Free energy of a copolymer in a micro-emulsion

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Abstract

In this paper we consider a two-dimensional model of a copolymer consisting of a random concatenation of hydrophilic and hydrophobic monomers, immersed in a micro-emulsion of random droplets of oil and water. The copolymer interacts with the micro-emulsion through an interaction Hamiltonian that favors matches and disfavors mismatches between the monomers and the solvents, in such a way that the interaction with the oil is stronger than with the water.

The configurations of the copolymers are directed self-avoiding paths in which only steps up, down and right are allowed. The configurations of the micro-emulsion are square blocks with oil and water arranged in percolation-type fashion. The only restriction imposed on the path is that in every column of blocks its vertical displacement on the block scale is bounded. The way in which the copolymer enters and exits successive columns of blocks is a directed self-avoiding path as well, but on the block scale. We refer to this path as the coarse-grained self-avoiding path. We are interested in the limit as the copolymer and the blocks become large, in such a way that the copolymer spends a long time in each block yet visits many blocks. This is a coarse-graining limit in which the space-time scales of the copolymer and of the micro-emulsion become separated.

We derive a *variational formula* for the *quenched free energy per monomer*, where quenched means that the disorder in the copolymer and the disorder in the micro-emulsion are both frozen. In a sequel paper we will analyze this variational formula and identify the phase diagram. It turns out that there are two regimes, *supercritical* and *subcritical*, depending on whether the oil blocks percolate or not along the coarse-grained self-avoiding path. The phase diagrams in the two regimes turn out to be completely different.

In earlier work we considered the same model, but with an unphysical restriction: paths could enter and exit blocks only at diagonally opposite corners. Without this restriction, the variational formula for the quenched free energy is more complicated, but in the sequel paper we will see that it is still tractable enough to allow for a qualitative analysis of the phase diagram.

Part of our motivation is that our model can be viewed as a coarse-grained version of the well-known *directed polymer with bulk disorder*. The latter has been studied intensively in the literature, but no variational formula is as yet available.

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1 Introduction and main result

In Section 1.1 we define the model. In Section 1.2 we state our main result, a *variational formula for the quenched free energy per monomer of a random copolymer in a random emulsion* (Theorem 1.1 below). In Section 1.3 we discuss the significance of this variational formula and place it in a broader context. Section 2 gives a precise definition of the various ingredients in the variational formula, and states some key properties of these ingredients formulated in terms of a number of propositions. The proof of these propositions is deferred to Section 3. The proof of the variational formula is given in Section 4. Appendices A–D contain a number of technical facts that are needed in Sections 2–4.

For a general overview on polymers with disorder, we refer the reader to the monographs by Giacomin [1] and den Hollander [2].

1.1 Model and free energy

To build our model, we distinguish between three scales: (1) the *microscopic* scale associated with the size of the monomers in the copolymer (= 1, by convention); (2) the *mesoscopic* scale associated with the size of the droplets in the micro-emulsion ($L_n \gg 1$); (3) the *macroscopic* scale associated with the size of the copolymer ($n \gg L_n$).

Copolymer configurations. Pick $n \in \mathbb{N} \cup \{\infty\}$ and let $\mathcal{W}_n$ be the set of n-step *directed self-avoiding paths* starting at the origin and being allowed to move *upwards, downwards and to the right*, i.e.,

$$\begin{aligned}\mathcal{W}_n = \big\{\pi = (\pi_i)_{i=0}^n \in (\mathbb{N}_0 \times \mathbb{Z})^{n+1}\colon\ \pi_0 = (0,1),\\ \pi_{i+1} - \pi_i \in \{(1,0),(0,1),(0,-1)\}\ \forall\, 0 \leq i < n,\ \pi_i \neq \pi_j\ \forall\, 0 \leq i < j \leq n\big\}.\end{aligned} \tag{1.1}$$

The copolymer is associated with the path π. The i-th monomer is associated with the bond (π_{i-1}, π_i). The starting point π_0 is located at $(0,1)$ for technical convenience only.

Microscopic disorder in the copolymer. Each monomer is randomly labelled A (hydrophobic) or B (hydrophilic), with probability $\frac{1}{2}$ each, independently for different monomers. The resulting labelling is denoted by

$$\omega = \{\omega_i\colon\ i \in \mathbb{N}\} \in \{A, B\}^{\mathbb{N}} \tag{1.2}$$

and represents the *randomness of the copolymer*, i.e., $\omega_i = A$ (respectively, $\omega_i = B$) means that the i-th monomer is of type A (respectively, B); see Fig. 1.

Mesoscopic disorder in the micro-emulsion. Fix $p \in (0,1)$ and $L_n \in \mathbb{N}$. Partition $(0,\infty) \times \mathbb{R}$ into square blocks of size L_n:

$$(0,\infty) \times \mathbb{R} = \bigcup_{x \in \mathbb{N}_0 \times \mathbb{Z}} \Lambda_{L_n}(x), \qquad \Lambda_{L_n}(x) = xL_n + (0, L_n]^2. \tag{1.3}$$

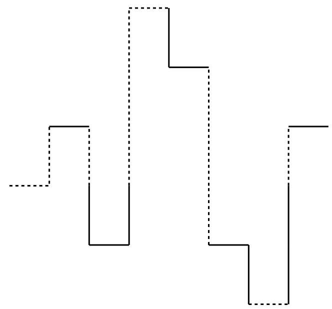

Figure 1: Microscopic disorder ω in the copolymer. Dashed edges represent monomers of type A (hydrophobic), drawn edges represent monomers of type B (hydrophilic).

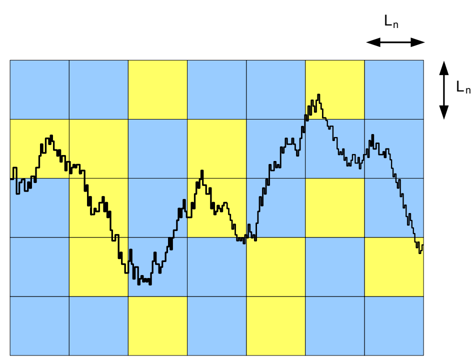

Figure 2: Mesoscopic disorder Ω in the micro-emulsion. Light shaded blocks represent droplets of type A (oil), dark shaded blocks represent droplets of type B (water). Drawn is also the copolymer, but without an indication of the microscopic disorder ω attached to it.

Each block is randomly labelled A (oil) or B (water), with probability p, respectively, $1-p$, independently for different blocks. The resulting labelling is denoted by

$$\Omega = \{\Omega(x)\colon\ x \in \mathbb{N}_0 \times \mathbb{Z}\} \in \{A, B\}^{\mathbb{N}_0\times\mathbb{Z}} \tag{1.4}$$

and represents the *randomness of the micro-emulsion*, i.e., $\Omega(x) = A$ (respectively, $\Omega(x) = B$) means that the x-th block is of type A (repectively, B); see Fig. 2. The size of the blocks L_n is assumed to be non-decreasing and to satisfy

$$\lim_{n\to\infty} L_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \tfrac{L_n}{n} = 0, \tag{1.5}$$

i.e., the blocks are large compared to the monomer size but (sufficiently) small compared to the copolymer size. For convenience we assume that if an A-block and a B-block are on top of each other, then the interface belongs to the A-block.

Path restriction. We bound the vertical displacement on the block scale in each column of blocks by $M \in \mathbb{N}$. The value of M will be arbitrary but fixed. In other words, instead of considering the full set of trajectories $\mathcal{W}_n$, we consider only trajectories that exit a column through a block at most M above or M below the block where the column was entered (see Fig. 3). Formally, we partition $(0,\infty)\times\mathbb{R}$ into columns of blocks of width L_n, i.e.,

$$(0,\infty)\times\mathbb{R} = \cup_{j\in\mathbb{N}_0}\mathcal{C}_{j,L_n}, \qquad \mathcal{C}_{j,L_n} = \cup_{k\in\mathbb{Z}}\Lambda_{L_n}(j,k), \tag{1.6}$$

where C_{j,L_n} is the j-th column. For each $\pi \in \mathcal{W}_n$, we let τ_j be the time at which π leaves the $(j-1)$-th column and enters the j-th column, i.e.,

$$\tau_j = \sup\{i \in \mathbb{N}_0\colon\ \pi_i \in \mathcal{C}_{j-1,n}\} = \inf\{i \in \mathbb{N}_0\colon\ \pi_i \in \mathcal{C}_{j,n}\} - 1, \qquad j = 1,\dots,N_\pi - 1, \tag{1.7}$$

where N_π indicates how many columns have been visited by π. Finally, we let $v_{-1}(\pi) = 0$ and, for $j \in \{0,\dots,N_\pi-1\}$, we let $v_j(\pi) \in \mathbb{Z}$ be such that the block containing the last step of the copolymer in $\mathcal{C}_{j,n}$ is labelled by $(j, v_j(\pi))$, i.e., $(\pi_{\tau_{j+1}-1}, \pi_{\tau_{j+1}}) \in \Lambda_{L_N}(j, v_j(\pi))$. Thus, we restrict $\mathcal{W}_n$ to the subset $\mathcal{W}_{n,M}$ defined as

$$\mathcal{W}_{n,M} = \big\{\pi \in \mathcal{W}_n\colon\ |v_j(\pi) - v_{j-1}(\pi)| \le M\ \forall\, j \in \{0,\dots,N_\pi-1\}\big\}. \tag{1.8}$$

Hamiltonian and free energy. Given ω, Ω, M and n, with each path $\pi \in \mathcal{W}_{n,M}$ we associate an *energy* given by the Hamiltonian

$$H^{\omega,\Omega}_{n,L_n}(\pi) = \sum_{i=1}^{n}\Big(\alpha\, 1\Big\{\omega_i = \Omega^{L_n}_{(\pi_{i-1},\pi_i)} = A\Big\} + \beta\, 1\Big\{\omega_i = \Omega^{L_n}_{(\pi_{i-1},\pi_i)} = B\Big\}\Big), \tag{1.9}$$

where $\Omega^{L_n}_{(\pi_{i-1},\pi_i)}$ denotes the label of the block the step (π_{i-1},π_i) lies in. What this Hamiltonian does is count the number of AA-matches and BB-matches and assign them energy α and β, respectively, where $\alpha, \beta \in \mathbb{R}$. (Note that the interaction is assigned to bonds rather than to sites, and that we do not follow the convention of putting a minus sign in front of the Hamiltonian.) Similarly to what was done in our earlier papers [3], [4], [5], [6], without loss of generality we may restrict the interaction parameters to the cone

$$\text{CONE} = \{(\alpha,\beta) \in \mathbb{R}^2\colon\ \alpha \ge |\beta|\}. \tag{1.10}$$

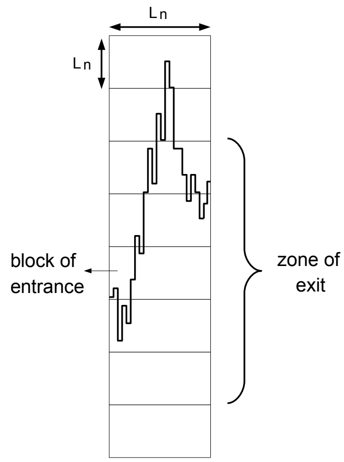

Figure 3: Example of a trajectory $\pi \in \mathcal{W}_{n,M}$ with $M = 2$ crossing the column $\mathcal{C}_{0,L_n}$ with $v_0(\pi) = 2$.

For $n \in \mathbb{N}$, the free energy per monomer is defined as

$$f_n^{\omega,\Omega}(M;\alpha,\beta) = \tfrac{1}{n} \log Z_{n,L_n}^{\omega,\Omega}(M;\alpha,\beta) \quad \text{with} \quad Z_{n,L_n}^{\omega,\Omega}(M) = \sum_{\pi \in \mathcal{W}_{n,M}} e^{H_{n,L_n}^{\omega,\Omega}(\pi)}, \tag{1.11}$$

and in the limit as $n \to \infty$ the free energy per monomer is given by

$$f(M;\alpha,\beta) = \lim_{n\to\infty} f_{n,L_n}^{\omega,\Omega}(M;\alpha,\beta), \tag{1.12}$$

provided this limit exists.

Henceforth, we subtract from the Hamiltonian the quantity $\alpha \sum_{i=1}^n 1\{\omega_i = A\}$, which by the law of large numbers is $\frac{\alpha}{2} n(1+o(1))$ as $n \to \infty$ and corresponds to a shift of $-\frac{\alpha}{2}$ in the free energy. The latter transformation allows us to lighten the notation, starting with the Hamiltonian, which becomes

$$H_{n,L_n}^{\omega,\Omega}(\pi) = \sum_{i=1}^n \Big(\beta\, 1\{\omega_i = B\} - \alpha\, 1\{\omega_i = A\}\Big)\, 1\Big\{\Omega^{L_n}_{(\pi_{i-1},\pi_i)} = B\Big\}. \tag{1.13}$$

1.2 Variational formula for the quenched free energy

Theorem 1.1 below is the main result of our paper. It expresses the quenched free energy per monomer in the form of a *variational formula.* To state this variational formula, we need to define some quantities that capture the way in which the copolymer moves inside single columns of blocks and samples different columns. A precise definition of these quantities will be given in Section 2.

Given $M \in \mathbb{N}$, the *type* of a column is denoted by Θ and takes values in a *type space* $\overline{\mathcal{V}}_M$, defined in Section 2.2.1. The type indicates both the vertical displacement of the copolymer in the column and the mesoscopic disorder seen relative to the block where the copolymer enters the column. In Section 2.2.1 we further associate with each $\Theta \in \overline{\mathcal{V}}_M$ a quantity $u_\Theta \in [t_\Theta, \infty)$

that indicates how many *steps on scale* L_n the copolymer makes in columns of type Θ, where t_Θ is the minimal number of steps required to cross a column of type Θ. These numbers are gathered into the set

$$\mathcal{B}_{\overline{\mathcal{V}}_M} = \big\{(u_\Theta)_{\Theta\in\overline{\mathcal{V}}_M} \in \mathbb{R}^{\overline{\mathcal{V}}_M}\colon\, u_\Theta \geq t_\Theta \;\forall\,\Theta\in\overline{\mathcal{V}}_M,\, \Theta\mapsto u_\Theta \text{ continuous}\big\}. \tag{1.14}$$

In Section 2.2.2 we introduce the *free energy per step* $\psi(\Theta, u_\Theta; \alpha, \beta)$ associated with the copolymer when crossing a column of type Θ in u_Θ steps, which depends on the parameters α, β. After that it remains to define the family of *frequencies with which successive pairs of different types of columns can be visited by the copolymer*. This is done in Section 2.3 and is given by a family of probability laws ρ in $\mathcal{M}_1(\overline{\mathcal{V}}_M)$, the set of probability measures on $\overline{\mathcal{V}}_M$, forming a set

$$\mathcal{R}_{p,M} \subset \mathcal{M}_1(\overline{\mathcal{V}}_M), \tag{1.15}$$

which depends on M and on the parameter p.

Theorem 1.1 *For every* $(\alpha,\beta)\in$ CONE*,* $M\in\mathbb{N}$ *and* $p\in(0,1)$ *the free energy in* (1.12) *exists for* $\mathbb{P}$*-a.e.* (ω,Ω) *and in* $L^1(\mathbb{P})$*, and is given by*

$$f(M;\,\alpha,\beta) = \sup_{\rho\in\mathcal{R}_{p,M}}\; \sup_{(u_\Theta)_{\Theta\in\overline{\mathcal{V}}_M}\in\mathcal{B}_{\overline{\mathcal{V}}_M}} V(\rho,u) \tag{1.16}$$

with

$$V(\rho,u) = \frac{\int_{\overline{\mathcal{V}}_M} u_\Theta\,\psi(\Theta,u_\Theta;\alpha,\beta)\,\rho(d\Theta)}{\int_{\overline{\mathcal{V}}_M} u_\Theta\,\rho(d\Theta)} \quad \textit{if} \quad \int_{\overline{\mathcal{V}}_M} u_\Theta\,\rho(d\Theta) = \infty, \tag{1.17}$$

and $V(\rho,u) = -\infty$ *otherwise.*

1.3 Discussion

Structure of the variational formula. The variational formula in (1.16) has a simple structure: each column type Θ has its own number of monomers u_Θ and its own free energy per monomer $\psi(\Theta,u_\Theta;\alpha,\beta)$ (both on the mesoscopic scale), and the total free energy per monomer is obtained by weighting each column type with the frequency $\rho_1(d\Theta)$ at which it is visited by the copolymer. The numerator is the total free energy, the denominator is the total number of monomers (both on the mesoscopic scale). The variational formula optimizes over $(u_\Theta)_{\Theta\in\overline{\mathcal{V}}_M}\in\mathcal{B}_{\overline{\mathcal{V}}_M}$ and $\rho\in\mathcal{R}_{p,M}$. The reason why these two suprema appear in (1.16) is that, as a consequence of assumption (1.5), the *mesoscopic scale carries no entropy*: all the entropy comes from the microscopic scale, through the free energy per monomer in single columns.

In Section 2 we will see that $\psi(\Theta,u_\Theta;\alpha,\beta)$ in turn is given by a variational formula that involves the entropy of the copolymer inside a single column (for which an explicit expression is available) and the quenched free energy per monomer of a copolymer near a *single linear interface* (for which there is an abundant literature). Consequently, the free energy of our model with a *random geometry* is directly linked to the free energy of a model with a *non-random geometry*. This will be crucial for our analysis of the free energy in the sequel paper.

Removal of the corner restriction. In our earlier papers [3], [4], [5], [6], we allowed the configurations of the copolymer to be given by the subset of $\mathcal{W}_n$ consisting of those paths that enter pairs of blocks through a common corner, exit them at one of the two corners diagonally opposite and in between stay confined to the two blocks that are seen upon entering. The latter is an *unphysical restriction* that was adopted to simplify the model. In these papers we derived a variational formula for the free energy per step that had a simpler structure. We analyzed this variational formula as a function of α, β, p and found that there are two regimes, *supercritical* and *subcritical*, depending on whether the oil blocks percolate or not along the coarse-grained self-avoiding path. In the supercritical regime the phase diagram turned out to have two phases, in the subcritical regime it turned out to have four phases, meeting at two tricritical points.

In a sequel paper we will show that the phase diagrams found in the restricted model are largely *robust* against the removal of the corner restriction, despite the fact that the variational formula is more complicated. In particular, there are again two types of phases: *localized phases* (where the copolymer spends a positive fraction of its time near the AB-interfaces) and *delocalized phases* (where it spends a zero fraction near the AB-interfaces). Which of these phases occurs depends on the parameters α, β, p. It is energetically favorable for the copolymer to stay close to the AB-interfaces, where it has the possibility of placing more than half of its monomers in their preferred solvent (by switching sides when necessary), but this comes with a loss of entropy. The competition between energy and entropy is controlled by the energy parameters α, β (determining the reward of switching sides) and by the density parameter p (determining the density of the AB-interfaces).

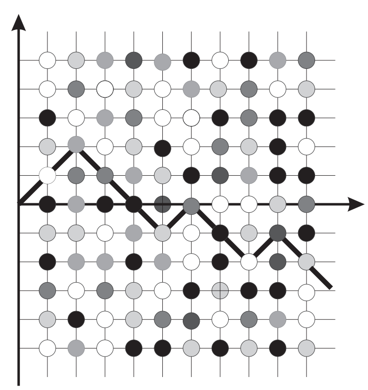

Figure 4: Picture of a directed polymer with bulk disorder. The different shades of black, grey and white represent different values of the disorder.

Comparison with the directed polymer with bulk disorder. A model of a polymer with disorder that has been studied intensively in the literature is the *directed polymer with bulk disorder*. Here, the set of paths is

$$\mathcal{W}_n = \left\{\pi = (i, \pi_i)_{i=0}^n \in (\mathbb{N}_0 \times \mathbb{Z}^d)^{n+1} \colon \pi_0 = 0, \, \|\pi_{i+1} - \pi_i\| = 1 \; \forall\, 0 \leq i < n\right\}, \tag{1.18}$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{Z}^d$, and the Hamiltonian is

$$H_n^\omega(\pi) = \lambda \sum_{i=1}^n \omega(i, \pi_i), \tag{1.19}$$

where $\lambda > 0$ is a parameter and $\omega = \{\omega(i,x) \colon i \in \mathbb{N}, \, x \in \mathbb{Z}^d\}$ is a field of i.i.d. $\mathbb{R}$-valued random variables with zero mean, unit variance and finite moment generating function, where

$\mathbb{N}$ is time and $\mathbb{Z}^d$ is space (see Fig. 4). This model can be viewed as a version of a copolymer in a micro-emulsion where the droplets are of the *same* size as the monomers. For this model *no variational formula is known for the free energy*, and the analysis relies on the application of martingale techniques (for details, see e.g. den Hollander [2], Chapter 12).

In our model (which is restricted to $d = 1$ and has self-avoiding paths that may move north, south and east instead of north-east and south-east), the droplets are much larger than the monomers. This causes a *self-averaging of the microscopic disorder*, both when the copolymer moves inside one of the solvents and when it moves near an interface. Moreover, since the copolymer is much larger than the droplets, also *self-averaging of the mesoscopic disorder* occurs. This is why the free energy can be expressed in terms of a variational formula, as in Theorem 1.1. In the sequel paper we will see that this variational formula acts as a *jumpboard* for a detailed analysis of the phase diagram. Such a detailed analysis is lacking for the directed polymer with bulk disorder.

The directed polymer in random environment has two phases: a *weak disorder phase* (where the quenched and the annealed free energy are asymptotically comparable) and a *strong disorder phase* (where the quenched free energy is asymptotically smaller than the annealed free energy). The strong disorder phase occurs in dimension $d = 1, 2$ for all $\lambda > 0$ and in dimension $d \geq 3$ for $\lambda > \lambda_c$, with $\lambda_c \in [0, \infty]$ a critical value that depends on d and on the law of the disorder. It is predicted that in the strong disorder phase the copolymer moves within a narrow corridor that carries sites with high energy (recall our convention of not putting a minus sign in front of the Hamiltonian), resulting in *superdiffusive* behavior in the spatial direction. We expect a similar behavior to occur in the localized phases of our model, where the polymer targets the AB-interfaces. It would be interesting to find out how far the coarsed-grained path in our model travels vertically as a function of n.

2 Key ingredients of the variational formula

In this section we give a precise definition of the various ingredients in Theorem 1.1. In Section 2.1 we define the entropy of the copolymer inside a single column (Proposition 2.1) and the quenched free energy per monomer for a random copolymer near a *single linear interface* (Proposition 2.2), which serve as the key *microscopic* ingredients. In Section 2.2 these quantities are used to derive variational formulas for the quenched free energy per monomer in a single column (Proposition 2.4). These variational formulas come in two varieties (Propositions 2.5 and 2.6). In Section 2.3 we define certain percolation frequencies describing how the copolymer samples the droplets in the emulsion (Proposition 2.8), which serve as the key *mesoscopic* ingredients. Propositions 2.4–2.6 will be proved in Section 3. The results in Sections 2.2–2.3 will be used in Section 4 to prove our variational formula in Theorem 1.1 for the copolymer in the emulsion, which is our main *macroscopic* object of interest.

2.1 Path entropies and free energy along a single linear interface

Path entropies. We begin by defining the entropy of a path crossing a single column. Let

$$\begin{aligned} \mathcal{H} &= \{(u,l) \in [0,\infty) \times \mathbb{R} \colon\ u \geq 1 + |l|\}, \\ \mathcal{H}_L &= \big\{(u,l) \in \mathcal{H} \colon\ l \in \tfrac{\mathbb{Z}}{L},\, u \in 1 + |l| + \tfrac{2\mathbb{N}}{L}\big\}, \qquad L \in \mathbb{N}, \end{aligned} \tag{2.1}$$

and note that $\mathcal{H} \cap \mathbb{Q}^2 = \cup_{L\in\mathbb{N}} \mathcal{H}_L$. For $(u,l) \in \mathcal{H}_L$, denote by $\mathcal{W}_L(u,l)$ (see Fig. 5) the set containing those paths $\pi = (0,-1) + \widetilde{\pi}$ with $\widetilde{\pi} \in \mathcal{W}_{uL}$ (recall (1.1)) for which $\pi_{uL} = (L, lL)$. The entropy per step associated with the paths in $\mathcal{W}_L(u,l)$ is given by

$$\tilde{\kappa}_L(u,l) = \tfrac{1}{uL} \log |\mathcal{W}_L(u,l)|. \tag{2.2}$$

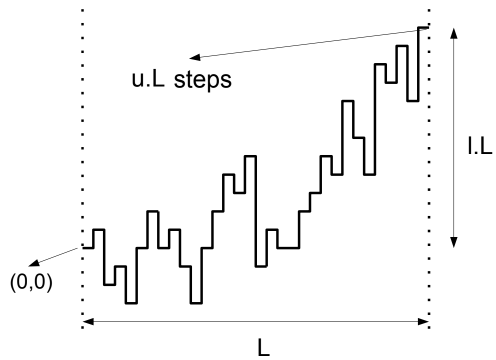

Figure 5: A trajectory in $\mathcal{W}_L(u,l)$.

The following proposition will be proved in Appendix A.

Proposition 2.1 *For all $(u,l) \in \mathcal{H} \cap \mathbb{Q}^2$ there exists a $\tilde{\kappa}(u,l) \in [0, \log 3]$ such that*

$$\lim_{\substack{L\to\infty \\ (u,l)\in\mathcal{H}_L}} \tilde{\kappa}_L(u,l) = \sup_{\substack{L\in\mathbb{N} \\ (u,l)\in\mathcal{H}_L}} \tilde{\kappa}_L(u,l) = \tilde{\kappa}(u,l). \tag{2.3}$$

An explicit formula is available for $\tilde{\kappa}(u,l)$, namely,

$$\tilde{\kappa}(u,l) = \begin{cases} \kappa(u/|l|, 1/|l|), & l \neq 0, \\ \hat{\kappa}(u), & l = 0, \end{cases} \tag{2.4}$$

where $\kappa(a,b)$, $a \geq 1+b$, $b \geq 0$, and $\hat{\kappa}(\mu)$, $\mu \geq 1$, are given in [3], Section 2.1, in terms of elementary variational formulas involving entropies (see [3], proof of Lemmas 2.1.1–2.1.2).

Free energy along a single linear interface. To analyze the free energy per monomer in a single column we need to first analyse the free energy per monomer when the path moves in the vicinity of an AB-interface. To that end we consider a *single linear interface* $\mathcal{I}$ separating a liquid B in the lower halfplane from a liquid A in the upper halfplane (including the interface itself).

For $L \in \mathbb{N}$ and $\mu \in 1 + \frac{2\mathbb{N}}{L}$, let $\mathcal{W}_L^{\mathcal{I}}(\mu) = \mathcal{W}_L(\mu, 0)$ denote the set of μL-step directed self-avoiding paths starting at $(0,0)$ and ending at $(L,0)$. Define

$$\phi_L^{\omega,\mathcal{I}}(\mu) = \frac{1}{\mu L} \log Z_{L,\mu}^{\omega,\mathcal{I}} \quad \text{and} \quad \phi_L^{\mathcal{I}}(\mu) = \mathbb{E}[\phi_L^{\omega,\mathcal{I}}(\mu)], \tag{2.5}$$

with

$$\begin{aligned} Z_{L,\mu}^{\omega,\mathcal{I}} &= \sum_{\pi\in\mathcal{W}_L^{\mathcal{I}}(\mu)} \exp\left[H_L^{\omega,\mathcal{I}}(\pi)\right], \\ H_L^{\omega,\mathcal{I}}(\pi) &= \sum_{i=1}^{\mu L} \big(\beta\, 1\{\omega_i = B\} - \alpha\, 1\{\omega_i = A\}\big)\ 1\{(\pi_{i-1}, \pi_i) < 0\}, \end{aligned} \tag{2.6}$$

where $(\pi_{i-1}, \pi_i) < 0$ means that the i-th step lies in the lower halfplane, strictly below the interface (see Fig. 6).

The following proposition was derived in [3], Section 2.2.2.

Proposition 2.2 *For all* $(\alpha, \beta) \in \text{CONE}$ *and* $\mu \in \mathbb{Q} \cap [1, \infty)$ *there exists a* $\phi^{\mathcal{I}}(\mu) = \phi^{\mathcal{I}}(\mu; \alpha, \beta) \in \mathbb{R}$ *such that*

$$\lim_{\substack{L \to \infty \\ \mu \in 1 + \frac{2\mathbb{N}}{L}}} \phi_L^{\omega, \mathcal{I}}(\mu) = \phi^{\mathcal{I}}(\mu) = \phi^{\mathcal{I}}(\mu; \alpha, \beta) \quad \text{for } \mathbb{P}\text{-a.e. } \omega \text{ and in } L^1(\mathbb{P}). \tag{2.7}$$

It is easy to check (via concatenation of trajectories) that $\mu \mapsto \mu\phi^{\mathcal{I}}(\mu; \alpha, \beta)$ is concave. For technical reasons we need to assume that it is *strictly concave*, a property which we believe to be true but are unable to verify:

Assumption 2.3 *For all* $(\alpha, \beta) \in \text{CONE}$ *the function* $\mu \mapsto \mu\phi^{\mathcal{I}}(\mu; \alpha, \beta)$ *is strictly concave on* $[1, \infty)$.

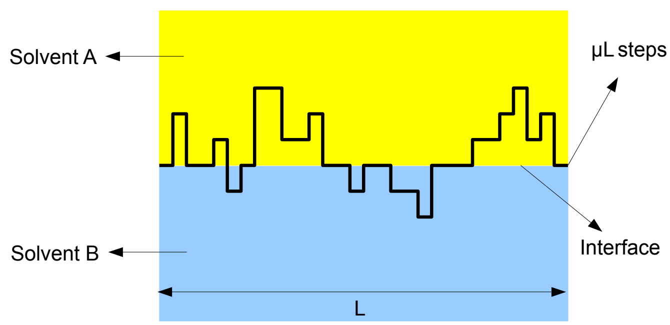

Figure 6: Copolymer near a single linear interface.

2.2 Free energy in a single column and variational formulas

In this section we use Propositions 2.1–2.2 to derive a variational formula for the free energy per step in a single column (Proposition 2.4). The variational formula comes in three varieties (Propositions 2.5 and 2.6), depending on *whether there is or is not an AB-interface between the heights where the copolymer enters and exits the column, and in the latter case whether an AB-interface is reached or not.*

In what follows we need to consider the randomness in a single column. To that aim, we recall (1.6), we pick $L \in \mathbb{N}$ and once Ω is chosen, we can record the randomness of $C_{j,L}$ as

$$\Omega_{(j,\,\cdot\,)} = \{\Omega_{(j,l)} \colon\ l \in \mathbb{Z}\}. \tag{2.8}$$

We will also need to consider the randomness of the j-th column seen by a trajectory that enters $\mathcal{C}_{j,L}$ through the block $\Lambda_{j,k}$ with $k \neq 0$ instead of $k = 0$. In this case, the randomness of $\mathcal{C}_{j,L}$ is recorded as

$$\Omega_{(j,k+\,\cdot\,)} = \{\Omega_{(j,k+l)} \colon\ l \in \mathbb{Z}\}. \tag{2.9}$$

Pick $L \in \mathbb{N}$, $\chi \in \{A, B\}^{\mathbb{Z}}$ and consider $\mathcal{C}_{0,L}$ endowed with the disorder χ, i.e., $\Omega(0, \cdot) = \chi$. Let $(n_i)_{i\in\mathbb{Z}} \in \mathbb{Z}^{\mathbb{Z}}$ be the successive heights of the AB-interfaces in $\mathcal{C}_{0,L}$ divided by L, i.e.,

$$\cdots < n_{-1} < n_0 \le 0 < n_1 < n_2 < \dots . \tag{2.10}$$

and the j-th interface of $\mathcal{C}_{0,L}$ is $\mathcal{I}_j = \{0, \dots, L\} \times \{n_j L\}$ (see Fig. 7). Next, for $r \in \mathbb{N}_0$ we set

$$k_{r,\chi} = 0 \text{ if } n_1 > r \text{ and } k_{r,\chi} = \max\{i \ge 1 \colon n_i \le r\} \text{ otherwise,} \tag{2.11}$$

while for $r \in -\mathbb{N}$ we set

$$k_{r,\chi} = 0 \text{ if } n_0 \le r \text{ and } k_{r,\chi} = \min\{i \le 0 \colon n_i \ge r+1\} - 1 \text{ otherwise.} \tag{2.12}$$

Thus, $|k_{r,\chi}|$ is the number of AB-interfaces between heigths 1 and rL in $\mathcal{C}_{0,L}$.

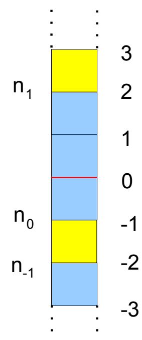

Figure 7: Example of a column with disorder $\chi = (\dots, \chi(-3), \chi(-2), \chi(-1), \chi(0), \chi(1), \chi(2), \dots) = (\dots, B, A, B, B, B, A, , \dots)$. In this example, for instance, $k_{-2,\chi} = -1$ and $k_{1,\chi} = 0$.

2.2.1 Free energy in a single column

Column crossing characteristics. Pick $L, M \in \mathbb{N}$, and consider the first column $\mathcal{C}_{0,L}$. The type of $\mathcal{C}_{0,L}$ is determined by $\Theta = (\chi, \Xi, x)$, where $\chi = (\chi_j)_{j\in\mathbb{Z}}$ encodes the type of each block in $\mathcal{C}_{0,L}$, i.e., $\chi_j = \Omega_{(0,j)}$ for $j \in \mathbb{Z}$, and (Ξ, x) indicates which trajectories π are taken into account. In the latter, Ξ is given by $(\Delta\Pi, b_0, b_1)$ such that the vertical increment in $\mathcal{C}_{0,L}$ on the block scale is $\Delta\Pi$ and satisfies $|\Delta\Pi| \le M$, i.e., π enters $\mathcal{C}_{0,L}$ at $(0, b_0 L)$ and exits $\mathcal{C}_{0,L}$ at $(L, (\Delta\Pi + b_1)L)$. As in (2.11) and (2.12), we set $k_\Theta = k_{\Delta\Pi,\chi}$ and we let $\mathcal{V}_{\text{int}}$ be the set containing those Θ satisfying $k_\Theta \neq 0$. Thus, $\Theta \in \mathcal{V}_{\text{int}}$ means that the trajectories crossing $\mathcal{C}_{0,L}$ from $(0, b_0 L)$ to $(L, (\Delta\Pi + b_1)L)$ necessarily hit an AB-interface, and in this case we set $x = 1$. If, on the other hand, $\Theta \in \mathcal{V}_{\text{nint}} = \mathcal{V} \setminus \mathcal{V}_{\text{int}}$, then we have $k_\Theta = 0$ and we set $x = 1$ when the set of trajectories crossing $\mathcal{C}_{0,L}$ from $(0, b_0 L)$ to $(L, (\Delta\Pi + b_1)L)$ is restricted to those that do not reach an AB-interface before exiting $\mathcal{C}_{0,L}$, while we set $x = 2$ when it is restricted to those trajectories that reach at least one AB-interface before exiting $\mathcal{C}_{0,L}$. To fix the possible values taken by $\Theta = (\chi, \Xi, x)$ in a column of width L, we put $\mathcal{V}_{L,M} = \mathcal{V}_{\text{int},L,M} \cup \mathcal{V}_{\text{nint},L,M}$ with

$$\begin{aligned}
\mathcal{V}_{\text{int},L,M} &= \Big\{(\chi, \Delta\Pi, b_0, b_1, x) \in \{A, B\}^{\mathbb{Z}} \times \mathbb{Z} \times \big\{\tfrac{1}{L}, \tfrac{2}{L}, \dots, 1\big\}^2 \times \{1\} \colon \\
&\qquad\qquad\qquad\qquad\qquad\qquad |\Delta\Pi| \le M,\ k_{\Delta\Pi,\chi} \neq 0\Big\}, \\
\mathcal{V}_{\text{nint},L,M} &= \Big\{(\chi, \Delta\Pi, b_0, b_1, x) \in \{A, B\}^{\mathbb{Z}} \times \mathbb{Z} \times \big\{\tfrac{1}{L}, \tfrac{2}{L}, \dots, 1\big\}^2 \times \{1, 2\} \colon \\
&\qquad\qquad\qquad\qquad\qquad\qquad |\Delta\Pi| \le M,\ k_{\Delta\Pi,\chi} = 0\Big\}.
\end{aligned} \tag{2.13}$$

Thus, the set of all possible values of Θ is $\mathcal{V}_M = \cup_{L\geq 1}\mathcal{V}_{L,M}$, which we partition into $\mathcal{V}_M = \mathcal{V}_{\mathrm{int},M} \cup \mathcal{V}_{\mathrm{nint},M}$ (see Fig. 8) with

$$\begin{aligned}\mathcal{V}_{\mathrm{int},M} &= \cup_{L\in\mathbb{N}}\, \mathcal{V}_{\mathrm{int},L,M}\\ &= \big\{(\chi,\Delta\Pi,b_0,b_1,x)\in\{A,B\}^{\mathbb{Z}}\times\mathbb{Z}\times(\mathbb{Q}_{(0,1]})^2\times\{1\}\colon\ |\Delta\Pi|\leq M,\, k_{\Delta\Pi,\chi}\neq 0\big\},\\ \mathcal{V}_{\mathrm{nint},M} &= \cup_{L\in\mathbb{N}}\, \mathcal{V}_{\mathrm{nint},L,M}\\ &= \big\{(\chi,\Delta\Pi,b_0,b_1,x)\in\{A,B\}^{\mathbb{Z}}\times\mathbb{Z}\times(\mathbb{Q}_{(0,1]})^2\times\{1,2\}\colon\ |\Delta\Pi|\leq M,\, k_{\Delta\Pi,\chi}= 0\big\},\end{aligned} \tag{2.14}$$

where, for all $I\subset\mathbb{R}$, we set $\mathbb{Q}_I = I\cap\mathbb{Q}$. We define the closure of $\mathcal{V}_M$ as $\overline{\mathcal{V}}_M = \overline{\mathcal{V}}_{\mathrm{int},M}\cup\overline{\mathcal{V}}_{\mathrm{nint},M}$ with

$$\begin{aligned}\overline{\mathcal{V}}_{\mathrm{int},M} &= \big\{(\chi,\Delta\Pi,b_0,b_1,x)\in\{A,B\}^{\mathbb{Z}}\times\mathbb{Z}\times[0,1]^2\times\{1\}\colon\ |\Delta\Pi|\leq M,\, k_{\Delta\Pi,\chi}\neq 0\big\},\\ \overline{\mathcal{V}}_{\mathrm{nint},M} &= \big\{(\chi,\Delta\Pi,b_0,b_1,x)\in\{A,B\}^{\mathbb{Z}}\times\mathbb{Z}\times[0,1]^2\times\{1,2\}\colon\ |\Delta\Pi|\leq M,\, k_{\Delta\Pi,\chi}= 0\big\}.\end{aligned} \tag{2.15}$$

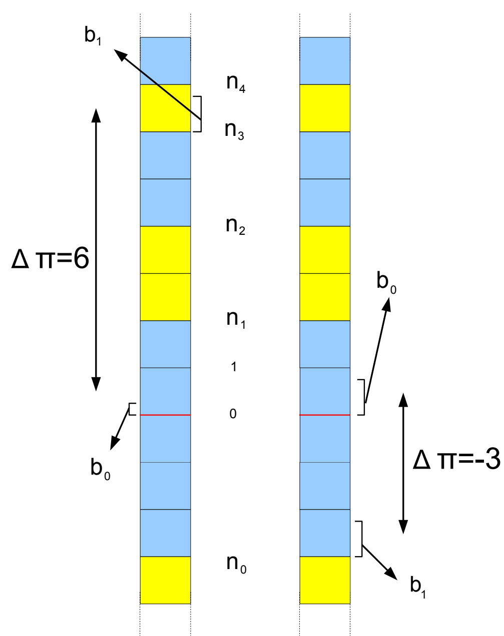

Figure 8: Labelling of coarse-grained paths and columns. On the left the type of the column is in $\mathcal{V}_{\mathrm{int},M}$, on the right it is in $\mathcal{V}_{\mathrm{nint},M}$ (with $M\geq 6$).

Time spent in columns. We pick $L,M\in\mathbb{N}$, $\Theta=(\chi,\Delta\Pi,b_0,b_1,x)\in\mathcal{V}_{L,M}$ and we specify the total number of steps that a trajectory crossing the column $\mathcal{C}_{0,L}$ of type Θ is allowed to make. For $\Theta=(\chi,\Delta\Pi,b_0,b_1,1)$, set

$$t_\Theta = 1+\mathrm{sign}(\Delta\Pi)\,(\Delta\Pi+b_1-b_0)\,1_{\{\Delta\Pi\neq 0\}}+|b_1-b_0|\,1_{\{\Delta\Pi=0\}}, \tag{2.16}$$

so that a trajectory π crossing a column of width L from $(0,b_0L)$ to $(L,(\Delta\Pi+b_1)L)$ makes a total of uL steps with $u\in t_\Theta+\frac{2\mathbb{N}}{L}$. For $\Theta=(\chi,\Delta\Pi,b_0,b_1,2)$ in turn, recall (2.10) and let

$$t_\Theta = 1+\min\{2n_1-b_0-b_1-\Delta\Pi,\, 2|n_0|+b_0+b_1+\Delta\Pi\}, \tag{2.17}$$

so that a trajectory π crossing a column of width L and type $\Theta \in \mathcal{V}_{\mathrm{nint},L,M}$ from $(0, b_0 L)$ to $(L, (\Delta\Pi + b_1)L)$ and reaching an AB-interface makes a total of uL steps with $u \in t_\Theta + \frac{2\mathbb{N}}{L}$.

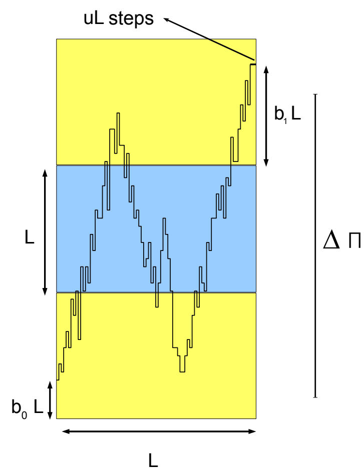

Figure 9: Example of a uL-step path inside a column of type $(\chi, \Delta\Pi, b_0, b_1, 1) \in \mathcal{V}_{\mathrm{int},L}$ with disorder $\chi = (\dots, \chi(0), \chi(1), \chi(2), \dots) = (\dots, A, B, A, \dots)$, vertical displacement $\Delta\Pi = 2$, entrance height b_0 and exit height b_1.

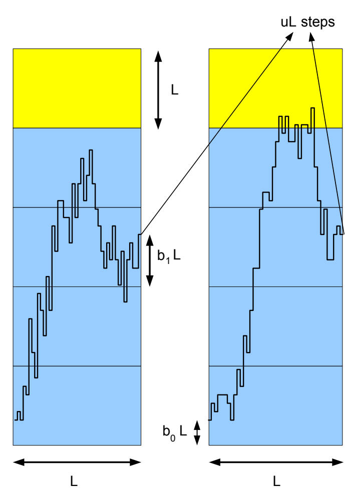

Figure 10: Two examples of a uL-step path inside a column of type $(\chi, \Delta\Pi, b_0, b_1, 1) \in \mathcal{V}_{\mathrm{nint},L}$ (left picture) and $(\chi, \Delta\Pi, b_0, b_1, 2) \in \mathcal{V}_{\mathrm{nint},L}$ (right picture) with disorder $\chi = (\dots, \chi(0), \chi(1), \chi(2), \chi(3), \chi(4), \dots) = (\dots, B, B, B, B, A, \dots)$, vertical displacement $\Delta\Pi = 2$, entrance height b_0 and exit height b_1.

At this stage, we can fully determine the set $\mathcal{W}_{\Theta,u,L}$ consisting of the uL-step trajectories π that are considered in a column of width L and type Θ. To that end, for $\Theta \in \mathcal{V}_{\mathrm{int},L,M}$ we map the trajectories $\pi \in \mathcal{W}_L(u, \Delta\Pi + b_1 - b_0)$ onto $\mathcal{C}_{0,L}$ such that π enters $\mathcal{C}_{0,L}$ at $(0, b_0 L)$ and exits $\mathcal{C}_{0,L}$ at $(L, (\Delta\Pi + b_1)L)$ (see Fig. 9), and for $\Theta \in \mathcal{V}_{\mathrm{nint},L,M}$ we remove, dependencing on $x \in \{1, 2\}$, those trajectories that reach or do not reach an AB-interface in the column (see Fig. 10). Thus, for $\Theta \in \mathcal{V}_{\mathrm{int},L,M}$ and $u \in t_\Theta + \frac{2\mathbb{N}}{L}$, we let

$$\mathcal{W}_{\Theta,u,L} = \big\{\pi = (0, b_0 L) + \widetilde{\pi} \colon\ \widetilde{\pi} \in \mathcal{W}_L(u, \Delta\Pi + b_1 - b_0)\big\}, \tag{2.18}$$

and, for $\Theta \in \mathcal{V}_{\mathrm{nint},L,M}$ and $u \in t_\Theta + \frac{2\mathbb{N}}{L}$,

$$\begin{aligned} \mathcal{W}_{\Theta,u,L} &= \big\{\pi \in (0,b_0L) + \mathcal{W}_L(u,\Delta\Pi + b_1 - b_0)\colon\ \pi \text{ reaches no } AB\text{-interface}\big\} \text{ if } x_\Theta = 1,\\ \mathcal{W}_{\Theta,u,L} &= \big\{\pi \in (0,b_0L) + \mathcal{W}_L(u,\Delta\Pi + b_1 - b_0)\colon\ \pi \text{ reaches an } AB\text{-interface}\big\} \text{ if } x_\Theta = 2, \end{aligned} \tag{2.19}$$

with x_Θ the last coordinate of $\Theta \in \mathcal{V}_M$. Next, we set

$$\begin{aligned} \mathcal{V}^*_{L,M} &= \Big\{(\Theta,u) \in \mathcal{V}_{L,M} \times [0,\infty)\colon\ u \in t_\Theta + \tfrac{2\mathbb{N}}{L}\Big\},\\ \mathcal{V}^*_{M} &= \big\{(\Theta,u) \in \mathcal{V}_{M} \times \mathbb{Q}_{[1,\infty)}\colon\ u \geq t_\Theta\big\},\\ \overline{\mathcal{V}}^*_{M} &= \big\{(\Theta,u) \in \overline{\mathcal{V}}_{M} \times [1,\infty)\colon\ u \geq t_\Theta\big\}, \end{aligned} \tag{2.20}$$

which we partition into $\mathcal{V}^*_{\mathrm{int},L,M} \cup \mathcal{V}^*_{\mathrm{nint},L,M}$, $\mathcal{V}^*_{\mathrm{int},M} \cup \mathcal{V}^*_{\mathrm{nint},M}$ and $\overline{\mathcal{V}}^*_{\mathrm{int},M} \cup \overline{\mathcal{V}}^*_{\mathrm{nint},M}$. Note that for every $(\Theta,u) \in \mathcal{V}^*_M$ there are infinitely many $L \in \mathbb{N}$ such that $(\Theta,u) \in \mathcal{V}^*_{L,M}$, because $(\Theta,u) \in \mathcal{V}^*_{qL,M}$ for all $q \in \mathbb{N}$ as soon as $(\Theta,u) \in \mathcal{V}^*_{L,M}$.

Restriction on the number of steps per column. In what follows, we set

$$\text{EIGH} = \{(M,m) \in \mathbb{N}\times\mathbb{N}\colon\ m \geq M+2\}, \tag{2.21}$$

and, for $(M,m) \in \text{EIGH}$, we consider the situation where the number of steps uL made by a trajectory π in a column of width $L \in \mathbb{N}$ is bounded by mL. Thus, we restrict the set $\mathcal{V}_{L,M}$ to the subset $\mathcal{V}^m_{L,M}$ containing only those types of columns Θ that can be crossed in less than mL steps, i.e.,

$$\mathcal{V}^m_{L,M} = \{\Theta \in \mathcal{V}_{L,M}\colon\ t_\Theta \leq m\}. \tag{2.22}$$

Note that the latter restriction only conconcerns those Θ satisfying $x_\Theta = 2$. When $x_\Theta = 1$ a quick look at (2.16) suffices to state that $t_\Theta \leq M+2 \leq m$. Thus, we set $\mathcal{V}^m_{L,M} = \mathcal{V}^m_{\mathrm{int},L,M} \cup \mathcal{V}^m_{\mathrm{nint},L,M}$ with $\mathcal{V}^m_{\mathrm{int},L,M} = \mathcal{V}_{\mathrm{int},L,M}$ and with

$$\begin{aligned} \mathcal{V}^m_{\mathrm{nint},L,M} = \Big\{\Theta \in \{A,B\}^{\mathbb{Z}} \times \mathbb{Z} \times \big\{\tfrac{1}{L},\tfrac{2}{L},\dots,1\big\}^2 \times \{1,2\}\colon&\\ |\Delta\Pi| \leq M,\ k_\Theta = 0 \ \text{ and } t_\Theta \leq m\Big\}.& \end{aligned} \tag{2.23}$$

The sets $\mathcal{V}^m_M = \mathcal{V}^m_{\mathrm{int},M} \cup \mathcal{V}^m_{\mathrm{nint},M}$ and $\overline{\mathcal{V}}^m_M = \overline{\mathcal{V}}^m_{\mathrm{int},M} \cup \overline{\mathcal{V}}^m_{\mathrm{nint},M}$ are obtained by mimicking (2.14–2.15). In the same spirit, we restrict $\mathcal{V}^*_{L,M}$ to

$$\mathcal{V}^{*,m}_{L,M} = \{(\Theta,u) \in \mathcal{V}^*_{L,M}\colon\ \Theta \in \mathcal{V}^m_{L,M}, u \leq m\} \tag{2.24}$$

and $\mathcal{V}^*_{L,M} = \mathcal{V}^*_{\mathrm{int},L,M} \cup \mathcal{V}^*_{\mathrm{nint},L,M}$ with

$$\begin{aligned} \mathcal{V}^{*,m}_{\mathrm{int},L,M} &= \Big\{(\Theta,u) \quad \in \mathcal{V}^m_{\mathrm{int},L,M} \times [1,m]\colon\ u \in t_\Theta + \tfrac{2\mathbb{N}}{L}\Big\},\\ \mathcal{V}^{*m}_{\mathrm{nint},L,M} &= \Big\{(\Theta,u) \quad \in \mathcal{V}^m_{\mathrm{nint},L,M} \times [1,m]\colon\ u \in t_\Theta + \tfrac{2\mathbb{N}}{L}\Big\}. \end{aligned} \tag{2.25}$$

We set also $\mathcal{V}^{*,m}_M = \mathcal{V}^{*,m}_{\mathrm{int},M} \cup \mathcal{V}^{*,m}_{\mathrm{nint},M}$ with $\mathcal{V}^{*,m}_{\mathrm{int},M} = \cup_{L\in\mathbb{N}} \mathcal{V}^{*,m}_{\mathrm{int},L,M}$ and $\mathcal{V}^{*,m}_{\mathrm{nint},M} = \cup_{L\in\mathbb{N}} \mathcal{V}^{*,m}_{\mathrm{nint},L,M}$, and rewrite these as

$$\begin{aligned} \mathcal{V}^{*,m}_{\mathrm{int},M} &= \big\{(\Theta,u) \in \mathcal{V}^m_{\mathrm{int},M} \times \mathbb{Q}_{[1,m]}\colon\ u \geq t_\Theta\big\},\\ \mathcal{V}^{*,m}_{\mathrm{nint},M} &= \big\{(\Theta,u) \in \mathcal{V}^m_{\mathrm{nint},M} \times \mathbb{Q}_{[1,m]}\colon\ u \geq t_\Theta\big\}. \end{aligned} \tag{2.26}$$

We further set $\overline{\mathcal{V}}^*_M = \overline{\mathcal{V}}^{*,m}_{\mathrm{int},M} \cup \overline{\mathcal{V}}^{*,m}_{\mathrm{nint},M}$ with

$$\begin{aligned} \overline{\mathcal{V}}^{*,m}_{\mathrm{int},M} &= \big\{(\Theta,u) \in \overline{\mathcal{V}}^{m}_{\mathrm{int},M} \times [1,m]\colon\ u \geq t_\Theta\big\}, \\ \overline{\mathcal{V}}^{*,m}_{\mathrm{nint},M} &= \Big\{(\Theta,u) \in \overline{\mathcal{V}}^{m}_{\mathrm{nint},M} \times [1,m]\colon\ u \geq t_\Theta\Big\}. \end{aligned} \tag{2.27}$$

Existence and uniform convergence of free energy per column. Recall (2.18), (2.19) and, for $L \in \mathbb{N}$, $\omega \in \{A,B\}^{\mathbb{N}}$ and $(\Theta,u) \in \mathcal{V}^*_{L,M}$, we associate with each $\pi \in \mathcal{W}_{\Theta,u,L}$ the energy

$$H^{\omega,\chi}_{uL,L}(\pi) = \sum_{i=1}^{uL} \big(\beta\, 1\{\omega_i = B\} - \alpha\, 1\{\omega_i = A\}\big)\, 1\Big\{\chi^L_{(\pi_{i-1},\pi_i)} = B\Big\}, \tag{2.28}$$

where $\chi^L_{(\pi_{i-1},\pi_i)}$ indicates the label of the block containing (π_{i-1},π_i) in a column with disorder χ of width L. (Recall that the disorder in the block is part of the type of the block.) The latter allows us to define the quenched free energy per monomer in a column of type Θ and size L as

$$\psi^\omega_L(\Theta,u) = \frac{1}{uL} \log Z^\omega_L(\Theta,u) \quad \text{with} \quad Z^\omega_L(\Theta,u) = \sum_{\pi \in \mathcal{W}_{\Theta,u,L}} e^{\,H^{\omega,\chi}_{uL,L}(\pi)}. \tag{2.29}$$

Abbreviate $\psi_L(\Theta,u) = \mathbb{E}[\psi^\omega_L(\Theta,u)]$, and note that for $M \in \mathbb{N}$, $m \geq M+2$ and $(\Theta,u) \in \mathcal{V}^{*,m}_{L,M}$ all $\pi \in \mathcal{W}_{\Theta,u,L}$ necessarily remain in the blocks $\Lambda_L(0,i)$ with $i \in \{-m+1,\dots,m-1\}$. Consequently, the dependence on χ of $\psi^\omega_L(\Theta,u)$ is restricted to those coordinates of χ indexed by $\{-m+1,\dots,m-1\}$. The following proposition will be proven in Section 3.

Proposition 2.4 *For every $M \in \mathbb{N}$ and $(\Theta,u) \in \mathcal{V}^*_M$ there exists a $\psi(\Theta,u) \in \mathbb{R}$ such that*

$$\lim_{\substack{L\to\infty \\ (\Theta,u)\in\mathcal{V}^*_{L,M}}} \psi^\omega_L(\Theta,u) = \psi(\Theta,u) = \psi(\Theta,u;\alpha,\beta) \quad \omega - a.s. \tag{2.30}$$

Moreover, for every $(M,m) \in$ EIGH the convergence is uniform in $(\Theta,u) \in \mathcal{V}^{*,m}_M$.

Uniform bound on the free energies. Pick $(\alpha,\beta) \in$ CONE, $n \in \mathbb{N}$, $\omega \in \{A,B\}^{\mathbb{N}}$, $\Omega \in \{A,B\}^{\mathbb{N}_0\times\mathbb{Z}}$, and let $\bar{\mathcal{W}}_n$ be any non-empty subset of $\mathcal{W}_n$ (recall (1.1)). Note that the quenched free energies per monomer introduced until now are all of the form

$$\psi_n = \tfrac{1}{n} \log \sum_{\pi \in \bar{\mathcal{W}}_n} e^{\,H_n(\pi)}, \tag{2.31}$$

where $H_n(\pi)$ may depend on ω and Ω and satisfies $-\alpha n \leq H_n(\pi) \leq \alpha n$ for all $\pi \in \bar{\mathcal{W}}_n$ (recall that $|\beta| \leq \alpha$ in CONE). Since $1 \leq |\bar{\mathcal{W}}_n| \leq |\mathcal{W}_n| \leq 3^n$, we have

$$|\psi_n| \leq \log 3 + \alpha =^{\mathrm{def}} C_{\mathrm{uf}}(\alpha). \tag{2.32}$$

The uniformity of this bound in n, ω and Ω allows us to average over ω and/or Ω or to let $n \to \infty$.

2.2.2 Variational formulas for the free energy in a single column

We next show how the free energies per column can be expressed in terms of two variational formulas involving the path entropy and the single interface free energy defined in Section 2.1. Note that $M \in \mathbb{N}$ is given until the end of the section.

Free energy in columns of class int. Pick $\Theta \in \mathcal{V}_{\mathrm{int},M}$ and put

$$\begin{aligned} l_1 &= 1_{\{\Delta\Pi>0\}}(n_1 - b_0) + 1_{\{\Delta\Pi<0\}}(b_0 - n_0), \\ l_j &= 1_{\{\Delta\Pi>0\}}(n_j - n_{j-1}) + 1_{\{\Delta\Pi<0\}}(n_{-j+2} - n_{-j+1}) \quad \text{for} \quad j \in \{2, \dots, |k_\Theta|\}, \\ l_{|k_\Theta|+1} &= 1_{\{\Delta\Pi>0\}}(\Delta\Pi + b_1 - n_{k_\Theta}) + 1_{\{\Delta\Pi<0\}}(n_{k_\Theta+1} - \Delta\Pi - b_1), \end{aligned} \tag{2.33}$$

i.e., l_1 is the vertical distance between the entrance point and the first interface, l_i is the vertical distance between the i-th interface and the $(i+1)$-th interface, and $l_{|k_\Theta|+1}$ is the vertical distance between the last interface and the exit point.

Denote by (h) and (a) the triples $(h_A, h_B, h^{\mathcal{I}})$ and $(a_A, a_B, a^{\mathcal{I}})$. For $(l_A, l_B) \in (0,\infty)^2$ and $u \geq l_A + l_B + 1$, put

$$\begin{aligned} \mathcal{L}(l_A, l_B; u) = \big\{(h),(a) \in [0,1]^3 \times [0,\infty)^3 \colon\ & h_A + h_B + h^{\mathcal{I}} = 1,\ a_A + a_B + a^{\mathcal{I}} = u \\ & a_A \geq h_A + l_A,\ a_B \geq h_B + l_B,\ a^{\mathcal{I}} \geq h^{\mathcal{I}}\big\}. \end{aligned} \tag{2.34}$$

With the help of (2.33) and (2.34) we can now provide a variational characterization of the free energy in columns of type Θ of class int. Let $l_A(\chi, \Delta\Pi, b_0, b_1)$ and $l_B(\chi, \Delta\Pi, b_0, b_1)$ correspond to the minimal vertical distance the copolymer must cross in blocks of type A and B, respectively, in a column with disorder χ when going from $(0, b_0)$ to $(1, \Delta\Pi + b_1)$, i.e.,

$$\begin{aligned} l_A(\chi, \Delta\Pi, b_0, b_1) &= 1_{\{\Delta\Pi>0\}} \sum_{j=1}^{|k_\Theta|+1} l_j 1_{\{\chi(n_{j-1})=A\}} + 1_{\{\Delta\Pi<0\}} \sum_{j=1}^{|k_\Theta|+1} l_j 1_{\{\chi(n_{-j+1})=A\}}, \\ l_B(\chi, \Delta\Pi, b_0, b_1) &= 1_{\{\Delta\Pi>0\}} \sum_{j=1}^{|k_\Theta|+1} l_j 1_{\{\chi(n_{j-1})=B\}} + 1_{\{\Delta\Pi<0\}} \sum_{j=1}^{|k_\Theta|+1} l_j 1_{\{\chi(n_{-j+1})=B\}}. \end{aligned} \tag{2.35}$$

The following proposition will be proven in Section 3.

Proposition 2.5 *For* $(\Theta, u) \in \mathcal{V}^*_{\mathrm{int},M}$,

$$\begin{aligned} \psi(\Theta, u) &= \psi_{\mathrm{int}}(u, l_A, l_B) \\ &= \sup_{(h),(a) \in \mathcal{L}(l_A, l_B; u)} \frac{a_A\, \tilde\kappa\big(\frac{a_A}{h_A}, \frac{l_A}{h_A}\big) + a_B\, \big[\tilde\kappa\big(\frac{a_B}{h_B}, \frac{l_B}{h_B}\big) + \frac{\beta-\alpha}{2}\big] + a^{\mathcal{I}}\, \phi^{\mathcal{I}}\big(\frac{a^{\mathcal{I}}}{h^{\mathcal{I}}}\big)}{u}. \end{aligned} \tag{2.36}$$

Free energy in columns of class nint. Pick $\Theta \in \mathcal{V}_{\mathrm{nint},M}$. In this case, there is no AB-interface between b_0 and $\Delta\Pi + b_1$, which means that $\Delta\Pi < n_1$ if $\Delta\Pi \geq 0$ and $\Delta\Pi \geq n_0$ if $\Delta\Pi < 0$ (n_0 and n_1 being defined in (2.10)). Let $l_{\mathrm{nint}}(\Delta\Pi, b_0, b_1)$ be the vertical distance between the entrance point $(0, b_0)$ and the exit point $(1, \Delta\Pi + b_1)$, i.e.,

$$l_{\mathrm{nint}}(\Delta\Pi, b_0, b_1) = 1_{\{\Delta\Pi\geq0\}}(\Delta\Pi - b_0 + b_1) + 1_{\{\Delta\Pi<0\}}(|\Delta\Pi| + b_0 - b_1) + 1_{\{\Delta\Pi=0\}}|b_1 - b_0|, \tag{2.37}$$

and let $l_{\rm int}(\chi, \Delta\Pi, b_0, b_1)$ be the minimal vertical distance a trajectory has to cross in a column with disorder χ, starting from $(0, b_0)$, to reach the closest AB-interface before exiting at $(1, \Delta\Pi + b_1)$, i.e.,

$$l_{\rm int}(\chi, \Delta\Pi, b_0, b_1) = \min\{2n_1 - b_0 - b_1 - \Delta\Pi, 2|n_0| + b_0 + b_1 + \Delta\Pi\}. \tag{2.38}$$

The following proposition will be proved in Section 3.

Proposition 2.6 *For $(\Theta, u) \in \mathcal{V}^*_{{\rm nint},M}$ such that $x_\Theta = 1$,*

$$\psi(\Theta, u) = \tilde{\kappa}(u, l_{\rm nint}) + \tfrac{\beta-\alpha}{2} 1_{\{\chi(0)=B\}}. \tag{2.39}$$

For $(\Theta, u) \in \mathcal{V}^*_{{\rm nint},M}$ such that $x_\Theta = 2$,

$$\begin{aligned} \psi(\Theta, u) &= \psi_{\rm nint}(u, l_{\rm int};\ \chi(0)) \\ &= \sup_{\substack{h^{\mathcal{I}} \in [0,1], \\ u^{\mathcal{I}} \in [h^{\mathcal{I}}, u+h^{\mathcal{I}}-1-l_{\rm int}]}} \frac{(u-u^{\mathcal{I}})\big[\tilde{\kappa}\big(\frac{u-u^{\mathcal{I}}}{1-h^{\mathcal{I}}}, \frac{l_{\rm int}}{1-h^{\mathcal{I}}}\big) + \frac{\beta-\alpha}{2} 1_{\{\chi(0)=B\}}\big] + u^{\mathcal{I}} \phi^{\mathcal{I}}\big(\frac{u^{\mathcal{I}}}{h^{\mathcal{I}}}\big)}{u}. \end{aligned} \tag{2.40}$$

The importance of Propositions 2.5–2.6 is that they *express the free energy in a single column in terms of the path entropy in a single column $\tilde{\kappa}$ and the free energy along a single linear interface $\phi^{\mathcal{I}}$*, which were defined in Section 2.1 and are well understood.

2.3 Mesoscopic percolation frequencies

In this section, we define a set of probability laws providing the frequencies with which each type of column can be crossed by the copolymer.

Coarse-grained paths. For $x \in \mathbb{N}_0 \times \mathbb{Z}$ and $n \in \mathbb{N}$, let $c_{x,n}$ denote the center of the block $\Lambda_{L_n}(x)$ defined in (1.3), i.e.,

$$c_{x,n} = xL_n + (\tfrac{1}{2}, \tfrac{1}{2})L_n, \tag{2.41}$$

and abbreviate

$$(\mathbb{N}_0 \times \mathbb{Z})_n = \{c_{x,n} \colon\ x \in \mathbb{N}_0 \times \mathbb{Z}\}. \tag{2.42}$$

Let $\widehat{\mathcal{W}}$ be the set of *coarse-grained paths* on $(\mathbb{N}_0 \times \mathbb{Z})_n$ that start at $c_{0,n}$, are self-avoiding and are allowed to jump up, down and to the right between neighboring sites of $(\mathbb{N}_0 \times \mathbb{Z})_n$, i.e., the increments of $\widehat{\Pi} = (\widehat{\Pi}_j)_{j \in \mathbb{N}_0} \in \widehat{\mathcal{W}}$ are $(0, L_n)$, $(0, -L_n)$ and $(L_n, 0)$. (These paths are the coarse-grained counterparts of the paths π introduced in (1.1).) For $l \in \mathbb{N} \cup \{\infty\}$, let $\widehat{\mathcal{W}}_l$ be the set of l-step coarse-grained paths.

Recall, for $\pi \in \mathcal{W}_n$, the definitions of N_π and $(v_j(\pi))_{j \leq N_\pi - 1}$ given below (1.7). With π we associate a coarse-grained path $\widehat{\Pi} \in \widehat{\mathcal{W}}_{N_\pi}$ that describes how π moves with respect to the blocks. The construction of $\widehat{\Pi}$ is done as follows: $\widehat{\Pi}_0 = c_{(0,0)}$, $\widehat{\Pi}$ moves vertically until it reaches $c_{(0,v_0)}$, moves one step to the right to $c_{(1,v_0)}$, moves vertically until it reaches $c_{(1,v_1)}$, moves one step to the right to $c_{(2,v_1)}$, and so on. The vertical increment of $\widehat{\Pi}$ in the j-th column is $\Delta\widehat{\Pi}_j = (v_j - v_{j-1})L_n$ (see Figs. 8–10).

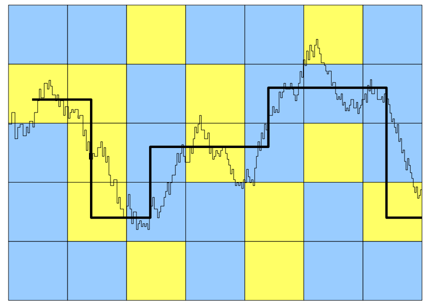

Figure 11: Example of a coarse-grained path.

To characterize a path π, we will often use the sequence of vertical increments of its associated coarse-grained path $\widehat{\Pi}$, modified in such a way that it does not depend on L_n anymore. To that end, with every $\pi \in \mathcal{W}_n$ we associate $\Pi = (\Pi_k)_{k=0}^{N_\pi - 1}$ such that $\Pi_0 = 0$ and,

$$\Pi_k = \sum_{j=0}^{k-1} \Delta\Pi_j \quad \text{with} \quad \Delta\Pi_j = \frac{1}{L_n}\Delta\widehat{\Pi}_j, \qquad j = 0, \dots, N_\pi - 1. \tag{2.43}$$

Pick $M \in \mathbb{N}$ and note that $\pi \in \mathcal{W}_{n,M}$ if and only if $|\Delta\Pi_j| \leq M$ for all $j \in \{0, \dots, N_\pi - 1\}$.

Percolation frequencies along coarse-grained paths. Given $M \in \mathbb{N}$, we denote by $\mathcal{M}_1(\overline{\mathcal{V}}_M)$ the set of probability measures on $\overline{\mathcal{V}}_M$. Pick $\Omega \in \{A, B\}^{\mathbb{N}_0 \times \mathbb{Z}}$, $\Pi \in \mathbb{Z}^{\mathbb{N}_0}$ such that $\Pi_0 = 0$ and $|\Delta\Pi_i| \leq M$ for all $i \geq 0$ and $b = (b_j)_{j \in \mathbb{N}_0} \in (\mathbb{Q}_{(0,1]})^{\mathbb{N}_0}$. Set $\Theta_{\text{traj}} = (\Xi_j)_{j \in \mathbb{N}_0}$ with

$$\Xi_j = \big(\Delta\Pi_j, b_j, b_{j+1}\big), \qquad j \in \mathbb{N}_0, \tag{2.44}$$

let

$$\mathcal{X}_{\Pi,\Omega} = \big\{x \in \{1,2\}^{\mathbb{N}_0} \colon\ (\Omega(i, \Pi_i + \cdot), \Xi_i, x_i) \in \mathcal{V}_M \ \ \forall\, i \in \mathbb{N}_0\big\}, \tag{2.45}$$

and for $x \in \mathcal{X}_{\Pi,\Omega}$ set

$$\Theta_j = \big(\Omega(j, \Pi_j + \cdot), \Delta\Pi_j, b_j, b_{j+1}, x_j\big), \qquad j \in \mathbb{N}_0. \tag{2.46}$$

With the help of (2.46), we can define the empirical distribution

$$\rho_N(\Omega, \Pi, b, x)(\Theta) = \frac{1}{N}\sum_{j=0}^{N-1} 1_{\{\Theta_j = \Theta\}}, \quad N \in \mathbb{N},\ \Theta \in \overline{\mathcal{V}}_M, \tag{2.47}$$

Definition 2.7 *For $\Omega \in \{A, B\}^{\mathbb{N}_0 \times \mathbb{Z}}$ and $M \in \mathbb{N}$, let*

$$\begin{aligned} \mathcal{R}^{\Omega}_{M,N} = \big\{ &\rho_N(\Omega, \Pi, b, x) \text{ with } b = (b_j)_{j \in \mathbb{N}_0} \in (\mathbb{Q}_{(0,1]})^{\mathbb{N}_0}, \\ &\Pi = (\Pi_j)_{j \in \mathbb{N}_0} \in \{0\} \times \mathbb{Z}^{\mathbb{N}} \colon\ |\Delta\Pi_j| \leq M \quad \forall\, j \in \mathbb{N}_0, \\ &x = (x_j)_{j \in \mathbb{N}_0} \in \{1,2\}^{N_0} \colon\ \big(\Omega(j, \Pi_j + \cdot), \Delta\Pi_j, b_j, b_{j+1}, x_j\big) \in \mathcal{V}_M \big\} \end{aligned} \tag{2.48}$$

and

$$\mathcal{R}^{\Omega}_M = \text{closure}\Big(\cap_{N' \in \mathbb{N}} \cup_{N \geq N'} \mathcal{R}^{\Omega}_{M,N} \Big), \tag{2.49}$$

both of which are subsets of $\mathcal{M}_1(\overline{\mathcal{V}}_M)$.

Proposition 2.8 *For every $p \in (0,1)$ and $M \in \mathbb{N}$ there exists a closed set $\mathcal{R}_{p,M} \subsetneq \mathcal{M}_1(\overline{\mathcal{V}}_M)$ such that*

$$\mathcal{R}_M^\Omega = \mathcal{R}_{p,M} \text{ for } \mathbb{P}\text{-a.e.}\,\Omega. \tag{2.50}$$

Proof. Note that, for every $\Omega \in \{A,B\}^{\mathbb{N}_0 \times \mathbb{Z}}$, the set $\mathcal{R}_M^\Omega$ does not change when finitely many variables in Ω are changed. Therefore $\mathcal{R}_M^\Omega$ is measurable with respect to the tail σ-algebra of Ω. Since Ω is an i.i.d. random field, the claim follows from Kolmogorov's zero-one law. Because of the constraint on the vertical displacement, $\mathcal{R}_{p,M}$ does not coincide with $\mathcal{M}_1(\overline{\mathcal{V}}_M)$. □

3 Proof of Propositions 2.4–2.6

In this section we prove Propositions 2.4 and 2.5–2.6, which were stated in Sections 2.1–2.3 and contain the precise definition of the key ingredients of the variational formula in Theorem 1.1. In Section 4 we will use these propositions to prove Theorem 1.1.

In Section 3.1 we associate with each trajectory π in a column a sequence recording the indices of the AB-interfaces successively visited by π. The latter allows us to state a key proposition, Proposition 3.1 below, from which Propositions 2.4 and 2.5–2.6 are straightforward consequences. In Section 3.2 we give an outline of the proof of Proposition 3.1, in Sections 3.3–3.5 we provide the details.

3.1 Column crossing characteristic

3.1.1 The order of the visits to the interfaces

Pick $(M,m) \in$ EIGH. To prove Proposition 2.4, instead of considering $(\Theta,u) \in \mathcal{V}_M^{*,m}$, we will restrict to $(\Theta,u) \in \mathcal{V}_{\text{int},M}^{*,m}$. Our proof can be easily extended to $(\Theta,u) \in \mathcal{V}_{\text{nint},M}^{*,m}$.

Pick $(\Theta,u) \in \mathcal{V}_{\text{int},M}^{*,m}$, recall (2.10) and set $\mathcal{J}_{\Theta,u} = \{\mathcal{N}_{\Theta,u}^{\downarrow}, \dots, \mathcal{N}_{\Theta,u}^{\uparrow}\}$, with

$$\begin{aligned} \mathcal{N}_{\Theta,u}^{\uparrow} &= \max\{i \geq 1 \colon n_i \leq u\} \quad \text{and} \quad \mathcal{N}_{\Theta,u}^{\uparrow} = 0 \quad \text{if} \quad n_1 > u. \\ \mathcal{N}_{\Theta,u}^{\downarrow} &= \min\{i \leq 0 \colon |n_i| \leq u\} \quad \text{and} \quad \mathcal{N}_{\Theta,u}^{\downarrow} = 1 \quad \text{if} \quad |n_0| > u. \end{aligned} \tag{3.1}$$

Next pick $L \in \mathbb{N}$ so that $(\Theta,u) \in \mathcal{V}_{\text{int},L,M}^{*}$ and recall that for $j \in \mathcal{J}_{\Theta,u}$ the j-th interface of the Θ-column is $\mathcal{I}_j = \{0,\dots,L\} \times \{n_j L\}$. Note also that $\pi \in \mathcal{W}_{\Theta,u,L}$ makes uL steps inside the column and therefore can not reach the AB-interfaces labelled outside $\{\mathcal{N}_{\Theta,u}^{\downarrow}, \dots, \mathcal{N}_{\Theta,u}^{\uparrow}\}$.

First, we associate with each trajectory $\pi \in \mathcal{W}_{\Theta,u,L}$ the sequence $J(\pi)$ that records the indices of the interfaces that are successively visited by π. Next, we pick $\pi \in \mathcal{W}_{\Theta,u,L}$, and define τ_1, J_1 as

$$\tau_1 = \inf\{i \in \mathbb{N} \colon \exists j \in \mathcal{J}_{\Theta,u} \colon \pi_i \in \mathcal{I}_j\}, \qquad \pi_{\tau_1} \in \mathcal{I}_{J_1}, \tag{3.2}$$

so that $J_1 = 0$ (respectively, $J_1 = 1$) if the first interface reached by π is $\mathcal{I}_0$ (respectively, $\mathcal{I}_1$). For $i \in \mathbb{N} \setminus \{1\}$, we define τ_i, J_i as

$$\tau_i = \inf\big\{t > \tau_{i-1} \colon \exists j \in \mathcal{J}_{\Theta,u} \setminus \{J_{i-1}\}, \pi_i \in \mathcal{I}_j\big\}, \qquad \pi_{\tau_i} \in \mathcal{I}_{J_i}, \tag{3.3}$$

so that the increments of $J(\pi)$ are restricted to -1 or 1. The length of $J(\pi)$ is denoted by $m(\pi)$ and corresponds to the number of jumps made by π between neighboring interfaces before time uL, i.e., $J(\pi)=(J_i)_{i=1}^{m(\pi)}$ with

$$m(\pi)=\max\{i\in\mathbb{N}\colon\ \tau_i\leq uL\}. \tag{3.4}$$

Note that $(\Theta,u)\in\mathcal{V}^{*,m}_{\mathrm{int},M}$ necessarily implies $k_\Theta\leq m(\pi)\leq u\leq m$. Set

$$\mathcal{S}_r=\{j=(j_i)_{i=1}^r\in\mathbb{Z}^{\mathbb{N}}\colon\ j_1\in\{0,1\},\, j_{i+1}-j_i\in\{-1,1\}\ \forall\, 1\leq i\leq r-1\},\qquad r\in\mathbb{N}, \tag{3.5}$$

and, for $\Theta\in\mathcal{V}$, $r\in\{1,\dots,m\}$ and $j\in\mathcal{S}_r$, define

$$\begin{aligned} l_1&=1_{\{j_1=1\}}(n_1-b_0)+1_{\{j_1=0\}}(b_0-n_0),\\ l_i&=|n_{j_i}-n_{j_{i-1}}|\ \text{for}\ i\in\{2,\dots,r\},\\ l_{r+1}&=1_{\{j_r=k_\Theta+1\}}(n_{k_\Theta+1}-\Delta\Pi-b_1)+1_{\{j_r=k_\Theta\}}(\Delta\Pi+b_1-n_{k_\Theta}), \end{aligned} \tag{3.6}$$

so that $(l_i)_{i\in\{1,\dots,r+1\}}$ depends on Θ and j. Set

$$\begin{aligned} \mathcal{A}_{\Theta,j}&=\{i\in\{1,\dots,r+1\}\colon\ A\ \text{between}\,\mathcal{I}_{j_{i-1}}\,\text{and}\,\mathcal{I}_{j_i}\},\\ \mathcal{B}_{\Theta,j}&=\{i\in\{1,\dots,r+1\}\colon\ B\ \text{between}\,\mathcal{I}_{j_{i-1}}\,\text{and}\,\mathcal{I}_{j_i}\}, \end{aligned} \tag{3.7}$$

and set $l_{\Theta,j}=(l_{A,\Theta,j},l_{B,\Theta,j})$ with

$$l_{A,\Theta,j}=\textstyle\sum_{i\in\mathcal{A}_{\Theta,j}}l_i,\ \ l_{B,\Theta,j}=\sum_{i\in\mathcal{B}_{\Theta,j}}l_i. \tag{3.8}$$

For $L\in\mathbb{N}$ and $(\Theta,u)\in\mathcal{V}^{*,m}_{\mathrm{int},L,M}$, we denote by $\mathcal{S}_{\Theta,u,L}$ the set $\{J(\pi),\pi\in\mathcal{W}_{\Theta,u,L}\}$. It is not difficult to see that a sequence $j\in\mathcal{S}_r$ belongs to $\mathcal{S}_{\Theta,u,L}$ if and only if it satisfies the two following conditions. First, $j_r\in\{k_\Theta,k_\Theta+1\}$, since j_r is the index of the interface last visited before the Θ-column is exited. Second, $u\geq 1+l_{A,\Theta,j}+l_{B,\Theta,j}$ because the number of steps taken by a trajectory $\pi\in\mathcal{W}_{\Theta,u,L}$ satisfying $J(\pi)=j$ must be large enough to ensures that all interfaces $\mathcal{I}_{j_s}$, $s\in\{1,\dots,r\}$, can be visited by π before time uL. Consequently, $\mathcal{S}_{\Theta,u,L}$ does not depend on L and can be written as $\mathcal{S}_{\Theta,u}=\cup_{r=1}^m\mathcal{S}_{\Theta,u,r}$, where

$$\mathcal{S}_{\Theta,u,r}=\{j\in\mathcal{S}_r\colon j_r\in\{k_\Theta,k_\Theta+1\},u\geq 1+l_{A,\Theta,j}+l_{B,\Theta,j}\}. \tag{3.9}$$

Thus, we partition $\mathcal{W}_{\Theta,u,L}$ according to the value taken by $J(\pi)$, i.e.,

$$\mathcal{W}_{\Theta,u,L}=\bigcup_{r=1}^m\ \bigcup_{j\in\mathcal{S}_{\Theta,u,r}}\mathcal{W}_{\Theta,u,L,j}, \tag{3.10}$$

where $\mathcal{W}_{\Theta,u,L,j}$ contains those trajectories $\pi\in\mathcal{W}_{\Theta,u,L}$ for which $J(\pi)=j$.

Next, for $j\in\mathcal{S}_{\Theta,u}$, we define (recall (2.28))

$$\psi_L^\omega(\Theta,u,j)=\frac{1}{uL}\log Z_L^\omega(\Theta,u,j),\qquad \psi_L(\Theta,u,j)=\mathbb{E}\big[\psi_L^\omega(\Theta,u,j)\big], \tag{3.11}$$

with

$$Z_L^\omega(\Theta,u,j)=\sum_{\pi\in\mathcal{W}_{\Theta,u,L,j}}e^{H^{\omega,\chi}_{uL,L}(\pi)}. \tag{3.12}$$

For each $L\in\mathbb{N}$ satisfying $(\Theta,u)\in\mathcal{V}^{*,m}_{\mathrm{int},L,M}$ and each $j\in\mathcal{S}_{\Theta,u}$, the quantity $l_{A,\Theta,j}L$ (respectively, $l_{B,\Theta,j}L$) corresponds to the minimal vertical distance a trajectory $\pi\in\mathcal{W}_{\Theta,u,L,j}$ has to cross in solvent A (respectively, B).

3.1.2 Key proposition

Recalling (2.36) and (3.8), we define the free energy associated with Θ, u, j as

$$\begin{aligned}\psi(\Theta,u,j) &= \psi_{\rm int}(u,l_{\Theta,j}) \\ &= \sup_{(h),(u)\in\mathcal{L}(l_{\Theta,j};\,u)} \frac{u_A\,\tilde\kappa\big(\frac{u_A}{h_A},\frac{l_{A,\Theta,j}}{h_A}\big)+u_B\,\big[\tilde\kappa\big(\frac{u_B}{h_B},\frac{l_{B,\Theta,j}}{h_B}\big)+\frac{\beta-\alpha}{2}\big]+u_I\,\phi\big(\frac{u^I}{h^I}\big)}{u}.\end{aligned} \tag{3.13}$$

Proposition 3.1 below states that $\lim_{L\to\infty}\psi_L(\Theta,u,j)=\psi(\Theta,u,j)$ uniformly in $(\Theta,u)\in\mathcal{V}^{*,m}_{{\rm int},M}$ and $j\in\mathcal{S}_{\Theta,u}$.

Proposition 3.1 *For every $M,m\in\mathbb{N}$ such that $m\geq M+2$ and every $\varepsilon>0$ there exists an $L_\varepsilon\in\mathbb{N}$ such that*

$$\big|\psi_L(\Theta,u,j)-\psi(\Theta,u,j)\big|\leq\varepsilon\quad\forall\,(\Theta,u)\in\mathcal{V}^{*,m}_{{\rm int},L,M},\;\; j\in\mathcal{S}_{\Theta,u},\;\; L\geq L_\varepsilon. \tag{3.14}$$

Proof of Propositions 2.4 and 2.5–2.6 subject to Proposition 3.1. Pick $\varepsilon>0$, $L\in\mathbb{N}$ and $(\Theta,u)\in\mathcal{V}^{*,m}_{{\rm int},L,M}$. Recall (2.35) and note that $l_A(\Theta)L$ and $l_B(\Theta)L$ are the minimal vertical distances the trajectories of $\mathcal{W}_{\Theta,u,L}$ have to cross in blocks of type A, respectively, B. For simplicity, in what follows the Θ-dependence of l_A and l_B will be suppressed. In other words, l_A and l_B are the two coordinates of $l_{\Theta,f}$ (recall (3.8)) with $f=(1,2,\dots,|k_\Theta|)$ when $\Delta\Pi\geq 0$ and $f=(0,-1,\dots,-|k_\Theta|+1)$ when $\Delta\Pi<0$, so (2.36) and (3.13) imply

$$\psi_{\rm int}(u,l_A,l_B)=\psi(\Theta,u,f). \tag{3.15}$$

Hence Propositions 2.4 and 2.5 will be proven once we show that $\lim_{L\to\infty}\psi_L(\Theta,u)=\psi(\Theta,u,f)$ uniformly in $(\Theta,u)\in\mathcal{V}^{*,m}_{{\rm int},L,M}$. Moreover, a look at (3.13), (3.15) and (2.36) allows us to assert that for every $j\in\mathcal{S}_{\Theta,u}$ we have $\psi(\Theta,u,j)\leq\psi(\Theta,u,f)$. The latter is a consequence of the fact that $l\mapsto\tilde\kappa(u,l)$ decreases on $[0,u-1]$ (see Lemma A.5(ii) in Appendix A) and that

$$\begin{aligned} l_A &= l_{A,\Theta,f}=\min\{l_{A,\Theta,j}\colon\, j\in\mathcal{S}_{\Theta,u}\},\\ l_B &= l_{B,\Theta,f}=\min\{l_{B,\Theta,j}\colon\, j\in\mathcal{S}_{\Theta,u}\}.\end{aligned} \tag{3.16}$$

By applying Proposition 3.1 we have, for $L\geq L_\varepsilon$,

$$\begin{aligned}\psi_L(\Theta,u,j)&\leq\psi(\Theta,u,f)+\varepsilon\qquad\forall\,(\Theta,u)\in\mathcal{V}^{*,m}_{{\rm int},L,M},\;\forall\,j\in\mathcal{S}_{\Theta,u},\\ \psi_L(\Theta,u,f)&\geq\psi(\Theta,u,f)-\varepsilon\qquad\forall\,(\Theta,u)\in\mathcal{V}^{*,m}_{{\rm int},L,M}.\end{aligned} \tag{3.17}$$

The second inequality in (3.17) allows us to write, for $L\geq L_\varepsilon$,

$$\psi(\Theta,u,f)-\varepsilon\leq\psi_L(\Theta,u,f)\leq\psi_L(\Theta,u)\qquad\forall\,(\Theta,u)\in\mathcal{V}^{*,m}_{{\rm int},L,M}. \tag{3.18}$$

To obtain the upper bound we introduce

$$\mathcal{A}_{L,\varepsilon}=\Big\{\omega\colon\,|\psi^\omega_L(\Theta,u,j)-\psi_L(\Theta,u,j)|\leq\varepsilon\quad\forall\,(\Theta,u)\in\mathcal{V}^{*,m}_{{\rm int},L,M},\,\forall\,j\in\mathcal{S}_{\Theta,u}\Big\}, \tag{3.19}$$

so that

$$\begin{aligned}\psi_L(\Theta,u)&\leq\mathbb{E}\big[1_{\mathcal{A}^c_{L,\varepsilon}}\,\psi^\omega_L(\Theta,u)\big]+\mathbb{E}\big[1_{\mathcal{A}_{L,\varepsilon}}\,\psi^\omega_L(\Theta,u)\big]\\ &\leq C_{\rm uf}(\alpha)\,\mathbb{P}(\mathcal{A}^c_{L,\varepsilon})+\tfrac{1}{uL}\mathbb{E}\Big[1_{\mathcal{A}_{L,\varepsilon}}\,\log\textstyle\sum_{j\in\mathcal{S}_{\Theta,u}}e^{uL(\psi_L(\Theta,u,j)+\varepsilon)}\Big],\end{aligned} \tag{3.20}$$

where we use (2.32) to bound the first term in the right-hand side, and the definition of $\mathcal{A}_{L,\varepsilon}$ to bound the second term. Next, with the help of the first inequality in (3.17) we can rewrite (3.20) for $L \geq L_\varepsilon$ and $(\Theta, u) \in \mathcal{V}^{*,m}_{\text{int},L,M}$ in the form

$$\psi_L(\Theta, u) \leq C_{\text{uf}}(\alpha)\, \mathbb{P}(\mathcal{A}^c_{L,\varepsilon}) + \tfrac{1}{uL} \log |\cup_{r=1}^m \mathcal{S}_r| + \psi(\Theta, u, f) + 2\varepsilon. \tag{3.21}$$

At this stage we want to prove that $\lim_{L\to\infty} \mathbb{P}(\mathcal{A}^c_{L,\varepsilon}) = 0$. To that end, we use the concentration of measure property in (C.3) in Appendix C with $l = uL$, $\Gamma = \mathcal{W}_{\Theta,u,L,j}$, $\eta = \varepsilon uL$, $\xi_i = -\alpha 1\{\omega_i = A\} + \beta 1\{\omega_i = B\}$ for all $i \in \mathbb{N}$ and $T(x, y) = 1\{\chi^{L_n}_{(x,y)} = B\}$. We then obtain that there exist $C_1, C_2 > 0$ such that, for all $L \in \mathbb{N}$, $(\Theta, u) \in \mathcal{V}^{*,m}_{\text{int},L,M}$ and $j \in \mathcal{S}_{\Theta,u}$,

$$\mathbb{P}\big(|\psi^\omega_L(\Theta, u, j) - \psi_L(\Theta, u, j)| > \varepsilon\big) \leq C_1\, e^{-C_2\, \varepsilon^2\, uL}. \tag{3.22}$$

The latter inequality, combined with the fact that $|\mathcal{V}^{*,m}_{\text{int},L,M}|$ grows polynomialy in L, allows us to assert that $\lim_{L\to\infty} \mathbb{P}(\mathcal{A}^c_{L,\varepsilon}) = 0$. Next, we note that $|\cup_{r=1}^m \mathcal{S}_r| < \infty$, so that for L_ε large enough we obtain from (3.21) that, for $L \geq L_\varepsilon$,

$$\psi_L(\Theta, u) \leq \psi(\Theta, u, f) + 3\varepsilon \qquad \forall\, (\Theta, u) \in \mathcal{V}^{*,m}_{\text{int},L,M}. \tag{3.23}$$

Now (3.18) and (3.23) are sufficient to complete the proof of Propositions 2.4–2.5. The proof of Proposition 2.6 follows in a similar manner after minor modifications. □

3.2 Structure of the proof of Proposition 3.1

Intermediate column free energies. Let

$$G^m_M = \big\{(L, \Theta, u, j)\colon\ (\Theta, u) \in \mathcal{V}^{*,m}_{\text{int},L,M},\ j \in \mathcal{S}_{\Theta,u}\big\}, \tag{3.24}$$

and define the following order relation.

Definition 3.2 *For $g, \widetilde{g}\colon\ G^m_M \mapsto \mathbb{R}$, write $g \prec \widetilde{g}$ when for every $\varepsilon > 0$ there exists an $L_\varepsilon \in \mathbb{N}$ such that*

$$g(L, \Theta, u, j) \leq \widetilde{g}(L, \Theta, u, j) + \varepsilon \qquad \forall\, (L, \Theta, u, j) \in G^m_M\colon\ L \geq L_\varepsilon. \tag{3.25}$$

Recall (3.11) and (3.13), set

$$\psi_1(L, \Theta, u, j) = \psi_L(\Theta, u, j), \qquad \psi_4(L, \Theta, u, j) = \psi(\Theta, u, j), \tag{3.26}$$

and note that the proof of Proposition 3.1 will be complete once we show that $\psi_1 \prec \psi_4$ and $\psi_4 \prec \psi_1$. In what follows, we will focus on $\psi_1 \prec \psi_4$. Each step of the proof can be adapted to obtain $\psi_4 \prec \psi_1$ without additional difficulty.

In the proof we need to define two intermediate free energies ψ_2 and ψ_3, in addition to ψ_1 and ψ_4 above. Our proof is divided into 3 steps, organized in Sections 3.3–3.5, and consists of showing that $\psi_1 \prec \psi_2 \prec \psi_3 \prec \psi_4$.

Additional notation. Before stating Step 1, we need some further notation. First, we partition $\mathcal{W}_{\Theta,u,L,j}$ according to the total number of steps and the number of horizontal steps made by a trajectory along and in between AB-interfaces. To that end, we assume that $j \in \mathcal{S}_{\Theta,u,r}$ with $r \in \{1, \dots, m\}$, we recall (3.6) and we let

$$\begin{aligned} \mathcal{D}_{\Theta,L,j} &= \big\{(d_i, t_i)_{i=1}^{r+1}\colon\ d_i \in \mathbb{N} \text{ and } t_i \in d_i + l_i L + 2\mathbb{N}_0\ \forall\, 1 \leq i \leq r+1\big\}, \\ \mathcal{D}^{\mathcal{I}}_r &= \big\{(d^{\mathcal{I}}_i, t^{\mathcal{I}}_i)_{i=1}^{r}\colon\ d^{\mathcal{I}}_i \in \mathbb{N} \text{ and } t^{\mathcal{I}}_i \in d^{\mathcal{I}}_i + 2\mathbb{N}_0\ \forall\, 1 \leq i \leq r\big\}, \end{aligned} \tag{3.27}$$

where d_i, t_i denote the number of horizontal steps and the total number of steps made by the trajectory between the $(i-1)$-th and i-th interfaces, and $d_i^{\mathcal{I}}, t_i^{\mathcal{I}}$ denote the number of horizontal steps and the total number of steps made by the trajectory along the i-th interface. For $(d,t) \in \mathcal{D}_{\Theta,L,j}$, $(d^{\mathcal{I}}, t^{\mathcal{I}}) \in \mathcal{D}_r^{\mathcal{I}}$ and $1 \leq i \leq r$, we set $T_0 = 0$ and

$$V_i = \sum_{j=1}^{i} t_j + \sum_{j=1}^{i-1} t_j^{\mathcal{I}}, \qquad i = 1, \dots, r,$$
$$T_i = \sum_{j=1}^{i} t_j + \sum_{j=1}^{i} t_j^{\mathcal{I}}, \qquad i = 1, \dots, r, \tag{3.28}$$

so that V_i, respectively, T_i indicates the number of steps made by the trajectory when reaching, respectively, leaving the i-th interface.

Next, we let $\theta \colon \mathbb{R}^{\mathbb{N}} \mapsto \mathbb{R}^{\mathbb{N}}$ be the left-shift acting on infinite sequences of real numbers and, for $u \in \mathbb{N}$ and $\omega \in \{A, B\}^{\mathbb{N}}$, we put

$$H_u^{\omega}(B) = \sum_{i=1}^{u} \big[\beta \, 1_{\{\omega_i = B\}} - \alpha \, 1_{\{\omega_i = A\}}\big]. \tag{3.29}$$

Finally, we recall that

$$\psi_1(L, \Theta, u, j) = \tfrac{1}{uL} \, \mathbb{E}[\log Z_1^{\omega}(L, \Theta, u, j)], \tag{3.30}$$

where the partition function defined in (2.29) has been renamed Z_1 and can be written in the form

$$Z_1^{\omega}(L, \Theta, u, j) = \sum_{(d,t) \in \mathcal{D}_{\Theta,L,j}} \; \sum_{(d^{\mathcal{I}}, t^{\mathcal{I}}) \in \mathcal{D}_r^{\mathcal{I}}} A_1 \, B_1 \, C_1, \tag{3.31}$$

where (recall (3.7) and (2.5))

$$A_1 = \prod_{i \in \mathcal{A}_{\Theta,j}} e^{t_i \, \tilde{\kappa}_{d_i}\left(\frac{t_i}{d_i}, \frac{l_i L}{d_i}\right)} \prod_{i \in \mathcal{B}_{\Theta,j}} e^{t_i \, \tilde{\kappa}_{d_i}\left(\frac{t_i}{d_i}, \frac{l_i L}{d_i}\right)} \, e^{H_{t_i}^{\theta^{T_{i-1}}(w)}(B)}, \tag{3.32}$$
$$B_1 = \prod_{i=1}^{r} e^{t_i^{\mathcal{I}} \, \phi_{d_i^{\mathcal{I}}}^{\theta^{V_i}(w)}\left(\frac{t_i^{\mathcal{I}}}{d_i^{\mathcal{I}}}\right)},$$
$$C_1 = 1_{\left\{\sum_{i=1}^{r+1} d_i + \sum_{i=1}^{r} d_i^{\mathcal{I}} = L\right\}} \, 1_{\left\{\sum_{i=1}^{r+1} t_i + \sum_{i=1}^{r} t_i^{\mathcal{I}} = uL\right\}}.$$

It is important to note that a simplification has been made in the term A_1 in (3.32). Indeed, this term is not $\tilde{\kappa}_{d_i}(\cdot, \cdot)$ defined in (2.2), since the latter does not take into account the vertical restrictions on the path when it moves from one interface to the next. However, the fact that two neighboring AB-interfaces are necessarily separated by a distance at least L allows us to apply Lemma A.6 in Appendix A.3, which ensures that these vertical restrictions can be removed at the cost of a negligible error.

To show that $\psi_1 \prec \psi_2 \prec \psi_3 \prec \psi_4$, we fix $(M, m) \in$ EIGH and $\varepsilon > 0$, and we show that there exists an $L_\varepsilon \in \mathbb{N}$s such that $\psi_k(L, \Theta, u, j) \leq \psi_{k+1}(L, \Theta, u, j) + \varepsilon$ for all $(L, \Theta, u, j) \in G_M^m$ and $L \geq L_\varepsilon$. The latter will complete the proof of Proposition 3.1.

3.3 Step 1

In this step, we remove the ω-dependence from $Z_1^\omega(L,\Theta,u,j)$. To that aim, we put

$$\psi_2(L,\Theta,u,j) = \frac{1}{uL}\log Z_2(L,\Theta,u,j) \tag{3.33}$$

with

$$Z_2(L,\Theta,u,j) = \sum_{(d,t)\in\mathcal{D}_{\Theta,L,j}} \sum_{(d^{\mathcal{I}},t^{\mathcal{I}})\in\mathcal{D}_r^{\mathcal{I}}} A_2\; B_2\; C_2, \tag{3.34}$$

where

$$\begin{aligned} A_2 &= \prod_{i\in\mathcal{A}_{\Theta,j}} e^{t_i\,\tilde\kappa_{d_i}\left(\frac{t_i}{d_i},\frac{l_iL}{d_i}\right)} \prod_{i\in\mathcal{B}_{\Theta,j}} e^{t_i\,\tilde\kappa_{d_i}\left(\frac{t_i}{d_i},\frac{l_iL}{d_i}\right)}\, e^{\frac{\beta-\alpha}{2}\,t_i}, \\ B_2 &= \prod_{i=1}^{r} e^{t_i^{\mathcal{I}}\,\phi_{d_i^{\mathcal{I}}}\left(\frac{t_i^{\mathcal{I}}}{d_i^{\mathcal{I}}}\right)}, \\ C_2 &= C_1. \end{aligned} \tag{3.35}$$

Next, for $n\in\mathbb{N}$ we define

$$\begin{aligned} \mathcal{A}_{\varepsilon,n} &= \Big\{\exists\, 0\le t,s\le n\colon\ t\ge\varepsilon n,\ \big|H_t^{\theta^s(\omega)}(B) - \tfrac{\beta-\alpha}{2}t\big| > \varepsilon t\Big\}, \\ \mathcal{B}_{\varepsilon,n} &= \Big\{\exists\, 0\le t,d,s\le n\colon\ t\in d+2\mathbb{N}_0,\ t\ge\varepsilon n,\ \big|\phi_d^{\theta^s(w)}(\tfrac{t}{d}) - \phi_d(\tfrac{t}{d})\big| > \varepsilon\Big\}. \end{aligned} \tag{3.36}$$

By applying Cramér's theorem for i.i.d. random variables (see e.g. den Hollander [2], Chapter 1), we obtain that there exist $C_1(\varepsilon),C_2(\varepsilon)>0$ such that

$$\mathbb{P}\Big(\big|H_t^{\theta^s(w)}(B) - \tfrac{\beta-\alpha}{2}t\big| > \varepsilon t\Big) \le C_1(\varepsilon)\,e^{-C_2(\varepsilon)t}, \qquad t,s\in\mathbb{N}. \tag{3.37}$$

By using the concentration of measure property in (C.3) in Appendix C with $l=t$, $\Gamma=\mathcal{W}_d^{\mathcal{I}}(\frac{t}{d})$, $T(x,y)=1\{(x,y)<0\}$, $\eta=\varepsilon t$ and $\xi_i = -\alpha 1\{\omega_i=A\}+\beta 1\{\omega_i=B\}$ for all $i\in\mathbb{N}$, we find that there exist $C_1,C_2>0$ such that

$$\mathbb{P}\Big(\big|\phi_d^{\theta^s(w)}(\tfrac{t}{d}) - \phi_d(\tfrac{t}{d})\big| > \varepsilon\big|\Big) \le C_1\,e^{-C_2\,\varepsilon^2 t}, \qquad t,d,s\in\mathbb{N},\ t\in d+2\mathbb{N}_0. \tag{3.38}$$

With the help of (2.32) and (3.30) we may write, for $(L,\Theta,u,j)\in G_M^m$,

$$\psi_1(L,\Theta,u,j) \le C_{\mathrm{uf}}(\alpha)\,\mathbb{P}\big(\mathcal{A}_{\varepsilon,mL}\cup\mathcal{B}_{\varepsilon,mL}\big) + \tfrac{1}{uL}\,\mathbb{E}\big[1_{\{\mathcal{A}^c_{\varepsilon,mL}\cap\mathcal{B}^c_{\varepsilon,mL}\}}\,\log Z_1^\omega(L,\Theta,u,j)\big]. \tag{3.39}$$

With the help of (3.37) and (3.38), we get that $\mathbb{P}(\mathcal{A}_{\varepsilon,mL})\to 0$ and $\mathbb{P}(\mathcal{B}_{\varepsilon,mL})\to 0$ as $L\to\infty$. Moreover, from ((3.31)-(3.36)) it follows that, for $(L,\Theta,u,j)\in G_M^m$ and $\omega\in\mathcal{A}^c_{\varepsilon,mL}\cap\mathcal{B}^c_{\varepsilon,ML}$,

$$Z_1^\omega(L,\Theta,u,j) \le Z_2(L,\Theta,u,j)\,e^{\varepsilon uL}. \tag{3.40}$$

The latter completes the proof of $\psi_1 \prec \psi_2$.

3.4 Step 2

In this step, we concatenate the pieces of trajectories that travel in A-blocks, respectively, B-blocks, respectively, along the AB-interfaces and replace the finite-size entropies and free energies by their infinite-size counterparts. Recall the definition of $l_{A,\Theta,j}$ and $l_{B,\Theta,j}$ in (3.8) and define, for $(L,\Theta,u,j) \in G_M^m$, the sets

$$\mathcal{J}_{\Theta,L,j} = \Big\{ \big(a_A, h_A, a_B, h_B\big) \in \mathbb{N}^4 \colon\ a_A \in l_{A,\Theta,j}L + h_A + 2\mathbb{N}_0,\ a_B \in l_{B,\Theta,j}L + h_B + 2\mathbb{N}_0 \Big\}, \tag{3.41}$$
$$\mathcal{J}^{\mathcal{I}} = \Big\{ \big(a^{\mathcal{I}}, h^{\mathcal{I}}\big) \in \mathbb{N}^2 \colon\ a^{\mathcal{I}} \in h^{\mathcal{I}} + 2\mathbb{N}_0 \Big\},$$

and put $\psi_3(L,\Theta,u,j) = \frac{1}{uL} \log Z_3(L,\Theta,u,j)$ with

$$Z_3(L,\Theta,u,j) = \sum_{(a,h)\in\mathcal{J}_{\Theta,L,j}} \ \sum_{(a^{\mathcal{I}},h^{\mathcal{I}})\in\mathcal{J}^{\mathcal{I}}} A_3\, B_3\, C_3, \tag{3.42}$$

where

$$\begin{aligned} A_3 &= e^{a_A\, \tilde{\kappa}\left(\frac{a_A}{h_A},\, \frac{l_{A,\Theta,j}L}{h_A}\right)}\, e^{a_B\, \tilde{\kappa}\left(\frac{a_B}{h_B},\, \frac{l_{B,\Theta,j}L}{h_B}\right)}\, e^{\frac{\beta-\alpha}{2}\, a_B}, \\ B_3 &= e^{a^{\mathcal{I}}\, \phi\left(\frac{a^{\mathcal{I}}}{h^{\mathcal{I}}}\right)}, \\ C_3 &= 1_{\{a_A + a_B + a^{\mathcal{I}} = uL\}}\, 1_{\{h_A + h_B + h^{\mathcal{I}} = L\}}. \end{aligned} \tag{3.43}$$

In order to establish a link between ψ_2 and ψ_3 we define, for $(a,h) \in \mathcal{J}_{\Theta,L,j}$ and $(a^{\mathcal{I}},h^{\mathcal{I}}) \in \mathcal{J}^{\mathcal{I}}$,

$$\begin{aligned} \mathcal{P}_{(a,h)} &= \big\{ (t,d) \in \mathcal{D}_{\Theta,L,j} \colon\ \textstyle\sum_{i\in\mathcal{A}_{\Theta,j}} (t_i,d_i) = (a_A,h_A),\ \sum_{i\in\mathcal{B}_{\Theta,j}} (t_i,d_i) = (a_B,h_B) \big\}, \\ \mathcal{Q}_{(a^{\mathcal{I}},h^{\mathcal{I}})} &= \big\{ (t^{\mathcal{I}},d^{\mathcal{I}}) \in \mathcal{D}_r^{\mathcal{I}} \colon\ \textstyle\sum_{i=1}^r (t_i^{\mathcal{I}},d_i^{\mathcal{I}}) = (a^{\mathcal{I}},h^{\mathcal{I}}) \big\}. \end{aligned} \tag{3.44}$$

Then we can rewrite Z_2 as

$$Z_2(L,\Theta,u,j) = \sum_{(a,h)\in\mathcal{J}_{\Theta,L,j}} \ \sum_{(a^{\mathcal{I}},h^{\mathcal{I}})\in\mathcal{J}^{\mathcal{I}}} C_3 \sum_{(t,d)\in\mathcal{P}_{(a,h)}} \ \sum_{(t^{\mathcal{I}},d^{\mathcal{I}})\in\mathcal{Q}_{(a^{\mathcal{I}},h^{\mathcal{I}})}} A_2\, B_2. \tag{3.45}$$

To prove that $\psi_2 \prec \psi_3$, we need the following lemma.

Lemma 3.3 *For every $\eta > 0$ there exists an $L_\eta \in \mathbb{N}$ such that, for every $(L,\Theta,u,j) \in G_M^m$ with $L \geq L_\eta$ and every $(d,t) \in \mathcal{D}_{\Theta,L,j}$ and $(d^{\mathcal{I}},t^{\mathcal{I}}) \in \mathcal{D}_r^{\mathcal{I}}$ satisfying $\sum_{i=1}^{r+1} d_i + \sum_{i=1}^r d_i^{\mathcal{I}} = L$ and $\sum_{i=1}^{r+1} t_i + \sum_{i=1}^r t_i^{\mathcal{I}} = uL$,*

$$\begin{aligned} t_i\, \tilde{\kappa}\big(\tfrac{t_i}{d_i},\, \tfrac{l_i L}{d_i}\big) - \eta uL \leq t_i\, \tilde{\kappa}_{d_i}\big(\tfrac{t_i}{d_i},\, \tfrac{l_i L}{d_i}\big) \leq t_i\, \tilde{\kappa}\big(\tfrac{t_i}{d_i},\, \tfrac{l_i L}{d_i}\big) + \eta uL \quad i = 1,\dots,r+1, \\ t_i^{\mathcal{I}} \phi\big(\tfrac{t_i^{\mathcal{I}}}{d_i^{\mathcal{I}}}\big) - \eta uL \leq t_i^{\mathcal{I}} \phi_{d_i^{\mathcal{I}}}\big(\tfrac{t_i^{\mathcal{I}}}{d_i^{\mathcal{I}}}\big) \leq t_i^{\mathcal{I}} \phi\big(\tfrac{t_i^{\mathcal{I}}}{d_i^{\mathcal{I}}}\big) + \eta uL \quad i = 1,\dots,r. \end{aligned} \tag{3.46}$$

Proof. By using Lemmas A.1 and B.2 in Appendix A, we have that there exists a $\tilde{L}_\eta \in \mathbb{N}$ such that, for $L \geq \tilde{L}_\eta$, $(u,l) \in \mathcal{H}_L$ and $\mu \in 1 + \frac{2\mathbb{N}}{L}$,

$$|\tilde{\kappa}_L(u,l) - \tilde{\kappa}(u,l)| \leq \eta, \qquad |\phi_L^{\mathcal{I}}(\mu) - \phi^{\mathcal{I}}(\mu)| \leq \eta. \tag{3.47}$$

Moreover, Lemmas 2.1, A.5(ii–iii), B.1(ii) and B.2 ensure that there exists a $v_\eta > 1$ such that, for $L \geq 1$, $(u,l) \in \mathcal{H}_L$ with $u \geq v_\eta$ and $\mu \in 1 + \frac{2\mathbb{N}}{L}$ with $\mu \geq v_\eta$,

$$0 \leq \tilde{\kappa}_L(u,l) \leq \eta, \qquad 0 \leq \phi_L(\mu) \leq \eta. \tag{3.48}$$

Note that the two inequalities in (3.48) remain valid when $L = \infty$. Next, we set $r_\eta = \eta/(2v_\eta C_{\mathrm{uf}})$ and $L_\eta = \tilde{L}_\eta / r_\eta$, and we consider $L \geq L_\eta$. Because of the left-hand side of (3.47), the two inequalities in the first line of (3.46) hold when $d_i \geq r_\eta L \geq \tilde{L}_\eta$. We deal with the case $d_i \leq r_\eta L$ by considering first the case $t_i \leq \eta u L/2C_{\mathrm{uf}}$, which is easy because $\tilde{\kappa}_{d_i}$ and $\tilde{\kappa}$ are uniformly bounded by C_{uf} (see (2.32)). The case $t_i \geq \eta uL/2C_{\mathrm{uf}}$ gives $t_i/d_i \geq u v_\eta \geq v_\eta$, which by the left-hand side of (3.48) completes the proof of the first line in (3.46). The same observations applied to $t_i^{\mathcal{I}}, d_i^{\mathcal{I}}$ combined with the right-hand side of (3.47) and (3.48) provide the two inequalities in the second line in (3.46). □

To prove that $\psi_2 \prec \psi_3$, we apply Lemma 3.3 with $\eta = \varepsilon/(2m+1)$ and we use (3.35) to obtain, for $L \geq L_{\varepsilon/(2m+1)}$, $(d,t) \in \mathcal{D}_{\Theta,L,j}$ and $(d^{\mathcal{I}}, t^{\mathcal{I}}) \in \mathcal{D}_r^{\mathcal{I}}$,

$$\begin{aligned} A_2 &\leq \prod_{i \in \mathcal{A}_{\Theta,j}} e^{t_i\, \tilde{\kappa}\left(\frac{t_i}{d_i}, \frac{l_i L}{d_i}\right) + \frac{\varepsilon u L}{2m+1}} \prod_{i \in \mathcal{B}_{\Theta,j}} e^{t_i\, \tilde{\kappa}\left(\frac{t_i}{d_i}, \frac{l_i L}{d_i}\right) + t_i\, \frac{\beta-\alpha}{2} + \frac{\varepsilon u L}{2m+1}}, \\ B_2 &\leq \prod_{i=1}^{r} e^{t_i^{\mathcal{I}}\, \phi\left(\frac{t_i^{\mathcal{I}}}{d_i^{\mathcal{I}}}\right) + \frac{\varepsilon u L}{2m+1}}. \end{aligned} \tag{3.49}$$

Next, we pick $(a,h) \in \mathcal{J}_{\Theta,L,j}$, $(a^{\mathcal{I}}, h^{\mathcal{I}}) \in \mathcal{J}^{\mathcal{I}}$, $(t,d) \in \mathcal{P}_{(a,h)}$ and $(t^{\mathcal{I}}, d^{\mathcal{I}}) \in \mathcal{Q}_{(a^{\mathcal{I}},h^{\mathcal{I}})}$, and we use the concavity of $(a,b) \mapsto a\tilde{\kappa}(a,b)$ and $\mu \mapsto \phi^{\mathcal{I}}(\mu)$ (see Lemma A.5 in Appendix A and Lemma B.1 in Appendix B) to rewrite (3.49) as

$$\begin{aligned} A_2 &\leq e^{a_A\, \tilde{\kappa}\left(\frac{a_A}{h_A}, \frac{l_{A,\Theta,j} L}{h_A}\right) + a_B\, \tilde{\kappa}\left(\frac{a_B}{h_B}, \frac{l_{B,\Theta,j} L}{h_B}\right) + \frac{\beta-\alpha}{2} a_B + \frac{\varepsilon (r+1) u L}{2m+1}} = A_3\, e^{\frac{\varepsilon (r+1) u L}{2m+1}}, \\ B_2 &\leq e^{a^{\mathcal{I}}\, \phi^{\mathcal{I}}\left(\frac{a^{\mathcal{I}}}{h^{\mathcal{I}}}\right) + \frac{\varepsilon r u L}{2m+1}} = B_3\, e^{\frac{\varepsilon r u L}{2m+1}}. \end{aligned} \tag{3.50}$$

Moreover, r, which is the number of AB interfaces crossed by the trajectories in $\mathcal{W}_{\Theta,u,j,L}$, is at most m (see (3.10)), so that (3.50) allows us to rewrite (3.45) as

$$Z_2(L,\Theta,u,j) \leq e^{\varepsilon u L} \sum_{(a,h) \in \mathcal{J}_{\Theta,L,j}} \sum_{(a^{\mathcal{I}},h^{\mathcal{I}}) \in \mathcal{J}^{\mathcal{I}}} C_3\, |\mathcal{P}_{(a,h)}|\, |\mathcal{Q}_{(a^{\mathcal{I}},h^{\mathcal{I}})}|\, A_3\, B_3. \tag{3.51}$$

Finally, it turns out that $|\mathcal{P}_{(a,h)}| \leq (uL)^{8r}$ and $|\mathcal{Q}_{(a^{\mathcal{I}},h^{\mathcal{I}})}| \leq (uL)^{8r}$. Therefore, since $r \leq m$, (3.42) and (3.51) allow us to write, for $(L,\Theta,u,j) \in G_M^m$ and $L \geq L_{\varepsilon/2m+1}$,

$$Z_2(L,\Theta,u,j) \leq (mL)^{16m} Z_3(L,\Theta,u,j). \tag{3.52}$$

The latter is sufficient to conclude that $\psi_2 \prec \psi_3$.

3.5 Step 3

For every $(L,\Theta,u,j) \in G_M^m$ we have, by the definition of $\mathcal{L}(l_{A,\Theta,j}, l_{B,\Theta,j}; u)$ in (2.34), that $(a,h) \in \mathcal{J}_{\Theta,L,j}$ and $(a^{\mathcal{I}}, h^{\mathcal{I}}) \in \mathcal{J}^{\mathcal{I}}$ satisfying $a_A + a_B + a^{\mathcal{I}} = uL$ and $h_A + h_B + h^{\mathcal{I}} = L$ also satisfy

$$\Big(\big(\tfrac{a_A}{L}, \tfrac{a_B}{L}, \tfrac{a^{\mathcal{I}}}{L}\big), \big(\tfrac{h_A}{L}, \tfrac{h_B}{L}, \tfrac{h^{\mathcal{I}}}{L}\big) \Big) \in \mathcal{L}(l_{A,\Theta,j}, l_{B,\Theta,j}; u). \tag{3.53}$$

Hence, (3.53) and the definition of $\psi_{\mathcal{I}}$ in (2.36) ensure that, for this choice of (a,h) and $(a^{\mathcal{I}},h^{\mathcal{I}})$,

$$A_3B_3 \leq e^{uL\psi_{\mathcal{I}}(u,l_{A,\Theta,j},l_{B,\Theta,j})}. \tag{3.54}$$

Because of C_3, the summation in (3.42) is restricted to those $(a,h)\in\mathcal{J}_{\Theta,L,j}$ and $(a^{\mathcal{I}},h^{\mathcal{I}})\in\mathcal{J}^{\mathcal{I}}$ for which $a_A,a_B,a^{\mathcal{I}}\leq uL$ and $h_A,h_B,h^{\mathcal{I}}\leq L$. Hence, the summation is restricted to a set of cardinality at most $(uL)^3L^3$. Consequently, for all $(L,\Theta,u,j)\in G_M^m$ we have

$$Z_3(L,\Theta,u,j) = \sum_{(a,h)\in\mathcal{J}_{\Theta,L,j}} \sum_{(a^{\mathcal{I}},h^{\mathcal{I}})\in\mathcal{J}^{\mathcal{I}}} A_4\,B_4\,C_4 \leq (mL)^3L^3\,e^{u\,L\,\psi_{\mathcal{I}}(u,l_{A,\Theta,j},l_{B,\Theta,j})}. \tag{3.55}$$

The latter implies that $\psi_3 \prec \psi_4$ since $\psi_4 = \psi_{\mathcal{I}}(u,\,l_{A,\Theta,j},\,l_{B,\Theta,j})$ by definition (recall (3.13) and (3.26)).

4 Proof of Theorem 1.1

This section is technically involved because it goes through a sequence of approximation steps in which the self-averaging of the free energy with respect to ω and Ω in the limit as $n\to\infty$ is proven, and the various ingredients of the variational formula in Theorem 1.1 that were constructed in Section 2 are put together.

In Section 4.1 we introduce additional notation and state Propositions 4.1, 4.2 and 4.11 from which Theorem 1.1 is a straightforward consequence. Proposition 4.1, which deals with $(M,m)\in$ EIGH, is proven in Section 4.2 and the details of the proof are worked out in Sections 4.2.1–4.2.5, organized into 5 Steps that link intermediate free energies. We pass to the limit $m\to\infty$ with Propositions 4.2 and 4.3 which are proven in Section 4.3 and 4.4, respectively.

4.1 Proof of Theorem 1.1

4.1.1 Additional notation

Pick $(M,m)\in$ EIGH and recall that Ω and ω are independent, i.e., $\mathbb{P}=\mathbb{P}_\omega\times\mathbb{P}_\Omega$. For $\Omega\in\{A,B\}^{\mathbb{N}_0\times\mathbb{Z}}$, $\omega\in\{A,B\}^{\mathbb{N}}$, $n\in\mathbb{N}$ and $(\alpha,\beta)\in$ CONE, define

$$f_{1,n}^{\omega,\Omega}(M,m;\alpha,\beta) = \tfrac{1}{n}\log Z_{1,n,L_n}^{\omega,\Omega}(M,m) \quad\text{with}\quad Z_{1,n,L_n}^{\omega,\Omega}(M,m) = \sum_{\pi\in\mathcal{W}_{n,M,}^m} e^{\,H_{n,L_n}^{\omega,\Omega}(\pi)}, \tag{4.1}$$

where $\mathcal{W}_{n,M}^m$ contains those paths in $\mathcal{W}_{n,M}$ that, in each column, make at most mL_n steps. We also restrict the set $\mathcal{R}_{p,M}$ in (2.7) to those limiting empirical measures whose support is included in $\overline{\mathcal{V}}_M^m$, i.e., those measures charging the types of column that can be crossed in less than mL_n steps only. To that aim we recall (2.48) and define, for $\Omega\in\{A,B\}^{\mathbb{N}_0\times\mathbb{Z}}$ and $N\in\mathbb{N}$,

$$\begin{aligned}\mathcal{R}_{M,N}^{\Omega,m} = \big\{&\rho_N(\Omega,\Pi,b,x)\text{ with } b=(b_j)_{j\in\mathbb{N}_0}\in(\mathbb{Q}_{(0,1]})^{\mathbb{N}_0},\\ &\Pi=(\Pi_j)_{j\in\mathbb{N}_0}\in\{0\}\times\mathbb{Z}^{\mathbb{N}}\colon\ |\Delta\Pi_j|\leq M\quad\forall\,j\in\mathbb{N}_0,\\ &x=(x_j)_{j\in\mathbb{N}_0}\in\{1,2\}^{N_0}\colon\ \big(\Omega(j,\Pi_j+\cdot),\Delta\Pi_j,b_j,b_{j+1},x_j\big)\in\mathcal{V}_M^m\big\}\end{aligned} \tag{4.2}$$

which is a subset of $\mathcal{R}_{M,N}^{\Omega}$ and allows us to define

$$\mathcal{R}_M^{\Omega,m} = \text{closure}\Big(\cap_{N'\in\mathbb{N}}\cup_{N\geq N'}\mathcal{R}_{M,N}^{\Omega,m}\Big), \tag{4.3}$$

which, for $\mathbb{P}$-a.e. Ω is equal to $\mathcal{R}^m_{p,M} \subsetneq \mathcal{R}_{p,M}$.

At this stage, we further define,

$$f(M,m;\alpha,\beta) = \sup_{\rho\in\mathcal{R}^m_{p,M}} \sup_{(u_\Theta)_{\Theta\in\overline{\mathcal{V}}^m_M}\in\mathcal{B}_{\overline{\mathcal{V}}^m_M}} V(\rho,u), \tag{4.4}$$

where

$$V(\rho,u) = \frac{\int_{\overline{\mathcal{V}}^m_M} u_\Theta\, \psi(\Theta,u_\Theta;\alpha,\beta)\, \rho(d\Theta)}{\int_{\overline{\mathcal{V}}^m_M} u_\Theta\, \rho(d\Theta)}, \tag{4.5}$$

where (recall (2.23))

$$\mathcal{B}_{\overline{\mathcal{V}}^m_M} = \Big\{(u_\Theta)_{\Theta\in\overline{\mathcal{V}}^m_M} \in \mathbb{R}^{\overline{\mathcal{V}}^m_M} \colon \Theta\mapsto u_\Theta \in \mathcal{C}^0\big(\overline{\mathcal{V}}^m_M,\mathbb{R}\big),\, t_\Theta \le u_\Theta \le m\ \forall\,\Theta\in\overline{\mathcal{V}}^m_M\Big\}, \tag{4.6}$$

and where $\overline{\mathcal{V}}^m_M$ is endowed with the distance d_M defined in (B.3) in Appendix B.2.

Let $\mathcal{W}^{*,m}_{n,M} \subset \mathcal{W}^m_{n,M}$ be the subset consisting of those paths whose endpoint lies at the boundary between two columns of blocks, i.e., satisfies $\pi_{n,1}\in\mathbb{N}L_n$. Recall (4.1), and define $Z^{*,\omega,\Omega}_{n,L_n}(M)$ and $f^{*,\omega,\Omega}_{1,n}(M,m;\alpha,\beta)$ as the counterparts of $Z^{\omega,\Omega}_{n,L_n}(M,m)$ and $f^{\omega,\Omega}_{1,n}(M,m;\alpha,\beta)$ when $\mathcal{W}^m_{n,M}$ is replaced by $\mathcal{W}^{*,m}_{n,M}$. Then there exists a constant $c>0$, depending on α and β only, such that

$$\begin{aligned} &Z^{\omega,\Omega}_{1,n,L_n}(M,m)e^{-cL_n} \le Z^{*,\omega,\Omega}_{1,n,L_n}(M,m) \le Z^{\omega,\Omega}_{1,n,L_n}(M,m),\\ &n\in\mathbb{N},\ \omega\in\{A,B\}^{\mathbb{N}},\ \Omega\in\{A,B\}^{\mathbb{N}_0\times\mathbb{Z}}. \end{aligned} \tag{4.7}$$

The left-hand side of the latter inequality is obtained by changing the last L_n steps of each trajectory in $\mathcal{W}^m_{n,M}$ to make sure that the endpoint falls in $L_n\mathbb{N}$. The energetic and entropic cost of this change are obviously $O(L_n)$. By assumption, $\lim_{n\to\infty} L_n/n = 0$, which together with (4.7) implies that the limits of $f^{\omega,\Omega}_{1,n}(M,m;\alpha,\beta)$ and $f^{*,\omega,\Omega}_{1,n}(M,m;\alpha,\beta)$ as $n\to\infty$ are the same. In the sequel we will therefore restrict the summation in the partition function to $\mathcal{W}^{*,m}_{n,M}$ and drop the $*$ from the notations.

Finally, let

$$\begin{aligned} f^{\Omega}_{1,n}(M,m;\alpha,\beta) &= \mathbb{E}_\omega\big[f^{\omega,\Omega}_{1,n}(M,m;\alpha,\beta)\big],\\ f_{1,n}(M,m;\alpha,\beta) &= \mathbb{E}_{\omega,\Omega}\big[f^{\omega,\Omega}_{1,n}(M,m;\alpha,\beta)\big], \end{aligned} \tag{4.8}$$

and recall (1.11) to set $f^{\Omega}_n(M;\alpha,\beta) = \mathbb{E}_\omega[f^{\omega,\Omega}_n(M;\alpha,\beta)]$.

4.1.2 Key Propositions

Theorem 1.1 is a consequence of Propositions 4.1, 4.2 and 4.3 stated below and proven in Sections 4.2.1–4.2.5, Sections 4.3.1–4.3.3 and Section 4.4, respectively.

Proposition 4.1 *For all $(M,m)\in$ EIGH,*

$$\lim_{n\to\infty} f^{\Omega}_{1,n}(M,m;\alpha,\beta) = f(M,m;\alpha,\beta) \quad \text{for } \mathbb{P}-a.e.\,\Omega. \tag{4.9}$$

Proposition 4.2 *For all $M\in\mathbb{N}$,*

$$\lim_{n\to\infty} f^{\Omega}_{n}(M;\alpha,\beta) = \sup_{m\ge M+2} f(M,m;\alpha,\beta) \quad \text{for } \mathbb{P}-a.e.\,\Omega. \tag{4.10}$$

Proposition 4.3 *For all $M \in \mathbb{N}$,*

$$\sup_{m \geq M+2} f(M, m; \alpha, \beta) = \sup_{\rho \in \mathcal{R}_{p,M}} \sup_{(u_\Theta)_{\Theta \in \overline{\mathcal{V}}_M} \in \mathcal{B}_{\overline{\mathcal{V}}_M}} V(\rho, u), \tag{4.11}$$

where, in the righthand side of (4.11), we recognize the variational formula of Theorem 1.1 and with $\mathcal{B}_{\overline{\mathcal{V}}_M}$ defined in (2.13).

Proof of Theorem 1.1 subject to Propositions 4.1, 4.2 and 4.3. The proof of Theorem 1.1 will be complete once we show that for all $(M, m) \in \text{EIGH}$

$$\lim_{n \to \infty} |f_n^{\omega,\Omega}(M, m; \alpha, \beta) - f_n^{\Omega}(M, m; \alpha, \beta)| = 0 \quad \text{ for } \mathbb{P} - a.e.\ (\omega, \Omega). \tag{4.12}$$

To that aim, we note that for all $n \in \mathbb{N}$ the Ω-dependence of $f_n^{\omega,\Omega}(M, m; \alpha, \beta)$ is restricted to $\{\Omega_x \colon\ x \in G_n\}$ with $G_n = \{0, \dots, \frac{n}{L_n}\} \times \{-\frac{n}{L_n}, \dots, \frac{n}{L_n}\}$. Thus, for $n \in \mathbb{N}$ and $\varepsilon > 0$ we set

$$A_{\varepsilon,n} = \{|f_n^{\omega,\Omega}(M; \alpha, \beta) - f_n^{\Omega}(M; \alpha, \beta)| > \varepsilon)\}, \tag{4.13}$$

and by independence of ω and Ω we can write

$$\begin{aligned} \mathbb{P}_{\omega,\Omega}(A_{\varepsilon,n}) &= \textstyle\sum_{\Upsilon \in \{A,B\}^{G_n}} \mathbb{P}_{\omega,\Omega}(A_{\varepsilon,n} \cap \{\Omega_{G_n} = \Upsilon\}) \\ &= \textstyle\sum_{\Upsilon \in \{A,B\}^{G_n}} \mathbb{P}_{\omega}(|f_n^{\omega,\Upsilon}(M; \alpha, \beta) - f_n^{\Upsilon}(M; \alpha, \beta)| > \varepsilon)\ \mathbb{P}_{\Omega}(\{\Omega_{G_n} = \Upsilon\}). \end{aligned} \tag{4.14}$$

At this stage, for each $n \in \mathbb{N}$ we can apply the concentration inequality (C.3) in Appendix C with $\Gamma = \mathcal{W}_{n,M}^m$, $l = n$, $\eta = \varepsilon n$,

$$\xi_i = -\alpha\, 1\{\omega_i = A\} + \beta\, 1\{\omega_i = B\}, \qquad i \in \mathbb{N}, \tag{4.15}$$

and with $T(x, y)$ indicating in which block step (x, y) lies in. Therefore, there exist $C_1, C_2 > 0$ such that for all $n \in \mathbb{N}$ and all $\Upsilon \in \{A, B\}^{G_n}$ we have

$$\mathbb{P}_{\omega}(|f_n^{\omega,\Upsilon}(M; \alpha, \beta) - f_n^{\Upsilon}(M; \alpha, \beta)| > \varepsilon) \leq C_1 e^{-C_2 \varepsilon^2 n}, \tag{4.16}$$

which, together with (4.14) yields $\mathbb{P}_{\omega,\Omega}(A_{\varepsilon,n}) \leq C_1 e^{-C_2 \varepsilon^2 n}$ for all $n \in \mathbb{N}$. By using the Borel-Cantelli Lemma, we obtain (4.12). □

4.2 Proof of Proposition 4.1

Pick $(M, m) \in \text{EIGH}$ and $(\alpha, \beta) \in \text{CONE}$. In Steps 1–2 in Sections 4.2.1–4.2.2 we introduce an intermediate free energy $f_{3,n}^{\Omega}(M, m; \alpha, \beta)$ and show that

$$\lim_{n \to \infty} |f_{1,n}^{\Omega}(M, m; \alpha, \beta) - f_{3,n}^{\Omega}(M, m; \alpha, \beta)| = 0 \qquad \forall\, \Omega \in \{A, B\}^{N_0 \times \mathbb{Z}}. \tag{4.17}$$

Next, in Steps 3–4 in Sections 4.2.3–4.2.4 we show that

$$\limsup_{n \to \infty} f_{3,n}^{\Omega}(M, m; \alpha, \beta) = f(M, m; \alpha, \beta) \qquad \text{for } \mathbb{P} - a.e.\ \Omega, \tag{4.18}$$

while in Step 5 in Section 4.2.5 we prove that

$$\liminf_{n \to \infty} f_{3,n}^{\Omega}(M, m; \alpha, \beta) = \limsup_{n \to \infty} f_{3,n}^{\Omega}(M, m; \alpha, \beta) \qquad \text{for } \mathbb{P} - a.e.\ \Omega. \tag{4.19}$$

Combing (4.17–4.19) we get

$$\liminf_{n\to\infty} f_{1,n}^{\Omega}(M,m;\alpha,\beta) = \limsup_{n\to\infty} f_{1,n}^{\Omega}(M,m;\alpha,\beta) = f(M,m;\alpha,\beta) \qquad \text{for } \mathbb{P}-a.e.\ \Omega, \tag{4.20}$$

which completes the proof of Proposition 4.1.

In the proof we need the following order relation.

Definition 4.4 *For $g, \widetilde{g}\colon \mathbb{N}^3 \times \text{CONE} \mapsto \mathbb{R}$, write $g \prec \widetilde{g}$ if for all $(M,m) \in \text{EIGH}$, $(\alpha,\beta) \in \text{CONE}$ and $\varepsilon > 0$ there exists an $n_\varepsilon \in \mathbb{N}$ such that*

$$g(n,M,m;\alpha,\beta) \le \widetilde{g}(n,M,m;\alpha,\beta) + \varepsilon \qquad \forall\, n \ge n_\varepsilon. \tag{4.21}$$

The proof of (4.17) will be complete once we show that $f_1^{\Omega} \prec f_3^{\Omega}$ and $f_3^{\Omega} \prec f_1^{\Omega}$ for all $\Omega \in \{A,B\}^{\mathbb{N}_0\times\mathbb{Z}}$. We will focus on $f_1^{\Omega} \prec f_3^{\Omega}$, since the proof of the latter can be easily adapted to obtain $f_3^{\Omega} \prec f_1^{\Omega}$. To prove $f_1^{\Omega} \prec f_3^{\Omega}$ we introduce another intermediate free energy f_2^{Ω}, and we show that $f_1^{\Omega} \prec f_2^{\Omega}$ and $f_2^{\Omega} \prec f_3^{\Omega}$.

For $L \in \mathbb{N}$, let

$$\mathcal{D}_L^M = \left\{\Xi = (\Delta\Pi, b_0, b_1) \in \{-M,\dots,M\} \times \{\tfrac{1}{L}, \tfrac{2}{L}, \dots, 1\}^2\right\}. \tag{4.22}$$

For $L, N \in \mathbb{N}$, let

$$\widetilde{\mathcal{D}}_{L,N}^M = \Big\{\Theta_{\text{traj}} = (\Xi_i)_{i\in\{0,\dots,N-1\}} \in (\mathcal{D}_L^M)^N\colon\ b_{0,0} = \tfrac{1}{L},\ b_{0,i} = b_{1,i-1}\ \forall\, 1 \le i \le N-1\Big\}, \tag{4.23}$$

and with each $\Theta_{\text{traj}} \in \widetilde{\mathcal{D}}_{L,N}^M$ associate the sequence $(\Pi_i)_{i=0}^N$ defined by $\Pi_0 = 0$ and $\Pi_i = \sum_{j=0}^{i-1} \Delta\Pi_j$ for $1 \le i \le N$. Next, for $\Omega \in \{A,B\}^{\mathbb{N}_0\times\mathbb{Z}}$ and $\Theta_{\text{traj}} \in \widetilde{\mathcal{D}}_{L,N}^M$, set

$$\mathcal{X}_{\Theta_{\text{traj}},\Omega}^{M,m} = \left\{x \in \{1,2\}^{\{0,\dots,N-1\}}\colon (\Omega(i,\Pi_i+\cdot),\Xi_i,x_i) \in \mathcal{V}_M^m\ \forall\, 0 \le i \le N-1\right\}, \tag{4.24}$$

and, for $x \in \mathcal{X}_{\Theta_{\text{traj}},\Omega}^{M,m}$, set

$$\Theta_i = (\Omega(i,\Pi_i+\cdot),\Xi_i,x_i) \quad \text{for} \quad i \in \{0,\dots,N-1\} \tag{4.25}$$

and

$$\mathcal{U}_{\Theta_{\text{traj}},x,n}^{M,m,L} = \Big\{u = (u_i)_{i\in\{0,\dots,N-1\}} \in [1,m]^N\colon u_i \in t_{\Theta_i} + \tfrac{2\mathbb{N}}{L}\ \ \forall\, 0 \le i \le N-1,\ \sum_{i=0}^{N-1} u_i = \tfrac{n}{L}\Big\}. \tag{4.26}$$

Note that $\mathcal{U}_{\Theta_{\text{traj}},x,n}^{M,m,L}$ is empty when $N \notin \left[\frac{n}{mL}, \frac{n}{L}\right]$.

For $\pi \in \mathcal{W}_{n,M}^m$, we let N_π be the number of columns crossed by π after n steps. We denote by $(u_0(\pi),\dots,u_{N_\pi-1}(\pi))$ the time spent by π in each column divided by L_n, and we set $\widetilde{u}_0(\pi) = 0$ and $\widetilde{u}_j(\pi) = \sum_{k=0}^{j-1} u_k(\pi)$ for $1 \le j \le N_\pi$. With these notations, the partition function in (4.1) can be rewritten as

$$Z_{1,n,L_n}^{\omega,\Omega}(M,m) = \sum_{N=n/mL_n}^{n/L_n} \sum_{\Theta_{\text{traj}}\in\widetilde{\mathcal{D}}_{L_n,N}^M} \sum_{x\in\mathcal{X}_{\Theta_{\text{traj}},\Omega}^{M,m}} \sum_{u\in\mathcal{U}_{\Theta_{\text{traj}},x,n}^{M,m,L_n}} A_1, \tag{4.27}$$

with (recall (2.29))

$$A_1 = \prod_{i=0}^{N-1} Z_{L_n}^{\theta^{\widetilde{u}_i L_n}(\omega)}(\Omega(i,\Pi_i+\cdot),\Xi_i,x_i,u_i). \tag{4.28}$$

4.2.1 Step 1

In this step we average over the disorder ω in each column. To that end, we set

$$f_{2,n}^{\Omega}(M,m;\alpha,\beta) = \tfrac{1}{n}\log Z_{2,n,L_n}^{\Omega}(M,m) \tag{4.29}$$

with

$$Z_{2,n,L_n}^{\Omega}(M,m) = \sum_{N=n/mL_n}^{n/L_n} \sum_{\Theta_{\rm traj}\in\widetilde{\mathcal{D}}_{L_n,N}^{M}} \sum_{x\in\mathcal{X}_{\Theta_{\rm traj},\Omega}^{M,m}} \sum_{u\in\mathcal{U}_{\Theta_{\rm traj},x,n}^{M,m,L_n}} A_2, \tag{4.30}$$

where

$$A_2 = \prod_{i=0}^{N-1} e^{\mathbb{E}_\omega\big[\log Z_{L_n}^{\theta^{\widetilde{u}_i(\omega)}}(\Omega(i,\Pi_i+\cdot),\Xi_i,x_i,u_i)\big]} = \prod_{i=0}^{N-1} e^{u_iL_n\psi_{L_n}(\Omega(i,\Pi_i+\cdot),\Xi_i,x_i,u_i)}. \tag{4.31}$$

Note that the ω-dependence has been removed from $Z_{2,n,L_n}^{\Omega}(M,m)$.

To prove that $f_1^{\Omega}\prec f_2^{\Omega}$, we need to show that for all $\varepsilon>0$ there exists an $n_\varepsilon\in\mathbb{N}$ such that, for $n\geq n_\varepsilon$ and all Ω,

$$\mathbb{E}_\omega\big[\log Z_{1,n,L_n}^{\omega,\Omega}(M,m)\big] \leq \log Z_{2,n,L_n}^{\Omega}(M,m)+\varepsilon n. \tag{4.32}$$

To this end, we rewrite $Z_{1,n,L_n}^{\omega,\Omega}(M,m)$ as

$$Z_{1,n,L_n}^{\omega,\Omega}(M,m) = \sum_{N=n/mL_n}^{n/L_n} \sum_{\Theta_{\rm traj}\in\widetilde{\mathcal{D}}_{L_n,N}^{M}} \sum_{x\in\mathcal{X}_{\Theta_{\rm traj},\Omega}^{M,m}} \sum_{u\in\mathcal{U}_{\Theta_{\rm traj},x,n}^{M,m,L_n}} A_2\,\frac{A_1}{A_2}, \tag{4.33}$$

where we note that

$$\frac{A_1}{A_2} = \prod_{i=0}^{N-1} e^{u_iL_n\big[\psi_{L_n}^{\theta^{\widetilde{u}_iL_n(\omega)}}(\Omega(i,\Pi_i+\cdot),\Xi_i,x_i,u_i)-\psi_{L_n}(\Omega(i,\Pi_i+\cdot),\Xi_i,x_i,u_i)\big]}. \tag{4.34}$$

In order to average over ω, we apply a concentration of measure inequality. Set

$$\mathcal{K}_n = \bigcup_{N=n/mL_n}^{n/L_n} \bigcup_{\Theta_{\rm traj}\in\widetilde{\mathcal{D}}_{L_n,N}^{M}} \bigcup_{x\in\mathcal{X}_{\Theta_{\rm traj},\Omega}^{M,m}} \bigcup_{u\in\mathcal{U}_{\Theta_{\rm traj},x,n}^{M,m,L_n}} \Big\{|\log A_1-\log A_2|\geq\varepsilon n\Big\}, \tag{4.35}$$

and note that $\omega\in\mathcal{K}_n^c$ implies that $Z_{1,n,L_n}^{\omega,\Omega}(M,m)\leq e^{\varepsilon n}Z_{2,n,L_n}^{\Omega}(M,m)$. Consequently, we can write

$$\begin{aligned}\mathbb{E}_\omega\big[\log Z_{1,n,L_n}^{\omega,\Omega}(M,m)\big] &= \mathbb{E}_\omega\big[\log Z_{1,n,L_n}^{\omega,\Omega}(M,m)\,1_{\{\mathcal{K}_n\}}\big]+\mathbb{E}_\omega\big[\log Z_{1,n,L_n}^{\omega,\Omega}(M,m)\,1_{\{\mathcal{K}_n^c\}}\big]\\ &\leq \mathbb{E}_\omega\big[\log Z_{1,n,L_n}^{\omega,\Omega}(M,m)\,1_{\{\mathcal{K}_n\}}\big]+\log Z_{2,n,L_n}^{\Omega}(M,m)+\varepsilon n.\end{aligned} \tag{4.36}$$

We can now use the uniform bound in (2.32) to control the first term in the right-hand side of (4.36), to obtain

$$\mathbb{E}_\omega\big[\log Z_{1,n,L_n}^{\omega,\Omega}(M,m)\big] \leq \log Z_{2,n,L_n}^{\Omega}(M,m)+\varepsilon n+C_{\rm uf}(\alpha)\,n\,\mathbb{P}_\omega(\mathcal{K}_n). \tag{4.37}$$

Therefore the proof of this step will be complete once we show that $\mathbb{P}_\omega(\mathcal{K}_n)$ vanishes as $n\to\infty$.

Lemma 4.5 *There exist $C_1, C_2 > 0$ such that, for all $\varepsilon > 0$, $n \in \mathbb{N}$, $N \in \{\frac{n}{mL_n}, \dots, \frac{n}{L_n}\}$, $\Omega \in \{A,B\}^{\mathbb{N}_0 \times \mathbb{Z}}$, $\Theta_{\rm traj} \in \widetilde{\mathcal{D}}^M_{L_n,N}$, $x \in \mathcal{X}^{M,m}_{\Theta_{\rm traj},\Omega}$ and $u \in \mathcal{U}^{M,m,L_n}_{\Theta_{\rm traj},x,n}$,*

$$\mathbb{P}_\omega(|\log A_1 - \log A_2| \geq \varepsilon n) \leq C_1 e^{-C_2 \varepsilon^2 n}. \tag{4.38}$$

Proof. Pick $\Theta_{\rm traj} \in \widetilde{\mathcal{D}}^M_{L_n,N}$, $x \in \mathcal{X}^{M,m}_{\Theta_{\rm traj},\Omega}$ and $u \in \mathcal{U}^{M,m,L_n}_{\Theta_{\rm traj},x,n}$, and consider the subset Γ of $\mathcal{W}^m_{n,M}$ consisting of those paths of length n that first cross the $(\Omega(0,\cdot), \Xi_0, x_0)$ column such that $\pi_0 = (0,1)$ and $\pi_{\widetilde{u}_1 L_n} = (1, \Pi_1 + b_{1,0}) L_n$, then cross the $(\Omega(1,\cdot), \Xi_1, x_1)$ column such that $\pi_{\widetilde{u}_1 L_n + 1} = (1 + 1/L_n, \Pi_1 + b_{1,0}) L_n$ and $\pi_{\widetilde{u}_2 L_n} = (2, \Pi_2 + b_{1,1}) L_n$, and so on. We can apply the concentration of measure inequality stated in (C.3) to the set Γ defined above, with $l = n$, $\eta = \varepsilon n$,

$$\xi_i = -\alpha\, 1\{\omega_i = A\} + \beta\, 1\{\omega_i = B\}, \qquad i \in \mathbb{N}, \tag{4.39}$$

and with $T(x,y)$ indicating in which block step (x,y) lies in. After noting that $\mathbb{E}_\omega(\log A_1) = \log A_2$, we obtain that there exist $C_1, C_2 > 0$ such that, for all $n \in \mathbb{N}$, $N \in \{\frac{n}{mL_n}, \dots, \frac{n}{L_n}\}$, $\Omega \in \{A,B\}^{\mathbb{N}_0 \times \mathbb{Z}}$, $\Theta_{\rm traj} \in \widetilde{\mathcal{D}}^M_{L_n,N}$, $x \in \mathcal{X}^{M,m}_{\Theta_{\rm traj},\Omega}$ and $u \in \mathcal{U}^{M,m,L_n}_{\Theta_{\rm traj},x,n}$,

$$\mathbb{P}\big(|\log A_1 - \log A_2| \geq \varepsilon\, n\big) \leq C_1\, e^{-C_2\, \varepsilon^3\, n}. \tag{4.40}$$

□

It now suffices to remark that

$$\big|\{(N, \Theta_{\rm traj}, x, u)\colon\ N \in \{\tfrac{n}{mL_n}, \dots, \tfrac{n}{L_n}\}, \Theta_{\rm traj} \in \widetilde{\mathcal{D}}^M_{L_n,N},\, x \in \mathcal{X}^{M,m}_{\Theta_{\rm traj},\Omega}, u \in \mathcal{U}^{M,m,L_n}_{\Theta_{\rm traj},x,n}\}\big| \tag{4.41}$$

grows subexponentially in n to obtain that $f_1^\Omega \prec f_2^\Omega$ for all Ω.

4.2.2 Step 2

In this step we replace the finite-size free energy ψ_{L_n} by its limit ψ. To do so we introduce a third intermediate free energy,

$$f^\Omega_{3,n}(M,m;\alpha,\beta) = \mathbb{E}\big[\tfrac{1}{n} \log Z^\Omega_{3,n,L_n}(M,m)\big], \tag{4.42}$$

where

$$Z^\Omega_{3,n,L_n}(M,m) = \sum_{N=n/mL_n}^{n/L_n} \sum_{\Theta_{\rm traj} \in \widetilde{\mathcal{D}}^M_{L_n,N}} \sum_{x \in \mathcal{X}^{M,m}_{\Theta_{\rm traj},\Omega}} \sum_{u \in \mathcal{U}^{M,m,L_n}_{\Theta_{\rm traj},x,n}} A_3 \tag{4.43}$$

with

$$A_3 = \prod_{i=0}^{N-1} e^{u_i L_n \psi(\Omega(i,\Pi_i + \cdot), \Xi_i, x_i, u_i)}. \tag{4.44}$$

For all Ω,

$$\frac{A_2}{A_3} = \prod_{i=0}^{N-1} e^{u_i L_n \big[\psi_{L_n}(\Omega(i,\Pi_i+\cdot),\Xi_i,x_i,u_i) - \psi(\Omega(i,\Pi_i+\cdot),\Xi_i,x_i,u_i)\big]}, \tag{4.45}$$

and, for all $i \in \{0, \dots, N-1\}$, we have $(\Omega(i, \Pi_i + \cdot), \Xi_i, x_i, u_i) \in \mathcal{V}^{*,m}_M$, so that Proposition 2.4 can be applied.

4.2.3 Step 3

In this step we want the variational formula (4.4) to appear. Recall (2.47) and define, for $n \in \mathbb{N}$, $(M,m) \in \mathrm{EIGH}$, $N \in \{\frac{n}{mL_n}, \dots, \frac{n}{L_n}\}$, $\Theta_{\mathrm{traj}} \in \widetilde{\mathcal{D}}^M_{L_n,N}$ and $x \in \mathcal{X}^{M,m}_{\Theta_{\mathrm{traj}},\Omega}$,

$$\Theta_j = (\Omega(j, \Pi_j + \cdot), \Xi_j, x_j), \qquad j = 0, \dots, N-1, \tag{4.46}$$

and

$$\rho^\Omega_{\Theta_{\mathrm{traj}},x}(\Theta, \Theta') = \frac{1}{N} \sum_{j=1}^{N} 1_{\left\{(\Theta_{j-1},\Theta_j)=(\Theta,\Theta')\right\}}, \tag{4.47}$$

and, for $u \in \mathcal{U}^{M,m,L_n}_{\Theta_{\mathrm{traj}},x,n}$,

$$H^\Omega(\Theta_{\mathrm{traj}}, x, u) = \sum_{j=0}^{N-1} u_j\, \psi(\Theta_j, u_j). \tag{4.48}$$

In terms of these quantities we can rewrite $Z^\Omega_{3,n,L_n}(M,m)$ in (4.43) as

$$Z^\Omega_{3,n,L_n}(M,m) = \sum_{N=n/mL_n}^{n/L_n} \sum_{\Theta_{\mathrm{traj}} \in \widetilde{\mathcal{D}}^M_{L_n,N}} \sum_{x \in \mathcal{X}^{M,m}_{\Theta_{\mathrm{traj}},\Omega}} \sum_{u \in \mathcal{U}^{M,m,L_n}_{\Theta_{\mathrm{traj}},x,n}} e^{L_n\, H^\Omega(\Theta_{\mathrm{traj}},x,u)}. \tag{4.49}$$

For $n \in \mathbb{N}$, denote by

$$N^\Omega_n, \qquad \Theta^\Omega_{\mathrm{traj},n} \in \widetilde{\mathcal{D}}^M_{L_n,N^\Omega_n}, \qquad x^\Omega_n \in \mathcal{X}^{M,m}_{\Theta^\Omega_{\mathrm{traj},n},\Omega}, \qquad u^\Omega_n \in \mathcal{U}^{M,m,L_n}_{\Theta^\Omega_{\mathrm{traj},n},x^\Omega_n,n}, \tag{4.50}$$

the indices in the summation set of (4.49) that maximize $H^\Omega(\Theta_{\mathrm{traj}}, x, u)$. For ease of notation we put

$$\Theta^\Omega_{\mathrm{traj},n} = (\Xi^n_j)_{j=0}^{N^\Omega_n-1}, \quad x^\Omega_n = (x^n_j)_{j=0}^{N^\Omega_n-1}, \quad u^\Omega_n = (u^n_j)_{j=0}^{N^\Omega_n-1}, \tag{4.51}$$

and

$$c_n = \left|\left\{(N, \Theta_{\mathrm{traj}}, x, u)\colon \tfrac{n}{mL_n} \le N \le \tfrac{n}{L_n},\ \Theta_{\mathrm{traj}} \in \widetilde{\mathcal{D}}^M_{L_n,N},\ x \in \mathcal{X}^{M,m}_{\Theta_{\mathrm{traj}},\Omega},\ u \in \mathcal{U}^{M,m,L_n}_{\Theta_{\mathrm{traj}},x,n}\right\}\right|. \tag{4.52}$$

Then we can estimate

$$\frac{1}{n} \log Z^\Omega_{3,n,L_n}(M,m) \le \frac{1}{n} \log c_n + \tfrac{L_n}{n} \sum_{j=0}^{N^\Omega_n-1} u^n_j\, \psi(\Theta^n_j, u^n_j). \tag{4.53}$$

We next note that $u \mapsto u\psi(\Theta, u)$ is concave for all $\Theta \in \overline{\mathcal{V}}_M$ (see Lemma B.4). Hence, after setting

$$v^n_\Theta = \sum_{j=0}^{N^\Omega_n-1} 1_{\{\Theta^n_j=\Theta\}}\, u^n_j, \qquad d^n_\Theta = \sum_{j=0}^{N^\Omega_n-1} 1_{\{\Theta^n_j=\Theta\}}, \qquad \Theta \in \overline{\mathcal{V}}^m_M, \tag{4.54}$$

we can estimate

$$\sum_{j=0}^{N^\Omega_n-1} 1_{\{\Theta^n_j=\Theta\}}\, u^n_j\, \psi(\Theta^n_j, u^n_j) \le v^n_\Theta\, \psi\big(\Theta, \tfrac{v^n_\Theta}{d^n_\Theta}\big) \quad \text{for} \quad \Theta \in \overline{\mathcal{V}}^m_M\colon d^n_\Theta \ge 1. \tag{4.55}$$

Next, we recall (4.47) and we set $\rho_n = \rho^{\Omega}_{\Theta^{\Omega}_{\mathrm{traj},n},x^{\Omega}_n}$, so that $\rho_{n,1}(\Theta) = d^n_\Theta / N^\Omega_n$ for all $\Theta \in \overline{\mathcal{V}}^m_M$. Since $\{\Theta \in \overline{\mathcal{V}}^m_M \colon d^n_\Theta \geq 1\}$ is a finite subset of $\overline{\mathcal{V}}^m_M$, we can easily extend $\Theta \mapsto v^n_\Theta / d^n_\Theta$ from $\{\Theta \in \overline{\mathcal{V}}_M \colon d^n_\Theta \geq 1\}$ to $\overline{\mathcal{V}}^m_M$ as a continuous function. Moreover, $\sum_{j=0}^{N^\Omega_n - 1} u^n_j = n/L_n$ implies that $N^\Omega_n \int_{\overline{\mathcal{V}}^m_M} v^n_\Theta / d^n_\Theta \, \rho_{n,1}(d\Theta) = n/L_n$, which, together with (4.53) and (4.55) gives

$$\tfrac{1}{n} \log Z^\Omega_{3,n,L_n}(M,m) \leq \sup_{u \in \mathcal{B}_{\overline{\mathcal{V}}^m_M}} \frac{\int_{\overline{\mathcal{V}}^m_M} u_\Theta \, \psi(\Theta, u_\Theta) \, \rho_n(d\Theta)}{\int_{\overline{\mathcal{V}}^m_M} u_\Theta \, \rho_n(d\Theta)} + o(1), \qquad n \to \infty, \tag{4.56}$$

where we use that $\lim_{n\to\infty} \frac{1}{n} \log c_n = 0$. In what follows, we abbreviate the first term in the right-hand side of the last display by l_n. We want to show that $\limsup_{n\to\infty} \frac{1}{n} \log Z^\Omega_{3,n,L_n}(M,m) \leq f(M,m;\alpha,\beta)$. To that end, we assume that $\frac{1}{n} \log Z^\Omega_{3,n,L_n}(M,m)$ converges to some $t \in \mathbb{R}$ and we prove that $t \leq f(M,m;\alpha,\beta)$. Since $(l_n)_{n\in\mathbb{N}}$ is bounded and $\overline{\mathcal{V}}^m_M$ is compact, it follows from the definition of l_n that along an appropriate subsequence both $l_n \to l_\infty \geq t$ and $\rho_n \to \rho_\infty \in \mathcal{R}^m_{p,M}$ as $n \to \infty$. Hence, the proof will be complete once we show that

$$l_\infty \leq \sup_{u \in \mathcal{B}_{\overline{\mathcal{V}}^m_M}} V(\rho_\infty, u), \tag{4.57}$$

because the right-hand side in (4.57) is bounded from above by $f(M,m;\alpha,\beta)$.

Recall (2.16) and, for $\Theta \in \overline{\mathcal{V}}^m_M$ and $y \in \mathbb{R}$, define

$$u^{M,m}_\Theta(y) = \begin{cases} t_\Theta & \text{if } \partial^+_u (u\,\psi(\Theta,u))(t_\Theta) \leq y, \\ m & \text{if } \partial^-_u (u\,\psi(\Theta,u))(m) \geq y, \\ z & \text{otherwise, with } z \text{ such that } \partial^-_u (u\,\psi(\Theta,u))(z) \geq y \geq \partial^+_u (u\,\psi(\Theta,u))(z), \end{cases} \tag{4.58}$$

where z is unique by strict concavity of $u \to u\psi(\Theta,u)$ (see Lemma B.2).

Lemma 4.6 *(i) For all $y \in \mathbb{R}$ and $(M,m) \in$ EIGH, $\Theta \mapsto u^{M,m}_\Theta(y)$ is continuous on $(\overline{\mathcal{V}}^m_M, d_M)$, where d_M is defined in* (B.3) *in Appendix* B.
(ii) For all $(M,m) \in$ EIGH and $\Theta \in \overline{\mathcal{V}}^m_M$, $y \mapsto u^{M,m}_\Theta(y)$ is continuous on $\mathbb{R}$.

Proof. The proof uses the strict concavity of $u \to u\psi(\Theta,u)$ (see Lemma B.2).

(i) The proof is by contradiction. Pick $y \in \mathbb{R}$, and pick a sequence $(\Theta_n)_{n\in\mathbb{N}}$ in $\overline{\mathcal{V}}^m_M$ such that $\lim_{n\to\infty} \Theta_n = \Theta_\infty \in \overline{\mathcal{V}}^m_M$. Suppose that $u^{M,m}_{\Theta_n}(y)$ does not tend to $u^{M,m}_{\Theta_\infty}(y)$ as $n \to \infty$. Then, by choosing an appropriate subsequence, we may assume that $\lim_{n\to\infty} u^{M,m}_{\Theta_n}(y) = u_1 \in [t_{\Theta_\infty}, m]$ with $u_1 < u^{M,m}_{\Theta_\infty}(y)$. The case $u_1 > u^{M,m}_{\Theta_\infty}(y)$ can be handled similarly.

Pick $u_2 \in (u_1, u^{M,m}_{\Theta_\infty}(y))$. For n large enough, we have $u^{M,m}_{\Theta_n}(y) < u_2 < u^{M,m}_{\Theta_\infty}(y)$. By the definition of $u^{M,m}_{\Theta_n}(y)$ in (4.58) and the strict concavity of $u \mapsto u\psi(\Theta_n,u)$ we have, for n large enough,

$$\partial^+_u (u\,\psi(\Theta_n,u))(u^{M,m}_{\Theta_n}(y)) > \frac{u^{M,m}_{\Theta_\infty}(y)\psi(\Theta_n, u^{M,m}_{\Theta_\infty}(y)) - u_2\psi(\Theta_n,u_2)}{u^{M,m}_{\Theta_\infty}(y) - u_2}. \tag{4.59}$$

Let $n \to \infty$ in (4.59) and use the strict concavity once again, to get

$$\liminf_{n\to\infty} \partial^+_u (u\,\psi(\Theta_n,u))(u^{M,m}_{\Theta_n}(y)) > \partial^-_u (u\,\psi(\Theta_\infty,u))(u^{M,m}_{\Theta_\infty}(y)). \tag{4.60}$$

If $u_{\Theta_\infty}^{M,m}(y) \in (t_{\Theta_\infty}, m]$, then (4.58) implies that the right-hand side of (4.60) is not smaller than y. Hence (4.60) yields that $\partial_u^+(u\,\psi(\Theta_n,u))(u_{\Theta_n}^{M,m}(y)) > y$ for n large enough, which implies that $u_{\Theta_n}^{M,m}(y) = m$ by (4.58). However, the latter inequality contradicts the fact that $u_{\Theta_n}^{M,m}(y) < u_2 < u_{\Theta_\infty}^{M,m}(y)$ for n large enough. If $u_{\Theta_\infty}^{M,m}(y) = t_{\Theta_\infty}$, then we note that $\lim_{n\to\infty} t_{\Theta_n} = t_{\Theta_\infty}$, which again contradicts that $t_{\Theta_n} \le u_{\Theta_n}^{M,m}(y) < u_2 < u_{\Theta_\infty}^{M,m}(y)$ for n large enough.

(ii) The proof is again by contradiction. Pick $\Theta \in \overline{\mathcal{V}}_M^m$, and pick an infinite sequence $(y_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} y_n = y_\infty \in \mathbb{R}$ and such that $u_\Theta^{M,m}(y_n)$ does not converge to $u_\Theta^{M,m}(y_\infty)$. Then, by choosing an appropriate subsequence, we may assume that there exists a $u_1 < u_\Theta^{M,m}(y_\infty)$ such that $\lim_{n\to\infty} u_\Theta^{M,m}(y_n) = u_1$. The case $u_1 > u_\Theta^{M,m}(y_\infty)$ can be treated similarly.

Pick $u_2, u_3 \in (u_1, u_\Theta^{M,m}(y_\infty))$ such that $u_2 < u_3$. Then, for n large enough, we have

$$t_\Theta \le u_\Theta^{M,m}(y_n) < u_2 < u_3 < u_\Theta^{M,m}(y_\infty) \le m. \tag{4.61}$$

Combining (4.58) and (4.61) with the strict concavity of $u \mapsto u\psi(\Theta,u)$ we get, for n large enough,

$$y_n > \partial_u^+(u\,\psi(\Theta,u))(u_2) > \partial_u^-(u\,\psi(\Theta,u))(u_3) > y_\infty, \tag{4.62}$$

which contradicts $\lim_{n\to\infty} y_n = y_\infty$. □

We resume the line of proof. Recall that $\rho_{n,1}$, $n \in \mathbb{N}$, charges finitely many $\Theta \in \overline{\mathcal{V}}_M^m$. Therefore the continuity and the strict concavity of $u \mapsto u\psi(\Theta,u)$ on $[t_\Theta, m]$ for all $\Theta \in \overline{\mathcal{V}}_M^m$ (see Lemma B.4) imply that the supremum in (4.56) is attained at some $u_n^{M,m} \in \mathcal{B}_{\overline{\mathcal{V}}_M^m}$ that satisfies $u_n^{M,m}(\Theta) = u_\Theta^{M,m}(l_n)$ for $\Theta \in \overline{\mathcal{V}}_M^m$. Set $u_\infty^{M,m}(\Theta) = u_\Theta^{M,m}(l_\infty)$ for $\Theta \in \overline{\mathcal{V}}_M^m$ and note that $(l_n)_{n\in\mathbb{N}}$ may be assumed to be monotone, say, non-decreasing. Then the concavity of $u \mapsto u\psi(\Theta,u)$ for $\Theta \in \overline{\mathcal{V}}_M^m$ implies that $(u_n^{M,m})_{n\in\mathbb{N}}$ is a non-increasing sequence of functions on $\overline{\mathcal{V}}_M^m$. Moreover, $\overline{\mathcal{V}}_M^m$ is a compact set and, by Lemma 4.6(ii), $\lim_{n\to\infty} u_n^{M,m}(\Theta) = u_\infty^{M,m}(\Theta)$ for $\Theta \in \overline{\mathcal{V}}_M^m$. Therefore Dini's theorem implies that $\lim_{n\to\infty} u_n^{M,m} = u_\infty^{M,m}$ uniformly on $\overline{\mathcal{V}}_M^m$. We estimate

$$\begin{aligned}
&\left| l_n - \int_{\overline{\mathcal{V}}_M^m} u_\infty^{M,m}(\Theta)\,\psi(\Theta, u_\infty^{M,m}(\Theta))\rho_\infty(d\Theta)\right| \\
&\quad \le \int_{\overline{\mathcal{V}}_M^m} \left| u_n^{M,m}(\Theta)\,\psi(\Theta, u_n^{M,m}(\Theta)) - u_\infty^{M,m}(\Theta)\,\psi(\Theta, u_\infty^{M,m}(\Theta))\right| \rho_n(d\Theta) \\
&\quad + \left| \int_{\overline{\mathcal{V}}_M^m} u_\infty^{M,m}(\Theta)\,\psi(\Theta, u_\infty^{M,m}(\Theta))\,\rho_n(d\Theta) - \int_{\overline{\mathcal{V}}_M^m} u_\infty^{M,m}(\Theta)\,\psi(\Theta, u_\infty^{M,m}(\Theta))\,\rho_\infty(d\Theta)\right|.
\end{aligned} \tag{4.63}$$

The second term in the right-hand side of (4.63) tends to zero as $n \to \infty$ because, by Lemma 4.6(i), $\Theta \mapsto u_\infty^{M,m}(\Theta)$ is continuous on $\overline{\mathcal{V}}_M^m$ and because ρ_n converges in law to ρ_∞ as $n \to \infty$. The first term in the right-hand side of (4.63) tends to zero as well, because $(\Theta,u) \mapsto u\psi(\Theta,u)$ is uniformly continuous on $\overline{\mathcal{V}}_M^{*,m}$ (see Lemma B.3) and because we have proved above that $u_n^{M,m}$ converges to $u_\infty^{M,m}$ uniformly on $\overline{\mathcal{V}}_M^m$. This proves (4.57), and so Step 3 is complete.

4.2.4 Step 4

In this step we prove that

$$\limsup_{n\to\infty} f_{3,n}^\Omega(M,m;\alpha,\beta) \ge f(M,m;\alpha,\beta) \;\text{ for } \mathbb{P}-a.e.\ \Omega. \tag{4.64}$$

Note that the proof will be complete once we show that

$$\limsup_{n\to\infty} f^{\Omega}_{3,n}(M,m,\alpha,\beta) \geq V(\rho,u) \text{ for } \rho\in\mathcal{R}^m_{p,M}, u\in\mathcal{B}_{\overline{\mathcal{V}}^m_M}. \tag{4.65}$$

Pick $\Omega\in\{A,B\}^{\mathbb{N}_0\times\mathbb{Z}}$, $\rho\in\mathcal{R}^{\Omega,m}_{p,M}$ and $u\in\mathcal{B}_{\overline{\mathcal{V}}^m_M}$. By the definition of $\mathcal{R}^{\Omega,m}_{p,M}$, there exists a strictly increasing subsequence $(n_k)_{k\in\mathbb{N}}\in\mathbb{N}^{\mathbb{N}}$ such that, for all $k\in\mathbb{N}$, there exists an

$$N_k\in\left\{\frac{n_k}{mL_{n_k}},\dots,\frac{n_k}{L_{n_k}}\right\}, \tag{4.66}$$

a $\Theta^k_{\text{traj}}\in\widetilde{\mathcal{D}}^M_{L_{n_k},N_k}$ and a $x^k\in\mathcal{X}^{M,m}_{\Theta^k_{\text{traj}},\Omega}$ such that $\rho_k=^{\text{def}}\rho^{\Omega}_{\Theta^k_{\text{traj}},x^k}$ (see (4.47)) converges in law to ρ as $k\to\infty$. Recall (4.23), and note that

$$\Xi^k_j=\big(\Delta\Pi^k_j,\, b^k_j,\, b^k_{j+1}\big),\quad j=0,\dots,N_k-1, \tag{4.67}$$

with $\Delta\Pi^k_j\in\{-M,\dots,M\}$ and $b^k_j\in(0,1]\cap\frac{\mathbb{N}}{L_{n_k}}$ for $j=0,\dots,N_k$. For ease of notation we define

$$\Theta^k_j=\big(\Omega(j,\Pi^k_j+\cdot),\Xi^k_j,x^k_j\big)\quad\text{with}\quad \Pi^k_j=\sum_{i=0}^{j-1}\Delta\Pi^k_i,\qquad j=0,\dots,N_k-1, \tag{4.68}$$

and

$$v_k=N_k\int_{\Theta\in\mathcal{V}^m_M}u_\Theta\,\rho_{k,1}(d\Theta)=\sum_{j=0}^{N_k-1}u_{\Theta^k_j}, \tag{4.69}$$

where we recall that $u=(u_\Theta)_{\Theta\in\overline{\mathcal{V}}^m_M}$ was fixed at the beginning of the section.

Next, we recall that $\lim_{n\to\infty} n/L_n=\infty$ and that L_n is non-decreasing. Together with the fact that $\lim_{n\to\infty} L_n/n=0$, this implies that L_n is constant on intervals. On those intervals, n/L_n takes constant increments. The latter implies that there exists an $\widetilde{n}_k\in\mathbb{N}$ satisfying

$$0\leq v_k-\tfrac{\widetilde{n}_k}{L_{\widetilde{n}_k}}\leq\tfrac{1}{L_{\widetilde{n}_k}}\quad\text{and therefore}\quad 0\leq v_kL_{\widetilde{n}_k}-\widetilde{n}_k\leq 1. \tag{4.70}$$

Next, for $j=0,\dots,N_k-1$ we pick $\overline{b^k_j}\in(0,1]\cap\frac{N}{L_{\widetilde{n}_k}}$ such that $|\overline{b^k_j}-b^k_j|\leq\frac{1}{L_{\widetilde{n}_k}}$, define

$$\overline{\Xi^k_j}=\big(\Delta\Pi^k_j,\overline{b^k_j},\overline{b^k_{j+1}}\,\big),\quad \overline{\Theta^k_j}=\big(\Omega(j,\Pi^k_j+\cdot),\overline{\Xi^k_j},x^k_j\,\big), \tag{4.71}$$

and pick

$$s^k_j\in t_{\overline{\Theta^k_j}}+\frac{2\mathbb{N}}{L_{\widetilde{n}_k}}\quad\text{such that}\quad |s^k_j-u_{\Theta^k_j}|\leq 2/L_{\widetilde{n}_k}. \tag{4.72}$$

We use (4.69) to write

$$L_{\widetilde{n}_k}\sum_{j=0}^{N_k-1}s^k_j=L_{\widetilde{n}_k}\left(v_k+\sum_{j=0}^{N_k-1}(s^k_j-u_{\Theta^k_j})\right)=L_{\widetilde{n}_k}(I+II). \tag{4.73}$$

Next, we note that (4.70) and (4.72) imply that $|L_{\widetilde{n}_k}I-\widetilde{n}_k|\leq 1$ and $|L_{\widetilde{n}_k}II|\leq 2N_k$. The latter in turn implies that, by adding or subtracting at most 3 steps per colum, the quantities s^k_j for $j=0,\dots,N_k-1$ can be chosen in such a way that $\sum_{j=0}^{N_k-1}s^k_j=\widetilde{n}_k/L_{\widetilde{n}_k}$.

Next, set

$$\overline{\Theta^k_{\rm traj}} = (\overline{\Xi^k_j})_{j=0}^{N_k-1} \in \widetilde{\mathcal{D}}^M_{L_{\widetilde{n}_k},N_k}, \quad s^k = (s^k_j)_{j=0}^{N_k-1} \in \mathcal{U}^{M,m,L_{\widetilde{n}_k}}_{\overline{\Theta^k_{\rm traj}},x^k,\widetilde{n}_k}, \tag{4.74}$$

and recall (4.43) to get $f_3^\Omega(\widetilde{n}_k, M) \geq R_k$ with

$$R_k = \frac{L_{\widetilde{n}_k} H^\Omega\big(\overline{\Theta^k_{\rm traj}}, x^k, s^k\big)}{\widetilde{n}_k} = \frac{\sum_{j=0}^{N_k-1} s^k_j\, \psi\Big(\overline{\Theta^k_j},\, s^k_j\Big)}{\sum_{j=0}^{N_k-1} s^k_j} = \frac{R^k_{\rm nu}}{R^k_{\rm de}}. \tag{4.75}$$

Further set

$$R'_k = \frac{R'^k_{\rm nu}}{R'^k_{\rm de}} = \frac{\int_{\overline{\mathcal{V}}^m_M} u_\Theta\, \psi(\Theta, u_\Theta)\rho_k(d\Theta)}{\int_{\overline{\mathcal{V}}^m_M} u_\Theta\, \rho_k(d\Theta)}, \tag{4.76}$$

and note that $\lim_{k\to\infty} R'_k = V(\rho, u)$, since $\lim_{k\to\infty} \rho_k = \rho$ by assumption and $\Theta \mapsto u_\Theta$ is continuous on $\mathcal{V}^m_M$. We note that R'_k can be rewritten in the form

$$R'_k = \frac{R'^k_{\rm nu}}{R'^k_{\rm de}} = \frac{\sum_{j=0}^{N_k-1} u_{\Theta^k_j}\, \psi\big(\Theta^k_j,\, u_{\Theta^k_j}\big)}{\sum_{j=0}^{N_k-1} u_{\Theta^k_j}}. \tag{4.77}$$

Now recall that $\lim_{k\to\infty} n_k = \infty$. Since $N_k \geq n_k/ML_{n_k}$, it follows that $\lim_{k\to\infty} N_k = \infty$ as well. Moreover, $N_k \leq \widetilde{n}_k/L_{\widetilde{n}_k}$ with $\lim_{k\to\infty} \widetilde{n}_k = \infty$. Therefore (4.69–4.70) allow us to conclude that $R^k_{\rm de} = \widetilde{n}_k/L_{\widetilde{n}_k} = R'^k_{\rm de}[1 + o(1)]$.

Next, note that $\mathcal{H}_M$ is compact, and that $(\Theta, u) \mapsto u\psi(\Theta, u)$ is continuous on $\mathcal{H}_M$ and therefore is uniformly continuous. Consequently, for all $\varepsilon > 0$ there exists an $\eta > 0$ such that, for all $(\Theta, u), (\Theta', u') \in \mathcal{H}_M$ satisfying $|\Theta - \Theta'| \leq \eta$ and $|u - u'| \leq \eta$,

$$|u\psi(\Theta, u) - u'\psi(\Theta', u')| \leq \varepsilon. \tag{4.78}$$

We recall (4.71), which implies that $d_M(\overline{\Theta^k_j}, \Theta_j) \leq 2/L_{\widetilde{n}_k}$ for all $j \in \{0, \dots, N_k - 1\}$, we choose k large enough to ensure that $2/L_{\widetilde{n}_k} \leq \eta$, and we use (4.78), to obtain

$$R^k_{\rm nu} = \sum_{j=0}^{N_k-1} s^k_j\, \psi\Big(\overline{\Theta^k_j},\, s^k_j\Big) = \sum_{j=0}^{N_k-1} u_{\Theta^k_j}\, \psi\big(\Theta^k_j,\, u_{\Theta^k_j}\big) + T = R'^k_{\rm nu} + T, \tag{4.79}$$

with $|T| \leq \varepsilon N_k$. Since $\lim_{k\to\infty} R'_k = V(\rho, u)$ and $\sum_{j=0}^{N_k-1} u_{\Theta^k_j} = v_k \geq \widetilde{n}_k/L_{\widetilde{n}_k}$ (see (4.70)), if $V(\rho, u) \neq 0$, then $\big|R'^k_{\rm nu}\big| \geq \text{Cst.}\ \widetilde{n}_k/L_{\widetilde{n}_k}$, whereas $|T| \leq \varepsilon N_k \leq \varepsilon \widetilde{n}_k/L_{\widetilde{n}_k}$ for k large enough. Hence $T = o(R'^k_{\rm nu})$ and

$$\frac{R^k_{\rm nu}}{R^k_{\rm de}} = \frac{R'^k_{\rm nu}\,[1 + o(1)]}{R'^k_{\rm de}\,[1 + o(1)]} \to V(\rho, u), \qquad k \to \infty. \tag{4.80}$$

Finally, if $V(\rho, u) = 0$, then $R'^k_{\rm nu} = o(R'^k_{\rm de})$ and $T = o(R'^k_{\rm de})$, so that R_k tends to 0. This completes the proof of Step 4.

4.2.5 Step 5

In this step we prove (4.19), suppressing the (α,β)-dependence from the notation. For $\Omega\in\{A,B\}^{\mathbb{N}_0\times\mathbb{Z}^2}$, $n\in\mathbb{N}$, $N\in\{n/mL_n,\dots,n/L_n\}$ and $r\in\{-NM,\dots,NM\}$, we recall (4.23) and define

$$\widetilde{\mathcal{D}}_{L,N}^{M,m,r}=\Big\{\Theta_{\text{traj}}\in\widetilde{\mathcal{D}}_{L,N}^{M,m}\colon \Pi_N=r\Big\},\tag{4.81}$$

where we recall that $\Pi_N=\sum_{j=0}^{N-1}\Delta\Pi_j$. We set

$$f_{3,n}^{\Omega}(M,m,N,r)=\tfrac{1}{n}\log Z_{3,n,L_n}^{\Omega}(N,M,m,r)\tag{4.82}$$

with

$$Z_{3,n,L_n}^{\Omega}(N,M,m,r)=\sum_{\Theta_{\text{traj}}\in\widetilde{\mathcal{D}}_{L_n,N}^{M,m,r}}\ \sum_{x\in\mathcal{X}_{\Theta_{\text{traj}},\Omega}^{M,m}}\ \sum_{u\in\mathcal{U}_{\Theta_{\text{traj}},n}^{M,m,L_n}}A_3,\tag{4.83}$$

where A_3 is defined in (4.44). We further set $f_3(\cdot)=\mathbb{E}_\Omega\big(f_3^{\Omega}(\cdot)\big)$.

4.2.6 Concentration of measure

In the first part of this step we prove that for all $(M,m,\alpha,\beta)\in$ EIGH $\times$ CONE there exist $c_1,c_2>0$ (depending on (M,m,α,β) only) such that, for all $n\in\mathbb{N}$, $N\in\{n/(mL_n),\dots n/L_n\}$ and $r\in\{-NM,\dots,NM\}$,

$$\begin{aligned}\mathbb{P}_\Omega\big(\big|f_{3,n}^{\Omega}(M,m)-f_{3,n}(M,m)\big|>\varepsilon\big)&\le c_1\,e^{-\frac{c_2\varepsilon^2 n}{L_n}},\\ \mathbb{P}_\Omega\big(\big|f_{3,n}^{\Omega}(M,m,N,r)-f_{3,n}(M,m,N,r)\big|>\varepsilon\big)&\le c_1\,e^{-\frac{c_2\varepsilon^2 n}{L_n}}.\end{aligned}\tag{4.84}$$

We only give the proof of the first inequality. The second inequality is proved in a similar manner. The proof uses Theorem C.1. Before we start we note that, for all $n\in\mathbb{N}$, $(M,m)\in$ EIGH and $\Omega\in\{A,B\}^{\mathbb{N}_0\times\mathbb{Z}}$, $f_{3,n}^{\Omega}(M,m)$ only depends on

$$\mathcal{C}_{0,L_n}^{\Omega},\dots,\mathcal{C}_{n/L_n,L_n}^{\Omega}\quad\text{with}\quad \mathcal{C}_{j,L_n}^{\Omega}=(\Omega(j,i))_{i=-n/L_n}^{n/L_n}.\tag{4.85}$$

We apply Theorem C.1 with $\mathcal{S}=\{0,\dots,n/L_n\}$, with $X_i=\{A,B\}^{\{-\frac{n}{L_n},\dots,\frac{n}{L_n}\}}$ and with μ_i the uniform measure on X_i for all $i\in\mathcal{S}$. Note that $|f_{3,n}^{\Omega_1}(M,m)-f_{3,n}^{\Omega_2}(M,m)|\le 2C_{\text{uf}}(\alpha)m\frac{L_n}{n}$ for all $i\in\mathcal{S}$ and all Ω_1,Ω_2 satisfying $\mathcal{C}_{j,n}^{\Omega_1}=\mathcal{C}_{j,n}^{\Omega_2}$ for all $j\neq i$. After we set $c=2C_{\text{uf}}(\alpha)m$ we can apply Theorem C.1 with $D=c^2L_n/n$ to get (4.84).

Next, we note that the first inequality in (4.84), the Borel-Cantelli lemma and the fact that $\lim_{n\to\infty}n/L_n\log n=\infty$ imply that, for all $(M,m)\in$ EIGH,

$$\lim_{n\to\infty}\Big[f_{3,n}^{\Omega}(M,m)-f_{3,n}(M,m)\Big]=0\quad\text{for }\mathbb{P}-a.e.\ \Omega.\tag{4.86}$$

Therefore (4.19) will be proved once we show that

$$\liminf_{n\to\infty}f_{3,n}(M,m)=\limsup_{n\to\infty}f_{3,n}(M,m).\tag{4.87}$$

To that end, we first prove that, for all $n\in\mathbb{N}$ and all $(M,m)\in$ EIGH, there exist an $N_n\in\{n/mL_n,\dots,n/L_n\}$ and an $r_n\in\{-MN_n,\dots,MN_n\}$ such that

$$\lim_{n\to\infty}\Big[f_{3,n}(M,m)-f_{3,n}(M,m,N_n,r_n)\Big]=0.\tag{4.88}$$

The proof of (4.88) is done as follows. Pick $\varepsilon > 0$, and for $\Omega \in \{A,B\}^{\mathbb{N}_0\times\mathbb{Z}}$, $n \in \mathbb{N}$ and $(M,m) \in$ EIGH, denote by N_n^Ω and r_n^Ω the maximizers of $f_{3,n}^\Omega(M,m,N,r)$. Then

$$f_{3,n}^\Omega\big(M,m,N_n^\Omega,r_n^\Omega\big) \leq f_{3,n}^\Omega(M,m) \leq \tfrac{1}{n}\log(\tfrac{n^2}{L_n^2}) + f_{3,n}^\Omega\big(M,m,N_n^\Omega,r_n^\Omega\big), \tag{4.89}$$

so that, for n large enough and every Ω,

$$0 \leq f_{3,n}^\Omega(M,m) - f_{3,n}^\Omega\big(M,m,N_n^\Omega,r_n^\Omega\big) \leq \varepsilon. \tag{4.90}$$

For $n \in \mathbb{N}$, $N \in \{n/mL_n,\dots,n/L_n\}$ and $r \in \{-NM,\dots,NM\}$, we set

$$A_{n,N,r} = \{\Omega\colon (N_n^\Omega, r_n^\Omega) = (N,r)\}. \tag{4.91}$$

Next, denote by N_n, r_n the maximizers of $\mathbb{P}(A_{n,N,r})$. Note that (4.88) will be proved once we show that, for all $\varepsilon > 0$, $|f_{3,n}(M,m) - f_{3,n}(M,m,N_n,r_n)| \leq \varepsilon$ for n large enough. Further note that $\mathbb{P}(A_{n,N_n,r_n}) \geq L_n^2/n^2$ for all $n \in \mathbb{N}$. For every Ω we can therefore estimate

$$|f_{3,n}(M,m) - f_{3,n}(M,m,N_n,r_n)| \leq I + II + III \tag{4.92}$$

with

$$\begin{aligned} I &= |f_{3,n}(M,m) - f_{3,n}^\Omega(M,m)|, \\ II &= |f_{3,n}^\Omega(M,m) - f_{3,n}^\Omega(M,m,N_n,r_n)|, \\ III &= |f_{3,n}^\Omega(M,m,N_n,r_n) - f_{3,n}(M,m,N_n,r_n)|. \end{aligned} \tag{4.93}$$

Hence, the proof of (4.88) will be complete once we show that, for n large enough, there exists an $\Omega_{\varepsilon,n}$ for which I, II and III in (4.93) are bounded from above by $\varepsilon/3$.

To that end, note that, because of (4.84), the probabilities $\mathbb{P}(\{I > \varepsilon/3\})$ and $\mathbb{P}(\{III > \varepsilon/3\})$ are bounded from above by $c_1 e^{-c_2\varepsilon^2 n/9L_n}$, while

$$\mathbb{P}(\{II > \varepsilon\}) \leq \mathbb{P}(A_{n,N_n,r_n}^c) \leq 1 - (L_n^2/n^2), \qquad n \in \mathbb{N}. \tag{4.94}$$

Since $\lim_{n\to\infty} n/L_n \log n = \infty$, we have $\mathbb{P}(\{I, II, III \leq \varepsilon/3\}) > 0$ for n large enough. Consequently, the set $\{I, II, III \leq \varepsilon/3\}$ is non-empty and (4.88) is proven.

4.2.7 Convergence

It remains to prove (4.87). Assume that there exist two strictly increasing subsequences $(n_k)_{k\in\mathbb{N}}$ and $(t_k)_{k\in\mathbb{N}}$ and two limits $l_2 > l_1$ such that $\lim_{k\to\infty} f_{3,n_k}(M,m) = l_2$ and $\lim_{k\to\infty} f_{3,t_k}(M,m) = l_1$. By using (4.88), we have that for every $k \in \mathbb{N}$ there exist $N_k \in \{n_k/mL_{n_k}, \dots, n_k/L_{n_k}\}$ and $r_k \in \{-MN_k,\dots,MN_k\}$ such that $\lim_{k\to\infty} f_{3,n_k}(M,m,N_k,r_k) = l_2$. Denote by

$$(\Theta_{\text{traj,max}}^{k,\Omega}, x_{\max}^{k,\Omega}, u_{\max}^{k,\Omega}) \in \widetilde{\mathcal{D}}_{L_{n_k},N_k}^{M,r_k} \times \mathcal{X}_{\Theta_{\text{traj,max}}^{k,\Omega},\Omega}^{M,m} \times \mathcal{U}_{\Theta_{\text{traj,max}}^{k,\Omega},x_{\max}^{k,\Omega},n_k}^{M,m,L_n} \tag{4.95}$$

the maximizer of $H^\Omega(\Theta_{\text{traj}},x,u)$. We recall that Θ_{traj}, x and u take their values in sets that grow subexponentially fast in n_k, and therefore

$$\lim_{k\to\infty} \tfrac{L_{n_k}}{n_k}\,\mathbb{E}_\Omega\big[H^\Omega(\Theta_{\text{traj,max}}^{k,\Omega}, x_{\max}^{k,\Omega}, u_{\max}^{k,\Omega})\big] = l_2. \tag{4.96}$$

Since $l_2 > l_1$, we can use (4.96) and the fact that $\lim_{k\to\infty} n_k/L_{n_k} = \infty$ to obtain, for k large enough,

$$\mathbb{E}_\Omega\big[H^\Omega(\Theta^{k,\Omega}_{\rm traj,max}, x^{k,\Omega}_{\max}, u^{k,\Omega}_{\max})\big] + (\beta-\alpha) \geq \tfrac{n_k}{L_{n_k}}\big(l_1 + \tfrac{l_2-l_1}{2}\big). \tag{4.97}$$

(The term $\beta-\alpha$ in the left-hand side of (4.97) is introduced for later convenience only.) Next, pick $k_0 \in \mathbb{N}$ satisfying (4.97), whose value will be specified later. Similarly to what we did in (4.72) and (4.73), for $\Omega \in \{A,B\}^{\mathbb{N}_0\times\mathbb{Z}}$ and $k \in \mathbb{N}$ we associate with

$$\Theta^{k_0,\Omega}_{\rm traj,max} = \big(\Delta\Pi^{k_0,\Omega}_j, b^{k_0,\Omega}_{0,j}, b^{k_0,\Omega}_{1,j}\big)_{j=0}^{N_{k_0}-1} \in \widetilde{\mathcal{D}}^{M,r_{k_0}}_{L_{n_{k_0}},N_{k_0}} \tag{4.98}$$

and

$$x^{k_0,\Omega}_{\max} = \big(x^{k_0,\Omega}_j\big)_{j=0}^{N_{k_0}-1} \in \mathcal{X}^{M,m}_{\Theta^{k_0,\Omega}_{\rm traj,max},\Omega} \tag{4.99}$$

and

$$u^{k_0,\Omega}_{\max} = \big(u^{k_0,\Omega}_j\big)_{j=0}^{N_{k_0}-1} \in \mathcal{U}^{M,m,L_{n_{k_0}}}_{\Theta^{k_0,\Omega}_{\rm traj,max},x^{k_0,\Omega}_{\max},n_{k_0}} \tag{4.100}$$

the quantities

$$\overline{\Theta}^{k,\Omega}_{\rm traj} = \big(\Delta\Pi^{k_0,\Omega}_j, \overline{b}^{\,k,\Omega}_{0,j}, \overline{b}^{\,k,\Omega}_{1,j}\big)_{j=0}^{N_{k_0}-1} \in \widetilde{\mathcal{D}}^{M,r_{k_0}}_{L_{t_k},N_{k_0}} \tag{4.101}$$

and

$$\overline{u}^{\,k,\Omega} = \big(\overline{u}^{\,k,\Omega}_j\big)_{j=0}^{N_{k_0}-1} \in \mathcal{U}^{M,m,L_{t_k}}_{\overline{\Theta}^{k,\Omega}_{\rm traj},x^{k_0,\Omega}_{\max},*} \tag{4.102}$$

(where $*$ will be specified later), so that

$$\big|\overline{b}^{\,k,\Omega}_{0,j} - b^{k_0,\Omega}_{0,j}\big| \leq \tfrac{1}{L_{t_k}}, \quad \big|\overline{b}^{\,k,\Omega}_{1,j} - b^{k_0,\Omega}_{1,j}\big| \leq \tfrac{1}{L_{t_k}}, \quad \big|\overline{u}^{\,k,\Omega}_j - u^{k_0,\Omega}_j\big| \leq \tfrac{2}{L_{t_k}}, \quad j=0,\dots,N_{k_0}-1. \tag{4.103}$$

Next, put $\overline{s}^{\,\Omega}_k = L_{t_k}\sum_{j=0}^{N_{k_0}-1} \overline{u}^{\,k,\Omega}_j$, which we substitute for $*$ above. The uniform continuity in Lemma B.3 allows us to claim that, for k large enough and for all Ω,

$$\Big|\overline{u}^{\,k,\Omega}_j\ \psi\Big(\overline{\Theta}^{k,\Omega}_j, \overline{u}^{\,k,\Omega}_j\Big) - u^{k_0,\Omega}_j\ \psi\Big(\Theta^{k_0,\Omega}_j, u^{k_0,\Omega}_j\Big)\Big| \leq \tfrac{l_2-l_1}{4}, \tag{4.104}$$

where we recall that, as in (4.68), for all $j=0,\dots,N_{k_0}-1$,

$$\begin{aligned}\overline{\Theta}^{k,\Omega}_j &= \Big(\Omega\big(j,\Pi^{k_0,\Omega}_j+\cdot\big),\, \Delta\Pi^{k_0,\Omega}_j,\, \overline{b}^{\,k,\Omega}_{0,j},\, \overline{b}^{\,k,\Omega}_{1,j},\, x^{k_0,\Omega}_j\Big),\\ \Theta^{k_0,\Omega}_j &= \Big(\Omega\big(j,\Pi^{k_0,\Omega}_j+\cdot\big),\, \Delta\Pi^{k_0,\Omega}_j,\, b^{k_0,\Omega}_{0,j},\, b^{k_0,\Omega}_{1,j},\, x^{k_0,\Omega}_j\Big).\end{aligned} \tag{4.105}$$

Recall (4.48). An immediate consequence of (4.104) is that

$$\big|H^\Omega(\overline{\Theta}^{k,\Omega}_{\rm traj}, x^{k_0,\Omega}_{\max}, \overline{u}^{\,k,\Omega}) - H^\Omega(\Theta^{k_0,\Omega}_{\rm traj,max}, x^{k_0,\Omega}_{\max}, u^{k_0,\Omega}_{\max})\big| \leq N_{k_0}\tfrac{l_2-l_1}{4}. \tag{4.106}$$

Hence we can use (4.97), (4.106) and the fact that $N_{k_0} \leq n_{k_0}/L_{n_{k_0}}$, to conclude that, for k large enough,

$$\mathbb{E}_\Omega\big[H^\Omega(\overline{\Theta}^{k,\Omega}_{\rm traj}, x^{k_0,\Omega}_{\max}, \overline{u}^{\,k,\Omega})\big] + (\beta-\alpha) \geq \tfrac{n_{k_0}}{L_{n_{k_0}}}\big(l_1 + \tfrac{l_2-l_1}{4}\big). \tag{4.107}$$

At this stage we add a column at the end of the group of N_{k_0} columns in such a way that the conditions $\widehat{b}^{k,\Omega}_{1,N_{k_0}-1} = \widehat{b}^{k,\Omega}_{0,N_{k_0}}$ and $\widehat{b}^{k,\Omega}_{1,N_{k_0}} = 1/L_{t_k}$ are satisfied. We put

$$\widehat{\Xi}^{k,\Omega}_{N_{k_0}} = \big(\Delta\Pi^{k_0,\Omega}_{N_{k_0}}, \widehat{b}^{k,\Omega}_{0,N_{k_0}}, \widehat{b}^{k,\Omega}_{1,N_{k_0}}\big) = \big(0, \widehat{b}^{k,\Omega}_{1,N_{k_0}-1}, \tfrac{1}{L_{t_k}}\big), \tag{4.108}$$

and we let $\widehat{\Theta}^{k,\Omega}_{\rm traj} \in \widetilde{\mathcal{D}}^{M,\,r_{k_0}}_{L_{t_k},\,N_{k_0}+1}$ be the concatenation of $\overline{\Theta}^{k,\Omega}_{\rm traj}$ (see (4.101)) and $\widehat{\Xi}^{k,\Omega}_{N_{k_0}}$. We let $\widehat{x}^{k_0,\Omega} \in \mathcal{X}^{M,m}_{\widehat{\Theta}^{k,\Omega}_{\rm traj},\Omega}$ be the concatenation of $x^{k_0,\Omega}_{\max}$ and 0. We further let

$$\widehat{s}^{\,\Omega}_k = \overline{s}^{\,\Omega}_k + \Big[1 + b^{k,\Omega}_{1,N_{k_0}-1} - \tfrac{1}{L_{t_k}}\Big] L_{t_k}, \tag{4.109}$$

and we let $\widehat{u}^{\,k,\Omega} \in \mathcal{U}^{M,m,\,L_{t_k}}_{\widehat{\Theta}^{k,\,\Omega}_{\rm traj},\,\widehat{x}^{k_0,\Omega},\,\widehat{s}^{\,\Omega}_k}$ be the concatenation of $\overline{u}^{\,k,\Omega}$ (see (4.102)) and

$$\widehat{u}^{\,k,\Omega}_{N_{k_0}} = 1 + (b^{k,\Omega}_{1,N_{k_0}-1} - \tfrac{1}{L_{t_k}}). \tag{4.110}$$

Next, we note that the right-most inequality in (4.103), together with the fact that

$$\sum_{j=0}^{N_{k_0}-1} u^{k_0,\Omega}_j = n_{k_0}/L_{n_{k_0}}, \tag{4.111}$$

allow us to asset that $|\overline{s}^{\,\Omega}_k - L_{t_k} n_{k_0}/L_{n_{k_0}}| \leq 2N_{k_0}$. Therefore the definition of $\widehat{s}^{\,\Omega}_k$ in (4.109) implies that

$$\widehat{s}^{\,\Omega}_k = L_{t_k} \frac{n_{k_0}}{L_{n_{k_0}}} + \widehat{m}^{\Omega}_k \quad \text{with} \quad |\widehat{m}^{\Omega}_k| \leq 2N_{k_0} + 2L_{t_k}. \tag{4.112}$$

Moreover,

$$H^{\Omega}\big(\widehat{\Theta}^{k,\Omega}_{\rm traj}, \widehat{x}^{k_0,\Omega}, \widehat{u}^{\,k,\Omega}\big) \geq H^{\Omega}\big(\overline{\Theta}^{k,\Omega}_{\rm traj}, x^{k_0,\Omega}_{\max}, \overline{u}^{\,k,\Omega}\big) + (\beta - \alpha), \tag{4.113}$$

because $\widehat{u}^{\,k,\Omega}_{N_{k_0}} \leq 2$ by definition (see (4.110)) and the free energies per columns are all bounded from below by $(\beta-\alpha)/2$. Hence, (4.107) and (4.113) give that for all Ω there exist a

$$\widehat{\Theta}^{k,\Omega}_{\rm traj} \in \widetilde{\mathcal{D}}^{M,\,r_{k_0}}_{L_{t_k},\,N_{k_0}+1}\colon\ b_{1,N_{k_0}} = \tfrac{1}{L_{t_k}}, \tag{4.114}$$

an $\widehat{x}^{k_0,\Omega} \in \mathcal{X}^{M,m}_{\widehat{\Theta}^{k,\Omega}_{\rm traj},\Omega}$ and a $\widehat{u}^{\,k,\Omega} \in \mathcal{U}^{M,m,\,L_{t_k}}_{\widehat{\Theta}^{k,\,\Omega}_{\rm traj},\,\widehat{x}^{k_0,\Omega},\,\widehat{s}^{\,\Omega}_k}$ such that, for k large enough,

$$\mathbb{E}_{\Omega}\big[H(\widehat{\Theta}^{k,\Omega}_{\rm traj}, \widehat{x}^{k_0,\Omega}, \widehat{u}^{\,k,\Omega})\big] \geq \tfrac{n_{k_0}}{L_{n_{k_0}}}\big(l_1 + \tfrac{l_2 - l_1}{4}\big). \tag{4.115}$$

Next, we subdivide the disorder Ω into groups of $N_{k_0}+1$ consecutive columns that are successively translated by r_{k_0} in the vertical direction, i.e., $\Omega = (\Omega_1, \Omega_2, \dots)$ with (recall (2.8))

$$\Omega_j = \big(\Omega(i,\,(j-1)\,r_{k_0} + \cdot)\big)_{i=(j-1)(N_{k_0}+1)}^{j(N_{k_0}+1)-1}, \tag{4.116}$$

and we let q^{Ω}_k be the unique integer satisfying

$$\widehat{s}^{\,\Omega_1}_k + \widehat{s}^{\,\Omega_2}_k + \dots + \widehat{s}^{\,\Omega_{q_k}}_k \leq t_k < \widehat{s}^{\,\Omega_1}_k + \dots + \widehat{s}^{\,\Omega_{q_k+1}}_k, \tag{4.117}$$

where we suppress the Ω-dependence of q_k. We recall that

$$f^{\Omega}_{3,t_k}(M,m) = \mathbb{E}\Bigg[\frac{1}{t_k} \log \sum_{N = t_k/mL_{t_k}}^{t_k/L_{t_k}} \ \sum_{\Theta_{\rm traj} \in \widetilde{\mathcal{D}}^{M}_{L_{t_k},N}} \ \sum_{x \in \mathcal{X}^{M,m}_{\Theta_{\rm traj},\Omega}} \ \sum_{u \in \mathcal{U}^{M,m,\,L_{t_k}}_{\Theta_{\rm traj},\,x,\,t_k}} e^{L_{t_k}\, H^{\Omega}(\Theta_{\rm traj},x,u)}\Bigg], \tag{4.118}$$

set $\widetilde{t}_k^\Omega = \widehat{s}_k^{\Omega_1} + \widehat{s}_k^{\Omega_2} + \cdots + \widehat{s}_k^{\Omega_{q_k}}$, and concatenate

$$\widehat{\Theta}^{k,\Omega}_{\mathrm{traj,tot}} = \Big(\widehat{\Theta}^{k,\Omega_1}_{\mathrm{traj}}, \widehat{\Theta}^{k,\Omega_2}_{\mathrm{traj}}, \dots, \widehat{\Theta}^{k,\Omega_{q_k}}_{\mathrm{traj}}\Big) \in \widetilde{\mathcal{D}}^{M,}_{L_{t_k},\, q_k(N_{k_0}+1),} \tag{4.119}$$

and

$$\widehat{x}^{k,\Omega}_{\mathrm{tot}} = \big(\widehat{x}^{k_0,\Omega_1}, \widehat{x}^{k_0,\Omega_2}, \dots, \widehat{x}^{k_0,\Omega_{q_k}}\big) \in \mathcal{X}^{M,m}_{\widehat{\Theta}^{k,\Omega}_{\mathrm{traj,tot}}\Omega}. \tag{4.120}$$

and

$$\widehat{u}^{k,\Omega}_{\mathrm{tot}} = \big(\widehat{u}^{k,\Omega_1}, \widehat{u}^{k,\Omega_2}, \dots, \widehat{u}^{k,\Omega_{q_k}}\big) \in \mathcal{U}^{M,m,L_{t_k}}_{\widehat{\Theta}^{k,\Omega}_{\mathrm{traj,tot}}, \widehat{x}^{k,\Omega}_{\mathrm{tot}}, \widetilde{t}_k^\Omega}. \tag{4.121}$$

It still remains to complete $\widehat{\Theta}^{k,\Omega}_{\mathrm{traj,tot}}$, $\widehat{x}^{k,\Omega}_{\mathrm{tot}}$ and $\widehat{u}^{k,\Omega}_{\mathrm{tot}}$ such that the latter becomes an element of $\mathcal{U}^{M,m,L_{t_k}}_{\widehat{\Theta}^{k,\Omega}_{\mathrm{traj,tot}}, \widehat{x}^{k,\Omega}_{\mathrm{tot}}, t_k}$. To that end, we recall (4.117), which gives $t_k - \widetilde{t}_k^\Omega \leq \widehat{s}_k^{\Omega_{q_k+1}}$. Then, using (4.112), we have that there exists a $c > 0$ such that

$$t_k - \widetilde{t}_k^\Omega \leq c L_{t_k} \tfrac{n_{k_0}}{L_{n_{k_0}}}. \tag{4.122}$$

Therefore we can complete $\widehat{\Theta}^{k,\Omega}_{\mathrm{traj,tot}}$, $\widehat{x}^{k,\Omega}_{\mathrm{tot}}$ and $\widehat{u}^{k,\Omega}_{\mathrm{tot}}$ with

$$\Theta_{\mathrm{rest}} \in \mathcal{D}^M_{L_{t_k}, g_k^\Omega}, \qquad x_{\mathrm{rest}} \in \mathcal{X}^{M,m}_{\Theta_{\mathrm{rest}},\Omega}, \qquad u_{\mathrm{rest}} \in \mathcal{U}^{M,m,L_{t_k}}_{\Theta_{\mathrm{rest}}, x_{\mathrm{rest}}, t_k - \widetilde{t}_k^\Omega}, \tag{4.123}$$

such that, by (4.122), the number of columns g_k^Ω involved in Θ_{rest} satisfies $g_k^\Omega \leq c n_{k_0}/L_{n_{k_0}}$. Henceforth $\widehat{\Theta}^{k,\Omega}_{\mathrm{traj,tot}}$, $\widehat{x}^{k,\Omega}_{\mathrm{tot}}$ and $\widehat{u}^{k,\Omega}_{\mathrm{tot}}$ stand for the quantities defined in (4.119) and (4.121), and concatenated with $\Theta_{\mathrm{rest}}, x_{\mathrm{rest}}$ and u_{rest} so that they become elements of

$$\mathcal{D}^M_{L_{t_k},\, q_k(N_{k_0}+1)+g_k^\Omega}, \qquad \mathcal{X}^{M,m}_{\widehat{\Theta}^{k,\Omega}_{\mathrm{traj,tot}},\Omega}, \qquad \mathcal{U}^{M,m,L_{t_k}}_{\widehat{\Theta}^{k,\Omega}_{\mathrm{traj,tot}}, \widehat{x}^{k,\Omega}_{\mathrm{tot}}, t_k}, \tag{4.124}$$

respectively. By restricting the summation in (4.42) to $\widehat{\Theta}^{k,\Omega}_{\mathrm{traj,tot}}$, $\widehat{x}^{k,\Omega}_{\mathrm{tot}}$ and $\widehat{u}^{k,\Omega}_{\mathrm{tot}}$, we get

$$f_{3,t_k}(M,m) \geq \frac{L_{t_k}}{t_k} \mathbb{E}_\Omega \Big[\sum_{j=1}^{q_k} H^{\Omega_j}(\widehat{\Theta}^{k,\Omega_j}_{\mathrm{traj}}, \widehat{x}^{k_0,\Omega_j}, \widehat{u}^{k,\Omega_j}) + H(\Theta_{\mathrm{rest}}, x_{\mathrm{rest}}, u_{\mathrm{rest}}) \Big], \tag{4.125}$$

where the term $H(\Theta_{\mathrm{rest}}, x_{\mathrm{rest}}, u_{\mathrm{rest}})$ is negligible because, by (4.122), $(t_k - \widetilde{t}_k^\Omega)/t_k$ vanishes as $k \to \infty$, while all free energies per column are bounded from below by $(\beta - \alpha)/2$. Pick $\varepsilon > 0$ and recall (4.112). Choose k_0 such that $2L_{n_{k_0}}/n_{k_0} \leq \varepsilon/2$ and note that, for k large enough,

$$\widehat{s}_k^\Omega \in \Big[L_{t_k} \tfrac{n_{k_0}}{L_{n_{k_0}}}(1-\varepsilon), L_{t_k} \tfrac{n_{k_0}}{L_{n_{k_0}}}(1+\varepsilon)\Big]. \tag{4.126}$$

By (4.117), we therefore have

$$q_k \in \Big[\tfrac{t_k L_{n_{k_0}}}{L_{t_k} n_{k_0}} \tfrac{1}{1+\varepsilon}, \tfrac{t_k L_{n_{k_0}}}{L_{t_k} n_{k_0}} \tfrac{1}{1-\varepsilon}\Big] = [a,b]. \tag{4.127}$$

Recalling (4.125), we obtain

$$f_{3,t_k}(M,m) \geq \frac{L_{t_k}}{t_k} \mathbb{E}_\Omega \Big[\sum_{j=1}^{a} H^{\Omega_j}(\widehat{\Theta}^{k,\Omega_j}_{\mathrm{traj}}, \widehat{x}^{k_0,\Omega_j}, \widehat{u}^{k,\Omega_j}) - \sum_{j=a}^{b} \Big| H^{\Omega_j}(\widehat{\Theta}^{k,\Omega_j}_{\mathrm{traj}}, \widehat{x}^{k_0,\Omega_j}, \widehat{u}^{k,\Omega_j}) \Big| \Big], \tag{4.128}$$

and, consequently,

$$f_{3,t_k}(M,m) \geq \tfrac{L_{n_{k_0}}}{n_{k_0}(1+\varepsilon)}\, \mathbb{E}_\Omega\Big[H^\Omega(\widehat{\Theta}^{k,\Omega}_{\mathrm{traj}}, \widehat{x}^{k_0,\Omega}, \widehat{u}^{k,\Omega})\Big] - \frac{L_{t_k}}{t_k}(b-a)(N_{k_0}+1)m^{\frac{\beta-\alpha}{2}}, \tag{4.129}$$

and, by (4.115),

$$f_{3,t_k}(M,m) \geq \tfrac{l_1+\frac{l_2-l_1}{4}}{1+\varepsilon} - (\tfrac{1}{1-\varepsilon} - \tfrac{1}{1+\varepsilon})(b-a)m^{\frac{\beta-\alpha}{2}}. \tag{4.130}$$

After taking ε small enough, we may conclude that $\liminf_{k\to\infty} f_{3,t_k}(M,m) > l_1$, which completes the proof.

4.3 Proof of Proposition 4.2

Pick $(M,m) \in$ EIGH and note that, for every $n \in \mathbb{N}$, the set $\mathcal{W}^m_{n,M}$ is contained in $\mathcal{W}_{n,M}$. Thus, by using Proposition 4.1 we obtain

$$\begin{aligned}\liminf_{n\to\infty} f^\Omega_{1,n}(M;\alpha,\beta) &\geq \sup_{m\geq M+2} \liminf_{n\to\infty} f^\Omega_{1,n}(M,m;\alpha,\beta)\\ &= \sup_{m\geq M+2} f(M,m;\alpha,\beta) \quad \text{for } \mathbb{P}-a.e.\,\Omega.\end{aligned} \tag{4.131}$$

Therefore, the proof of Proposition 4.2 will be complete once we show that

$$\limsup_{n\to\infty} f^\Omega_{1,n}(M;\alpha,\beta) \leq \sup_{m\geq M+2} \limsup_{n\to\infty} f^\Omega_{1,n}(M,m;\alpha,\beta) \quad \text{for } \mathbb{P}-a.e.\,\Omega. \tag{4.132}$$

We will not prove (4.132) in full detail, but only give the main steps in the proof. The proof consists in showing that, for m large enough, the pieces of the trajectory in a column that exeed mL_n steps do not contribute substantially to the free energy.

Recall (4.22–4.27) and use (4.27) with $m = \infty$, i.e.,

$$Z^{\omega,\Omega}_{n,L_n}(M) = \sum_{N=1}^{n/L_n} \sum_{\Theta_{\mathrm{traj}} \in \widetilde{\mathcal{D}}^M_{L_n,N}} \sum_{x\in\mathcal{X}^{M,\infty}_{\Theta_{\mathrm{traj}},\Omega}} \sum_{u\in\mathcal{U}^{M,\infty,L_n}_{\Theta_{\mathrm{traj}},x,n}} A_1. \tag{4.133}$$

With each $(N,\Theta_{\mathrm{traj}},x,u)$ in (4.133), we associate the trajectories obtained by concatenating N shorter trajectories $(\pi_i)_{i\in\{0,\dots,N-1\}}$ chosen in $(\mathcal{W}_{\Theta_i,u_i,L_n})_{i\in\{0,\dots,N-1\}}$, respectively. Thus, the quantity A_1 in (4.133) corresponds to the restriction of the partition function to the trajectories associated with $(N,\Theta_{\mathrm{traj}},x,u)$. In order to discriminate between the columns in which more than mL_n steps are taken and those in which less are taken, we rewrite A_1 as $A_2\widetilde{A}_2$ with

$$A_2 = \prod_{i\in V_{u,m}} Z^{\omega_{I_i}}_{L_n}(\Theta_i,u_i), \qquad \widetilde{A}_2 = \prod_{i\in\widetilde{V}_{u,m}} Z^{\omega_{I_i}}_{L_n}(\Theta_i,u_i), \tag{4.134}$$

with $\widetilde{u}_i = \sum_{k=0}^{i-1} u_k$, $\Theta_i = (\Omega(i,\Pi_i+\cdot),\Xi_i,x_i)$ and $I_i = \{\widetilde{u}_i L_n,\dots,\widetilde{u}_{i+1}L_n - 1\}$ for $i \in \{0,\dots,N-1\}$, with $\omega_I = (\omega_i)_{i\in I}$ for $I \subset \mathbb{N}$, where $\{0,\dots,N-1\}$ is partitioned into

$$\widetilde{V}_{u,m} \cup V_{u,m} \quad \text{with} \quad \widetilde{V}_{u,m} = \{i \in \{0,\dots,N-1\}\colon\ u_i > m\}. \tag{4.135}$$

For all $(N,\Theta_{\mathrm{traj}},x,u)$, we rewrite $\widetilde{V}_{u,m}$ in the form of an increasing sequence $\{i_1,\dots,i_{\widetilde{k}}\}$ and we drop the (u,m)-dependence of $\widetilde{k}$ for simplicity. We also set $\widetilde{u} = u_{i_1} + \dots + u_{i_{\widetilde{k}}}$, which is the total number of steps taken by a trajectory associated with $(N, \Theta_{\rm traj}, x, u)$ in those columns where more than mL_n steps are taken. Finally, for $s \in \{1, \dots, \widetilde{k}\}$ we partition I_{i_s} into

$$J_{i_s} \cup \widetilde{J}_{i_s} \quad \text{with} \quad J_{i_s} = \{\widetilde{u}_{i_s} L_n, \dots, (\widetilde{u}_{i_s} + M + 2)L_n\}, \tag{4.136}$$

$$\widetilde{J}_{i_s} = \{(\widetilde{u}_{i_s} + M + 2)L_n + 1, \dots, \widetilde{u}_{i_s+1} L_n - 1\}, \tag{4.137}$$

and we partition $\{1, \dots, n\}$ into

$$J \cup \widetilde{J} \quad \text{with} \quad \widetilde{J} = \cup_{s=1}^{\widetilde{k}} \widetilde{J}_{i_s}, \qquad J = \{1, \dots, n\} \setminus \widetilde{J}, \tag{4.138}$$

so that $\widetilde{J}$ contains the label of the steps constituting the pieces of trajectory exeeding $(M+2)L_n$ steps in those columns where more than mL_n steps are taken.

4.3.1 Step 1

In this step we replace the pieces of trajectories in the columns indexed in $\widetilde{V}_{u,m}$ by shorter trajectories of length $(M+2)L_n$. To that aim, for every $(N, \Theta_{\rm traj}, x, u)$ we set

$$\widehat{A}_2 = \prod_{i \in \widetilde{V}_{u,m}} Z_{L_n}^{\omega_{J_i}}(\Theta_i', M+2) \tag{4.139}$$

with $\Theta_i' = (\Omega(i, \Pi_i + \cdot), \Xi_i, 1)$. We will show that for all $\varepsilon > 0$ and for m large enough, the event

$$B_n = \{\omega \colon\ \widetilde{A}_2 \le \widehat{A}_2\, e^{3\varepsilon n} \text{ for all } (N, \Theta_{\rm traj}, x, u)\} \tag{4.140}$$

satisfies $\mathbb{P}_\omega(B_n) \to 1$ as $n \to \infty$.

Pick, for each $s \in \{1, \dots, \widetilde{k}\}$, a trajectory π_s in the set $\mathcal{W}_{\Theta_{i_s}, u_{i_s}, L_n}$. By concatenating them we obtain a trajectory in $\mathcal{W}_{\widetilde{u}L_n}$ satisfying $\pi_{\widetilde{u}L_n,1} = \widetilde{k}L_n$. Thus, the total entropy carried by those pieces of trajectories crossing the columns indexed in $\{i_1, \dots, i_{\widetilde{k}}\}$ is bounded above by

$$\textstyle\prod_{s=1}^{\widetilde{k}} |\mathcal{W}_{\Theta_{i_s}, u_{i_s}, L_n}| \le \big|\{\pi \in \mathcal{W}_{\widetilde{u}L_n} \colon\ \pi_{\widetilde{u}L_n,1} = \widetilde{k}L_n\}\big|. \tag{4.141}$$

Since $\widetilde{u}/\widetilde{k} \ge m$, we can use Lemma A.2 in Appendix A to assert that, for m large enough, the right-hand side of (4.141) is bounded above by $e^{\varepsilon n}$.

Moreover, we note that an $\widetilde{u}L_n$-step trajectory satisfying $\pi_{\widetilde{u}L_n,1} = \widetilde{k}L_n$ makes at most $\widetilde{k}L_n + \widetilde{u}$ excursions in the B solvent because such an excursion requires at least one horizontal step or at least L_n vertical steps. Therefore, by using the inequalities $\widetilde{k}L_n \le n/m$ and $\widetilde{u} \le n/L_n$ we obtain that, for n large enough, the sum of the Hamiltonians associated with $(\pi_1, \dots, \pi_{\widetilde{k}})$ is bounded from above, uniformly in $(N, \Theta_{\rm traj}, x, u)$ and $(\pi_1, \dots, \pi_{\widetilde{k}})$, by

$$\textstyle\sum_{s=1}^{\widetilde{k}} H_{u_{i_s} L_n, L_n}^{\omega_{I_{i_s}}, \Omega(i_s, \Pi_{i_s} + \cdot)}(\pi_s) \le \max\{\sum_{i \in I} \xi_i \colon\ I \in \cup_{r=1}^{2n/m} \mathcal{E}_{n,r}\}, \tag{4.142}$$

with $\mathcal{E}_{n,r}$ defined in (D.1) in Appendix D and $\xi_i = \beta 1_{\{\omega_i = A\}} - \alpha 1_{\{\omega_i = B\}}$ for $i \in \mathbb{N}$. At this stage we use the definition in (D.3) and note that, for all $\omega \in \mathcal{Q}_{n,m}^{\varepsilon/\beta, (\alpha-\beta)/2+\varepsilon}$, the right-hand side in (4.142) is smaller than εn. Consequently, for m and n large enough we have that, for all $\omega \in \mathcal{Q}_{n,m}^{\varepsilon/\beta, (\alpha-\beta)/2+\varepsilon}$,

$$\widetilde{A}_2 \le e^{2\varepsilon n} \quad \text{for all} \quad (N, \Theta_{\rm traj}, x, u). \tag{4.143}$$

Recalling (2.32) and noting that $\widetilde{k}L_n \leq n/m$, we can write

$$\widehat{A}_2 \geq e^{-\widetilde{k}(M+2)L_n C_{\rm uf}(\alpha)} \geq e^{-n\frac{M+2}{m}C_{\rm uf}(\alpha)}, \tag{4.144}$$

and therefore, for m large enough, for all n and all $(N, \Theta_{\rm traj}, x, u)$ we have $\widehat{A}_2 \geq e^{-\varepsilon n}$.

Finally, use (4.143) and (4.144) to conclude that, for m and n large enough, $\mathcal{Q}_{n,m}^{\varepsilon/\beta,(\alpha-\beta)/2+\varepsilon}$ is a subset of B_n. Thus, Lemma D.1 ensures that, for m large enough, $\lim_{n\to\infty} P_\omega(B_n) = 1$.

4.3.2 Step 2

Let $(\widetilde{w}_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of Bernouilli trials, independent of ω, Ω. For $(N, \Theta_{\rm traj}, x, u)$ we set $\widehat{u} = \widetilde{u} - \widetilde{k}(M+2)$. In Step 1 we have removed $\widehat{u}L_n$ steps from the trajectories associated with $(N, \Theta_{\rm traj}, x, u)$ so that they have become trajectories associated with $(N, \Theta_{\rm traj}, x', u)$. In this step, we will concatenate the trajectories associated with $(N, \Theta_{\rm traj}, x', u)$ with an $\widehat{u}L_n$-step trajectory to recover a trajectory that belongs to $\mathcal{W}_{n,M}^m$.

For $\Omega \in \{A, B\}^{\mathbb{N}_0\times\mathbb{Z}}$, $t, N \in \mathbb{N}$ and $k \in \mathbb{Z}$, let

$$P_A^\Omega(N,k)(t) = \frac{1}{t}\sum_{j=0}^{t-1} 1_{\{\Omega(N+j,k)=A\}} \tag{4.145}$$

be the proportion of A-blocks on the $k^{\rm th}$ line and between the $N^{\rm th}$ and the $(N+t-1)^{\rm th}$ column of Ω. Pick $\eta > 0$ and $j \in \mathbb{N}$, and set

$$S_{\eta,j} = \bigcup_{N=0}^{j} \bigcup_{k=-m_1 N}^{m_1 N} \bigcup_{t\geq \eta j} \Big\{P_A^\Omega(N,k)(t) \leq \frac{p}{2}\Big\}. \tag{4.146}$$

By a straightforward application of Cramer's Theorem for i.i.d. random variables, we have that $\sum_{j\in\mathbb{N}} P_\Omega(S_{\eta,j}) < \infty$. Therefore, using the Borel-Cantelli Lemma, it follows that for $\mathbb{P}_\Omega$-a.e. Ω, there exists a $j_\eta(\Omega) \in \mathbb{N}$ such that $\Omega \notin S_{\eta,j}$ as soon as $j \geq j_\eta(\Omega)$. In what follows, we consider $\eta = \varepsilon/\alpha m$ and we take n large enough so that $n/L_n \geq j_{\varepsilon/\alpha m}(\Omega)$, and therefore $\Omega \notin S_{\frac{n}{L_n},\frac{\varepsilon}{\alpha m}}$.

Pick (N, Θ, x, u) and consider one trajectory $\widehat{\pi}$, of length $\widehat{u}L_n$, starting from $(N, \Pi_N + b_N)L_n$, staying in the coarsed-grained line at height Π_N, crossing the B-blocks in a straight line and the A-blocks in mL_n steps. The number of columns crossed by $\widehat{\pi}$ is denoted by $\widehat{N}$ and satisfies $\widehat{N} \geq \widehat{u}/m$. If $\widehat{u}L_n \leq \varepsilon n/\alpha$, then the Hamiltonian associated with $\widehat{\pi}$ is clearly larger than $-\varepsilon n$. If $\widehat{u}L_n \geq \varepsilon n/\alpha$ in turn, then

$$H_{\widehat{u}L_n,L_n}^{\widetilde{w},\Omega(N+\cdot,\Pi_N)}(\widehat{\pi}) \geq -\alpha L_n \widehat{N}\big[1 - P_A^\Omega(N,\Pi_N)(\widehat{N})\big]. \tag{4.147}$$

Since $N \leq n/L_n$, $|\Pi_N| \leq m_1 N$ and $\widehat{N} \geq \varepsilon n/(\alpha m L_n)$, we can use the fact that $\Omega \notin S_{\frac{n}{L_n},\frac{\varepsilon}{\alpha m}}$ to obtain

$$P_A^\Omega(N,\Pi_N)(\widehat{N}) \geq \frac{p}{2}. \tag{4.148}$$

At this point it remains to bound $\widehat{N}$ from above, which is done by noting that

$$\widehat{N}\big[mP_A^\Omega(N,\Pi_N)(\widehat{N}) + 1 - P_A^\Omega(N,\Pi_N)(\widehat{N})\big] = \widehat{u} \leq \tfrac{n}{L_n}. \tag{4.149}$$

Hence, using (4.148) and (4.149), we obtain $\widehat{N} \leq 2n/pmL_n$ and therefore the right-hand side of (4.147) is bounded from below by $-\alpha(2-p)n/pm$, which for m large enough is larger than $-\varepsilon n$.

Thus, for n and m large enough and for all (N, Θ, x, u), we have a trajectory $\widehat{\pi}$ at which the Hamiltonian is bounded from below by $-\varepsilon n$ that can be concatenated with all trajectories associated with (N, Θ, x', u) to obtain a trajectory in $\mathcal{W}^m_{n,M}$. Consequently, recalling (4.136), for n and m large enough we have

$$A_2\widehat{A_2} \leq e^{\varepsilon n} Z^{(\omega_J, \widetilde{\omega}),\Omega}_{n,L_n}(M, m) \qquad \forall\, (N, \Theta, x, u). \tag{4.150}$$

4.3.3 Step 3

In this step, we average over the microscopic disorders $\omega, \widetilde{\omega}$. Use (4.150) to note that, for n and m large enough and all $\omega \in B_n$, we have

$$Z^{\omega,\Omega}_{n,L_n}(M) \leq e^{4\varepsilon n} \sum_{N=1}^{n/L_n} \sum_{\Theta_{\rm traj} \in \widetilde{\mathcal{D}}^M_{L_n,N}} \sum_{x \in \mathcal{X}^{M,\infty}_{\Theta_{\rm traj},\Omega}} \sum_{u \in \mathcal{U}^{M,\infty,L_n}_{\Theta_{\rm traj},x,n}} Z^{(\omega_J, \widetilde{\omega}),\Omega}_{n,L_n}(M, m). \tag{4.151}$$

We use (C.3) to claim that there exists $C_1, C_2 > 0$ so that for all $n \in \mathbb{N}$, all $m \in \mathbb{N}$ and all J,

$$\mathbb{P}_{\omega,\widetilde{\omega}}\Big(\Big|\tfrac{1}{n} \log Z^{(\omega_J, \widetilde{\omega}),\Omega}_{n,L_n}(M, m) - f^{\Omega}_{1,n}(M, m)\Big| \geq \varepsilon\Big) \leq C_1 e^{-C_2\varepsilon^2 n}. \tag{4.152}$$

We set also

$$D_n = \bigcap_{(N,\Theta_{\rm traj},x,u)} \Big\{\Big|\tfrac{1}{n} \log Z^{(\omega_J, \widetilde{\omega}),\Omega}_{n,L_n}(M, m) - f^{\Omega}_{1,n}(M, m)\Big| \leq \varepsilon\Big\}, \tag{4.153}$$

recall the definition of c_n in (4.52) (used with (M, ∞)), and use (4.152) and the fact that c_n grows subexponentially, to obtain $\lim_{n\to\infty} \mathbb{P}_{\omega,\widetilde{\omega}}(D^c_n) = 0$. For all $(\omega, \widetilde{\omega})$ satisfying $\omega \in B_n$ and $(\omega, \widetilde{\omega}) \in D_n$, we can rewrite (4.151) as

$$Z^{\omega,\Omega}_{n,L_n}(M) \leq c_n\ e^{n f^{\Omega}_{1,n}(M,m)+5\varepsilon n}. \tag{4.154}$$

As a consequence, recalling (2.32), for m large enough we have

$$f^{\Omega}_n(M; \alpha, \beta) \leq \mathbb{P}(B^c_n \cup D^c_n)\, C_{\rm uf}(\alpha) + \frac{\log c_n}{n} + \frac{1}{n}\mathbb{E}\Big(1_{\{B_n \cup D_n\}}\big(n f^{\Omega}_{1,n}(M, m) + 5\varepsilon n\big)\Big). \tag{4.155}$$

Since $\mathbb{P}(B^c_n \cup D^c_n)$ and $(\log c_n)/n$ vanish when $n \to \infty$, it suffices to apply Proposition 4.1 and to let $\varepsilon \to 0$ to obtain (4.132). This completes the proof of Proposition 4.2.

4.4 Proof of Proposition 4.3

Note that, for all $m \geq M + 2$, we have $\mathcal{R}^m_{p,M} \subset \mathcal{R}_{p,M}$. Moreover, any $(u_\Theta)_{\Theta \in \overline{\mathcal{V}}^m_M} \in \mathcal{B}_{\overline{\mathcal{V}}^m_M}$ can be extended to $\overline{\mathcal{V}}_M$ so that it belongs to $\mathcal{B}_{\overline{\mathcal{V}}_M}$. Thus,

$$\sup_{m \geq M+2} f(M, m; \alpha, \beta) \leq \sup_{\rho \in \mathcal{R}_{p,M}} \sup_{(u) \in \mathcal{B}_{\overline{\mathcal{V}}_M}} V(\rho, u). \tag{4.156}$$

As a consequence, it suffices to show that for all $\rho \in \mathcal{R}_{p,M}$ and $(u_\Theta)_{\Theta\in\overline{\mathcal{V}}_M} \in \mathcal{B}_{\overline{\mathcal{V}}_M}$,

$$V(\rho,u) \le \sup_{m\ge M+2} \sup_{\rho\in\mathcal{R}^m_{p,M}} \sup_{(u)\in\mathcal{B}_{\overline{\mathcal{V}}^m_M}} V(\rho,u). \tag{4.157}$$

If $\int_{\overline{\mathcal{V}}_M} u_\Theta\,\rho(d\Theta) = \infty$, then (4.157) is trivially satisfied since $V(\rho,u) = -\infty$. Thus, we can assume that $\rho(\overline{\mathcal{V}}_M \setminus D_M) = 1$, where $D_M = \{\Theta \in \overline{\mathcal{V}}_M\colon\, \chi_\Theta \in \{A^{\mathbb{Z}}, B^{\mathbb{Z}}\}, x_\Theta = 2\}$. Since $\int_{\overline{\mathcal{V}}_M} u_\Theta\,\rho(d\Theta) < \infty$ and since (recall (2.32)) $\psi(\Theta,u)$ is uniformly bounded by $C_{\mathrm{uf}}(\alpha)$ on $(\Theta,u) \in \overline{\mathcal{V}}^*_M$, we have by dominated convergence that for all $\varepsilon > 0$ there exists an $m_0 \ge M+2$ such that, for all $m \ge m_0$,

$$V(\rho,u) \le \frac{\int_{\overline{\mathcal{V}}^m_M} u_\Theta \psi(\Theta,u_\Theta)\rho(d\Theta)}{\int_{\overline{\mathcal{V}}^m_M} u_\Theta \rho(d\Theta)} + \tfrac{\varepsilon}{2}. \tag{4.158}$$

Since $\rho(\overline{\mathcal{V}}_M \setminus D_M) = 1$ and since $\cup_{m\ge M+2}\overline{\mathcal{V}}^m_M = \overline{\mathcal{V}}_M \setminus D_M$, we have $\lim_{m\to\infty} \rho(\overline{\mathcal{V}}^m_M) = 1$. Moreover, for all $m \ge m_0$ there exists a $\widehat{\rho}_m \in \mathcal{R}^m_{p,M}$ such that $\widehat{\rho}_m = \rho_m + \overline{\rho}_m$, with ρ_m the restriction of ρ to $\overline{\mathcal{V}}^m_M$ and $\overline{\rho}_m$ charging only those Θ satisfying $x_\Theta = 1$. Since all $\Theta \in \overline{\mathcal{V}}_M$ with $x_\Theta = 1$ also belong to $\overline{\mathcal{V}}^{M+2}_M$, we can state that $\overline{\rho}_m$ only charges $\overline{\mathcal{V}}^{M+2}_M$. Therefore

$$V(\widehat{\rho}_m,u) = \frac{\int_{\overline{\mathcal{V}}^m_M} u_\Theta \psi(\Theta,u_\Theta)\rho(d\Theta) + \int_{\overline{\mathcal{V}}^{M+2}_M} u_\Theta \psi(\Theta,u_\Theta)\overline{\rho}_m(d\Theta)}{\int_{\overline{\mathcal{V}}^m_M} u_\Theta \rho(d\Theta) + \int_{\overline{\mathcal{V}}^{M+2}_M} u_\Theta \overline{\rho}_m(d\Theta)}. \tag{4.159}$$

Since $\Theta \mapsto u_\Theta$ is continuous on $\overline{\mathcal{V}}_M$, there exists an $R > 0$ such that $u_\Theta \le R$ for all $\Theta \in \overline{\mathcal{V}}^{M+2}_M$. Therefore we can use (4.158) and (4.159) to obtain, for $m \ge m_0$,

$$V(\widehat{\rho}_m,u) \ge (V(\rho,u) - \tfrac{\varepsilon}{2}) \frac{\int_{\overline{\mathcal{V}}^m_M} u_\Theta \rho(d\Theta)}{\int_{\overline{\mathcal{V}}^m_M} u_\Theta \rho(d\Theta) + \int_{\overline{\mathcal{V}}^{M+2}_M} u_\Theta \overline{\rho}_m(d\Theta)} - R\,C_{\mathrm{uf}}(\alpha)\,(1 - \rho(\overline{\mathcal{V}}^m_M)). \tag{4.160}$$

The fact that $\overline{\rho}_m(\mathcal{V}^{M+2}_M) = \rho(\overline{\mathcal{V}}_M \setminus \overline{\mathcal{V}}^m_M)$ for all $m \ge m_0$ impliess that $\lim_{m\to\infty} \overline{\rho}_m(\mathcal{V}^{M+2}_M) = 0$. Consequently, the right-hand side in (4.160) tends to $V(\rho,u) - \varepsilon/2$ as $m \to \infty$. Thus, there exists a $m_1 \ge m_0$ such that $V(\widehat{\rho}_{m_1},u) \ge V(\rho,u) - \varepsilon$. Finally, we note that there exists a $m_2 \ge m_1 + 1$ such that $u_\Theta \le m_2$ for all $\Theta \in \overline{\mathcal{V}}^{m_1}_M$, which allows us to extend $(u_\Theta)_{\Theta\in\overline{\mathcal{V}}^{m_1}_M}$ to $\overline{\mathcal{V}}^{m_2}_M$ such that $(u_\Theta)_{\Theta\in\overline{\mathcal{V}}^{m_2}_M} \in \mathcal{B}_{\overline{\mathcal{V}}^{m_2}_M}$. It suffices to note that $\widehat{\rho}_{m_1} \in \mathcal{R}^{m_1}_{p,M} \subset \mathcal{R}^{m_2}_{p,M}$ to conclude that

$$V(\rho,u) \le f(M,m_2;\,\alpha,\beta) + \varepsilon. \tag{4.161}$$

A Properties of path entropies

In Appendix A.1 we state a basic lemma (Lemma A.1) about uniform convergence of path entropies in a single column. This lemma is proved with the help of three additional lemmas (Lemmas A.2–A.4), which are proved in Appendix A.2. The latter ends with an elementary lemma (Lemma A.5) that allows us to extend path entropies from rational to irrational parameter values. In Appendix A.3, we extend Lemma A.1 to entropies associated with sets of paths fullfilling certain restrictions on their vertical displacement.

A.1 Basic lemma

We recall the definition of $\widetilde{\kappa}_L$, $L \in \mathbb{N}$, in (2.2) and $\widetilde{\kappa}$ in (2.3).

Lemma A.1 *For every $\varepsilon > 0$ there exists an $L_\varepsilon \in \mathbb{N}$ such that*

$$|\tilde{\kappa}_L(u,l) - \tilde{\kappa}(u,l)| \leq \varepsilon \text{ for } L \geq L_\varepsilon \text{ and } (u,l) \in \mathcal{H}_L. \tag{A.1}$$

Proof. With the help of Lemma A.2 below we get rid of those $(u,l) \in \mathcal{H} \cap \mathbb{Q}^2$ with u large, i.e., we prove that $\lim_{u\to\infty} \kappa_L(u,l) = 0$ uniformly in $L \in \mathbb{N}$ and $(u,l) \in \mathcal{H}_L$. Lemma A.3 in turn deals with the moderate values of u, i.e., u bounded away from infinity and $1 + |l|$. Finally, with Lemma A.4 we take into account the small values of u, i.e., u close to $1+|l|$. To ease the notation we set, for $\eta \geq 0$ and $M > 1$,

$$\mathcal{H}_{L,\eta,M} = \{(u,l) \in \mathcal{H}_L \colon 1 + |l| + \eta \leq u \leq M\}, \qquad \mathcal{H}_{\eta,M} = \{(u,l) \in \mathcal{H} \colon 1 + |l| + \eta \leq u \leq M\}. \tag{A.2}$$

Lemma A.2 *For every $\varepsilon > 0$ there exists an $M_\varepsilon > 1$ such that*

$$\tfrac{1}{uL} \log \left|\{\pi \in \mathcal{W}_{uL} \colon \pi_{uL,1} = L\}\right| \leq \varepsilon \qquad \forall\, L \in \mathbb{N},\, u \in 1 + \tfrac{\mathbb{N}}{L} \colon u \geq M_\varepsilon. \tag{A.3}$$

Lemma A.3 *For every $\varepsilon > 0$, $\eta > 0$ and $M > 1$ there exists an $L_{\varepsilon,\eta,M} \in \mathbb{N}$ such that*

$$|\tilde{\kappa}_L(u,l) - \tilde{\kappa}(u,l)| \leq \varepsilon \qquad \forall\, L \geq L_{\varepsilon,\eta,M},\, (u,l) \in \mathcal{H}_{L,\eta,M}. \tag{A.4}$$

Lemma A.4 *For every $\varepsilon > 0$ there exist $\eta_\varepsilon \in (0, \frac{1}{2})$ and $L_\varepsilon \in \mathbb{N}$ such that*

$$|\tilde{\kappa}_L(u,l) - \tilde{\kappa}_L(u+\eta,l)| \leq \varepsilon \qquad \forall\, L \geq L_\varepsilon,\, (u,l) \in \mathcal{H}_L,\, \eta \in (0,\eta_\varepsilon) \cap \tfrac{2\mathbb{N}}{L}. \tag{A.5}$$

Note that, after letting $L \to \infty$ in Lemma A.4, we get

$$|\tilde{\kappa}(u,l) - \tilde{\kappa}(u+\eta,l)| \leq \varepsilon \qquad \forall\, (u,l) \in \mathcal{H} \cap \mathbb{Q}^2,\, \eta \in (0,\eta_\varepsilon) \cap \mathbb{Q}. \tag{A.6}$$

Pick $\varepsilon > 0$ and $\eta_\varepsilon \in (0, \frac{1}{2})$ as in Lemma A.4. Note that Lemmas A.2–A.3 yield that, for L large enough, (A.1) holds on $\{(u,l) \in \mathcal{H}_L \colon u \geq 1 + |l| + \frac{\eta_\varepsilon}{2}\}$. Next, pick $L \in \mathbb{N}$, $(u,l) \in \mathcal{H}_L \colon u \leq 1 + |l| + \frac{\eta_\varepsilon}{2}$ and $\eta_L \in (\frac{\eta_\varepsilon}{2}, \eta_\varepsilon) \cap \frac{2\mathbb{N}}{L}$, and write

$$|\tilde{\kappa}_L(u,l) - \tilde{\kappa}(u,l)| \leq A + B + C, \tag{A.7}$$

where

$$A = |\tilde{\kappa}_L(u,l) - \tilde{\kappa}_L(u+\eta_L,l)|, \quad B = |\tilde{\kappa}_L(u+\eta_L,l) - \tilde{\kappa}(u+\eta_L,l)|, \quad C = |\tilde{\kappa}(u+\eta_L,l) - \tilde{\kappa}(u,l)|. \tag{A.8}$$

By (A.6), it follows that $C \leq \varepsilon$. As mentioned above, the fact that $(u + \eta_L, l) \in \mathcal{H}_L$ and $u + \eta_L \geq |l| + \frac{\eta_\varepsilon}{2}$ implies that, for L large enough, $B \leq \varepsilon$ uniformly in $(u,l) \in \mathcal{H}_L \colon u \leq 1 + |l| + \frac{\eta_\varepsilon}{2}$. Finally, from Lemma A.4 we obtain that $A \leq \varepsilon$ for L large enough, uniformly in $(u,l) \in \mathcal{H}_L \colon u \leq 1 + |l| + \frac{\eta_\varepsilon}{2}$. This completes the proof of Lemma A.1. □

A.2 Proofs of Lemmas A.2–A.4

A.2.1 Proof of Lemma A.2

The proof relies on the following expression:

$$v_{u,L} = \big|\{\pi \in \mathcal{W}_{uL} \colon \pi_{uL,1} = L\}\big| = \sum_{r=1}^{L+1} \binom{L+1}{r}\binom{(u-1)L}{r} 2^r, \tag{A.9}$$

where r stands for the number of vertical stretches made by the trajectory (a vertical stretch being a maximal sequence of consecutive vertical steps). Stirling's formula allows us to assert that there exists a $g\colon\ [1,\infty) \to (0,\infty)$ satisfying $\lim_{u\to\infty} g(u) = 0$ such that

$$\binom{uL}{L} \le e^{g(u)uL}, \qquad u \ge 1,\ L \in \mathbb{N}. \tag{A.10}$$

Equations (A.9–A.10) complete the proof.

A.2.2 Proof of Lemma A.3

We first note that, since u is bounded from above, it is equivalent to prove (A.4) with $\tilde{\kappa}_L$ and $\tilde{\kappa}$, or with G_L and G given by

$$G(u,l) = u\tilde{\kappa}(u,l), \qquad G_L(u,l) = u\tilde{\kappa}_L(u,l), \quad (u,l) \in \mathcal{H}_L. \tag{A.11}$$

Via concatenation of trajectories, it is straightforward to prove that G is $\mathbb{Q}$-concave on $\mathcal{H}\cap\mathbb{Q}^2$, i.e.,

$$G(\lambda(u_1,l_1)+(1-\lambda)(u_2,l_2)) \ge \lambda G(u_1,l_1)+(1-\lambda)G(u_2,l_2),\ \ \lambda \in \mathbb{Q}_{[0,1]},\ (u_1,l_1),(u_2,l_2) \in \mathcal{H}\cap\mathbb{Q}^2. \tag{A.12}$$

Therefore G is Lipschitz on every $K\cap\mathcal{H}\cap\mathbb{Q}^2$ with $K \subset \mathcal{H}^0$ (the interior of $\mathcal{H}$) compact. Thus, G can be extended on $\mathcal{H}^0$ to a function that is Lipschitz on every compact subset in $\mathcal{H}^0$.

Pick $\eta > 0$, $M > 1$, $\varepsilon > 0$, and choose $L_\varepsilon \in \mathbb{N}$ such that $1/L_\varepsilon \le \varepsilon$. Since $\mathcal{H}_{\eta,M} \subset \mathcal{H}^0$ is compact, there exists a $c > 0$ (depending on η, M) such that G is c-Lipschitz on $\mathcal{H}_{\eta,M}$. Moreover, any point in $\mathcal{H}_{\eta,M}$ is at distance at most ε from the finite lattice $\mathcal{H}_{L_\varepsilon,\eta,M}$. Lemma 2.1 therefore implies that there exists a $q_\varepsilon \in \mathbb{N}$ satisfying

$$|G_{qL_\varepsilon}(u,l) - G(u,l)| \le \varepsilon \qquad \forall\,(u,l) \in \mathcal{H}_{L_\varepsilon,\eta,M},\ q \ge q_\varepsilon. \tag{A.13}$$

Let $L' = q_\varepsilon L_\varepsilon$, and pick $q \in \mathbb{N}$ to be specified later. Then, for $L \ge qL'$ and $(u,l) \in \mathcal{H}_{L,\eta,M}$, there exists an $(u',l') \in \mathcal{H}_{L_\varepsilon,\eta,M}$ such that $|(u,l)-(u',l')|_\infty \le \varepsilon$, $u > u'$, $|l| \ge |l'|$ and $u - u' \ge |l| - |l'|$. We recall (2.3) and write

$$0 \le G(u,l) - G_L(u,l) \le A + B + C, \tag{A.14}$$

with

$$A = |G(u,l) - G(u',l')|, \quad B = |G(u',l') - G_{L'}(u',l')|, \quad C = G_{L'}(u',l') - G_L(u,l). \tag{A.15}$$

Since G is c-Lipschitz on $\mathcal{H}_{\eta,M}$, and since $|(u,l)-(u',l')|_\infty \le \varepsilon$, we have $A \le c\varepsilon$. By (A.13) we have that $B \le \varepsilon$. Therefore only C remains to be considered. By Euclidean division, we get that $L = sL' + r$, where $s \geq q$ and $r \in \{0, \ldots, L'-1\}$. Pick $\pi_1, \pi_2, \ldots, \pi_s \in \mathcal{W}_{L'}(u', |l'|)$, and concatenate them to obtain a trajectory in $\mathcal{W}_{sL'}(u', |l'|)$. Moreover, note that

$$\begin{aligned} uL - u'sL' &= (u-u')sL' + ur \\ &\geq (|l| - |l'|)sL' + (1+|l|)r = (L - sL') + (|l|L - s|l'|L'), \end{aligned} \tag{A.16}$$

where we use that $L - sL' = r$, $u - u' \geq |l| - |l'|$ and $u \geq 1 + |l|$. Thus, (A.16) implies that any trajectory in $\mathcal{W}_{L'}(u', |l'|)$ can be concatenated with an $(uL - u'sL')$-step trajectory, starting at $(sL', s|l'|L')$ and ending at $(L, |l|L)$, to obtain a trajectory in $\mathcal{W}_L(u, |l|)$. Consequently,

$$G_L(u,l) \geq \tfrac{s}{L} \log \kappa_{L'}(u', l') \geq \tfrac{s}{s+1} G_{L'}(u', l'). \tag{A.17}$$

But $s \geq q$ and therefore $G_{L'}(u', l') - G_L(u, l) \leq \frac{1}{q} G_{L'}(u', l') \leq \frac{1}{q} M \log 3$ (recall that $\log 3$ is an upper bound for all entropies per step). Thus, by taking q large enough, we complete the proof.

A.2.3 Proof of Lemma A.4

Pick $L \in \mathbb{N}$, $(u,l) \in \mathcal{H}_L$, $\eta \in \frac{2\mathbb{N}}{L}$, and define the map $T \colon \mathcal{W}_L(u,l) \mapsto \mathcal{W}_L(u+\eta, l)$ as follows. Pick $\pi \in \mathcal{W}_L(u,l)$, find its first vertical stretch, and extend this stretch by $\frac{\eta L}{2}$ steps. Then, find the first vertical stretch in the opposite direction of the stretch just extended, and extend this stretch by $\frac{\eta L}{2}$ steps. The result of this map is $T(\pi) \in \mathcal{W}_L(u+\eta, l)$, and it is easy to verify that T is an injection, so that $|\mathcal{W}_L(u,l)| \leq |\mathcal{W}_L(u+\eta, l)|$.

Next, define a map $\widetilde{T} \colon \mathcal{W}_L(u+\eta, l) \mapsto \mathcal{W}_L(u,l)$ as follows. Pick $\pi \in \mathcal{W}_L(u+\eta, l)$ and remove its first $\frac{\eta L}{2}$ steps north and its first $\frac{\eta L}{2}$ steps south. The result is $\widetilde{T}(\pi) \in \mathcal{W}_L(u,l)$, but $\widetilde{T}$ is not injective. However, we can easily prove that for every $\varepsilon > 0$ there exist $\eta_\varepsilon > 0$ and $L_\varepsilon \in \mathbb{N}$ such that, for all $\eta < \eta_\varepsilon$ and all $L \geq l_\varepsilon$, the number of trajectories in $\mathcal{W}_L(u+\eta, l)$ that are mapped by $\widetilde{T}$ to a particular trajectory in $\pi \in \mathcal{W}_L(u,l)$ is bounded from above by $e^{\varepsilon L}$, uniformly in $(u,l) \in \mathcal{H}_L$ and $\pi \in \mathcal{W}_L, (u,l)$.

This completes the proof of Lemmas A.2–A.4.

A.2.4 Observation

We close this appendix with the following observation. Recall Lemma 2.1, where $(u,l) \mapsto \tilde{\kappa}(u,l)$ is defined on $\mathcal{H} \cap \mathbb{Q}^2$.

Lemma A.5 *(i) $(u,l) \mapsto u\tilde{\kappa}(u,l)$ extends to a continuous and strictly concave function on $\mathcal{H}$.*
(ii) $l \mapsto \tilde{\kappa}(u,l)$ is increasing on $[-u+1, 0]$ and decreasing on $[0, u-1]$,
(iii) $\lim_{u\to\infty} \tilde{\kappa}(u,0) = 0$.
(iv) $u \mapsto u\tilde{\kappa}(u,l)$ is strictly increasing on $[1+|l|, \infty)$ and $\lim_{u\to\infty} u\tilde{\kappa}(u,l) = \infty$.

Proof. (i) In the proof of Lemma A.1 we have shown that $\tilde{\kappa}$ can be extended to $\mathcal{H}^0$ in such a way that $(u,l) \mapsto u\tilde{\kappa}(u,l)$ is continuous and concave on $\mathcal{H}^0$. Lemma A.4 allows us to extend $\tilde{\kappa}$ to the boundary of $\mathcal{H}$, in such a way that continuity and concavity of $(u,l) \mapsto u\tilde{\kappa}(u,l)$ hold on all of $\mathcal{H}$. To obtain the strict concavity, we recall the formula in (2.4), i.e.,

$$u\tilde{\kappa}(u,l) = \begin{cases} u\kappa(u/|l|, 1/|l|), & l \neq 0, \\ u\hat{\kappa}(u), & l = 0, \end{cases} \tag{A.18}$$

where $(a,b) \mapsto a\kappa(a,b)$, $a \geq 1+b$, $b \geq 0$, and $\mu \mapsto \mu\hat{\kappa}(\mu)$, $\mu \geq 1$, are given in [3], Section 2.1, and are strictly concave. In the case $l \neq 0$, (A.18) provides strict concavity of $(u,l) \mapsto u\tilde{\kappa}(u,l)$ on $\mathcal{H}^+ = \{(u,l) \in \mathcal{H} \colon l > 0\}$ and on $\mathcal{H}^- = \{(u,l) \in \mathcal{H} \colon l < 0\}$, while in the case $l = 0$ it provides strict concavity on $\overline{\mathcal{H}} = \{(u,0), u \geq 1\}$. We already know that $(u,l) \mapsto u\tilde{\kappa}(u,l)$ is concave on $\mathcal{H}$, which, by the strict concavity on $\mathcal{H}^+$, $\mathcal{H}^-$ and $\overline{\mathcal{H}}$, implies strict concavity of $(u,l) \mapsto u\tilde{\kappa}(u,l)$ on $\mathcal{H}$.

(ii) This follows from concavity of $l \mapsto \tilde{\kappa}(u,l)$ and the fact that $\tilde{\kappa}(u,l) = \tilde{\kappa}(u,-l)$.

(iii) This is a direct consequence of Lemma A.2.

(iv) By (i) we have that $u \mapsto u\tilde{\kappa}(u,l)$ is strictly concave on $[1+|l|,\infty)$. Therefore, proving that $\lim_{u\to\infty} u\tilde{\kappa}(u,l) = \infty$ is sufficient to obtain that $u \mapsto u\tilde{\kappa}(u,l)$ is strictly increasing. It is proven in [3], Lemma 2.1.2 (iii), that $\lim_{\mu\to\infty} u\hat{\kappa}(u) = \infty$, so that (A.18) completes the proof for $l = 0$. If $l \neq 0$, then we use (A.18) again and the variational formula in the proof of [3], Lemma 2.1.1, to check that $\lim_{a\to\infty} a\kappa(a,b) = \infty$ for all $b > 0$. □

A.3 A generalization of Lemma A.1

In Section 4 we sometimes needed to deal with subsets of trajectories of the following form. Recall (2.1), pick $L \in \mathbb{N}$, $(u,l) \in \mathcal{H}_L$ and $B_0, B_1 \in \frac{Z}{L}$ such that

$$B_1 \geq 0 \vee l \geq 0 \wedge l \geq B_0 \quad \text{and} \quad B_1 - B_0 \geq 1. \tag{A.19}$$

Denote by $\widetilde{\mathcal{W}}_L(u,l,B_0,B_1)$ the subset of $\mathcal{W}_L(u,l)$ containing those trajectories that are constrained to remain above B_0L and below B_1L (see Fig. 12), i.e.,

$$\widetilde{\mathcal{W}}_L(u,l,B_0,B_1) = \big\{\pi \in \mathcal{W}_L(u,l) \colon\; B_0L < \pi_{i,2} < B_1L \text{ for } i \in \{1,\dots,uL-1\}\big\}, \tag{A.20}$$

and let

$$\widetilde{\kappa}_L(u,l,B_0,B_1) = \frac{1}{uL} \log |\widetilde{\mathcal{W}}_L(u,l,B_0,B_1)| \tag{A.21}$$

be the entropy per step carried by the trajectories in $\widetilde{\mathcal{W}}_L(u,l,B_0,B_1)$. With Lemma A.6 below we prove that the effect on the entropy of the restriction induced by B_0 and B_1 in the set $\widetilde{\mathcal{W}}_L(u,l)$ vanishes uniformly as $L \to \infty$.

Lemma A.6 *For every $\varepsilon > 0$ there exists an $L_\varepsilon \in \mathbb{N}$ such that, for $L \geq L_\varepsilon$, $(u,l) \in \mathcal{H}_L$ and $B_0, B_1 \in \mathbb{Z}/L$ satisfying $B_1 - B_0 \geq 1$, $B_1 \geq \max\{0,l\}$ and $B_0 \leq \min\{0,l\}$,*

$$|\tilde{\kappa}_L(u,l,B_0,B_1) - \tilde{\kappa}_L(u,l)| \leq \varepsilon. \tag{A.22}$$

Proof. The key fact is that $B_1 - B_0 \geq 1$. The vertical restrictions $B_1 \geq \max\{0,l\}$ and $B_0 \leq \min\{0,l\}$ gives polynomial corrections in the computation of the entropy, but these corrections are harmless because $(B_1 - B_0)L$ is large. □

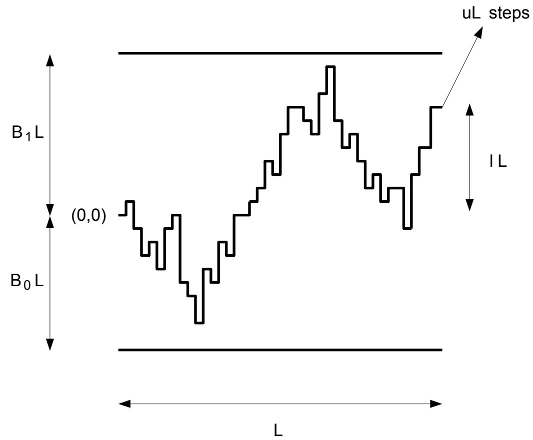

Figure 12: A trajectory in $\widetilde{\mathcal{W}}_L(u, l, B_0, B_1)$.

B Properties of free energies

B.1 Free energy along a single linear interface

Also the free energy $\mu \mapsto \phi^{\mathcal{I}}(\mu; \alpha, \beta)$ defined in Proposition 2.2 can be extended from $\mathbb{Q} \cap [1, \infty)$ to $[1, \infty)$, in such a way that $\mu \mapsto \mu\phi^{\mathcal{I}}(\mu; \alpha, \beta)$ is concave and continous on $[1, \infty)$. By concatenating trajectories, we can indeed check that $\mu \mapsto \mu\phi^{\mathcal{I}}(\mu; \alpha, \beta)$ is concave on $\mathbb{Q} \cap [1, \infty)$. Therefore it is Lipschitz on every compact subset of $(1, \infty)$ and can be extended to a concave and continuous function on $(1, \infty)$. The continuity at $\mu = 1$ comes from the fact that $\phi^{\mathcal{I}}(1; \alpha, \beta) = 0$ and $\lim_{\mu \downarrow 1} \phi^{\mathcal{I}}(\mu) = 0$, which is obtained by using Lemma D.1 below.

Lemma B.1 *For all* $(\alpha, \beta) \in \text{CONE}$*:*
(i) $\mu \mapsto \mu\phi^{\mathcal{I}}(\mu; \alpha, \beta)$ *is strictly increasing on* $[1, \infty)$ *and* $\lim_{\mu \to \infty} \mu\phi^{\mathcal{I}}(\mu; \alpha, \beta) = \infty$.
(ii) $\lim_{\mu \to \infty} \phi^{\mathcal{I}}(\mu; \alpha, \beta) = 0$.

Proof. (i) Clearly, $\phi^{\mathcal{I}}(\mu; \alpha, \beta) \geq \widetilde{\kappa}(\mu, 0)$ for $\mu \geq 1$. Therefore Lemma A.5(iv) implies that $\lim_{\mu \to \infty} \mu\phi^{\mathcal{I}}(\mu; \alpha, \beta) = \infty$. Thus, the concavity of $\mu \mapsto \mu\phi^{\mathcal{I}}(\mu; \alpha, \beta)$ is sufficient to obtain that it is strictly increasing on $[1, \infty)$.
(ii) See [4], Lemma 2.4.1(i). □

Recall Assumption 2.3, in which we assumed that $\mu \mapsto \mu\phi^{\mathcal{I}}(\mu; \alpha, \beta)$ is strictly concave on $[1, \infty)$. The next lemma states that the convergence of the average quenched free energy $\phi_L^{\mathcal{I}}$ to $\phi^{\mathcal{I}}$ as $L \to \infty$ is uniform on $\mathbb{Q} \cap [1, \infty)$.

Lemma B.2 *For every* $(\alpha, \beta) \in \text{CONE}$ *and* $\varepsilon > 0$ *there exists an* $L_\varepsilon \in \mathbb{N}$ *such that*

$$|\phi_L(\mu) - \phi(\mu)| \leq \varepsilon \qquad \forall\, \mu \in 1 + \tfrac{2\mathbb{N}}{L},\, L \geq L_\varepsilon. \tag{B.1}$$

Proof. Similarly to what we did for Lemma A.1, the proof can be done by treating separately the cases μ large, moderate and small. We leave the details to the reader. □

B.2 Free energy in a single column

We can extend $(\Theta, u) \mapsto \psi(\Theta, u)$ from $\mathcal{V}_M^*$ to $\overline{\mathcal{V}}_M^*$ by using the variational formulas in (2.36) and (2.39) and by recalling that $\widetilde{\kappa}$ and $\phi^{\mathcal{I}}$ have been extended to $\mathcal{H}$ and $[1,\infty)$ in Appendices A.2 and B.1.

Pick $M \in \mathbb{N}$ and recall (2.13). Define a distance d_M on $\overline{\mathcal{V}}_M$ as follows. Pick $\Theta_1, \Theta_2 \in \overline{\mathcal{V}}_M$, abbreviate

$$\Theta_1 = (\chi_1, \Delta\Pi_1, b_{0,1}, b_{1,1}, x_1), \qquad \Theta_2 = (\chi_2, \Delta\Pi_2, b_{0,2}, b_{1,2}, x_2), \tag{B.2}$$

and define

$$d_M(\Theta_1, \Theta_2) = \sum_{j\in\mathbb{Z}} \frac{1_{\{\chi_1(j)\neq\chi_2(j)\}}}{2^{|j|}} + |\Delta\Pi_1 - \Delta\Pi_2| + |b_{0,1} - b_{0,2}| + |b_{1,1} - b_{1,2}| \tag{B.3}$$

so that $\widetilde{d}_M((\Theta_1, u_1), (\Theta_2, u_2)) = \max\{|u_1 - u_2|, d_M(\Theta_1, \Theta_2)\}$ is a distance on $\overline{\mathcal{V}}_M^{*,m}$ for which $\overline{\mathcal{V}}_M^{*,m}$ is compact.

Lemma B.3 *For every* $(M, m) \in \text{EIGH}$ *and* $(\alpha, \beta) \in \text{CONE}$*,*

$$(u, \Theta) \mapsto u\, \psi(\Theta, u; \alpha, \beta) \tag{B.4}$$

is uniformly continuous on $\overline{\mathcal{V}}_M^{*,m}$ *endowed with* $\widetilde{d}_M$*.*

Proof. Pick $(M, m) \in \text{EIGH}$. By the compactness of $\overline{\mathcal{V}}_M^{*,m}$, it suffices to show that $(u, \Theta) \mapsto u\, \psi(\Theta, u)$ is continuous on $\overline{\mathcal{V}}_M^{*,m}$. Let $(\Theta_n, u_n) = (\chi_n, \Delta\Pi_n, b_{0,n}, b_{1,n}, u_n)$ be the general term of an infinite sequence that tends to $(\Theta, u) = (\chi, \Delta\Pi, b_0, b_1, u)$ in $(\overline{\mathcal{V}}_M^{*,m}, \widetilde{d}_M)$. We want to show that $\lim_{n\to\infty} u_n \psi(\Theta_n, u_n) = u\psi(\Theta, u)$. By the definition of $\widetilde{d}_M$, we have $\chi_n = \chi$ and $\Delta\Pi_n = \Delta\Pi$ for n large enough. We assume that $\Theta \in \mathcal{V}_{\text{int}}$, so that $\Theta_n \in \mathcal{V}_{\text{int}}$ for n large enough as well. The case $\Theta \in \mathcal{V}_{\text{nint}}$ can be treated similarly.

Set

$$\mathcal{R}_m = \{(a, h, l) \in [0, m] \times [0, 1] \times \mathbb{R} \colon h + |l| \le a\} \tag{B.5}$$

and note that $\mathcal{R}_m$ is a compact set. Let $g\colon \mathcal{R}_m \mapsto [0, \infty)$ be defined as $g(a, h, l) = a\, \widetilde{\kappa}(\frac{a}{h}, \frac{l}{h})$ if $h > 0$ and $g(a, h, l) = 0$ if $h = 0$. The continuity of $\widetilde{\kappa}$, stated in Lemma A.5(i), ensures that g is continuous on $\{(a, h, l) \in \mathcal{R}_m \colon h > 0\}$. The continuity at all $(a, 0, l) \in \mathcal{R}_m$ is obtained by recalling that $\lim_{u\to\infty} \tilde{\kappa}(u, l) = 0$ uniformly in $l \in [-u + 1, u - 1]$ (see Lemma A.5(ii-iii)) and that $\widetilde{\kappa}$ is bounded on $\mathcal{H}$.

In the same spirit, we may set $\mathcal{R}'_m = \{(u, h) \in [0, m] \times [0, 1] \colon\ h \le u\}$ and define $g'\colon \mathcal{R}'_m \mapsto [0, \infty)$ as $g'(u, h) = u\, \phi^{\mathcal{I}}(\frac{u}{h})$ for $h > 0$ and $g'(u, h) = 0$ for $h = 0$. With the help of Lemma B.1 we obtain the continuity of g' on $\mathcal{R}'_m$ by mimicking the proof of the continuity of g on $\mathcal{R}_m$.

Note that the variational formula in (2.36) can be rewriten as

$$u\, \psi(\Theta, u) = \sup_{(h),(a)\in\mathcal{L}(l_A, l_B; u)} Q((h), (a), l_A, l_B), \tag{B.6}$$

with

$$Q((h), (a), l_A, l_B) = g(a_A, h_A, l_A) + g(a_B, h_B, l_B) + a_B\, \tfrac{\beta-\alpha}{2} + g^{'}(a^{\mathcal{I}}, h^{\mathcal{I}}), \tag{B.7}$$

and with l_A and l_B defined in (2.35). Note that $\mathcal{L}(l_A,\, l_B;\, u)$ is compact, and that $(h), (a) \mapsto Q((h), (a), l_A, l_B)$ is continuous on $\mathcal{L}(l_A,\, l_B;\, u)$ because g and g' are continuous on $\mathcal{R}_m$ and $\mathcal{R}_m^{'}$, respectively. Hence, the supremum in (B.6) is attained.

Pick $\varepsilon > 0$, and note that g and g' are uniformly continuous on $\mathcal{R}_m$ and $\mathcal{R}'_m$, which are compact sets. Hence there exists an $\eta_\varepsilon > 0$ such that $|g(a,h,l) - g(a',h',l')| \leq \varepsilon$ and $|g'(u,b) - g'(u',b')| \leq \varepsilon$ when $(a,h,l),(a',h',l') \in \mathcal{R}_m$ and $(u,b),(u',b') \in \mathcal{R}'_m$ are such that $|a-a'|, |h-h'|, |l-l'|, |u-u'|$ and $|b-b'|$ are bounded from above by η_ε.

Since $\lim_{n\to\infty}(\Theta_n, u_n) = (\Theta, u)$ we also have that $\lim_{n\to\infty} b_{0,n} = b_0$, $\lim_{n\to\infty} b_{1,n} = b_1$ and $\lim_{n\to\infty} u_n = u$. Thus, $\lim_{n\to\infty} l_{A,n} = l_A$ and $\lim_{n\to\infty} l_{B,n} = l_B$, and therefore $|l_{A,n} - l_A| \leq \eta_\varepsilon$, $|l_{B,n} - l_B| \leq \eta_\varepsilon$ and $|u_n - u| \leq \eta_\varepsilon$ for $n \geq n_\varepsilon$ large enough.

For $n \in \mathbb{N}$, let $(h_n),(a_n) \in \mathcal{L}(l_{A,n},\, l_{B,n};\, u_n)$ be a maximizer of (B.6) at (Θ_n, u_n), and note that, for $n \geq n_\varepsilon$, we can choose $(\widetilde{h}_n),(\widetilde{a}_n) \in \mathcal{L}(l_A,\, l_B;\, u)$ such that $|\widetilde{a}_{A,n} - a_{A,n}|$, $|\widetilde{a}_{B,n} - a_{B,n}|$, $|\widetilde{a}^{\mathcal{I}}_n - a^{\mathcal{I}}_n|$, $|\widetilde{h}_{A,n} - h_{A,n}|$, $|\widetilde{h}_{B,n} - h_{B,n}|$ and $|\widetilde{h}^{\mathcal{I}}_n - h^{\mathcal{I}}_n|$ are bounded above by η_ε. Consequently,

$$u_n\psi(\Theta_n, u_n) - u\psi(\Theta, u) \leq Q((h_n),(a_n), l_{A,n}, l_{B,n}) - Q((\widetilde{h}_n),(\widetilde{a}_n), l_A, l_B) \leq 3\varepsilon. \tag{B.8}$$

We bound $u\psi(\Theta, u) - u_n\psi(\Theta_n, u_n)$ from above in a similar manner, and this suffices to obtain the claim. □

Lemma B.4 *For every $\Theta \in \overline{\mathcal{V}}_M$, the function $u \mapsto u\psi(\Theta, u)$ is continuous and strictly concave on $[t_\Theta, \infty)$.*

Proof. The continuity is a straightforward consequence of Lemma B.3: simply fix Θ and let $m \to \infty$. To prove the strict concavity, we note that the cases $\Theta \in \mathcal{V}_{\mathrm{int}}$ and $\Theta \in \mathcal{V}_{\mathrm{nint}}$ can be treated similarly. We will therefore focus on $\Theta \in \mathcal{V}_{\mathrm{int}}$.

For $l \in \mathbb{R}$, let

$$\mathcal{N}_l = \{(a,h) \in [0,\infty) \times [0,1] \colon a \geq h + |l|\}, \quad \mathcal{N}_l^+ = \{(a,h) \in \mathcal{N}_l \colon h > 0\}, \tag{B.9}$$

and let $g_l \colon \mathcal{N}_l \mapsto [0,\infty)$ be defined as $g_l(a,h) = a\,\widetilde{\kappa}(\frac{a}{h}, \frac{l}{h})$ for $h > 0$ and $g_l(a,h) = 0$ for $h = 0$. The strict concavity of $(u,l) \mapsto u\widetilde{\kappa}(u,l)$ on $\mathcal{H}$, stated in Lemma A.5(i), immediately yields that g_l is strictly concave on $\mathcal{N}_l^+$ and concave on $\mathcal{N}_l$. Consequently, for all $(a_1,h_1) \in \mathcal{N}_l^+$ and $(a_2,h_2)\mathcal{N}_l \setminus \mathcal{N}_l^+$, g_l is strictly concave on the segment $[(u_1,h_1),(u_2,h_2)]$.

Let $\widetilde{\mathcal{N}} = \{(u,h) \in [0,\infty) \times [0,1] \colon\, h \leq u\}$ and define $\widetilde{g} \colon \widetilde{\mathcal{N}} \mapsto [0,\infty)$ as $\widetilde{g}(u,h) = u\,\phi^{\mathcal{I}}(\frac{u}{h})$ for $h > 0$ and $\widetilde{g}(u,h) = 0$ for $h = 0$. The strict concavity of $u \mapsto u\phi^{\mathcal{I}}(u)$ on $[1,\infty)$, stated in Assumption 2.3, immediately yields that $\widetilde{g}$ is strictly concave on $\widetilde{\mathcal{N}}^+ = \{(u,h) \in \widetilde{\mathcal{N}} \colon h > 0\}$ and concave on $\widetilde{\mathcal{N}}$. Consequently, for all $(u_1,h_1) \in \widetilde{\mathcal{N}}^+$ and $(u_2,h_2) \in \widetilde{\mathcal{N}} \setminus \widetilde{\mathcal{N}}^+$, $\widetilde{g}$ is strictly concave on the segment $[(u_1,h_1),(u_2,h_2)]$.

Similarly to what we did in (B.6), we can rewrite the variational formula in (2.36) as

$$u\,\psi(\Theta, u) = \sup_{(h),(a) \in \mathcal{L}(l_A,\, l_B;\, u)} \widetilde{Q}((h),(a)) \tag{B.10}$$

with

$$\widetilde{Q}((h),(a)) = g_{l_A}(a_A, h_A) + g_{l_B}(a_B, h_B) + a_B\,\tfrac{\beta-\alpha}{2} + \widetilde{g}(u - a_A - a_B, 1 - h_A - h_B), \tag{B.11}$$

and the supremum in (B.10) is attained. Next we show that if $(h),(a) \in \mathcal{L}(l_A,\, l_B;\, u)$ realizes the maximum in (B.10), then $(h),(a) \notin \widetilde{\mathcal{L}}(l_A,\, l_B;\, u)$ with

$$\widetilde{\mathcal{L}}(l_A,\, l_B;\, u) = \widetilde{\mathcal{L}}_A(l_A,\, l_B;\, u) \cup \widetilde{\mathcal{L}}_B(l_A,\, l_B;\, u) \cup \widetilde{\mathcal{L}}^{\mathcal{I}}(l_A,\, l_B;\, u) \tag{B.12}$$

and

$$
\begin{aligned}
\widetilde{\mathcal{L}}_A(l_A,\,l_B;\,u) &= \{(h),(a) \in \mathcal{L}(l_A,\,l_B;\,u)\colon\ h_A = 0 \ \text{ and } \ a_A > l_A\},\\
\widetilde{\mathcal{L}}_B(l_A,\,l_B;\,u) &= \{(h),(a) \in \mathcal{L}(l_A,\,l_B;\,u)\colon\ h_B = 0 \ \text{ and } \ a_B > l_B\},\\
\widetilde{\mathcal{L}}^{\mathcal{I}}(l_A,\,l_B;\,u) &= \{(h),(a) \in \mathcal{L}(l_A,\,l_B;\,u)\colon\ h_I = 0 \ \text{ and } \ a_I > 0\}.
\end{aligned}
\tag{B.13}
$$

Assume that $(h),(a) \in \widetilde{\mathcal{L}}(l_A,\,l_B;\,u)$, and that $h_A > 0$ or $h^{\mathcal{I}} > 0$. For instance, $(h),(a) \in \widetilde{\mathcal{L}}^{\mathcal{I}}(l_A,\,l_B;\,u)$ and $h_A > 0$. Then, by Lemma A.5(iv), $\widetilde{Q}$ strictly increases when a_A is replaced by $a_A + a^{\mathcal{I}}$ and $a^{\mathcal{I}}$ by 0. This contradicts the fact that $(h),(a)$ is a maximizer. Next, if $(h),(a) \in \widetilde{\mathcal{L}}(l_A,\,l_B;\,u)$ and $h_A = h^{\mathcal{I}} = 0$, then $h_B = 1$, and the first case is $(h),(a) \in \widetilde{\mathcal{L}}_A(l_A,\,l_B;\,u)$, while the second case is $(h),(a) \in \widetilde{\mathcal{L}}^{\mathcal{I}}(l_A,\,l_B;\,u)$. In the second case, as before, we replace a_A by $a_A + a^{\mathcal{I}}$ and $a^{\mathcal{I}}$ by 0, which does not change $\widetilde{Q}$ but yields that $a_A > l_A$ and therefore brings us back to the first case. In this first case, we are left with an expression of the form

$$
Q((h),(a)) = g_{l_B}(a_B, 1) + a_B\,\tfrac{\beta-\alpha}{2}
\tag{B.14}
$$

with $h_A = h^{\mathcal{I}} = 0$ and $a_A > l_A$. Thus, if we can show that there exists an $x \in (0,1)$ such that

$$
g_{l_A}(a_A, x) + g_{l_B}(a_B, 1-x) > g_{l_B}(a_B, 1),
\tag{B.15}
$$

then we can claim that $(h),(a)$ is not a maximizer of (B.10) and the proof for $(h),(a) \notin \widetilde{\mathcal{L}}(l_A,\,l_B;\,u)$ will be complete.

To that end, we recall (2.4), which allows us to rewrite the left-hand side in (B.15) as

$$
g_{l_A}(a_A, x) + g_{l_B}(a_B, 1-x) = a_A\,\kappa\big(\tfrac{a_A}{l_A}, \tfrac{x}{l_A}\big) + a_B\,\kappa\big(\tfrac{a_B}{l_B}, \tfrac{1-x}{l_B}\big) + a_B\,\tfrac{\beta-\alpha}{2}.
\tag{B.16}
$$

We recall [3], Lemma 2.1.1, which claims that κ is defined on $\text{DOM} = \{(a,b)\colon a \geq 1+b, b \geq 0\}$, is analytic on the interior of DOM and is continuous on DOM. Moreover, in the proof of this lemma, an expression for $\partial_b\,\kappa(a,b)$ is provided, which is valid on the interior of DOM. From this expression we can easily check that if $a > 1$, then $\lim_{b\to 0}\partial_b\,\kappa(a,b) = \infty$. Therefore, by the continuity of κ on $(a_A/l_A, 0)$ with $a_A/l_A > 1$ we can assert that the derivative with respect to x of the left-hand side in (B.16) at $x = 0$ is infinite, and therefore there exists an $x > 0$ such that (B.15) is satisfied.

Pick $u_1 > u_2 \geq t_\Theta$, and let $(h_1),(a_1) \in \mathcal{L}(l_A,\,l_B;\,u_1)$ and $(h_2),(a_2) \in \mathcal{L}(l_A,\,l_B;\,u_2)$ be maximizers of (B.10) for u_1 and u_2, respectively. We can write

$$
\begin{aligned}
(a_1),(h_1) &= \big(a_{A,1}, a_{B,1}, a_1^{\mathcal{I}}\big), (h_{A,1}, h_{B,1}, h_1^{\mathcal{I}}\big),\\
(a_2),(h_2) &= \big(a_{A,2}, a_{B,2}, a_2^{\mathcal{I}}\big), (h_{A,2}, h_{B,2}, h_2^{\mathcal{I}}\big).
\end{aligned}
\tag{B.17}
$$

Thus, $(\frac{a_1+a_2}{2}),(\frac{h_1+h_2}{2}) \in \mathcal{L}(l_A,\,l_B;\,\frac{u_1+u_2}{2})$ and, with the help of the concavity of $g_{l_A}, g_{l_B}, \widetilde{g}$ proven above, we can write

$$
\tfrac{u_1+u_2}{2}\,\psi(\Theta, \tfrac{u_1+u_2}{2}) \geq \widetilde{Q}\big((\tfrac{a_1+a_2}{2}),(\tfrac{h_1+h_2}{2})\big) \geq \tfrac{1}{2}\big(u_1\,\psi(\Theta,u_1) + u_2\,\psi(\Theta,u_2)\big).
\tag{B.18}
$$

We have proven above that $(a_1),(h_1) \notin \widetilde{\mathcal{L}}(l_A,\,l_B;\,u_1)$ and $(a_2),(h_2) \notin \widetilde{\mathcal{L}}(l_A,\,l_B;\,u_2)$. Thus, we can use (B.11) and the strict concavity of $g_{l_A}, g_{l_B}, \widetilde{g}$ on $\mathcal{N}_{l_A}^+, \mathcal{N}_{l_B}^+ \widetilde{\mathcal{N}}^+$, to conclude that the right-most inequality in (B.18) is an equality only if

$$
\begin{aligned}
&(a_{A,1}, h_{A,1}) = (a_{A,2}, h_{A,2}), \quad (a_{B,1}, h_{B,1}) = (a_{B,2}, h_{B,2}),\\
&(u_1 - a_{A,1} - a_{B,1}, 1 - h_{A,1} - h_{B,1}) = (u_2 - a_{A,2} - a_{B,2}, 1 - h_{A,2} - h_{B,2}),
\end{aligned}
\tag{B.19}
$$

which clearly is not possible because $u_1 > u_2$. □

C Concentration of measure

Let $\mathcal{S}$ be a finite set and let $(X_i, \mathcal{A}_i, \mu_i)_{i\in\mathcal{S}}$ be a family of probability spaces. Consider the product space $X = \prod_{i\in\mathcal{S}} X_i$ endowed with the product σ-field $\mathcal{A} = \otimes_{i\in\mathcal{S}}\mathcal{A}_i$ and with the product probability measure $\mu = \otimes_{i\in\mathcal{S}}\mu_i$.

Theorem C.1 (Talagrand [7]) *Let $f\colon X \mapsto \mathbb{R}$ be integrable with respect to $(\mathcal{A}, \mu)$ and, for $i \in \mathcal{S}$, let $d_i > 0$ be such that $|f(x) - f(y)| \le d_i$ when $x, y \in X$ differ in the i-th coordinate only. Let $D = \sum_{i\in\mathcal{S}} d_i^2$. Then, for all $\varepsilon > 0$,*

$$\mu\left\{x \in X\colon \left|f(x) - \int f d\mu\right| > \varepsilon\right\} \le 2e^{-\frac{\varepsilon^2}{2D}}. \tag{C.1}$$

The following corollary of Theorem C.1 was used several times in the paper. Let $(\alpha, \beta) \in$ CONE and let $(\xi_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of Bernouilli trials taking the values $-\alpha$ and β with probability $\frac{1}{2}$ each. Let $l \in \mathbb{N}$, $T\colon\ \{(x, y) \in \mathbb{Z}^2 \times \mathbb{Z}^2\colon |x - y| = 1\} \to \{0, 1\}$ and $\Gamma \subset \mathcal{W}_l$ (recall (1.1)). Let $F_l\colon\ [-\alpha, \alpha]^l \to \mathbb{R}$ be such that

$$F_l(x_1, \dots, x_l) = \log \sum_{\pi\in\Gamma} e^{\sum_{i=1}^{l} x_i\, T((\pi_{i-1},\pi_i))}. \tag{C.2}$$

For all $x, y \in [-\alpha, \alpha]^l$ that differ in one coordinate only we have $|F_l(x)-F_l(y)| \le 2\alpha$. Therefore we can use Theorem C.1 with $\mathcal{S} = \{1, \dots, l\}$, $X_i = [-\alpha, \alpha]$ and $\mu_i = \frac{1}{2}(\delta_{-\alpha} + \delta_\beta)$ for all $i \in \mathcal{S}$, and $D = 4\alpha^2 l$, to obtain that there exist $C_1, C_2 > 0$ such that, for every $l \in \mathbb{N}$, $\Gamma \subset \mathcal{W}_n$ and $T\colon\ \{(x, y) \in \mathbb{Z}^2 \times \mathbb{Z}^2\colon |x - y| = 1\} \to \{0, 1\}$,

$$\mathbb{P}\big(|F_l(\xi_1, \dots, \xi_m) - \mathbb{E}(F_l(\xi_1, \dots, \xi_m))| > \eta\big) \le C_1 e^{-\frac{C_2\eta^2}{l}}. \tag{C.3}$$

D Large deviation estimate

Let $(\xi_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of Bernouilli trials taking values β and $-\alpha$ with probability $\frac{1}{2}$ each. For $N \le n \in \mathbb{N}$, denote by $\mathcal{E}_{n,N}$ the set of all ordered sequences of N disjoint and non-empty intervals included in $\{1, \dots, n\}$, i.e.,

$$\begin{aligned}\mathcal{E}_{n,N} = \big\{(I_j)_{1\le j\le N} \subset \{1, \dots, n\}\colon I_j = \{\min I_j, \dots, \max I_j\}\ \forall\, 1 \le j \le N,\\ \max I_j < \min I_{j+1}\ \forall\, 1 \le j \le N - 1 \text{ and } I_j \neq \emptyset\ \forall\, 1 \le j \le N\big\}.\end{aligned} \tag{D.1}$$

For $(I) \in \mathcal{E}_{n,N}$, let $T(I) = \sum_{j=1}^{N} |I_j|$ be the cumulative length of the intervals making up (I). Pick $\gamma > 0$ and $M \in \mathbb{N}$, and denote by $\widehat{\mathcal{E}}^{\gamma}_{n,M}$ the set of those (I) in $\cup_{1\le N\le(n/M)}\, \mathcal{E}_{n,N}$ that have a cumulative length larger than γn, i.e.,

$$\widehat{\mathcal{E}}^{\gamma}_{n,M} = \cup_{N=1}^{n/M}\big\{(I) \in \mathcal{E}_{n,N}\colon\ T(I) \ge \gamma n\big\}. \tag{D.2}$$

Next, for $\eta > 0$ set

$$\mathcal{Q}^{\gamma,\eta}_{n,M} = \bigcap_{(I)\in\widehat{\mathcal{E}}^{\gamma}_{n,M}} \left\{\sum_{j=1}^{N}\sum_{i\in I_j} \xi_i \le \left(\tfrac{\beta-\alpha}{2} + \eta\right) T(I)\right\}. \tag{D.3}$$

Lemma D.1 *For all* $(\alpha,\beta)\in$ CONE, $\gamma>0$ *and* $\eta>0$ *there exists an* $\widehat{M}\in\mathbb{N}$ *such that, for all* $M\geq\widehat{M}$,

$$\lim_{n\to\infty} P((\mathcal{Q}_{n,M}^{\gamma,\eta})^c)=0. \tag{D.4}$$

Proof. An application of Cramér's theorem for i.i.d. random variables gives that there exists a $c_\eta>0$ such that, for every $(I)\in\widehat{\mathcal{E}}_{n,M}^{\gamma}$,

$$\mathbb{P}_\xi\Big(\sum_{j=1}^{N}\sum_{i\in I_j}\xi_i\geq\big(\tfrac{\beta-\alpha}{2}+\eta\big)\,T(I)\Big)\leq e^{-c_\eta T(I)}\leq e^{-c_\eta\gamma n}, \tag{D.5}$$

where we use that $T(I)\geq\gamma n$ for every $(I)\in\widehat{\mathcal{E}}_{n,M}^{\gamma}$. Therefore

$$\mathbb{P}_\xi((\mathcal{Q}_{n,M}^{\gamma,\eta})^c)\leq|\widehat{\mathcal{E}}_{n,M}^{\gamma}|e^{-c(\eta)\gamma n}, \tag{D.6}$$

and it remains to bound $|\widehat{\mathcal{E}}_{n,M}^{\gamma}|$ as

$$\widehat{\mathcal{E}}_{n,M}^{\gamma}=\sum_{N=1}^{n/M}\big|\big\{(I)\in\mathcal{E}_{n,N}\colon T(I)\geq\gamma n\big\}\big|\leq\sum_{N=1}^{n/M}\binom{n}{2N}, \tag{D.7}$$

where we use that choosing $(I)\in\mathcal{E}_{n,N}$ amounts to choosing in $\{1,\ldots,n\}$ the end points of the N disjoint intervals. Thus, the right-hand side of (D.7) is at most $(n/M)\binom{n}{2n/M}$, which for M large enough is $o(e^{c(\eta)\gamma n})$ as $n\to\infty$. □

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