Kolmogorov forward equation and explosiveness in countable state Markov processes

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Abstract

In countable state non-explosive minimal Markov processes the Kolmogorov forward equations hold under sufficiently weak conditions. However, a precise description of the functions that one may integrate with respect to these equations seems to be absent in the literature. This problem arises for instance when studying the Poisson equation, as well as the average cost optimality equation in a Markov decision process.

We will show that the class of non-negative functions for which an associated transformed Markov process is non-explosive do have this desirable property. This characterisation easily allows to construct counter-examples of functions for which the functional form of the Kolmogorov forward equations does not hold.

Another approach of the problem is to study the transition operator as a transition semi-group on Banach space. The domain of the generator is a collection of functions that can be integrated with respect to the Kolmogorov forward equations. We focus on Banach spaces equipped with a weighted supremum norm, and we identify subsets of the domain of the generator.

Keywords Minimal process, explosiveness, regularity, transition semigroup, weighted supremum norm.

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1 Introduction

Let us consider a Markov process $X = \{X_t\}_{t\geq 0}$ on the countable state space $\boldsymbol{S}$, with transition function $\{ P_t = (p_{t,xy})_{x,y\in S}\}_{t\geq 0}$. Suppose for the moment that X is the minimal Markov process. Additionally suppose X to be standard and stable, with (everywhere) right-continuous sample paths. Then each function $t \mapsto p_{t,xy}$ has a derivative q_{xy} at $t = 0$. Under the assumptions above, the matrix of derivatives, Q say, has the properties that for all $x \in \boldsymbol{S}$

1. $0 \leq q_x = -q_{x,x} < \infty$;
2. $\sum_y q_{xy} \leq 0$.

In accordance with [1], we will call the matrix of derivatives Q a q-matrix, whenever it has the two properties above. The transition function (cf. [1] Theorem 2.2.2) $\{P_t\}_t$ satisfies the Kolmogorov forward equation

$$p_{t,xy} = \delta_{xy} + \int_0^t (P_s Q)_{xy} ds, \quad t \geq 0, \quad x, y \in \boldsymbol{S}, \tag{1.1}$$

with δ_{xy} the Kronecker delta, as well as the backward equation, and it is the minimal non-negative solution to both equations.

Our question of interest is the following: for which functions $f : \boldsymbol{S} \to \mathbf{R}$ does the Kolmogorov equation hold:

$$\mathsf{E}_x f(X_t) = f(x) + \int_0^t P_s(Qf)(x)ds. \tag{1.2}$$

Interpret $P_s h(x) = \sum_y p_{s,xy} h(y)$ whenever this expectation is well-defined. Our interest in equation (1.2) lies mainly in the fact that we may rewrite it as follows

$$\mathsf{E}_x f(X_t) = f(x) + \mathsf{E}_x \int_0^t Qf(X_s)ds, \tag{1.3}$$

provided e.g. $\mathsf{E}_x \int_0^t |Qf(X_s)|ds < \infty$. This then implies that the process $M_t(f) = f(X_t) - \int_0^t Qf(X_s)ds - f(x)$ is a martingale. The optional stopping theorem provides conditions allowing to replace the deterministic time t by a stopping time. The resulting relation is the well-known Dynkin formula.

Regarding our question, clearly equation (1.1) is equal to equation (1.2) for the function δ_y, the indicator function of the set $\{y\}$. Hence (1.2) holds for functions with finite support.

Other cases are not quite as clear. The only reference we have found so far, attacking this problem, is Appendix C.3 of [8]. There it it is stated (without proof) that (1.3) holds for all functions f, such that Qf is defined, $\int_0^t P_s|Qf|ds < \infty$ for all $t > 0$, and $P_t|f| < \infty$ for all $t \geq 0$. The statement unfortunately is not true without further assumptions, as we will show in this paper.

Strongly continuous semigroup An alternative approach is to consider P_t, $t \geq 0$, as a semigroup of bounded linear operators on some suitable Banach space $\mathcal{B}(\boldsymbol{S})$ of real-valued functions on $\boldsymbol{S}$. Following the set-up in [1] section 1.4, [7], the transition function P_t, $t \geq 0$, associated with a Markov process is a semigroup of operators, by virtue of the Chapman-Kolmogorov equations. It is a strongly continuous semi-group of bounded linear operators on the Banach space $\mathcal{B}(\boldsymbol{S})$, equipped with norm $\|\cdot\|$, if $\| P_t\| = \sup_f \| P_t f\|/\|f\| < \infty$, for each $t \geq 0$, and $\lim_{t\downarrow 0} \| P_t f - f\| = 0$, for each $f \in \mathcal{B}(\boldsymbol{S})$. In [7] Proposition 1.1 it is shown that this implies the existence of constants $M \geq 1, \alpha \geq 0$, such that $\| P_t\| \leq M e^{\alpha t}$.

Suppose that P_t, $t \geq 0$, is a strongly continuous semigroup on $\mathcal{B}(\boldsymbol{S})$. A linear operator $\mathcal{Q}$ is called the generator of the semigroup, if there exist a subspace $\mathcal{D}(\mathcal{Q}) \subset \mathcal{B}(\boldsymbol{S})$, such that to each $f \in \mathcal{D}(\mathcal{Q})$ there exists an element $g = \mathcal{Q}f \in \mathcal{B}(\boldsymbol{S})$, with

$$\lim_{t\downarrow 0} \| \frac{P_t f - f}{t} - g\| = 0.$$

$\mathcal{D}(\mathcal{Q})$ is called the domain of the generator. The following main properties are well-known.

Theorem 1.1 *Let $t \geq 0$.*

i) *If $f \in \mathcal{B}(\boldsymbol{S})$ then $\int_0^t P_s f ds \in \mathcal{D}(\mathcal{Q})$ and $P_t f = f + \mathcal{Q}\int_0^t P_s f ds$.*

ii) *If $f \in \mathcal{D}(\mathcal{Q})$ then $P_t f \in \mathcal{D}(\mathcal{Q})$ and*

$$\frac{d}{dt} P_t f = \mathcal{Q}\, P_t f = P_t \mathcal{Q} f.$$

iii) *If $f \in \mathcal{D}(\mathcal{Q})$ then the Kolmogorov backward and forward equation hold:*

$$P_t f = f + \int_0^t \mathcal{Q}(P_s f)ds = f + \int_0^t P_s(\mathcal{Q}f)ds.$$

iv) $\overline{\mathcal{D}(\mathcal{Q})} = \mathcal{B}(\mathcal{S})$.

The domain $\mathcal{D}(\mathcal{Q})$ is non-empty. However, it is difficult to assess whether a given function belongs to it.

The particular Banach spaces that we are interested in are function spaces equipped with the weighted supremum norm. Let $V : \boldsymbol{S} \to \mathbf{R}_+$ be a given function. It generates a weighted supremum norm $\|\cdot\|_V$, given by

$$\|f\|_V = \sup_{x\in S} \frac{|f(x)|}{V(x)}.$$

The space

$$\ell^\infty(E, V) = (\{f : E \to \mathbf{R} \,|\, \|f\|_V < \infty\}, \|\cdot\|_V)$$

is a Banach space. Such spaces play an increasing role in the stability and control of Markov chains and processes (cf. [4, 5, 6, 9, 8]). However, it can be deduced from [1] Lemma 1.4.7 and Proposition 1.4.8 that strongly continuous semigroups on $\ell^\infty(V,E)$ necessarily have bounded q-matrices. A more delicate approach is therefore required, see Section 6.

The set-up of the paper is as follows. Since explosiveness properties play a role in the analysis, we will provide different characterisations of explosiveness in Section 2 from the literature. The section also provides sufficient conditions for explosiveness, and for the lack of it, all gathered from the literature. Then we will discuss alternative conditions for the Kolmogorov forward equation to hold in Section 3.

Explosiveness is equivalent to the existence of bounded, non-negative eigenvectors to positive eigenvalues of the q-matrix (cf. [1], see Section 2). Section 4 will study how to transform these eigenvectors, when putting a taboo on one state and vice versa. In Section 5 we will provide counterexamples to the conditions of [8]. Finally we will study the transition function as an operator on Banach space in Section 6.

In a subsequent paper [14], we will use the results obtained here to connect uniform drift conditions and uniform exponential ergodicity conditions for a Markov decision process, cf. [8].

2 Explosiveness or the lack of it

For simplicity we will use the following condition.

Assumption 2.1 *X is a minimal, standard and stable process with q-matrix Q.*

Recursively define $\tau_0 = 0$ and

$$\tau_{n+1} = \inf\{t > \tau_n \mid X_t \neq X_{t-}\},$$

if $X(\tau_n)$ is not an absorbing state (i.e. $q_{X(\tau_n)} \neq 0$). We put $\tau_k = \infty$, $k > n$, if $X(\tau_n)$ is an absorbing state and then $X(\tau_k) = X(\tau_n)$. The sequence $\{\tau_n\}_n$ is a non-decreasing sequence of stopping times, representing the successive jump times. Put further $J_\infty = \lim_{n\to\infty} \tau_n$.

Definition 2.1 X is said to be *explosive*, if there exists a state $x \in \boldsymbol{S}$, such that $\mathsf{P}_x\{J_\infty < \infty\} > 0$.

We will recall a number of criteria for explosiveness from the literature.

Substochasticity criterion ([1], Theorem 2.2.2, Propositions 2.2.3, 2.2.4). *Under the assumption that Q be conservative and Assumption 2.1, X is explosive if and only if $\sum_y p_{t,xy} < 1$ for at least one state $x \in \boldsymbol{S}$ and one value of t (hence for all time points).*

Eigenvectors ([1] Lemma 2.2.6, Theorem 2.2.7, Propositions 4.1.7, 4.1.12, 4.1.13). For describing interesting properties associated with the existence of eigenvectors, we need the resolvent matrix: for any $\lambda > 0$

$$R(\lambda) = \int_0^\infty e^{-\lambda t}\, P_t dt. \tag{2.1}$$

The resolvent matrix $\lambda R(\lambda)$ is a stochastic matrix if and only if P_t is stochastic for some $t > 0$ (and hence for all $t \geq 0$), cf. [1], Proposition 2.1.1. Consider the set of inequalities

$$\lambda f \leq Qf \quad 0 \leq f(x) \leq 1, \quad x \in \boldsymbol{S}.$$

The maximum solution is the function

$$r_\lambda(x) = \lambda \int_0^\infty e^{-\lambda t} \mathsf{P}_x\{J_\infty \leq t\} dt.$$

The vector r_λ solves this system with equality and is hence a bounded, non-negative eigenvector to eigenvalue λ.

Eigenvector criterion *Under Assumption 2.1, X is explosive if and only if there exists $\lambda > 0$, such that Q has a* bounded *non-negative, non-zero eigenvector f_λ to eigenvalue λ.*

An eigenvector to eigenvalue λ of Q is the same object as a λ-invariant vector of Q. We will use the latter terminology henceforth. Given any λ-subinvariant non-negative vector f_λ of Q, $\lambda > 0$, one can obtain a μ-invariant non-negative vector of Q, $\mu > 0$, $\mu \neq \lambda$, by the transformation

$$f_\mu = (\mathbf{I} + (\lambda - \mu)R(\mu))f_\lambda.$$

One has $f_\mu \le f_\lambda$, if $\mu \ge \lambda$, with $f_\mu \downarrow 0$ componentwise, as $\mu \to \infty$.

If $f_0 = \lim_{\mu\downarrow 0} f_\mu$ is finite (it always exists by monotonicity!), then we can write $f_\mu = (\mathbf{I} - \mu R(\mu)) f_0$.

Sojourn sets To describe this criterion we need two further concepts. The first is the *jump chain* X^J: it is a discrete time Markov chain on $\boldsymbol{S}$ with transition probabilities $p_{J,xy} = q_{xy}/q_x$, for $y \ne x$. The transition matrix is denoted by P_J and we allow it to be substochastic. This clearly can be easily remedied by adding a coffin state, to which transitions take place with probability $1 - \sum_y p_{J,xy}$, when the present state equals x.

The second concept is the *taboo matrix*. Let P be the transition matrix of a discrete time Markov chain $X^d = \{X^d_t\}_{t=0,1,\dots}$ on the countable state space $\boldsymbol{S}$, and let $A \subset, \ne \boldsymbol{S}$ a given subset. With P we can associate the taboo matrix $_AP$ with taboo set A, which has elements $_Ap_{xy} = p_{xy}$ if $y \notin A$, and 0 otherwise. This means that all transitions leading into the set A are ignored. By $_AP^{(t)}$ we denote the t-th iterate, whereas $_AP^{(0)}$ is the identity matrix.

A set $A \subset \boldsymbol{S}$ is called a sojourn set (cf.[3] §I.17, [2]) if there exist $x \in A$ and $T \ge 0$, such that

$$\mathsf{P}_x\{X^d_t \in A, \text{for all } t \ge T \mid X_T = x\} > 0.$$

Sojourn set criterion ([3] Theorem II.19.3) *Suppose that Assumption 2.1 holds and that Q is conservative. X is explosive if and only if there exists a sojourn set A for the jump chain X^J and a state x such that*

$$\sum_{t=0}^{\infty}\sum_{y\in A} p^{(t)}_{J,xy}\frac{1}{q_y} < \infty,$$

and $\lim_{t\to\infty}\sum_{y}\,{}_{S\setminus A}p^{(t)}_{J,xy} > 0$.

These conditions are generally hard to check. The following sufficient conditions for either explosiveness or the lack of it, are easier to handle, although clearly not exhaustive.

Sufficient conditions for (non)-explosiveness First we need the concept of a moment function.

The function $f : \boldsymbol{S} \to \mathbf{R}_+$ is called a moment function, if there exists an increasing sequence of finite sets $\{K_n\}_n$, $K_n \uparrow \infty$, with the property that

$$\liminf_{n\to\infty}\inf_{x\notin K_n} f(x) = \infty.$$

Lemma 2.2 *Suppose that Assumption 2.1 holds and that Q is conservative. For X to be non-explosive, it is sufficient that*

i) $\sup_x q_x < \infty$*; or*

ii) *([1] Corollary 2.2.16, [10]) there exist a moment function $V : \boldsymbol{S} \to \mathbf{R}_+$ and a constant c, such that $QV \le cV$; or*

iii) *(analogously to [11] Theorem 2.7.1) X is irreducible and the jump chain X^J is recurrent.*

Lemma 2.3 *(cf. [11] Theorem 3.5.3) Suppose that Assumption 2.1 holds and that Q is conservative. For X to be explosive it is sufficient that X^J is irreducible transient, but there exists a finite 0-invariant measure μ for Q, i.e. $\mu Q = 0$.*

3 Main results

This section connects the validity of the Kolmogorov forward equation (1.2) to non-explosiveness of a certain associated Markov process. First we introduce some notation. Let $f : \boldsymbol{S} \to \mathbf{R}_+$ be a (strictly) positive function with

$$Qf \le cf \tag{3.1}$$

for some constant c. This condition is a weak form of well-known drift conditions for ergodicity and exponential ergodicity [10]. By [1] Proposition 2.2.3 it implies that $P_t f \leq e^{ct} f$, provided X is a minimal process. As a consequence $t \mapsto P_t f$ is continuous on $[0,\infty)$. The fact that $0 \leq (Qf)^+ \leq cf$ implies

$$\int_0^t P_s(Qf)^+ ds < \infty. \tag{3.2}$$

Inequalities applied to vectors are meant to hold componentwise. We will first derive the following simple, useful lemma.

Lemma 3.1 *Suppose that X satisfies Assumption 2.1. Let $f : \boldsymbol{S} \to \mathbf{R}_{>0}$ satisfy (3.1) as well as the Kolmogorov forward equation (1.2). Then for any constant $d \in \mathbf{R}$*

$$e^{dt} P_t f(x) = f(x) + \int_0^t e^{du}[P_u(Qf)(x) + d\, P_u f(x)]du. \tag{3.3}$$

Proof. By (3.2) also $\int_0^t P_s(Qf)^- ds < \infty$. Hence $\int_0^t P_s|Qf|(x)ds < \infty$. This justifies the use Fubini's theorem for the interchange of integrals in the second equality below: for any constant d,

$$\begin{aligned}
\int_0^t e^{ds} P_s(Qf)(x)ds &= \int_0^t [\int_0^s de^{du}du + 1]\, P_s(Qf)(x)ds \\
&= \int_0^t de^{du} \int_u^t P_s(Qf)(x)dsdu + P_t f(x) - f(x) \\
&= (e^{dt} - 1)\, P_t f(x) - \int_0^t de^{du}\, P_u f(x)du + P_t f(x) - f(x) \\
&= e^{dt}\, P_t f(x) - f(x) - \int_0^t de^{du}\, P_u f(x)du.
\end{aligned}$$

In the third equality we have used (1.2). Rewriting yields (3.3). QED

[10] uses the concept of the 'extended generator', and then by definition Eqn. (3.3) applies all functions belonging to the domain of the extended generator. The problem there is to check whether a function belongs to the domain of the extended generator.

With X we associate the minimal Markov process X^f with q-matrix Q^f. First extend $\boldsymbol{S}$ with a coffin state $\delta \notin \boldsymbol{S}$, i.e. $\boldsymbol{S}_\delta := \boldsymbol{S} \cup \{\delta\}$. Then define

$$q^f_{xy} = \begin{cases} q_{xy} f(y)/f(x), & x \neq y, x, y \neq \delta \\ q_{xx} - c, & x = y, x, y \neq \delta \\ c - \sum_{y \in S} q_{xy} f(y)/f(x), & x \neq \delta, y = \delta \\ 0, & x = \delta, y \in \boldsymbol{S}_\delta, \end{cases}$$

with δ_{xy} the Kronecker delta. This makes Q^f a conservative q-matrix. Denote by $\{P^f_t\}_t$ again the (minimum) transition function on the enlarged state space $\boldsymbol{S} \cup \{\delta\}$. It holds (cf. [1] Lemma 5.4.2) that

$$p^f_{t,xy} = e^{-ct}\, p_{t,xy} f(y)/f(x), \quad x, y \neq \delta. \tag{3.4}$$

It follows that X^f is standard, whenever X is; it is stable, whenever X is. It is immediate that $\{P^f_t\}_t$ satisfy the Kolmogorov forward and backward equations, if X is standard and stable. Note that X may not have a conservative q-matrix, whereas X^f does.

The process X^f is not uniquely defined, since with each constant c in (3.1), a larger constant suffices as well.

Theorem 3.2 *Let X satisfy Assumption 2.1. Let $f : \boldsymbol{S} \to \mathbf{R}_{>0}$ satisfy (3.1),*

Then f satisfies (1.2) if and only if the minimal Markov process X^f is non-explosive. If this is the case, $\mathsf{E}_x \int_0^t |Qf(X_s)|ds < \infty$ and f satisfies (1.3) as well.

Proof. Since f satisfies (3.1), Qf and $Q^f ind\boldsymbol{S}$ are well-defined. In particular

$$e^{-cu}\frac{[\,P_u(Qf)(x) - c\,P_u f(x)]}{f(x)} \;=\; P^f_u(Q^f \mathbf{1}_{\{S\}})(x), \quad x \in \boldsymbol{S}. \tag{3.5}$$

Suppose that f satisfies (1.2). By the previous Lemma 3.1, f satisfies (3.3). So plugging in $d = -c$ into (3.3) and using (3.5), we get

$$P^f_t \mathbf{1}_{\{S\}}(x) = \frac{1}{f(x)} e^{-ct}\, P_t f(x) = 1 + \int_0^t P^f_u(Q^f \mathbf{1}_{\{S\}})(x) du. \tag{3.6}$$

Since the Kolmogorov forward equations hold for X^f and indicator functions of states, we have for $x \neq \delta$ that $u \mapsto \sum p^f_{u,xy} q^f_{y\delta} = d(\,p^f_{u,x\delta})/du$ is a finite, continuous function on $[0,\infty)$ ([1], Lemma 1.2.4). Hence

$$p^f_{t,x\delta} = \int_0^t \sum_{y\neq\delta} p^f_{u,xy} q^f_{y\delta} du, \quad x \neq \delta.$$

We can then add both equations to yield

$$\sum_y p^f_{t,xy} = 1 + \int_0^t P_s(Q^f \mathbf{1}_{\{S\cup\{\delta\}\}})(x) ds = 1,$$

as $Q^f \mathbf{1}_{\{S\cup\{\delta\}\}} = 0$. Hence X^f is non-explosive.

For the reverse statement, we assume that X^f is non-explosive and so the Kolmogorov forward equation holds for the function $\mathbf{1}_{\{S\cup\{\delta\}\}}$ identically equal to 1 and the indicator $\mathbf{1}_{\{\delta\}}$ of state δ. It therefore holds for the indicator $\mathbf{1}_{\{S\}}$ of $\boldsymbol{S}$. Lemma 3.1 is applicable, with constant $d = c$, and so

$$e^{ct} \sum_{y\in S} p^f_{t,xy} = 1 + \int_0^t e^{cu}\big(\,P^f_u(Q^f \mathbf{1}_{\{S\}})(x) + c\,P^f_u \mathbf{1}_{\{S\}}(x)\big) du, \quad x \in \boldsymbol{S}.$$

By virtue of (3.5) and using the first equation in (3.6) we get (1.2).

Combination of (3.2) and (1.2) yields $\int_0^t P_u(Qf)^-(x)du < \infty$. Hence $\int_0^t P_u|Qf|du < \infty$. (1.3) follows. QED

Since I have found that the validity of the Kolmogorov forward equation for Markov processes with bounded jumps and functions satisfying an elementary integrability condition is not clear to everyone, I have included an interchange argument for completeness. It not unimportant neither for our analysis that the validity of the bounded jump case be rigorously established.

Lemma 3.3 *Let X satisfy Assumption 2.1. Suppose that Q has bounded jumps $\sup_x q_x < \infty$. Let $f : \boldsymbol{S} \to \mathbf{R}$ satisfy the integrability condition $P_t|f| < \infty$ for all $t \geq 0$. Then (1.2) holds, and so does (1.3).*

Proof. We use the representation

$$P_t = \sum_n e^{-\tau t} \frac{(\tau t)^n}{n!} P^{(n)},$$

where $P = (\mathbf{I} + \tau^{-1}Q)$ and $\tau \geq \sup_x q_x$, given in [1] Proposition 2.2.10.

It is sufficient to show the result for $f \geq 0$. This follows from the fact that the condition $P_t|f| < \infty$ implies $P_t f^+ < \infty$ and $P_t f^- < \infty$.

We will first show that $t \mapsto P_t f(x)$ is continuous for all $x \in \boldsymbol{S}$. Let $h > 0$. Then

$$|\,P_t f(x) - P_{t+h} f(x)| \leq e^{-\lambda t} \sum_n \tau^n \frac{(t+h)^n - t^n}{n!} P^{(n)} f(x) + e^{-\lambda t}(1 - e^{-\lambda h}) \sum_n \frac{(t+h)^n}{n!} P^{(n)} f(x).$$

The second term converges to 0 as $h \downarrow 0$. For the first term, notice for $h<1$ that

$$\begin{aligned}\sum_{n\geq 0}\tau^n\frac{(t+h)^n-t^n}{n!}\,P^{(n)}f(x) &\leq \sum_{n\geq 1}\tau^n\frac{\sum_{k=0}^{n-1}\binom{n}{k}t^kh^{n-k}}{n!}\,P^{(n)}f(x)\\ &\leq h\sum_{n\geq 1}\frac{(\tau(t+1))^n}{n!}\,P^{(n)}f(x)\\ &\leq he^{\lambda(t+1)}\,P_{t+1}f(x).\end{aligned}$$

It follows that $t\mapsto P_tf(x)$ is right-continuous. Left-continuity is proved in a similar manner. Hence $t\mapsto P_tf(x)$ is integrable for each $x\in\boldsymbol{S}$.

The rest is a simple interchange argument. Since the Kolmogorov backward equations apply to indicator functions, we may write

$$\begin{aligned}P_tf(x) &= f(x)+\sum_y\int_0^t(Q\,P)_{xy}ds f(y)\\ &= f(x)-\sum_y\int_0^t q_x\,p_{s,xy}dsf(y)+\sum_y\int_0^t\sum_z q_{xz}\,p_{s,zy}dsf(y).\end{aligned}$$

The second equality holds, because $s\mapsto P_sf(x)$ is integrable, and so the first sum is finite. By Fubini's theorem and nonnegativity of all terms involved we may interchange integral and summation signs, and so we get

$$P_tf(x)=f(x)+\int_0^t\sum_y p'_{s,xy}f(y)ds=f(x)+\int_0^t(\,P_sQ)f(x)ds.$$

Since Q is bounded, f is non-negative and $s\mapsto\sum_y p_{s,xy}f(y)$ is integrable, it follows that $\int_0^t\sum_y p_{s,xy}q_yf(y)ds$ is finite, and non-negative. We may substract this term from the above integral to obtain that

$$\int_0^t\sum_y(\sum_{z\neq y}p_{s,xz}q_{zy})f(y)ds$$

is finite. By non-negativity of all terms involved we may now swap summations and get that this integral

$$\int_0^t\sum_z p_{s,xz}\sum_{y\neq z}q_{zy}f(y)ds.$$

The final result follows by combination. QED

We can now prove our main result on the validity of the Kolmogorov forward equation (1.2).

Theorem 3.4 *Let X satisfy Assumption 2.1. Let $V:\boldsymbol{S}\to\mathbf{R}_{>0}$ be a function such that (3.1) holds for some constant c, i.e. $QV\leq cV$. Suppose that the minimal process X^V is non-explosive.*

Let $f\in\ell^\infty(\boldsymbol{S},V)$ with $\mathsf{E}_x\int_0^t|Qf(X_s)|ds<\infty$. Then f satisfies (1.2) and (1.3). In particular V satisfies (1.2) and (1.3).

Proof. By Theorem 3.2 V satisfies the above condition as well as (1.2). Notice that we may assume $c\geq 0$ without loss of generality, by simply enlarging the right-hand side of (3.1).

Let $f\in\ell^\infty(\boldsymbol{S},V)$ satisfy the above condition. We will use an approximation argument. Let $\boldsymbol{S}_n\uparrow\boldsymbol{S}$, be an increasing sequence of finite sets, converging to the whole space. Define approximating Q-matrices $Q^{(n)}$ by

$$q^{(n)}_{xy}=\begin{cases}q_{xy}, & x\in\boldsymbol{S}_n\\ 0, & \text{otherwise.}\end{cases}$$

The states outside $\boldsymbol{S}_n$ are absorbing. Index the associated minimal Markov process and transition kernel by n. X^n are conservative, standard processes with bounded jumps. Since $Q^{(n)}V\leq cV$, one has that

$P_t^{(n)}|f| \le e^{ct}\|f\|_V$. Hence the Kolmogorov forward equation (1.3) holds for the function f, i.e.

$$\mathsf{E}_x f(X_t^n) = f(x) + \mathsf{E}_x \int_0^t Q^{(n)} f(X_s^n) ds, \tag{3.7}$$

for all $x \in \boldsymbol{S}$. We will first show convergence of the right-hand side to $\mathsf{E}_x \int_0^t Qf(X_s)ds$. In other words, we will show that $\int_0^t P_s^{(n)} Q^{(n)} f(x)ds \to \int_0^t P_s Qf(x)ds$, for any $x \in \boldsymbol{S}$.

Notice that $|Q^{(n)}f| \le |Qf|$, $P_s^{(n)}|Q^{(n)}f| \le P_s|Qf|$, componentwise. Hence $|P_s^{(n)}Q^{(n)}f| \le P_s|Qf|$. The desired result will follow from the dominated convergence theorem if we can show that

$$P_s^{(n)} Q^{(n)} f(x) \to P_s Qf(x), n \to \infty, \quad \text{for } x \in \boldsymbol{S}, s \ge 0. \tag{3.8}$$

Fix $x \in \boldsymbol{S}$ and $s \ge 0$. Let $\epsilon > 0$. There exists a finite set $K_\epsilon \supset x$, such that $P_s \mathbf{1}_{\{K_\epsilon^C\}}|Qf|(x) \le \epsilon$, where K_ϵ^C denotes the complement of K_ϵ in $\boldsymbol{S}$. Hence $P_s^{(n)} \mathbf{1}_{\{K_\epsilon^C\}}|Q^{(n)}f|(x) \le \epsilon$ for all n. By [1] Proposition 2.2.14, $p_{s,xy}^{(n)} \uparrow p_{s,xy}$, $n \to \infty$, for $y \in K_\epsilon$, provided n is large enough so that $\boldsymbol{S}_n \supset K_\epsilon$. Hence

$$P_s^{(n)} \mathbf{1}_{\{K_\epsilon\}} Q^{(n)} f(x) \to P_s \mathbf{1}_{\{K_\epsilon\}} Qf(x), \quad n \to \infty.$$

Choose N_ϵ, with $\boldsymbol{S}_{N_\epsilon} \supset K_\epsilon$, and $|P_s^{(n)} \mathbf{1}_{\{K_\epsilon\}} Q^{(n)} f(x) - P_s \mathbf{1}_{\{K_\epsilon\}} Qf(x)| \le \epsilon$, $n \ge N_\epsilon$. Then $|P_s^{(n)} Q^{(n)} f(x) - P_s Qf(x)| \le 3\epsilon$, $n \ge N_\epsilon$. Eqn. (3.8) follows.

For convergence of the left-hand side of (3.7), we use the existence of a constant $\gamma > 0$ such that $f \le \gamma V$. Fix $x \in \boldsymbol{S}$. Consider the transformed chains $X^{n,V}$, $n = 1, \ldots, X^V$.

As in [1] Proposition 2.2.14 $p_{s,xy}^{(n),V} \to p_{s,xy}^V$, $n \to \infty$, for $x, y \in \boldsymbol{S}$. In view of the fact that V satisfies (1.3) and (3.7), and that the right-hand side of (3.7) for V converges to the right-hand side of (1.3), $\mathsf{E}_x V(X_t^n) \to \mathsf{E}_x V(X_t)$, $n \to \infty$. By relation (3.4) we have

$$P_t^{(n),V} \mathbf{1}_{\{S\}}(x) = \frac{e^{-ct}}{V(x)} \mathsf{E}_x V(X_t^n) \to \frac{e^{-ct}}{V(x)} \mathsf{E}_x V(X_t) = P_t^V \mathbf{1}_{\{S\}}(x).$$

Since $p_{s,x\cdot}^{(n),V}$, $p_{s,x\cdot}^V$ are probability distributions, this implies that $p_{s,x\delta}^{(n),V} \to p_{s,x\delta}^V$, $n \to \infty$. Then it is an easy consequence that $p_{s,x\cdot}^{(n),V} \to p_{s,x\cdot}^V$, $n \to \infty$, setwise. We now may apply the generalised dominated convergence theorem Proposition 11.18 [13] , and obtain that $P_t^{(n),V} g(x) \to P_t^V g(x)$, $n \to \infty$, for g a bounded function. Hence $P_t^{(n)} g \cdot V(x) \to P_t g \cdot V(x)$, $n \to \infty$. Choose $g(x) = f(x)/V(x)$, $x \in \boldsymbol{S}$, $g(\delta) = 0$. Using (3.4) this gives required convergence of the left-hand side of (3.7) to the left-hand side of (1.3). QED

To check (non)-explosiveness of X^f it is sometimes helpful to perturb the transitions rates from a finite set of states. We will justify this procedure in the next section.

4 Perturbation of transitions

Assume that X satisfies Assumption 2.1. Recall that explosiveness of X^V is equivalent to the existence of a λ-invariant, non-negative, bounded vector for Q^V. In particular, for $\lambda > 0$ there exists a function $f_\lambda : \boldsymbol{S} \to \mathbf{R}_+$, $\sup_x f_\lambda(x) < \infty$, such that $Q^V f_\lambda = \lambda f_\lambda$. What does this imply for the q-matrix of the process X itself?

Since the additional coffin state (if any) is absorbing, necessarily $f_\lambda(\delta) = 0$. Hence $\sum_{y \in S} q_{xy}^V f_\lambda(y) = \lambda f_\lambda(x)$. In other words,

$$\sum_{y \in S} q_{xy} f_\lambda(y) V(y) = \lambda f_\lambda(x) V(x), \quad x \in \boldsymbol{S}.$$

Hence $f_\lambda V$ is a V-bounded non-negative λ-invariant vector for Q, with $\lambda > 0$. The reverse implication clearly holds trivially. We summarise this:

Let X satisfy Assumption 2.1 and suppose that $QV \le cV$ for some constant c. Let $\lambda > 0$. f_λ is a bounded non-negative, (non-trivial) non-negative λ-invariant vector for Q^V, if and only if $f_\lambda V$ is a V-bounded non-negative

λ*-invariant vector for* Q*. In either case,* X^V *is explosive.*

We will study the effect of perturbations. To this end, let us consider the taboo matrix ${}_KQ$ with finite set $K \subset \boldsymbol{S}$, defined by

$$ {}_Kq_{xy} = \begin{cases} q_{xy}, & y \notin K, \text{ or } x = y \in K \\ 0, & y \in K, x \notin K. \end{cases} $$

We impose a taboo on jumps *towards the set* K. Note that ${}_KQ$ is non-conservative in general. The associated minimal taboo transition function will be denoted by $\{{}_KP_t\}_t$.

Define $\tau_x = \inf\{t \geq 0 \,|\, X_t = x\}$. Then clearly $\mathsf{P}_x\{\tau_x = 0\} = 1$ for a stable process. Further define the following map T_λ acting on finite functions g on E:

$$ T_\lambda g(x) = g(x)\mathbf{1}_{\{x \neq 0\}} + (q_0 + \lambda) R_{x0}(\lambda) g(0). \tag{4.1} $$

Theorem 4.1 *Suppose that* X *satisfies Assumption 2.1. Further, suppose that* $0 \in \boldsymbol{S}$ *has the property that* $\mathsf{P}_x\{\tau_0 < \infty\} > 0$ *for each* $x \in \boldsymbol{S}$*. The following holds.*

i) $R_{x0}(\lambda) = \mathsf{E}_x e^{-\lambda\tau_0} R_{00}(\lambda)$ *(cf. Eqn. (2.1) for the definition of the resolvent).*

ii) $T_\lambda g(x) = g(x)\mathbf{1}_{\{x\neq 0\}} + \mathsf{E}_x e^{-\lambda\tau_0} T_\lambda g(0)$.

iii) *The inverse* T_λ^{-1} *exists and is defined by*

$$ T_\lambda^{-1} g(x) = \begin{cases} g(0)/(q_0+\lambda)R_{00}(\lambda), & x = 0 \\ g(x) - \mathsf{E}_x e^{-\lambda\tau_0} g(0), & x \neq 0. \end{cases} $$

iv) *Let* $\lambda > 0$*. If* f_λ *is a non-zero, non-negative* λ*-invariant vector of* Q*, then* ${}_0f_\lambda = T_\lambda^{-1} f_\lambda$ *is a non-zero non-negative* λ*-invariant vector of* ${}_0Q$*. Vice versa, if* ${}_0f_\lambda$ *is a non-zero, non-negative* λ*-invariant vector of* ${}_0Q$*, then* $f_\lambda = T_\lambda {}_0f_\lambda$ *is a non-zero non-negative* λ*-invariant vector of* Q*. Furthermore,* $f_\lambda(0) > 0$ *if and only if* ${}_0f_\lambda(0) > 0$*, and then* f_λ *is strictly positive on* $\boldsymbol{S}$*.*

Proof. We will first prove (i). To this end, note for every $T > 0$

$$ \begin{aligned} R_{x0}(\lambda) &= \int_0^\infty e^{-\lambda t} p_{t,x0} dt \\ &= \mathsf{E}_x \int_0^\infty e^{-\lambda t} \mathbf{1}_{\{0\}}(X_t) dt \\ &= \mathsf{E}_x \int_{\tau_0 \wedge T}^\infty e^{-\lambda t} \mathbf{1}_{\{0\}}(X_t) dt \\ &= \mathsf{E}_x e^{-\lambda(\tau_0\wedge T)} R_{X_{\tau_0\wedge T}0}(\lambda). \end{aligned} $$

The last equation follows from the strong Markov property. On the event $\{\tau_0 < \infty\}$ one has

$$ e^{-\lambda(\tau_0\wedge T)} R_{X_{\tau_0\wedge T}0}(\lambda) \to e^{-\lambda\tau_0} R_{00}(\lambda), \quad T \to \infty. $$

On the event $\{\tau_0 = \infty\}$

$$ e^{-\lambda(\tau_0\wedge T)} R_{X_{\tau_0\wedge T}0}(\lambda) \to 0, \quad T \to \infty, $$

since $R_{X_{\tau_0\wedge T}0}(\lambda) \leq 1/\lambda$ is bounded, and the first term converges to 0. A straightforward application of the dominated convergence theorem gives the desired result. Assertions (ii, iii) follow directly from the definition of the map T_λ.

Next we prove (iv). Suppose that f_λ is a non-negative eigenvector of Q to eigenvalue $\lambda > 0$. By assumption $R_{00}(\lambda) > 0$. Put ${}_0f_\lambda = T_\lambda^{-1} f_\lambda$. One has $f_\lambda = T_\lambda {}_0f_\lambda$. In other words

$$ f_\lambda(x) = T_\lambda {}_0f_\lambda(x) = {}_0f_\lambda(x)\mathbf{1}_{\{x\neq 0\}} + (q_0+\lambda) R_{x0}(\lambda) {}_0f_\lambda(0). $$

Multiplying both sides of the above by Q yields for all x

$$\begin{aligned} Qf_\lambda(x) &= \sum_{y\neq 0} q_{xy}\,{}_0f_\lambda(y) + (q_0+\lambda)(QR(\lambda))_{x0}\,{}_0f_\lambda(0) \\ &= \sum_{y\neq 0} q_{xy}\,{}_0f_\lambda(y) + (q_0+\lambda)(\lambda R_{x0}(\lambda)-\delta_{x0})\,{}_0f_\lambda(0) \\ &= {}_0Q\,{}_0f_\lambda(x) + \lambda(q_0+\lambda)R_{x0}(\lambda)\,{}_0f_\lambda(0) - \lambda\,{}_0f_\lambda(0)\mathbf{1}_{\{0\}}(x). \end{aligned} \tag{4.2}$$

In the second equality we have used that $QR(\lambda) = \lambda R(\lambda) - \mathbf{I}$. By assumption

$$Qf_\lambda(x) = \lambda f_\lambda(x) = \lambda\,{}_0f_\lambda(x)\mathbf{1}_{\{x\neq 0\}} + \lambda(q_0+\lambda)R_{x0}(\lambda){}_0f_\lambda(0). \tag{4.3}$$

Equating the right-hand sides of (4.2) and (4.3) and cancelling common terms yields

$${}_0Q\,{}_0f_\lambda(x) - \lambda{}_0f_\lambda(0)\mathbf{1}_{\{0\}}(x) = \lambda\,{}_0f_\lambda(x)\mathbf{1}_{\{x\neq 0\}}.$$

In other words,

$${}_0Q\,{}_0f_\lambda(x) = \lambda\,{}_0f_\lambda(x).$$

We will next show that ${}_0f_\lambda(x) \geq 0$ for all x. By construction ${}_0f_\lambda(0) \geq 0$. In [1] Ch.2 Proposition 2.13 it is shown that

$$e^{-\lambda t}P_tf_\lambda(x) \leq f_\lambda(x).$$

By assumed regularity, this implies that the stochastic process $M_t = e^{-\lambda t}f_\lambda(X_t)$, $t \geq 0$ is a non-negative, right-continuous supermartingale for each initial condition M_0 with $\mathsf{E}|M_0| < \infty$. By the martingale convergence theorem M_t converges a.s. to a non-negative random variable M_∞ say. By Fatou's lemma, $\mathsf{E}\{M_0\} \geq \mathsf{E}\{M_\infty\}$, so that M_∞ is everywhere finite. Similarly, the stopped process $(M_t^{\tau_0})_t$ is a right-continuous, non-negative supermartingale that converges to the limit

$$M_\infty^{\tau_0} = M_\infty\mathbf{1}_{\{\tau_0=\infty\}} + e^{-\lambda\tau_0}f_\lambda(0)\mathbf{1}_{\{\tau_0<\infty\}}.$$

An analogous application of Fatou's lemma yields

$$\mathsf{E}\{M_0\} = \mathsf{E}\{M_0^{\tau_0}\} \geq \mathsf{E}\{M_\infty^{\tau_0}\} \geq \mathsf{E}\{e^{-\lambda\tau_0}\}\mathbf{1}_{\{\tau_0<\infty\}}f_\lambda(0) = \mathsf{E}\{e^{-\lambda\tau_0}\}f_\lambda(0),$$

the latter being valid since $e^{-\lambda\tau_0} = 0$ when $\tau = \infty$. For initial condition $M_0 \equiv x$ this implies

$$f_\lambda(x) = \mathsf{E}\{M_0\} \geq \mathsf{E}_x e^{-\lambda\tau_0}f_\lambda(0). \tag{4.4}$$

Together with assertion (iii) this imply for $x \neq 0$ that

$${}_0f_\lambda(x) = f_\lambda(x) - (\lambda+q_0)R(x0)\lambda{}_0f_\lambda(0) = f_\lambda(x) - \mathsf{E}_x e^{-\lambda\tau_0}f_\lambda(0) \geq 0.$$

This shows that ${}_0f_\lambda$ is non-negative.

Finally assume that ${}_0f_\lambda$ is a non-negative λ-invariant vector of ${}_0Q$. Then $f_\lambda = T_\lambda{}_0f_\lambda$ is a λ-invariant vector of Q. This follows by inserting ${}_0Q{}_0f_\lambda = \lambda{}_0f_\lambda$ in Eqn. (4.2) and using the second equality of Eqn. (4.3).

By construction $f_\lambda(0) \geq 0$. It follows as well that $f_\lambda(0) > 0$ if and only if ${}_0f_\lambda(0) > 0$. By definition of T_λ, f_λ is then strictly positive on $\boldsymbol{S}$. QED

The above theorem can be used to construct λ-invariant non-negative and non-trivial vectors for finite perturbations of a given q-matrix Q, from a λ-invariant vector for Q.

Let Q' and Q both be q-matrices on the same state space. Q' will be called a *K-perturbation of* Q, $K \subset \boldsymbol{S}$, if $q_{xy} = q'_{xy}$ for $x \notin K$, $y \in \boldsymbol{S}$. That is, only the transitions from states in K may differ.

Suppose that $QV \leq cV$, for the function $V : \boldsymbol{S} \to \mathbf{R}_{>0}$ and a constant c. Further assume that Q' is a $\{0\}$-perturbation of Q, such that $Q'V$ is well-defined and finite. Then clearly there exists a constant c', such that $Q'V \leq c'V$. We assume that state 0 is reachable from any other state.

If Q has a λ-invariant vector f_λ for some $\lambda > 0$, and if the conditions of the above theorem are satisfied, then we may construct the taboo λ-invariant vector ${}_0f_\lambda$ of ${}_0Q$. It is immediate that $f_\lambda \in \ell^\infty(\boldsymbol{S}, V)$ implies ${}_0f_\lambda = T_\lambda^{-1}f_\lambda \in \ell^\infty(\boldsymbol{S}, V)$ by virtue of the expression for T_λ^{-1} in Theorem 4.1 (iii).

Now, from ${}_0f_\lambda$ we may construct a λ-invariant vector $f'_\lambda \in \ell^\infty(\boldsymbol{S}, V)$ of Q' in the following manner. Put ${}_0f'_\lambda(x) = {}_0f_\lambda(x)$ for $x \neq 0$, and put

$$ {}_0f'_\lambda(0) = \frac{1}{q_0 + \lambda} \sum_{y \neq 0} q'_{0y} {}_0f'_\lambda(y). $$

Next, put $f'_\lambda = T'_\lambda {}_0f'_\lambda$, where T'_λ is the operator in (4.1) corresponding to the minimal process associated with Q'. Then $f'_\lambda \in \ell^\infty(\boldsymbol{S}, V)$ as well, for $\lambda > c'$.

To see the latter, index transition operators etc. for the minimal Markov process generated by Q', by $'$. Since $P'_t V \leq e^{c't} V$, it follows that $R'(\lambda)V(x) \in \ell^\infty(\boldsymbol{S}, V)$ for $\lambda > c'$. But $R'(\lambda)V(x) \geq R'_{x0}(\lambda)V(0)$. Since $V(0) > 0$ by assumption, for $g(x) = R'_{x0}(\lambda){}_0f'_\lambda(0)$ it holds that $g \in \ell^\infty(\boldsymbol{S}, V)$. By virtue of Theorem 4.1 one has $f'_\lambda(x) = {}_0f'_\lambda(x)\mathbf{1}_{\{x\neq 0\}} + (q'_0 + \lambda R'_{x0}(\lambda){}_0f'_\lambda(0) \in \ell^\infty(\boldsymbol{S}, V)$ for $\lambda > c'$.

Again, by considering the V-transformed process with q-matrix $Q'^{,V}$, f'_λ generates a bounded non-negative non-trivial λ-invariant vector for $Q'^{,V}$. As indicated in the paragraph on eigenvectors Section 2, this implies the existence of a bounded non-negative, non-trivial μ-invariant vector for $Q'^{,V}$, for any $\mu > 0$. In turn, we obtain a V-bounded non-negative, non-trivial μ-invariant vector for Q'. We have proved the following assertion.

Corollary 4.2 *Assume the conditions of Theorem 4.1 to hold. Suppose that there exists a V-bounded λ-invariant vector f for Q for some $\lambda > 0$. Then there exists a V-bounded λ-invariant vector for any $\{0\}$-perturbation Q', provided that $\sum_{y\neq 0} q'_{0y} V(y) < \infty$. Hence the minimal processes associated with Q^V and $Q'^{,V}$ are either both explosive or both non-explosive. Taking $V \equiv 1$, it follows that process and perturbation are either both explosive or both non-explosive.*

Clearly the above corollary is not very surprising. Further, the assertion holds for $\{K\}$-perturbations as well, with $K \subset \boldsymbol{S}$ a finite set, provided $Q^K V$ is well-defined. Here Q^K denotes the q-matrix of the K-perturbed process. The main novelty is the explicit construction of the eigenvector of the perturbed process.

5 Examples

We will consider three transformations of the following example. Let $\boldsymbol{S} = \mathbf{Z}_+$. Let X be the minimal Markov process associated with the Q-matrix defined by

$$ q_{xy} = \begin{cases} p2^x, & y = x+1, x \neq 0 \\ (1-p)2^x, & y = x-1, x \neq 0 \\ -2^x, & y = x \neq 0 \\ q_{0y} = 0, & y \in \boldsymbol{S}. \end{cases} $$

Hence state 0 is absorbing. We further assume that $p < 1/2$.

If, instead, we were to put $q_{01} = p = -q_{00}$, then the associated jump chain would become an irreducible, ergodic Markov chain and X non-explosive by Lemma 2.2 (iii). This perturbed process has stationary distribution π given by

$$ \pi(x) = \Big(\frac{p}{2(1-p)}\Big)^x \Big(1 - \frac{p}{2(1-p)}\Big). $$

It follows from Corollary 4.2 that our basic example is non-explosive.

Non-explosive transformation Let $V(x) = \alpha^x$, with $1 < \alpha < (1-p)/p$. Then

$$ QV(x) = \big(\alpha p + \frac{1-p}{\alpha} - 1\big) \cdot 2^x \cdot V(x)\mathbf{1}_{\{x\neq 0\}}. $$

For $1 < \alpha < (1-p)/p$ one has $c = 1 - \alpha p - \frac{1-p}{\alpha} > 0$. Consequently

$$ QV(x) \leq -c \cdot V(x)\mathbf{1}_{\{x\neq 0\}}. $$

This implies [10] that X is a so-called V-exponentially ergodic Markov process with stationary distribution concentrated on state 0.

The Q-matrix Q^V equals

$$q^V_{xy} = \begin{cases} \alpha \cdot p2^x, & y = x+1, x \neq 0, \delta \\ \frac{(1-p)}{\alpha} 2^x & y = x-1, x \neq 0, \delta \\ -2^x, & y = x \neq 0, \delta \\ (1 - \alpha p - \frac{1-p}{\alpha}) 2^x, & y = \delta, x \neq 0, \delta \end{cases}$$

with δ an added coffin state. The associated jump process has transition matrix with non-zero entries

$$p^{V,J}_{xy} = \begin{cases} \alpha p, & y = x+1, x \neq 0, \delta \\ \frac{1-p}{\alpha}, & y = x-1, x \neq 0, \delta \\ 1 - \alpha p - \frac{1-p}{\alpha}, & y = \delta, x \neq 0, \delta \\ 1, & x = y, x \in \{0, \delta\}. \end{cases}$$

So the transformed chain has two absorbing classes, $\{0\}$ and $\{\delta\}$. The set $\boldsymbol{S} \setminus \{0\}$ is a collection of transient states, but in finite expected time the coffin state is reached from any other state. This follows from the fact that the probability of jumping to the coffin state δ is bounded away from 0 as a function of state. Hence the sojourn set criterion cannot be satisfied and so the V-transformation is non-explosive.

Alternatively, $W(x) = \beta^x$ with $\beta \in (\alpha, (1-p)/p)$ satisfies $QW \leq dW$ for a constant d as well. Consequently, $x \mapsto W'(x) = W(x)/V(x)$, $x \in \boldsymbol{S}$, $W'(\delta) = 0$, is a moment function for Q^V, with $Q^V W' \leq d'W'$ for a constant d'. This provides an alternative argument for showing non-explosiveness.

Explosive transformation This is inspired by Example 3.5.4 from [11]. Let $V(x) = \alpha^x$, with $\alpha = (1-p)/p$. Then $\alpha p + (1-p)/\alpha = 1$ and the V-transformation has the birth and death rates of X interchanged. Moreover, $QV = 0$, and the transitions leading to the coffin state from $x \in \boldsymbol{S}$ all have probability 0.

To show that X^V is explosive is simplest by means of the following argument. We perturb the transitions in state 0: put $q^V_{01} = (1-p) = -q^V_{00}$. Then the perturbed process, that we will call X^V again, has become irreducible. Since the associated jump process is transient, X^V is transient as well.

The explosiveness properties are not affected, by virtue of the analysis in the previous section, since 0 can be reached from any other state. However, there exists a 0-invariant finite measure m to Q^V given by

$$m(x) = \Big(\frac{1-p}{2p}\Big)^x \Big(1 - \frac{1-p}{2p}\Big),$$

provided $p \in (1/3, 1/2)$. Hence the perturbed process X^V must be explosive by virtue of Lemma 2.3. This applies to the original process as well, and so by virtue of Theorem 3.2 V does *not* satisfy the Kolmogorov forward equations (1.2) and (1.3) for the original process.

Notice that this implies Theorem Appendix C.3 of [8] not to be true without further conditions. Indeed, $QV(x) = 0$ implies that

$$\int_0^t P_s |QV(X_s)|(x) ds = 0 < \infty.$$

Consequently, $P_t V \leq V$ is finite. These are precisely the conditions required in [8].

Explosiveness for V with $QV \leq -cV + d\mathbf{1}_{\{K\}}$ The final example is related to the following question. Suppose that there exist V, positive constants c, d and a finite set K, such that $QV \leq -cV + d\mathbf{1}_{\{K\}}$. By virtue of [10] it is known that X is V-exponentially ergodic. This result has been mentioned earlier in our discussion of the first transformation. A question of interest is whether this strong stability property implies non-explosiveness of the transformed process X^V. This question has been my original motivation for this research.

We will construct a function V showing that unfortunately this is not necessarily true. Let $p = 2/5$. Then $(1-p)/p = 3/2$. We will determine numbers $\alpha_1, \alpha_2, \ldots \in (1, 3/2)$, such that

i) $\alpha_n \uparrow 3/2$;

ii) $QV(x) = -cV(x)$ for all states $x \neq 0$, for $V(x) = \prod_{n=1}^x \alpha_n$, $V(0) = 1$; and

iii) X^V explosive.

One has $QV(x) = -cV(x)$ for $x \neq 0$, if

$$\alpha_{x+1} = \frac{5}{2}(1 - c \cdot 2^{-x}) - \frac{3}{2\alpha_x}. \tag{5.1}$$

Then $\alpha_2 > \alpha_1$ if and only if

$$\frac{5}{2}(1 - c \cdot 2^{-1}) > \frac{3}{2\alpha_1} + \alpha_1, \tag{5.2}$$

and $c < 2$. The function $f(\alpha) = 3/(2\alpha) + \alpha$ takes values less than $5/2$ for $\alpha \in (1, 3/2)$. Choose $\alpha_1 \in (1, 3/2)$ and c accordingly so that (5.2) is satisfied.

An induction argument yields that $\{\alpha_n\}_{n=1,\ldots}$ is an increasing sequence. Indeed, by (5.1) for $n > 2$

$$\alpha_{n+1} - \alpha_n = c\frac{5}{2}(\frac{1}{2^{n-1}} - \frac{1}{2^n}) + \frac{3}{2}(\frac{1}{\alpha_{n-1}} - \frac{1}{\alpha_n}) > 0,$$

since $\alpha_n > \alpha_{n-1}$ by the induction assumption. On the other hand, by (5.1) $\alpha_n \leq 5/2$. As a consequence, $\{\alpha_n\}_n$ is an increasing, bounded sequence, and so it has a limit, α^*. α^* satisfies

$$\alpha^* = \frac{5}{2} - \frac{3}{2\alpha^*}.$$

Solving gives the roots 1 and $3/2$. Since $\alpha^* > \alpha_1 > 1$, $\alpha^* = 3/2$.

Next transform the Markov process by V. Use that $QV \leq 0$. This yields the process

$$q^V_{xy} = \begin{cases} \frac{2}{5}\alpha_{x+1}2^x, & y = x+1, x \neq 0 \\ \frac{3}{5\alpha_x}2^x, & y = x-1, x \neq 0 \\ c - 2^x, & y = x \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

We need to check explosiveness of this process. A direct construction of a bounded λ-invariant vector is gruesome. The simplest road is to use the same trick as in the previous explosiveness example. To this end, we need that the Markov process be irreducible.

Set the transition rates from state 0 to $q_{01} = 1 = -q_{00}$ and denote the new process again by X^V. As in the previous example, the explosiveness properties are not affected. However, the resulting jump chain has the following transition probabilities

$$p^{V,J}_{xy} = \begin{cases} \dfrac{2\alpha_{x+1}}{5(1 - c \cdot 2^{-x})}, & y = x+1, x \neq 0 \\ \dfrac{3}{5(1 - c \cdot 2^{-x})\alpha_x}, & y = x-1, x \neq 0 \\ 1 & y = 1, x = 0, \end{cases}$$

and all other transitions are equal to 0. This is a transient chain, since for all x large enough, $p^{V,J}_{x\,x+1} > p^{V,J}_{x\,x-1} + 1/10$. On the other hand, Q^V has the left 0-invariant measure m with

$$m(x) = m(0)\frac{5\alpha_1}{6}\prod_{k=1}^{x-1}\frac{\alpha^2_{k+1}}{3} \leq m(0)\frac{5\alpha_1}{6}\left(\frac{3}{4}\right)^{x-1}, \quad x \geq 1.$$

The measure m is finite and so the Markov process X^V must be explosive. Consequently, V does not satisfy the Kolmogorov forward equations (1.2) and (1.3).

6 Domain of the generator

As has been mentioned in the introduction, strong continuity of a semigroup on the Banach space $\ell^\infty(\boldsymbol{S}, V)$ is a property implying the jumps to be bounded. The reverse statement is true as well. We have the following relation.

Lemma 6.1 *Suppose that X is a minimal Markov process on the countable state space $\boldsymbol{S}$. Assume that X has a q-matrix Q with $\sup_x q_x < \infty$. Let $\tau \geq \sup_x q_x$. Suppose that $P = \mathbf{I} + \tau^{-1} Q$ is a bounded linear operator on the space $\ell^\infty(\boldsymbol{S}, V)$. Then $\{ P_t \}_t$ a strongly continuous transition semi-group on $\ell^\infty(\boldsymbol{S}, V)$, in particular $\| P_t - \mathbf{I} \|_V \to 0$, $t \downarrow 0$.*

Proof. Note that

$$\| P_t - \mathbf{I} \|_V \leq e^{-\tau t} \sum_{n \geq 1} \frac{(\lambda \tau \| P \|_V)^n}{n!} = e^{-\tau t} (e^{\tau t \| P \|_V} - 1) \to 0, \quad t \downarrow 0.$$

QED

Clearly $\| P_t \|_V < \infty$ implies $\| P \|_V < \infty$ and so the condition in the above lemma follows naturally, if $\{P_t\}_t$ is a transition semi-group on $\ell^\infty(\boldsymbol{S}, V)$.

Suppose next that X is a stable, conservative, minimal Markov process. For convenience we assume that $\boldsymbol{S}$ is equipped with the discrete topology, so as to make all functions on $\boldsymbol{S}$ continuous. Let $W : \boldsymbol{S} \to \mathbf{R}_+$ be a function satisfying (3.1) for the constant d, i.e. $QW \leq dW$. Define

$$C_0(\boldsymbol{S}, W) = \{ f : \boldsymbol{S} \to \mathbf{R} \;\Big|\; \begin{array}{l} \|f\|_W < \infty, \\ \text{for each } \epsilon > 0 \, \exists \text{ a finite set } K \subset \boldsymbol{S}, \text{ such that } \sup_{x \notin K} |f(x)|/W(x) < \epsilon \end{array} \Big\}$$

as the collection of functions with finite norm w.r.t. W, that become arbitrarily small outside compact sets. Working on $C_0(\boldsymbol{S}, W)$ has one big prerogative. If $\{ P_t \}_t$ is a transition semigroup on $C_0(\boldsymbol{S}, W)$, then pointwise continuity $P_t f(x) \to f(x)$, $t \downarrow 0$, for $x \in \boldsymbol{S}$ and each $f \in C_0(\boldsymbol{S}, W)$, implies strong continuity as a semigroup (see [12] Lemma III.6.7).

Lemma 6.2 *Let X satisfy Assumption 2.1, and suppose that Q is conservative. If Q has a λ-invariant vector $f \geq 0$, for some $\lambda > 0$, then $f \notin C_0(\boldsymbol{S}, W)$.*

Proof. Suppose $f \in C_0(\boldsymbol{S}, W)$. Then the function $x \mapsto W(x)/f(x)$ is a moment function for X^f. Since Q^f is conservative, X^f is non-explosive by virtue of Lemma 2.2 (ii). Theorem 3.4 therefore applies with the function $V = f$ and constant $c = \lambda$.

For any $\mu > \lambda$, Q has a non-negative eigenvector $f_\mu \leq f$ to eigenvalue μ, with $P_t f_\mu(x) \leq e^{ct} W(x) \cdot \|f_\mu\|_W$. Since $f_\mu \in \ell^\infty(\boldsymbol{S}, f)$, by virtue of Theorem 3.4 the Kolmogorov forward equation applies to f_μ. By virtue of Lemma 3.1, using the constant $d = -\mu$, this implies

$$P_t f_\mu(x) = f_\mu(x) e^{\mu t}, \quad t \geq 0.$$

On the other hand, $P_t f_\mu \leq e^{\lambda t} f \cdot \|f_\mu\|_f$, $t \geq 0$. A contradiction, since $\mu > \lambda$. QED

Denote by $\ell^\infty(\boldsymbol{S}, V, W)$ the Banach space of functions $f : \boldsymbol{S} \to \mathbf{R}$, with $\|f\|_V < \infty$, equipped with the norm $\| \cdot \|_W$. Clearly, if $V \in C_0(\boldsymbol{S}, W)$ then $\ell^\infty(\boldsymbol{S}, V, W) \subset C_0(\boldsymbol{S}, W)$.

Theorem 6.3 *Let X satisfy Assumption 2.1, and suppose that Q is conservative. $\{P_t\}_t$ is a strongly continuous semigroup on $C_0(\boldsymbol{S}, W)$ if and only there exist a function $V : \boldsymbol{S} \to \mathbf{R}_+$, and a constant c such that $V \in C_0(\boldsymbol{S}, W)$, and $QV \leq cV$. Under either condition X^V is non-explosive, and $\mathcal{D}(\mathcal{Q}) \supset \{f \in \ell^\infty(\boldsymbol{S}, V, W) \,|\, \|Qf\|_W < \infty\}$.*

Proof. Suppose that $\{P_t\}_{t \geq 0}$ is a strongly continuous semigroup on $C_0(\boldsymbol{S}, W)$. Choose any $f \geq 0$, $f \in C_0(\boldsymbol{S}, W)$. Let $g = R_\lambda f$, then $g \in \mathcal{D}(\mathcal{Q})$ ([12] p.236 (4.14)), for $\lambda > 0$, g is non-negative, and $(\lambda \mathbf{I} - \mathcal{Q}) g = f$. In other words $\lambda g = f + \mathcal{Q} g$, and hence $\lambda g \geq \mathcal{Q} g$. As in [1] Proposition 4.6, we can derive that $\mathcal{Q} g = Q g$. The conclusion follows by putting $V = g$.

Next we suppose that there exist a function $V \in C_0(\boldsymbol{S}, W)$ and a constant c such that $QV \leq cV$. As has been pointed out already, $\ell^\infty(\boldsymbol{S}, V, W) \subset C_0(\boldsymbol{S}, W)$. Since $P_t |f| \leq e^{ct} V \cdot \|f\|_V$, it follows that $P_t f \in C_0(\boldsymbol{S}, W)$ for any function $f \in \ell^\infty(\boldsymbol{S}, V, W)$.

We prove that $P_t f \in C_0(\boldsymbol{S}, W)$ for all $f \in C_0(\boldsymbol{S}, W)$. By assumption $QW \leq dW$, so that $P_t W \leq e^{dt} W$. I.o.w. $\| P_t \|_W \leq e^{dt} < \infty$. Let $f \in C_0(\boldsymbol{S}, W)$. Then $\| Pf \|_W \leq \| P_t \|_W \|f\|_W < \infty$ and so $P_t f \in \ell^\infty(\boldsymbol{S}, W)$.

Let next $\{K_n\}_n$ be an increasing sequence of finite sets, with $\lim_n K_n = \boldsymbol{S}$. Let $f_n = \mathbf{1}_{\{K_n\}} f$ be the projection of f on K_n. Then $f_n \in \ell^\infty(\boldsymbol{S}, V)$, $\|f_n\|_V$, $\|f_n\|_W < \infty$ and $\|f_n - f\|_W \to 0$ as $n \to \infty$.

Since $C_0(\boldsymbol{S}, W)$ is a Banach space, this implies that $\overline{\ell^\infty(\boldsymbol{S}, V, W)} = C_0(\boldsymbol{S}, W)$. Further, $\| P_t f_n - P_t f\|_W \leq \| P_t\|_W \|f_n - f\|_W \to 0$, $n \to \infty$ and hence $P_t f \in C_0(\boldsymbol{S}, W)$.

This proves that $\{ P_t\}_t$ is a transition semigroup on $C_0(\boldsymbol{S}, W)$. Next we will show pointwise continuity. For the function W

$$\limsup_{t\downarrow 0} P_t W(x) \leq \limsup_{t\downarrow 0} e^{ct} W(x) \leq W(x), \quad x \in \boldsymbol{S}.$$

On the other hand, by Fatou's lemma

$$\liminf_{t\downarrow 0} P_t W(x) \geq \sum_y \liminf_{t\downarrow 0} p_{t,xy} W(y) = W(x).$$

Consequently, $\lim_{t\downarrow 0} P_t W(x) = W(x)$. The result for $f \geq 0$, $f \in \ell^\infty(\boldsymbol{S}, W)$ follows by an application of the generalised dominated convergence Theorem (cf.[13]), analogously to the proof of Theorem 3.4.

Non-explosiveness of X^V follows as in the proof of Lemma 6.2, since $W(x)/V(x)$ is a moment function for X^V, and X^V is conservative. For the final statement, by definition one has that $\mathcal{D}(\mathcal{Q}) \subset \{f \in C_0(\boldsymbol{S}, W) \,|\, \|\mathcal{Q}f\|_W < \infty\}$. Note that $Qf = \mathcal{Q}f$ on $\mathcal{D}(\mathcal{Q})$ (cf. [1] Section 1.4).

Let next $f \in \ell^\infty(\boldsymbol{S}, V, W)$ with $\|Qf\|_W < \infty$. Then f satisfies the conditions of Theorem 3.4 with W playing the role of the bounding vector. Hence the Kolmogorov forward equation applies. This easily can be shown to imply $f \in \mathcal{D}(\mathcal{Q})$. QED

One may wonder to what extent the Banach space setting is useful for the denumerable state space case compared to setting up the analysis from the q-matrix, apart from elegance of the approach. The main result that Banach space techniques provide us, seems to me to be strong continuity of the transition semigroup as a consequence of pointwise continuity, when the space is $C_0(\boldsymbol{S}, W)$.

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