A DYNAMICAL SYSTEM PERTURBED BY STOCHASTIC INTERVENTIONS

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Abstract. Central to the paper is an improved form of general lower bound technique for showing asymptotic stability of ergodic measures for Markov operators on Polish spaces, having the e-property. By elaborating on the simplest example conceivable in connection to modeling population dynamics in mathematical biology, i.e. regular random harvesting from a logistically growing population, we demonstrate its utility in studying the long-term behaviour of deterministic dynamical systems that are perturbed at fixed time points by random jumps in state space, where the laws of these jumps can depend on the system's state just before intervention. In the presentation generally valid arguments and results have been separated carefully from those specific to the running example, such that their domain of application can be assessed easily. Thus, it is shown that tracing the change of supports of measures under application of an associated Markov operator suffices to establish the lower bound condition. Moreover, in the abstract part of the paper we obtain the equivalence of uniform and global stability of ergodic measures for Markov-Feller operators on compact spaces that satisfy the e-property to a novel condition involving the successive iterates of these operators.

1. Introduction

This paper considers the long-term behaviour of a system that results from a particular type of stochastic perturbations of a deterministic system, which occurs naturally in the modeling and analysis of population dynamics in biology. That is,

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the change in state of a population of individuals is changing in time, adequately described by some deterministic dynamical system, while at discrete points in time quick interventions change its state after which it continues to evolve deterministically till the next time of intervention. On one hand, the time of next intervention may be random, with law dependent on the population state just after the last intervention, but the change in state is deterministic, as in models for cell division (cf. Lasota & Mackey [15]). On the other hand intervention times may be fixed, but the jumps are random with state-dependent distribution, as in a growing population of bacteria where samples are drawn regularly from the system in an experiment, or a large fish tank from which fish are caught daily using a net as we consider here. Both types of stochastic interventions may be combined too. It appears that such types of systems lacked mathematical attention over the past years.

Stochastic perturbations of deterministic systems of a different type have been studied extensively: stochastic differential equations, for instance

$$dX = F\left(X_t\right) dt + G\left(X_t\right) dU_t,$$

where U_t is a stochastic process, e.g. a Wiener process or a jump process and G is a certain coefficient function (c.f. Protter, [20], Øksendal, [18]). However, in such equations only the amplitude $G\left(X_t\right)$ of the stochastic perturbations is allowed to be state dependent. Ji et. al. [14] considers such a random perturbation of the logistic equation. The current state of the theory cannot accommodate for models in which also the *shape* of the distribution may depend on state.

In this paper we make a step towards analysis of deterministic systems stochastically perturbed by a jump process where the shape of the distribution of the jumps is state dependent. To that end we focus on a particular example, which is linked to application in biological population dynamics. That is, we consider a population of individuals, fish say, that reproduce, grow and die and that is subject to occasional external intervention ('fishing') at a discrete set of time points in which part of the population is removed. The 'catch size' is random, but limited to a maximal number of individuals. The distribution for the catch size depends on the total number of individuals present in the population at time just before the intervention. Population behavior in-between interventions is modeled deterministically. Under conditions on the parameters of the model, we show existence of invariant distributions for the population state and establish stability of a non-trivial invariant measure. We employ equicontinuity conditions on the Markov operator (the 'e-property') and a lower bound technique to establish stability of an ergodic measure associated to a Markov-Feller operator linked to the model. It fits into a series of approaches to studying ergodic properties of Markov operators, semigroups and the associated Markov processes. Doeblin, [6] employed a *uniform* lower-bound condition for transition probabilities. According to Lasota and Szarek [17], Markov essentially used a version of the technique already in the years 1906-1908. Harris, [13] established a generalization of Doeblin's condition. A similar approach was used recently in Hairer and Mattingly [12]. Szarek and co-authors substantially extended the applicability of the technique to Markov operators and semigroups with the e-property on Polish spaces, (c.f. [25], [23]). Worm [26], subsequently weakened the conditions in the results for Markov semigroups of the latter: e.g. the e-property could be replaced by eventual e-property, while the lower-bound condition itself could be simplified. In Section 4.1. we will formulate and prove a version of the latter result for Markov *operators,* which is not stated in [26], Chapter 7, nor it can be found in the work by Szarek and co-authors (c.f. [25]).

As for the abstract results introduced in Section 4, we present the following theorem as a conclusion to this section, which summarizes most of the results there.

Theorem 1. *Let (S,d) be a compact metric space and let P be a Markov operator on S with the* e-property. *If P has a unique ergodic measure μ^*, then the following are equivalent:*

(1) There exist $x \in \operatorname{supp}(\mu^*)$ and $z \in S$, such that

$$\liminf_{n\to\infty} P^n \delta_x \left(B\left(z, r\right)\right) > 0$$

for every $r > 0$.

(2) μ^* *is uniformly stable, i.e.* $\lim_{n\to\infty} P^n \mu = \mu^*$ *uniformly for all* $\mu \in \mathcal{P}(S)$.

Moreover, $z \in \operatorname{supp}(\mu^*)$ *necessarily.*

The proof follows Theorem 12 directly.

The structure of the paper is as follows. In Section 2 we describe the two parts of the model; the deterministic part and the stochastic interventions, we introduce the transition operator and we give an example of a distribution that fits our model. The deterministic part is characterized by a logistic equation, and the Beta distribution is proposed as a distribution of the interventions; the catch size. Section 3 summarizes several results about Markov operators for the reader's convenience. Section 4 contains the improved and new results concerning the stability of invariant measures of Markov-Feller operators with the e-property. The subsequent Section 5 demonstrates the applicability of the theory, by means of a detailed study of the 'fishery model'.

Before proceeding, let us introduce the following notational conventions. Unless otherwise mentioned, (S, d) will denote a complete separable metric space, viewed as a measurable space with respect to its Borel σ-algebra $\mathcal{B}(S)$. We write $\mathcal{M}(S)$ to denote the real vector space of all signed finite Borel measures on S, containing $\mathcal{M}^+(S)$, the cone of positive measures. $\mathcal{P}(S)$ is the set of probability measures in $\mathcal{M}^+(S)$. We denote the total variation norm on $\mathcal{M}(S)$ by $\|\cdot\|_{TV}$ and write $\mathcal{M}(S)_{TV}$ for the Banach space consisting of $\mathcal{M}(S)$ endowed with this norm. We denote by $\mathrm{BM}(S)$ the real vector space of all bounded Borel measurable functions from S to $\mathbb{R}$ (equipped with the supremum norm) and by $\mathrm{BL}(S)$ the Banach space of bounded Lipschitz functions from S to $\mathbb{R}$, with the norm $\| f \|_{\mathrm{BL}} = | f |_{\mathrm{Lip}} + \| f \|_\infty$, where $| f |_{\mathrm{Lip}}$ denotes the Lipschitz constant of f. We write 1_E for the indicator function of $E \subset S$. For $f : S \to \mathbb{R}$ measurable and $\mu \in \mathcal{M}(S)$, we write $\langle f, \mu \rangle$ for $\int_S f \, d\mu$. $C_b(S)$ denotes the Banach space of bounded continuous functions from S to $\mathbb{R}$, endowed with the supremum norm $\| \cdot \|_\infty$. For $x \in S$ and $r > 0$, $B(x, r)$ denotes the open ball around x with radius r, and δ_x denotes the Dirac measure (point measure) at x.

2. Model Description

Essentially, we will consider a deterministic dynamical system with stochastic interventions at fixed times with **equal length time intervals**. That is, a system which is subject to interventions at fixed, equally spaced discrete times $0 < t_1 < t_2 \ldots < t_n < \ldots$ at which the state is changed to a new state randomly. The law describing this new state depends on the state just before the intervention. We aim to analyze the long term dynamics of this system.

2.1. The Deterministic Population Model.

In this paper we have selected a model that is simple to describe, has some relationship to natural systems, but allows to exhibit the strength of the abstract results obtained. It will pave the way to further investigations.

The illustrative case that we will consider is the *logistic equation,* which is the simplest, though realistic, model for a developing population, given by

$$\frac{dv}{dt} = rv\left(1 - \frac{v}{K}\right). \tag{1}$$

Here $v(t)$ is the expected number of individuals in the population at time t (often called 'victims' in a predator prey model), $r > 0$ is its maximum per-capita rate of change, and $K > 0$ is the so-called *carrying capacity* of the environment. The proof of the following Lemma is straightforward, thus omitted.

Lemma 2. *The logistic equation* (1) *has the following properties:*

(1) *The unique solution with* $v(0) = v_0$ *is explicitly given by*

$$v(t) = \left(\frac{1}{K} + \left(\frac{1}{v_0} - \frac{1}{K}\right)\mathrm{e}^{-rt}\right)^{-1}. \tag{2}$$

(2) The steady states *of the logistic equation* (1) are $v = 0$ and $v = K$. Furthermore, K is stable, while 0 is unstable.

(3) *The interval* $[0, K]$ *is positively invariant (i.e. if* $v(0) \in [0, K]$*, then* $v(t) \in [0, K]$ *for all* $t \geq 0$*). Also,* $[K, \infty)$ *is positively invariant.*

We will denote by ϕ_t the semi-flow (solution operator) associated to (2): $\phi_t(v_0) := v(t)$, given by (2).

2.2. **Stochastic Interventions.** We will consider stochastic interventions at fixed equally spaced times. The distribution for the jump to the new state after intervention will be state dependent. We suppose that at the intervention times the number of individuals is diminished by a random amount, the *catch size*. The catch size is constrained: it cannot exceed the maximum m_c (a 'full net'). In our illustrative example we aim to model a fishery that has fish only and is subject to occasional fishing. We need the following crucial assumptions on the law Q_x for the catch size, where x is the population size just before the intervention. We call Q_x the *catch size distribution.*

A1): Q_x is fully supported on $[0, m_c]$.

A2): Q_x has density function $y \mapsto q(x, y)$ with respect to Lebesgue measure on $\mathbb{R}$ for each $x > m_c$.

A3): The map $x \mapsto q(x, \cdot) : [m_c, \infty) \to \mathrm{L}^1(\mathbb{R})$ is continuous.

Lemma 3. *If* $x \mapsto q(x, y)$ *is continuous on* $[m_c, \infty)$ *for almost every* $y \in \mathbb{R}$*, then* q *satisfies A3).*

Proof. Since $q(x,y)$ and $q(x_0,y)$ are densities, Due to the Vitali-Scheffé Theorem (Scheffé [21], Bogachev [3]) a sufficient condition that

$$\lim_{x\to x_0}\int_{\mathbb{R}} \mid q(x,y)-q(x_0,y)\mid dy=0,$$

in $[m_c,\infty)$ is that $\lim_{x\to x_0} q(x,y)=q(x_0,y)$ for almost every y. (i.e. we only need to prove that q is pointwise continuous) which we have by assumption. □

A suitable example of such a distribution Q_x is a *Beta* distribution with parameters depending on x, which we will discuss below. The abstract conditions actually evolved out of the analysis of this case.

2.3. **Introduction of the transition probability P.** The evolution of the system between the interventions is given by the logistic equation, so the population just before the next intervention will be

$$\phi_{\Delta t}(x)=\left(\frac{1}{K}+\left(\frac{1}{x}-\frac{1}{K}\right)\mathrm{e}^{-r\Delta t}\right)^{-1},$$

where x is the state just after an intervention. If the population size just before an intervention equals x, then the population y just after the intervention will be in a Borel set $A\subseteq[0,\infty)$ with probability $Q_x(x-A)$. (The result y is in A if and only if the subtracted amount is $x-y$ with $y\in A$, this happens with probability $Q_x(x-A)$). So, if the population size at time 0 equals x, then the population size just before the first intervention equals $\phi_{\Delta t}(x)$, thus, the distribution of the population size just after the first intervention is $Q_{\phi_{\Delta t}(x)}(\phi_{\Delta t}(x)-\cdot)$. Furthermore, if the population size just after the n-th intervention would have distribution μ, then the distribution just after the $(n+1)$- intervention equals $P\mu$ given by

$$(P\mu)(A)=\int_{[0,\infty)} Q_{\phi_{\Delta t}(x)}(\phi_{\Delta t}(x)-A)\,d\mu(x). \tag{3}$$

Put $p(x,A) := Q_{\phi_{\Delta t}(x)}(\phi_{\Delta t}(x) - A)$, with $A \subseteq [0,\infty)$ Borel. Observe that $P : \mathcal{M}^+(S) \to \mathcal{M}^+(S)$ is a Markov operator with transition kernel[1] $p(x,\cdot) = P\delta_x = Q_{\phi_{\Delta t}(x)}(\phi_{\Delta t}(x) - \cdot)$. Therefore it suffices to understand the iterates $(P^n\delta_x)_n$ in order to determine the dynamics of P.

Alternatively, one could consider the population size just after the n-th intervention as a random variable X_n. Then $(X_n)_n$ is a Markov chain and P is its *corresponding* Markov operator. This means, $p(x,E) = P\delta_x(E)$ and for $\mu \in \mathcal{P}(S)$ we have

$$\text{Prob}(X_0 \in A) = \mu(A) \quad \text{and} \quad \text{Prob}(X_{n+1} \in A \mid X_n = x) = p(x,A).$$

2.4. **An Example of distribution that satisfies A1)-A3).** Let us consider a specific example of a distribution Q_x which is appropriate for the fishing model. If there are many individuals a catch is expected to be large, if there are few, there will be hardly any fish in a net. A distribution which captures these properties and satisfies assumptions A1)-A3) is a suitably scaled Beta distribution;

$$Q_x \sim \text{Beta}\left(0, m_c, \beta, \beta\frac{x}{x^*}\right), \tag{4}$$

for some parameters β and x^*. The meaning of x^* is that it is the population size just before intervention at which the distribution Q_{x^*} has expectation $\frac{1}{2}m_c$ (i.e. $\mathbb{E}[Q_{x^*}] = \frac{1}{2}m_c$). We begin by recalling some facts about the Beta distribution.

The Beta distribution $\text{Beta}(a,b,\alpha,\beta)$ with parameters a,b,α,β is defined on $[a,b]$ and given by

$$\frac{1}{B(\alpha,\beta)}(x-a)^{\alpha-1}\frac{(b-x)^{\beta-1}}{(b-a)^{\alpha+\beta-1}}, \quad a < x < b,$$

[1]A function $p : S \times \mathcal{B}(S) \to [0,1]$, defined as $p(x,E) = P\delta_x(E)$ for $x \in S$ and $E \in \mathcal{B}(S)$ is called the *transition function* (transition kernel). (c.f. Ethier and Kurtz, [11]).

where $B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ is the Beta function. Thus, the density of the catch size distribution (4) is $q(x,y)=0$ if $y\in\mathbb{R}\setminus(0,m_c)$ and

$$
\begin{aligned}
q(x,y) =& \operatorname{Beta}\left(0,m_c,\beta,\beta\frac{x}{x^*}\right)(y)\\
=& \frac{1}{B\left(\beta,\beta\frac{x}{x^*}\right)}y^{\beta-1}\frac{(m_c-y)^{\beta\frac{x}{x^*}-1}}{m_c^{\beta\left(1+\frac{x}{x^*}\right)-1}},\quad 0<y<m_c.
\end{aligned}
\tag{5}
$$

Remark 4. q given by (5) is continuous on $[0,\infty)\times(0,m_c)$, but may have singularities for y at 0 and m_c.

For this distribution of the catch size we can compute the distribution $P\delta_x$ of the population size just after fishing, if the population size just after the previous fishing equals x. Indeed, the density of this distribution is given by

$$
H(x,y)=\begin{cases} q\left(\phi_{\Delta t}(x),\phi_{\Delta t}(x)-y\right) & \text{for } y \text{ such that } \phi_{\Delta t}(x)-y\in(0,m_c)\\ 0 & \text{otherwise}\end{cases}. \tag{6}
$$

Hence, $P\delta_x$ could be given by

$$
P\delta_x=\begin{cases} \mathrm{F}_x(y)\,dy & \text{for } x:\phi_{\Delta t}(x)\geq m_c\\ \mathrm{F}_x(y)\,dy+d_x\delta_0 & \text{for } x:\phi_{\Delta t}(x)<m_c\end{cases},
$$

where

$$
\mathrm{F}_x(y)=\begin{cases} \operatorname{Beta}\left(\phi_{\Delta t}(x)-m_c,\phi_{\Delta t}(x),\beta\frac{\phi_{\Delta t}(x)}{x^*},\beta\right)(y) & \text{for } 0\leq y\leq\phi_{\Delta t}(x)\\ 0 & \text{elsewhere}\end{cases},
$$

and

$$
d_x=1-\int_0^{\phi_{\Delta t}(x)}\operatorname{Beta}\left(\phi_{\Delta t}(x)-m_c,\phi_{\Delta t}(x),\beta\frac{\phi_{\Delta t}(x)}{x^*},\beta\right)dy.
$$

Observe that if x is so small that $\phi_{\Delta t}(x)<m_c$, then the law of the population size after fishing must have an atomic part supported at 0. If $\phi_{\Delta t}(x)\geq m_c$, then $P\delta_x$ has a density function $y\mapsto\mathrm{F}_x(y)=H(x,y)$. So, $p(x,A)=\int_A H(x,y)\,dy$, and $H(x,y)=q\left(\phi_{\Delta t}(x),\phi_{\Delta t}(x)-y\right)$ whenever Q_x has density $q(x,\cdot)$.

Clearly Q_x given by (5) satisfies A1) and A2). We next highlight that it also satisfies A3).

Lemma 5. *If q is given by* (5)*, then $x \mapsto q(x,\cdot)$ is a continuous map from $[m_c,\infty)$ into $L^1(\mathbb{R})$, so, our specific choice of the Beta distribution for Q_x satisfies A3).*

3. Fundamentals of Markov Operators and Semigroups

Let us firstly recall some related facts. An operator $P : \mathcal{M}^+(S) \to \mathcal{M}^+(S)$ is called a *Markov operator* (on measures) if it preserves the mass (or total variation norm); $P\mu(S) = \mu(S)$, for $\mu \in \mathcal{M}^+(S)$ and it is positive linear, that is

$$P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2$$

for $\lambda_1, \lambda_2 \geq 0$ and $\mu_1, \mu_2 \in \mathcal{M}^+(S)$, (c.f. [23, 26]).

A Markov operator P is said to be *regular* if there exists an operator U which maps $\mathrm{BM}(S)$, into itself such that $\langle Uf, \mu\rangle = \langle f, P\mu\rangle$, for all $f \in \mathrm{BM}(S)$ and $\mu \in \mathcal{M}^+(S)$. In that case U is unique and is called the *dual* of P. A regular Markov operator on S is *Markov-Feller* if its dual maps $C_b(S)$ into itself. It is *strong Feller* if its dual maps $\mathrm{BM}(S)$ into $C_b(S)$. Finally, a Markov operator P is *ultra-Feller* if the map $x \to P\delta_x$ from S to $\mathcal{M}(S)_{TV}$ is continuous (c.f. [22, 26]).

Remark 6. Under weak hypotheses on the topology of the state space, which are always satisfied if the state space is a separable metric space, every strong Feller Markov operator is ultra-Feller. (Seidler, [22]).

A measure $\mu \in \mathcal{M}^+(S)$ is called *invariant* under P if $P\mu = \mu$. In other words, if μ_0 is a distribution of our process at time 0, this distribution does not change in time. A set E is P-invariant (or invariant with respect to P) if for all $x \in E$, $P\delta_x$ is concentrated on E (i.e. $P\delta_x(E) = 1$, or equivalently $p(x,E) = 1$ whenever $x \in E$). Also, μ is ergodic (with respect to P) if $\mu(E) = 0$ or $\mu(E) = 1$ for every P-invariant set E. (c.f. Zaharopol [27]. p18).

Based on [26] we define, for $f \in \mathrm{BL}(S)$, the dual norm, $\|\cdot\|_{BL}^{*}$, as follows:

$$\| \mu \|_{BL}^{*} := \sup_{f \in BL,\, \|f\|_{BL} \leq 1} \left| \int f d(\mu) \right| = \sup_{f \in BL} \frac{|\langle f, \mu \rangle|}{\| f \|_{BL}}.$$

Thus, $|\langle f, \mu \rangle| \leq \| \mu \|_{BL}^{*} \cdot \| f \|_{BL}$. One says that (μ_n), $\mu_n \in \mathcal{M}(S)$, converges weakly to $\mu \in \mathcal{M}(S)$ if $\lim_{n\to\infty} \langle f, \mu_n \rangle = \langle f, \mu \rangle$ for every continuous bounded function $f : \mathbb{R} \to \mathbb{R}$. The convergence

$$\lim_{n\to\infty} \| \mu_n - \mu \|_{BL}^{*} = 0 \quad \text{for } \mu_n, \mu \in \mathcal{P}(S)$$

is equivalent to the weak convergence of $(\mu_n)_{n\geq 1}$ to μ (see Dudley, [7], [8], and [9]). We denote by $\mathcal{S}_{\mathrm{BL}}$ the closure of D in $\mathrm{BL}(S)^{*}$, where $D := \mathrm{span}\{\delta_x : x \in S\}$.

Let X and Y be two metric spaces, and F a family of functions from X to Y. The family$F$ is equicontinuous at a point $x_0 \in X$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d(f(x_0), f(x)) < \epsilon$ for all$f \in F$ and all x such that $d(x_0, x) < \delta$. The family is *equicontinuous* if it is equicontinuous at each point of X. Let $P : \mathcal{M}^{+}(S) \to \mathcal{M}^{+}(S)$ be a regular Markov operator. Following Lasota, Szarek and others, e.g. [26], we say that P has the *e-property* if for each $f \in \mathrm{BL}(S)$ the family $(U^n f)_{n\in\mathbb{N}_0}$ is equicontinuous. Note that having the e-property depends on the choice of metric; the same P may have the e-property for one metric, but not for another that yields the same metric topology. Moreover, P has the e-property if and only if the family $(P^n)_{n\in\mathbb{N}}$ is an equicontinuous family in $C_b\left(\mathcal{S}_{\mathrm{BL}}^{+}, \mathcal{S}_{\mathrm{BL}}^{+}\right)$. (c.f. [26], Theorem 7.2.2., p139). Every regular Markov operator with the e-property is Markov-Feller ([26], Lemma 7.2.1 p.138). Also, we will use the following results, which are Corollary 3.2.8 on page 48 and Corollary 3.2.6 on page 47 in [26],

Remark 7. For any $\mu \in \mathcal{M}^{+}(S)$, $\mu = \int_S \delta_x d\mu(x)$, as a Bochner integral in $\mathcal{S}_{\mathrm{BL}}$.

It follows that, for a regular Markov operator P,

$$P\mu = P\left(\int_S \delta_x d\mu(x)\right) = \int_S P\delta_x d\mu(x). \tag{7}$$

Remark 8. Assume that $p:\Omega\to\mathcal{M}^{+}(S)$ is strongly measurable as map into $\mathcal{S}_{\mathrm{BL}}$, such that $\upsilon:=\int_{\Omega}p(\omega)\,d\mu(\omega)$ exists in $\mathcal{S}_{\mathrm{BL}}$ for $f\in\mathrm{BM}(S)$. Then

$$\int_S f d\upsilon = \int_\Omega \langle f, p(\omega)\rangle d\mu(\omega)\,. \tag{8}$$

For a particular class of Markov operators, uniqueness of invariant probability measures may be established by employing the following result, initially proved in [24] for P with the e-property and extended in [26] to P having the weaker Cesàro e-property. (Theorem 7.4.6. p.156 in [26]):

Theorem 9. *Let P be a regular Markov operator on (S,d) with the (Cesàro)* e-*property. Suppose that for every $x,y\in S$ there exists a $z\in S$, such that for every $\delta>0$ there are $n_1,n_2\in\mathbb{N}$ such that*

$$P^{n_1}\delta_x\left(B\left(z,\delta\right)\right)>0 \text{ and } P^{n_2}\delta_y\left(B\left(z,\delta\right)\right)>0.$$

Then P has at most one invariant probability measure on S.

4. New results on Stability of Invariant Measures

While developing the required theory for the illustrative case we are considering-explained in Section 2, an interesting result was established. This *new* result (Theorem 21) provides a novel equivalence of uniform and global stability of unique ergodic measures for regular Markov operators on compact metric spaces, that satisfy the e-*property* to a condition involving the successive iterates of these operators. It is worthy to mention that there exist various notions of asymptotic stability in the theory of Markov operators and semigroups originating from a probabilistic view. We advocate a view from dynamical systems theory: an invariant measure μ^* for a Markov operator P is called *locally asymptotically stable*, if there exists an open neighborhood U of μ^* in $\mathcal{P}(S)$ such that $P^n\nu\to\mu^*$ for all $\nu\in U$ as $n\to\infty$. If U can be taken equal to $\mathcal{P}(S)$ we call μ^* *globally asymptotically stable*. Throughout this section let (S,d) be a complete separable metric space. The results here are of general interest for applications in the theory of Markov-Feller operators.

4.1. **A Lower Bound Result for Asymptotic Stability.** In this section we present the result that is our main tool in establishing asymptotic stability of invariant measures for the Markov operator P associated to the 'fishery model'. We singled out particular partial results in the proof of the main theorem (Theorem 12) that deserve separate attention.

Let $P^{(n)} := \frac{1}{n}\sum_{k=0}^{n-1} P^k$ denote the Cesaro averages of P^n and define the following sets:

$$S_1 = \left\{ x \in S : \left(P^{(n)}\delta_x\right) \text{ is tight} \right\},$$
$$S_2 = \left\{ x \in S : \left(P^{(n)}\delta_x\right) \text{ converges in } \mathcal{S}_{BL} \right\},$$
$$S_3 = \left\{ x \in S_2 : \epsilon_x = \lim_{n\to\infty} P^{(n)}\delta_x \text{ is invariant} \right\},$$

and for any P-invariant measure μ^*, let

$$S_{\mu^*} := \left\{ x \in S : P^{(n)}\delta_x \to \mu^* \;\text{ as }\; n \to \infty \right\}. \tag{9}$$

The following result is due to Worm [26], Lemma 5.3.2:

Lemma 10. *If P is a Markov-Feller operator and P has a unique invariant probability measure μ^*, then $S_1 = S_2 = S_3 = S_{\mu^*}$.*

Remark 11. If S is compact, and P is a Markov-Feller operator with a unique invariant probability measure μ^*, then $S_{\mu^*} = S$. Indeed, since $\mathcal{P}(S)$ is compact in this case, $S_1 = S$.

The lower bound result alluded to in the introduction of this section is:

Theorem 12. *Let P be a Markov operator with the* e*-property. Assume that P has an ergodic measure μ^* and let S_{μ^*} be defined by* (9)*. If there exist $x \in \operatorname{supp}(\mu^*)$*

and $z \in S$, such that

$$\liminf_{n\to\infty} P^n \delta_x \left(B\left(z,r\right)\right) > 0 \tag{10}$$

for every $r > 0$, then for every $\nu_1, \nu_2 \in \mathcal{P}(S)$ with $\nu_1 (S_{\mu^*}) = \nu_2 (S_{\mu^*}) = 1$,

$$\lim_{n\to\infty} \| P^n \nu_1 - P^n \nu_2 \|_{BL}^* = 0.$$

In particular, $P^n \nu \to \mu^*$ for every $\nu \in \mathcal{P}(S)$ such that $\nu (S_{\mu^*}) = 1$. Moreover, $z \in \operatorname{supp}(\mu^*)$ necessarily.

As a direct corollary to Theorem 12 we prove Theorem 1 from the introduction.

Proof. (Proof of Theorem 1) If (1) holds, then global stability follows from Remark 11 and Theorem 12 and uniform stability follows from Theorem 21. If (2) holds, then (1) follows from the Portmanteau Theorem and the fact that $z \in \operatorname{supp}(\mu^*)$. In particular, we get for any $x \in S$ and $r > 0$, $\liminf_{n\to\infty} P^n \delta_x (B(z,r)) \geq \mu^* (B(z,r)) > 0$. □

The proof of Theorem 12 relies on several lemmas:

Lemma 13. *Let P be a Markov operator that has the* e*-property. Assume that P has an ergodic measure μ^*. If there exist $x \in \operatorname{supp}(\mu^*)$ and $z \in S$, such that* $\liminf_{n\to\infty} P^n \delta_x (B(z,r)) > 0$ *for every $r > 0$, then, for each $r > 0$ there exists $\alpha = \alpha(r) > 0$, such that*

$$\liminf_{n\to\infty} P^n \nu \left(B\left(z,r\right)\right) > \alpha$$

for every $\nu \in \mathcal{P}(S)$ satisfying $\nu (S_{\mu^*}) = 1$. (Note that α is independent of ν).

Proof. Let $r > 0$. Define $\gamma := \frac{1}{2} \liminf_{n\to\infty} P^n \delta_x \left(B\left(z, \frac{r}{2}\right)\right)$. By assumption $\gamma > 0$. Let $f := \left[1 - \frac{2}{r} d\left(x, B\left(z, \frac{r}{2}\right)\right)\right]^+$. Then $1_{B\left(z,\frac{r}{2}\right)} \leq f \leq 1_{B(z,r)}$ and $f \in \mathrm{BL}(S)$. By the e-property, there exists an $s > 0$ such that $| \langle P^n \delta_y - P^n \delta_x, f \rangle |< \gamma$ for every

$y \in B(x,s)$ and n. Thus, for all $y \in B(x,s)$ and n,

$$\begin{aligned} P^n\delta_y(B(z,r)) \geq& \langle P^n\delta_y, f\rangle \ \geq \langle P^n\delta_x, f\rangle - |\langle P^n\delta_y - P^n\delta_x, f\rangle| \\ >& P^n\delta_x(B(z,r/2)) - \gamma. \end{aligned}$$

Hence,

$$\liminf_{n\to\infty} P^n\delta_y(B(z,r)) \geq \liminf_{n\to\infty} P^n\delta_x(B(z,r/2)) - \gamma = \gamma \tag{11}$$

for every $y \in B(x,s)$. Set $\theta := \frac{1}{2}\mu^*(B(x,s))$. Then $\theta > 0$, since $x \in \operatorname{supp}(\mu^*)$. Fix $\nu \in \mathcal{P}(S)$ such that $\nu(S_{\mu^*}) = 1$. Thus, $P^{(n)}\nu \to \mu^*$. By the Portmanteau Theorem,

$$\liminf_{n\to\infty} P^{(n)}\nu(B(x,s)) \geq \mu^*(B(x,s)) \geq 2\theta > 0.$$

Then there exists $n_0 > 0$ such that $P^{n_0}\nu(B(x,s)) > \theta$. Thus, by Fatou's Lemma

$$\begin{aligned} \liminf_{n\to\infty} P^n\nu(B(z,r)) =& \liminf_{n\to\infty} P^{n+n_0}\nu(B(z,r)) \\ \geq& \int_S \liminf_{n\to\infty} P^n\delta_y(B(z,r))\, d[P^{n_0}\nu](y) \\ \geq& \int_{B(x,s)} \liminf_{n\to\infty} P^n\delta_y(B(z,r))\, d[P^{n_0}\nu](y) > \gamma\theta, \end{aligned}$$

using (11). Taking $\alpha = \gamma\theta$ completes the proof. □

Lemma 14. *Let P be a Markov operator and E a P-invariant set in S. If $\mu_0 \in \mathcal{P}(S)$ is such that $\mu_0(E) = 1$ and there exist $n \in \mathbb{N}$, $\alpha > 0$ and $B \subset S$ measurable such that $P^n\mu_0(B) > \alpha$, then there exist $\nu_1, \mu_1 \in \mathcal{P}(S)$ with* $\operatorname{supp}(\nu_1) \subset \overline{B}$, *$\mu_1(E) = 1$ and*

$$P^n\mu_0 = \alpha\nu_1 + (1-\alpha)\mu_1.$$

Proof. Define $\alpha_1 := (P^n\mu_0)(B) > \alpha$. Denote by $(P^n\mu_0)_B$ the restriction of $P^n\mu_0$ to B, i.e. $(P^n\mu_0)_B(F) := P^n\mu_0(F \cap B)$. Clearly

$$P^n\mu_0 = \alpha\left(\frac{1}{\alpha_1}(P^n\mu_0)_B\right) + \left(P^n\mu_0 - \frac{\alpha}{\alpha_1}(P^n\mu_0)_B\right).$$

Define $\nu_1 := \frac{1}{\alpha_1}(P^n\mu_0)_B$ and $\mu_1 := \frac{1}{1-\alpha}\left(P^n\mu_0 - \frac{\alpha}{\alpha_1}(P^n\mu_0)_B\right)$. Then $\nu_1, \mu_1 \in \mathcal{P}(S)$ and $\mathrm{supp}(\nu_1) \subset \overline{B}$, while $P^n\mu_0 = \alpha\nu_1 + (1-\alpha)\mu_1$. Note that $\mu_1 \leq \frac{1}{1-\alpha_1}(P^n\mu_0)$. Since E is P-invariant and $\mu_0(E) = 1$, one has $P^n\mu_0(E) = 1$, or $P^n\mu_0(S \setminus E) = 0$. Then $\mu_1(S \setminus E) = 0$, hence $\mu_1(E) = 1$. □

Note that one may take for P the identity map in Lemma 14.

Lemma 15. *If $\nu_1, \nu_2 \in \mathcal{P}(S)$ have* $\mathrm{supp}(\nu_i) \subset B(z,r)$*, then* $\| \nu_1 - \nu_2 \|^*_{\mathrm{BL}} \leq 2r$. *Consequently, if P satisfies the* e*-property, then for every $\epsilon > 0$, there exists $r > 0$ such that for all $\nu_i \in \mathcal{P}(S)$ with* $\mathrm{supp}(\nu_i) \subset B(z,r)$*, for $i = \{1,2\}$, one has $\| P^n\nu_1 - P^n\nu_2 \|^*_{BL} < \epsilon$ for all n.*

Proof. Fix $\epsilon > 0$. If $f \in \mathrm{BL}(S)$ with $| f |_{Lip} \leq \| f \|_{BL} \leq 1$, define

$$f_{\inf} := \inf_{x \in B(z,r)} f(x) \quad \text{and} \quad f_{\sup} := \sup_{x \in B(z,r)} f(x).$$

Then $f_{\sup} - f_{\inf} \leq 2r$, since by $| f |_{Lip} \leq 1$, $f_{\sup} \leq f(z) + r$ and $f_{\inf} \leq f(z) - r$. We have $f_{\inf} \leq \langle \nu_i, f \rangle \leq f_{\sup}$, for $i = \{1,2\}$, thus, $| \langle \nu_1 - \nu_2, f \rangle | \leq 2r$. So, $\| \nu_1 - \nu_2 \|^*_{BL} \leq 2r$. Now, let $\epsilon > 0$. Since P^n satisfies the e-property, there exists $\delta > 0$ such that $\| P^n\nu_1 - P^n\delta_z \|^*_{BL} < \epsilon/2$ for all ν: $\| \nu - \delta_z \| < \delta$. Put $r = \frac{1}{2}\delta$. If ν_1, ν_2 are such that $\mathrm{supp}(\nu_i) \subset B(z,r)$ then,

$$\| P^n\nu_1 - P^n\nu_2 \|^*_{BL} \leq \| P^n\nu_1 - P^n\delta_z \|^*_{BL} + \| P^n\nu_2 - P^n\delta_z \|^*_{BL} < \epsilon.$$

□

Lemma 16. *Let Q be a Markov-Feller operator on S. Suppose that there exist $x, z \in S$, $m \in \mathbb{N}$ and $r > 0$ such that $Q\delta_x(B(z,r)) \geq \beta > 0$. Then for each $R > r$ and $0 < \beta' < \beta$ there exists an open neighborhood U_x of x in S, such that*

$$Q\delta_y(B(z,R)) \geq \beta' \quad \textit{for all } y \in U_x.$$

Proof. Let $R > r$, $0 < \beta' < \beta$. Put $\epsilon := \beta - \beta'$. There exists $f \in BL(S)$ such that $1_{B(z,r)} \leq f \leq 1_{B(z,R)}$(see the proof of Lemma 13). Since Q is continuous, there exists an open neighborhood V_x of δ_x in $\mathcal{P}(S)$ such that $\| Q\mu - Q\delta_x \|_{BL}^* < \frac{\epsilon}{\|f\|_{BL}}$ for all $\mu \in V_x$. Let $U_x := \{y \in S : \delta_y \in V_x\}$. Since $y \longmapsto \delta_y : S \to \mathcal{P}(S)$ is continuous, U_x is open and $x \in U_x$. Moreover,

$$|\langle Q\delta_y, f\rangle - \langle Q\delta_x, f\rangle| \leq \| Q\delta_y - Q\delta_x \|_{BL}^* \cdot \| f \|_{BL} < \epsilon$$

for all $y \in U_x$. Since $\langle Q\delta_y, f\rangle \leq Q\delta_y (B(z,R))$ and $\langle Q\delta_x, f\rangle \geq Q\delta_x (B(z,r))$, we find for $y \in U_x$:

$$\begin{aligned} Q\delta_y (B(z,R)) \geq &\langle Q\delta_y, f\rangle \ \geq \langle Q\delta_x, f\rangle - |\langle Q\delta_y, f\rangle - \langle Q\delta_x, f\rangle| \\ &> Q\delta_x (B(z,r)) - \epsilon \geq \beta - \epsilon = \beta'. \end{aligned}$$

□

Lemma 17. *Let P be a Markov-Feller on S, with invariant measure μ^*. If $y \in S$ is such that for each $r > 0$, there exist n and $x \in \operatorname{supp}(\mu^*)$ satisfying $P^n\delta_x\left(B\left(y, \frac{1}{2}r\right)\right) > 0$, then $y \in \operatorname{supp}(\mu^*)$.*

Proof. We have to show $\mu^*(B(y,r)) > 0$ for each $r > 0$. Fix $r > 0$. Let $\beta := P^n\delta_x\left(B\left(y, \frac{1}{2}r\right)\right) > 0$. According to Lemma 16 with $\beta' = \frac{1}{2}\beta$ and $R = r > \frac{1}{2}r$, there exist U_x, an open neighborhood of x, such that $P^n\delta_z(B(y,r)) \geq \beta' > 0$ for all $z \in U_x$. Thus, using $\mu^* = P^n\mu^* = \int_S P^n\delta_z d\mu^*(z)$, and Remark 8,

$$\begin{aligned} \mu^*(B(y,r)) &= \int_S P^n\delta_z (B(y,r))\, d\mu^*(z) \\ &\geq \int_{U_x} P^n\delta_z (B(y,r))\, d\mu^*(z) \geq \beta' \mu^*(U_x) > 0. \end{aligned}$$

□

We now use Lemmas 13-17 to present the proof of Theorem 12.

Proof. (Proof of Theorem 12) Let $\nu_i \in \mathcal{P}(S)$ such that $\nu_i(S_{\mu^*}) = 1$. Let $\epsilon > 0$ and let r be as in the second part of Lemma 15 with ϵ replaced by $\frac{\epsilon}{2}$. Put $B =$

$B\left(z,\frac{r}{2}\right)$. Using Lemma 13, there exist α such that $\liminf_{n\to\infty} P^n\nu\left(B\right) > \alpha$ for all $\nu \in \mathcal{P}\left(S\right)$ satisfying $\nu\left(S_{\mu^*}\right) = 1$. Thus, there exists a sequence (n_k) in $\mathbb{N}$ such that $P^{n_k}\nu_1\left(B\right) > \alpha$ for all k. Since

$$\liminf_{k\to\infty} P^{n_k}\nu_2\left(B\right) \geq \liminf_{n\to\infty} P^n\nu_2\left(B\right) > \alpha,$$

we obtain a subsequence (n_{k_m}) such that $P^{n_{k_m}}\nu_2\left(B\right) > \alpha$ for all m. Thus, there exists $n'_1 \in \mathbb{N}$ with $P^{n'_1}\nu_i\left(B\right) > \alpha$ $(i = 1, 2)$. Since P has the e-property, S_{μ^*} is P-invariant (c.f. [26, Corrollary 7.3.12]). Lemma 14 yields $\nu_i^1, \mu_i^1 \in \mathcal{P}\left(S\right)$, $\mu_i^1\left(S_{\mu^*}\right) = 1$ and $P^{n'_1}\nu_i = \alpha\nu_i^1 + \left(1-\alpha\right)\mu_i^1$. Replacing ν_i by μ_i^1 and proceeding inductively, we obtain a sequence $\left(n'_j\right)$ such that $P^{n'_{j+1}}\mu_i^j = \alpha\nu_i^{j+1} + \left(1-\alpha\right)\mu_i^{j+1}$ with $\nu_i^{j+1}, \mu_i^{j+1} \in \mathcal{P}\left(S\right)$, $\mu_i^{j+1}\left(S_{\mu^*}\right) = 1$, while $\operatorname{supp}\left(\nu_i^{j+1}\right) \subset \overline{B}$. Fix k such that $\left(1-\alpha\right)^k < \frac{\epsilon}{4}$. Write $m_i = \sum_{j=i}^k n'_j$. Then

$$P^{m_1+n}\nu_i = \alpha\sum_{j=1}^k \left(1-\alpha\right)^{j-1} P^{m_{j+1}+n}\nu_i^j + \left(1-\alpha\right)^k P^n\mu_i^k.$$

Consequently, using Lemma 15,

$$\begin{aligned}\| P^{m_1+n}\nu_1 - P^{m_1+n}\nu_2 \|_{BL}^* &\leq \alpha\sum_{j=1}^k \left(1-\alpha\right)^{j-1} \| P^{m_{j+1}+n}\nu_1^j - P^{m_{j+1}+n}\nu_2^j \|_{BL}^* \\ &\quad + \left(1-\alpha\right)^k \| P^n\mu_1^k - P^n\mu_2^k \|_{BL}^* \\ &\leq \alpha\sum_{j=1}^k \left(1-\alpha\right)^{j-1} \frac{\epsilon}{2} + \frac{\epsilon}{4}\cdot 2 < \epsilon,\end{aligned}$$

for all n. Lemma 17 yields $z \in \operatorname{supp}\left(\mu^*\right)$. □

4.2. **The e-property and Asymptotic Stability in Compact Spaces.** In order to apply Theorem 12 and obtain uniform stability of P, it should be verified that

$$\liminf_{n\to\infty} P^n\delta_x\left(B\left(z,r\right)\right) > 0. \tag{12}$$

The latter condition turns out to hold if there is a point such that each ball around this point is eventually reached with positive probability, uniformly from all points in the space. That is,

Lemma 18. *let P be a Markov-Feller operator on a compact metric space S. Suppose that there exists a $K \in S$ such that for each $r > 0$ there exists $N = N_r \in \mathbb{N}$ such that for all $x \in S$,*

$$\left(\operatorname{supp} P^N \delta_x\right) \cap B\left(K, \frac{1}{2}r\right) \neq \phi. \tag{13}$$

Then there exists $\beta > 0$ such that for all $m \geq N$ and $x \in S$, one has

$$P^m \delta_x \left(B\left(K, r\right)\right) \geq \beta > 0.$$

Consequently, for each $r > 0$, and for all $x \in S$, we have (12) with $z = K$.

Proof. Let $r > 0$. Property (13) implies that there exists $N = N_{\frac{1}{2}r}$ such that for each $x \in S$ there exists $\beta_x > 0$ such that $P^N \delta_x \left(B\left(K, \frac{1}{2}r\right)\right) \geq \beta_x$. Let $0 < \beta_x^{'} < \beta_x$. Fix an $x \in S$. Due to Lemma 16 there exists an open ball $B\left(x, R_x\right)$ such that $P^N \delta_y \left(B\left(K, r\right)\right) \geq \beta_x^{'}$ for all $y \in B\left(x, R_x\right)$. As $\left\{B\left(x, R_x\right) : x \in S\right\}$ covers S and S is compact, there exist $x_1, x_2, \ldots, x_k \in S$ such that the balls $B\left(x_i, R_{x_i}\right) =: U_i$ cover S. Take $\beta := \min\left\{\beta_{x_1}^{'}, \ldots, \beta_{x_k}^{'}\right\}$. Then $P^N \delta_y \left(B\left(K, r\right)\right) \geq \beta > 0$ for all $y \in S$. Finally, for $s \in S$ and $m \geq N$ we infer

$$\begin{aligned} P^m \delta_s \left(B\left(K, r\right)\right) &= \int_S P^N \delta_y \left(B\left(K, r\right)\right) d\left[P^{m-N} \delta_s\right](y) \\ &\geq \beta \cdot P^{m-N} \delta_s (S) = \beta \cdot 1 = \beta. \end{aligned}$$

□

Theorem 21 below asserts that the concepts of global and uniform asymptotic stability coincide on compact spaces for Markov operators having the e-property. We also obtained a remarkable technical condition that is equivalent to global and uniform stability in the compact case. The proof of this main result leans on Lemma 19 and the below auxiliary Proposition-20- of separate interest.

Lemma 19. *If P is a Markov-Feller operator with the* e*-property, and Q is a Markov-Feller operator such that there exists an open set $U \subset \mathcal{P}(S)$ such that $P^n\mu \to Q\mu$ for each $\mu \in U$, then $P^n\mu \to Q\mu$ uniformly on each compact subset of U.*

Proof. Let $K \subset \mathcal{P}(S)$ be compact and let $\epsilon > 0$. First, observe that Q is a Markov-Feller operator if and only if $Q : \mathcal{S}^+_{BL} \to \mathcal{S}^+_{BL}$ is continuous ([26], Proposition 3.3.2., p.52). Thus, for each $\mu \in K$ there exists V_μ open, contained in U, such that $\| Q\mu - Q\nu \|^*_{BL} < \epsilon$ for all $\nu \in V_\mu$. Since P has the e-property, for all $\mu \in K$, there exist U_μ open, $\mu \in U_\mu$ such that $\| P^n\nu - P^n\mu \|^*_{BL} < \epsilon$ for all $\nu \in U_\mu$ and for all $n \in \mathbb{N}$. Since K is compact, there exist $\mu_1, \mu_2, ..., \mu_k$ such that $K \subset \bigcup_{i=1}^k U_{\mu_i} \bigcap V_{\mu_i}$. For each j there exists N_j such that $\| P^n\mu_j - Q\mu_j \|^*_{BL} < \epsilon$ for all $n \geq N_j$. Let $N := \max(N_1, \dots, N_k)$. For $\nu \in K$, there exists j such that $\nu \in U_{\mu_j} \bigcap V_{\mu_j}$ and for all $n \geq N$

$$\| P^n\nu - Q\nu \|^*_{BL} \leq \| P^n\nu - P^n\mu_j \|^*_{BL} + \| P^n\mu_j - Q\mu_j \|^*_{BL} + \| Q\mu_j - Q\nu \|^*_{BL} < 3\epsilon.$$

□

Proposition 20. *Let S be compact and P a Markov operator with the* e*-property. Then $(P^n)_n$ is precompact in $(C_b(\mathcal{P}(S), \mathcal{S}_{BL}), d_\infty)$.*

Proof. Let $\mathcal{F} = \{P^n : n \in \mathbb{N}\} \subset C_b(\mathcal{P}(S), \mathcal{S}_{BL})$. Since P has the e-property, $\mathcal{F}$ is equicontinuous in $C_b(\mathcal{P}(S), \mathcal{S}_{BL})$ (c.f. Theorem 7.2.2 from [26]). Denote $\mathcal{F}(\mu) = \{P^n\mu : \mu \in \mathcal{P}(S), n \in \mathbb{N}\} \subset \mathcal{P}(S)$, $\mu \in \mathcal{P}(S)$. As $\| P^n\mu \|^*_{BL} = \| P^n\mu \|_{TV} = 1$, $\mathcal{F}(\mu)$ is bounded in $\mathcal{S}_{BL}$. Let κ stand for the Kuratowski measure of non-compactness. According to Ambrosetti ([1], or [5, Proposition 1.2.10, p17]),

$$\kappa(\mathcal{F}) = \sup_{\mu \in \mathcal{P}(S)} \kappa_{\mathcal{S}_{BL}}(\mathcal{F}(\mu)).$$

Since $\mathcal{F}(\mu) \subset \mathcal{P}(S)$ and $\mathcal{P}(S)$ is compact, $\kappa_{\mathcal{S}_{BL}}(\mathcal{F}(\mu)) \leq \kappa_{\mathcal{S}_{BL}}(\mathcal{P}(S)) = 0$. Hence, $\kappa(\mathcal{F}) = 0$. Thus, $\mathcal{F}$ is precompact in $(C_b(\mathcal{P}(S), \mathcal{S}_{BL}), d_\infty)$. □

Theorem 21. *Let (S,d) be compact and let P be a Markov operator on S with the* e*-property. If P has a unique ergodic measure μ^*, then the following are equivalent:*

(1) *μ^* is globally asymptotically stable.*
(2) *μ^* is uniformly stable, i.e. $\lim_{n\to\infty} P^n\mu = \mu^*$ uniformly for all $\mu \in \mathcal{P}(S)$.*
(3) *For all $\mu \in \mathcal{P}(S)$, $\| P^{n+1}\mu - P^n\mu \|_{BL}^* \to 0$ as $n \to \infty$.*

Proof. For the implication (3) implies (1), we have, since S is compact, that Proposition 20 yields that, $(P^n) \subset C_b(\mathcal{P}(S), \mathcal{S}_{BL})$ is precompact. Thus, there exists a subsequence (n_k) and an operator $Q \in C_b(\mathcal{P}(S), \mathcal{S}_{BL})$ such that $P^{n_k} \to Q$ with respect to d_∞. Hence, by Lemma 19, $P^{n_k}\mu \to Q\mu$ uniformly for all $\mu \in \mathcal{P}(S)$. Now, because P is Markov-Feller, hence continuous on $\mathcal{S}_{BL}^+$, condition (3) implies

$$\| (P - I) Q\mu \|_{BL}^* = \lim_{k\to\infty} \| (P - I) P^{n_k}\mu \|_{BL}^* = 0.$$

Hence, $Q\mu$ is an invariant measure for P. Since μ^* is the unique ergodic measure for P, $Q\mu = \mu^*$. So, $P^{n_k}\mu \to \mu^*$ uniformly. Since (P^n) is equicontinuous at μ^*, for all $\epsilon > 0$, there exists U_{μ^*} open with $\mu^* \in U_{\mu^*}$ such that $\| P^n\nu - \mu^* \|_{BL}^* = \| P^n\nu - P^n\mu^* \|_{BL}^* < \epsilon$ for all n and for all $\nu \in U_{\mu^*}$. Thus, by uniform convergence of $P^{n_k}\mu$ to μ^*, there exists $N \in \mathbb{N}$ such that

$$(14) \quad \| P^n P^{n_k}\mu - P^n\mu^* \|_{BL}^* < \epsilon \quad \text{for all } n, \quad \text{for all } k \geq N, \quad \text{and} \quad \text{for all } \mu \in U_{\mu^*}.$$

Also observe that (3) implies that, for all m, $\| P^n P^m\mu - P^n\mu \|_{BL}^* \to 0$ as $n \to \infty$. Thus, there exists $N' \in \mathbb{N}$ such that

$$\begin{aligned} \| P^n\mu - \mu^* \|_{BL}^* &\leq \| P^n\mu - P^n P^{n_N}\mu \|_{BL}^* + \| P^n P^{n_N}\mu - \mu^* \|_{BL}^* \\ &< 2\epsilon \quad \text{for all } n \geq N'. \end{aligned}$$

Hence, $P^n\mu \to \mu^*$ as $n \to \infty$ for all $\mu \in \mathcal{P}(S)$. (i.e. (1)). For the implication (1) implies (2) we have $Q\mu = \mu^*$, $P^n\mu \to Q\mu$

Thus, by Lemma 19, $P^n\mu \to \mu^*$ uniformly. For the implication (2) implies (3), we have (2) implies that $(P^n\mu)$ is a Cauchy sequence, so there exists N such that $\| P^nP^m\mu - P^n\mu \|_{BL}^* < \epsilon$ for all $n \geq N$ and $m \geq 0$. □

5. Existence and stability of a non-trivial ergodic Measure in the Fishery Model

In connection to the previous section, we proceed to give the specific details of the illustrative example –the model introduced in Section 2– we analyse the cases where no fishing occur or a full net is catched. Further, by first proving P defined by (3) is ultra Feller, we show uniform asymptotic stability for a nontrivial invariant measure μ^* for the operator P on a suitable interval $[v^*, K]$, for some $v^* > 0$. Also, we prove that P preserves the (positive) measures on $[v^*, K]$ (i.e. P^n is supported on $[v^*, K]$ for all n). The idea is that if we could find $v^* > 0$ such that $\phi_{\Delta t}(v^*) - m_c \geq v^*$, then by monotonicity of $\phi_{\Delta t}$, if $v \in [v^*, K]$, then $\phi_{\Delta t}(v) \in [v^* + m_c, K]$ (note that $v^* + m_c < K$). Then P will preserve $\mathcal{M}^+([v^*, K])$. Observe that Corollary 7.4.12 on page 161 in [26] (or Theorem 7.4.11 there) is sufficient to obtain the global asymptotic stability of μ^* (this corollary could be used after establishing that $\liminf > 0$).

5.1. **A Prelude: Two Extreme Deterministic Cases.**

At this point we would like to analyze what happens in the extreme deterministic case; at time steps where nothing is caught or more interestingly, if it is 'always' the maximum catch size m_c. If at each time step nothing is caught and the initial population size is x, then the population size at $t = n$ equals $a_n^+(x) := \phi_{n\Delta t}(x)$. If at each time the maximum size m_c is caught, then the population size at time n will be $a_n^-(x) := \psi^n(x)$, where $\psi(x) = (\phi_{\Delta t}(x) - m_c)^+$. Here we set the population size to be equal zero if the size just before fishing is smaller than m_c. The reason for our interest in these deterministic cases is that it will turn out that $\operatorname{supp}(P^n\delta_x) = [a_n^-(x), a_n^+(x)]$, see Proposition 25.

For the latter we have to analyze the iterates of the map $\psi : [0,\infty) \to [0,\infty)$ defined by $\psi(x) := (\phi_{\Delta t}(x) - m_c)^+$ since the iterates of this map determine the dynamics of the model.

In order to formulate our results conveniently, let us define

$$m_c^* := K \cdot \frac{\left(e^{\frac{1}{2}r\Delta t} - 1\right)}{\left(e^{\frac{1}{2}r\Delta t} + 1\right)},$$

the *critical catch size*, and for m_c satisfying $(K - m_c)^2 \geq \frac{4Km_c}{e^{r\Delta t}-1}$,

$$v_\pm^* := \frac{1}{2}(K - m_c) \pm \frac{1}{2}\sqrt{(K - m_c)^2 - \frac{4Km_c}{e^{r\Delta t} - 1}}.$$

Note that for $m_c \leq K$, one has

$$(K - m_c)^2 \geq \frac{4Km_c}{e^{r\Delta t} - 1} \quad \text{iff} \quad m_c \leq m_c^*.$$

Proposition 22. *(Fixed points of ψ) Consider the map $\psi : [0,\infty) \to [0,\infty)$, $x \mapsto (\phi_{\Delta t}(x) - m_c)^+$. We distinguish the following cases:*

1. *If $m_c > K$, then ψ has one fixed point, which is 0.*
2. *If $m_c^* < m_c \leq K$, then ψ has one fixed point, which is 0.*
3. *If $0 < m_c < m_c^* \leq K$, then ψ has three distinct fixed points in $[0,\infty)$ which are $0, v_-^*, v_+^*$ given by equation (15).*
4. *If $m_c = m_c^*$, then v_-^* and v_+^* coincide and ψ has two fixed points in $[0,\infty)$; 0 and $v_\pm^*$.*

Proof. The explicit calculation of $\phi_{\Delta t}(v^*)$ when taken equal to $v^* + m_c$ is as follows:

$$\phi_{\Delta t}(v^*) = v(\Delta t, v^*) = \left(\frac{1}{K} + \left(\frac{1}{v^*} - \frac{1}{K}\right)e^{-r\Delta t}\right)^{-1} = v^* + m_c.$$

This in turn gives

$$(v^*)^2\left(1-\mathrm{e}^{-r\Delta t}\right)+(m_c-K)\left(1-\mathrm{e}^{-r\Delta t}\right)v^*+\left(Km_c\mathrm{e}^{-r\Delta t}\right)=0.$$

The determinant of this equation is given by

$$D=(m_c-K)^2\left(1-\mathrm{e}^{-r\Delta t}\right)^2-4\left(1-\mathrm{e}^{-r\Delta t}\right)\left(Km_c\mathrm{e}^{-r\Delta t}\right).$$

By the assumption the roots are real. Then, the sum of the roots of the above quadratic equation is $(K-m_c)$ and the product is $\frac{Km_c}{\mathrm{e}^{r\Delta t}-1}$, the latter is always positive since $r,\Delta t>0$, hence, the roots have the same sign. So, as $K>m_c$ then the solutions $v^*_-,v^*_+>0$. Hence, we obtain explicitly

$$\begin{aligned} v^*_\pm &= \frac{-(m_c-K)\left(1-\mathrm{e}^{-r\Delta t}\right)\pm\sqrt{D}}{2\left(1-\mathrm{e}^{-r\Delta t}\right)} \\ &= \frac{1}{2}(K-m_c)\pm\frac{1}{2}\sqrt{(K-m_c)^2-\frac{4Km_c}{\mathrm{e}^{r\Delta t}-1}}. \end{aligned} \tag{15}$$

For the last case (where m_c^* is such that $m_c>0$ and $(K-m_c)^2=\frac{4Km_c}{\mathrm{e}^{r\Delta t}-1}$) we have solving the equation $(K-m_c^*)^2=\frac{4Km_c^*}{\mathrm{e}^{r\Delta t}-1}$ yields

$$(m_c^*)^2-2K\left(1+\frac{2}{\mathrm{e}^{r\Delta t}-1}\right)m_c^*+K^2=0.$$

The discriminant of the above equation is $D'=16K^2\left(1+\frac{1}{\mathrm{e}^{r\Delta t}-1}\right)\frac{1}{\mathrm{e}^{r\Delta t}-1}>0$. Thus, the solutions are given by:

$$\begin{aligned} m^*_{c,\pm} =&K\left(1+\frac{2}{\mathrm{e}^{r\Delta t}-1}\right)\pm\sqrt{16K^2\left(1+\frac{1}{\mathrm{e}^{r\Delta t}-1}\right)\frac{1}{\mathrm{e}^{r\Delta t}-1}} \\ =&K\left[\frac{\mathrm{e}^{r\Delta t}+1\pm 2\mathrm{e}^{\frac{1}{2}r\Delta t}}{\mathrm{e}^{r\Delta t}-1}\right]=K\cdot\frac{\left(\mathrm{e}^{\frac{1}{2}r\Delta t}\pm1\right)^2}{\left(\mathrm{e}^{\frac{1}{2}r\Delta t}-1\right)\left(\mathrm{e}^{\frac{1}{2}r\Delta t}+1\right)}. \end{aligned}$$

That is, there are two values for $m_c^*>0$ such that $(K-m_c)^2=\frac{4Km_c}{\mathrm{e}^{r\Delta t}-1}$ one of them is less than K and the other is greater than K; $m^*_{c,-}=K\frac{\mathrm{e}^{\frac{1}{2}r\Delta t}-1}{\mathrm{e}^{\frac{1}{2}r\Delta t}+1}<K$ and $m^*_{c,+}=K\frac{\mathrm{e}^{\frac{1}{2}r\Delta t}+1}{\mathrm{e}^{\frac{1}{2}r\Delta t}-1}>K$. Note that $m^*_{c,-}$ is the one we called the critical catch size. □

Remark 23. If $m_c = m_{c,+}^*$, then also $v_-^* = v_+^*$ but in this case $v_\pm^* < 0$ as can be seen from (15).

Proposition 24. *(Stability of fixed points of ψ) Suppose that $0 < m_c < m_c^* \leq K$. Then one has the following cases:*

(1) If $0 < x < v_-^*$, then $a_n^-(x) \downarrow 0$ as $n \to \infty$.
(2) If $v_-^* < x < v_+^*$, then $a_n^-(x) \uparrow v_+^*$ as $n \to \infty$.
(3) If $x \geq v_+^*$, then $a_n^-(x) \downarrow v_+^*$.

Proof. The results follow easily from the interpretation of the graph of ψ. □

5.2. **A Restriction of P is Ultra Feller.** Because of assumption A1) we have the following result on the support of $P^n\delta_x$:

Proposition 25. *For each $x \in [0, \infty)$, and $n \in \mathbb{N}$,* $\operatorname{supp}(P^n\delta_x) = [a_n^-(x), a_n^+(x)]$.

Proof. According to assumptions A1) and A3), we see that $P\delta_x$ has as support the interval

$$[\psi(x), \phi_{\Delta t}(x)] = \left[a_1^-(x), a_1^+(x)\right].$$

If $P^n\delta_x$ is supported on the interval $I_n = [\alpha, \beta]$, then

$$P^{n+1}\delta_x = P(P^n\delta_x) = P\left(\int_{[0,\infty)} \delta_y [P^n\delta_x](dy)\right) = \int_{I_n} P\delta_y [P^n\delta_x](dy),$$

which, by monotonicity of ψ and $\phi_{\Delta t}$, is supported on

$$\overline{\bigcup_{y\in I_n} [\psi(y), \phi_{\Delta t}(y)]} = [\psi(\alpha), \phi_{\Delta t}(\beta)].$$

Induction yields the result. □

Corollary 26. *If $0 < m_c < m_c^* \leq K$, then the sets $\left[v_-^*, K\right]$, $\left[v_+^*, K\right]$, $[v_-^*, \infty)$ and $[v_+^*, \infty)$ are P-invariant. In particular, P leaves $\mathcal{M}^+\left(\left[v_+^*, K\right]\right)$, $\mathcal{M}^+\left(\left[v_-^*, K\right]\right)$, $\mathcal{M}^+\left([v_+^*, \infty)\right)$ and $\mathcal{M}^+\left([v_-^*, \infty)\right)$ invariant.*

Proof. The first part follows from Propositions 24 and 25. For the second part, take a measure μ such that $\text{supp}\mu \subset [v_-^*, \infty)$. If $y < v_-^*$ and $r > 0$ such that $y + r < v_-^*$, then, with $E = (y - r, y + r)$, $P\mu(E) = \int_{[v_-^*,\infty)} P\delta_x(E)\, d\mu(x) = 0$, according to Propositions 24 and 25. Hence, $y \notin \text{supp}P\mu$ and we conclude $\text{supp}P\mu \subset [v_-^*, \infty)$. The other cases are shown by similar reasoning. □

Theorem 27. *Let $0 < m_c < m_c^* \leq K$. The Markov operator P defined by* (3) *with Q_x satisfying A1)-A3) is ultra-Feller on $[v_-^*, \infty)$.*

Proof. We need to show that the map $x \mapsto P\delta_x : [v_-^*, \infty) \to \mathcal{M}(S)_{TV}$ is continuous. Fix x_0 in $[v_-^*, \infty)$. Then $\| P\delta_x - P\delta_{x_0} \|_{TV} \to 0$ as $x \to x_0$ if and only if $\| p(x, \cdot) - p(x_0, \cdot) \|_{TV} \to 0$ or $\| H(x, .) - H(x_0, .) \|_{L^1([v_-^*,\infty),dy)} \to 0$ as $x \to x_0$, with H defined as in (6). Let us write $\hat{x} = \phi_{\Delta t}(x)$. Consider

$$f : (x, z) \mapsto q(\hat{x}, \hat{z} - \cdot) = T_{\hat{z}}[q(\hat{x}, \cdot)] : [v_-^*, \infty) \times [v_-^*, \infty) \to \mathrm{L}^1(\mathbb{R}),$$

where $T_{\hat{z}}$ denotes the translation-reflection map $T_{\hat{z}}g(y) := g(\hat{z} - y)$ on $\mathrm{L}^1(\mathbb{R})$. f is separately continuous. Since $T_{\hat{z}}$ is strongly continuous on $\mathrm{L}^1(\mathbb{R})$, the map $(z, f) \mapsto T_{\hat{z}}f$ is jointly continuous on $[v_-^*, \infty) \times K$, for any $K \subset \mathrm{L}^1(\mathbb{R})$ compact (e.g. [10], Lemma I.5.2). Now let $x_0 \in [v_-^*, \infty)$ and $(x_n)_n$ be a sequence in $[v_-^*, \infty)$ such that $x_n \to x_0$, then $(q(\hat{x_n}, \cdot))_{n=0}^{\infty}$ are contained in a compact subset of $\mathrm{L}^1(\mathbb{R})$, according to assumption A3). (Note that $x \in [v_-^*, \infty)$ implies that $\hat{x} > m_c$). So, $f(x_n, x_n) \to f(x_0, x_0)$, as desired. □

5.3. **Existence and Stability of Ergodic Measures.** We now consider the general case with random catch size.

Theorem 28. *If $0 < m_c < m_c^* \leq K$, and A1)-A3) hold, then the Markov operator P defined by (3) is uniquely ergodic on $[v_-^*, \infty)$ and strictly ergodic[2] on $\left[v_+^*, K\right]$. Thus, the ergodic measure μ^* has support $\left[v_+^*, K\right]$.*

[2]We call P *uniquely ergodic* if P has exactly one invariant probability measure, which is then an ergodic measure. If P is uniquely ergodic and the support of the unique invariant probability is the entire space S, then P is called *strictly ergodic.* (Zaharopol [27], p18).

Proof. We first show that P restricted to $\left[v_-^*, K\right]$ is uniquely ergodic. Since $S = \left[v_-^*, K\right]$ is compact, $\mathcal{P}(S)$ is compact. If $\mu_0 \in \mathcal{P}\left(\left[v_-^*, K\right]\right)$, then by the converse of Prokhorov Theorem, (see e.g. K. P. Parthasarathy [19]), the set of measures $\{P^n \mu_0 : n \in \mathbb{N}\} = \{\mu_n : n \in \mathbb{N}\}$ is tight. Since P is Markov-Feller (follows from being ultra-Feller), by the Krylov-Bogoliubov Theorem (c.f. Da Prato [4]), P has an invariant measure μ^* on $\left[v_-^*, K\right]$. Since P satisfies the e-property, it suffices to check the other condition of Theorem 9. For this purpose we propose $z = K$. Since $P^n \delta_x$ is supported on $[a_n^-(x), \phi_{n\Delta t}(x)]$, see Proposition 25, and $\phi_{n\Delta t}(x) \to K$ as $n \to \infty$, for $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\phi_{N\Delta t}(x) \in B(K, \epsilon)$, i.e. $P^N \delta_x (B(K, \epsilon)) > 0$. Therefore, μ^* is the unique invariant measure on $\left[v_-^*, K\right]$. For the extension to $\left[v_-^*, \infty\right)$, note that, for $x_0 \in (K, \infty)$, $P^n \delta_{x_0}$ is supported on $[a_n^-(x_0), \phi_{n\Delta t}(x_0)]$, $\phi_{n\Delta t}(x_0) \downarrow K$, and $a_n^-(x_0) \geq v_-^*$, so by the same arguments there exists a unique invariant measure on $\left[v_-^*, x_0\right]$, which then equals the unique invariant measure on $\left[v_-^*, K\right]$. In order to prove that $\operatorname{supp}(\mu^*) = \left[v_+^*, K\right]$, first note that for any $x \in \left[v_-^*, K\right]$, $a_n(x)$ converges to v_+^* (see Proposition 24) and $\phi_{n\Delta t}(x) \to K$. For any probability measure μ on $\left[v_-^*, K\right]$ and $0 \leq k \leq n$ one has

$$(P^n \mu)(A) = \int\limits_{\left[v_-^*, K\right]} \left(P^{n-k} \delta_x\right)(A) \, d\left[P^k \mu\right](x),$$

for any $A \subseteq \left[v_-^*, K\right]$ Borel. In particular, $P^n \mu^* = \int_{\left[v_-^*, K\right]} P^n \delta_x \, d\mu^*(x)$. Hence,

$$\operatorname{supp}(P^n \mu^*) \subseteq \overline{\bigcup_{x \in \left[v_-^*, K\right]} \operatorname{supp}(P^n \delta_x)}.$$

Since

$$\operatorname{supp}(P^n \delta_x) \subseteq \left[a_n^-(x), \phi_{n\Delta t}(x)\right] \subseteq \left[a_n^-\left(v_-^*\right), \phi_{n\Delta t}(K)\right]$$

for all $x \in \left[v_-^*, K\right]$, $\operatorname{supp}(P^n \mu^*) \subseteq \left[a_n^-\left(v_-^*\right), \phi_{n\Delta t}(K)\right]$. Since $P^n \mu^* = \mu^*$ for all n, we obtain

$$\operatorname{supp}(\mu^*) = \operatorname{supp}(P^n \mu^*) \subseteq \bigcap_{n=1}^{\infty} \left[a_n^-\left(v_-^*\right), \phi_{n\Delta t}(K)\right] = \left[v_+^*, K\right].$$

For the other inclusion, let $x \in \left(v_+^*, K\right)$ and $U \subseteq \left[v_-^*, K\right]$ open with $x \in U$. Take $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq U$ and its closure $[x-\delta, x+\delta] \subseteq \left(v_+^*, K\right)$. Take n so large that $a_n^-(K) < x - \delta$ and $\phi_{n\Delta t}\left(v_+^*\right) > x + \delta$. Then for every $y \in \left[v_+^*, K\right]$,

$$\operatorname{supp}\left(P^n \delta_y\right) = \left[a_n^-(y), \phi_{n\Delta t}(y)\right] \supseteq \left[a_n^-(K), \phi_{n\Delta t}\left(v_+^*\right)\right] \supseteq [x-\delta, x+\delta],$$

so $P^n \delta_y((x-\delta, x+\delta)) > 0$. Hence, using invariance of μ^*,

$$\begin{aligned}\mu^*((x-\delta, x+\delta)) &= \int_{\left[v_-^*, K\right]} P^n \delta_y((x-\delta, x+\delta))\, d\mu^*(y) \\ &\geq \int_{\left[v_+^*, K\right]} P^n \delta_y((x-\delta, x+\delta))\, d\mu^*(y) > 0,\end{aligned}$$

so that $\mu^*(U) > 0$. Therefore, $x \in \operatorname{supp}(\mu^*)$. It follows that $\left[v_+^*, K\right] \subseteq \operatorname{supp}(\mu^*)$. Hence, $\operatorname{supp}(\mu^*) = \left[v_+^*, K\right]$. □

In order to relate our model to the above theorems and lemmas, we have the following lemma:

Lemma 29. *The Markov operator P defined by (3) on $S = [v_0, R]$, where $v_0 \in \left[v_-^*, K\right]$ and $R \geq K$, satisfies the conditions of Lemma 18. Thus, there exists m and there exists β such that for all $x \in [v_0, R]$, $P^m \delta_x(B(z, r)) \geq \beta > 0$. Thus, there is $z \in \operatorname{supp}(\mu^*)$ such that for all $x \in [v_0, R]$ we have $\liminf_{n\to\infty} P^n \delta_x(B(z, r)) \geq \beta > 0$. (Choose $z = K$.)*

Proof. For any $x \in \left[v_-^*, R\right]$, we have $\operatorname{supp}(P^n \delta_x) = \left[a_n^-(x), \phi_{n\Delta t}(x)\right]$, by Proposition 25. Fix $R \geq K$. Since $\phi_{n\Delta t}(x) \to K$ uniformly for $x \in \left[v_-^*, R\right]$, there exists $N = N_r$ such that

$$B\left(K, \frac{1}{2} r\right) \cap \operatorname{supp}(P^n \delta_x) \neq \phi,$$

for all $n \geq N$ and for all $x \in \left[v_-^*, R\right]$. □

The following theorem is the major conclusion for this section.

Theorem 30. *If* $0 < m_c < m_c^* \leq K$*, and A1)-A3) hold, then the unique ergodic measure* μ^* *of the restriction of the Markov operator* P *to* $[v_-^*, \infty)$ *is globally asymptotically stable on* $\left(v_-^*, \infty\right)$*. Moreover,* μ^* *is uniformly stable for the restriction of* P *to* $[r, R]$*, for any* $v_-^* < r \leq v_+^*$ *and any* $R \geq K$*.*

Proof. For $v_-^* < r \leq v_+^*$ and any $R \geq K$, $[r, R]$ is compact and invariant under P. By Lemma 29 and Theorem 1, μ^* is uniformly stable on $[r, R]$, hence globally stable on the same interval. Since r, R are arbitrary, μ^* is globally stable on $\left(v_-^*, \infty\right)$. For if we take $S = [v_-^*, \infty)$, and if we prove that $\left(v_-^*, \infty\right) \subset S_0$ then the result follows from Theorem 12. Pick $x > v_-^*$, then $x \in (v_-^*, R]$ for any $R \geq K$. Thus, $P^n \delta_x \to \mu^*$ as $n \to \infty$ in $\left(\mathcal{M}^+ \left[v_-^*, R\right]\right)$. Since $\left(\mathcal{M}^+ \left[v_-^*, R\right]\right)$ is embedded in $\left(\mathcal{M}^+[v_-^*, \infty)\right)$, and the convergence implies the convergence of the Cesaro averages, $P^{(n)} \delta_x \to \mu^*$ as $n \to \infty$ in $\left(\mathcal{M}^+[v_-^*, \infty)\right)$. □

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