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On Computing Optimal (Q,r) Replenishment Policies under Quantity Discounts

The all - units and incremental discount cases

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Abstract This article studies the classical reorder quantity, order point (Q,r) continuous review stochastic inventory model with Poisson arrivals and a fixed lead time. This model has been extensively studied in the literature and its use in practice is widespread. This work extends previous research in this area by providing efficient algorithms for the computation of the optimal (Q^*,r^*) values when there is a multi-breakpoint discount pricing structure.

Keywords Inventory · production · lot sizing · backorders · quantity discounts

1 Introduction

This article treats the classical order quantity, reorder point (Q,r) continuous review stochastic inventory model with Poisson arrivals and a fixed lead time under quantity discount pricing, cf. [6]. Procedures for both all-units and incremental quantity discount schedules are provided. The term all-units refers to the discount schedule where a discount is given for all ordered products, once a quantity exceeding a given price breakpoint is ordered. Incremental

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refers to the discount schedule where a discount is given only for the products exceeding the price break quantity. In both cases multiple break points are possible.

For the case of a single ordering price, an efficient algorithm for this problem has been described in [2]. This article generalizes previous work by constructing algorithms for both all-units and incremental quantity discount schedules. The quantity discounts studied herein are discussed, in the context of the "EOQ" model, in [6], pp. 62 to 68.

For other literature related to inventory management with quantity discounts, we refer to [8], [9] and [7]. For a review of this general area we refer to [1] and [5].

The paper is organized as follows. In Section 2 we review the work of [2] and [6] and give some of the proofs, others are given in appendix A. Section 3 contains the main results of this paper regarding algorithms and complexity. Further, in this section it is shown that a computational improvement of the order of $O(r^*)$ is possible when one uses a normal approximation to the Poisson distribution. Section 4 contains the algorithms and Section 5 provides computational and graphical illustrations.

Finally, we note that many generalizations regarding the arrival process are possible.

2 Assumptions, Notation and Preliminaries

The basic *order quantity* - *reorder point* model studied herein was first introduced by [4] and it is described in [6] (pages 181-194) as follows. A system under consideration consists of a single installation that uses a transactions based inventory management ("reporting") system for a single item. There are no interactions between other items that the system may handle.

We assume that there is a fixed, positive, procurement lead time τ. Let $N_{\tau t}$ denote the number of arrivals in the time interval $(t-\tau, t]$. Arrivals of customers occur according to a Poisson Process with rate λ, the expected value of arrivals during a unit of time. We will use the notation $p_k = P(N_{\tau t} = k) = e^{-\lambda\tau}(\lambda\tau)^k/k!$ and $P_j = \sum_{k=0}^{j} p_k = P(N_{\tau t} \leq j)$. The transaction system monitors continuously the inventory position, defined as the inventory on hand plus the quantity on order - if any. The policy employed is determined by two integer values Q (the order quantity) and r (the reorder level) such that an order of size Q is placed when the inventory position reaches the reorder level r. In this model the demand variable is discrete, the order quantity Q, the reorder point r, and all the inventory levels will also be treated as discrete.

Let c_h and c_p denote respectively the holding cost per inventory unit per time unit and the penalty costs per backordered unit per time unit. Let $c_{\tilde{p}}$, c_K denote respectively the fixed single costs per backordered unit and the costs per replenishment order.

Finally let c_i denote the unit ordering cost (unit price) per product when the quantity ordered is in the price interval $[b_i, b_{i+1})$, for $i = 1, \ldots, M$. Here M is the fixed number of pricing intervals, with $b_i \leq b_{i+1}$, $b_1 = 0$ and $b_{M+1} = \infty$. We will refer to the price interval $[b_i, b_{i+1})$ as the i^{th} price interval.

The time unit is taken to be a year and the optimal values of Q and r are those that minimize the annual expected cost. To avoid trivialities c_h, c_K and c_i are assumed to be positive as well as that at least one of the costs: c_p, $c_{\tilde{p}}$ is positive. Further, without loss of generality we will assume that

$$c_i > c_{i+1} \text{ for all } i = 0, 1, \ldots, M.$$

Also for notational convenience we define:

$$\bar{c}_i = c_i\lambda$$
$$\bar{c}_K = c_K\lambda$$
$$\bar{c}_{\tilde{p}} = c_{\tilde{p}}\lambda.$$

Definitions.
a) A sequence of real numbers $\{d_1, d_2, \ldots\}$ is said to be *unimodal* if there exists a finite index k^* called the *mode* such that d_j *decreases* as j increases up to $j = k^*$ and increases from then on, i.e., $d_1 \geq d_2 \geq \ldots \geq d_{k^*} < d_{k^*+1} \leq d_{k^*+2} \leq \ldots$

b) We call a real function g on the integers *unimodal* if there exists a minimizing point x^* such that $\ldots \geq g(x^*-1) \geq g(x^*) < g(x^*+1) \leq g(x^*+2) \leq \ldots.$

c) For a unimodal function g we define the $x-mode$ to be the point x^g:

$$x^g = \min\{x \,:\, g(x-1) \geq g(x) < g(x+1)\}.$$

d) For a fixed unimodal function g and for any $k \geq 1$ we define the set of points $\mathscr{L}_k^g = \{x_1, \ldots, x_k\}$ as follows:

$$\begin{aligned}
x_1 &= x^g, \\
x_2 &= argmin_x\{g(x),\, x \notin \{x_1\}\}, \\
\vdots \;\; &\;\; \vdots \\
x_k &= argmin_x\{g(x),\, x \notin \{x_1, \ldots, x_{k-1}\}\}.
\end{aligned}$$

Remark 1 The definition of unimodality implies that each set $\mathscr{L}_k^g$ contains k *adjacent* points.

Theorem 1 and Lemma 1 below are due to [2] (see also, [9]). Theorem 1 provides an expression of the long-run average annual cost per product $C(Q,r)$, without quantity discounts, in terms of a unimodal function G. Further, it is shown that for this G, its $x-mode$ point x^G exists (i.e. the "min" is well defined above) and it is the maximal minimizing point of G. Lemma 1 provides a simple characterization for $r^*(Q)$ that minimizes $C(Q,r)$, with respect to r for a fixed $Q > 0$. Also, Lemma 1 gives the minimum value of $C(Q^*, r^*(Q^*))$ of $C(Q,r)$ with respect to Q and r, in terms of G. Further it provides an implicit solution for Q^* (and thus $r^* = r^*(Q^*)$) in terms of a condition on G and the following function of Q: $C^*(Q) := \min_r C(Q,r)$. A proof of Theorem 1 is included in appendix A.

Theorem 1 *The following are true:*
a)

$$C(Q,r) = \bar{c}_K/Q + \sum_{x=r+1}^{r+Q} G(x)/Q, \tag{1}$$

where,

$$G(x) = (c_h + c_p)\sum_{i=0}^{x-1} P_i + c_p(\lambda\tau - x) + \bar{c}_{\tilde{p}}(1 - P_{x-1}). \tag{2}$$

b) The function $G(x)$ is unimodal and its $x-mode$ point x^G is its maximal minimizing point.

Remark 2 If $c_p > 0$, simple algebra implies for all $x < 0$, we have $G(x+1) - G(x) = -c_p < 0$. It then follows that x^G is the unique minimizing point of $G(x)$.

In the sequel x^G will always refer to the $x-mode$ of G, and $\mathscr{L}_Q^G = \{x_1, \dots, x_Q\}$ are the sets constructed as per Definition d) above for G.

Lemma 1 *a) For any fixed positive integer Q, $C(Q,r)$, is minimized with respect to r at $r^* = \min \mathscr{L}_Q^G - 1$, and $C^*(Q) := \min_r C(Q,r)$ is given by:*

$$C^*(Q) = \frac{\bar{c}_K + \sum_{i=1}^{Q} G(x_i)}{Q}. \tag{3}$$

b) The value $Q^* = \min\{Q : G(x_{Q+1}) \geq C^*(Q)\}$ is the optimal order quantity i.e., $\min_{Q,r}\{C(Q,r)\} = C(Q^*, r^*(Q^*))$.

Proof For a) note that the unimodality of G and Remark 1 imply that $\mathscr{L}_Q^G$ contains Q adjacent integers and $G(x_1), \dots, G(x_Q)$ are the smallest values of $G(x)$. Then $\sum_{i=1}^{Q} G(x_i)$ is the summation of the Q smallest values of $G(x)$, corresponding to $C(Q,r)$ attaining a minimum at $r = r^* = (\min \mathscr{L}_Q^G) - 1$.

For part b) we first show that the following inequalities are equivalent:

$$C^*(Q+1) < C^*(Q) \tag{4}$$

$$G(x_{Q+1}) < C^*(Q). \tag{5}$$

The proof of the equivalence is by noticing that

$$\begin{aligned} C^*(Q+1) - C^*(Q) &= (QC^*(Q) + G(x_{Q+1}))/(Q+1) - C^*(Q) \\ &= (G(x_{Q+1}) - C^*(Q))/(Q+1), \end{aligned}$$

and the equivalence follows directly. This equivalence proves that $C(Q^*) \leq C(Q)$ for $Q < Q^*$.

Now for $Q > Q^*$ we have:

$$\begin{aligned} C^*(Q) - C^*(Q^*) &= \frac{\bar{c}_K + \sum_{i=1}^{Q} G(x_i)}{Q} - C^*(Q^*) \\ &= \frac{\bar{c}_K + \sum_{i=1}^{Q^*} G(x_i)}{Q} + \frac{\sum_{i=Q^*+1}^{Q} G(x_i)}{Q} - C^*(Q^*) \\ &= \frac{1}{Q}\left(Q^*C^*(Q^*) + \sum_{i=Q^*+1}^{Q} G(x_i) - QC^*(Q^*)\right) \\ &\geq \frac{Q-Q^*}{Q}\left(G(x_{Q^*+1}) - C^*(Q^*)\right) \qquad (6) \\ &\geq 0, \end{aligned}$$

where Eq. (6) above follows using $\sum_{i=Q^*+1}^{Q} G(x_i) \geq \sum_{i=Q^*+1}^{Q} G(x_{Q^*+1}) = (Q-Q^*)G(x_{Q^*+1})$. These calculations show that $C^*(Q^*) \leq C^*(Q)$ for $Q^* < Q$. This completes the proof of part b). □

Remark 3 Note that Q^* is *not* the x-mode: Q^* is the size of the set $\mathscr{L}^G_{Q^*}$, which has the property that adding an extra point will lead to a higher value of $C^*(Q^*+1)$.

Now, using the previous lemma, a process for finding (Q^*,r^*) works as follows: first find x^G, by comparing $G(x+1)$ with $G(x)$, starting at $x=0$, and stop when $G(x+1)>G(x)$.

After finding x_1, we continue to the second stage with initializing $Q=1$, $C^*(Q)=\bar{c}_K+G(x^G)$ and also $\mathscr{L}^G_1=x^G$.

Next, we compare $G(x_{Q+1})$ with $C^*(Q)$, with x_{Q+1} defined as in the previous. If $G(x_{Q+1})\geq C^*(Q)$ we stop, and $C^*=C^*(Q)$, $Q^*=Q$ and $r^*=(\min\mathscr{L}^G_Q)-1$.
If not, $C^*(Q+1)=(QC^*(Q)+G(x_{Q+1}))/(Q+1)$, $Q=Q+1$, and we repeat the process.

3 Quantity Discounts

In this section we consider the case when the unit ordering cost (unit price) depends on the quantity ordered Q and it is c_i per unit of the product when Q is in the price interval $[b_i,b_{i+1})$, for $i=1,\ldots,M$, where $M+1\geq 2$, is a fixed number of pricing intervals, with $b_1=0$ and $b_{M+1}=\infty$. Both all-units and incremental discounts are considered. Note that in the previous section, the cost function $C(Q,r)$ did not contain unit ordering costs, since they were independent of the order size Q and the reorder point r and therefore they did not influence the values of Q^* and r^*.

Let $C_D(Q,r)$ denote the expected annual cost function, including the average unit price, ie., $C_A(Q,r)$, respectively $C_I(Q,r)$, will refer to the all units case, respectively to the incremental case. For $C_D(Q,r)$ we state the following lemma (with $G(x)$ as in the previous section):

Lemma 2

$$C_D(Q,r)=\frac{(\bar{c}_K+R_{i(Q)})+\sum_{x=r+1}^{r+Q}G(x)}{Q}+\bar{c}_{i(Q)}, \tag{7}$$

where $i(Q)$ is the unique i for which $Q\in[b_i,b_{i+1})$ and

$$R_i=\begin{cases}\sum_{j=1}^{i}b_j(\bar{c}_{j-1}-\bar{c}_j), & \text{if } D\text{=}I,\\ 0, & \text{if } D\text{=}A.\end{cases}$$

Proof Let $c^{av}_D(Q)$ be the average unit ordering cost, for $D=A,I$. Then:

$$\begin{aligned}c^{av}_A(Q) &= (Q\bar{c}_{i(Q)})/Q\\ &= \bar{c}_{i(Q)}, \text{ for the all-units case,} \end{aligned}\tag{8}$$

$$\begin{aligned}c^{av}_I(Q) &= \left(\bar{c}_{i(Q)}(Q-b_{i(Q)})+\bar{c}_{i(Q)-1}(b_{i(Q)}-b_{i(Q)-1})+\ldots+\bar{c}_0(b_1-0)\right)/Q\\ &= \left(Q\bar{c}_{i(Q)}+b_{i(Q)}(\bar{c}_{i(Q)-1}-\bar{c}_{i(Q)})+\ldots+b_1(\bar{c}_0-\bar{c}_1)\right)/Q\\ &= \sum_{j=1}^{i(Q)}b_j(\bar{c}_{j-1}-\bar{c}_j)/Q+\bar{c}_{i(Q)}, \text{ for the incremental case.}\end{aligned}\tag{9}$$

Furthermore, $C_D(Q,r)$ is by definition:

$$C_D(Q,r)=\bar{c}_K\frac{1}{Q}+c_h\sum_{x=0}^{r+Q}x\pi_x-c_p\sum_{x=-\infty}^{0}x\pi_x+\bar{c}_{\tilde{p}}\sum_{x=-\infty}^{0}\pi_x+c^{av}_D(Q)$$

The proof can be now completed using Lemma A.2 and Eqs. (8), (9). □

3.1 The All Units Discount Case.

In this case we have:

$$C_A(Q,r) = C(Q,r) + \bar{c}_{i(Q)},$$

where $C(Q,r)$ is the cost function of Section 2.
Furthermore, define

$$C_A^*(i) = \min_{Q,r}\{C_A(Q,r),\ Q \in [b_i, b_{i+1})\},$$

and

$$C_A^* = \min_i C_A^*(i).$$

In Section 2 the unimodality of G was used for creating a stopping criterion for finding Q^* that minimizes $C^*(Q) = \min_r C(Q,r)$, which was in turn used to minimize $C(Q,r)$.
However in the present case the unimodality of G does not suffice directly to construct stopping criteria for an algorithm. This is achieved by the following two lemmata, Lemma 3 describes a useful property of $C^*(Q)$.

Lemma 3 *For any integers Q' and Q'' with $Q'' > Q' > Q^*$ the following holds:*

$$C^*(Q'') \geq C^*(Q').$$

Proof First we will look at the difference between $C^*(Q'')$ and $C^*(Q')$. By the same arguments as for Eq. (6) in Lemma 1 we get that:

$$C^*(Q'') - C^*(Q') \geq \frac{Q''-Q'}{Q''}\left(G(x_{Q'+1}) - C^*(Q')\right). \tag{10}$$

Now, we will prove by contradiction that $G(x_{Q'+1}) \geq C^*(Q')$.
Suppose that $G(x_{Q'+1}) < C^*(Q')$. If this is true, then:

$$G(x_{Q'+1}) < \frac{\bar{c}_K + \sum_{i=1}^{Q'} G(x_i)}{Q'}.$$

It is clear that this can only hold if $\bar{c}_K > Q'G(x_{Q'+1}) - \sum_{i=1}^{Q'} G(x_i)$.
We look at $C^*(Q^*)$ with this lower bound for $\bar{c}_K$.

$$\begin{aligned}
C^*(Q^*) &= \frac{\bar{c}_K + \sum_{i=1}^{Q^*} G(x_i)}{Q^*} \\
&> \frac{Q'G(x_{Q'+1}) - \sum_{i=1}^{Q'} G(x_i) + \sum_{i=1}^{Q^*} G(x_i)}{Q^*} \\
&= \frac{Q^*G(x_{Q'+1}) + (Q'-Q^*)G(x_{Q'+1}) - \sum_{i=Q^*+1}^{Q'} G(x_i)}{Q^*} \\
&\geq G(x_{Q'+1}) \\
&\geq G(x_{Q^*+1})
\end{aligned}$$

The above implies $C^*(Q^*) > G(Q^*+1)$ and that is a contradiction to Lemma 1. Thus, $G(x_{Q'+1}) \geq C^*(Q')$ and by (10) we have $C^*(Q'') \geq C^*(Q')$. □

Recall, $C_A(Q,r)$ reduces to $C(Q,r)+\bar{c}_{i(Q)}$ in the all-units discounts case. The results from Lemma 1 (b) and Lemma 3 will form the basis for the next lemma, with Q^* the previously defined optimal order quantity of $C(Q,r)$ and $r^*(Q) = (\min \mathscr{L}_Q^G) - 1$.

Lemma 4 *For the expected all-units discount cost function $C_A(Q,r)$, the following hold:*

a) $C_A^*(i) > C_A^*(i(Q^*))$ *for all* $i < i(Q^*)$,
b) $C_A^*(i) = C_A(b_i, r^*(b_i))$ *for all* $i > i(Q^*)$.

Proof Note that for every $Q < b_{i(Q^*)}$ by definition $\bar{c}_{i(Q^*)} < \bar{c}_{i(Q)}$. This implies that, for every $Q < Q^*$:

$$\begin{aligned} C_A(Q,r) &= C(Q,r)+\bar{c}_{i(Q)} \\ &> C(Q,r)+\bar{c}_{i(Q^*)} \\ &\geq C^*(Q^*)+\bar{c}_{i(Q^*)} \\ &= C_A^*(i(Q^*)). \end{aligned}$$

The proof of part a) is complete by noting that the validity of the above inequality for every $Q < Q^*$ implies its validity for $C_A^*(i)$ with $i < i(Q^*)$.

For the proof of part b) it suffices to note that Lemma 3 implies:

$$\begin{aligned} C_A(b_i, r^*(b_i)) &= C^*(b_i)+\bar{c}_i \\ &\leq C^*(Q)+\bar{c}_i \text{ for all } Q \in [b_i, b_{i+1}). \end{aligned}$$

□

We next state the following.

Corollary 1 *In the all units discount case the following are true:*

$$\begin{aligned} C_A^* &= min_{i>i(Q^*)}\{C^*(Q^*)+\bar{c}_{i(Q^*)},\, C^*(b_i)+\bar{c}_i\}, \\ Q_A^* &= argmin_Q\ (C_A(Q, r_A^*(Q)),\ with\ Q \in \{Q^*, b_i,\ for\ i > i(Q^*)\}), \\ r_A^* &= r_A(Q_A^*). \end{aligned}$$

Proof It follows immediately from Lemma 4. □

Corollary 1 suggests a process for determining the minimal value of C_A^*.

i) We start with finding C^*, Q^* and r^*, according to the process described for the case without quantity discounts. Then $C_A^*(i(Q^*)) = C^*(Q^*)+\bar{c}_{i(Q^*)}$.

ii) Next, determine $r^*(b_{i(Q^*)+j})$ and compute $C_A^*(i(Q^*)+j) = C^*(b_{i(Q)+j})+\bar{c}_{i(Q^*)+j}$ for all $j = 1,\ldots,M-i(Q^*)$.

iii) Compute the minimum of all these values. The overall minimum is C_A^*, with (Q_A^*, r_A^*) the corresponding order quantity and reorder level.

3.2 The Incremental Discount Case

In this case we have:

$$C_I(Q,r) = C_{i(Q)}(Q,r) + \bar{c}_{i(Q)}, \tag{11}$$

where $C_{i(Q)}(Q,r)$ is the general cost function for the case without quantity discounts, were, instead of $\bar{c}_K$, we use $\bar{c}_K + R_{i(Q)}$ as the order costs. As before $i(Q)$ is the unique i for which $Q \in [b_i, b_{i+1})$ and

$$R_i = \sum_{j=1}^{i} b_j(\bar{c}_{j-1} - \bar{c}_j).$$

Again, the unimodality of G does not suffice to construct a stopping criterion for an algorithm to determine the minimal costs. As in the all units discounts case, we will derive a different but sufficient procedure for finding (Q_I^*, r_I^*) to minimize $C_I(Q,r)$.

Towards this end we introduce the functions:

$$\widehat{C}_I(Q,r,i) = C_i(Q,r) + \bar{c}_i, \tag{12}$$

where $C_i(Q,r)$ is the general cost function for the case without quantity discounts, where, instead of $\bar{c}_K$, we use $\bar{c}_K + R_i$ as the order costs, *independent* of Q. The idea behind these functions, is that these functions can be treated as the general cost function for the case without quantity discounts plus (different) constants. In this way the results for the case without quantity discounts can be used directly.

We also define the following:

$$\widehat{C}_I^*(i) = \min_{Q,r}\{\widehat{C}_I(Q,r,i)\}.$$

A minimizing point for the unconstrained problem $\widehat{C}_I^*(i) = \min_{Q,r}\{\widehat{C}_I(Q,r,i)\}$ will be denoted by (Q_i^*, r_i^*). This minimizing point will be called *achievable* if $Q_i^* \in [b_i, b_{i+1})$. Let $\mathscr{A}$ be the set of all i's such that (Q_i^*, r_i^*) is achievable.

Note that for any $i \in \mathscr{A}$ we have by definition $\widehat{C}_I^*(i) = \widehat{C}_I(Q_i^*, r_i^*, i) = C_I(Q_i^*, r_i^*)$.

Lemma 5 *For the "cost" function $\widehat{C}_I(Q,r,i)$ there is a price domain $[b_{i_o}, b_{i_o+1})$ for which $i_o \in \mathscr{A}$.*

Proof Note that the assumption $0 \le \bar{c}_{i+1} < \bar{c}_i$, for all $i < M$ implies that the set up "costs" $\bar{c}_K + R_i$ are increasing in $i : R_{i+1} > R_i$. Because $G(x)$ is independent of i, this $G(x)$ is the same for $C_i(Q,r)$ and $C_{i+1}(Q,r)$. Therefore we get:

$$C_{i+1}^*(Q) = C_i^*(Q) + \frac{(R_{i+1} - R_i)}{Q}.$$

And thus:

$$C_{i+1}^*(Q) > C_i^*(Q).$$

Now, because the stopping criterion depends on comparing $C^*(Q)$ with the function $G(x)$ it is easy to see that

$$Q_{i+1}^* \ge Q_i^*. \tag{13}$$

Now, suppose that $\mathscr{A} = \emptyset$. Because Q_i^* is increasing over i and the intervals are positioned in an increasing way as well, there are four cases possible if the claim $\mathscr{A} = \emptyset$ is true. It is easy to see that the cases below are the only possible cases. We will show that each case leads to a contradiction.

Case 1: $\exists i_1, i_2$, with $i_1 < i_2$, such that $Q_i^* > b_{i+1}$, for all $i \le i_1$ and $Q_i^* \le b_i$, for all $i \ge i_2$.

In this case, there is a j with $i_1 \le j < i_2$ for which $Q_{j-1}^* > b_j$ and $Q_{j+1}^* \le b_{j+1}$, by the increasing property of Q_i^*. This means that Q_j^* is achievable, a contradiction to our assumption that $\mathscr{A} = \emptyset$.

Case 2: $\exists j$ such that $Q_i^* < b_i, \forall i \le j$ and $Q_i^* \ge b_{i+1}, \forall i > j$.

Recall that $b_1 = 0$ and $b_{M+1} = \infty$. In this case the only solution can be a $Q_0^* < 0$ However, for all i Theorem 1 implies $0 < Q_i^* < \infty$ which is a contradiction.

Case 3: For all $i: \; Q_i^* < b_i$.

A contradiction can be obtained as in case 2.

Case 4: For all $i: \; Q_i^* \ge b_{i+1}$.

In this case $Q_M^* > \infty$ a contradiction.

All the cases lead to a contradiction and therefore there is an i_o for which $i_o \in \mathscr{A}$, i.e. $\mathscr{A}$ is not the empty set. □

Now we know, $\widehat{C}_I^*(Q_i^*, r_i^*, i) = C_I(Q_i^*, r_i^*)$ for some i and we next show that C_I^* is one of these points.

Lemma 6 *In the incremental discounts case, C_I^* is the minimum of all achievable solutions of $\widehat{C}_I^*(i)$, ie.,*

$$C_I^* = \min_{Q,r}\{C_I(Q,r)\} = \min_{i \in \mathscr{A}}\{\widehat{C}_I^*(i)\}.$$

Proof First note that since $Q_i^* > 0$ for all i, Q_I^* can not be equal to $b_1 = 0$.

We next show that the claim that Q_I^* is not equal to Q_i^* for some $i \in \mathscr{A}$, ie., Q_I^* is equal to a boundary point b_i or to $b_i - 1$ for some i, leads to a contradiction.

Indeed, if $Q_I^* = b_i$ $(i > 0)$ then b_i is a local minimum of $C_I(Q, r^*(Q))$ and hence $C_I(Q, r^*(Q))$ is increasing for Q in the entire price interval $[b_i, b_{i+1})$ and strictly decreasing for Q in the price interval $[b_{i-1}, b_i)$. This is clear from Lemma 3. Then by Theorem 1 we obtain that $Q_i^* < b_i$ and $Q_{i-1}^* \ge b_i$, therefore $Q_i^* < Q_{i-1}^*$ which is a contradiction by virtue of Lemma 5.

A similar argument shows that $Q_I^* = b_{i-1}$ can not be the global minimum if b_{i-1} is not the minimum of $C_{i-1}^*(Q, r^*(Q))$.

The above imply that Q_I^* occurs at a minimizing point of C_i^* for some i, which is achievable for at least one i, since by Lemma 5: $\mathscr{A} \neq \emptyset$. □

Now, a process of computing (Q_I^*, r_I^*) works by finding all the (Q_i^*, r_i^*) (the minimizers of $C_i(Q,r)$, found as in the no discount case), checking if they are achievable and comparing the corresponding $C_I(Q_i^*, r_i^*)$. The minimum of these values is C_I^*.

3.3 Efficient computation of the $x-mode$ of G when $c_{\tilde{p}}=0$

Next we point out that when $c_{\tilde{p}}=0$, the computation of the $x-mode$ point x^G, cf. Definition (c), can be done more efficiently using the observation of the lemma below, where P^{-1} is the inverse cumulative distribution function of the Poisson distribution P.

Lemma 7 *If* $c_{\tilde{p}}=0$ *then* $x^G=P^{-1}(c_p/(c_p+c_h))$.

Proof Since $c_{\tilde{p}}=0$, simple algebra using Eq. (2) shows that

$$G(x+1)-G(x)=(c_h+c_p)P_x-c_p. \tag{14}$$

The above and the definition of x^G implies the following:

$$\begin{aligned} P_{x^G-1} &\le c_p/(c_h+c_p), \\ P_{x^G} &> c_p/(c_h+c_p). \end{aligned}$$

The proof is easy to complete. □

Note that in the above proof the argument for the difference $G(x+1)-G(x)$ is analogous to that used in the context of the newsvendor model, cf. [6], p. 297.

Further, one can use the Normal approximation to P_{x^G} to obtain the easy to compute expression for $x^G=P^{-1}(c_p/(c_p+c_h))\approx F^{-1}(c_p/(c_p+c_h))$, where F is the normal cumulative distribution with mean $\mu=\lambda\tau$ and variance $\sigma^2=\lambda\tau$, cf. [3].

Indeed, as Table 1 displays the exact value of x^G and the corresponding value of $F^{-1}(c_p/(c_p+c_h))$. We see that in all cases when $F^{-1}(c_p/(c_p+c_h))$ is rounded to its closest integer we obtain the exact value for x^G. In the table belowf ρ denotes the fraction $c_p/(c_p+c_h)$.

	$\lambda\tau=10$		$\lambda\tau=50$		$\lambda\tau=100$		$\lambda\tau=250$	
ρ	x^G	$F^{-1}(\rho)$	x^G	$F^{-1}(\rho)$	x^G	$F^{-1}(\rho))$	x^G	$F^{-1}(\rho)$
0.1	6	5.9	41	40.9	87	87.2	230	229.7
0.3	8	8.3	46	46.3	95	94.8	242	241.7
0.5	10	10.0	50	50.0	100	100.0	250	250.0
0.7	12	11.7	54	53.7	105	105.2	258	258.3
0.9	14	14.1	59	59.1	113	112.8	270	270.3

Table 1: Display of x^G with corresponding values of $F^{-1}(c_p/(c_p+c_h))$.

4 Algorithms

The procedures described in the previous section can be presented as statement algorithms. The NoDiscounts-algorithm is meant for the computation of a global optimum when there are no quantity discounts. The parameter K is used for the order costs $\bar{c}_K$, or when Incremental calls this algorithm, K takes the value $\bar{c}_K+R_i$. The AllUnits and Incremental algorithms work according to the procedures described in Corollary 1 and Lemma 6.

ALLUNITS first calls NODISCOUNTS to find the value of C^*, then finds the corresponding interval and adds $G(x_1-1)$ or $G(x_Q+1)$ until $\mathscr{L}_Q^G$ contains b_i points, so $Q=b_i$. Then it compares the current lowest value with the corresponding costs. This is done for all b_i, with $i>i(Q^*)$.

INCREMENTAL calls NODISCOUNTS once for each price interval with the corresponding value R_i, $i=0,\ldots,M$ and it computes the minimum of $\widehat{C_I}(i)$. It checks whether $i \in \mathscr{A}$, and if indeed $i \in \mathscr{A}$ it compares $C^*(Q_i^*)+\bar{c}_i$ with the up to then minimum value of $C_I(Q,r)$, denoted in the algorithm as C_I. When $i=M$, we have $C_I^*=C_I$.

For notational convenience, all parameters in the algorithms are not defined explicitly as inputs but we assume they are global. Also, the results returned by NODISCOUNTS are used with the same notation in ALLUNITS and INCREMENTAL. Finally computations for $G(x)$ are done by using Eq. (15) below. (easily derived from Eq. (2).)

$$
\begin{aligned}
G(x) &= (c_h+c_p)\sum_{i=0}^{x-1} P_i + c_p(\lambda\tau - x) + \bar{c}_{\tilde{p}}(1-P_{x-1}) \\
&= (c_h+c_p)(xP_{x-1}-\lambda\tau P_{x-2}) + c_p(\lambda\tau - x) + \bar{c}_{\tilde{p}}(1-P_{x-1}).
\end{aligned} \tag{15}
$$

NODISCOUNTS(K)
- **If** $c_{\tilde{p}}>0$
  - $i=0$
  - **While** $G(i+1)<G(i)$
    - $i=i+1$
  - **End**
  - $x_1=i$
- **Else**
  - $x_1=F^{-1}(c_p/(c_h+c_p))$
  - $G(x_1)=(c_h+c_p)(x_1P_{x_1-1}-\lambda\tau P_{x_1-2})+c_p(\lambda\tau-x_1)$
- **End**
- $SumG=G(x_1)$; $C=K+G(x_1)$; $Q=1$; $\underline{r}=x_1-1$; $\overline{r}=x_1+1$
- **While** $C>G(\underline{r})$ and $C>G(\overline{r})$
  - **If** $G(\underline{r})<G(\overline{r})$
    - $SumG=SumG+G(\underline{r})$; $\underline{r}=\underline{r}-1$
  - **Else**
    - $SumG=SumG+G(\overline{r})$; $\overline{r}=\overline{r}+1$
  - **End**
  - $Q=Q+1$; $C=(K+SumG)/Q$
- **End**
- $C^*=C$; $Q^*=Q$; $r^*=\underline{r}$;

Return C^*; Q^*; r^*

ALLUNITS
- **Call** NODISCOUNTS($\bar{c}_K$)
- $i=1$
- **While** $b_{i+1}\leq Q^*$
  - $i=i+1$
- **End**
- $C_A=C^*+\bar{c}_i$; $Q_A=Q^*$; $\underline{r_A}=r^*$; $\overline{r_A}=r^*+Q^*+1$; $b_i=Q^*$;

For $j = i : M$
For $k = 1 : b_{j+1} - b_j$
If $G(\underline{r_A}) < G(\overline{r_A})$
$SumG = SumG + G(\underline{r_A})$; $\underline{r_A} = \underline{r_A} - 1$;
Else
$SumG = SumG + G(\overline{r_A})$; $\overline{r_A} = \overline{r_A} + 1$;
End
End
If $(\bar{c}_K + SumG)/b_{j+1} + \bar{c}_{j+1} < C_A$
$C_A = (\bar{c}_K + SumG)/b_{j+1} + \bar{c}_{j+1}$; $Q_A = b_j$; $r_A = \underline{r_A}$;
End
End
$C_A^* = C_A$; $Q_A^* = Q_A$; $r_A^* = r_A$;
Return C_A^*; Q_A^*; r_A^*

INCREMENTAL
$C_I = \infty$
For $i = 0 : M$
Call NODISCOUNTS($\bar{c}_K + R_i$)
If $b_i \le Q^* < b_{i+1}$ and $C^* + \bar{c}_i < C_I$
$C_I = C^* + \bar{c}_i$; $Q_I = Q^*$; $r_I = r^*$;
End
End
$C_I^* = C_I$; $Q_I^* = Q_I$; $r_I^* = r_I$;
Return C_I^*; Q_I^*; r_I^*

In Theorem 2 below we assume that the computation of P_x can be done in $O(1)$.

Theorem 2 *Algorithms* ALLUNITS *and* INCREMENTAL *have complexity* $O(Q^* + |r^*| + b_m M)$ *and* $O((Q^* + |r^*|)M)$ *respectively.*

Proof The complexity of NODISCOUNTS is $O(Q^* + |r^*|)$: it consists of two while loops, where every loop has length $O(1)$. The first while-loop runs in at most $|r^*| + i$, iterations, with $i < Q^*$ (since $x^G \le |r^*| + Q^*$), and the second loop in Q^* steps. Together both loops take at most $Q^* + |r^*| + i$ iterations with complexity $O(Q^* + |r^*|)$.

ALLUNITS runs NODISCOUNTS once and afterwards a loop of $i(Q^*)$ iterations and a double loop with at most $(M - b_{i(Q^*)})b_m$ iterations. So ALLUNITS has complexity $O(Q^* + |r^*| + b_m M)$. Furthermore, ALLUNITS is correct since Corollary 1.

INCREMENTAL calls NODISCOUNTS and makes 2 single computations each loops. Therefore, NODISCOUNTS has complexity $O((Q^* + r^*)M)$. □

Remarks

a) If $c_{\tilde{p}} = 0$, one can use the Normal Approximation to replace the first loop of NODISCOUNTS, see proof of Theorem 2 above, by a single computation. Then, the complexities of the ALLUNITS and the INCREMENTAL algorithms become respectively $O(Q^* + b_m M)$ and $O(Q^* M)$.

b) It is preferable to express the complexity in terms of exogenous parameters c_h, c_p, $c_{\tilde{p}}$, M, b, c, λ, τ. However, in this model this is not possible, because there are too many parameters and they have correlated effects.

5 Computations

To test the algorithms and demonstrate some of the issues discussed above, some simple computations have been done, both for the all-units and incremental discount schedules. The data used for both cases is summarized in the Table 2 below.

λ	τ	c_h	c_p	c_K	$\bar{c}_{\bar{p}}$	M
1	15	2	5	100	5	3

Table 2: Parameter values

Tables 3 and 4 below summarize the results. Note that an entry of the form: "-" means that the corresponding value is not required to be computed because it can not be optimal (for all-units) or it is not achievable (for incremental). The boldface entry is the corresponding minimal value of C_A^* and C_I^* respectively. The values of **c** are chosen differently for the all-units and the incremental for emphasizing different issues regarding both cases.

In the all units table, C^* is computed, and the lower bounds of the above price intervals are compared since these values can be optimal as well, exactly as the algorithm has described. The minimal value can be any value of these.

	$\mathbf{b} = [0,10,20,30]$, $\mathbf{c} = [10,7,6,1.5]$			
τ	$(C_A^*(0),Q_0^*,r_0^*)$	$(C_A^*(1),Q_1^*,r_1^*)$	$(C_A^*(2),Q_2^*,r_2^*)$	$(C_A^*(3),Q_3^*,r_3^*)$
5	-	**(25.00, 12, 0)**	(25.95, 20, -2)	(26.70, 30, -5)
15	-	(27.63, 14, 11)	**(27.84, 20, 9)**	(27.98, 30, 6)
25	-	(29.58, 15, 21)	(29.43, 20, 19)	**(28.63, 30, 16)**
	$\mathbf{b} = [0,20,40,50]$, $\mathbf{c} = [10,7,6,1.5]$			
5	(28.00, 12, 0)	**(26.95, 20, -2)**	(37.40, 40, -8)	(40.48, 50, -11)
15	(30.63, 14, 11)	**(28.84, 20, 9)**	(38.36, 40, 3)	(40.25, 50, 0)
25	(32.58, 15, 21)	**(30.43, 20, 19)**	(39.24, 40, 13)	(40.95, 50, 10)

Table 3: All Units discounts

The next table, for the incremental case, shows that the differences between the optimal values can be very small, even for large discounts, as in this case. However, we still need to compute every value since all the values can be achievable, especially with an uneven distribution of boundary points or discount prices.

Figure 1 and 2 are graphical representations of both cases. They show C_A and C_I respectively as function of Q and r.

The figures show the characteristic described in the lemmata. In the all-units case, local minima can be found at the "no discounts" minimum in the second interval and on the lower bounds of the third and fourth interval.
In the incremental case, the figure shows that the minimum can be either the "achievable" minimum in the second, third or fourth interval.

	$\mathbf{b} = [0,10,20,30]$, $\mathbf{c} = [60,50,40,30]$			
τ	$(\widehat{C}_I^*(0), Q_0^*, r_0^*)$	$(\widehat{C}_I^*(1), Q_1^*, r_1^*)$	$(\widehat{C}_I^*(2), Q_2^*, r_2^*)$	$(\widehat{C}_I^*(3), Q_3^*, r_3^*)$
3	(-, 12, -1)	(74.50, 17, 3)	**(74.23, 24, -4)**	(75.05, 32, -7)
10	(-, 13, 6)	(75.88, 18, 8)	**(75.28, 25, 2)**	(75.81, 32, 0)
15	(-, 14, 11)	(76.77, 19, 9)	**(75.94, 25, 7)**	(76.35, 33, 5)
	$\mathbf{b} = [0,20,40,50]$ $\mathbf{c} = [60,50,40,30]$			
3	**(77.71, 12, -1)**	(79.78, 29, -6)	(-, 29, -6)	(-, 43, -10)
10	**(79.52, 13, 6)**	(80.63, 22, 3)	(-, 30, 1)	(-, 43, -3)
15	**(80.63, 14, 11)**	(81.69, 22, 8)	(-, 33, 5)	(-, 44, 2)

Table 4: Incremental discounts

To emphasize the forms of the surfaces the values of the parameters **c** and **b** are chosen as in the table below, all other parameters are chosen as in Table 2.

c	$[30,20,10,0]$ for all-units
c	$[75,50,25,0]$ for incremental
b	$[0,20,40,60]$ for both

Table 5: Parameter values

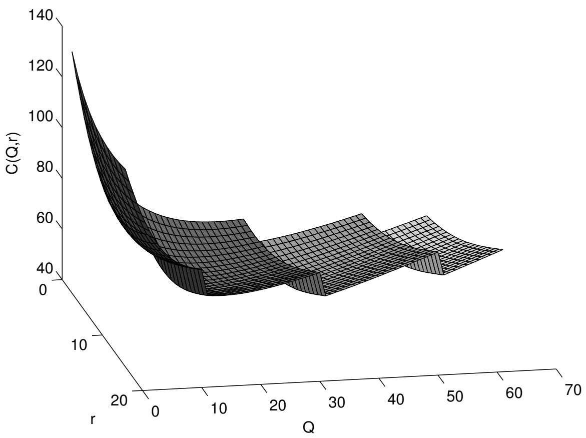

Fig. 1: The Expected cost function of under all units discount pricing.

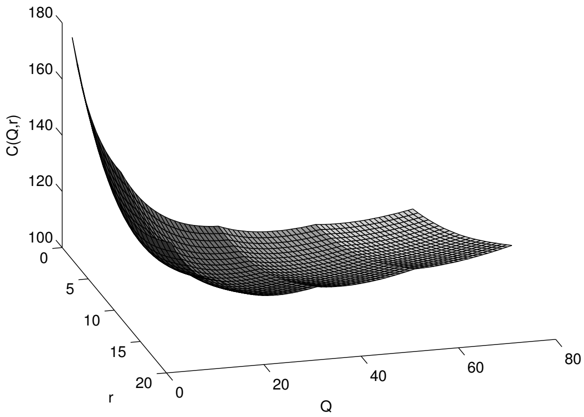

Fig. 2: The Expected cost function of under incremental discount pricing.

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Appendix A.

This appendix summarizes some of the previous work in this area. The proofs are sometimes along different lines.

In this model, the *inventory position* (defined as the inventory at hand plus outstanding orders) provides a suitable state description variable. This is not the case with the *inventory level* (defined as inventory at hand or net inventory). Indeed, when there is heavy demand during some cycle resulting in a large number of backorders, then the arrival of outstanding orders might never bring the on hand inventory back up to the reorder point again, and hence another order would never be placed under a (Q,r) system that is based on the inventory at hand. However, when a (Q,r) system is based on the inventory position, the holding costs cannot be computed directly. If during some cycle there is a considerable number of backorders, then a large number of orders will be placed, for the reorder point in terms of the inventory position will be crossed a large number of times. If r is the reorder point in terms of the inventory position, then immediately after an order is placed the inventory position is $Q+r$. Using the Poisson demand arrival assumption we see that the time evolution of the inventory position can be described by a continuous time Markov Chain with state space $\mathscr{S}=\{r+1,\dots,r+Q\}$ and transition diagram given by figure 3 below.

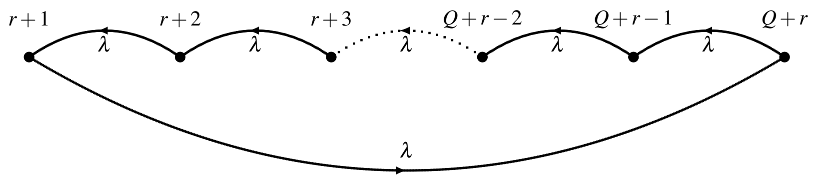

Fig. 3: inventory position

Since all rates of this Markov chain are equal it follows that the steady state probabilities of the inventory position, $\pi_{ip}(x)=\lim_{t\to\infty}P(X(t)=x)=1/Q$, for all $x=r+1,\dots,r+Q$, i.e., in equilibrium the inventory position is uniformly distributed over $r+1,\dots,r+Q$.

Recall from Section 2 that there is a fixed, positive, procurement lead time τ. Also $N_{\tau t}$ denotes the number of arrivals in the time interval $(t-\tau,t]$. Arrivals of customers occur according to a Poisson Process with rate λ, the expected value of arrivals during a unit of time. We denote: $p_k=P(N_{\tau t}=k)=e^{-\lambda\tau}(\lambda\tau)^k/k!$. and $P_j=\sum_{k=0}^{j}p_k=P(N_{\tau t}\le j)$. Further, let $X(t)$, $Y(t)$ denote respectively the *inventory position* and the *inventory level* at time t. Note that $X(t)=Y(t)+O_tQ$, all $t\ge 0$, where O_t denotes the number of outstanding orders at time t. Since orders placed after $t-\tau$ have not arrived by time t, the following equation holds:

$$Y(t)=Y(t-\tau)+O_{t-\tau}Q-N_{\tau t}. \tag{16}$$

Note also that $X(t-\tau)=Y(t-\tau)+O_{t-\tau}Q$, hence

$$Y(t)=X(t-\tau)-N_{\tau t}. \tag{17}$$

From Eq. (17) it follows that $Y(t)$ is also a continuous time Markov chain. Its state space is the set $\{\dots,-2,-1,0,1,2,\dots,Q+r\}$ and transition diagram given by figure 4 below.

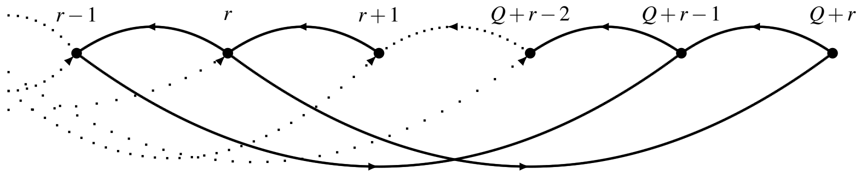

Fig. 4: inventory level

Even though, the transition rates and diagram of the *inventory level* process $Y(t)$ are more complex, Eq. (17) above allows the computation of the steady state probabilities $\pi(x) = \lim_{t\to\infty} P(Y(t)=x)$ in terms of those of the probabilities $\pi_{ip}(x) = 1/Q$ and $P_j = P(N_{\tau t} \leq j)$ as follows.

Lemma A.1 For any integer inventory position x the following are true:

$$\pi(x) = \begin{cases} (P_{r+Q-x} - P_{r-x})/Q, & \text{for } -\infty < x \leq r, \\ P_{r+Q-x}/Q, & \text{for } r+1 \leq x \leq r+Q. \end{cases}$$

Proof Since it is easy to see that both $X(t)$ and $Y(t)$ Markov chains are ergodic, we can assume that for $t=\infty$, $P(X(t)=k) = \pi_{ip}(k) = 1/Q$ and $P(Y(t)=x) = \pi(x)$. For $-\infty < x \leq r$ we have:

$$\begin{aligned}
\pi(x) = P(Y(t)=x) &= \sum_{j=r+1}^{Q+r} P(Y(t)=x \,|\, X(t-\tau)=j)\, P(X(t-\tau)=j) \\
&= \sum_{j=r+1}^{Q+r} P(N_{\tau t} = j-x \,|\, X(t-\tau)=j) P(X(t-\tau)=j) \\
&= \sum_{j=r+1}^{Q+r} P(N_{\tau t} = j-x)\pi_{ip}(j) \\
&= \sum_{k=r+1-x}^{r+Q-x} p_k/Q,
\end{aligned}$$

where in the above we have used the independence of $X(t-\tau)$ and $N_{\tau t}$ as well as the observation that conditional on $X(t-\tau)=j$, $Y(t)=x$ if and only if $N_{\tau t} = j-x$. Similarly, for $r+1 \leq x \leq r+Q$ we have:

$$\begin{aligned}
\pi(x) = P(Y(t)=x) &= \sum_{j=r+1}^{Q+r} P(Y(t)=x \,|\, X(t-\tau)=j)\, P(X(t-\tau)=j) \\
&= \sum_{j=x}^{Q+r} P(Y(t)=x \,|\, X(t-\tau)=j)\, P(X(t-\tau)=j) \\
&= \sum_{j=x}^{Q+r} P(N_{\tau t}=j-x \,|\, X(t-\tau)=j) P(X(t-\tau)=j) \\
&= \sum_{j=x}^{Q+r} p_{j-x}\pi_{ip}(j) \\
&= \sum_{y=0}^{r+Q-x} p_y/Q.
\end{aligned}$$

where the first equality above follows from the observation that $P(Y(t)=x \,|\, X(t-\tau)=j) = 0$ if $j<x$, since it is not possible for the *inventory level* at time t to be $Y(t)=x \geq r+1$, if the *inventory position* at time $t-\tau$ is smaller then x. The proof is now complete. □

Next, we assume for the moment for simplicity, that there are no quantity discounts. Now the expected annual cost function $C(Q,r)$, is written as follows.

$$C(Q,r) = \bar{c}_K \frac{1}{Q} + c_h \sum_{x=0}^{r+Q} x\pi_x - c_p \sum_{x=-\infty}^{0} x\pi_x + \bar{c}_{\bar{p}} \sum_{x=-\infty}^{0} \pi_x \tag{18}$$

One can simplify $C(Q,r)$ using the lemma below.

Lemma A.2

The following are true:

i) $\sum_{x=0}^{r+Q} x\pi(x) = \sum_{x=r+1}^{r+Q} \sum_{i=0}^{x-1} P_i/Q$.

ii) $-\sum_{x=-\infty}^{-1} x\pi(x) = \left(\sum_{x=r+1}^{r+Q} \left(\sum_{i=0}^{x-1} P_i + \lambda\tau - x\right)\right)/Q$.

iii) $\sum_{x=-\infty}^{0} \lambda\pi(x) = \left(\sum_{x=r+1}^{r+Q} \lambda(1-P_{x-1})\right)/Q$.

Proof For i) we have:

$$\begin{aligned}
\sum_{x=0}^{r+Q} x\pi(x) &= \left(\sum_{x=0}^{r} x(P_{r+Q-x}-P_{r-x}) + \sum_{x=r+1}^{r+Q} xP_{r+Q-x}\right)/Q \\
&= \left(\sum_{x=0}^{r+Q}(r+Q-x)P_x - \sum_{x=0}^{r}(r-x)P_x\right)/Q \\
&= \left(\sum_{x=r+1}^{r+Q}(r+Q-x)P_x + Q\sum_{x=0}^{r}P_x + \sum_{x=0}^{r}(r-x)P_x - \sum_{x=0}^{r}(r-x)P_x\right)/Q \\
&= \sum_{x=r+1}^{r+Q}\left(\sum_{i=r+1}^{x-1}P_i + \sum_{i=0}^{r}P_i\right)/Q \\
&= \sum_{x=r+1}^{r+Q}\sum_{i=0}^{x-1}P_i/Q.
\end{aligned}$$

Similarly, for ii) we use the following property of a Poisson distribution: $xp_x = \lambda\tau p_{x-1}$ for any $x = 1,2,\ldots$. We obtain:

$$\begin{aligned}
-\sum_{x=-\infty}^{0} x\pi(x) &= -\sum_{x=-\infty}^{-1} x(P_{r+Q-x}-P_{r-x})/Q \\
&= \sum_{x=1}^{\infty} x((1-P_{r+x})-(1-P_{r+Q+x}))/Q \\
&= \left(\sum_{x=r+Q+1}^{\infty}(r+Q-x)(1-P_x) - \sum_{x=r+1}^{\infty}(r-x)(1-P_x)\right)/Q \\
&= \left(Q\sum_{x=r+Q+1}^{\infty}(1-P_x) - \sum_{x=r+1}^{r+Q}(r-x)(1-P_x)\right)/Q \\
&= \left(Q\sum_{x=0}^{\infty}(1-P_x) - \sum_{x=r+1}^{r+Q}(r+Q-x)(1-P_x) - Q\sum_{x=0}^{r}(1-P_x)\right)/Q \\
&= \left(Q\sum_{x=0}^{\infty}xp_x - \sum_{x=r+1}^{r+Q}\left((r+Q-x)(1-P_x) + \sum_{i=0}^{r}(1-P_i)\right)\right)/Q \\
&= \left(Q\lambda\tau - \sum_{x=r+1}^{r+Q}\left(\sum_{i=0}^{x-1}(1-P_i) - \sum_{i=0}^{r}(1-P_i) + \sum_{i=0}^{r}(1-P_i)\right)\right)/Q \\
&= \left(\sum_{x=r+1}^{r+Q}\left(\sum_{i=0}^{x-1}P_i + \lambda\tau - x\right)\right)/Q
\end{aligned}$$

Finally, for iii) we have

$$\begin{aligned}\sum_{x=-\infty}^{0} \pi(x) &= \sum_{x=-\infty}^{0} (P_{r+Q-x} - P_{r-x})/Q \\ &= \left(\sum_{x=r}^{\infty}(1-P_x) - \sum_{x=r+Q}^{\infty}(1-P_x)\right)/Q \\ &= \left(\sum_{x=r+1}^{r+Q}(1-P_{x-1})\right)/Q\end{aligned}$$

□

Proof of Theorem 1:
It is important to remember the definition of $\bar{c}_K, \bar{c}_i$ and $\bar{c}_{\tilde{p}}$ on page 3 so the factor λ is not repeated separately in the calculations below.
a) From Lemma A.2 we have:

$$\begin{aligned}C(Q,r) &= \bar{c}_K \frac{1}{Q} + c_h \sum_{x=0}^{r+Q} x\pi(x) - c_p \sum_{x=-\infty}^{0} x\pi(x) + \bar{c}_{\tilde{p}} \sum_{x=-\infty}^{0} \pi(x) \\ &= \frac{\bar{c}_K + \sum_{x=r+1}^{r+Q}\left(c_h \sum_{i=0}^{x-1} P_i + c_p\left(\sum_{i=0}^{x-1} P_i + \lambda\tau - x\right) + \bar{c}_{\tilde{p}}(1-P_{x-1})\right)}{Q} \\ &= \frac{\bar{c}_K + \sum_{x=r+1}^{r+Q}\left((c_h + c_p)\sum_{i=0}^{x-1} P_i + c_p(\lambda\tau - x) + \bar{c}_{\tilde{p}}(1-P_{x-1})\right)}{Q}\end{aligned}$$

b) First, we will establish that x^G exists. Then the uniqueness of x^G will be established by showing that $G(x)$ is strictly increasing for $x \ge x^G + 1$.

Simple algebra shows that

$$G(x+1) - G(x) = (c_h + c_p)P_x - \bar{c}_{\tilde{p}} p_x - c_p. \tag{19}$$

The above implies the following:

$$G(x+1) - G(x) = \begin{cases} \le 0, \text{ iff } (c_h + c_p)P_x - \bar{c}_{\tilde{p}} p_x \le c_p \\ > 0, \text{ iff } (c_h + c_p)P_x - \bar{c}_{\tilde{p}} p_x > c_p. \end{cases}$$

We notice that for every $x < 0$, $G(x+1) - G(x) = -c_p \le 0$. Thus, x^G is positive if it exists. Since $G(x+1) - G(x) > 0$ for large x, because then $G(x+1) - G(x) \approx c_h > 0$, the x-mode x^G exists.

First we prove uniqueness when $c_{\tilde{p}}$ is positive. Therefore, we show that $G(x)$ is strictly increasing for $x \ge x^G + 1$, by first finding a lower bound for x^G, using its definition and Eq. (19), as follows.

$$\begin{aligned}G(x^G) - G(x^G - 1) &\le 0 < G(x^G + 1) - G(x^G) \\ G(x^G) - G(x^G - 1) &< G(x^G + 1) - G(x^G) \\ -\bar{c}_{\tilde{p}} p_{x^G - 1} &< (c_h + c_p - \bar{c}_{\tilde{p}}) p_{x^G} \\ p_{x^G - 1}/p_{x^G} &> (\bar{c}_{\tilde{p}} - c_h - c_p)/\bar{c}_{\tilde{p}}\end{aligned}$$

i.e.,

$$x^G > \frac{\lambda\tau(\bar{c}_{\tilde{p}} - c_h - c_p)}{\bar{c}_{\tilde{p}}}.$$

Using this bound in (20), we obtain for $G(x^G+2) - G(x^G+1)$:

$$\begin{aligned} G(x^G+2) - G(x^G+1) &= (c_h + c_p)P_{x^G+1} - \bar{c}_{\tilde{p}} p_{x^G+1} - c_p \\ &= (c_h + c_p)P_{x^G} - \bar{c}_{\tilde{p}} p_{x^G} - c_p \\ &\quad + \bar{c}_{\tilde{p}} p_{x^G} + (c_h + c_p - \bar{c}_{\tilde{p}}) p_{x^G+1} \\ &> \bar{c}_{\tilde{p}} p_{x^G} - \frac{\lambda\tau(\bar{c}_{\tilde{p}} - c_h - c_p)}{x^G+1} p_{x^G} \\ &> \bar{c}_{\tilde{p}} \left(1 - \frac{\lambda\tau(\bar{c}_{\tilde{p}} - c_h - c_p)}{\lambda\tau(\bar{c}_{\tilde{p}} - c_h - c_p) + \bar{c}_{\tilde{p}}}\right) p_{x^G} \\ &> 0. \end{aligned} \tag{20}$$

Thus, $G(x^G+2) > G(x^G+1)$ and by an induction argument on $x > x^G$ we have $G(x+2) > G(x+1) > 0$ for all $x > x^G$ and we see that x^G is the maximal minimizing point. This also implies that $G(x)$ is unimodal.

If $c_{\tilde{p}} = 0$ then $G(x+1) - G(x) = (c_h + c_p)P_x - c_p$. Thus, for x^G we have:

$$\begin{aligned} P_{x^G-1} &\le c_p/(c_h + c_p), \\ P_{x^G} &> c_p/(c_h + c_p). \end{aligned}$$

Since P_x is increasing in x, we have that $P_x \le P_{x^G-1}$ for $x < x^G$ and $P_x > P_{x^G}$ for $x > x^G$, thus it follows that x^G is the maximum minimizing point of the function $G(x)$, i.e, $G(x)$ is unimodal.

□