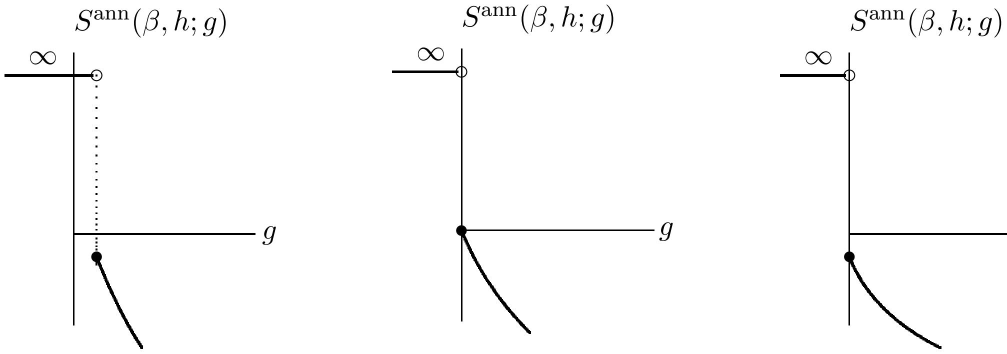
which settles (3.30) in view of (3.28).

(II) $g < [M(2\beta) - 2\beta h]$. It follows from (3.28–3.29) that $\bar{S}^{\mathrm{ann}}(\beta, h, g) = \infty$. We therefore need to show that $S^{\mathrm{ann}}(\beta, h; g) = \infty$ as well. For $L \in \mathbb{N}$, let $q_\beta^L \in \mathcal{P}(\widetilde{E})$ be defined by

$$q_\beta^L(dx_1, \dots, dx_n) = \delta_{nL}\, \nu_\beta^{\otimes n}(dx_1, \dots, dx_n), \qquad n \in \mathbb{N},\, x_1, \dots, x_n \in \mathbb{R}, \tag{3.37}$$

where $\nu_\beta \in \mathcal{P}(\mathbb{R})$ is defined by

$$\nu_\beta(dx) = e^{-2\beta x - M(2\beta)}\, \nu(dx), \qquad x \in \mathbb{R}. \tag{3.38}$$



(1) $h < h_c^{\mathrm{ann}}(\beta)$ (2) $h = h_c^{\mathrm{ann}}(\beta)$ (3) $h > h_c^{\mathrm{ann}}(\beta)$

Figure 7: Qualitative picture of $g \mapsto S^{\mathrm{ann}}(\beta, h; g)$ for $\beta, h > 0$. Compare with Fig. 5.

Put $Q_\beta^L = (q_\beta^L)^{\otimes \mathbb{N}}$. Then $m_{Q_\beta^L} = L$, while

$$\begin{aligned} I^{\mathrm{ann}}(Q_\beta^L) &= H(Q_\beta^L \mid q_{\rho,\nu}^{\otimes \mathbb{N}}) \\ &= h(q_\beta^L \mid q_{\rho,\nu}) \\ &= \int_{\widetilde{E}} q_\beta^L(dy)\, \frac{dq_\beta^L}{dq_{\rho,\nu}}(y) \\ &= -\log \rho(L) + L h(\nu_\beta \mid \nu) \\ &= -\log \rho(L) + L \int_{\mathbb{R}^L} \nu_\beta(dx)\, \log\left(e^{-2\beta x - M(2\beta)}\right) \\ &= -\log \rho(L) - L\left[2\beta\, \mathbb{E}_{\nu_\beta}(\omega_1) + M(2\beta)\right] \end{aligned} \tag{3.39}$$

and

$$\begin{aligned} \Phi_{\beta,h}(Q_\beta^L) &= \int_{\widetilde{E}} q_\beta^L(dy)\, \log \phi_{\beta,h}(y) \\ &= \int_{\mathbb{R}^L} \nu_\beta^{\otimes L}(dx_1, \dots, dx_L)\, \log\left(\tfrac{1}{2}\left[1 + e^{-2\beta \sum_{k=1}^L (x_k + h)}\right]\right) \\ &\geq \log(\tfrac{1}{2}) - L\left[2\beta \mathbb{E}_{\nu_\beta}(\omega_1) + 2\beta h\right]. \end{aligned} \tag{3.40}$$

It follows that

$$\Phi_{\beta,h}(Q^L_\beta) - g\, m_{Q^L_\beta} - I^{\mathrm{ann}}(Q^L_\beta) \geq \log(\tfrac{1}{2}) + \log \rho(L) + L\,[M(2\beta) - 2\beta h - g]\,, \tag{3.41}$$

which tends to infinity as $L \to \infty$ (use (1.6)).

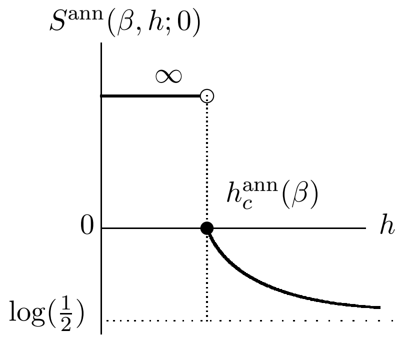


Figure 8: Qualitative picture of $h \mapsto S^{\mathrm{ann}}(\beta, h; 0)$ for $\beta > 0$. Compare with Fig. 6.

(III) $M(2\beta) - 2\beta h < 0$ and $g \in [M(2\beta) - 2\beta h, 0)$. Repeat the argument in (3.39–3.41) with Q^L_β replaced by Q^L_0 and keep only the first term in the right-hand side of (3.41). This gives

$$\Phi_{\beta,h}(Q^L_0) - g\, m_{Q^L_0} - I^{\mathrm{ann}}(Q^L_0) \geq \log(\tfrac{1}{2}) + \log \rho(L) - Lg, \tag{3.42}$$

which tends to infinity as $L \to \infty$ for $g < 0$. ∎

Fig. 7 provides a sketch of $g \mapsto S^{\mathrm{ann}}(\beta, h; g)$ for (β, h) drawn from $\mathcal{L}^{\mathrm{ann}}$, $\partial\mathcal{D}^{\mathrm{ann}}$ and $\mathrm{int}(\mathcal{D}^{\mathrm{ann}})$, respectively, and completes the variational characterization in Theorem 3.2. Fig. 8 provides a sketch of $h \mapsto S^{\mathrm{ann}}(\beta, h; 0)$.

3.3 Proof of Theorem 1.1

Theorems 3.1 and 3.2 complete the proof of part (i) of Theorem 1.1. From the computations carried out in Section 3.2 we also get parts (ii) and (iii) for the annealed model, but to get parts (ii) and (iii) for the quenched model we need some further information.

Theorem 3.1 provides no information on $S^{\mathrm{que}}(\beta, h; 0)$. We know that, for every $\beta > 0$, $h \mapsto S^{\mathrm{que}}(\beta, h; 0)$ is lower semi-continuous, convex and non-increasing on $(0, \infty)$. Indeed, $h \mapsto \phi_{\beta,h}(k, l)$ is continuous, convex and non-increasing for all $k \in \mathbb{N}$ and $l \in \mathbb{R}$, hence $h \mapsto \Phi_{\beta,h}(Q)$ is lower semi-continuous, convex and non-increasing for every $Q \in \mathcal{C}^{\mathrm{fin}}$, and these properties are preserved under taking suprema. We know that $h \mapsto S^{\mathrm{que}}(\beta, h; 0)$ is strictly negative on $(h_c^{\mathrm{que}}(\beta), \infty)$. In Section 6 we prove the following theorem, which corroborates the picture drawn in Fig. 6 and completes the proof of parts (ii) and (iii) of Theorem 1.1 for the quenched model.

Theorem 3.3 *For every $\beta > 0$,*

$$S^{\mathrm{que}}(\beta, h; 0) \begin{cases} = \infty & \textit{for } h < h_c^{\mathrm{ann}}(\beta/\alpha), \\ > 0 & \textit{for } h = h_c^{\mathrm{ann}}(\beta/\alpha), \\ < \infty & \textit{for } h > h_c^{\mathrm{ann}}(\beta/\alpha). \end{cases} \tag{3.43}$$

We close this section with the following remark. The difference between the variational formulas in (3.5) (quenched model) and (3.25) (annealed model) is that the supremum in the former runs over $\mathcal{C}^{\mathrm{fin}} \cap \mathcal{R}$ while the supremum in the latter runs over $\mathcal{C}^{\mathrm{fin}}$. Both involve the annealed rate function I^{ann}. However, the restriction to $\mathcal{R}$ for the quenched model allows us to replace I^{ann} by I^{que} (recall (2.18)). After passing to the limit $g \downarrow 0$, we can remove the restriction to $\mathcal{R}$ to obtain the alternative variational formula for the quenched model given in (3.8). The latter turns out to be crucial in Sections 5 and 6.

Note that the two variational formulas for $g \neq 0$ are different even when $\alpha = 1$, although in that case $I^{\mathrm{ann}} = I^{\mathrm{que}}$ (compare Theorems 2.2 and 2.3). For $\alpha = 1$ the quenched and the annealed critical curves coincide, but the free energies do not.

4 Proof of Corollary 1.2

Proof. The claim is trivial for $h_c^{\mathrm{que}}(\beta) \leq h < h_c^{\mathrm{ann}}(\beta)$ because $g^{\mathrm{que}}(\beta,h) = 0 < g^{\mathrm{ann}}(\beta,h)$. Therefore we may assume that $0 < h < h_c^{\mathrm{que}}(\beta)$. Since $I^{\mathrm{que}}(Q) \geq I^{\mathrm{ann}}(Q)$, (3.5) and (3.25) yield

$$S^{\mathrm{que}}(\beta,h;0) \leq S^{\mathrm{ann}}(\beta,h;0) \tag{4.1}$$

which, via (3.4) and (3.24), implies that $g^{\mathrm{que}}(\beta,h) \leq g^{\mathrm{ann}}(\beta,h)$, a property that is also evident from (1.9) and (1.17). To prove that $g^{\mathrm{que}}(\beta,h) < g^{\mathrm{ann}}(\beta,h)$ for $0 < h < h_c^{\mathrm{que}}(\beta)$, we combine (4.1) with Figs. 5 and 7. First note that

$$S^{\mathrm{que}}(\beta,h;g^{\mathrm{ann}}(\beta,h)) \leq S^{\mathrm{ann}}(\beta,h;g^{\mathrm{ann}}(\beta,h)) < 0, \qquad 0 < h < h_c^{\mathrm{ann}}(\beta). \tag{4.2}$$

Next, for $0 < h < h_c^{\mathrm{ann}}(\beta)$, $g \mapsto S^{\mathrm{ann}}(\beta,h;g)$ blows up at $g = g^{\mathrm{ann}}(\beta,h) > 0$ by jumping from a strictly negative value to infinity (see Fig. 7). Since $S^{\mathrm{que}}(\beta,h;g^{\mathrm{ann}}(\beta,h)) < 0$, and $g \mapsto S^{\mathrm{que}}(\beta,h;g)$ is strictly decreasing and continuous when finite, the claim is immediate from Theorem 1.1(ii), which says that $S^{\mathrm{que}}(\beta,h;g^{\mathrm{que}}(\beta,h)) = 0$. ∎

5 Proof of Corollary 1.3

Proof. Throughout the proof, $\alpha > 1$ and $\beta > 0$ are fixed. The proof comes in 4 steps.

1. We begin with a truncation approximation.

Lemma 5.1 *For every $\beta > 0$, there exists a sequence $(Q_{\mathrm{tr}})_{\mathrm{tr}\in\mathbb{N}}$ with $Q_{\mathrm{tr}} \in \mathcal{C}^{\mathrm{fin}}$ for all $\mathrm{tr} \in \mathbb{N}$ such that*

$$\lim_{\mathrm{tr}\to\infty} \Big[\Phi_{\beta,h_c^{\mathrm{que}}(\beta)}(Q_{\mathrm{tr}}) - I^{\mathrm{fin}}(Q_{\mathrm{tr}})\Big] = 0. \tag{5.1}$$

Proof. Note that

$$\sup_{Q\in\mathcal{C}^{\mathrm{fin}}} \Big[\Phi_{\beta,h}(Q) - I^{\mathrm{fin}}(Q)\Big] = \sup_{\mathrm{tr}\in\mathbb{N}} \sup_{Q\in\mathcal{C}} \Big[\Phi_{\beta,h}([Q]_{\mathrm{tr}}) - I^{\mathrm{fin}}([Q]_{\mathrm{tr}})\Big] \qquad \forall\, \beta, h > 0. \tag{5.2}$$

Indeed, trivially the left-hand side is $\geq$ the right-hand side, but the reverse inequality is also true because

$$\liminf_{\mathrm{tr}\to\infty} \Phi_{\beta,h}([Q]_{\mathrm{tr}}) \geq \Phi_{\beta,h}(Q), \qquad \lim_{\mathrm{tr}\to\infty} I^{\mathrm{fin}}([Q]_{\mathrm{tr}}) = I^{\mathrm{fin}}(Q), \qquad \forall\, Q \in \mathcal{P}^{\mathrm{inv,fin}}(\widetilde{E}^{\mathbb{N}}). \tag{5.3}$$

The former follows from the fact that $Q \mapsto \Phi_{\beta,h}(Q)$ is lower semi-continuous on $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$, while the latter is the second half of (2.19). Because $I^{\mathrm{que}} = I^{\mathrm{fin}}$ on $\mathcal{P}^{\mathrm{inv,fin}}(\widetilde{E}^{\mathbb{N}}) \supseteq \mathcal{C}^{\mathrm{fin}}$, the left-hand side of (5.2) equals $S_*^{\mathrm{que}}(\beta,h)$ defined in (3.8). We know from Theorem 3.1(ii) and Fig. 6 that $S_*^{\mathrm{que}}(\beta,h_c^{\mathrm{que}}(\beta)) = S^{\mathrm{que}}(\beta,h_c^{\mathrm{que}}(\beta);0) = 0$. Combine this with (5.2) to get the claim. ∎

In steps 2-4 below we exploit Lemma 5.1.

2. Let $q_\beta \in \mathcal{P}(\widetilde{E})$ be defined by

$$\begin{aligned} &q_\beta(dx_1,\dots,dx_n) = \tfrac12\,\rho(n)\,\big[\nu^{\otimes n}(dx_1,\dots,dx_n) + \nu_\beta^{\otimes n}(dx_1,\dots,dx_n)\big],\\ &n\in\mathbb{N},\ x_1,\dots,x_n\in\mathbb{R}, \end{aligned} \tag{5.4}$$

where $\nu_\beta \in \mathcal{P}(\mathbb{R})$ is given in (3.38). The projection of q_β from $\widetilde{E}$ to $\mathbb{N}\times\mathbb{R}$ is precisely $q_{\beta,h_c^{\mathrm{ann}}(\beta);0}$ defined in (3.34). Define

$$Q_\beta = q_\beta^{\otimes\mathbb{N}}. \tag{5.5}$$

We saw in Section 3.2 that Q_β is the unique maximizer of the variational expression for $S^{\mathrm{ann}}(\beta,h;0)$ at $h = h_c^{\mathrm{ann}}(\beta)$ when $\sum_{k\in\mathbb{N}} k\rho(k) < \infty$ and $h(q_\beta|q_{\rho,\nu}) < \infty$, and the unique limit of any maximizing sequence when $\sum_{k\in\mathbb{N}} k\rho(k) = \infty$ and/or $h(q_\beta|q_{\rho,\nu}) = \infty$.

3. The key to proving the strict inequality in Corollary 1.3 is the following lemma.

Lemma 5.2 *For every $\beta > 0$ there exists a $\delta(\beta) > 0$ such that*

$$I^{\mathrm{que}}\big([Q_\beta]_{\mathrm{tr}}\big) - I^{\mathrm{ann}}\big([Q_\beta]_{\mathrm{tr}}\big) \geq \delta(\beta)\, m_{[Q_\beta]_{\mathrm{tr}}} \qquad \forall\, \mathrm{tr}\in\mathbb{N}. \tag{5.6}$$

Proof. By (2.9) and (2.17), we have

$$I^{\mathrm{que}}([Q_\beta]_{\mathrm{tr}}) - I^{\mathrm{ann}}([Q_\beta]_{\mathrm{tr}}) = (\alpha-1)\, m_{[Q_\beta]_{\mathrm{tr}}}\, H\big(\Psi_{[Q_\beta]_{\mathrm{tr}}} \mid \nu^{\otimes\mathbb{N}}\big), \tag{5.7}$$

where we recall (2.11–2.12). Let $\rho_{\mathrm{tr}} \in \mathcal{P}(\mathbb{N})$ be defined by

$$\rho_{\mathrm{tr}}(k) = \begin{cases} \rho(k) & \text{if } k < \mathrm{tr},\\ \sum_{l=\mathrm{tr}}^\infty \rho(l) & \text{if } k = \mathrm{tr},\\ 0 & \text{if } k > \mathrm{tr}. \end{cases} \tag{5.8}$$

It is immediate from (5.4–5.5) that

$$\begin{aligned} m_{[Q_\beta]_{\mathrm{tr}}} &= \sum_{k\in\mathbb{N}} k\rho_{\mathrm{tr}}(k),\\ m_{[Q_\beta]_{\mathrm{tr}}}\,\Psi_{[Q_\beta]_{\mathrm{tr}}}\big(\{dx\}\times E^{\mathbb{N}\setminus\{1\}}\big) &= \tfrac12\,[\nu(dx)+\nu_\beta(dx)] \sum_{k\in\mathbb{N}} k\rho_{\mathrm{tr}}(k). \end{aligned} \tag{5.9}$$

Putting

$$\tfrac12[\nu(dx)+\nu_\beta(dx)] = \mu_\beta(dx), \qquad x\in\mathbb{R}, \tag{5.10}$$

we get from (5.9) that

$$\Psi_{[Q_\beta]_{\mathrm{tr}}}\big(\{dx\}\times E^{\mathbb{N}\setminus\{1\}}\big) = \mu_\beta(dx), \qquad x\in\mathbb{R}, \tag{5.11}$$

which is *independent* of the truncation level tr. Hence (recall that the limit in (2.5) is non-decreasing)

$$H\big(\Psi_{[Q_\beta]_{\mathrm{tr}}} \mid \nu^{\otimes\mathbb{N}}\big) \geq h(\mu_\beta \mid \nu) \qquad \forall\, \mathrm{tr}\in\mathbb{N}. \tag{5.12}$$

But $\mu_\beta \neq \nu$, and so (5.7) yields the claim with $\delta(\beta) = (\alpha - 1)h(\mu_\beta \mid \nu) > 0$. ∎

4. We finish by showing that Lemma 5.2 implies Corollary 1.3. The proof is by contradiction and uses (5.1). Suppose that $h_c^{\mathrm{que}}(\beta) = h_c^{\mathrm{ann}}(\beta)$. Then, with $(Q_{\mathrm{tr}})_{\mathrm{tr}\in\mathbb{N}}$ as in Lemma 5.1, we have

$$\begin{aligned} \Phi_{\beta,h_c^{\mathrm{que}}(\beta)}(Q_{\mathrm{tr}}) - I^{\mathrm{que}}(Q_{\mathrm{tr}}) &= \Phi_{\beta,h_c^{\mathrm{ann}}(\beta)}(Q_{\mathrm{tr}}) - I^{\mathrm{que}}(Q_{\mathrm{tr}}) \\ &\leq \Phi_{\beta,h_c^{\mathrm{ann}}(\beta)}(Q_{\mathrm{tr}}) - I^{\mathrm{ann}}(Q_{\mathrm{tr}}) \\ &\leq \Phi_{\beta,h_c^{\mathrm{ann}}(\beta)}(\pi_1 Q_{\mathrm{tr}}) - h\big(\pi_1 Q_{\mathrm{tr}} \mid q_{\rho,\nu}\big) \\ &\leq \sup_{\substack{q\in\mathcal{P}([\widetilde{E}]_{\mathrm{tr}}):\\ h(q|q_\beta)<\infty}} \big[\Phi_{\beta,h_c^{\mathrm{ann}}(\beta)}(q) - h(q \mid q_{\rho,\nu})\big] \\ &= \log \mathcal{N}_{\mathrm{tr}}\big(\beta, h_c^{\mathrm{ann}}(\beta); 0\big) - \inf_{\substack{q\in\mathcal{P}([\widetilde{E}]_{\mathrm{tr}}):\\ h(q|q_\beta)<\infty}} h(q \mid q_\beta) \\ &\leq - \inf_{\substack{q\in\mathcal{P}(\widetilde{E}):\\ h(q|q_\beta)<\infty}} h(q \mid q_\beta) = 0. \end{aligned} \tag{5.13}$$

The first inequality uses that $I^{\mathrm{que}} \geq I^{\mathrm{ann}}$, the second inequality that

$$I^{\mathrm{ann}}(Q_{\mathrm{tr}}) = H(Q_{\mathrm{tr}} \mid q_{\rho,\nu}^{\otimes\mathbb{N}}) \geq h(\pi_1 Q_{\mathrm{tr}} \mid q_{\rho,\nu}), \tag{5.14}$$

the third inequality that $h(\pi_1 Q_{\mathrm{tr}} \mid q_{\rho,\nu}) < \infty$ because $I^{\mathrm{ann}}(Q_{\mathrm{tr}}) < \infty$ (note from Lemma 5.1 that $Q_{\mathrm{tr}} \in \mathcal{C}^{\mathrm{fin}}$), the second equality uses the computation carried out in Section 3.2 (recall from (3.34) and (5.4) that $q_\beta = q_{\beta,h_c^{\mathrm{ann}}(\beta);0}$), and the fourth inequality uses that $\mathcal{N}_{\mathrm{tr}}(\beta, h_c^{\mathrm{ann}}(\beta); 0) = 1$ for all $\mathrm{tr} \in \mathbb{N}$, where $\mathcal{N}_{\mathrm{tr}}\big(\beta, h_c^{\mathrm{ann}}(\beta); 0\big)$ is as in (3.29) but with ρ replaced by ρ_{tr}.

Since, according to (5.1), the left-hand side of (5.13) tends to zero as $\mathrm{tr} \to \infty$, it follows from (5.13) that

$$\mathrm{w} - \lim_{\mathrm{tr}\to\infty} Q_{\mathrm{tr}} = q_\beta^{\otimes\mathbb{N}}, \tag{5.15}$$

where we use that the inequality in (5.14) is an equality if and only if Q_{tr} is a product measure. It now also follows from (5.13) that

$$\lim_{\mathrm{tr}\to\infty} \big[I^{\mathrm{que}}(Q_{\mathrm{tr}}) - I^{\mathrm{ann}}(Q_{\mathrm{tr}})\big] = 0, \tag{5.16}$$

which contradicts Lemma 5.2 because $m_{Q_{\mathrm{tr}}} \geq 1$ for all $\mathrm{tr} \in \mathbb{N}$. ∎

We close this section with the following remark. As (2.17) shows, $I^{\mathrm{fin}}(Q)$ depends on $q_{\rho,\nu}$, the reference law defined in (2.3). Since the latter depends on the full law $\rho \in \mathcal{P}(\mathbb{N})$ of the excursion lengths, it is evident from Lemma 5.1 that the quenched critical curve is *not* a function of the exponent α in (1.6) alone. This supports the statement made in Section 1.5, item 6.

6 Proof of Corollary 1.4

The proof is immediate from Theorem 3.3 (recall Fig. 6), which is proved in Sections 6.1–6.3.

6.1 Proof for $h > h_c^{\rm ann}(\beta/\alpha)$

Proof. Recall from (3.15–3.16) and (3.22) that

$$\begin{aligned} S^{\rm que}(\beta,h;g) &= \limsup_{N\to\infty} \frac{1}{N} \log F_N^{\beta,h,\omega}(g) \\ &= \log \mathcal{N}(g) + \limsup_{N\to\infty} \frac{1}{N} \log E_g^*\big(\exp\big[N\Phi_{\beta,h}(R_N^\omega)\big]\big). \end{aligned} \tag{6.1}$$

Abbreviate

$$S_N^\omega(g) = E_g^*\big(\exp\big[N\Phi_{\beta,h}(R_N^\omega)\big]\big) \tag{6.2}$$

and pick

$$t = [0,1], \qquad h = h_c^{\rm ann}(\beta t). \tag{6.3}$$

Then the t-th moment of $S_N^\omega(g)$ can be estimated as (recall (3.10–3.11))

$$\begin{aligned} \mathbb{E}\left([S_N^\omega(g)]^t\right) &= \mathbb{E}\left(\left[E_g^*\left(\exp\left[\sum_{i=1}^N \log\left(\tfrac12\left[1+e^{-2\beta\sum_{k\in I_i}(\omega_k+h)}\right]\right)\right]\right)\right]^t\right) \\ &= \mathbb{E}\left(\left[E_g^*\left(\prod_{i=1}^N \tfrac12\left[1+e^{-2\beta\sum_{k\in I_i}(\omega_k+h)}\right]\right)\right]^t\right) \\ &= \mathbb{E}\left(\left[\sum_{0<k_1<\cdots<k_N<\infty}\left\{\prod_{i=1}^N \rho_g(k_i-k_{i-1})\right\}\left\{\prod_{i=1}^N \tfrac12\left[1+e^{-2\beta\sum_{k\in(k_{i-1},k_i]}(\omega_k+h)}\right]\right\}\right]^t\right) \\ &\le \mathbb{E}\left(\sum_{0<k_1<\cdots<k_N<\infty}\left\{\prod_{i=1}^N \rho_g(k_i-k_{i-1})^t\right\}\left\{\prod_{i=1}^N 2^{-t}\left[1+e^{-2\beta t\sum_{k\in(k_{i-1},k_i]}(\omega_k+h)}\right]\right\}\right) \\ &= \sum_{0<k_1<\cdots<k_N<\infty}\left\{\prod_{i=1}^N \rho_g(k_i-k_{i-1})^t\right\}\left\{\prod_{i=1}^N 2^{-t}\left[1+e^{(k_i-k_{i-1})[M(2\beta t)-2\beta t h]}\right]\right\} \\ &= 2^{(1-t)N}\sum_{0<k_1<\cdots<k_N<\infty}\left\{\prod_{i=1}^N \rho_g(k_i-k_{i-1})^t\right\} \\ &= \left(2^{1-t}\sum_{k\in\mathbb{N}}\rho_g(k)^t\right)^N. \end{aligned} \tag{6.4}$$

The inequality uses that $(u+v)^t \le u^t + v^t$ for $u,v\ge 0$ and $t\in[0,1]$, while the fifth equality uses that $M(2\beta t) - 2\beta t h = 0$ for the choice of t and h in (6.3) (recall (1.21)).

Let $K(g)$ denote the term between round brackets in the last line of (6.4). Then, for every $\epsilon > 0$, we have

$$\begin{aligned} \mathbb{P}\left(\frac{1}{N}\log S_N^\omega(g) \ge \frac{1}{t}\left[\log K(g)+\epsilon\right]\right) &= \mathbb{P}\left([S_N^\omega(g)]^t \ge K(g)^N e^{N\epsilon}\right) \\ &\le \mathbb{E}\left([S_N^\omega(g)]^t\right) K(g)^{-N} e^{-N\epsilon} \le e^{-N\epsilon}. \end{aligned} \tag{6.5}$$

Since this bound is summable it follows from the Borel-Cantelli lemma that

$$\limsup_{N\to\infty} \frac{1}{N} \log S_N^{\omega}(g) \leq \frac{1}{t} \log K(g) \qquad \omega - a.s. \tag{6.6}$$

Combine (6.1–6.2) and (6.6) to obtain

$$\begin{aligned} S^{\mathrm{que}}(\beta, h; g) &\leq \log \mathcal{N}(g) + \frac{1-t}{t} \log 2 + \frac{1}{t} \log \left(\sum_{k\in\mathbb{N}} \rho_g(k)^t \right) \\ &= \frac{1-t}{t} \log 2 + \frac{1}{t} \log \left(\sum_{k\in\mathbb{N}} e^{-gtk} \rho(k)^t \right). \end{aligned} \tag{6.7}$$

We see from (6.7) that $S^{\mathrm{que}}(\beta, h_c^{\mathrm{ann}}(\beta t); g) < \infty$ for $g > 0$ and $t \in (0, 1]$, and also for $g = 0$ and $t \in (1/\alpha, 1]$, i.e., $S^{\mathrm{que}}(\beta, h; 0) < \infty$ for $h \in (h_c^{\mathrm{ann}}(\beta/\alpha), h_c^{\mathrm{ann}}(\beta)]$. This completes the proof because we already know that $S^{\mathrm{que}}(\beta, h; 0) < 0$ for $h \in (h_c^{\mathrm{ann}}(\beta), \infty)$. ∎

Note that if $\sum_{k\in\mathbb{N}} \rho(k)^{1/\alpha} < \infty$, then $S^{\mathrm{que}}(\beta, h_c^{\mathrm{ann}}(\beta/\alpha); 0) < \infty$. This explains the remark made below (3.23).

6.2 Proof for $h < h_c^{\mathrm{ann}}(\beta/\alpha)$

Proof. For $L \in \mathbb{N}$, define (recall (3.38))

$$q_\beta^L(k, dl) = \delta_{kL}\, \nu_{\beta/\alpha}^{\circledast L}(dl), \qquad (k, l) \in \mathbb{N} \times \mathbb{R}, \tag{6.8}$$

and

$$Q_\beta^L = (q_\beta^L)^{\otimes\mathbb{N}} \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}), \tag{6.9}$$

with $q_\beta^L(y)$, $y \in \widetilde{E}$, linked to $q_\beta^L(k, l)$, $(k, l) \in \mathbb{N} \times \mathbb{R}$, in the same manner as in (5.4). We will show that

$$h < h_c^{\mathrm{ann}}(\beta/\alpha) \quad \Longrightarrow \quad \liminf_{L\to\infty} \frac{1}{L} \left[\Phi_{\beta,h}(Q_\beta^L) - I^{\mathrm{que}}(Q_\beta^L) \right] > 0, \tag{6.10}$$

which will imply the claim because $Q_\beta^L \in \mathcal{C}^{\mathrm{fin}}$. (Recall (3.3) and note that both $m_{Q_\beta^L} = L$ and $I^{\mathrm{ann}}(Q_\beta^L) = h(q_\beta^L \mid q_{\rho,\nu}) = -\log \rho(L) + h(\nu_{\beta/\alpha} \mid \nu_\beta)$ are finite.)

We have (recall (3.6) and (3.27))

$$\begin{aligned} \Phi_{\beta,h}(Q_\beta^L) &= \sum_{k\in\mathbb{N}} \int_{l\in\mathbb{R}} q_\beta^L(k, dl) \log \phi_{\beta,h}(k, l), \\ H(Q_\beta^L \mid q_{\rho,\nu}^{\otimes\mathbb{N}}) = h(q_\beta^L \mid q_{\rho,\nu}) &= \sum_{k\in\mathbb{N}} \int_{l\in\mathbb{R}} q_\beta^L(k, dl) \log \Big(\frac{q_\beta^L(k, dl)}{q_{\rho,\nu}(k, dl)} \Big). \end{aligned} \tag{6.11}$$

Dropping the 1 in front of the exponential in (3.27), we obtain (similarly as in (3.39–3.41))

$$
\begin{aligned}
&\Phi_{\beta,h}(Q^L_\beta) - H(Q^L_\beta \mid q_{\rho,\nu}^{\otimes\mathbb{N}}) \\
&\quad\geq \log(\tfrac12) + \sum_{k\in\mathbb{N}} \int_{l\in\mathbb{R}} q^L_\beta(k,dl)\, \log\left[\frac{e^{-2\beta hk-2\beta l}\, q_{\rho,\nu}(k,dl)}{q^L_\beta(k,dl)}\right] \\
&\quad= \log(\tfrac12) + \int_{l\in\mathbb{R}} \nu^{\circledast L}_{\beta/\alpha}(dl)\, \log\left[e^{-2\beta hL}\, e^{-2\beta l}\, \frac{\nu^{\circledast L}(dl)}{\nu^{\circledast L}_{\beta/\alpha}(dl)}\, \rho(L)\right] \\
&\quad= \log(\tfrac12) + \int_{l\in\mathbb{R}} \nu^{\circledast L}_{\beta/\alpha}(dl)\, \log\left[e^{[M(2\beta)-2\beta h]L}\, \frac{\nu^{\circledast L}_{\beta}(dl)}{\nu^{\circledast L}_{\beta/\alpha}(dl)}\, \rho(L)\right] \\
&\quad= \log(\tfrac12) + [M(2\beta)-2\beta h]\, L - h\big(\nu_{\beta/\alpha} \mid \nu_\beta\big)\, L + \log\rho(L).
\end{aligned}
\tag{6.12}
$$

Furthermore, from (6.8) we have (recall (2.11))

$$
m_{Q^L_\beta} = L, \qquad \Psi_{Q^L_\beta} = \nu^{\otimes\mathbb{N}}_{\beta/\alpha}, \tag{6.13}
$$

which gives

$$
(\alpha-1)\, m_{Q^L_\beta}\, H\big(\Psi_{Q^L_\beta} \mid \nu^{\otimes\mathbb{N}}\big) = (\alpha-1)\, L\, h(\nu_{\beta/\alpha} \mid \nu). \tag{6.14}
$$

Combining (6.12–6.14), recalling (2.16–2.17) and using that $\lim_{L\to\infty} L^{-1}\log\rho(L) = 0$ by (1.6), we arrive at

$$
\begin{aligned}
\liminf_{L\to\infty} \frac{1}{L}\left[\Phi_{\beta,h}(Q^L_\beta) - I^{\mathrm{que}}(Q^L_\beta)\right] &\geq [M(2\beta)-2\beta h] - h\big(\nu_{\beta/\alpha} \mid \nu_\beta\big) - (\alpha-1)\, h(\nu_{\beta/\alpha} \mid \nu) \\
&= \alpha M(\tfrac{2\beta}{\alpha}) - 2\beta h = 2\beta\, [h^{\mathrm{ann}}_c(\beta/\alpha) - h],
\end{aligned}
\tag{6.15}
$$

where the first equality uses the relation (recall (1.21) and (3.38))

$$
\begin{aligned}
&h\big(\nu_{\beta/\alpha} \mid \nu_\beta\big) + (\alpha-1)\, h(\nu_{\beta/\alpha} \mid \nu) \\
&\quad= \int_{l\in\mathbb{R}} \nu_{\beta/\alpha}(dl)\, \Big(\Big[-\tfrac{2\beta}{\alpha}\, l - M\big(\tfrac{2\beta}{\alpha}\big)\Big] + [2\beta\, l + M(2\beta)] + (\alpha-1)\Big[-\tfrac{2\beta}{\alpha} l - M\big(\tfrac{2\beta}{\alpha}\big)\Big]\Big) \\
&\quad= M(2\beta) - \alpha M\big(\tfrac{2\beta}{\alpha}\big).
\end{aligned}
\tag{6.16}
$$

Note that (6.15) proves (6.10). ∎

6.3 Proof for $h = h^{\mathrm{ann}}_c(\beta/\alpha)$

Proof. Our starting point is (3.8), where (recall Theorem 2.3)

$$
I^{\mathrm{que}}(Q) = I^{\mathrm{fin}}(Q) = H(Q \mid q_{\rho,\nu}^{\otimes\mathbb{N}}) + (\alpha-1)\, m_Q\, H(\Psi_Q \mid \nu^{\otimes\mathbb{N}}), \qquad Q \in \mathcal{C}^{\mathrm{fin}}. \tag{6.17}
$$

The proof comes in 4 steps.

1. As shown in Birkner, Greven and den Hollander [2], Equation (1.32),

$$
H(Q \mid q_{\rho,\nu}^{\otimes\mathbb{N}}) = m_Q\, H(\Psi_Q \mid \nu^{\otimes\mathbb{N}}) + R(Q), \tag{6.18}
$$

where $R(Q) \geq 0$ is the "specific relative entropy w.r.t. $\rho^{\otimes\mathbb{N}}$ of the word length process under Q conditional on the concatenation". Combining (6.17–6.18), we have $I^{\mathrm{que}}(Q) \leq \alpha\, H(Q \mid q_{\rho,\nu}^{\otimes\mathbb{N}})$, which yields

$$S_*^{\mathrm{que}}(\beta,h) \geq \sup_{Q\in\mathcal{C}^{\mathrm{fin}}} \big[\Phi_{\beta,h}(Q) - \alpha\, H(Q \mid q_{\rho,\nu}^{\otimes\mathbb{N}})\big]. \tag{6.19}$$

2. The variational formula in the right-hand side of (6.19) can be computed similarly as in part (I) of Section 3.2. Indeed,

$$\text{r.h.s. (6.19)} = \sup_{\substack{q\in\mathcal{P}(\widetilde{E})\\ m_q<\infty,\, h(q|q_{\rho,\nu})<\infty}} \left[\int_{\widetilde{E}} q(k,dl)\log\phi_{\beta,h}(k,l) - \alpha\, h(q \mid q_{\rho,\nu})\right]. \tag{6.20}$$

Define

$$q_{\beta,h}(k,dl) = \frac{1}{\mathcal{N}(\beta,h)}\,[\phi_{\beta,h}(k,l)]^{1/\alpha}\, q_{\rho,\nu}(k,dl), \tag{6.21}$$

where $\mathcal{N}(\beta,h)$ is the normalizing constant. Then the term between square brackets in the right-hand side of (6.20) equals $\alpha\log\mathcal{N}(\beta,h) - \alpha h(q \mid q_{\beta,h})$, and hence

$$S_*^{\mathrm{que}}(\beta,h) \geq \alpha\log\mathcal{N}(\beta,h), \tag{6.22}$$

provided $\mathcal{N}(\beta,h) < \infty$ so that $q_{\beta,h}$ is well-defined.

3. Abbreviate $\mu = 2\beta/\alpha$. Since $h_c^{\mathrm{ann}}(\beta/\alpha) = M(\mu)/\mu$, we have

$$\mathcal{N}\big(\beta,h_c^{\mathrm{ann}}(\beta/\alpha)\big) = \sum_{k\in\mathbb{N}}\rho(k)\int_{l\in\mathbb{R}}\nu^{\circledast k}(dl)\,\left\{\tfrac{1}{2}\left(1+e^{-\alpha[M(\mu)k+\mu l]}\right)\right\}^{1/\alpha}. \tag{6.23}$$

Let Z be the random variable on $(0,\infty)$ with law P that is equal in distribution to the random variable $e^{-[M(\mu)k+\mu l]}$ with law $\rho(k)\,\nu^{\circledast k}(dl)$. Let $f(z) = \{\frac{1}{2}(1+z^\alpha)\}^{1/\alpha}$, $z>0$. Then

$$\text{r.h.s. (6.23)} = E(f(Z)). \tag{6.24}$$

We have $E(Z) = 1$. Moreover, an easy computation gives

$$\begin{aligned} f'(z) &= (\tfrac{1}{2})^{1/\alpha}\,(1+z^\alpha)^{(1/\alpha)-1}\,z^{\alpha-1},\\ f''(z) &= (\tfrac{1}{2})^{1/\alpha}\,(1+z^\alpha)^{(1/\alpha)-2}\,z^{\alpha-2}\,(\alpha-1), \end{aligned} \tag{6.25}$$

so that f is strictly convex. Therefore, by Jensen's inquality and the fact that P is not a point mass, we have

$$E(f(Z)) > f(E(Z)) = f(1) = 1. \tag{6.26}$$

Combining (6.22–6.24) and (6.26), we arrive at

$$S_*^{\mathrm{que}}\big(\beta,h_c^{\mathrm{ann}}(\beta/\alpha)\big) > 0, \tag{6.27}$$

which proves the claim.

4. It remains to check that $\mathcal{N}(\beta,h_c^{\mathrm{ann}}(\beta/\alpha)) < \infty$. But $f(z) \leq (\frac{1}{2})^{1/\alpha}(1+z)$, $z>0$, and so we have

$$\mathcal{N}\big(\beta,h_c^{\mathrm{ann}}(\beta/\alpha)\big) \leq (\tfrac{1}{2})^{1/\alpha}(1+E(Z)) \leq 2^{1-(1/\alpha)} < \infty. \tag{6.28}$$

∎

7 Proof of Corollaries 1.5 and 1.6

Corollaries 1.5 and 1.6 are proved in Sections 7.1 and 7.2, respectively.

7.1 Proof of Corollary 1.5

Proof. Fix $(\beta,h)\in \mathrm{int}(\mathcal{D}^{\mathrm{que}})$. We know that $S^{\mathrm{que}}(\beta,h;0)<0$ (recall Fig. 6) and $\sum_{n\in\mathbb{N}}\widetilde{Z}_n^{\beta,h,\omega}<\infty$. It follows from (3.16) and (3.22) that for every $\epsilon>0$ and ω-a.s. there exists an $N_0=N_0(\omega,\epsilon)<\infty$ such that

$$F_N^{\beta,h,\omega}(0)\le e^{N[S^{\mathrm{que}}(\beta,h;0)+\epsilon]},\qquad N\ge N_0. \tag{7.1}$$

For E an arbitrary event, write $\widetilde{Z}_n^{\beta,h,\omega}(E)$ to denote the constrained partition restricted to E. Estimate, for $M\in\mathbb{N}$ and ϵ small enough such that $S^{\mathrm{que}}(\beta,h;0)+\epsilon<0$,

$$\begin{aligned}\widetilde{\mathcal{P}}_n^{\beta,h,\omega}(\mathcal{M}_n\ge M)&=\frac{\widetilde{Z}_n^{\beta,h,\omega}(\mathcal{M}_n\ge M)}{\widetilde{Z}_n^{\beta,h,\omega}}\le\frac{\sum_{n\in\mathbb{N}}\widetilde{Z}_n^{\beta,h,\omega}(\mathcal{M}_n\ge M)}{\widetilde{Z}_n^{\beta,h,\omega}}\\&=\frac{1}{\widetilde{Z}_n^{\beta,h,\omega}}\sum_{N\ge M}F_N^{\beta,h,\omega}(0)\le\frac{2}{\rho(n)}\frac{e^{M[S^{\mathrm{que}}(\beta,h;0)+\epsilon]}}{1-e^{[S^{\mathrm{que}}(\beta,h;0)+\epsilon]}},\end{aligned}\tag{7.2}$$

where the second equality follows from (3.11–3.13). The second inequality follows from (7.1) and the bound $\widetilde{Z}_n^{\beta,h,\omega}\ge\frac12\rho(n)$, the latter being immediate from (1.15) and the fact that every excursion has probability $\frac12$ of lying below the interface. Since $\rho(n)=n^{-\alpha+o(1)}$, we get the claim by choosing $M=\lceil c\log n\rceil$ with c such that $\alpha+c[S^{\mathrm{que}}(\beta,h;0)+\epsilon]<0$, and letting $n\to\infty$ followed by $\epsilon\downarrow 0$. ■

7.2 Proof of Corollary 1.6

Proof. Fix $(\beta,h)\in\mathcal{L}^{\mathrm{que}}$. We know that $g^{\mathrm{que}}(\beta,h)>0$ and $S^{\mathrm{que}}(\beta,h;g^{\mathrm{que}}(\beta,h))=0$. It follows from (3.16) and (3.22) that for every $\epsilon,\delta>0$ and ω-a.s. there exist $n_0=n_0(\omega,\epsilon)<\infty$ and $M_0=M_0(\omega,\delta)<\infty$ such that

$$\begin{aligned}\widetilde{Z}_n^{\beta,h,\omega}&\ge e^{n[g^{\mathrm{que}}(\beta,h)-\epsilon]}, && n\ge n_0,\\ F_M^{\beta,h,\omega}(g^{\mathrm{que}}(\beta,h)+\delta)&\le e^{M[S^{\mathrm{que}}(\beta,h;g^{\mathrm{que}}(\beta,h)+\delta)+\delta^2]}, && M\ge M_0,\\ F_M^{\beta,h,\omega}(g^{\mathrm{que}}(\beta,h)-\delta)&\le e^{M[S^{\mathrm{que}}(\beta,h;g^{\mathrm{que}}(\beta,h)-\delta)+\delta^2]}, && M\ge M_0.\end{aligned}\tag{7.3}$$

For every $M_1,M_2\in\mathbb{N}$ with $M_1<M_2$ we have

$$\widetilde{\mathcal{P}}_n^{\beta,h,\omega}(M_1<\mathcal{M}_n<M_2)=1-\Big[\widetilde{\mathcal{P}}_n^{\beta,h,\omega}(\mathcal{M}_n\ge M_2)+\widetilde{\mathcal{P}}_n^{\beta,h,\omega}(\mathcal{M}_n\le M_1)\Big]. \tag{7.4}$$

Below we show that the probabilities in the right-hand side of (7.4) vanish as $n\to\infty$ when $M_1=\lceil c_1 n\rceil$ with $c_1<C_-$ and $M_2=\lceil c_2 n\rceil$ with $c_2>C_+$, respectively, where

$$\begin{aligned}-\frac{1}{C_-}&=\left(\frac{\partial}{\partial g}\right)^-S^{\mathrm{que}}\big(\beta,h;g^{\mathrm{que}}(\beta,h)\big),\\-\frac{1}{C_+}&=\left(\frac{\partial}{\partial g}\right)^+S^{\mathrm{que}}\big(\beta,h;g^{\mathrm{que}}(\beta,h)\big),\end{aligned}\tag{7.5}$$

are the left-derivative and right-derivative of $g \mapsto S^{\mathrm{que}}(\beta, h; g)$ at $g = g^{\mathrm{que}}(\beta, h)$, which exist by convexity, are strictly negative (recall Fig. 5) and satisfy $C_- \leq C_+$. Throughout the proof we assume that $M_1 \geq M_0$.

• Put $M_2 = \lceil c_2 n \rceil$, and abbreviate

$$a(\beta, h, \delta) = S^{\mathrm{que}}(\beta, h; g^{\mathrm{que}}(\beta, h) + \delta) + \delta^2, \tag{7.6}$$

where we choose δ small enough such that $a(\beta, h, \delta) < 0$ (recall Fig. 6). Estimate

$$\begin{aligned}
\widetilde{\mathcal{P}}_n^{\beta,h,\omega}(\mathcal{M}_n \geq M_2) &= \frac{\widetilde{Z}_n^{\beta,h,\omega}\,(\mathcal{M}_n \geq M_2)}{\widetilde{Z}_n^{\beta,h,\omega}} \\
&\leq e^{n[\epsilon+\delta]}\, \widetilde{Z}_n^{\beta,h,\omega}\,(\mathcal{M}_n \geq M_2)\; e^{-n[g^{\mathrm{que}}(\beta,h)+\delta]} \\
&\leq e^{n[\epsilon+\delta]} \sum_{n' \in \mathbb{N}} \widetilde{Z}_{n'}^{\beta,h,\omega}(\mathcal{M}_{n'} \geq M_2)\, e^{-n'[g^{\mathrm{que}}(\beta,h)+\delta]} \\
&= e^{n[\epsilon+\delta]} \sum_{N \geq M_2} F_N^{\beta,h,\omega}(g^{\mathrm{que}}(\beta, h) + \delta) \\
&\leq e^{n[\epsilon+\delta]} \sum_{N \geq M_2} e^{N a(\beta,h,\delta)} \\
&= \frac{e^{n[\epsilon+\delta+c_2 a(\beta,h,\delta)]}}{1 - e^{a(\beta,h,\delta)}}.
\end{aligned} \tag{7.7}$$

The first inequality follows from the first line in (7.3), the second equality from (3.11–3.13), and the third inequality from (7.3). The claim follows by picking c_2 such that

$$\epsilon + \delta + c_2 a(\beta, h, \delta) < 0, \tag{7.8}$$

letting $n \to \infty$ followed by $\epsilon \downarrow 0$ and $\delta \downarrow 0$, and using that

$$\lim_{\delta \downarrow 0} \frac{1}{\delta}\, a(\beta, h, \delta) = \left(\frac{\partial}{\partial g}\right)^{+} S^{\mathrm{que}}(\beta, h; g^{\mathrm{que}}(\beta, h)) = -\frac{1}{C_+} < -\frac{1}{c_2}. \tag{7.9}$$

• Put $M_1 = \lceil c_1 n \rceil$ and abbreviate

$$b(\beta, h, \delta) = S^{\mathrm{que}}(\beta, h; g^{\mathrm{que}}(\beta, h) - \delta) + \delta^2, \tag{7.10}$$

where we choose δ small enough such that $b(\beta, h, \delta) > 0$. Split

$$\widetilde{\mathcal{P}}_n^{\beta,h,\omega}(\mathcal{M}_n \leq M_1) = I + II \tag{7.11}$$

with

$$I = \frac{\widetilde{Z}_n^{\beta,h,\omega}\,(\mathcal{M}_n < M_0)}{\widetilde{Z}_n^{\beta,h,\omega}}, \qquad II = \frac{\widetilde{Z}_n^{\beta,h,\omega}\,(M_0 \leq \mathcal{M}_n \leq M_1)}{\widetilde{Z}_n^{\beta,h,\omega}}. \tag{7.12}$$

Since

$$I \leq e^{-n[g^{\mathrm{que}}(\beta,h)-\epsilon]}\, \widetilde{Z}_n^{\beta,h,\omega}\,(\mathcal{M}_n < M_0) = e^{-n[g^{\mathrm{que}}(\beta,h)-\epsilon]} \sum_{N < M_0} F_N^{\beta,h,\omega}(g^{\mathrm{que}}(\beta, h) - \delta), \tag{7.13}$$

this term is harmless as $n \to \infty$. Repeat the arguments leading to (7.7), to estimate

$$\begin{aligned} II &\leq e^{n[\epsilon-\delta]} \sum_{n'\in\mathbb{N}} \widetilde{Z}_{n'}^{\beta,h,\omega}\,(M_0 \leq \mathcal{M}_{n'} \leq M_1)\; e^{-n'[g^{\mathrm{que}}(\beta,h)-\delta]} \\ &= e^{n[\epsilon-\delta]} \sum_{M_0\leq N\leq M_1} F_N^{\beta,h,\omega}(g^{\mathrm{que}}(\beta,h)-\delta) \\ &\leq e^{n[\epsilon-\delta]} \sum_{M_0\leq N\leq M_1} e^{Nb(\beta,h,\delta)} \\ &\leq e^{n[\epsilon-\delta+c_1 b(\beta,h,\delta)]} \sum_{N\leq M_1} e^{[N-M_1]b(\beta,h,\delta)} \\ &\leq \frac{e^{n[\epsilon-\delta+c_1 b(\beta,h,\delta)]}}{1-e^{-b(\beta,h,\delta)}}. \end{aligned} \tag{7.14}$$

Therefore the assertion follows by choosing c_1 such that

$$\epsilon - \delta + c_1 b(\beta,h,\delta) < 0, \tag{7.15}$$

letting $n \to \infty$ followed by $\epsilon \downarrow 0$ and $\delta \downarrow 0$, and using that

$$\lim_{\delta\downarrow 0} \frac{1}{\delta}\, b(\beta,h,\delta) = -\left(\frac{\partial}{\partial g}\right)^{-} S^{\mathrm{que}}(\beta,h;g^{\mathrm{que}}(\beta,h)) = \frac{1}{C_-} < \frac{1}{c_1}. \tag{7.16}$$

Recalling (7.4), we have now proved that

$$\lim_{n\to\infty} \widetilde{\mathcal{P}}_n^{\beta,h,\omega}(\lceil c_1 n\rceil < \mathcal{M}_n < \lceil c_2 n\rceil) = 1 \qquad \forall\, c_1 < C_-,\; c_2 > C_+. \tag{7.17}$$

Finally, if (1.28) holds, then $C_- = C_+$, and we get the law of large numbers in (1.27). ∎

A Control of $\Phi_{\beta,h}$

In this Appendix we prove that $h(\pi_1 Q|q_{\rho,\nu}) < \infty$ implies that $\Phi_{\beta,h}(Q) < \infty$ for all $\beta, h > 0$. In the proof we make use of a concentration of measure estimate for the disorder ω whose proof is given in Appendix C.

Lemma A.1 *Fix $\beta, h > 0$, $\rho \in \mathcal{P}(\mathbb{N})$ and $\nu \in \mathcal{P}(\mathbb{R})$. Then, for all $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{\mathbb{R}}^{\mathbb{N}})$ with $h(\pi_1 Q|q_{\rho,\nu}) < \infty$, there are finite constants $C > 0$, $\gamma > \frac{2\beta}{C}$ and $K = K(\beta,h,\rho,\nu,\gamma)$ such that*

$$\Phi_{\beta,h}(Q) \leq \gamma\, h(\pi_1 Q|q_{\rho,\nu}) + K. \tag{A.1}$$

Proof. Abbreviate

$$f(y) = \frac{d(\pi_1 Q)}{dq_{\rho,\nu}}(y), \quad u(y) = -2\beta[\tau(y)h + \sigma(y)], \qquad y \in \widetilde{\mathbb{R}} = \cup_{n\in\mathbb{N}}\mathbb{R}^n. \tag{A.2}$$

Fix $\gamma > 2\beta/C$, with $C > 0$ as in (C.8), and for $n, m \in \mathbb{N}$ define

$$\begin{aligned} A_{m,n} &= \{y \in \mathbb{R}^n\colon\ m-1 \leq \gamma \log f(y) < m\}, \\ A_{0,n} &= \{y \in \mathbb{R}^n\colon\ 0 \leq f(y) < 1\}, \\ B_{m,n} &= \{y \in \mathbb{R}^n\colon\ m-1 \leq u(y) < m\}. \end{aligned} \tag{A.3}$$

Note that

$$\mathbb{R}^n = A_{0,n} \cup [\cup_{m\in\mathbb{N}} A_{m,n}], \quad n \in \mathbb{N}, \tag{A.4}$$

and that

$$B_n = \bigcup_{m\in\mathbb{N}} B_{m,n}, \quad n \in \mathbb{N}, \tag{A.5}$$

is the set of points $y \in \mathbb{R}^n$ for which $u(y) \geq 0$. This gives rise to the decomposition

$$\begin{aligned}
\Phi_{\beta,h}(Q) &= \sum_{n\in\mathbb{N}} \int_{\mathbb{R}^n} \log\left(\tfrac{1}{2}\left[1 + e^{u(y)}\right]\right) (\pi_1 Q)(dy) \\
&\leq \sum_{n\in\mathbb{N}} \int_{\mathbb{R}^n} \log\left(1 \vee e^{u(y)}\right) (\pi_1 Q)(dy) \\
&= \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \int_{B_{m,n}} u(y) f(y)\, q_{\rho,\nu}(dy) \\
&= I + II + III
\end{aligned} \tag{A.6}$$

with

$$\begin{aligned}
I &= \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \int_{[\cup_{l\in\mathbb{N}_0} B_{m+l,n}]\cap A_{m,n}} u(y) f(y)\, q_{\rho,\nu}(dy) \\
II &= \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \int_{A_{m,n}\cap[\cup_{l=1}^{m-1} B_{l,n}]} u(y) f(y)\, q_{\rho,\nu}(dy), \\
III &= \sum_{n\in\mathbb{N}} \int_{A_{0,n}\cap[\cup_{m\in\mathbb{N}}\, B_{m,n}]} u(y)\, f(y)\, q_{\rho,\nu}(dy).
\end{aligned} \tag{A.7}$$

The terms I and II deal with the set $B_n \cap \bigcup_{m\in\mathbb{N}} A_{m,n}$, while III deals with the set $B_n \cap A_{0,n}$. Note that

$$\begin{aligned}
I &\leq \sum_{n\in\mathbb{N}} \rho(n) \sum_{m\in\mathbb{N}} e^{m/\gamma} \sum_{l\in\mathbb{N}_0} (m+l)\, \mathbb{P}(B_{m+l,n}), \\
III &\leq \sum_{n\in\mathbb{N}} \rho(n) \sum_{m\in\mathbb{N}} m\, \mathbb{P}(B_{m,n}),
\end{aligned} \tag{A.8}$$

where we recall that $\mathbb{P} = \nu^{\otimes\mathbb{N}}$. The upper bound on I uses that $f \leq e^{m\gamma}$ on $A_{m,n}$ and $u < m$ on $B_{m,n}$. The upper bound on III uses that $f \leq 1$ on $A_{0,n}$ and $u < m$ on $B_{m,n}$. We need to show that each of the three terms is finite. Observe from (A.8) that $III \leq I$. Hence it suffices to show that I and II are finite.

I: Estimate

$$\begin{aligned}
I &\leq \sum_{n\in\mathbb{N}} \rho(n) \sum_{m\in\mathbb{N}} e^{m/\gamma} \sum_{l\in\mathbb{N}_0} (m+l)\, \mathbb{P}(B_{m+l,n}) \\
&\leq \sum_{n\in\mathbb{N}} \rho(n) \sum_{m\in\mathbb{N}} e^{m/\gamma} \sum_{l\in\mathbb{N}_0} (l+m)\, \mathbb{P}\left(\sum_{i=1}^{n} \omega_i \leq -\left[nh + \frac{l+m-1}{2\beta}\right]\right) \\
&\leq \sum_{n\in\mathbb{N}} \rho(n)\, e^{-Cn} \sum_{m\in\mathbb{N}} e^{m/\gamma} \sum_{l\in\mathbb{N}_0} (l+m) \exp\left[-C(l+m-1)\right] < \infty,
\end{aligned} \tag{A.9}$$

where the third inequality follows from Lemma C.1, with $A = \frac{l+m-1}{2\beta}$, $B = h$ and $C > 0$ (depending on β, h; see (C.6–C.8)).

II: Use that $u(y) < m - 1 \leq \gamma \log f(y)$ for $y \in A_{m,n} \cap [\cup_{l=1}^{m-1} B_{l,n}]$, to estimate

$$\begin{aligned} II &\leq \gamma \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \int_{A_{m,n}\cap[\cup_{l=1}^{m-1} B_{l,n}]} f(y) \log f(y)\, q_{\rho,\nu}(dy) \\ &\leq \gamma \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} \int_{A_{m,n}} f(y) \log f(y)\, q_{\rho,\nu}(dy) \\ &= \gamma \sum_{n\in\mathbb{N}} \int_{\mathbb{R}^n\setminus A_{0,n}} f(y) \log f(y)\, q_{\rho,\nu}(dy) < \infty. \end{aligned} \tag{A.10}$$

The finiteness of the last term stems from the fact that

$$h(\pi_1 Q|q_{\rho,\nu}) = \sum_{n\in\mathbb{N}} \int_{\mathbb{R}^n\setminus A_{0,n}} f(y) \log f(y)\, q_{\rho,\nu}(dy) + \sum_{n\in\mathbb{N}} \int_{A_{0,n}} f(y) \log f(y)\, q_{\rho,\nu}(dy) \tag{A.11}$$

is assumed to be finite, while the second term in the right-hand side of (A.11) lies in $[-1/e, 0]$. ■

B Application of Varadhan's lemma

This Appendix settles (3.20) for $\beta, h > 0$ and $g \geq 0$. We first establish the variational formula on $\mathcal{C}$ (Lemma B.1 in Appendix B.1), and then on $\mathcal{C}^{\text{fin}}$ (Lemma B.2 in Appendix B.2), proving also the identity $S^{\text{que}}(\beta, h; 0) = S^{\text{que}}_*(\beta, h)$ (Lemma B.3 in Appendix B.2). The proof uses a truncation argument (Lemma B.4 in Appendix B.3).

B.1 Variational formula on $\mathcal{C}$

Lemma B.1 *For all $\beta, h > 0$ and $g \geq 0$,*

$$\bar{S}^{\text{que}}(\beta, h; g) = \begin{cases} \sup_{Q\in\mathcal{C}\cap\mathcal{R}} [\Phi_{\beta,h}(Q) - g m_Q - I^{\text{ann}}(Q)], & \text{if } g > 0, \\ \sup_{Q\in\mathcal{C}} [\Phi_{\beta,h}(Q) - I^{\text{que}}(Q)], & \text{if } g = 0, \end{cases} \tag{B.1}$$

where $\bar{S}^{\text{que}}(\beta, h; g)$ is the ω-a.s. constant limit defined in (3.16).

Proof. Throughout the proof, $\beta, h > 0$ and $g \geq 0$ are fixed. Note that, since $h(\pi_1 Q|q_{\rho,\nu}) \leq H(Q|q_{\rho,\nu}^{\otimes\mathbb{N}}) = I^{\text{ann}}(Q) < \infty$, it follows from (3.6–3.7) and Lemma A.1 that $\Phi_{\beta,h}(Q)$ is finite on $\mathcal{C} = \{Q \in \mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}})\colon I^{\text{ann}}(Q) < \infty\}$. We treat the cases $g > 0$ and $g = 0$ separately.

Case 1: $g > 0$.

Lower bound: Because $\Phi_{\beta,h}$ is lower semi-continuous on $\mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}})$ and finite on $\mathcal{C}$, the set

$$\mathcal{A}_\epsilon = \left\{Q' \in \mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}})\colon \Phi_{\beta,h}(Q') > \Phi_{\beta,h}(Q) - \epsilon\right\} \tag{B.2}$$

is open for every $Q \in \mathcal{C}$ and $\epsilon > 0$. Fix $Q \in \mathcal{C} \cap \mathcal{R}$ and $\epsilon > 0$, and use (3.15–3.16) to estimate

$$\begin{aligned} \bar{S}^{\text{que}}(\beta, h; g) &= \log \mathcal{N}(g) + \limsup_{N\to\infty} \frac{1}{N} \log E^*_g\Big(e^{N\Phi_{\beta,h}(R^\omega_N)}\Big) \\ &\geq \log \mathcal{N}(g) + \liminf_{N\to\infty} \frac{1}{N} \log E^*_g\Big(e^{N\Phi_{\beta,h}(R^\omega_N)}\, 1_{\mathcal{A}_\epsilon}(R^\omega_N)\Big) \\ &\geq \log \mathcal{N}(g) + \inf_{Q'\in\mathcal{A}_\epsilon} \Phi_{\beta,h}(Q') + \liminf_{N\to\infty} \frac{1}{N} \log P^*_g(\mathcal{A}_\epsilon) \\ &\geq \log \mathcal{N}(g) + \inf_{Q'\in\mathcal{A}_\epsilon} \Phi_{\beta,h}(Q') - \inf_{Q'\in\mathcal{A}_\epsilon} I^{\text{que}}_g(Q') \\ &\geq \log \mathcal{N}(g) + \Phi_{\beta,h}(Q) - I^{\text{que}}_g(Q) - \epsilon, \end{aligned} \tag{B.3}$$

where in the third inequality we use the quenched LDP in Theorem 2.3. Next, note that $I_g^{\rm que}(Q) = I_g^{\rm ann}(Q)$ for $Q \in \mathcal{R}$ by Theorem 2.3 and $I_g^{\rm ann}(Q) = I^{\rm ann}(Q) + \log \mathcal{N}(g) + g m_Q$ for $Q \in \mathcal{C}$ by Lemma 2.1. Insert these identities, take the supremum over $Q \in \mathcal{C} \cap \mathcal{R}$ and let $\epsilon \downarrow 0$, to arrive at the desired lower bound. Note that Q's in $\mathcal{C}$ with $m_Q = \infty$ do not contribute to the supremum.

Upper bound: The proof of the upper bound uses a truncation argument and comes in 3 steps.

1. For $M > 0$, let

$$\Phi_{\beta,h}^M(Q) = \int_{\widetilde{E}} (\pi_1 Q)(dy) \, [\log \phi_{\beta,h}(y) \wedge M] \tag{B.4}$$

(compare with (3.6–3.7)). Since $\phi_{\beta,h} \geq \frac{1}{2}$, $Q \mapsto \Phi_{\beta,h}^M(Q)$ is bounded and continuous. Our approach will be to compare $\bar{S}^{\rm que}(\beta,h;g)$ with its truncated analogue

$$\bar{S}_{M,p}^{\rm que}(\beta,h;g) = \log \mathcal{N}(g) + \frac{1}{p} \limsup_{N\to\infty} \frac{1}{N} \log E_g^*\big(e^{pN\Phi_{\beta,h}^M(R_N^\omega)}\big), \qquad M>0,\, p>1, \tag{B.5}$$

and afterwards let $M \to \infty$ and $p \downarrow 1$.

2. Abbreviate $\chi(y) = \log \phi_{\beta,h}(y)$ and note that

$$\Phi_{\beta,h}(Q) \leq \Phi_{\beta,h}^M(Q) + \int_{\widetilde{E}} (\pi_1 Q)(dy) \, 1_{\{\chi(y)>M\}} \, \chi(y). \tag{B.6}$$

Therefore, for any $g > 0$ and $p,q > 1$ such that $p^{-1} + q^{-1} = 1$, Hölder's inequality gives

$$\begin{aligned} \frac{1}{N} \log \Big[E_g^* \Big(e^{N\Phi_{\beta,h}(R_N^\omega)} \Big) \Big] &\leq \frac{1}{pN} \log \Big[E_g^* \Big(e^{pN\Phi_{\beta,h}^M(R_N^\omega)} \Big) \Big] \\ &\quad + \frac{1}{qN} \log \Big[E_g^* \Big(e^{q\sum_{i=1}^N \chi(y_i)\, 1_{\{\chi(y_i)>M\}}} \Big) \Big]. \end{aligned} \tag{B.7}$$

Here we use that $N \int_{\widetilde{E}} (\pi_1 R_N^\omega)(dy) \, 1_{\{\chi(y)>M\}} \, \chi(y) = \sum_{i=1}^N \chi(y_i) \, 1_{\{\chi(y_i)>M\}}$. We next claim that ω-a.s. there exist $M'(\omega) < \infty$ and $N'(\omega) < \infty$ such that, for all $N \geq N'(\omega)$ and $M \geq M'(\omega)$,

$$\frac{1}{qN} \log \Big[E_g^* \Big(e^{q\sum_{i=1}^N \chi(y_i)\, 1_{\{\chi(y_i)>M\}}} \Big) \Big] \leq 0. \tag{B.8}$$

This claim will be proved in Lemma B.4 in Appendix B.3.

3. We apply Varadhan's lemma to (B.5) using Theorem 2.3 and the fact that $\Phi_{\beta,h}^M$ is bounded and continuous on $\mathcal{P}^{\rm inv}(\widetilde{E}^{\mathbb{N}})$. This gives

$$\begin{aligned} \bar{S}_{M,p}^{\rm que}(\beta,h;g) &= \log \mathcal{N}(g) + \frac{1}{p} \sup_{Q\in\mathcal{R}} \big[p\Phi_{\beta,h}^M(Q) - I_g^{\rm que}(Q) \big] \\ &= \log \mathcal{N}(g) + \frac{1}{p} \sup_{Q\in\mathcal{R}} \big[p\, \Phi_{\beta,h}^M(Q) - I_g^{\rm ann}(Q) \big] \\ &= \Big(1 - \frac{1}{p}\Big) \log \mathcal{N}(g) + \frac{1}{p} \sup_{Q\in\mathcal{R}} \big[p\, \Phi_{\beta,h}^M(Q) - g\, m_Q - I^{\rm ann}(Q) \big] \\ &= \Big(1 - \frac{1}{p}\Big) \log \mathcal{N}(g) + \frac{1}{p} \sup_{Q\in\mathcal{C}\cap\mathcal{R}} \big[p\Phi_{\beta,h}^M(Q) - g\, m_Q - I^{\rm ann}(Q) \big], \end{aligned} \tag{B.9}$$

where the last equality uses that $\Phi^M_{\beta,h} \leq M < \infty$ and the fact that the Q's with $I^{\mathrm{ann}}(Q) = m_Q = \infty$ do not contribute to the supremum. Combining (3.15–3.16), (B.5) and (B.7–B.9) we find that, for every $p > 1$ and $M > M'(\omega)$,

$$\begin{aligned} \bar{S}^{\mathrm{que}}(\beta, h; g) &= \log \mathcal{N}(g) + \limsup_{N\to\infty} \frac{1}{N} \log E^*_g \left(e^{N\Phi_{\beta,h}(R^\omega_N)} \right) \\ &\leq \bar{S}^{\mathrm{que}}_{M,p}(\beta, h; g) = \left(1 - \frac{1}{p}\right) \log \mathcal{N}(g) + \frac{1}{p} A^M(p) \end{aligned} \tag{B.10}$$

with

$$A^M(p) = \sup_{Q\in\mathcal{C}\cap\mathcal{R}} \left[p\Phi^M_{\beta,h}(Q) - g\, m_Q - I^{\mathrm{ann}}(Q) \right]. \tag{B.11}$$

Since $p \mapsto A^M(p)$ is a supremum of linear functions with a slope that is bounded from above by M, it is convex and finite on $[0, \infty)$ and therefore is continuous. Hence, letting $p \downarrow 1$ we get that, for every $M > M'(\omega)$,

$$\begin{aligned} \bar{S}^{\mathrm{que}}(\beta, h; 0) \leq A^M(1) &= \sup_{Q\in\mathcal{C}\cap\mathcal{R}} \left[\Phi^M_{\beta,h}(Q) - g\, m_Q - I^{\mathrm{ann}}(Q) \right] \\ &\leq \sup_{Q\in\mathcal{C}\cap\mathcal{R}} \left[\Phi_{\beta,h}(Q) - g\, m_Q - I^{\mathrm{ann}}(Q) \right], \end{aligned} \tag{B.12}$$

since $\Phi^M_{\beta,h} \leq \Phi_{\beta,h}$. Note that the Q's with $m_Q = \infty$ do not contribute to the above suprema.

Case 2: $g = 0$. The proof follows from that of Case 1 upon choosing $Q \in \mathcal{C}$ for the open set $\mathcal{A}_\epsilon$ in (B.3) and applying the version of Theorem 2.3 for $g = 0$ to (B.3) and (B.9). ∎

B.2 Reduction from $\mathcal{C}$ to $\mathcal{C}^{\mathrm{fin}}$

Next we restrict our variational formula in (B.1) from $\mathcal{C}$ to $\mathcal{C}^{\mathrm{fin}}$. This reduction is trivial for the case $g > 0$, since then the $Q \in \mathcal{C}$ with $m_Q = \infty$ do not contribute.

Lemma B.2 *For $\beta, h > 0$ and $g \geq 0$, the variational formula* (B.1) *remains the same upon replacing $\mathcal{C}$ by $\mathcal{C}^{\mathrm{fin}}$.*

Proof. Recall Lemma B.1. First we observe that

$$\begin{aligned} \bar{S}^{\mathrm{que}}(\beta, h; 0) &= \sup_{Q\in\mathcal{C}} [\Phi_{\beta,h}(Q) - I^{\mathrm{que}}(Q)] \\ &= \sup_{\substack{Q\in\mathcal{C}:\\ I^{\mathrm{que}}(Q)<\infty}} [\Phi_{\beta,h}(Q) - I^{\mathrm{que}}(Q)] \\ &\geq \sup_{\substack{Q\in\mathcal{C}^{\mathrm{fin}}:\\ I^{\mathrm{que}}(Q)<\infty}} [\Phi_{\beta,h}(Q) - I^{\mathrm{que}}(Q)] \\ &= \sup_{Q\in\mathcal{C}^{\mathrm{fin}}} [\Phi_{\beta,h}(Q) - I^{\mathrm{que}}(Q)]. \end{aligned} \tag{B.13}$$

To obtain the reverse inequality we note that, for any $\epsilon > 0$, we can find a Q^ϵ with $I^{\mathrm{que}}(Q^\epsilon) < \infty$ (and hence $\Phi_{\beta,h}(Q^\epsilon) < \infty$ by Lemma A.1) such that

$$\Phi_{\beta,h}(Q^\epsilon) - I^{\mathrm{que}}(Q^\epsilon) \geq \bar{S}^{\mathrm{que}}(\beta, h; 0) - \epsilon. \tag{B.14}$$

For $\mathrm{tr} \in \mathbb{N}$, let $f_{\mathrm{tr}}\colon \widetilde{E} \to \mathbb{R}$ be defined by

$$f_{\mathrm{tr}}(y) = \log\left(1 + e^{-2\beta h\tau(y) - 2\beta\sigma(y)}\right) 1_{\{\tau(y)<\mathrm{tr}\}}(y) + \log\left(1 + e^{-2\beta h \mathrm{tr} - 2\beta\sigma_{\mathrm{tr}}(y)}\right) 1_{\{\tau(y)\geq \mathrm{tr}\}}(y), \quad \text{(B.15)}$$

where $\sigma_{\mathrm{tr}}(y) = \sum_{i=1}^{\mathrm{tr}} \omega_i$. Then

$$\Phi_{\beta,h}([Q]_{\mathrm{tr}}) = \log(\tfrac{1}{2}) + \int_{\widetilde{E}} f_{\mathrm{tr}}(y)(\pi_1 Q)(dy), \qquad Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}). \quad \text{(B.16)}$$

Note that f_{tr} is non-negative and $\lim_{\mathrm{tr}\to\infty} f_{\mathrm{tr}}(y) = \log(1 + e^{-2\beta\tau(y) - 2\beta\sigma(y)})$ for all $y \in \widetilde{E}^{\mathbb{N}}$. It therefore follows from Fatou's lemma that

$$\liminf_{\mathrm{tr}\to\infty} \Phi_{\beta,h}([Q]_{\mathrm{tr}}) \geq \Phi_{\beta,h}(Q), \qquad Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}). \quad \text{(B.17)}$$

Hence, for any $\epsilon > 0$, there is a $\mathrm{tr}_0 = \mathrm{tr}_0(\epsilon) < \infty$ such that, for all $\mathrm{tr} \geq \mathrm{tr}_0$,

$$\Phi_{\beta,h}([Q^\epsilon]_{\mathrm{tr}}) \geq \Phi_{\beta,h}(Q^\epsilon) - \epsilon. \quad \text{(B.18)}$$

Since $I^{\mathrm{que}}([Q^\epsilon]_{\mathrm{tr}}) \leq I^{\mathrm{que}}(Q^\epsilon)$ by (2.19), we can combine (B.14) and (B.18) to get that, for $\mathrm{tr} \geq \mathrm{tr}_0$,

$$\begin{aligned} \sup_{Q\in\mathcal{C}^{\mathrm{fin}}} [\Phi_{\beta,h}(Q) - I^{\mathrm{que}}(Q)] &\geq \Phi_{\beta,h}([Q^\epsilon]_{\mathrm{tr}}) - I^{\mathrm{que}}([Q^\epsilon]_{\mathrm{tr}}) \\ &\geq \Phi_{\beta,h}(Q^\epsilon) - I^{\mathrm{que}}(Q^\epsilon) - \epsilon \\ &\geq \bar{S}^{\mathrm{que}}(\beta,h;0) - 2\epsilon. \end{aligned} \quad \text{(B.19)}$$

The first inequality holds because $[Q]_{\mathrm{tr}} \in \mathcal{C}^{\mathrm{fin}}$ whenever $Q \in \mathcal{C}$. Now let $\epsilon \downarrow 0$. ∎

Lemma B.3 *$S^{\mathrm{que}}(\beta,h;g)$ (= the first line of (B.1)) evaluated at $g=0$ equals $S_*^{\mathrm{que}}(\beta,h)$ (= the second line of (B.1)).*

Proof. By copying the proof of Lemma 3.2 in Appendix B of Cheliotis and den Hollander [12], we obtain that

$$\bar{S}^{\mathrm{que}}(\beta,h;0+) \geq S^{\mathrm{que}}(\beta,h;0), \qquad \bar{S}^{\mathrm{que}}(\beta,h;0+) \geq S_*^{\mathrm{que}}(\beta,h), \quad \text{(B.20)}$$

where $\bar{S}^{\mathrm{que}}(\beta,h;0+) = \lim_{g\downarrow 0} \bar{S}^{\mathrm{que}}(\beta,h;g)$. (The idea behind these inequalities is that, for any $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ and in the limit as $N\to\infty$, R_N^ω can be arbitrarily close to Q with probability $\exp[-NI^{\mathrm{que}}(Q) + o(N)]$ while $m_{R_N^\omega}$ remains bounded by a large constant.) By (3.5) and (3.20), we have

$$\bar{S}^{\mathrm{que}}(\beta,h;0+) = S^{\mathrm{que}}(\beta,h;0+) \leq S^{\mathrm{que}}(\beta,h;0). \quad \text{(B.21)}$$

Moreover, from (3.5) and (3.8) it follows that

$$S_*^{\mathrm{que}}(\beta,h) = \sup_{Q\in\mathcal{C}^{\mathrm{fin}}} [\Phi_{\beta,h}(Q) - I^{\mathrm{que}}] \geq \sup_{Q\in\mathcal{C}^{\mathrm{fin}}\cap\mathcal{R}} [\Phi_{\beta,h}(Q) - I^{\mathrm{que}}] = S^{\mathrm{que}}(\beta,h;0), \quad \text{(B.22)}$$

where the last equality uses that $I^{\mathrm{que}} = I^{\mathrm{ann}}$ on $\mathcal{C}^{\mathrm{fin}} \cap \mathcal{R}$ (recall (2.18)). Combining (B.20–B.22), we obtain

$$\bar{S}^{\mathrm{que}}(\beta,h;0+) = S^{\mathrm{que}}(\beta,h;0) = S_*^{\mathrm{que}}(\beta,h). \quad \text{(B.23)}$$

∎

B.3 Control of truncation error

We finally prove the claim in (B.8) to complete the proof of Lemma B.1.

Lemma B.4 *For $\beta, h > 0$, $g \geq 0$ and $q > 1$,*

$$\limsup_{N\to\infty} \frac{1}{qN} \log \left[E_g^* \left(e^{q \sum_{i=1}^N \chi(y_i)\, 1_{\{\chi(y_i)>M\}}} \right) \right] \leq \frac{1}{qt} \log C_t^M \qquad \forall\, 0 \leq t < \frac{C}{2\beta q}, \tag{B.24}$$

where $C = C(h) > 0$ is given in (C.8)*, and*

$$C_t^M = 2^{-t} \left[e^{-CM/2\beta} + \frac{e^{-M(-qt+C/2\beta)+qt}}{1 - e^{-qt+C/2\beta}} \right] \sum_{n\in\mathbb{N}} \rho_g(n)^t\, e^{-Cn}. \tag{B.25}$$

Proof. The proof follows from a fractional moment estimate. Define

$$S_{N,M}^\omega(g) = E_g^* \left(e^{q \sum_{i=1}^N \chi(y_i)\, 1_{\{\chi(y_i)>M\}}} \right). \tag{B.26}$$

Recalling that $\chi(y) = \log \phi_{\beta,h}(y)$, we have

$$\begin{aligned} S_{N,M}^\omega(g) &= \sum_{0<k_1<\cdots<k_N<\infty} \left\{ \prod_{i=1}^N \rho_g(k_i - k_{i-1}) \right\} \left\{ \prod_{\substack{i=1\\ 1_{\{\chi(y_i)>M\}}}}^N \left(\tfrac{1}{2} \left[1 + e^{-2\beta \sum_{k\in I_i}(\omega_k+h)} \right] \right)^q \right\} \\ &\leq \sum_{0<k_1<\cdots<k_N<\infty} \left\{ \prod_{i=1}^N \rho_g(k_i - k_{i-1}) \right\} \left\{ \prod_{\substack{i=1\\ 1_{\{\chi(y_i)>M\}}}}^N \tfrac{1}{2} \left[1 + e^{-2\beta q \sum_{k\in I_i}(\omega_k+h)} \right] \right\}, \end{aligned} \tag{B.27}$$

where $I_i = (k_{i-1}, k_i] \cap \mathbb{N}$, ρ_g is given in (2.4), and the inequality uses the convexity of $x \mapsto x^q$, $x > 0$, for $q > 1$. Note that for $M > 0$ the constraint $\chi(y_i) > M$ implies that $\sum_{k\in I_i} \omega_k < -M/2\beta - (k_i - k_{i-1})h$. Define

$$\begin{aligned} A_n^M &= \left\{ \omega\colon\ \sum_{k=1}^n \omega_k < -\frac{M}{2\beta} - nh \right\}, \qquad n \in \mathbb{N},\ M > 0, \\ B_n^m &= \left\{ \omega\colon\ -\frac{(m+1)}{2\beta} - nh \leq \sum_{k=1}^n \omega_k < -\frac{m}{2\beta} - nh \right\}, \qquad m, n \in \mathbb{N}. \end{aligned} \tag{B.28}$$

Then $A_n^M \subseteq \bigcup_{m\geq\lfloor M\rfloor} B_n^m$, and so it follows from (6.5) and (B.27) that, for $t \in [0,1]$,

$$\begin{aligned} &\mathbb{E}\left(\left[S_{N,M}^\omega(g) \right]^t \right) \\ &\leq \sum_{0<k_1<\cdots<k_N<\infty} \left\{ \prod_{i=1}^N \rho_g(k_i - k_{i-1})^t \right\} \left\{ \prod_{i=1}^N \left(\tfrac{1}{2} \left[\mathbb{P}(A_{k_i-k_{i-1}}^M) + \sum_{m\geq\lfloor M\rfloor} e^{qt(m+1)}\, \mathbb{P}(B_{k_i-k_{i-1}}^m) \right] \right) \right\}, \\ &= \left(2^{-t} \sum_{n\in\mathbb{N}} \rho_g(n)^t \left[\mathbb{P}(A_n^M) + \sum_{m\geq\lfloor M\rfloor} e^{q\,t(m+1)}\, \mathbb{P}(B_n^m) \right] \right)^N \leq \left(C_t^M \right)^N. \end{aligned} \tag{B.29}$$

The first inequality uses that $\sum_{k=1}^{n} \omega_k \geq -(m+1)/2\beta - nh$ on B_n^m, while the second inequality uses the estimates on the probabilities $\mathbb{P}(A_n^M)$ and $\mathbb{P}(B_n^m)$ in Lemma C.1. Follow the argument in the proof of Theorem 3.3 in Section 6.1 to conclude the proof. ∎

Picking $t = C/4\beta q$ in (B.25) and noting that $t \in [0,1]$ for q large enough, we get

$$\text{r.h.s. (B.24)} \leq \frac{C}{4\beta} \log \left(2^{-C/4\beta q} \left[e^{-CM/2\beta} + \frac{e^{-(M-1)(C/4\beta)}}{1-e^{-C/4\beta}} \right] \frac{e^{-C}}{1-e^{-C}} \right), \tag{B.30}$$

which is $\leq -(C^2/4\beta^2 q)\log 2 < 0$ for M large enough. This implies the estimate we made in (B.8). (In Section B.1 the limits were taken in the order $N \to \infty$, $q \to \infty$, $M \to \infty$.)

C Concentration of measure estimates for the disorder

First we introduce some notation. After that we state and prove the concentration of measure estimate for the disorder ω that was used in the proof of Lemmas A.1 and B.4 (Lemmas C.1–C.2 below).

Recall (1.2). The cumulant generating function $\lambda \mapsto M(\lambda)$ is analytic, non-negative and strictly convex on $\mathbb{R}$, with $M(0) = M'(0) = 0$ (recall (1.1)). In particular, $G = M'$ and its inverse $H = G^{-1}$ are both analytic and strictly increasing on $[0,\infty)$.

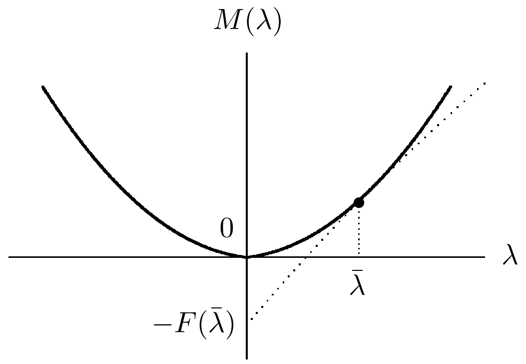


Figure 9: Qualitative picture of $\lambda \mapsto M(\lambda)$. The slope at $\bar{\lambda}$ equals $G(\bar{\lambda})$.

For $W, x > 0$, define

$$f_{W,x}(\lambda) = x\left[M(\lambda) - \lambda \tfrac{W}{x}\right], \qquad \lambda \in \mathbb{R}, \tag{C.1}$$

and note that $\lambda \mapsto f_{W,x}(\lambda)$ is strictly convex on $\mathbb{R}$, with $f_{W,x}(0) = 0$ and $f'_{W,x}(0) = -W < 0$. Putting

$$\chi = \lim_{\lambda\to\infty} G(\lambda) \in (0,\infty] \tag{C.2}$$

(which, by (1.2), equals the supremum of the support of the law of $-\omega_1$), we have $\lim_{\lambda\to\infty} f_{W,x}(\lambda)/\lambda = \chi x - W$, and so there are two cases:

(I) If $\frac{W}{x} \leq \chi$, then $f_{W,x}$ has a unique minimizer at some $\lambda_* \in (0,\infty]$. Note that $\lambda^* = \infty$ if and only if $\frac{W}{x} = l_*$.

(II) If $\frac{W}{x} > \chi$, then $f_{W,x}$ attains it minimum at infinity. In this case $f_{W,x}(\infty) = -\infty$, since

$$-\lambda W \leq f_{W,x}(\lambda) = -\lambda x \left[\frac{W}{x} - \frac{M(\lambda)}{\lambda}\right] \leq -\lambda x \left[\frac{W}{x} - \chi\right], \tag{C.3}$$

where we use that $0 \leq \frac{M(\lambda)}{\lambda} \leq \chi$.

In case (I), we have

$$\lambda_* = \lambda_*(W,x) = H(\tfrac{W}{x}), \qquad f_{W,x}(\lambda_*) = -x\,[\lambda_*\,G(\lambda_*) - M(\lambda_*)]\,. \tag{C.4}$$

Since $H(y)$ is well defined only for $y \leq \chi$, in what follows we will always assume that the arguments of H are at most χ.

Our concentration of measure estimate is the following. Let

$$F(\lambda) = \lambda\,G(\lambda) - M(\lambda), \qquad \lambda \in [0,\infty). \tag{C.5}$$

Lemma C.1 *For $n \in \mathbb{N}$ and $A, B > 0$,*

$$\mathbb{P}\left(\sum_{i=1}^{n} \omega_i \leq -A - nB\right) \begin{cases} \leq \exp\left[-n\,F(H(\frac{A}{n}+B))\right] & \text{when } A/n + B \leq \chi, \\ = 0 & \text{when } A/n + B > \chi, \end{cases} \tag{C.6}$$

where

$$n\,F(H(\tfrac{A}{n}+B)) \geq C[A+n], \qquad \text{when } A/n + B \leq \chi, \tag{C.7}$$

with

$$C = \tfrac{1}{2}[F(H(B)) \wedge F(H(1))] > 0. \tag{C.8}$$

Proof. Estimate

$$\begin{aligned} \mathbb{P}\left(\sum_{k=1}^{n} \omega_k \leq -W\right) &= \inf_{\lambda>0} \mathbb{P}\left(e^{-\lambda\sum_{k=1}^{n}\omega_k} \geq e^{\lambda W}\right) \\ &\leq \inf_{\lambda>0} e^{-\lambda W}\left[\mathbb{E}\big(e^{-\lambda\omega_1}\big)\right]^n = \inf_{\lambda>0} e^{-\lambda W + n\,M(\lambda)} = e^{\inf_{\lambda>0} f_{W,n}(\lambda)} \end{aligned} \tag{C.9}$$

with $\lambda \mapsto f_{W,n}(\lambda)$ the function defined in (C.1). In Case (I), (C.4) shows that the minimal value of $f_{W,n}$ is $-nF(\lambda_*(W,n)) = -nF(H(\frac{W}{n}))$. Together with the lower bound on $nF(H(\frac{W}{n}))$ that is derived in Lemma C.2 below, this proves the first line of (C.6) with the estimates in (C.7–C.8). In Case (II), $f_{W,n}$ attains its infimum at infinity, with $f_{W,n}(\infty) = -\infty$, which proves the second line of (C.6). ∎

Lemma C.2 *For every $A, B > 0$ and $x \in [1,\infty)$ with $A/x + B \leq \chi$ there exists a $C > 0$ (depending on B only) such that*

$$x\,F(H(\tfrac{A}{x}+B)) \geq C(A+x), \qquad x \in [1,\infty). \tag{C.10}$$

Proof. For $x \geq A$, estimate

$$x\,F(H(\tfrac{A}{x}+B)) \geq x\,F(H(B)) \geq \tfrac{1}{2}(A+x)\,F(H(B)). \tag{C.11}$$

For $x \leq A$, on the other hand, estimate

$$x\,F(H(\tfrac{A}{x}+B)) \geq A\,(\tfrac{A}{x})^{-1}\,F(H(\tfrac{A}{x})) \geq A\,F(H(1)) \geq \tfrac{1}{2}(A+x)\,F(H(1)), \tag{C.12}$$

where the second inequality uses that $y \mapsto y^{-1}F(H(y))$ is strictly increasing on $(0,\chi)$. Combining the two estimates, we get the claim with C given by (C.8). ∎

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