# Copolymer with pinning: variational characterization of the phase diagram

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**Abstract**

This paper studies a polymer chain in the vicinity of a linear interface separating two immiscible solvents. The polymer consists of *random monomer types*, while the interface carries *random charges*. Both the monomer types and the charges are given by i.i.d. sequences of random variables. The configurations of the polymer are directed paths that can make i.i.d. excursions of finite length above and below the interface. The Hamiltonian has two parts: a monomer-solvent interaction ("copolymer") and a monomer-interface interaction ("pinning"). The quenched and the annealed version of the model each undergo a transition from a *localized phase* (where the polymer stays close to the interface) to a *delocalized phase* (where the polymer wanders away from the interface). We exploit the approach developed in [5] and [3] to derive *variational formulas* for the quenched and the annealed free energy per monomer. These variational formulas are analyzed to obtain detailed information on the critical curves separating the two phases and on the typical behavior of the polymer in each of the two phases. Our main results settle a number of open questions.

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# 1 Introduction and main results

## 1.1 The model

**1. Polymer configuration.** The polymer is modeled by a directed path drawn from the set

$$\Pi = \Big\{\pi = (k, \pi_k)_{k\in\mathbb{N}_0} \colon\ \pi_0 = 0,\ \mathrm{sign}(\pi_{k-1}) + \mathrm{sign}(\pi_k) \neq 0,\ \pi_k \in \mathbb{Z}\ \forall\, k \in \mathbb{N}\Big\} \tag{1.1}$$

of directed paths in $\mathbb{N}_0 \times \mathbb{Z}$ that start at the origin and visit the interface $\mathbb{N}_0 \times \{0\}$ when switching from the lower halfplane to the upper halfplane, and vice versa. Let $P^*$ be the path measure on $\Pi$ under which the excursions away from the interface are i.i.d., lie above and below the interface with equal probability, and have a length distribution $\rho$ on $\mathbb{N}$ with $\sum_{n\in\mathbb{N}} \rho(n) = 1$, with infinite support and with a *polynomial tail*:

$$\lim_{\substack{n\to\infty \\ \rho(n)>0}} \frac{\log \rho(n)}{\log n} = -\alpha \text{ for some } \alpha \in [1, \infty). \tag{1.2}$$

Denote by $\Pi_n, P*_n$ the restriction of $\Pi, P^*$ to $n$-step paths that end at the interface.

**2. Disorder.** Let $\hat{E}$ and $\bar{E}$ be subsets of $\mathbb{R}$. The edges of the paths in $\Pi$ are labeled by an i.i.d. sequence of $\hat{E}$-valued random variables $\hat{\omega} = (\hat{\omega}_i)_{i\in\mathbb{N}}$ with common law $\hat{\mu}$, modeling the random monomer types. The sites at the interface are labeled by an i.i.d. sequence of $\bar{E}$-valued random variables $\bar{\omega} = (\bar{\omega}_i)_{i\in\mathbb{N}}$ with common law $\bar{\mu}$, modeling the random charges. In the sequel we abbreviate $\omega = (\omega_i)_{i\in\mathbb{N}}$ with $\omega_i = (\hat{\omega}_i, \bar{\omega}_i)$ and assume that $\hat{\omega}$ and $\bar{\omega}$ are independent. We further assume, without loss of generality, that both $\hat{\omega}_1$ and $\bar{\omega}_1$ have zero mean, unit variance, and satisfy

$$\hat{M}(t) = \log \int_{\hat{E}} e^{-t\hat{\omega}_1} \hat{\mu}(d\hat{\omega}_1) < \infty \quad \forall\, t \in \mathbb{R}, \qquad \bar{M}(t) = \log \int_{\bar{E}} e^{-t\bar{\omega}_1} \bar{\mu}(d\bar{\omega}_1) < \infty \quad \forall\, t \in \mathbb{R}. \tag{1.3}$$

We write $\mathbb{P}$ for the law of $\omega$, and $\mathbb{P}_{\hat{\omega}}$ and $\mathbb{P}_{\bar{\omega}}$ for the laws of $\hat{\omega}$ and $\bar{\omega}$.

**3. Path measure.** Given $n \in \mathbb{N}$ and $\omega$, the *quenched copolymer with pinning* is the path measure given by

$$\widetilde{P}_n^{\hat{\beta},\hat{h},\bar{\beta},\bar{h},\omega}(\pi) = \frac{1}{\widetilde{Z}_n^{\hat{\beta},\hat{h},\bar{\beta},\bar{h},\omega}} \exp\left[\widetilde{H}_n^{\hat{\beta},\hat{h},\bar{\beta},\bar{h},\omega}(\pi)\right] P_n^*(\pi), \qquad \pi \in \Pi_n, \tag{1.4}$$

where $\hat{\beta}, \hat{h}, \bar{\beta} \geq 0$ and $\bar{h} \in \mathbb{R}$ are parameters, $\widetilde{Z}_n^{\hat{\beta},\hat{h},\bar{\beta},\bar{h},\omega}$ is the normalizing partition sum, and

$$\widetilde{H}_n^{\hat{\beta},\hat{h},\bar{\beta},\bar{h},\omega}(\pi) = \hat{\beta} \sum_{i=1}^{n} (\hat{\omega}_i + \hat{h})\, \Delta_i + \sum_{i=1}^{n} (\bar{\beta}\, \bar{\omega}_i - \bar{h})\delta_i \tag{1.5}$$

is the interaction Hamiltonian, where $\delta_i = 1_{\{\pi_i=0\}} \in \{0, 1\}$ and $\Delta_i = \mathrm{sign}(\pi_{i-1}, \pi_i) \in \{-1, 1\}$ (the $i$-th edge is below or above the interface).

**Key example:** The choice $\hat{E} = \bar{E} = \{-1, 1\}$ corresponds to the situation where the upper halfplane consists of oil, the lower halfplane consists of water, the monomer types are either hydrophobic ($\hat{\omega}_i = 1$) or hydrophilic ($\hat{\omega}_i = -1$), and the charges are either positive ($\bar{\omega}_i = 1$) or negative ($\bar{\omega}_i = -1$); see Fig. 1. In (1.5), $\hat{\beta}$ and $\bar{\beta}$ are the *strengths* of the monomer-solvent and monomer-interface interactions, while $\hat{h}$ and $\bar{h}$ are the *biases* of these interactions. If $P^*$ is the law of the directed simple random walk on $\mathbb{Z}$, i.e., the uniform distribution on $\Pi$, then (1.2) holds with $\alpha = \frac{3}{2}$.

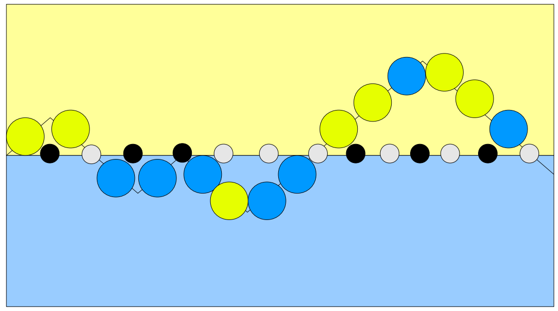


Figure 1: A directed polymer near a linear interface, separating oil in the upper halfplane and water in the lower halfplane. Hydrophobic monomers in the polymer are light shaded, hydrophilic monomers are dark shaded. Positive charges at the interface are light shaded, negative charges are dark shaded.

In the literature, the model without the monomer-interface interaction ($\bar\beta = \bar h = 0$) is called the *copolymer model*, while the model without the monomer-solvent interaction ($\hat h = \hat\beta = 0$) is called the *pinning model* (see Giacomin [11] and den Hollander [12] for an overview). The model with both interactions is referred to as the *copolymer with pinning model*. In the sequel, if $k$ is a quantity associated with the combined model, then $\hat k$ and $\bar k$ denote the analogous quantities in the copolymer model, respectively, the pinning model.

### 1.2 Quenched excess free energy and critical curve

The *quenched free energy* per monomer

$$f^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta,\bar h) = \lim_{n\to\infty} \frac{1}{n} \log \widetilde Z_n^{\hat\beta,\hat h,\bar\beta,\bar h,\omega} \tag{1.6}$$

exists $\omega$-a.s. and in $\mathbb{P}$-mean (see e.g. Giacomin [7]). By restricting the partition sum $\widetilde Z_n^{\hat\beta,\hat h,\bar\beta,\bar h,\omega}$ to paths that stay above the interface up to time $n$, we obtain, using the law of large numbers for $\hat\omega$, that $f^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta,\bar h) \geq \hat\beta\,\hat h$. The *quenched excess free energy* per monomer

$$g^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta,\bar h) = f^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta,\bar h) - \hat\beta\,\hat h \tag{1.7}$$

corresponds to the Hamiltonian

$$H_n^{\hat\beta,\hat h,\bar\beta,\bar h,\omega}(\pi) = \hat\beta \sum_{i=1}^{n} (\hat\omega_i + \hat h)\ [\Delta_i - 1] + \sum_{i=1}^{n} (\bar\beta\,\bar\omega_i - \bar h)\delta_i \tag{1.8}$$

and has two phases

$$\begin{aligned} \mathcal{L}^{\mathrm{que}} &= \left\{(\hat\beta,\hat h,\bar\beta,\bar h) \in [0,\infty)^3 \times \mathbb{R}\colon\ g^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta,\bar h) > 0\right\},\\ \mathcal{D}^{\mathrm{que}} &= \left\{(\hat\beta,\hat h,\bar\beta,\bar h) \in [0,\infty)^3 \times \mathbb{R}\colon\ g^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta,\bar h) = 0\right\}, \end{aligned} \tag{1.9}$$

called the *quenched localized phase* (where the strategy of staying close to the interface is optimal) and the *quenched delocalized phase* (where the strategy of wandering away from the interface is optimal). The map $\hat h \mapsto g^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta,\bar h)$ is non-increasing and convex for every $\hat\beta,\bar\beta \geq 0$ and $\bar h \in \mathbb{R}$. Hence, $\mathcal{L}^{\mathrm{que}}$ and $\mathcal{D}^{\mathrm{que}}$ are separated by a single curve (or rather single surface)

$$h_c^{\mathrm{que}}(\hat\beta,\bar\beta,\bar h) = \inf\left\{\hat h \geq 0\colon\ g^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta,\bar h) = 0\right\}, \tag{1.10}$$

called the *quenched critical curve*.

In the sequel we write $\hat{g}^{\mathrm{que}}(\hat{\beta},\hat{h})$, $\hat{h}_c^{\mathrm{que}}(\hat{\beta})$, $\hat{\mathcal{L}}^{\mathrm{que}}$, $\hat{\mathcal{D}}^{\mathrm{que}}$ for the analogous quantities in the copolymer model ($\bar{\beta}=\bar{h}=0$), and $\bar{g}^{\mathrm{que}}(\bar{\beta},\bar{h})$, $\bar{h}_c^{\mathrm{que}}(\bar{\beta})$, $\bar{\mathcal{L}}^{\mathrm{que}}$, $\bar{\mathcal{D}}^{\mathrm{que}}$ for the analogous quantities in the pinning model ($\hat{\beta}=\hat{h}=0$).

## 1.3 Annealed excess free energy and critical curve

The *annealed excess free energy* per monomer is given by

$$g^{\mathrm{ann}}(\hat{\beta},\hat{h},\bar{\beta},\bar{h}) = \lim_{n\to\infty}\frac{1}{n}\log Z_n^{\hat{\beta},\hat{h},\bar{\beta},\bar{h}} = \lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}\left(Z_n^{\hat{\beta},\hat{h},\bar{\beta},\bar{h},\omega}\right), \tag{1.11}$$

where $\mathbb{E}$ is the expectation w.r.t. the joint disorder distribution $\mathbb{P}$. This also has two phases,

$$\begin{aligned}\mathcal{L}^{\mathrm{ann}} &= \left\{(\hat{\beta},\hat{h},\bar{\beta},\bar{h})\in[0,\infty)^3\times\mathbb{R}\colon\ g^{\mathrm{ann}}(\hat{\beta},\hat{h},\bar{\beta},\bar{h})>0\right\},\\ \mathcal{D}^{\mathrm{ann}} &= \left\{(\hat{\beta},\hat{h},\bar{\beta},\bar{h})\in[0,\infty)^3\times\mathbb{R}\colon\ g^{\mathrm{ann}}(\hat{\beta},\hat{h},\bar{\beta},\bar{h})=0\right\},\end{aligned} \tag{1.12}$$

called the *annealed localized phase* and the *annealed delocalized phase*, respectively. The two phases are separated by the *annealed critical curve*

$$h_c^{\mathrm{ann}}(\hat{\beta},\bar{\beta},\bar{h}) = \inf\left\{\hat{h}\geq 0\colon\ g^{\mathrm{ann}}(\hat{\beta},\hat{h},\bar{\beta},\bar{h})=0\right\}. \tag{1.13}$$

Let $\mathcal{N}(g)=\sum_{n\in\mathbb{N}}e^{-ng}\,\rho(n)$. We will show in Section 3.2 that

$$\begin{aligned}&g^{\mathrm{ann}}(\hat{\beta},\hat{h},\bar{\beta},\bar{h}) \text{ is the unique } g\text{-value at which}\\ &\log\left[\tfrac{1}{2}\mathcal{N}(g)+\tfrac{1}{2}\mathcal{N}\big(g-[\hat{M}(2\hat{\beta})-2\hat{\beta}\hat{h}]\big)\right]+\bar{M}(-\bar{\beta})-\bar{h} \text{ changes sign.}\end{aligned} \tag{1.14}$$

It follows from (1.14) that for the copolymer model ($\bar{\beta}=\bar{h}=0$)

$$\begin{aligned}\hat{g}^{\,\mathrm{ann}}(\hat{\beta},\hat{h}) &= 0\vee[\hat{M}(2\hat{\beta})-2\hat{\beta}\hat{h}],\\ \hat{h}_c^{\mathrm{ann}}(\hat{\beta}) &= (2\hat{\beta})^{-1}\hat{M}(2\hat{\beta}),\end{aligned} \tag{1.15}$$

and for the pinning model ($\hat{\beta}=\hat{h}=0$)

$$\begin{aligned}&\bar{g}^{\mathrm{ann}}(\bar{\beta},\bar{h}) \text{ is the unique } g\text{-value for which } \mathcal{N}(g)=e^{-(0\vee[\bar{M}(-\bar{\beta})-\bar{h}])},\\ &\bar{h}_c^{\mathrm{ann}}(\bar{\beta})=\bar{M}(-\bar{\beta}).\end{aligned} \tag{1.16}$$

For more details on these special cases, see Giacomin [11] and den Hollander [12], and references therein.

## 1.4 Main results

Our variational characterization of the excess free energies and the critical curves is contained in the following theorem. For technical reasons, in the sequel we *exclude* the case $\hat{\beta}>0$, $\hat{h}=0$ for the quenched version.

**Theorem 1.1** *Assume* (1.2) *and* (1.3).
*(i) For every $\alpha \geq 1$ and $\hat\beta, \hat h, \bar\beta \geq 0$, there are lower semi-continuous, convex and non-increasing functions*

$$\begin{aligned} g &\mapsto S^{\mathrm{que}}(\hat\beta, \hat h, \bar\beta; g), \\ g &\mapsto S^{\mathrm{ann}}(\hat\beta, \hat h, \bar\beta; g), \end{aligned} \tag{1.17}$$

*given by explicit variational formulas such that, for every $\bar h \in \mathbb{R}$,*

$$\begin{aligned} g^{\mathrm{que}}(\hat\beta, \hat h, \bar\beta, \bar h) &= \inf\{g \in \mathbb{R}\colon\ S^{\mathrm{que}}(\hat\beta, \hat h, \bar\beta; g) - \bar h < 0\}, \\ g^{\mathrm{ann}}(\hat\beta, \hat h, \bar\beta, \bar h) &= \inf\{g \in \mathbb{R}\colon\ S^{\mathrm{ann}}(\hat\beta, \hat h, \bar\beta; g) - \bar h < 0\}. \end{aligned} \tag{1.18}$$

*(ii) For every $\alpha \geq 1$, $\hat\beta > 0$, $\bar\beta \geq 0$ and $\bar h \in \mathbb{R}$,*

$$\begin{aligned} h_c^{\mathrm{que}}(\hat\beta, \bar\beta, \bar h) &= \inf\left\{\hat h > 0\colon\ S^{\mathrm{que}}(\hat\beta, \hat h, \bar\beta; 0) - \bar h \leq 0\right\}, \\ h_c^{\mathrm{ann}}(\hat\beta, \bar\beta, \bar h) &= \inf\left\{\hat h \geq 0\colon\ S^{\mathrm{ann}}(\hat\beta, \hat h, \bar\beta; 0) - \bar h \leq 0\right\}. \end{aligned} \tag{1.19}$$

The variational formulas for $S^{\mathrm{que}}(\hat\beta, \hat h, \bar\beta; g)$ and $S^{\mathrm{ann}}(\hat\beta, \hat h, \bar\beta; g)$ are given in Theorems 3.1–3.2 in Section 3. Figs. 6–9 in Sections 3 and 5 show how these functions depend on $\hat\beta$, $\hat h$, $\bar\beta$ and $g$, which is crucial for our analysis.

Next, we state seven corollaries that are consequences of the variational formulas. The content of these corollaries will be discussed in Section 1.5. The first corollary looks at the excess free energies. Put

$$\begin{aligned} \bar h_*(\hat\beta, \hat h, \bar\beta) &= \bar M(-\bar\beta) + \log\left(\tfrac12\left[1 + \mathcal{N}\big(|\hat M(2\hat\beta) - 2\hat\beta\hat h|\big)\right]\right), \\ \mathcal{L}_1^{\mathrm{ann}} &= \left\{(\hat\beta, \hat h, \bar\beta, \bar h) \in [0,\infty)^3 \times \mathbb{R}\colon\ (\hat\beta, \hat h) \in \hat{\mathcal{L}}^{\mathrm{ann}}\right\}, \\ \mathcal{L}_2^{\mathrm{ann}} &= \left\{(\hat\beta, \hat h, \bar\beta, \bar h) \in [0,\infty)^3 \times \mathbb{R}\colon\ (\hat\beta, \hat h) \in \hat{\mathcal{D}}^{\mathrm{ann}}, \bar h < \bar h_*(\hat\beta, \hat h, \bar\beta)\right\}. \end{aligned} \tag{1.20}$$

**Corollary 1.2** *(i) For every $\alpha \geq 1$, $\hat\beta > 0$ and $\bar\beta \geq 0$, $g^{\mathrm{que}}(\hat\beta, \hat h, \bar\beta, \bar h)$ and $g^{\mathrm{ann}}(\hat\beta, \hat h, \bar\beta, \bar h)$ are the unique $g$-values that solve the equations*

$$\begin{aligned} S^{\mathrm{que}}(\hat\beta, \hat h, \bar\beta; g) &= \bar h, \quad \text{if } \bar h \in \mathbb{R},\ 0 < \hat h \leq h_c^{\mathrm{que}}(\hat\beta, \bar\beta, \bar h), \\ S^{\mathrm{ann}}(\hat\beta, \hat h, \bar\beta; g) &= \bar h, \quad \text{if } \hat h \geq 0,\ \bar h \leq \bar h_*(\hat\beta, \hat h, \bar\beta). \end{aligned} \tag{1.21}$$

*(ii) The annealed localized phase $\mathcal{L}^{\mathrm{ann}}$ admits the decomposition $\mathcal{L}^{\mathrm{ann}} = \mathcal{L}_1^{\mathrm{ann}} \cup \mathcal{L}_2^{\mathrm{ann}}$.*
*(iii) On $\mathcal{L}^{\mathrm{ann}}$,*

$$g^{\mathrm{que}}(\hat\beta, \hat h, \bar\beta, \bar h) < g^{\mathrm{ann}}(\hat\beta, \hat h, \bar\beta, \bar h), \tag{1.22}$$

*with the possible exception of the case where $m_\rho = \sum_{n\in\mathbb{N}} n\rho(n) = \infty$ and $\bar h = \bar h_*(\hat\beta, \hat h, \bar\beta)$.*
*(iv) For every $\alpha \geq 1$ and $\hat\beta, \hat h, \bar\beta \geq 0$,*

$$g^{\mathrm{ann}}(\hat\beta, \hat h, \bar\beta, \bar h) \begin{cases} = \hat g^{\mathrm{ann}}(\hat\beta, \hat h), & \text{if } \bar h \geq \bar h_*(\hat\beta, \hat h, \bar\beta), \\ > \hat g^{\mathrm{ann}}(\hat\beta, \hat h), & \text{otherwise.} \end{cases} \tag{1.23}$$

The next four corollaries look at the critical curves.

**Corollary 1.3** *For every $\alpha \geq 1$, $\hat{\beta} > 0$ and $\bar{\beta} \geq 0$ , the maps*

$$\begin{aligned} \hat{h} &\mapsto S^{\text{que}}(\hat{\beta}, \hat{h}, \bar{\beta}; 0), \\ \hat{h} &\mapsto S^{\text{ann}}(\hat{\beta}, \hat{h}, \bar{\beta}; 0), \end{aligned} \tag{1.24}$$

*are convex and non-increasing on $(0, \infty)$. Both critical curves are continuous and non-increasing in $\bar{h}$. Moreover (see Figs. 2–3),*

$$h_c^{\text{que}}(\hat{\beta}, \bar{\beta}, \bar{h}) = \begin{cases} \infty, & \text{if } \bar{h} \leq \bar{h}_c^{\text{que}}(\bar{\beta}) - \log 2, \\ \hat{h}_c^{\text{ann}}(\hat{\beta}/\alpha), & \text{if } \bar{h} > s^*(\hat{\beta}, \bar{\beta}, \alpha), \\ h_*^{\text{que}}(\hat{\beta}, \bar{\beta}, \bar{h}), & \text{otherwise,} \end{cases} \tag{1.25}$$

*and*

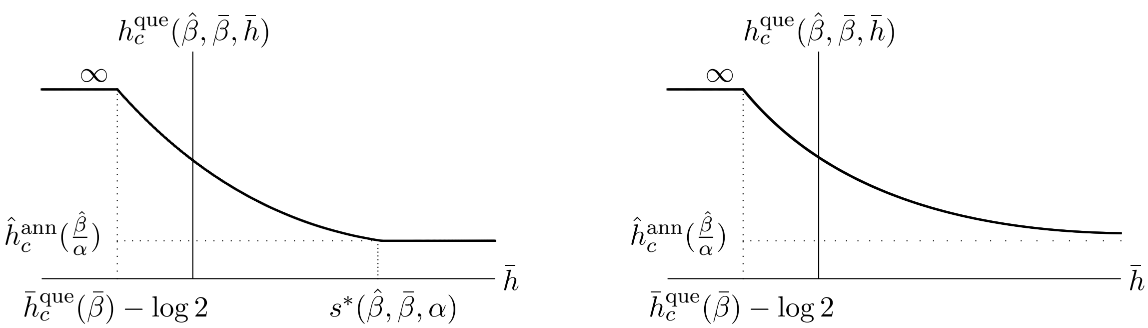
$$h_c^{\text{ann}}(\hat{\beta}, \bar{\beta}, \bar{h}) = \begin{cases} \infty, & \text{if } \bar{h} \leq \bar{h}_c^{\text{ann}}(\bar{\beta}) - \log 2, \\ \hat{h}_c^{\text{ann}}(\hat{\beta}), & \text{if } \bar{h} > \bar{h}_c^{\text{ann}}(\bar{\beta}), \\ h_*^{\text{ann}}(\hat{\beta}, \bar{\beta}, \bar{h}), & \text{otherwise,} \end{cases} \tag{1.26}$$

*where*

$$s^*(\hat{\beta}, \bar{\beta}, \alpha) = s^{\text{que}}\left(\hat{\beta}, \hat{h}_c^{\text{ann}}(\hat{\beta}/\alpha), \bar{\beta}; 0\right) \in (\bar{h}_c^{\text{que}}(\bar{\beta}) - \log 2, \infty] \tag{1.27}$$

*is defined in (3.15), and $h_*^{\text{que}}(\hat{\beta}, \bar{\beta}, \bar{h})$ and $h_*^{\text{ann}}(\hat{\beta}, \bar{\beta}, \bar{h})$ are the unique $\hat{h}$-values that solve the equations*

$$\begin{aligned} S^{\text{que}}(\hat{\beta}, \hat{h}, \bar{\beta}; 0) &= \bar{h}, \\ S^{\text{ann}}(\hat{\beta}, \hat{h}, \bar{\beta}; 0) &= \bar{h}. \end{aligned} \tag{1.28}$$




(a) (b)

Figure 2: Qualitative picture of the map $\bar{h} \mapsto h_c^{\text{que}}(\hat{\beta}, \bar{\beta}, \bar{h})$ for $\hat{\beta} > 0$ and $\bar{\beta} \geq 0$ when: (a) $s^*(\hat{\beta}, \bar{\beta}, \alpha) < \infty$; (b) $s^*(\hat{\beta}, \bar{\beta}, \alpha) = \infty$.

**Corollary 1.4** *For every $\alpha > 1$, $\hat{\beta} > 0$ and $\bar{\beta} \geq 0$,*

$$h_c^{\text{que}}(\hat{\beta}, \bar{\beta}, \bar{h}) \begin{cases} < h_c^{\text{ann}}(\hat{\beta}, \bar{\beta}, \bar{h}) \leq \infty, & \text{if } \bar{h} > \bar{h}_c^{\text{que}}(\bar{\beta}) - \log 2, \\ = h_c^{\text{ann}}(\hat{\beta}, \bar{\beta}, \bar{h}) = \infty, & \text{otherwise.} \end{cases} \tag{1.29}$$

**Corollary 1.5** *For every $\alpha > 1$, $\hat{\beta} > 0$ and $\bar{\beta} \geq 0$,*

$$h_c^{\text{que}}(\hat{\beta}, \bar{\beta}, \bar{h}) \begin{cases} > \hat{h}_c^{\text{ann}}(\hat{\beta}/\alpha), & \text{if } \bar{h} < s^*(\hat{\beta}, \bar{\beta}, \alpha), \\ = \hat{h}_c^{\text{ann}}(\hat{\beta}/\alpha), & \text{otherwise.} \end{cases} \tag{1.30}$$

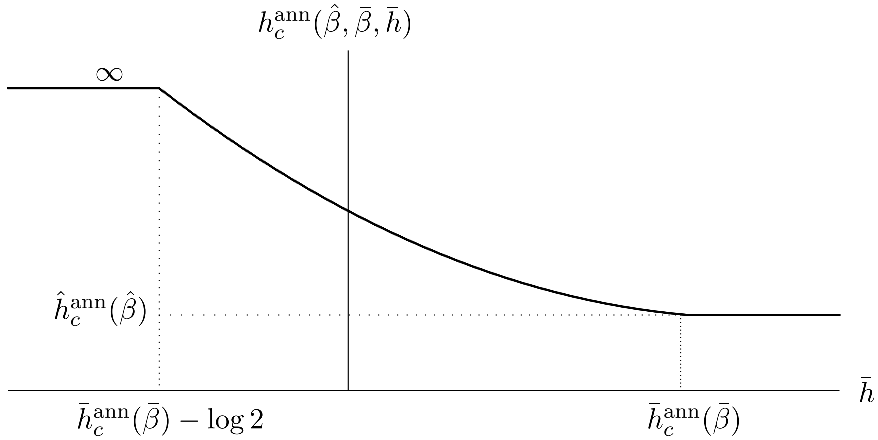



Figure 3: Qualitative picture of the map $\bar h \mapsto h_c^{\mathrm{ann}}(\hat\beta,\bar\beta,\bar h)$ for $\hat\beta,\bar\beta \geq 0$.

**Corollary 1.6** *(i) For every $\alpha \geq 1$ and $\hat\beta,\bar\beta \geq 0$,*

$$\begin{aligned}\inf\Big\{\bar h\in\mathbb{R}\colon\ g^{\mathrm{ann}}(\hat\beta,\hat h_c^{\mathrm{ann}}(\hat\beta),\bar\beta,\bar h)=0\Big\}&=\bar h_c^{\mathrm{ann}}(\bar\beta),\\ \inf\Big\{\hat h\geq 0\colon\ g^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta,\bar h_c^{\mathrm{ann}}(\bar\beta))=0\Big\}&=\hat h_c^{\mathrm{ann}}(\hat\beta).\end{aligned}\tag{1.31}$$

*(ii) For every $\alpha \geq 1$, $\hat\beta > 0$ and $\bar\beta = 0$,*

$$\inf\Big\{\bar h\in\mathbb{R}\colon g^{\mathrm{que}}(\hat\beta,\hat h_c^{\mathrm{que}}(\hat\beta),\bar\beta,\bar h)=0\Big\}=\hat h_c^{\mathrm{ann}}(\hat\beta).\tag{1.32}$$

The last two corollaries concern the typical path behavior. Let $P_n^{\hat\beta,\hat h,\bar\beta,\bar h,\omega}$ denote the path measure associated with the Hamiltonian $H_n^{\hat\beta,\hat h,\bar\beta,\bar h,\omega}$ defined in (1.8). Write $\mathcal{M}_n=\mathcal{M}_n(\pi)=|\{1\leq i\leq n\colon\ \pi_i=0\}|$ to denote the number of times the polymer returns to the interface up to time $n$. Define

$$\mathcal{D}_1^{\mathrm{que}}=\Big\{(\hat\beta,\hat h,\bar\beta,\bar h)\in\mathcal{D}^{\mathrm{que}}\colon\ \bar h\leq s^*(\hat\beta,\bar\beta,\alpha)\Big\}.\tag{1.33}$$

**Corollary 1.7** *For every $(\hat\beta,\hat h,\bar\beta,\bar h)\in\mathrm{int}(\mathcal{D}_1^{\mathrm{que}})\cup(\mathcal{D}^{\mathrm{que}}\setminus\mathcal{D}_1^{\mathrm{que}})$ and $c>\alpha/[-(S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0)-\bar h)]\in(0,\infty)$,*

$$\lim_{n\to\infty}P_n^{\hat\beta,\hat h,\bar\beta,\bar h,\omega}\left(\mathcal{M}_n\geq c\log n\right)=0\qquad \omega-a.s.\tag{1.34}$$

**Corollary 1.8** *For every $(\hat\beta,\hat h,\bar\beta,\bar h)\in\mathcal{L}^{\mathrm{que}}$,*

$$\lim_{n\to\infty}P_n^{\hat\beta,\hat h,\bar\beta,\bar h,\omega}\left(|\tfrac{1}{n}\mathcal{M}_n-C|\leq\varepsilon\right)=1\qquad\omega-a.s.\quad\forall\,\varepsilon>0,\tag{1.35}$$

*where*

$$-\frac{1}{C}=\frac{\partial}{\partial g}\,S^{\mathrm{que}}\big(\hat\beta,\hat h,\bar\beta;g^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta,\bar h)\big)\in(-\infty,0),\tag{1.36}$$

*provided this derivative exists. (By convexity, at least the left-derivative and the right-derivative exist.)*

### 1.5 Discussion

**1.** The copolymer and pinning versions of Theorem 1.1 are obtained by putting $\bar{\beta} = \bar{h} = 0$ and $\hat{\beta} = \hat{h} = 0$, respectively. The copolymer version of Theorem 1.1 was proved in Bolthausen, den Hollander and Opoku [3].

**2.** Corollary 1.2(i) identifies the range of parameters for which the free energies given by (1.18) are the $g$-values where the variational formulas equal $\bar{h}$. Corollary 1.2(ii) shows that the annealed combined model is localized when the annealed copolymer model is localized. On the other hand, if the annealed copolymer model is delocalized, then a sufficiently attractive pinning interaction is needed for the annealed combined model to become localized, namely, $\bar{h} < \bar{h}_*(\hat{\beta}, \hat{h}, \bar{\beta})$. It is an open problem to identify a similar threshold for the quenched combined model.

**3.** In Bolthausen, den Hollander and Opoku [3] it was shown with the help of the variational approach that for the copolymer model there is a gap between the quenched and the annealed excess free energy in the localized phase of the annealed copolymer model. It was argued that this gap can also be deduced with the help of a result in Giacomin and Toninelli [9, 10], namely, the fact that the map $\hat{h} \mapsto \hat{g}^{\mathrm{que}}(\hat{\beta}, \hat{h})$ drops below a quadratic as $\hat{h} \uparrow \hat{h}_c^{\mathrm{que}}(\hat{\beta})$ (i.e., the phase transition is "at least of second order"). Indeed, $g^{\mathrm{que}} \leq g^{\mathrm{ann}}$, $\hat{h} \mapsto \hat{g}^{\mathrm{que}}(\hat{\beta}, \hat{h})$ is convex and strictly decreasing on $(0, \hat{h}_c^{\mathrm{que}}(\hat{\beta})]$, and $\hat{h} \mapsto \hat{g}^{\mathrm{ann}}(\hat{\beta}, \hat{h})$ is linear and strictly decreasing on $(0, \hat{h}_c^{\mathrm{ann}}(\hat{\beta})]$. The quadratic bound implies that the gap is present for $\hat{h}$ slightly below $\hat{h}_c^{\mathrm{ann}}(\hat{\beta})$, and therefore it must be present for all $\hat{h}$ below $\hat{h}_c^{\mathrm{ann}}(\hat{\beta})$. Now, the same arguments as in [9, 10] show that also $\hat{h} \mapsto g^{\mathrm{que}}(\hat{\beta}, \hat{h}, \bar{\beta}, \bar{h})$ drops below a quadratic as $\hat{h} \uparrow h_c^{\mathrm{que}}(\hat{\beta}, \bar{\beta}, \bar{h})$. However, $\hat{h} \mapsto g^{\mathrm{ann}}(\hat{\beta}, \hat{h}, \bar{\beta}, \bar{h})$ is *not* linear on $(0, h_c^{\mathrm{ann}}(\hat{\beta}, \bar{\beta}, \bar{h})]$ (see (1.14)), and so there is no similar proof of Corollary 1.2(iii). Our proof underscores the *robustness* of the variational approach. We expect the gap to be present also when $m_\rho = \infty$ and $\bar{h} = \bar{h}_*(\hat{\beta}, \hat{h}, \bar{\beta})$, but this remains open.

**4.** Corollary 1.2(iv) gives a natural interpretation for $\bar{h}_*(\hat{\beta}, \hat{h}, \bar{\beta})$, namely, this is the critical value below which the pinning interaction has an effect in the annealed model and above which it has not.

**5.** The precise shape of the quenched critical curve for the combined model was not well understood (see e.g. Giacomin [11], Section 6.3.2, and Caravenna, Giacomin and Toninelli [4], last paragraph of Section 1.5). In particular, in [11] two possible shapes were suggested for $\bar{\beta} = 0$, as shown in Fig. 4. Corollary 1.3 rules out line 2, while it proves line 1 in the following sense: (1) this line holds for all $\bar{\beta} \geq 0$; (2) for $\bar{h} < \bar{h}_c^{\mathrm{que}}(\bar{\beta}) - \log 2$, the combined model is *fully localized*; (3) conditionally on $s^*(\hat{\beta}, \bar{\beta}, \alpha) < \infty$, for $\bar{h} \geq s^*(\hat{\beta}, \bar{\beta}, \alpha)$ the quenched critical curve concides with $\hat{h}_c^{\mathrm{ann}}(\hat{\beta}/\alpha)$ (see Fig. 2). In the literature $\hat{h}_c^{\mathrm{ann}}(\hat{\beta}/\alpha)$ is called the *Monthus-line*. Thus, when we sit at the far ends of the $\bar{h}$-axis, the critical behavior of the quenched combined model is determined either by the copolymer interaction (on the far right) or by the pinning interaction (on the far left). Only in-between is there a non-trivial competition between the two interactions.

**6.** The threshold values $\bar{h} = \bar{h}^{\mathrm{que}}(\bar{\beta}) - \log 2$ and $\bar{h} = \bar{h}^{\mathrm{ann}}(\bar{\beta}) - \log 2$ (see Figs. 2–3) are the critical points for the quenched and the annealed pinning model when the polymer is allowed to stay in the upper halfplane only. In the literature this restricted pinning model is called the *wetting model* (see Giacomin [11], den Hollander [12]). These values of $\bar{h}$ are the transition points at which the quenched and the annealed critical curves of the combined model change from being finite to being infinite. Thus, we recover the critical curves for the wetting model from those of the combined model by putting $\hat{h} = \infty$.

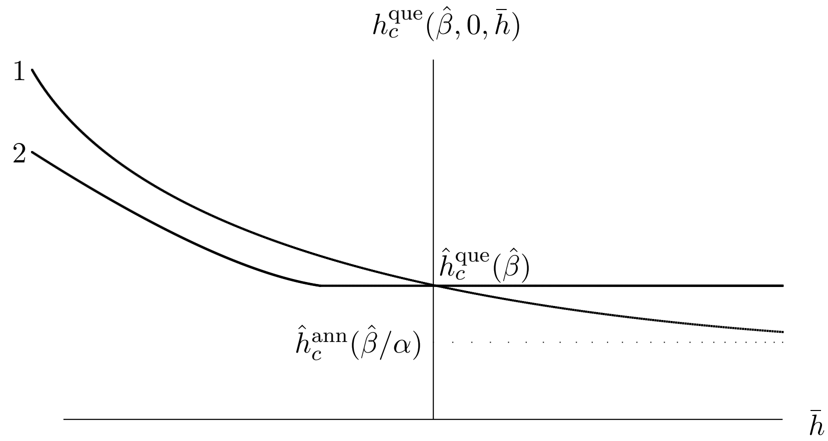



Figure 4: Possible qualitative pictures of the map $\bar h \mapsto h_c^{\mathrm{que}}(\hat\beta, 0, \bar h)$ for $\hat\beta > 0$.

**7.** It is known from the literature that the pinning model undergoes a transition between *disorder relevance* and *disorder irrelevance*. In the former regime, there is a gap between the quenched and the annealed critical curve, in the latter there is not. The transition depends on $\alpha$, $\bar\beta$ and $\bar\mu$. In particular, if $\alpha > \frac{3}{2}$, then the disorder is relevant for all $\bar\beta > 0$, while if $\alpha \in (1, \frac{3}{2})$, then there is a critical threshold $\bar\beta_c \in (0,\infty]$ such that the disorder is irrelevant for $\bar\beta \le \bar\beta_c$ and relevant for $\bar\beta > \bar\beta_c$. The transition is absent in the copolymer model, where the disorder is relevant for all $\alpha > 1$. However, Corollary 1.4 shows that in the combined model the transition occurs for all $\alpha > 1$, $\hat\beta > 0$ and $\bar\beta \ge 0$. Indeed, the disorder is relevant for $\bar h > \bar h^{\mathrm{que}}(\bar\beta) - \log 2$ and is irrelevant for $\bar h \le \bar h^{\mathrm{que}}(\bar\beta) - \log 2$.

**8.** The quenched critical curve is bounded from below by the Monthus-line (as the critical curve moves closer to the Monthus-line, the copolymer interaction more and more dominates the pinning interaction). Corollary 1.5 and Fig. 2 show that the critical curve stays above the Monthus-line as long as $\bar h < s^*(\hat\beta, \bar\beta, \alpha)$. If $s^*(\hat\beta, \bar\beta, \alpha) = \infty$, then the quenched critical curve is everywhere above the Monthus-line (see Fig. 2(b)). A *sufficient* condition for $s^*(\hat\beta, \bar\beta, \alpha) < \infty$ is

$$\sum_{n\in\mathbb{N}} \rho(n)^{\frac{1}{\alpha}} < \infty. \tag{1.37}$$

We do not know whether $s^*(\hat\beta, \bar\beta, \alpha) < \infty$ always. For $\bar\beta = 0$, Toninelli [14] proved that, under condition (1.37), the quenched critical curve coincides with the Monthus-line for $\bar h$ large enough.

**9.** Corollary 1.6(i) shows that the critical curve for the annealed combined model taken at the $\bar h$-value where the annealed copolymer model is critical coincides with the annealed critical curve of the pinning model, and vice versa. For the quenched combined model a similar result is expected, but this remains open. One of the questions that was posed in Giacomin [11], Section 6.3.2, for the quenched combined model is whether an arbitrary small pinning bias $-\bar h > 0$ can lead to localization for $\bar\beta = 0$, $\hat\beta > 0$ and $\hat h = \hat h_c^{\mathrm{que}}(\hat\beta)$. This question is answered in the affirmative by Corollary 1.6(ii).

**10.** Giacomin and Toninelli [8] showed that in $\mathcal{L}^{\mathrm{que}}$ the longest excursion under the quenched path measure $P_n^{\hat\beta,\hat h,\bar\beta,\bar h,\omega}$ is of order $\log n$. No information was obtained about the path behavior in $\mathcal{D}^{\mathrm{que}}$. Corollary 1.7 says that in $\mathcal{D}^{\mathrm{que}}$ (which is the region on or above the critial curve in Fig. 2), with the exception of the piece of the critical curve over the interval $(-\infty, s_*(\hat\beta, \bar\beta, \alpha))$, the total number of visits to the interface up to time $n$ is at most of order $\log n$. On this piece, the number may very well be of larger order. Corollary 1.8 says that in $\mathcal{L}^{\rm que}$ this number is proportional to $n$, with a variational formula for the proportionality constant. Since on the piece of the critical curve over the interval $[s_*(\hat\beta,\bar\beta,\alpha),\infty)$ the number is of order $\log n$, the phase transition is expected to be first order on this piece.

**11.** Smoothness of the free energy in the localized phase, finite-size corrections, and a central limit theorem for the free energy can be found in [8]. Pétrélis [13] studies the weak interaction limit of the combined model.

### 1.6 Outline

The present paper uses ideas from Cheliotis and den Hollander [5] and Bolthausen, den Hollander and Opoku [3]. The proof of Theorem 1.1 uses large deviation principles derived in Birkner [1] and Birkner, Greven and den Hollander [2]. The quenched variational formula and its proof are given in Section 3.1, the annealed variational formula and its proof in Section 3.2. Section 3.3 contains the proof of Theorem 1.1. The proofs of Corollaries 1.2–1.8 are given in Sections 4–6. The latter require certain technical results, which are proved in Appendices A–C.

## 2 Large Deviation Principle (LDP)

Let $E$ be a Polish space, playing the role of an alphabet, i.e., a set of *letters*. Let $\widetilde{E}=\cup_{k\in\mathbb{N}}E^k$ be the set of *finite words* drawn from $E$, which can be metrized to become a Polish space.

Fix $\nu\in\mathcal{P}(E)$, and $\rho\in\mathcal{P}(\mathbb{N})$ satisfying (1.2). Let $X=(X_k)_{k\in\mathbb{N}}$ be i.i.d. $E$-valued random variables with marginal law $\nu$, and $\tau=(\tau_i)_{i\in\mathbb{N}}$ i.i.d. $\mathbb{N}$-valued random variables with marginal law $\rho$. Assume that $X$ and $\tau$ are independent, and write $\mathbb{P}\otimes P^*$ to denote their joint law. Cut words out of the letter sequence $X$ according to $\tau$ (see Fig. 5), i.e., put

$$T_0=0 \quad\text{and}\quad T_i=T_{i-1}+\tau_i,\quad i\in\mathbb{N}, \tag{2.1}$$

and let

$$Y^{(i)}=\big(X_{T_{i-1}+1},X_{T_{i-1}+2},\dots,X_{T_i}\big),\quad i\in\mathbb{N}. \tag{2.2}$$

Under the law $\mathbb{P}\otimes P^*$, $Y=(Y^{(i)})_{i\in\mathbb{N}}$ is an i.i.d. sequence of words with marginal distribution $q_{\rho,\nu}$ on $\widetilde{E}$ given by

$$\begin{aligned}\mathbb{P}\otimes P^*\big(Y^{(1)}\in(dx_1,\dots,dx_n)\big)&=q_{\rho,\bar\mu}\big((dx_1,\dots,dx_n)\big)\\&=\rho(n)\,\nu(dx_1)\times\cdots\times\nu(dx_n),\qquad n\in\mathbb{N},\,x_1,\dots,x_n\in E.\end{aligned}\tag{2.3}$$

The reverse operation of *cutting* words out of a sequence of letters is *glueing* words together into a sequence of letters. Formally, this is done by defining a *concatenation* map $\kappa$ from $\widetilde{E}^{\mathbb{N}}$ to $E^{\mathbb{N}}$. This map induces in a natural way a map from $\mathcal{P}(\widetilde{E}^{\mathbb{N}})$ to $\mathcal{P}(E^{\mathbb{N}})$, the sets of probability measures on $\widetilde{E}^{\mathbb{N}}$ and $E^{\mathbb{N}}$ (endowed with the topology of weak convergence). The concatenation $q_{\rho,\nu}^{\otimes\mathbb{N}}\circ\kappa^{-1}$ of $q_{\rho,\nu}^{\otimes\mathbb{N}}$ equals $\nu^{\otimes\mathbb{N}}$, as is evident from (2.3).

### 2.1 Annealed LDP

Let $\mathcal{P}^{\rm inv}(\widetilde{E}^{\mathbb{N}})$ be the set of probability measures on $\widetilde{E}^{\mathbb{N}}$ that are invariant under the left-shift $\widetilde{\theta}$ acting on $\widetilde{E}^{\mathbb{N}}$. For $N\in\mathbb{N}$, let $(Y^{(1)},\dots,Y^{(N)})^{\rm per}$ be the periodic extension of the $N$-tuple

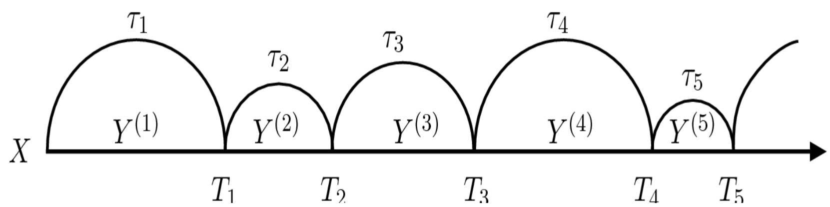



Figure 5: Cutting words out of a sequence of letters according to renewal times.

$(Y^{(1)},\dots,Y^{(N)}) \in \widetilde{E}^N$ to an element of $\widetilde{E}^{\mathbb{N}}$. The *empirical process of $N$-tuples of words* is defined as

$$R_N^X = \frac{1}{N}\sum_{i=0}^{N-1} \delta_{\widetilde{\theta}^i(Y^{(1)},\dots,Y^{(N)})^{\mathrm{per}}} \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}), \tag{2.4}$$

where the supercript $X$ indicates that the words $Y^{(1)},\dots,Y^{(N)}$ are cut from the latter sequence $X$. For $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$, let $H(Q \mid q_{\rho,\nu}^{\otimes\mathbb{N}})$ be the *specific relative entropy of $Q$ w.r.t. $q_{\rho,\nu}^{\otimes\mathbb{N}}$* defined by

$$H(Q \mid q_{\rho,\nu}^{\otimes\mathbb{N}}) = \lim_{N\to\infty} \frac{1}{N}\, h(\pi_N Q \mid q_{\rho,\nu}^N), \tag{2.5}$$

where $\pi_N Q \in \mathcal{P}(\widetilde{E}^N)$ denotes the projection of $Q$ onto the first $N$ words, $h(\,\cdot\mid\cdot\,)$ denotes relative entropy, and the limit is non-decreasing.

For the applications below we will need the following tilted version of $\rho$:

$$\rho_g(n) = \frac{e^{-gn}\,\rho(n)}{\mathcal{N}(g)} \qquad \text{with} \qquad \mathcal{N}(g) = \sum_{n\in\mathbb{N}} e^{-gn}\,\rho(n), \quad g \geq 0. \tag{2.6}$$

Note that, for $g > 0$, $\rho_g$ has a tail that is exponentially bounded. The following result relates the relative entropies with $q_{\rho_g,\nu}^{\otimes\mathbb{N}}$ and $q_{\rho,\nu}^{\otimes\mathbb{N}}$ as reference measures.

**Lemma 2.1** [3] *For $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ and $g \geq 0$,*

$$H(Q \mid q_{\rho_g,\nu}^{\otimes\mathbb{N}}) = H(Q \mid q_{\rho,\nu}^{\otimes\mathbb{N}}) + \log\mathcal{N}(g) + g\,\mathbb{E}_Q(\tau_1). \tag{2.7}$$

This result shows that, for $g \geq 0$, $m_Q = \mathbb{E}_Q(\tau_1) < \infty$ whenever $H(Q \mid q_{\rho_g,\nu}^{\otimes\mathbb{N}}) < \infty$, which is a special case of [1], Lemma 7.

The following *annealed LDP* is standard (see e.g. Dembo and Zeitouni [6], Section 6.5).

**Theorem 2.2** *For every $g \geq 0$, the family $(\mathbb{P}\otimes P_g^*)(R_N^{\cdot} \in \cdot)$, $N \in \mathbb{N}$, satisfies the LDP on $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ with rate $N$ and with rate function $I_g^{\mathrm{ann}}$ given by*

$$I_g^{\mathrm{ann}}(Q) = H\big(Q \mid q_{\rho_g,\nu}^{\otimes\mathbb{N}}\big), \qquad Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}). \tag{2.8}$$

*This rate function is lower semi-continuous, has compact level sets, has a unique zero at $q_{\rho_g,\nu}^{\otimes\mathbb{N}}$, and is affine.*

It follows from Lemma 2.1 that

$$I_g^{\mathrm{ann}}(Q) = I^{\mathrm{ann}}(Q) + \log\mathcal{N}(g) + g m_Q, \tag{2.9}$$

where $I^{\mathrm{ann}}(Q) = H(Q \mid q_{\rho,\nu}^{\otimes\mathbb{N}})$, the annealed rate function for $g = 0$.

## 2.2 Quenched LDP

To formulate the quenched analogue of Theorem 2.2, we need some more notation. Let $\mathcal{P}^{\mathrm{inv}}(E^{\mathbb{N}})$ be the set of probability measures on $E^{\mathbb{N}}$ that are invariant under the left-shift $\theta$ acting on $E^{\mathbb{N}}$. For $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ such that $m_Q < \infty$, define

$$\Psi_Q = \frac{1}{m_Q} E_Q\left(\sum_{k=0}^{\tau_1 - 1} \delta_{\theta^k \kappa(Y)}\right) \in \mathcal{P}^{\mathrm{inv}}(E^{\mathbb{N}}). \tag{2.10}$$

Think of $\Psi_Q$ as the shift-invariant version of $Q \circ \kappa^{-1}$ obtained after *randomizing* the location of the origin. This randomization is necessary because a shift-invariant $Q$ in general does not give rise to a shift-invariant $Q \circ \kappa^{-1}$.

For $\mathrm{tr} \in \mathbb{N}$, let $[\cdot]_{\mathrm{tr}} \colon \widetilde{E} \to [\widetilde{E}]_{\mathrm{tr}} = \cup_{n=1}^{\mathrm{tr}} E^n$ denote the *truncation map* on words defined by

$$y = (x_1, \dots, x_n) \mapsto [y]_{\mathrm{tr}} = (x_1, \dots, x_{n \wedge \mathrm{tr}}), \qquad n \in \mathbb{N},\, x_1, \dots, x_n \in E, \tag{2.11}$$

i.e., $[y]_{\mathrm{tr}}$ is the word of length $\leq \mathrm{tr}$ obtained from the word $y$ by dropping all the letters with label $> \mathrm{tr}$. This map induces in a natural way a map from $\widetilde{E}^{\mathbb{N}}$ to $[\widetilde{E}]_{\mathrm{tr}}^{\mathbb{N}}$, and from $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ to $\mathcal{P}^{\mathrm{inv}}([\widetilde{E}]_{\mathrm{tr}}^{\mathbb{N}})$. Note that if $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$, then $[Q]_{\mathrm{tr}}$ is an element of the set

$$\mathcal{P}^{\mathrm{inv,fin}}(\widetilde{E}^{\mathbb{N}}) = \{Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}) \colon\, m_Q < \infty\}. \tag{2.12}$$

Define (w-lim means weak limit)

$$\mathcal{R} = \left\{ Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}) \colon\, \mathrm{w} - \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \delta_{\theta^k \kappa(Y)} = \nu^{\otimes \mathbb{N}} \quad Q - a.s. \right\}, \tag{2.13}$$

i.e., the set of probability measures in $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ under which the concatenation of words almost surely has the same asymptotic statistics as a typical realization of $X$.

**Theorem 2.3** (Birkner [1]; Birkner, Greven and den Hollander [2]) *Assume* (1.2–1.3)*. Then, for $\nu^{\otimes \mathbb{N}}$–a.s. all $X$ and all $g \in [0, \infty)$, the family of (regular) conditional probability distributions $P_g^*(R_N^X \in \cdot \mid X)$, $N \in \mathbb{N}$, satisfies the LDP on $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ with rate $N$ and with deterministic rate function $I_g^{\mathrm{que}}$ given by*

$$I_g^{\mathrm{que}}(Q) = \begin{cases} I_g^{\mathrm{ann}}(Q), & \text{if } Q \in \mathcal{R}, \\ \infty, & \text{otherwise}, \end{cases} \quad \text{when } g > 0, \tag{2.14}$$

*and*

$$I^{\mathrm{que}}(Q) = \begin{cases} I^{\mathrm{fin}}(Q), & \text{if } Q \in \mathcal{P}^{\mathrm{inv,fin}}(\widetilde{E}^{\mathbb{N}}), \\ \lim_{\mathrm{tr} \to \infty} I^{\mathrm{fin}}\big([Q]_{\mathrm{tr}}\big), & \text{otherwise}, \end{cases} \quad \text{when } g = 0, \tag{2.15}$$

*where*

$$I^{\mathrm{fin}}(Q) = H(Q \mid q_{\rho,\nu}^{\otimes \mathbb{N}}) + (\alpha - 1)\, m_Q\, H\big(\Psi_Q \mid \nu^{\otimes \mathbb{N}}\big). \tag{2.16}$$

*This rate function is lower semi-continuous, has compact level sets, has a unique zero at $q_{\rho_g,\nu}^{\otimes \mathbb{N}}$, and is affine.*

It was shown in [1], Lemma 2, that

$$\Psi_Q = \nu^{\otimes\mathbb{N}} \quad \Longleftrightarrow \quad Q \in \mathcal{R} \qquad \text{on } \mathcal{P}^{\mathrm{inv,fin}}(\widetilde{E}^{\mathbb{N}}), \tag{2.17}$$

which explains why the restriction $Q \in \mathcal{R}$ appears in (2.14). For more background, see [2].

Note that $I^{\mathrm{que}}(Q)$ requires a truncation approximation when $m_Q = \infty$, for which case there is no closed form expression like in (2.16). As we will see later on, the cases $m_Q < \infty$ and $m_Q = \infty$ need to be separated. For later reference we remark that, for all $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$,

$$\begin{aligned} I^{\mathrm{ann}}(Q) &= \lim_{\mathrm{tr}\to\infty} I^{\mathrm{ann}}([Q]_{\mathrm{tr}}) = \sup_{\mathrm{tr}\in\mathbb{N}} I^{\mathrm{ann}}([Q]_{\mathrm{tr}}),\\ I^{\mathrm{que}}(Q) &= \lim_{\mathrm{tr}\to\infty} I^{\mathrm{que}}([Q]_{\mathrm{tr}}) = \sup_{\mathrm{tr}\in\mathbb{N}} I^{\mathrm{que}}([Q]_{\mathrm{tr}}), \end{aligned} \tag{2.18}$$

as shown in [2], Lemma A.1.

# 3 Variational formulas for excess free energies

This section uses the LDP of Section 2 to derive variational formulas for the excess free energy of the quenched and the annealed version of the combined model. The quenched version is treated in Section 3.1, the annealed version in Section 3.2. The results in Sections 3.1–3.2 are used in Section 3.3 to prove Theorem 1.1.

In the combined model words are made up of letters from the alphabet $E = \hat{E} \times \bar{E}$, where $\hat{E}$ and $\bar{E}$ are subsets of $\mathbb{R}$, and are cut from the letter sequence $\omega = ((\hat{\omega}_i, \bar{\omega}_i))_{i\in\mathbb{N}}$, where $\hat{\omega} = (\hat{\omega}_i)_{i\in\mathbb{N}}$ and $\bar{\omega} = (\bar{\omega}_i)_{i\in\mathbb{N}}$ are i.i.d. sequences of $\hat{E}$-valued and $\bar{E}$-valued random variables with joint common law $\nu = \hat{\mu} \otimes \bar{\mu}$. Let $\hat{\pi}$ and $\bar{\pi}$ be the projection maps from $E$ onto $\hat{E}$ and $\bar{E}$, respectively, i.e $\hat{\pi}((\hat{\omega}_1, \bar{\omega}_1)) = \hat{\omega}_1$ and $\bar{\pi}((\hat{\omega}_1, \bar{\omega}_1)) = \bar{\omega}_1$ for $(\hat{\omega}_1, \bar{\omega}_1) \in E$. These maps extend naturally to $E^{\mathbb{N}}$, $\widetilde{E}$, $\widetilde{E}^{\mathbb{N}}$, $\mathcal{P}(\widetilde{E})$ and $\mathcal{P}(\widetilde{E}^{\mathbb{N}})$. For instance, if $\xi \in E^{\mathbb{N}}$, i.e., $\xi = ((\hat{\omega}_i, \bar{\omega}_i))_{i\in\mathbb{N}}$, then $\hat{\pi}\xi = \hat{\omega} = (\hat{\omega}_i)_{i\in\mathbb{N}}$ and $\bar{\pi}\xi = \bar{\omega} = (\bar{\omega}_i)_{i\in\mathbb{N}}$.

As before, we will write $k$, $\hat{k}$ and $\bar{k}$ for a quantity $k$ associated with the copolymer with pinning model, the copolymer model, respectively, the pinning model. For instance, if $Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$, $\hat{Q} \in \mathcal{P}^{\mathrm{inv}}(\widetilde{\hat{E}}^{\mathbb{N}})$ and $\bar{Q} \in \mathcal{P}^{\mathrm{inv}}(\widetilde{\bar{E}}^{\mathbb{N}})$, then the rate functions $I^{\mathrm{ann}}(Q) = H(Q|q_{\rho,\hat{\mu}\otimes\bar{\mu}}^{\otimes\mathbb{N}})$, $\hat{I}^{\mathrm{ann}}(\hat{Q}) = H(\hat{Q}|q_{\rho,\hat{\mu}}^{\otimes\mathbb{N}})$, $\bar{I}^{\mathrm{ann}}(\bar{Q}) = H(\bar{Q}|q_{\rho,\bar{\mu}}^{\otimes\mathbb{N}})$ and the sets $\mathcal{R}$, $\hat{\mathcal{R}}$, $\bar{\mathcal{R}}$ are defined as in (2.13).

The LDPs of the laws of the empirical processes $R_N^{\hat{\omega}} = \hat{\pi} R_N^{\omega}$ and $R_N^{\bar{\omega}} = \bar{\pi} R_N^{\omega}$ can be derived from those of $R_N^{\omega}$ via the contraction principle (see e.g. Dembo and Zeutouni [6], Theorem 4.2.1), because the projection maps $\hat{\pi}$ and $\bar{\pi}$ are continuous. In particular, for any $\hat{Q} \in \mathcal{P}^{\mathrm{inv}}(\widetilde{\hat{E}}^{\mathbb{N}})$ and $\bar{Q} \in \mathcal{P}^{\mathrm{inv}}(\widetilde{\bar{E}}^{\mathbb{N}})$

$$\hat{I}^{\mathrm{que}}(\hat{Q}) = \inf_{\substack{Q\in\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}):\\ \hat{\pi}Q=\hat{Q}}} I^{\mathrm{que}}(Q), \qquad \bar{I}^{\mathrm{que}}(\bar{Q}) = \inf_{\substack{Q\in\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}}):\\ \bar{\pi}Q=\bar{Q}}} I^{\mathrm{que}}(Q), \tag{3.1}$$

where $\hat{\pi}Q = Q \circ (\hat{\pi})^{-1}$ and $\bar{\pi}Q = Q \circ (\bar{\pi})^{-1}$. Similarly, we may express $\hat{I}^{\mathrm{ann}}$ and $\bar{I}^{\mathrm{ann}}$ in terms of $I^{\mathrm{ann}}$.

### 3.1 Quenched excess free energy

Abbreviate

$$\mathcal{C}^{\rm fin} = \left\{ Q \in \mathcal{P}^{\rm inv}(\widetilde{E}^{\mathbb{N}})\colon\ I^{\rm ann}(Q) < \infty,\, m_Q < \infty \right\}. \tag{3.2}$$

**Theorem 3.1** *Assume* (1.2) *and* (1.3)*. Fix $\hat\beta$, $\hat h > 0$, $\bar\beta \geq 0$ and $\bar h \in \mathbb{R}$.*
*(i) The quenched excess free energy is given by*

$$g^{\rm que}(\hat\beta,\hat h,\bar\beta,\bar h) = \inf\left\{ g \in \mathbb{R}\colon\ S^{\rm que}(\hat\beta,\hat h,\bar\beta;g) - \bar h < 0 \right\}, \tag{3.3}$$

*where*

$$S^{\rm que}(\hat\beta,\hat h,\bar\beta;g) = \sup_{Q\in\mathcal{C}^{\rm fin}\cap\mathcal{R}} \left[ \bar\beta\Phi(Q) + \Phi_{\hat\beta,\hat h}(Q) - g m_Q - I^{\rm ann}(Q) \right] \tag{3.4}$$

*with*

$$\Phi(Q) = \int_{\bar E} \bar\omega_1\, (\bar\pi_{1,1}Q)(d\bar\omega_1) \tag{3.5}$$

$$\Phi_{\hat\beta,\hat h}(Q) = \int_{\widetilde{\hat E}} (\hat\pi_1 Q)(d\hat\omega)\, \log\phi_{\hat\beta,\hat h}(\hat\omega), \tag{3.6}$$

$$\phi_{\hat\beta,\hat h}(\hat\omega) = \tfrac12\left(1 + \exp\left[-2\hat\beta\hat h\,\tau_1 - 2\hat\beta\sum_{k=1}^{\tau_1}\hat\omega_k\right]\right). \tag{3.7}$$

*Here, the map $\bar\pi_{1,1}\colon\ \widetilde{E}^{\mathbb{N}} \to \bar E$ is the projection onto the first letter of the first word in the sentence consisting of words cut out from $\bar\omega$, i.e., $\bar\pi_{1,1}Q = Q\circ(\bar\pi_{1,1})^{-1}$, while the map $\hat\pi_1\colon\ \widetilde{E}^{\mathbb{N}} \to \widetilde{\hat E}$ is the projection onto the first word in the sentence consisting of words cut out from $\hat\omega$, i.e., $\hat\pi_1 Q = Q\circ(\hat\pi_1)^{-1}$, and $\tau_1$ is the length of the first word.*
*(ii) An alternative variational formula at $g=0$ is $S^{\rm que}(\hat\beta,\hat h,\bar\beta;0) = S^{\rm que}_*(\hat\beta,\hat h,\bar\beta)$ with*

$$S^{\rm que}_*(\hat\beta,\hat h,\bar\beta) = \sup_{Q\in\mathcal{C}^{\rm fin}} \left[ \bar\beta\Phi(Q) + \Phi_{\hat\beta,\hat h}(Q) - I^{\rm que}(Q) \right]. \tag{3.8}$$

*(iii) The map $g \mapsto S^{\rm que}(\hat\beta,\hat h,\bar\beta;g)$ is lower semi-continuous, convex and non-increasing on $\mathbb{R}$, is infinite on $(-\infty,0)$, and is finite, continuous and strictly decreasing on $(0,\infty)$.*

*Proof.* The proof is an adaptation of the proof of Theorem 3.1 in [3] and comes in 3 steps.

**1.** Suppose that $\pi \in \Pi_n$ has $t_n = t_n(\pi)$ excursions away from the interface. If $k_i$ denote the times at which $\pi$ visits the interface, then the Hamiltonian reads

$$\begin{aligned} H_n^{\hat\beta,\hat h,\bar\beta,\bar h,\omega}(\pi) &= \hat\beta\sum_{k=1}^n (\hat\omega_k + \hat h)\,[\mathrm{sign}(\pi_{k-1},\pi_k) - 1] + \sum_{k=1}^n (\bar\beta\bar\omega_k - \bar h)1_{\{\pi_k=0\}} \\ &= \sum_{i=1}^{t_n}\left[ \bar\beta\bar\omega_{k_i} - \bar h - 2\,\hat\beta\,1_{A_i^-}\sum_{k\in I_i}(\hat\omega_k + \hat h) \right], \end{aligned} \tag{3.9}$$

where $A_i^-$ is the event that the $i$-th excursion is below the interface and $I_i = (k_{i-1}, k_i] \cap \mathbb{N}$. Since each excursion has equal probability to lie below or above the interface, the $i$-th excursion contributes

$$\phi_{\hat\beta,\hat h}(\hat\omega_{I_i})\, e^{\bar\beta\bar\omega_{k_i} - \bar h} = \tfrac12\left(1 + \exp\left[-2\,\hat\beta\sum_{k\in I_i}(\hat\omega_k + \hat h)\right]\right)\, e^{\bar\beta\bar\omega_{k_i} - \bar h} \tag{3.10}$$

to the partition sum $Z_n^{\hat\beta,\hat h,\bar\beta,\bar h,\omega}$, where $\hat\omega_{I_i}$ is the word in $\widetilde{\widetilde{E}}$ cut out from $\hat\omega$ by the $i$-th excursion interval $I_i$. Consequently, we have

$$Z_n^{\hat\beta,\hat h,\bar\beta,\bar h,\omega} = \sum_{N\in\mathbb{N}} \sum_{0=k_0<k_1<\cdots<k_N=n} \prod_{i=1}^N \rho(k_i-k_{i-1})\, e^{(\bar\beta\bar\omega_{k_i}-\bar h)} e^{\log\phi_{\hat\beta,\hat h}(\hat\omega_{I_i})}. \tag{3.11}$$

Therefore, summing over $n$, we get

$$\sum_{n\in\mathbb{N}} Z_n^{\hat\beta,\hat h,\bar\beta,\bar h,\omega} e^{-gn} = \sum_{N\in\mathbb{N}} F_N^{\hat\beta,\hat h,\bar\beta,\bar h,\omega}(g), \qquad g\geq 0, \tag{3.12}$$

with

$$\begin{aligned} F_N^{\hat\beta,\hat h,\bar\beta,\bar h,\omega}(g) &= \Big(\mathcal{N}(g)\, e^{-\bar h}\Big)^N \sum_{0=k_0<k_1<\cdots<k_N<\infty} \left(\prod_{i=1}^N \rho_g(k_i-k_{i-1})\right) \\ &\qquad\times \exp\left[\sum_{i=1}^N \Big(\log\phi_{\hat\beta,\hat h}(\hat\omega_{I_i}) + \bar\beta\;\bar\omega_{k_i}\Big)\right] \\ &= \Big(\mathcal{N}(g)\, e^{-\bar h}\Big)^N\; E_g^*\Big(\exp\Big[N\Big(\Phi_{\hat\beta,\hat h}(R_N^\omega) + \bar\beta\,\Phi(R_N^\omega)\Big)\Big]\Big), \end{aligned} \tag{3.13}$$

where

$$R_N^\omega\big((k_i)_{i=0}^N\big) = \frac{1}{N}\sum_{i=1}^N \delta_{\widetilde\theta^i(\omega_{I_1},\ldots,\omega_{I_N})^{\rm per}} \tag{3.14}$$

denotes the *empirical process of $N$-tuples of words* cut out from $\omega$ by the $N$ successive excursions, and $\Phi_{\hat\beta,\hat h}, \Phi$ are defined in (3.5–3.7).

**2.** The left-hand side of (3.12) is a power series with radius of convergence $g^{\rm que}(\hat\beta,\hat h,\bar\beta,\bar h)$ (recall (1.7)). Define

$$s^{\rm que}(\hat\beta,\hat h,\bar\beta;g) = \log\mathcal{N}(g) + \limsup_{N\to\infty}\frac{1}{N}\log E_g^*\Big(\exp\Big[N\Big(\Phi_{\hat\beta,\hat h}(R_N^\omega) + \bar\beta\,\Phi(R_N^\omega)\Big)\Big]\Big) \tag{3.15}$$

and note that the limsup exists and is constant (possibly infinity) $\omega$-a.s. because it is measurable w.r.t. the tail sigma-algebra of $\omega$ (which is trivial). Note from (3.13) and (3.15) that

$$s^{\rm que}(\hat\beta,\hat h,\bar\beta;g) - \bar h = \limsup_{N\to\infty}\frac{1}{N}\log F_N^{\hat\beta,\hat h,\bar\beta,\bar h,\omega}(g). \tag{3.16}$$

By (1.7), the left-hand side of (3.12) is a power series that converges for $g > g^{\rm que}(\hat\beta,\hat h,\bar\beta,\bar h)$ and diverges for $g < g^{\rm que}(\hat\beta,\hat h,\bar\beta,\bar h)$. Hence we have

$$g^{\rm que}(\hat\beta,\hat h,\bar\beta,\bar h) = \inf\Big\{g\in\mathbb{R}\colon\; s^{\rm que}(\hat\beta,\hat h,\bar\beta;g) - \bar h < 0\Big\}. \tag{3.17}$$

**3.** We claim that, for any $\hat\beta,\hat h>0$ and $\bar\beta\geq 0$, the map $g\mapsto \bar S^{\rm que}(\hat\beta,\hat h,\bar\beta;g)$ is finite on $(0,\infty)$ and infinite on $(-\infty,0)$ (see Fig. 6), and

$$s^{\rm que}(\hat\beta,\hat h,\bar\beta;g) = S^{\rm que}(\hat\beta,\hat h,\bar\beta;g) \quad \forall\, g\in\mathbb{R}. \tag{3.18}$$

Note from the contraction principle in (3.1) that $\hat I^{\rm ann}(\hat\pi Q)$ and $\bar I^{\rm ann}(\bar\pi Q)$ are finite whenever $I^{\rm ann}(Q)<\infty$. Therefore, for any $\hat\beta>0$, $\bar\beta\geq 0$ and $\hat h>0$, it follows from Lemmas A.1 and

A.3 in Appendix A that $\bar\beta\Phi(Q)+\Phi_{\hat\beta,\hat h}(Q)<\infty$ whenever $I^{\mathrm{ann}}(Q)<\infty$. This implies that the map $g\mapsto S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g)$ is convex and lower-semicontinuous, since, by (3.4), $S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g)$ is the supremum of a family of functions that are finite and linear (and hence continuous) in $g$. This and the above claim prove part (iii) of the theorem (since convexity and finiteness imply continuity). The rest of the proof follows from the claim in (3.18), whose proof we defer to Appendix B. ∎

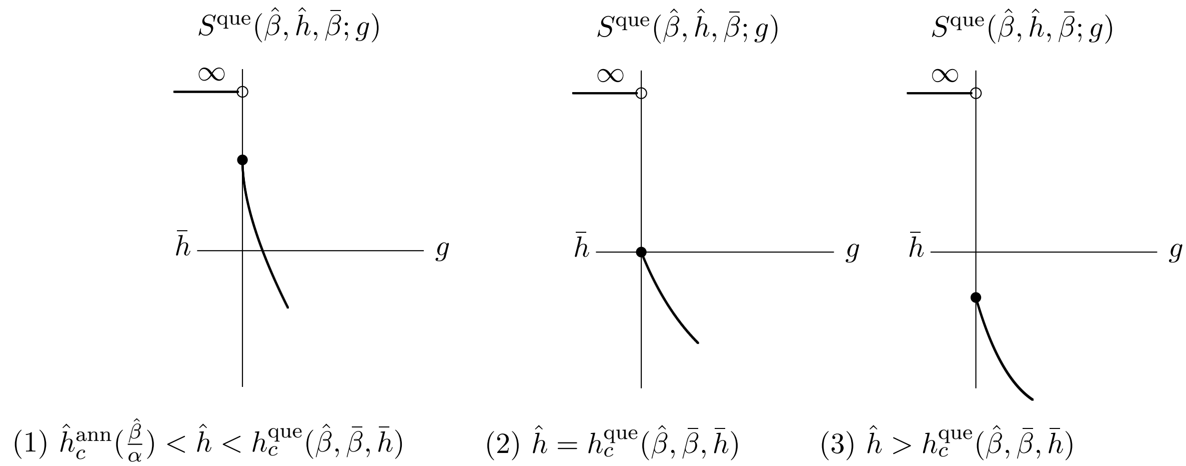



Figure 6: Qualitative picture of the map $g\mapsto S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g)$ for $\hat\beta,\hat h>0$ and $\bar\beta\geq 0$.

Analogues of Theorem 3.1 also hold for the copolymer model and the pinning model. The copolymer analogue is obtained by putting $\bar\beta=\bar h=0$, which leads to analogous variational formulas for $\hat S^{\mathrm{que}}(\hat\beta,\hat h;g)$ and $\hat g^{\mathrm{que}}(\hat\beta,\hat h)$. In the variational formula for $\hat S^{\mathrm{que}}(\hat\beta,\hat h;g)$ we replace $\mathcal{C}^{\mathrm{fin}}\cap\mathcal{R}$ by $\hat{\mathcal{C}}^{\mathrm{fin}}\cap\hat{\mathcal{R}}$ in (3.4). This replacement is a consequence of the contraction principle in (3.1). Although the contraction principle holds on $\mathcal{P}^{\mathrm{inv}}(\widetilde E^{\mathbb{N}})$, it turns out that the $Q\notin\mathcal{C}^{\mathrm{fin}}\cap\mathcal{R}$ play no role in (3.4). Similarly, Theorem 3.1 reduces to the pinning model upon putting $\hat\beta=\hat h=0$. The variational formula for $\bar S^{\mathrm{que}}(\bar\beta;g)$ is the same as that in (3.4), with $\mathcal{C}^{\mathrm{fin}}\cap\mathcal{R}$ replaced by $\bar{\mathcal{C}}^{\mathrm{fin}}\cap\bar{\mathcal{R}}$.

### 3.2 Annealed excess free energy

We next present the variational formula for the annealed excess free energy. This will serve as an *object of comparison* in our study of the quenched model. Define

$$\mathcal{N}(\hat\beta,\hat h,\bar\beta;g)=\tfrac12 e^{\bar M(-\bar\beta)}\left(\sum_{n\in\mathbb{N}}\rho(n)\,e^{-ng}+\sum_{n\in\mathbb{N}}\rho(n)\,e^{-n\left(g-[\hat M(2\hat\beta)-2\hat\beta\hat h]\right)}\right) \tag{3.19}$$

(recall (1.3)).

**Theorem 3.2** *Assume* (1.2) *and* (1.3). *Fix* $\hat\beta,\hat h,\bar\beta\geq 0$ *and* $\bar h\in\mathbb{R}$.
*(i) The annealed excess free energy is given by*

$$g^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta,\bar h)=\inf\left\{g\in\mathbb{R}\colon\ S^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta;g)-\bar h<0\right\}, \tag{3.20}$$

*where*

$$S^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta;g)=\sup_{Q\in\mathcal{C}^{\mathrm{fin}}}\left[\bar\beta\Phi(Q)+\Phi_{\hat\beta,\hat h}(Q)-gm_Q-I^{\mathrm{ann}}(Q)\right]. \tag{3.21}$$

*(ii) The map $g \mapsto S^{\text{ann}}(\hat\beta,\hat h,\bar\beta;g)$ is lower semi-continuous, convex and non-increasing on $\mathbb{R}$. Furthermore, it is infinite on $(-\infty,\hat g^{\text{ann}}(\hat\beta,\hat h))$, and finite, continuous and strictly decreasing on $[\hat g^{\text{ann}}(\hat\beta,\hat h),\infty)$ (recall (1.15)).*

*Proof.* The proof comes in 3 steps.

Replacing $Z_n^{\hat\beta,\hat h,\bar\beta,\bar h,\omega}$ by $Z_n^{\hat\beta,\hat h,\bar\beta,\bar h} = \mathbb{E}(Z_n^{\hat\beta,\hat h,\bar\beta,\bar h,\omega})$ in (3.12), we obtain from (3.13) that

$$F_N^{\hat\beta,\hat h,\bar\beta,\bar h}(g) = \mathbb{E}\left(F_N^{\hat\beta,\hat h,\bar\beta,\bar h,\omega}(g)\right) = \mathcal{N}(\hat\beta,\hat h,\bar\beta;g)^N e^{-\bar h N}. \tag{3.22}$$

It therefore follows from (3.16) and (3.22) that

$$s^{\text{ann}}(\hat\beta,\hat h,\bar\beta;g) - \bar h = \limsup_{N\to\infty}\frac{1}{N}\log F_N^{\hat\beta,\hat h,\bar\beta,\bar h}(g) = \log\mathcal{N}(\hat\beta,\hat h,\bar\beta;g) - \bar h, \tag{3.23}$$

where

$$s^{\text{ann}}(\hat\beta,\hat h,\bar\beta;g) = \limsup_{N\to\infty}\frac{1}{N}\log\left(e^{N\bar h}F_N^{\hat\beta,\hat h,\bar\beta,\bar h}(g)\right) = \log\mathcal{N}(\hat\beta,\hat h,\bar\beta;g). \tag{3.24}$$

Note from (3.19) and (3.24) that the map $g\mapsto s^{\text{ann}}(\hat\beta,\hat h,\bar\beta;g)$ is non-increasing. Moreover, for any $\hat\beta,\hat h,\bar\beta\geq 0$ and $\bar h\in\mathbb{R}$, we see from (3.12) after replacing $Z_n^{\hat\beta,\hat h,\bar\beta,\bar h,\omega}$ by $Z_n^{\hat\beta,\hat h,\bar\beta,\bar h}$ that $g^{\text{ann}}(\hat\beta,\hat h,\bar\beta,\bar h)$ is the smallest $g$-value at which $s^{\text{ann}}(\hat\beta,\hat h,\bar\beta;g)-\bar h$ changes sign, i.e.,

$$g^{\text{ann}}(\hat\beta,\hat h,\bar\beta,\bar h) = \inf\left\{g\in\mathbb{R}\colon s^{\text{ann}}(\hat\beta,\hat h,\bar\beta;g)-\bar h<0\right\}. \tag{3.25}$$

The proof of (i) and (ii) will follow once we show that

$$S^{\text{ann}}(\hat\beta,\hat h,\bar\beta;g) = s^{\text{ann}}(\hat\beta,\hat h,\bar\beta;g)\quad\forall\, g\in\mathbb{R}, \tag{3.26}$$

since (3.19), (3.24) and (3.26) show that the map $g\mapsto S^{\text{ann}}(\hat\beta,\hat h,\bar\beta;g)$ is infinite whenever $g<\hat g^{\,\text{ann}}(\hat\beta,\hat h) = 0\vee[M(2\hat\beta)-2\hat\beta\hat h]$, and is finite otherwise. Lower semi-continuity and convexity of this map follow from (3.21), because the function under the supremum is linear and finite in $g$, while convexity and finiteness imply continuity. The proof of (3.26) follows from the arguments in [3], Theorem 3.2, as we show in steps 2–3.

**2.** For the case $g<\hat g^{\text{ann}}(\hat\beta,\hat h)$, note from (3.19) that $\mathcal{N}(\hat\beta,\hat h,\bar\beta;g)=\infty$ for all $\hat\beta,\hat h,\bar\beta\geq 0$ and $\bar h\in\mathbb{R}$. To show that $S^{\text{ann}}(\hat\beta,\hat h,\bar\beta;g)=\infty$ for this case, we proceed as in steps (II) and (III) of the proof of [3], Theorem 3.2, by evaluating the functional under the supremum in (3.21) at $Q_{\hat\beta}^L = (q_{\hat\beta}^L)^{\otimes\mathbb{N}}$ with

$$q_{\hat\beta}^L(d(\hat\omega_1,\bar\omega_1),\dots,d(\hat\omega_n,\bar\omega_n)) = \delta_{Ln}\left[\hat\mu_{\hat\beta}(d\hat\omega_1)\times\cdots\times\hat\mu_{\hat\beta}(d\hat\omega_n)\right]\times\left[\bar\mu(d\bar\omega_1)\times\cdots\times\bar\mu(d\bar\omega_n)\right], \tag{3.27}$$

where $L,n\in\mathbb{N}$, $\hat\omega_1,\dots,\hat\omega_n\in\hat E$, $\bar\omega_1,\dots,\bar\omega_n\in\bar E$, and (recall (1.3))

$$\hat\mu_{\hat\beta}(d\hat\omega_1) = e^{-2\hat\beta\hat\omega_1-\hat M(2\hat\beta)}\hat\mu(d\hat\omega_1). \tag{3.28}$$

Note from (3.5) that $\Phi(Q_{\hat\beta}^L)=0$ because $\bar\mu$ has zero mean. This leads to a lower bound on $S^{\text{ann}}(\hat\beta,\hat h,\bar\beta;g)$ that tends to infinity as $L\to\infty$. To get the desired lower bound, we have to distinguish between the cases $\hat{g}^{\mathrm{ann}}(\hat\beta,\hat h)=0$ and $\hat{g}^{\mathrm{ann}}(\hat\beta,\hat h)>0$. For $\hat{g}^{\mathrm{ann}}(\hat\beta,\hat h)=0$ use $Q_0^L$, for $\hat{g}^{\mathrm{ann}}(\hat\beta,\hat h)>0$ with $\hat\beta>0$ use $Q^L_{\hat\beta}$.

**3.** For the case $g\geq \hat{g}^{\mathrm{ann}}(\hat\beta,\hat h)$, we proceed as in step 1 and 2 of the proof of Theorem 3.2 of [3]. Note that $\Phi_{\hat\beta,\hat h}(Q)$ and $\Phi(Q)$ defined in (3.5–3.7) are functionals of $\pi_1 Q$, where $\pi_1 Q$ is the first-word marginal of $Q$. Moreover, by (2.5),

$$\inf_{\substack{Q\in\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})\\ \pi_1 Q=q}} H(Q\mid q^{\otimes\mathbb{N}}_{\rho,\hat\mu\otimes\bar\mu}) = h(q\mid q_{\rho,\hat\mu\otimes\bar\mu}) \qquad \forall\, q\in\mathcal{P}(\widetilde{E}) \tag{3.29}$$

with the infimum *uniquely* attained at $Q=q^{\otimes\mathbb{N}}$, where the right-hand side denotes the relative entropy of $q$ w.r.t. $q_{\rho,\hat\mu\otimes\bar\mu}$. (The uniqueness of the minimum is easily deduced from the strict convexity of relative entropy on finite cylinders.) Consequently, the variational formula in (3.21) becomes

$$\begin{aligned}
S^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta;g) &= \sup_{\substack{q\in\mathcal{P}(\widetilde{E})\\ m_q<\infty,\, h(q|q_{\rho,\hat\mu\otimes\bar\mu})<\infty}} \left\{\int_{\widetilde{E}} q(d\omega)\,[\bar\beta\bar\omega_1+\log\phi_{\hat\beta,\hat h}(\hat\omega)]-gm_Q-h(q\mid q_{\rho,\hat\mu\otimes\bar\mu})\right\}\\
&= \sup_{\substack{q\in\mathcal{P}(\widetilde{E})\\ m_q<\infty,\, h(q|q_{\rho,\hat\mu\otimes\bar\mu})<\infty}} \left\{\int_{\widetilde{E}} q(d\omega)\,[\bar\beta\bar\omega_1+\log\phi_{\hat\beta,\hat h}(\hat\omega)-g\tau(\omega)]\right.\\
&\qquad\qquad \left. -\int_{\widetilde{E}} q(d\omega)\log\left(\frac{q(d\omega)}{q_{\rho,\hat\mu\otimes\bar\mu}(d\omega)}\right)\right\}\\
&= \bar M(-\bar\beta)+\log\hat{\mathcal{N}}(\hat\beta,\hat h;g)-\inf_{\substack{q\in\mathcal{P}(\widetilde{E})\\ m_q<\infty,\, h(q|q_{\rho,\hat\mu\otimes\bar\mu})<\infty}} h(q\mid q_{\hat\beta,\hat h,\bar\beta;g}),
\end{aligned} \tag{3.30}$$

where (by an abuse of notation) $\omega=((\hat\omega_i,\bar\omega_i))_{i=1}^{\tau(\omega)}$ is the disorder in the first word, $\phi_{\hat\beta,\hat h}(\hat\omega)$ is defined in (3.7), $m_q=\int_{\widetilde{E}} q(d\omega)\tau(\omega)$, $\tau(\omega)$ is the length of the word $\omega$, and

$$\begin{aligned}
q_{\hat\beta,\hat h,\bar\beta;g}(d(\hat\omega_1,\bar\omega_1),\cdots,d(\hat\omega_n,\bar\omega_n)) &= \frac{\rho(n)\phi_{\hat\beta,\hat h}(\hat\omega)e^{\bar\beta\bar\omega_1-ng}}{\hat{\mathcal{N}}(\hat\beta,\hat h;g)e^{\bar M(-\bar\beta)}}(\hat\mu\otimes\bar\mu)^n(d(\hat\omega_1,\bar\omega_1),\cdots,d(\hat\omega_n,\bar\omega_n)),\\
\hat{\mathcal{N}}(\hat\beta,\hat h;g) &= \tfrac12\left[\sum_{n\in\mathbb{N}}\rho(n)e^{-ng}+\sum_{n\in\mathbb{N}}\rho(n)e^{-n(g-[M(2\hat\beta)-2\hat\beta\hat h])}\right].
\end{aligned} \tag{3.31}$$

Note from (3.19) that $\mathcal{N}(\hat\beta,\hat h,\bar\beta;g)=\hat{\mathcal{N}}(\hat\beta,\hat h;g)e^{\bar M(-\bar\beta)}$. The infimum in the last equality of (3.30) is uniquely attained at $q=q_{\hat\beta,\hat h,\bar\beta;g}$. Therefore the variational problem in (3.21) for $g\geq\hat{g}^{\mathrm{ann}}(\hat\beta,\hat h)$ takes the form

$$\begin{aligned}
S^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta;g) &= \log\left(\tfrac12\left(\sum_{n\in\mathbb{N}}\rho(n)e^{-gn}+\sum_{n\in\mathbb{N}}\rho(n)e^{-n\left(g-[\hat M(2\hat\beta)-2\hat\beta\hat h]\right)}\right)e^{\bar M(-\bar\beta)}\right)\\
&= \log\mathcal{N}(\hat\beta,\hat h,\bar\beta;g)=s^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta;g).
\end{aligned} \tag{3.32}$$

The last formula proves (1.14). ∎

As in the quenched model, there are analogous versions of Theorem 3.2 for the annealed copolymer model and the annealed pinning model. These are obtained by putting either $\bar\beta=\bar h=0$ or $\hat\beta=\hat h=0$, replacing $\mathcal{C}^{\mathrm{fin}}$ by $\hat{\mathcal{C}}^{\mathrm{fin}}$ and $\bar{\mathcal{C}}^{\mathrm{fin}}$, respectively. The copolymer version of Theorem 3.2 was derived in [3], Theorem 3.2, and the pinning version (for $g=0$ only) in [5], Theorem 1.3.

Putting $\bar\beta = \bar h = 0$, we get the copolymer analogue of (3.32):

$$\hat S^{\rm ann}(\hat\beta,\hat h;g) = \log\left(\tfrac12\left[\sum_{n\in\mathbb{N}}\rho(n)\,e^{-ng} + \sum_{n\in\mathbb{N}}\rho(n)\,e^{-n(g-[\hat M(2\hat\beta)-2\hat\beta\hat h])}\right]\right). \tag{3.33}$$

This expression, which was obtained in [3], is plotted in Fig. 7. Putting $\hat\beta = \hat h = 0$, we get the pinning analogue:

$$\bar S^{\rm ann}(\bar\beta;g) = \bar M(-\bar\beta) + \log\left(\sum_{n\in\mathbb{N}}\rho(n)\,e^{-ng}\right). \tag{3.34}$$

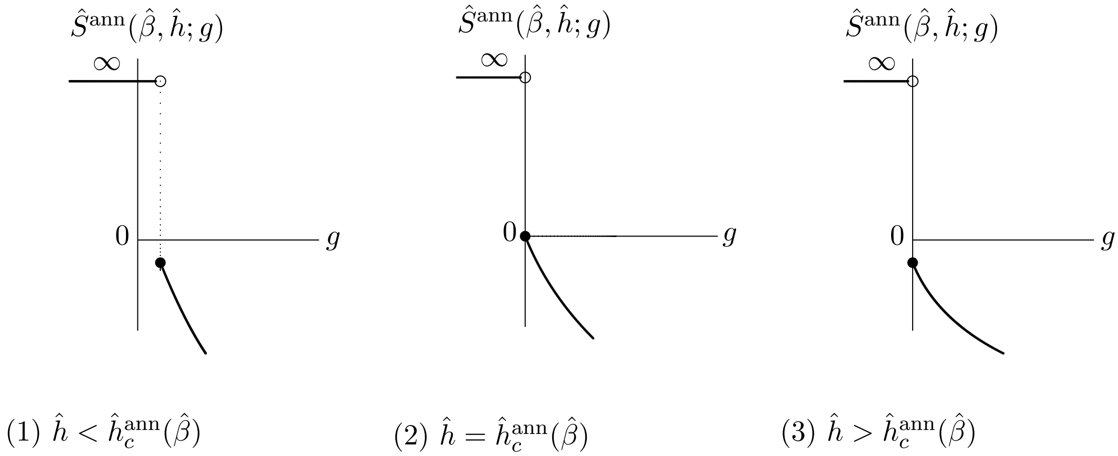



Figure 7: Qualitative picture of the map $g \mapsto \hat S^{\rm ann}(\hat\beta,\hat h;g)$ for $\hat\beta,\hat h \ge 0$. Compare with Fig. 6.

The map $g \mapsto S^{\rm ann}(\hat\beta,\hat h,\bar\beta;g)$ has the same qualitative picture as in Fig. 7, with the following changes: the horizontal axis is located at $\bar h$ instead of zero, and $\hat h_c^{\rm ann}(\hat\beta)$ is replaced by $h_c^{\rm ann}(\hat\beta,\bar\beta,\bar h)$.

Subtracting $\bar h$ from (3.33) and (3.34), we get from (3.20) that the excess free energies $\hat g^{\rm ann}(\hat\beta,\hat h)$ and $\bar g^{\rm ann}(\bar\beta,\bar h)$ take the form given in (1.15) and (1.16), respectively. The following lemma summarizes their relationship.

**Lemma 3.3** *For every $\bar\beta,\hat h,\hat\beta \ge 0$ and $\bar h \in \mathbb{R}$ (recall (1.20))*

$$g^{\rm ann}(\hat\beta,\hat h,\bar\beta,\bar h)\begin{cases} = \hat g^{\rm ann}(\hat\beta,\hat h), & \text{if} \quad \bar h \ge \bar h_*(\hat\beta,\hat h,\bar\beta),\\ > \hat g^{\rm ann}(\hat\beta,\hat h), & \text{if} \quad \bar h < \bar h_*(\hat\beta,\hat h,\bar\beta),\\ \le \bar g^{\rm ann}(\bar\beta,\bar h), & \text{if} \quad \hat h > \hat h_c^{\rm ann}(\hat\beta),\\ = \bar g^{\rm ann}(\bar\beta,\bar h), & \text{if} \quad \hat h = \hat h_c^{\rm ann}(\hat\beta),\\ \ge \bar g^{\rm ann}(\bar\beta,\bar h), & \text{if} \quad \hat h < \hat h_c^{\rm ann}(\hat\beta). \end{cases} \tag{3.35}$$

*Proof.* Note from (3.21) and (3.32–3.33) that $S^{\rm ann}(\hat\beta,\hat h,\bar\beta;g)-\bar h$ is $\hat S^{\rm ann}(\hat\beta,\hat h;g)$ shifted by $\bar M(-\bar\beta)-\bar h$. We see from Fig. 7 that if $\bar h \ge \bar h_*(\hat\beta,\hat h,\bar\beta)$, then the map $g \mapsto S^{\rm ann}(\hat\beta,\hat h,\bar\beta;g)-\bar h$ changes sign at the same value of $g$ as the map $g \mapsto \hat S^{\rm ann}(\hat\beta,\hat h;g)$ does. Hence $g^{\rm ann}(\hat\beta,\hat h,\bar\beta,\bar h) = \hat g^{\rm ann}(\hat\beta,\hat h)$ whenever $\bar h \ge \bar h_*(\hat\beta,\hat h,\bar\beta)$. On the other hand, if $\bar h < \bar h_*(\hat\beta,\hat h,\bar\beta)$, then the map $g \mapsto \hat S^{\rm ann}(\hat\beta,\hat h;g)$

changes sign before the map $g \mapsto S^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta;g)-\bar h$ does, i.e., $S^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta;\hat g^{\mathrm{ann}}(\hat\beta,\hat h))-\bar h>0$, and hence $g^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta,\bar h)>\hat g^{\mathrm{ann}}(\hat\beta,\hat h)$.

The rest of the proof follows from a comparison of (3.32) and (3.34). Note that, for $\hat h>\hat h_c^{\mathrm{ann}}(\hat\beta)$, we have $S^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta;g)-\bar h<\bar S^{\mathrm{ann}}(\bar\beta;g)-\bar h$, which implies that $g^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta,\bar h)\le\bar g^{\mathrm{ann}}(\bar\beta,\bar h)$. For $\hat h=\hat h_c^{\mathrm{ann}}(\hat\beta)$, we have $S^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta;g)-\bar h=\bar S^{\mathrm{ann}}(\bar\beta;g)-\bar h$, which implies that $g^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta,\bar h)=\bar g^{\mathrm{ann}}(\bar\beta,\bar h)$. Finally, for $\hat h<\hat h_c^{\mathrm{ann}}(\hat\beta)$ we have $S^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta;g)-\bar h>\bar S^{\mathrm{ann}}(\bar\beta;g)-\bar h$, which implies that $g^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta,\bar h)\ge\bar g^{\mathrm{ann}}(\bar\beta,\bar h)$. ∎

### 3.3 Proof of Theorem 1.1

*Proof.* Throughout the proof $\hat\beta>0$, $\bar\beta\ge0$ and $\bar h\in\mathbb{R}$ are fixed.
(i) Use Theorems 3.1(i,iii).
(ii) Recall from (1.10) and (3.3) that

$$\begin{aligned} h_c^{\mathrm{que}}(\hat\beta,\bar\beta,\bar h) &= \inf\left\{\hat h>0\colon\ g^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta,\bar h)=0\right\}\\ &= \inf\left\{\hat h>0\colon\ S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0)-\bar h\le 0\right\}. \end{aligned} \tag{3.36}$$

Indeed, it follows from (3.3) that $g^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta,\bar h)=0$ is equivalent to saying that the map $g\mapsto S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g)-\bar h$ changes sign at zero. This sign change can happen while $S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g)-\bar h$ is either zero or negative (see Fig. 6(2–3)). The corresponding expression for $h_c^{\mathrm{ann}}(\hat\beta,\bar\beta,\bar h)$ is obtained in a similar way. ∎

# 4 Key lemma and proof of Corollary 1.2

The following lemma will be used in the proof of Corollary 1.2.

**Lemma 4.1** *Fix $\alpha\ge1$, $\hat\beta,\hat h>0$ and $\bar\beta\ge0$. Then, for $g>0$,*

$$S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g)<S^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta;g)\begin{cases} \text{if } (\hat\beta,\hat h)\in\hat{\mathcal{D}}^{\mathrm{ann}},\\ \text{if } (\hat\beta,\hat h)\in\hat{\mathcal{L}}^{\mathrm{ann}} \text{ and } m_\rho<\infty,\\ \text{if } g\neq\hat g^{\mathrm{ann}}(\hat\beta,\hat h), (\hat\beta,\hat h)\in\hat{\mathcal{L}}^{\mathrm{ann}} \text{ and } m_\rho=\infty. \end{cases} \tag{4.1}$$

Lemma 4.1 is proved in Section 4.2. In Section 4.1 we use Lemma 4.1 to prove Corollary 1.2.

### 4.1 Proof of Corollary 1.2

*Proof.* (ii) Throughout the proof, $\alpha\ge1$, $\hat\beta,\hat h>0$ and $\bar\beta\ge0$. Note that, for $(\hat\beta,\hat h)\in\hat{\mathcal{L}}^{\mathrm{ann}}$, the map $g\mapsto S^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta;g)-\bar h$ changes sign at some $g\ge\hat g^{\mathrm{ann}}(\hat\beta,\hat h)>0$, i.e., $g^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta,\bar h)\ge\hat g^{\mathrm{ann}}(\hat\beta,\hat h)>0$ for all $\bar\beta\ge0$ and $\bar h\in\mathbb{R}$. Hence $\mathcal{L}_1^{\mathrm{ann}}\subset\mathcal{L}^{\mathrm{ann}}$.

Note from (3.32) and (3.33) that

$$S^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta;g)-\bar h=\hat S^{\mathrm{ann}}(\hat\beta,\hat h;g)+\bar M(-\bar\beta)-\bar h. \tag{4.2}$$

Furthermore, note from Fig. 7(2–3) that, for $(\hat\beta,\hat h)\in\hat{\mathcal{D}}^{\mathrm{ann}}$, the map $g\mapsto\hat S^{\mathrm{ann}}(\hat\beta,\hat h;g)$ changes sign at $g=0$ while $\hat S^{\mathrm{ann}}(\hat\beta,\hat h;0)$ is either negative or zero. In either case, we need

$$\bar h<\bar M(-\bar\beta)+\log\left(\tfrac12\left[1+\sum_{n\in\mathbb{N}}\rho(n)e^{n[\hat M(2\hat\beta)-2\hat\beta\hat h]}\right]\right)=\bar h_*(\hat\beta,\hat h,\bar\beta) \tag{4.3}$$

to ensure that the map $g \mapsto S^{\rm ann}(\hat\beta,\hat h,\bar\beta;g) - \bar h$ changes sign at a positive $g$-value. This concludes the proof that $\mathcal{L}^{\rm ann} = \mathcal{L}_1^{\rm ann} \cup \mathcal{L}_2^{\rm ann}$.

(i) As we saw in the proof of (ii), for the map $g \mapsto S^{\rm ann}(\hat\beta,\hat h,\bar\beta;g) - \bar h$ to reach zero we need that $\bar h \leq \bar h_*(\hat\beta,\hat h,\bar\beta)$. Thus, for this range of $\bar h$-values, we know that the map $g \mapsto S^{\rm ann}(\hat\beta,\hat h,\bar\beta;g) - \bar h$ changes sign when it is zero. The proof for $g^{\rm que}(\hat\beta,\hat h,\bar\beta,\bar h)$ follows from Fig. 6.

(iii) We first consider the cases: (a) $(\hat\beta,\hat h,\bar\beta,\bar h) \in \mathcal{L}_2^{\rm ann}$; (b) $(\hat\beta,\hat h,\bar\beta,\bar h) \in \mathcal{L}_1^{\rm ann}$ and $m_\rho < \infty$. In these cases we have that $\hat h < h_c^{\rm ann}(\hat\beta,\bar\beta,\bar h)$ by (ii). It follows from (3.32–3.33) and Fig. 7 that the map $g \mapsto S^{\rm ann}(\hat\beta,\hat h,\bar\beta;g) - \bar h$ changes sign at some $g > 0$ while it is either zero or negative. In either case the finiteness of the map $g \mapsto S^{\rm que}(\hat\beta,\hat h,\bar\beta;g) - \bar h$ on $(0,\infty)$ and (4.1) imply that $g \mapsto S^{\rm que}(\hat\beta,\hat h,\bar\beta;g) - \bar h$ changes sign at a smaller value of $g$ than $g \mapsto S^{\rm ann}(\hat\beta,\hat h,\bar\beta;g) - \bar h$ does. This concludes the proof for cases (a–b).

We next consider the case: (c) $(\hat\beta,\hat h,\bar\beta,\bar h) \in \mathcal{L}^{\rm ann}$ with $(\hat\beta,\hat h) \in \hat{\mathcal{L}}^{\rm ann}$, $\bar h \neq \bar h_*(\hat\beta,\hat h,\bar\beta)$ and $m_\rho = \infty$. We know from (4.1) that $S^{\rm que}(\hat\beta,\hat h,\bar\beta;g) < S^{\rm ann}(\hat\beta,\hat h,\bar\beta;g)$ for $g > 0$ and $g \neq \hat g^{\rm ann}(\hat\beta,\hat h)$. If $\bar h > \bar h_*(\hat\beta,\hat h,\bar\beta)$, then the map $g \mapsto S^{\rm ann}(\hat\beta,\hat h,\bar\beta;g) - \bar h$ changes sign at $\hat g^{\rm ann}(\hat\beta,\hat h)$ while jumping from $< 0$ to $\infty$. By the continuity of the map $g \mapsto S^{\rm que}(\hat\beta,\hat h,\bar\beta;g)$ on $(0,\infty)$, this implies that the map $g \mapsto S^{\rm que}(\hat\beta,\hat h,\bar\beta;g) - \bar h$ changes sign at a $g$-value smaller than $\hat g^{\rm ann}(\hat\beta,\hat h)$. Furthermore, if $\bar h < \bar h_*(\hat\beta,\hat h,\bar\beta)$, then the map $g \mapsto S^{\rm ann}(\hat\beta,\hat h,\bar\beta;g) - \bar h$ changes sign at a $g$-value larger than $\hat g^{\rm ann}(\hat\beta,\hat h)$, while it is zero. Since $S^{\rm que}(\hat\beta,\hat h,\bar\beta;g) < S^{\rm ann}(\hat\beta,\hat h,\bar\beta;g)$ for $g > \hat g^{\rm ann}(\hat\beta,\hat h)$, we have that $g^{\rm que}(\hat\beta,\hat h,\bar\beta,\bar h) < g^{\rm ann}(\hat\beta,\hat h,\bar\beta,\bar h)$.

(iv) The proof follows from Lemma 3.3. ∎

### 4.2 Proof of Lemma 4.1

*Proof.* The proof comes in five steps. Step 1 proves the strict inequality in (4.1), using a claim about the finiteness of $I^{\rm ann}$ at some specific $Q$ in combination with arguments from Birker [1]. Steps 2-5 are used to prove the claim about the finiteness of $I^{\rm ann}$. Note that for $0 < g < \hat g^{\rm ann}(\hat\beta,\hat h)$ the claim trivially follows from Theorems 3.1(iii) and 3.2(ii), since $S^{\rm que}(\hat\beta,\hat h,\bar\beta;g) < \infty$ and $S^{\rm ann}(\hat\beta,\hat h,\bar\beta;g) = \infty$ for this range of $g$-values. Thus, what remains to be considered is the case $g \geq \hat g^{\rm ann}(\hat\beta,\hat h)$.

**1.** For $g \geq \hat g^{\rm ann}(\hat\beta,\hat h)$, note from (3.31) and the remark below it that there is a unique maximizer $Q_{\hat\beta,\hat h,\bar\beta;g} = (q_{\hat\beta,\hat h,\bar\beta;g})^{\otimes\mathbb{N}}$ for the variational formula for $S^{\rm ann}(\hat\beta,\hat h,\bar\beta;g)$ in (3.21), where

$$\begin{aligned} &q_{\hat\beta,\hat h,\bar\beta;g}(d(\hat\omega_1,\bar\omega_1),\dots,d(\hat\omega_n,\bar\omega_n)) \\ &= \frac{\frac12\,\rho(n)e^{-gn}(1+e^{-2\hat\beta\left[n\hat h+\sum_{i=1}^n\hat\omega_i\right]})e^{\bar\beta\bar\omega_1}}{\mathcal{N}(\hat\beta,\hat h,\bar\beta;g)}(\hat\mu\otimes\bar\mu)^{\otimes n}(d(\hat\omega_1,\bar\omega_1),\dots,d(\hat\omega_n,\bar\omega_n)), \end{aligned} \tag{4.4}$$

where

$$\mathcal{N}(\hat\beta,\hat h,\bar\beta;g) = \tfrac12 e^{\bar M(-\bar\beta)}\left[\sum_{n\in\mathbb{N}}\rho(n)e^{-gn} + \sum_{n\in\mathbb{N}}\rho(n)e^{-n(g-[\hat M(2\hat\beta)-2\hat\beta\hat h])}\right] = e^{\bar M(-\bar\beta)}\hat{\mathcal{N}}(\hat\beta,\hat h;g). \tag{4.5}$$

Note further that $Q_{\hat\beta,\hat h,\bar\beta;g} \notin \mathcal{R}$. We claim that, for $g \geq \hat g^{\rm ann}(\hat\beta,\hat h)$ and under the conditions in (4.1),

$$H(Q_{\hat\beta,\hat h,\bar\beta;g} \mid q_{\rho,\hat\mu\otimes\bar\mu}^{\otimes\mathbb{N}}) = h(q_{\hat\beta,\hat h,\bar\beta;g} \mid q_{\rho,\hat\mu\otimes\bar\mu}) < \infty. \tag{4.6}$$

This will be proved in Step 2. Let $M<\infty$ be such that $h(q_{\hat\beta,\hat h,\bar\beta;g}\mid q_{\rho,\hat\mu\otimes\bar\mu})<M$. Then the set

$$\mathcal{A}_M=\Big\{Q\in\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})\colon\ H(Q\mid q^{\otimes\mathbb{N}}_{\rho,\hat\mu\otimes\bar\mu})\le M\Big\} \tag{4.7}$$

is compact in the weak topology, and contains $Q_{\hat\beta,\hat h,\bar\beta;g}$ in its interior. It follows from Birkner [1], Remark 8, that $\mathcal{A}_M\cap\mathcal{R}$ is a closed subset of $\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$. This in turn implies that there exists a $\delta>0$ such that $B_\delta(Q_{\hat\beta,\hat h,\bar\beta;g})$ (the $\delta$-ball around $Q_{\hat\beta,\hat h,\bar\beta;g}$) satisfies $B_\delta(Q_{\hat\beta,\hat h,\bar\beta;g})\cap\mathcal{A}_M\subset\mathcal{R}^c$. Let

$$\bar\delta=\sup\Big\{0\le\delta'\le\delta\colon\ B_{\delta'}(Q_{\hat\beta,\hat h,\bar\beta;g})\cap\mathcal{A}_M=B_{\delta'}(Q_{\hat\beta,\hat h,\bar\beta;g})\Big\}. \tag{4.8}$$

Then $\mathcal{R}\subset B_{\bar\delta}(Q_{\hat\beta,\hat h,\bar\beta;g})^c$. Therefore, for $g\ge\hat g^{\mathrm{ann}}(\hat\beta,\hat h)$ and under the conditions in (4.1), we get that

$$\begin{aligned}
S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g)&=\sup_{Q\in\mathcal{C}^{\mathrm{fin}}\cap\mathcal{R}}\Big[\bar\beta\Phi(Q)+\Phi_{\hat\beta,\hat h}(Q)-gm_Q-I^{\mathrm{ann}}(Q)\Big]\\
&\le\sup_{Q\in\mathcal{C}^{\mathrm{fin}}\cap B_{\bar\delta}(Q_{\hat\beta,\hat h,\bar\beta;g})^c}\Big[\bar\beta\Phi(Q)+\Phi_{\hat\beta,\hat h}(Q)-gm_Q-I^{\mathrm{ann}}(Q)\Big]\\
&<\sup_{Q\in\mathcal{C}^{\mathrm{fin}}}\Big[\bar\beta\Phi(Q)+\Phi_{\hat\beta,\hat h}(Q)-gm_Q-I^{\mathrm{ann}}(Q)\Big]\\
&=S^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta;g)=\log\mathcal{N}(\hat\beta,\hat h,\bar\beta;g).
\end{aligned} \tag{4.9}$$

The strict inequality follows because no maximizing sequence in $\mathcal{C}^{\mathrm{fin}}\cap B_{\bar\delta}(Q_{\hat\beta,\hat h,\bar\beta;g})^c$ can have $Q_{\hat\beta,\hat h,\bar\beta;g}$ as its limit ($Q_{\hat\beta,\hat h,\bar\beta;g}$ being the unique maximizer of the variational problem in the second inequality).

**2.** Let us now turn to the proof of the claim in (4.6). For $g\ge\hat g^{\mathrm{ann}}(\hat\beta,\hat h)$, it follows from (4.4) and (4.5) that

$$h(q_{\hat\beta,\hat h\bar\beta;g}\mid q_{\rho,\hat\mu\otimes\bar\mu})\le I+II, \tag{4.10}$$

where

$$\begin{aligned}
I&=\bar\beta\int_{\bar E}\bar\omega_1 e^{\bar\beta\bar\omega_1-\bar M(-\bar\beta)}\bar\mu(d\bar\omega_1)-\log\mathcal{N}(\hat\beta,\hat h,\bar\beta;g),\\
II&=\frac{1}{\hat{\mathcal{N}}(\hat\beta,\hat h;g)}\sum_{n\in\mathbb{N}}\rho(n)e^{-ng}A(n),\\
A(n)&=\tfrac12\int_{\hat E^n}\Big[1+e^{-2\hat\beta\sum_{i=1}^n(\hat\omega_i+\hat h)}\Big]\log\Big(\tfrac12\Big[1+e^{-2\hat\beta\sum_{i=1}^n(\hat\omega_i+\hat h)}\Big]\Big)\hat\mu^{\otimes n}(d\hat\omega).
\end{aligned} \tag{4.11}$$

The inequality in (4.10) follows from (4.4) after replacing $e^{-gn}$ by 1. It is easy to see that $I<\infty$, because for $g\ge\hat g^{\mathrm{ann}}(\hat\beta,\hat h)$ we have that $\mathcal{N}(\hat\beta,\hat h,\bar\beta;g)<\infty$. Furthermore, since $\bar\mu$ has a finite moment generating function, it follows from the Hölder inequality that $\int_{\mathbb{R}}\bar\omega_1 e^{\bar\beta\bar\omega_1-\bar M(-\bar\beta)}\bar\mu(d\bar\omega_1)<\infty$. We proceed to show that $II<\infty$.

**3.** We first estimate $A(n)$. Note that

$$\begin{aligned}
A(n)&=\tfrac12\int_{\hat E^n}\Big[1+e^{-2\hat\beta\sum_{i=1}^n(\hat\omega_i+\hat h)}\Big]\log\Big(\tfrac12\Big[1+e^{-2\hat\beta\sum_{i=1}^n(\hat\omega_i+\hat h)}\Big]\Big)\hat\mu^{\otimes n}(d\hat\omega)\\
&=\tfrac12\int_{\hat E^n}\Big[1+e^{-2\hat\beta\sum_{i=1}^n(\hat\omega_i+\hat h)}\Big]\log\left(\frac{e^{-2\hat\beta\sum_{i=1}^n(\hat\omega_i+\hat h)}}{2}\Big[1+e^{2\hat\beta\sum_{i=1}^n(\hat\omega_i+\hat h)}\Big]\right)\hat\mu^{\otimes n}(d\hat\omega)\\
&=A_1(n)+A_2(n),
\end{aligned} \tag{4.12}$$

where

$$
\begin{aligned}
A_1(n) &= -\hat\beta \sum_{k=1}^{n} \int_{\hat E^n} (\hat\omega_k + \hat h)\left[1 + e^{-2\hat\beta \sum_{i=1}^n (\hat\omega_i + \hat h)}\right] \hat\mu^{\otimes n}(d\hat\omega),\\
A_2(n) &= \tfrac12 \int_{\hat E^n} \left[1 + e^{-2\hat\beta \sum_{i=1}^n (\hat\omega_i + \hat h)}\right] \log\left(\tfrac12 \left[1 + e^{2\hat\beta \sum_{i=1}^n (\hat\omega_i + \hat h)}\right]\right) \hat\mu^{\otimes n}(d\hat\omega).
\end{aligned}
\tag{4.13}
$$

The finiteness of $II$ will follow once we show that

$$
\sum_{n\in\mathbb{N}} \rho(n) e^{-gn}[A_1(n) + A_2(n)] < \infty. \tag{4.14}
$$

**4.** We start with the estimation of $A_2(n)$. Put $u_n(\hat\omega) = -2\hat\beta \sum_{i=1}^n (\hat\omega_i + h)$ and, for $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$, define

$$
B_{m,n} = \left\{\hat\omega \in \hat E^n \colon -(m+1) < u_n(\hat\omega) \le -m\right\}, \qquad m_n = m_n(\hat\beta, \hat h) = \lceil 4\hat\beta\hat h n\rceil. \tag{4.15}
$$

Then note that

$$
\begin{aligned}
A_2(n) &= \tfrac12 \int_{\hat E^n} \left[1 + e^{u_n(\hat\omega)}\right] \log\left(\tfrac12\left[1 + e^{-u_n(\hat\omega)}\right]\right) \hat\mu^{\otimes n}(d\hat\omega)\\
&\le \int_{\hat E^n} \left[1 \vee e^{u_n(\hat\omega)}\right] \log\left(1 \vee e^{-u_n(\hat\omega)}\right) \hat\mu^{\otimes n}(d\hat\omega)\\
&= -\int_{u_n \le 0} u_n(\hat\omega) \hat\mu^{\otimes n}(d\hat\omega)\\
&= -\sum_{m\in\mathbb{N}_0} \int_{B_{m,n}} u_n(\hat\omega) \hat\mu^{\otimes n}(d\hat\omega)\\
&\le \sum_{m=0}^{m_n} (m+1) + \sum_{m>m_n} (m+1)\mathbb{P}_{\hat\omega}(B_{m,n})\\
&\le 4m_n^2 + \sum_{v\in\mathbb{N}} (m_n + v + 1)\mathbb{P}_{\hat\omega}(B_{m_n+v,n}).
\end{aligned}
\tag{4.16}
$$

The second inequality uses that $-(m+1) < u_n \le -m$ on $B_{m,n}$ and $\mathbb{P}_{\hat\omega}(B_{m_n+v,n}) \le 1$. Estimate

$$
\begin{aligned}
\mathbb{P}_{\hat\omega}(B_{m_n+v,n}) &= \mathbb{P}_{\hat\omega}\left(\frac{m_n+v}{2\hat\beta} \le \sum_{k=1}^n (\hat\omega_k + \hat h) < \frac{m_n+v+1}{2\hat\beta}\right)\\
&\le \mathbb{P}_{\hat\omega}\left(\sum_{k=1}^n \hat\omega_k \ge \frac{m_n+v}{2\hat\beta} - n\hat h\right)\\
&\le \mathbb{P}_{\hat\omega}\left(\sum_{k=1}^n \hat\omega_k \ge \frac{4\hat\beta n\hat h + v}{2\hat\beta} - n\hat h\right)\\
&= \mathbb{P}_{\hat\omega}\left(\sum_{k=1}^n \hat\omega_k \ge \frac{v}{2\hat\beta} + n\hat h\right)\\
&\le e^{-C\left(\frac{v}{2\hat\beta} + n\right)}.
\end{aligned}
\tag{4.17}
$$

The last inequality uses [3], Lemma D.1, where $C$ is a positive constant depending on $\hat{h}$ only. Inserting (4.17) into (4.16), we get

$$A_2(n) \leq 4m_n^2 + (m_n+1)e^{-Cn}\frac{e^{-1/2\hat{\beta}}}{1-e^{-1/2\hat{\beta}}} + e^{-Cn}\sum_{v\in\mathbb{N}} v\, e^{-v/2\hat{\beta}}. \tag{4.18}$$

Furthermore, using that $g>0$, we get

$$\begin{aligned}\sum_{n\in\mathbb{N}}\rho(n)e^{-ng}A_2(n) \leq 4\sum_{n\in\mathbb{N}}\rho(n)e^{-ng}m_n^2 + \frac{e^{-1/2\hat{\beta}}}{1-e^{-1/2\hat{\beta}}}\sum_{n\in\mathbb{N}}\rho(n)(m_n+1)e^{-n[g+C]}\\ + \sum_{n\in\mathbb{N}}\rho(n)e^{-n[g+C]}\sum_{v\in\mathbb{N}} v\, e^{-v/2\hat{\beta}} < \infty.\end{aligned} \tag{4.19}$$

**5.** We proceed with the estimation of $A_1(n)$:

$$\begin{aligned} A_1(n) &= -\hat{\beta}\sum_{k=1}^{n}\int_{\hat{E}^n}(\hat{\omega}_k+\hat{h})\left[1+e^{-2\hat{\beta}\sum_{i=1}^n(\hat{\omega}_i+\hat{h})}\right]\hat{\mu}^{\otimes n}(d\hat{\omega})\\ &\leq -\hat{\beta}\sum_{k=1}^{n}\int_{\hat{E}^n}\hat{\omega}_k\left[1+e^{-2\hat{\beta}\sum_{i=1}^n(\hat{\omega}_i+\hat{h})}\right]\hat{\mu}^{\otimes n}(d\hat{\omega})\\ &= -n\hat{\beta}\; e^{n[\hat{M}(2\hat{\beta})-2\hat{\beta}\hat{h}]}\;\mathbb{E}_{\hat{\mu}_{\hat{\beta}}}(\hat{\omega}_1),\end{aligned} \tag{4.20}$$

where $\hat{\mu}_{\hat{\beta}}(d\hat{\omega}_1) = e^{-2\hat{\beta}\hat{\omega}_1 - M(2\hat{\beta})}\hat{\mu}(d\hat{\omega}_1)$. The right-hand side is non-negative because $\mathbb{E}_{\hat{\mu}_{\hat{\beta}}}(\hat{\omega}_1)\leq 0$, and so

$$\sum_{n\in\mathbb{N}}\rho(n)e^{-ng}A_1(n) \leq -\hat{\beta}\mathbb{E}_{\hat{\mu}_{\hat{\beta}}}(\hat{\omega}_1)\sum_{n\in\mathbb{N}} n\rho(n)\, e^{-n(g-[M(2\hat{\beta})-2\hat{\beta}\hat{h}])}. \tag{4.21}$$

This bound is finite if

1. $g > \hat{g}^{\mathrm{ann}}(\hat{\beta},\hat{h}) = \hat{M}(2\hat{\beta}) - 2\hat{\beta}\hat{h}$;
2. $g = \hat{g}^{\mathrm{ann}}(\hat{\beta},\hat{h})$ and $m_\rho<\infty$.

This concludes the proof since, if $(\hat{\beta},\hat{h})\in\hat{\mathcal{D}}^{\mathrm{ann}}$, then $\hat{g}^{\mathrm{ann}}(\hat{\beta},\hat{h})=0$ and we only want the finiteness for $g>0$. ∎

For the pinning model, the associated unique maximizer $\bar{Q}_{\bar{\beta};g}$ for the variational formula for $\bar{S}^{\mathrm{ann}}(\bar{\beta};g)$ satisfies $H(\bar{Q}_{\bar{\beta};g}|q_{\rho,\bar{\mu}}^{\otimes\mathbb{N}})<\infty$ for $g\geq 0$. However, this does not imply separation between $\bar{S}^{\mathrm{que}}(\bar{\beta};0)$ and $\bar{S}^{\mathrm{ann}}(\bar{\beta};0)$, since we may have $\bar{Q}_{\bar{\beta};0}\in\bar{\mathcal{R}}$ for $m_\rho=\infty$. The separation occurs at $g=0$ as soon as $m_\rho<\infty$, since this will imply that $\bar{Q}_{\bar{\beta};0}\notin\bar{\mathcal{R}}$.

# 5 Proof of Corollary 1.3

To prove Corollary 1.3 we need some further preparation, formulated as Lemmas 5.1–5.3 below. These lemmas, together with the proof of Corollary 1.3, are given in Section 5.1. Section 5.2 contains the proof of the first two lemmas, and Appendix C the proof of the third lemma.

### 5.1 Key lemmas and proof of Corollary 1.3

**Lemma 5.1** *For $\hat\beta, \bar\beta \geq 0$,*

$$S^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta;0)=\begin{cases}\bar M(-\bar\beta), & \text{if} \quad \hat h=\hat h_c^{\mathrm{ann}}(\hat\beta),\\ \infty, & \text{if} \quad \hat h<\hat h_c^{\mathrm{ann}}(\hat\beta),\\ \bar M(-\bar\beta)-\log 2, & \text{if} \quad \hat h=\infty.\end{cases} \tag{5.1}$$

*Furthermore, the map $\hat h \mapsto S^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta;0)$ is strictly convex and strictly decreasing on $[\hat h_c^{\mathrm{ann}}(\hat\beta),\infty)$.*

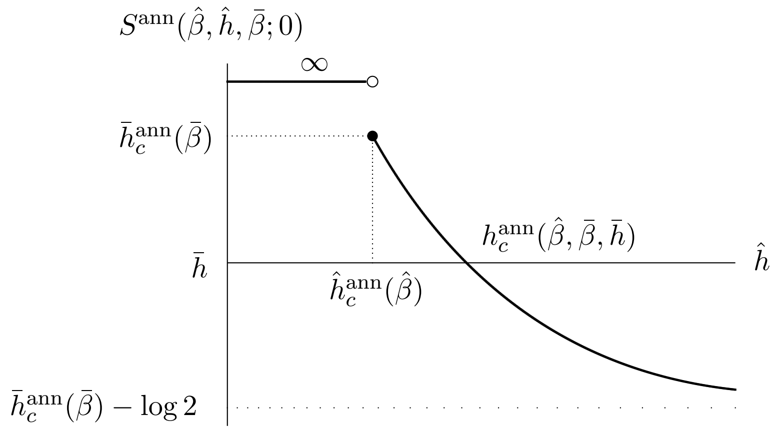



Figure 8: Qualitative picture of $\hat h \mapsto S^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta;0)$ for $\hat\beta>0$ and $\bar\beta\geq 0$.

**Lemma 5.2** *For every $\hat\beta>0$ and $\bar\beta\geq 0$ (see Fig. 8),*

$$S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0)=S_*^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta)\begin{cases}=\infty, & \text{for} \quad \hat h<\hat h_c^{\mathrm{ann}}(\hat\beta/\alpha)\\ >0, & \text{for} \quad \hat h=\hat h_c^{\mathrm{ann}}(\hat\beta/\alpha)\\ <\infty, & \text{for} \quad \hat h>\hat h_c^{\mathrm{ann}}(\hat\beta/\alpha).\end{cases} \tag{5.2}$$

**Lemma 5.3** *For every $\hat\beta>0$ and $\bar\beta\geq 0$ (see Fig. 9),*

$$S^{\mathrm{que}}(\hat\beta,\infty-,\bar\beta;0)=\lim_{\hat h\to\infty}S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0)=S^{\mathrm{que}}(\hat\beta,\infty,\bar\beta;0)=\bar h_c^{\mathrm{que}}(\bar\beta)-\log 2. \tag{5.3}$$

We now give the proof of Corollary 1.3.

*Proof.* Throughout the proof $\hat\beta>0$, $\bar\beta\geq 0$ and $\bar h\in\mathbb{R}$ are fixed. Note from (3.7) that the map $\hat h\mapsto\log\phi_{\hat\beta,\hat h}(\hat\omega)$ is strictly decreasing and convex for all $\hat\omega\in\widetilde{\hat E}$. It therefore follows from (3.4) and (3.21) that the maps $\hat h\mapsto S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0)$ and $\hat h\mapsto S^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta;0)$ are strictly decreasing when finite (because $\tau(\omega)\geq 1$) and convex (because sums and suprema of convex functions are convex).

Recall from (1.13) and (3.3) that

$$\begin{aligned} h_c^{\mathrm{ann}}(\hat\beta,\bar\beta,\bar h) &= \inf\left\{\hat h\geq 0\colon\ g^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta,\bar h)=0\right\}\\ &= \inf\left\{\hat h\geq 0\colon\ S^{\mathrm{ann}}(\hat\beta,\hat h,\bar\beta;0)-\bar h\leq 0\right\}. \end{aligned} \tag{5.4}$$

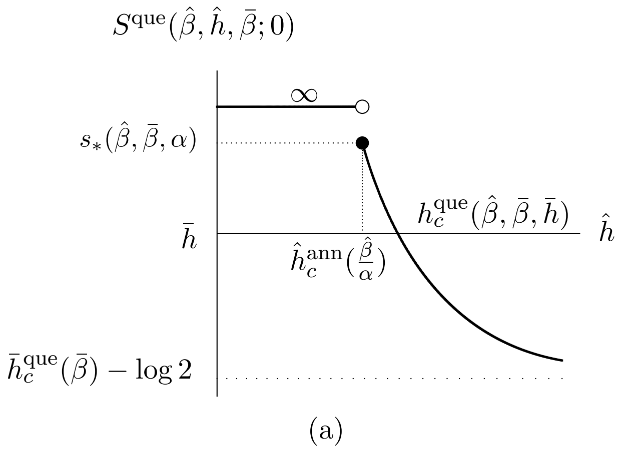



Figure 9: Qualitative picture of $\hat h \mapsto S^{\rm que}(\hat\beta,\bar\beta,\hat h;0)$ for $\hat\beta>0$ and $\bar\beta\geq 0$: (a) $s_*(\hat\beta,\bar\beta,\alpha)<\infty$; (b) $s_*(\hat\beta,\bar\beta,\alpha)=\infty$.

Indeed, it follows from (3.3) that $g^{\rm ann}(\hat\beta,\hat h,\bar\beta,\bar h)=0$ is equivalent to saying that the map $g\mapsto S^{\rm ann}(\hat\beta,\hat h,\bar\beta;g)-\bar h$ changes sign at zero. This change of sign can happen while $S^{\rm ann}(\hat\beta,\hat h,\bar\beta;g)-\bar h$ is either zero or negative (see e.g. Fig. 6(2–3)).

For $\bar h\geq \bar h_c^{\rm ann}(\bar\beta)$, it follows from Lemma 5.1 and Fig. 8 that $\hat h=\hat h_c^{\rm ann}(\hat\beta)$ is the smallest value of $\hat h$ at which $S^{\rm ann}(\hat\beta,\hat h,\bar\beta;0)-\bar h\leq 0$ and hence $h_c^{\rm ann}(\hat\beta,\bar\beta,\bar h)=\hat h_c^{\rm ann}(\hat\beta)$. Furthermore, note from Fig. 8 that the map $\hat h\mapsto S^{\rm ann}(\hat\beta,\hat h,\bar\beta;0)$ is strictly decreasing and convex on $[\hat h_c^{\rm ann}(\hat\beta),\infty)$ and has the interval $\big(\bar h_c^{\rm ann}(\bar\beta)-\log 2,\bar h_c^{\rm ann}(\bar\beta)\big]$ as its range. In particular, $S^{\rm ann}(\hat\beta,\hat h_c^{\rm ann}(\hat\beta),\bar\beta;0)=\bar h_c^{\rm ann}(\bar\beta)$ and $S^{\rm ann}(\hat\beta,\infty,\bar\beta;0)=\bar h_c^{\rm ann}(\bar\beta)-\log 2$. Therefore, for $\bar h\in\big(\bar h_c^{\rm ann}(\bar\beta)-\log 2,\bar h_c^{\rm ann}(\bar\beta)\big]$, the map $\hat h\mapsto S^{\rm ann}(\hat\beta,\hat h,\bar\beta;0)-\bar h$ changes sign at the unique value of $\hat h$ at which $S^{\rm ann}(\hat\beta,\hat h,\bar\beta;0)=\bar h$.

For $\bar h\leq \bar h_c^{\rm ann}(\bar\beta)-\log 2$, it follows from Fig. 8 that $S^{\rm ann}(\hat\beta,\hat h,\bar\beta;0)-\bar h>0$ for all $\hat h\in[0,\infty)$. It therefore follows from (5.4) that

$$h_c^{\rm ann}(\hat\beta,\bar\beta,\bar h)=\inf\Big\{\hat h\geq 0\colon\ S^{\rm ann}(\hat\beta,\hat h,\bar\beta;0)-\bar h\leq 0\Big\}=\infty. \tag{5.5}$$

The proof for $h_c^{\rm que}(\hat\beta,\bar\beta,\bar h)$ follows from that of $h_c^{\rm ann}(\hat\beta,\bar\beta,\bar h)$ after replacing $S^{\rm ann}(\hat\beta,\hat h,\bar\beta;0)$, $\bar h_c^{\rm ann}(\bar\beta)-\log 2$ and $\bar h_c^{\rm ann}(\bar\beta)$ by $S^{\rm que}(\hat\beta,\hat h,\bar\beta;0)$, $\bar h_c^{\rm que}(\bar\beta)-\log 2$ and $s^*(\hat\beta,\bar\beta,\alpha)$, respectively. ∎

### 5.2 Proof of Lemmas 5.1–5.2

**Proof of Lemma 5.1:**
*Proof.* Note from (3.32) that

$$S^{\rm ann}(\hat\beta,\hat h,\bar\beta;0)=\bar M(-\bar\beta)+\log\left(\tfrac12\left[1+\sum_{n\in\mathbb N}\rho(n)e^{n[\hat M(2\hat\beta)-2\hat\beta\hat h]}\right]\right), \tag{5.6}$$

which implies the claim. ∎

**Proof of Lemma 5.2:**
*Proof.* Throughout the proof $\hat\beta,\hat h>0$ and $\bar\beta\geq 0$ are fixed. The proof uses arguments from [3], Theorem 3.3 and Section 6. Note from (3.16), (3.18) and Lemma B.1 that

$$S^{\rm que}(\hat\beta,\hat h,\bar\beta;g)=\bar h+\limsup_{N\to\infty}\frac1N\log F_N^{\hat\beta,\hat h,\bar\beta,\bar h,\omega}(g)=\log\mathcal N(g)+\limsup_{N\to\infty}\frac1N\log S_N^{\omega}(g), \tag{5.7}$$

where

$$S_N^\omega(g) = E_g^*\left(\exp\left[N\left(\Phi_{\hat\beta,\hat h}(R_N^\omega) + \bar\beta\,\Phi(R_N^\omega)\right)\right]\right). \tag{5.8}$$

It follows from the fractional-moment argument in [3], Eq. (6.4), that

$$\mathbb{E}([S_N^\omega(g)]^t) \le \left(2^{1-t}\sum_{n\in\mathbb{N}} \rho_g(n)^t e^{\bar M(-\bar\beta t)}\right)^N < \infty, \quad g \ge 0, \tag{5.9}$$

where $t \in [0,1]$ is chosen such that $\hat M(2\hat\beta t) - 2\hat\beta\hat h t \le 0$. Abbreviate the term inside the brackets of (5.9) by $K_t$ and note that

$$\begin{aligned}\mathbb{P}\left(\frac{t}{N}\log S_N^\omega(g) \ge \log K_t + \epsilon\right) &= \mathbb{P}\left([S_N^\omega(g)]^t \ge K_t^N e^{N\epsilon}\right)\\ &\le \mathbb{E}([S_N^\omega(g)]^t) K_t^{-N} e^{-N\epsilon}\\ &\le e^{-N\epsilon}, \qquad \epsilon > 0.\end{aligned} \tag{5.10}$$

Therefore, for $g > 0$, this estimate together with the Borel-Cantelli lemma shows that $\omega$-a.s. (recall (2.6))

$$\begin{aligned} S^{\rm que}(\hat\beta,\hat h,\bar\beta; g) &= \log\mathcal{N}(g) + \limsup_{N\to\infty}\frac{1}{N}\log S_N^\omega(g) \le \frac{1}{t}\log K_t + \log\mathcal{N}(g)\\ &= \frac{1-t}{t}\log 2 + \frac{1}{t}\log\left(\sum_{n\in\mathbb{N}}\rho_g(n)^t\right) + \frac{1}{t}\bar M(-\bar\beta t) + \log\mathcal{N}(g) < \infty.\end{aligned} \tag{5.11}$$

This estimate also holds for $g = 0$ when $\sum_{n\in\mathbb{N}}\rho(n)^t < \infty$. This is the case for any pair $t \in (1/\alpha, 1]$ and $\hat h > \hat h_c^{\rm ann}(\hat\beta/\alpha)$ satisfying $\hat M(2\hat\beta t) - 2\hat\beta\hat h t \le 0$ (recall (1.2)). Therefore we conclude that $S^{\rm que}(\hat\beta,\hat h,\bar\beta; 0) < \infty$ whenever $\hat h > \hat h_c^{\rm ann}(\hat\beta/\alpha)$.

To prove that $S^{\rm que}(\hat\beta,\hat h,\bar\beta; 0) = \infty$ for $\hat h < \hat h_c^{\rm ann}(\hat\beta/\alpha)$, we replace $q_{\hat\beta}^L$ in [3], Eq. (6.8), by

$$q_{\hat\beta}^L(d(\hat\omega_1,\bar\omega_1),\dots,d(\hat\omega_n,\bar\omega_n)) = \delta_{n,L}\left[\hat\mu_{\hat\beta/\alpha}(\hat\omega_1)\times\cdots\times\hat\mu_{\hat\beta/\alpha}(d\hat\omega_n)\right]\times\left[\bar\mu(d\bar\omega_1)\times\cdots\times\bar\mu(d\bar\omega_n)\right], \tag{5.12}$$

where

$$\hat\mu_{\hat\beta/\alpha}(d\hat\omega_1) = e^{-(2\hat\beta/\alpha)\hat\omega_1 - \hat M(2\hat\beta/\alpha)}\hat\mu(d\hat\omega_1). \tag{5.13}$$

With this choice the rest of the argument in [3], Section 6.2, goes through easily.

Finally, to prove that $S^{\rm que}(\hat\beta,\hat h,\bar\beta; 0) > 0$ at $\hat h = \hat h_c^{\rm ann}(\hat\beta/\alpha)$ we proceed as follows. Adding, respectively, $\bar\beta\Phi(Q)$ and $\bar\beta\sum_{n\in\mathbb{N}}\int_{\bar E^n}\bar\omega_1 q(d(\hat\omega_1,\bar\omega_1),\dots,d(\hat\omega_n,\bar\omega_n))$ to the functionals being optimized in [3], Eqs. (6.19–6.20), we get the following analogue of [3], Eq. (6.21),

$$\begin{aligned} &q_{\hat\beta,\hat h,\bar\beta}(d(\hat\omega_1,\bar\omega_1),\dots,d(\hat\omega_n,\bar\omega_n))\\ &= \frac{1}{\hat{\mathcal{N}}(\hat\beta,\hat h)\,e^{\bar M(-\bar\beta/\alpha)}}\left[\phi_{\hat\beta,\hat h}((\hat\omega_1,\dots\hat\omega_n))\,e^{\bar\beta\bar\omega_1}\right]^{1/\alpha} q_{\rho,\hat\mu\otimes\bar\mu}(d(\hat\omega_1,\bar\omega_1),\dots,d(\hat\omega_n,\bar\omega_n)),\end{aligned} \tag{5.14}$$

where

$$\hat{\mathcal{N}}(\hat\beta,\hat h) = \sum_{n\in\mathbb{N}}\rho(n)\int_{\hat E^n}\hat\mu(d\hat\omega_1)\times\cdots\times\hat\mu(\hat\omega_n)\left\{\tfrac12\left(1+e^{-2\hat\beta\sum_{k=1}^n(\hat\omega_k+\hat h)}\right)\right\}^{1/\alpha}. \tag{5.15}$$

Note from [3], Eqs. (6.23–6.29), that $1 < \hat{\mathcal{N}}(\hat\beta, \hat h_c^{\rm ann}(\hat\beta/\alpha)) \leq 2^{1-(1/\alpha)}$. Therefore it follows from [3], Steps 1 and 2 in Section 6.3, that

$$S^{\rm que}(\hat\beta, \hat h_c^{\rm ann}(\hat\beta/\alpha), \bar\beta; 0) = S_*^{\rm que}(\hat\beta, \hat h_c^{\rm ann}(\hat\beta/\alpha), \bar\beta) \geq \alpha \log \hat{\mathcal{N}}(\hat\beta, \hat h_c^{\rm ann}(\hat\beta/\alpha)) + \alpha \bar M(-\bar\beta/\alpha) > 0. \tag{5.16}$$

∎

# 6 Proofs of Corollaries 1.4–1.8

## 6.1 Proof of Corollary 1.4

*Proof.* Throughout the proof, $\alpha > 1$, $\hat\beta > 0$, $\bar\beta \geq 0$ and $\bar h \in \mathbb{R}$ are fixed. The proof for $\bar h_c^{\rm que}(\bar\beta) - \log 2 < \bar h \leq \bar h_c^{\rm ann}(\bar\beta) - \log 2$ is trivial, since $h_c^{\rm ann}(\hat\beta, \bar\beta, \bar h) = \infty$ and $h_c^{\rm que}(\hat\beta, \bar\beta, \bar h) < \infty$. The rest of the proof will follow once we show that $h_c^{\rm que}(\hat\beta, \bar\beta, \bar h) < h_c^{\rm ann}(\hat\beta, \bar\beta, \bar h)$ for $\bar h_c^{\rm ann}(\bar\beta) - \log 2 < \bar h \leq \bar h_c^{\rm ann}(\bar\beta)$. This is because, for $\bar h \geq \bar h_c^{\rm ann}(\bar\beta)$, $h_c^{\rm ann}(\hat\beta, \bar\beta, \bar h) = \hat h_c^{\rm ann}(\hat\beta)$ and $\bar h \mapsto h^{\rm que}(\hat\beta, \bar\beta, \bar h)$ is non-increasing. Furthermore, the map $\bar h \mapsto h^{\rm ann}(\hat\beta, \bar\beta, \bar h)$ is also non-increasing, and so $h_c^{\rm ann}(\hat\beta, \bar\beta, \bar h) \geq \hat h_c^{\rm ann}(\hat\beta)$.

For $\bar h_c^{\rm ann}(\bar\beta) - \log 2 < \bar h \leq \bar h_c^{\rm ann}(\bar\beta)$, it follows from Corollary 1.3 that $h_c^{\rm ann}(\hat\beta, \bar\beta, \bar h)$ is the unique $\hat h$-value that solves the equation $S^{\rm ann}(\hat\beta, \hat h, \bar\beta; 0) = \bar h$. Note that for $\hat h \geq \hat h_c^{\rm ann}(\hat\beta) = \hat M(2\hat\beta)/2\hat\beta$, which is the range of $\hat h$-values attainable by $h_c^{\rm ann}(\hat\beta, \bar\beta, \bar h)$, the measure $q_{\hat\beta,\hat h,\bar\beta;0}$ (recall (3.31)) is well-defined and is the unique minimizer of the last variational formula in (3.30), for $g = 0$. Hence, for $\bar h_c^{\rm ann}(\bar\beta) - \log 2 < \bar h \leq \bar h_c^{\rm ann}(\bar\beta)$, $h_c^{\rm ann}(\hat\beta, \bar\beta, \bar h)$ is the unique $\hat h$-value that solves the equation

$$S^{\rm ann}(\hat\beta, \hat h, \bar\beta; 0) - \bar h = \bar M(-\bar\beta) + \log \hat{\mathcal{N}}(\hat\beta, \hat h; 0) - \bar h = 0. \tag{6.1}$$

Again, it follows from (2.18) that, for any $Q \in \mathcal{P}^{\rm inv}(\widetilde{E}^{\mathbb{N}})$,

$$\begin{aligned} I^{\rm que}(Q) &= \sup_{{\rm tr}\in\mathbb{N}} I^{\rm que}([Q]_{\rm tr}) = \sup_{{\rm tr}\in\mathbb{N}} \left[ H\left([Q]_{\rm tr} | q_{\rho,\hat\mu\otimes\bar\mu}^{\otimes\mathbb{N}}\right) + (\alpha-1) m_{[Q]_{\rm tr}} H\left(\Psi_{[Q]_{\rm tr}} | (\hat\mu\otimes\bar\mu)^{\otimes\mathbb{N}}\right)\right] \\ &\geq H\left(Q \mid q_{\rho,\hat\mu\otimes\bar\mu}^{\otimes\mathbb{N}}\right) + (\alpha-1) m_{[Q]_{\rm tr}} H\left(\Psi_{[Q]_{\rm tr}} | (\hat\mu\otimes\bar\mu)^{\otimes\mathbb{N}}\right), \quad {\rm tr} \in \mathbb{N}. \end{aligned} \tag{6.2}$$

Furthermore, it follows from (2.5) and the remark below it that

$$H(Q \mid q_{\rho,\hat\mu\otimes\bar\mu}^{\otimes\mathbb{N}}) \geq h(\pi_1 Q \mid q_{\rho,\hat\mu\otimes\bar\mu}), \qquad H(\Psi_{[Q]_{\rm tr}} \mid (\hat\mu\otimes\bar\mu)^{\otimes\mathbb{N}}) \geq h(\widetilde\pi_1 \Psi_{[Q]_{\rm tr}} \mid \hat\mu\otimes\bar\mu), \tag{6.3}$$

where $\widetilde\pi_1$ is the projection onto the first *letter* and ${\rm tr} \in \mathbb{N}$. Moreover, it follows from (2.10) that

$$\widetilde\pi_1 \Psi_Q = \widetilde\pi_1 \Psi_{(\pi_1 Q)^{\otimes\mathbb{N}}}. \tag{6.4}$$

Since $m_Q = m_{(\pi_1 Q)^{\otimes\mathbb{N}}} = m_{\pi_1 Q}$, (6.3–6.4) combine with (3.8) to give

$$\begin{aligned} &S_*^{\rm que}(\hat\beta, \hat h, \bar\beta) \\ &\leq \sup_{\substack{q\in\mathcal{P}(\widetilde E) \\ h(q|q_{\rho,\hat\mu\otimes\bar\mu})<\infty;\, m_q<\infty}} \left[\int_{\widetilde E} q(d\omega)[\bar\beta\bar\omega_1 + \log\phi_{\hat\beta,\hat h}(\hat\omega)] - h(q \mid q_{\rho,\hat\mu\otimes\bar\mu}) \right. \\ &\qquad\qquad \left. -(\alpha-1) m_{[q]_{\rm tr}} h(\widetilde\pi_1 \Psi_{[q]_{\rm tr}} \mid \hat\mu\otimes\bar\mu)\right], \end{aligned} \tag{6.5}$$

where

$$\phi_{\hat\beta,\hat h}(\hat\omega) = \tfrac{1}{2}\left(1 + e^{-2\hat\beta\hat h m - 2\hat\beta[\hat\omega_1+\cdots+\hat\omega_m]}\right) \tag{6.6}$$

and

$$(\widetilde{\pi}_1\Psi_{[q]_{\rm tr}})(d(\hat\omega_1,\bar\omega_1)) = \frac{1}{m_{[q]_{\rm tr}}}\sum_{m\in\mathbb{N}}[r]_{\rm tr}(m)\sum_{k=1}^{m} q_m(E^{k-1},d(\hat\omega_1,\bar\omega_1),E^{m-k}) \tag{6.7}$$

with the notation

$$q(d\omega) = r(m)q_m(d(\hat\omega_1,\bar\omega_1),\dots,d(\hat\omega_m,\bar\omega_m)), \qquad \omega = ((\hat\omega_1,\bar\omega_1),\dots,(\hat\omega_m,\bar\omega_m)), \tag{6.8}$$

and

$$[r]_{\rm tr}(m) = \begin{cases} r(m) & \text{if } 1\le m<{\rm tr} \\ \sum_{n={\rm tr}}^{\infty} r(n) & \text{if } m={\rm tr} \\ 0 & \text{otherwise.}\end{cases} \tag{6.9}$$

Therefore, for $\hat h \ge \hat h_c^{\rm ann}(\hat\beta)$, after combining the first two terms in the supremum in (6.5), as in (3.30), we obtain

$$\begin{aligned} S_*^{\rm que}(\hat\beta,\hat h,\bar\beta) \le{}& \bar M(-\bar\beta) + \log\hat{\mathcal N}(\hat\beta,\hat h;0) \\ &- \inf_{\substack{q\in\mathcal{P}(\widetilde E)\\ h(q|q_{\rho,\hat\mu\otimes\bar\mu})<\infty;\, m_q<\infty}} \Big[h(q\mid q_{\hat\beta,\hat h,\bar\beta}) + (\alpha-1)m_{[q]_{\rm tr}}h(\widetilde\pi_1\Psi_{[q]_{\rm tr}}\mid \hat\mu\otimes\bar\mu)\Big]. \end{aligned} \tag{6.10}$$

Hence, for $\hat h \ge \hat h_c^{\rm ann}(\hat\beta)$, it follows from (3.30) that

$$\begin{aligned} S_*^{\rm que}(\hat\beta,\hat h,\bar\beta) \le{}& S^{\rm ann}(\hat\beta,\hat h,\bar\beta;0) \\ &- \inf_{\substack{q\in\mathcal{P}(\widetilde E)\\ h(q|q_{\rho,\hat\mu\otimes\bar\mu})<\infty;\ m_q<\infty}} \Big[h(q\mid q_{\hat\beta,\hat h,\bar\beta}) + (\alpha-1)m_{[q]_{\rm tr}}h(\widetilde\pi_1\Psi_{[q]_{\rm tr}}\mid \hat\mu\otimes\bar\mu)\Big]. \end{aligned} \tag{6.11}$$

The first term in the variational formula achieves its minimal value zero at $q = q_{\hat\beta,\hat h,\bar\beta}$ (or along a minimizing sequence converging to $q_{\hat\beta,\hat h,\bar\beta}$). However, via some simple computations we obtain

$$\widetilde\pi_1\Psi_{[q_{\hat\beta,\hat h,\bar\beta}]_{\rm tr}}(d\hat\omega_1,d\bar\omega_1) = \frac{C_{\rm tr}(\hat\omega_1,\hat\beta,\hat h)}{A_{\rm tr}(\hat\beta,\hat h)}\hat\mu(d\hat\omega_1)\bar\mu_{\bar\beta}(d\bar\omega_1) + \frac{B_{\rm tr}(\hat\omega_1,\hat\beta,\hat h)}{A_{\rm tr}(\hat\beta,\hat h)}\hat\mu(d\hat\omega_1)\bar\mu(d\bar\omega_1), \tag{6.12}$$

where

$$\begin{aligned} A_{\rm tr}(\hat\beta,\hat h) &= \tfrac12\left(\sum_{n=1}^{{\rm tr}-1} n[1+e^{n[\hat M(2\hat\beta)-2\hat\beta\hat h]}]\rho(n) + {\rm tr}\sum_{n={\rm tr}}^{\infty}[1+e^{n[\hat M(2\hat\beta)-2\hat\beta\hat h]}]\rho(n)\right), \\ B_{\rm tr}(\hat\omega_1,\hat\beta,\hat h) &= \sum_{m=1}^{{\rm tr}-1}(m-1)\rho(m)\left[1+e^{(m-1)[\hat M(2\hat\beta)-2\hat\beta\hat h]-2\hat\beta(\hat\omega_1+\hat h)}\right] \\ &\quad + ({\rm tr}-1)\frac{1+e^{({\rm tr}-1)[\hat M(2\hat\beta)-2\hat\beta\hat h]-2\hat\beta(\hat\omega_1+\hat h)}}{1+e^{{\rm tr}[\hat M(2\hat\beta)-2\hat\beta\hat h]}}\sum_{m={\rm tr}}^{\infty}\rho(m)\left[1+e^{m[\hat M(2\hat\beta)-2\hat\beta\hat h]}\right], \\ C_{\rm tr}(\hat\omega_1,\hat\beta,\hat h) &= \sum_{m=1}^{{\rm tr}-1}\rho(m)\left[1+e^{(m-1)[\hat M(2\hat\beta)-2\hat\beta\hat h]-2\hat\beta(\hat\omega_1+\hat h)}\right] \\ &\quad + \frac{1+e^{({\rm tr}-1)[\hat M(2\hat\beta)-2\hat\beta\hat h]-2\hat\beta(\hat\omega_1+\hat h)}}{1+e^{{\rm tr}[\hat M(2\hat\beta)-2\hat\beta\hat h]}}\sum_{m={\rm tr}}^{\infty}\rho(m)\left[1+e^{m[\hat M(2\hat\beta)-2\hat\beta\hat h]}\right], \end{aligned} \tag{6.13}$$

and $\bar\mu_{\bar\beta}(d\bar\omega_1) = e^{\bar\beta\bar\omega_1-\bar M(-\bar\beta)}\bar\mu(d\bar\omega_1)$. Here we use that

$$q_{\hat\beta,\hat h,\bar\beta}(d(\hat\omega_1,\bar\omega_1),\dots,d(\hat\omega_m,\bar\omega_m)) = r(m)q_m(d(\hat\omega_1,\bar\omega_1),\dots,d(\hat\omega_m,\bar\omega_m)) \tag{6.14}$$

with

$$
\begin{aligned}
r(m) &= \frac{1+e^{m[\hat{M}(2\hat{\beta})-2\hat{\beta}\hat{h}]}}{2\hat{\mathcal{N}}(\hat{\beta},\hat{h};0)}\rho(m),\\
q_m(d(\hat{\omega}_1,\bar{\omega}_1),\dots,d(\hat{\omega}_m,\bar{\omega}_m)) &= \frac{\left(1+e^{-2\hat{\beta}\sum_{i=1}^m(\hat{\omega}_i+\hat{h})}\right)e^{\bar{\beta}\bar{\omega}_1}}{e^{\bar{M}(-\bar{\beta})}\left(1+e^{m[\hat{M}(2\hat{\beta})-2\hat{\beta}\hat{h}]}\right)}\prod_{i=1}^m \hat{\mu}(d\hat{\omega}_i)\bar{\mu}(d\bar{\omega}_i).
\end{aligned}
\tag{6.15}
$$

Note that $\widetilde{\pi}_1\Psi_{[q_{\hat{\beta},\hat{h}_c^{\mathrm{ann}}(\hat{\beta}),0}]_{\mathrm{tr}}} = (\frac{1}{2}[\hat{\mu}+\hat{\mu}_{\hat{\beta}}])\otimes\bar{\mu}$, where $\hat{\mu}_{\hat{\beta}}(d\hat{\omega}_1) = e^{-2\hat{\beta}\hat{\omega}_1-\hat{M}(2\hat{\beta})}\hat{\mu}(d\hat{\omega}_1)$. Thus $\widetilde{\pi}_1\Psi_{[q_{\hat{\beta},\hat{h},\bar{\beta}}]_{\mathrm{tr}}} \neq \hat{\mu}\otimes\bar{\mu}$ for $\hat{h}\geq \hat{h}_c^{\mathrm{ann}}(\hat{\beta})$, and so we have

$$
\begin{aligned}
&S_*^{\mathrm{que}}(\hat{\beta},h_c^{\mathrm{ann}}(\hat{\beta},\bar{\beta},\bar{h}),\bar{\beta})-\bar{h} \leq S^{\mathrm{ann}}(\hat{\beta},h_c^{\mathrm{ann}}(\hat{\beta},\bar{\beta},\bar{h}),\bar{\beta};0)-\bar{h}\\
&-\inf_{\substack{q\in\mathcal{P}(\widetilde{E})\\ h(q|q_{\rho,\hat{\mu}\otimes\bar{\mu}})<\infty;\, m_q<\infty}}\Big[h(q\mid q_{\hat{\beta},h_c^{\mathrm{ann}}(\hat{\beta},\bar{\beta},\bar{h}),\bar{\beta}})+(\alpha-1)m_{[q]_{\mathrm{tr}}}h(\widetilde{\pi}_1\Psi_{[q]_{\mathrm{tr}}}\mid\hat{\mu}\otimes\bar{\mu})\Big]\\
&= \quad -\inf_{\substack{q\in\mathcal{P}(\widetilde{E})\\ h(q|q_{\rho,\hat{\mu}\otimes\bar{\mu}})<\infty;\, m_q<\infty}}\Big[h(q\mid q_{\hat{\beta},h_c^{\mathrm{ann}}(\hat{\beta},\bar{\beta},\bar{h}),\bar{\beta}})+(\alpha-1)m_{[q]_{\mathrm{tr}}}h(\widetilde{\pi}_1\Psi_{[q]_{\mathrm{tr}}}\mid\hat{\mu}\otimes\bar{\mu})\Big]<0.
\end{aligned}
\tag{6.16}
$$

Since $S_*^{\mathrm{que}}(\hat{\beta},h_c^{\mathrm{que}}(\hat{\beta},\bar{\beta},\bar{h}),\bar{\beta})-\bar{h}=0$ and since $\hat{h}\mapsto S_*^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta})$ is strictly decreasing on $(\hat{h}_c^{\mathrm{ann}}(\hat{\beta}/\alpha),\infty)$, it follows that $h_c^{\mathrm{que}}(\hat{\beta},\bar{\beta},\bar{h}) < h_c^{\mathrm{ann}}(\hat{\beta},\bar{\beta},\bar{h})$ for $\bar{h}_c^{\mathrm{ann}}(\bar{\beta})-\log 2 < \bar{h} \leq \bar{h}_c^{\mathrm{ann}}(\bar{\beta})$. ∎

### 6.2 Proof of Corollary 1.5

*Proof.* The map $\hat{h}\mapsto S^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta};0)$ is strictly decreasing and convex on $(\hat{h}_c^{\mathrm{ann}}(\hat{\beta}/\alpha),\infty)$ (recall Fig. 9). Therefore, for $\bar{h}<s^*(\hat{\beta},\bar{\beta},\alpha)$, the $\hat{h}$-value that solves the equation $S^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta};0)=\bar{h}$ is strictly greater than $\hat{h}_c^{\mathrm{ann}}(\hat{\beta}/\alpha)$, which proves that $h_c^{\mathrm{que}}(\hat{\beta},\bar{\beta},\bar{h})>\hat{h}_c^{\mathrm{ann}}(\hat{\beta}/\alpha)$. The proof for $\bar{h}\geq s^*(\hat{\beta},\bar{\beta},\alpha)$ follows from Corollary 1.3 and (1.25). ∎

### 6.3 Proof of Corollary 1.6

*Proof.* (i) Note from (3.32) that

$$
S^{\mathrm{ann}}(\hat{\beta},\hat{h},\bar{\beta};0)-\bar{h} = \bar{M}(-\bar{\beta})-\bar{h}+\log\left(\tfrac{1}{2}\left[1+\sum_{n\in\mathbb{N}}\rho(n)e^{n[\hat{M}(2\hat{\beta})-2\hat{\beta}\hat{h}]}\right]\right). \tag{6.17}
$$

Note from (6.17) and (3.33–3.34) that $S^{\mathrm{ann}}(\hat{\beta},\hat{h}_c^{\mathrm{ann}}(\hat{\beta}),\bar{\beta};0)-\bar{h}=\bar{S}^{\mathrm{ann}}(\bar{\beta};0)-\bar{h}$ and $S^{\mathrm{ann}}(\hat{\beta},\hat{h},\bar{\beta};0)$ $-\bar{h}_c^{\mathrm{ann}}(\bar{\beta}) = \hat{S}^{\mathrm{ann}}(\hat{\beta},\hat{h};0)$. These observations, together with the remark below Theorem 3.2, conclude the proof for (i).

(ii) Recall from (3.8) that

$$
\begin{aligned}
S_*^{\mathrm{que}}(\hat{\beta},\hat{h},0) &= \sup_{Q\in\mathcal{C}^{\mathrm{fin}}}\Big[\Phi_{\hat{\beta},\hat{h}}(Q)-I^{\mathrm{que}}(Q)\Big]\\
&= \sup_{\hat{Q}\in\hat{\mathcal{C}}^{\mathrm{fin}}}\Big[\Phi_{\hat{\beta},\hat{h}}(\hat{Q})-\hat{I}^{\mathrm{que}}(\hat{Q})\Big] = \hat{S}^{\mathrm{que}}(\hat{\beta},\hat{h};0).
\end{aligned}
\tag{6.18}
$$

The second equality uses the remark below Theorem 3.1. Hence $\tilde{h}_c^{\mathrm{que}}(\hat{\beta},0) = S_*^{\mathrm{que}}(\hat{\beta},\hat{h}_c^{\mathrm{que}}(\hat{\beta}),0) = \hat{S}^{\mathrm{que}}(\hat{\beta},\hat{h}_c^{\mathrm{que}}(\hat{\beta});0)=0$ by [3], Theorem 1.1(ii). ∎

### 6.4 Proofs of Corollaries 1.7 and 1.8

*Proof.* The proofs are similar to those of Corollaries 1.6–1.7 in [3], Section 8. For the former, all that is needed is $S^{\rm que}(\hat\beta,\hat h,\bar\beta;0)-\bar h<0$, which holds for $(\hat\beta,\hat h,\bar\beta,\bar h)\in {\rm int}(\mathcal{D}_1^{\rm que})\cup(\mathcal{D}^{\rm que}\setminus\mathcal{D}_1^{\rm que})$. ■

# A Finiteness of $\Phi$

**Lemma A.1** *Fix $\delta>0$ and $\bar\mu\in\mathcal{P}(\bar E)$ satisfying (1.3). Then, for all $\bar Q\in\mathcal{P}^{\rm inv}(\widetilde{\bar E}^{\mathbb{N}})$ with $h(\pi_{1,1}\bar Q\mid\bar\mu)<\infty$, there are constants $\gamma\in(\delta^{-1},\infty)$ and $K(\delta,\gamma,\bar\mu)\in(0,\infty)$ such that*

$$|\Phi(\bar Q)|<\gamma\, h(\pi_{1,1}\bar Q\mid\bar\mu)+K(\delta,\gamma,\bar\mu). \tag{A.1}$$

*Proof.* The proof comes in 3 steps.

**1.** Abbreviate

$$f(\bar\omega_1)=\frac{d(\pi_{1,1}\bar Q)}{d\bar\mu}(\bar\omega_1),\qquad \bar\omega_1\in\bar E. \tag{A.2}$$

Fix $\gamma\in(\delta^{-1},\infty)$. For $m\in\mathbb{N}$ and $l\in\mathbb{Z}$, define

$$\begin{aligned} A_m&=\{\bar\omega_1\in\bar E\colon\ m-1\le\gamma\log f(\bar\omega_1)<m\},\\ A_0&=\{\bar\omega_1\in\bar E\colon\ 0\le f(\bar\omega_1)<1\},\\ B_l&=\{\bar\omega_1\in\bar E\colon\ l-1\le\bar\omega_1<l\}.\end{aligned} \tag{A.3}$$

Note that the $A_m$'s and the $B_l$'s are pairwise disjoint, and that

$$\bar E=A_0\cup[\cup_{m\in\mathbb{N}}A_m],\quad \bar E_+\cup\{0\}=\cup_{l\in\mathbb{N}}B_l,\quad \bar E_-=\cup_{l\in-\mathbb{N}_0}B_l, \tag{A.4}$$

where $\bar E_+$ and $\bar E_-$ denote the set of positive and negative real numbers in $\bar E$. Also note that

$$\begin{aligned}\Phi(\bar Q)&=\int_{\bar E}\bar\omega_1\,(\pi_{1,1}\bar Q)(d\bar\omega_1)\le\int_{\bar E}(0\vee\bar\omega_1)\ f(\bar\omega_1)\ \bar\mu(d\bar\omega_1)\\ &=\sum_{m\in\mathbb{N}_0}\int_{A_m}(0\vee\bar\omega_1)\ f(\bar\omega_1)\ \bar\mu(d\bar\omega_1)=I+II+III,\end{aligned} \tag{A.5}$$

where

$$\begin{aligned} I&=\int_{A_0\cap[\cup_{l\in\mathbb{N}}B_l]}\bar\omega_1\ f(\bar\omega_1)\ \bar\mu(d\bar\omega_1)\le\sum_{l\in\mathbb{N}}l\,\mathbb{P}_{\bar\omega}(B_l),\\ II&=\sum_{m\in\mathbb{N}}\int_{A_m\cap[\cup_{l=1}^{m-1}B_l]}\bar\omega_1\ f(\bar\omega_1)\ \bar\mu(d\bar\omega_1),\\ III&=\sum_{m\in\mathbb{N}}\int_{A_m\cap[\cup_{l\in\mathbb{N}_0}B_{m+l}]}\bar\omega_1\ f(\bar\omega_1)\ \bar\mu(d\bar\omega_1)\le\sum_{m\in\mathbb{N}}e^{m/\gamma}\sum_{l\in\mathbb{N}_0}(m+l)\mathbb{P}_{\bar\omega}(B_{m+l}).\end{aligned} \tag{A.6}$$

The term $I$ follows from the restriction of the $\bar\mu$-integral to the set $A_0\cap\bar E_+$. The terms $II$ and $III$ follow from the restrictions to the sets $\cup_{m\in\mathbb{N}}[A_m\cap\cup_{l=1}^{m-1}B_l]$ and $\cup_{m\in\mathbb{N}}[A_m\cap\cup_{l\in\mathbb{N}_0}B_{m+l}]$. The bound on $I$ uses that $f < 1$ on $A_0$ and $\bar{\omega}_1 < l$ on $B_l$. The bound on $III$ follows from the fact that $f < e^{m/\gamma}$ on $A_m$ and $\bar{\omega}_1 < m + l$ on $B_{m+l}$. It follows from (A.6) that

$$\begin{aligned} I + III &\leq 2 \sum_{m\in\mathbb{N}} e^{m/\gamma} \sum_{l\in\mathbb{N}_0} (m+l)\mathbb{P}_{\bar{\omega}}(B_{m+l}) \\ &\leq 2 \sum_{m\in\mathbb{N}} e^{m/\gamma} \sum_{l\in\mathbb{N}_0} (m+l)\mathbb{P}_{\bar{\omega}}(\bar{\omega}_1 \geq m+l-1) \\ &\leq 2C(\delta) \sum_{m\in\mathbb{N}} e^{m/\gamma} \sum_{l\in\mathbb{N}_0} (m+l) e^{-\delta(m+l-1)} \\ &\leq 2C(\delta)\, e^{\delta} \sum_{m\in\mathbb{N}} e^{-m(\delta-1/\gamma)} \sum_{l\in\mathbb{N}_0} (m+l) e^{-\delta l} = k_+(\delta,\gamma,\bar{\mu}) < \infty, \end{aligned} \tag{A.7}$$

where the third inequality uses (1.3). Moreover, use that $\bar{\omega}_1 < m - 1 \leq \gamma \log f$ on $A_m \cap \cup_{l=1}^{m-1} B_l$, to estimate

$$\begin{aligned} II &\leq \gamma \sum_{m\in\mathbb{N}} \int_{\bar{\omega}_1 \in A_m \cap [\cup_{l=1}^{m-1} B_l]} f(\bar{\omega}_1) \log f(\bar{\omega}_1)\, \bar{\mu}(d\bar{\omega}_1) \\ &\leq \gamma \sum_{m\in\mathbb{N}} \int_{\bar{\omega}_1 \in A_m} f(\bar{\omega}_1) \log f(\bar{\omega}_1)\, \bar{\mu}(d\bar{\omega}_1) \\ &= \gamma \int_{\bar{E}\setminus A_0} f(\bar{\omega}_1) \log f(\bar{\omega}_1)\, \bar{\mu}(d\bar{\omega}_1) \\ &= \gamma h(\pi_{1,1}\bar{Q} \mid \bar{\mu}) - \gamma \int_{A_0} f(\bar{\omega}_1) \log f(\bar{\omega}_1)\, \bar{\mu}(d\bar{\omega}_1) \\ &\leq \gamma h(\pi_{1,1}\bar{Q} \mid \bar{\mu}) + \gamma\, e^{-1} < \infty, \end{aligned} \tag{A.8}$$

where the third inequality uses that $f \log f \geq -e^{-1}$ on $A_0$, and the second equality that

$$h(\pi_{1,1}\bar{Q}|\bar{\mu}) = \int_{\bar{E}\setminus A_0} f(\bar{\omega}_1) \log f(\bar{\omega}_1)\, \bar{\mu}(d\bar{\omega}_1) + \int_{A_0} f(\bar{\omega}_1) \log f(\bar{\omega}_1)\, \bar{\mu}(d\bar{\omega}_1) < \infty. \tag{A.9}$$

Put

$$K_+(\delta,\gamma,\bar{\mu}) = k_+(\delta,\gamma,\bar{\mu}) + \gamma\, e^{-1}. \tag{A.10}$$

**2.** Similarly, we have

$$\begin{aligned} \Phi(\bar{Q}) &= \int_{\bar{E}} \bar{\omega}_1\ (\pi_{1,1}\bar{Q})(d\bar{\omega}_1) \geq \int_{\bar{E}} (0 \wedge \bar{\omega}_1)\ f(\bar{\omega}_1)\ \bar{\mu}(d\bar{\omega}_1) \\ &= \sum_{m\in\mathbb{N}_0} \int_{A_m} (0 \wedge \bar{\omega}_1)\ f(\bar{\omega}_1)\ \bar{\mu}(d\bar{\omega}_1) = I' + II' + III', \end{aligned} \tag{A.11}$$

where

$$\begin{aligned} I' &= \int_{A_0 \cap [\cup_{l\in-\mathbb{N}_0} B_l]} \bar{\omega}_1\ f(\bar{\omega}_1)\ \bar{\mu}(d\bar{\omega}_1) \geq \sum_{l\in-\mathbb{N}_0} (l-1)\, \mathbb{P}_{\bar{\omega}}(B_l), \\ II' &= \sum_{m\in\mathbb{N}} \int_{A_m \cap [\cup_{l=-m+1}^{0} B_l]} \bar{\omega}_1\ f(\bar{\omega}_1)\ \bar{\mu}(d\bar{\omega}_1), \\ III' &= \sum_{m\in\mathbb{N}} \int_{A_m \cap [\cup_{l\in-\mathbb{N}_0} B_{l-m}]} \bar{\omega}_1\ f(\bar{\omega}_1)\ \bar{\mu}(d\bar{\omega}_1) \geq \sum_{m\in\mathbb{N}} e^{m/\gamma} \sum_{l\in-\mathbb{N}_0} (l-m-1)\mathbb{P}_{\bar{\omega}}(B_{l-m}). \end{aligned} \tag{A.12}$$

The bounds on $I'$ and $III'$ use that $\bar\omega_1 \geq l-1$ on $B_l$ and $f < e^{m/\gamma}$ on $A_m$. Note that

$$
\begin{aligned}
I' + III' &\geq 2 \sum_{m\in\mathbb{N}_0} e^{m/\gamma} \sum_{l\in-\mathbb{N}_0} (l-m-1)\mathbb{P}_{\bar\omega}(B_{l-m}) \\
&\geq 2 \sum_{m\in\mathbb{N}_0} e^{m/\gamma} \sum_{l\in-\mathbb{N}_0} (l-m-1)\mathbb{P}_{\bar\omega}(\bar\omega_1 \leq l-m) \\
&\geq 2C(\delta) \sum_{m\in\mathbb{N}_0} e^{m/\gamma} \sum_{l\in-\mathbb{N}_0} (l-m-1)e^{\delta(l-m)} = k_-(\delta,\gamma,\bar\mu) > -\infty.
\end{aligned}
\tag{A.13}
$$

Also use that $\bar\omega_1 \geq -m \geq -[\gamma \log f + 1]$ on $A_m \cap \cup_{l=-m+1}^{0} B_l$, to estimate

$$
\begin{aligned}
II' &\geq -\sum_{m\in\mathbb{N}} \int_{\bar\omega_1 \in A_m \cap [\cup_{l=-m+1}^{0} B_l]} f(\bar\omega_1)[\gamma \log f(\bar\omega_1) + 1]\, \bar\mu(d\bar\omega_1) \\
&\geq -\gamma \sum_{m\in\mathbb{N}} \int_{A_m} f(\bar\omega_1) \log f(\bar\omega_1)\, \bar\mu(d\bar\omega_1) - 1 \\
&= -\gamma \int_{\bar E \setminus A_0} f(\bar\omega_1) \log f(\bar\omega_1)\, \bar\mu(d\bar\omega_1) - 1 \\
&= -\gamma h(\pi_{1,1}\bar Q \mid \bar\mu) + \gamma \int_{A_0} f(\bar\omega_1) \log f(\bar\omega_1)\, \bar\mu(d\bar\omega_1) - 1 \\
&\geq -[\gamma h(\pi_{1,1}\bar Q \mid \bar\mu) + \gamma e^{-1} + 1] > -\infty.
\end{aligned}
\tag{A.14}
$$

**3.** Put $K_-(\delta,\gamma,\bar\mu) = 1 + \gamma e^{-1} - k_-(\delta,\gamma,\bar\mu)$. Then the claim follows with $K(\delta,\gamma,\bar\mu) = K_+(\delta,\gamma,\bar\mu) \vee K_-(\delta,\gamma,\bar\mu)$. ∎

For the sake of completeness we state the follow finiteness results for $\Phi_{\hat\beta,\hat h}$ that were proved in [3], Appendix A.

**Lemma A.2** *Fix $\hat\beta, \hat h, g > 0$. Then $\hat\omega$-a.s. there exists a $K(\hat\omega, \hat\beta, \hat h, g) < \infty$ such that, for all $N \in \mathbb{N}$ and for all sequences $0 = k_0 < k_1 < \cdots < k_N < \infty$,*

$$
-gk_N + \sum_{i=1}^{N} \log \phi_{\hat\beta,\hat h}\big(\hat\omega_{(k_{i-1},k_i]}\big) \leq K(\hat\omega, \hat\beta, \hat h, g)N,
\tag{A.15}
$$

*where $\hat\omega_{(k_{i-1},k_i]}$ is the word cut out from $\hat\omega$ by the ith excursion interval $(k_{i-1}, k_i]$.*

**Lemma A.3** *Fix $\hat\beta, \hat h > 0$, $\rho \in \mathcal{P}(\mathbb{N})$ and $\hat\mu \in \mathcal{P}(\mathbb{R})$ satisfying (1.2) and (1.3). Then, for all $\hat Q \in \mathcal{P}^{\mathrm{inv}}(\widetilde{\hat E}^{\mathbb{N}})$ with $h(\pi_1\hat Q \mid q_{\rho,\hat\mu}) < \infty$, there are finite constants $C > 0$, $\gamma > 2\hat\beta/C$ and $K = K(\hat\beta, \hat h, \rho, \hat\mu, \gamma)$ such that*

$$
\Phi_{\hat\beta,\hat h}(\hat Q) \leq \gamma\; h(\pi_1\hat Q | q_{\rho,\hat\mu}) + K.
\tag{A.16}
$$

# B Application of Varadhan's lemma

In this appendix we prove (3.18) and the claim above it. This was used in Section 3 to complete the proof of Theorem 3.1.

**Lemma B.1** *For every $\hat\beta,\hat h>0$ and $\bar\beta\geq 0$,*

$$s^{\rm que}(\hat\beta,\hat h,\bar\beta;g)=S^{\rm que}(\hat\beta,\hat h,\bar\beta;g)\quad\forall\,g\in\mathbb{R}, \tag{B.1}$$

*with the possible exception of $g=0$, $\hat h=\hat h_c^{\rm ann}(\hat\beta/\alpha)$ and $s^{\rm que}(\hat\beta,\hat h,\bar\beta;0)=\infty$, where $s^{\rm que}(\hat\beta,\hat h,\bar\beta;g)$ is the $\omega$-a.s. constant limit defined in* (3.16)*, and $S^{\rm que}(\hat\beta,\hat h,\bar\beta;g)$ is as in* (3.4)*. In particular, the map $g\mapsto S^{\rm que}(\hat\beta,\hat h,\bar\beta;g)$ is finite on $(0,\infty)$ and infinite on $(-\infty,0)$.*

*Proof.* Throughout the proof $\hat\beta,\hat h>0$, $\bar\beta\geq 0$ and $\bar h\in\mathbb{R}$ are fixed. The proof comes in 3 steps, where we establish the equality in (B.1) for the cases $g<0$, $g=0$ and $g>0$ separately.

**Step 1.** For $g<0$ the proof of (B.1) is given in two steps.

**1a.** In this step we show that $S^{\rm que}(\hat\beta,\hat h,\bar\beta;g)=\infty$ when $g<0$. Fix $L\in\mathbb{N}$ and let $Q^L=(q^L_{\hat\mu\otimes\bar\mu})^{\otimes\mathbb{N}}$, with

$$q^L_{\hat\mu\otimes\bar\mu}(d\omega_1,\dots,d\omega_n)=\delta_{Ln}(\hat\mu\otimes\bar\mu)^{\otimes n}(d\omega_1,\dots,d\omega_n) \tag{B.2}$$

and $(\omega_1,\dots,\omega_n)=((\hat\omega_1,\bar\omega_1),\dots,(\hat\omega_n,\bar\omega_n))\in E^n$. It follows from (3.4) that

$$S^{\rm que}(\hat\beta,\hat h,\bar\beta;g)\geq\bar\beta\Phi(Q^L)+\Phi_{\hat\beta,\hat h}(Q^L)-I^{\rm ann}(Q^L)-gL\geq-\log 2-gL+\log\rho(L). \tag{B.3}$$

The second inequality uses that $\Phi(Q^L)=0$, $I^{\rm ann}(Q^L)=-\log\rho(L)$ and $\Phi_{\hat\beta,\hat h}(Q^L)\geq-\log 2$. Letting $L\to\infty$ and using that $\rho$ has a polynomial tail by (1.2), we get the claim.

**1b.** In this step we show that $s^{\rm que}(\hat\beta,\hat h,\bar\beta;g)=\infty$ when $g<0$. The proof follows from a moment estimate. We start by showing that, for each $\bar\beta\in\mathbb{R}$,

$$\limsup_{N\to\infty}\frac{1}{N}\log E_0^*\left(e^{N\bar\beta\Phi(R_N^{\bar\omega})}\right)\leq\bar M(-\bar\beta) \tag{B.4}$$

(recall (1.3)). Indeed, for any $\bar\beta\in\mathbb{R}$, by the Markov inequality,

$$\begin{aligned}\mathbb{P}_{\bar\omega}\left(\frac{1}{N}\log E_0^*\left(e^{N\bar\beta\Phi(R_N^{\bar\omega})}\right)\geq\bar M(-\bar\beta)+\epsilon\right)&=\mathbb{P}_{\bar\omega}\left(E_0^*\left(e^{N\bar\beta\Phi(R_N^{\bar\omega})}\right)\geq e^{N(\bar M(-\bar\beta)+\epsilon)}\right)\\&\leq e^{-N\bar M(-\bar\beta)}e^{-\epsilon N}\mathbb{E}_{\bar\omega}\left(E_0^*\left(e^{N\bar\beta\Phi(R_N^{\bar\omega})}\right)\right)\\&=e^{-N\bar M(-\bar\beta)}e^{-\epsilon N}E_0^*\left[\mathbb{E}_{\bar\omega}\left(e^{\bar\beta\sum_{i=1}^N\bar\omega_{k_{i-1}}}\right)\right]=e^{-\epsilon N}.\end{aligned} \tag{B.5}$$

The claim therefore follows from the Borel-Cantelli lemma.

Let $\tau_i$ be the length of the $i$-th word, let $L\in\mathbb{N}$, and put

$$k_N=\sum_{i=1}^N\tau_i\quad\text{and}\quad k_N(L)=\sum_{i=1}^N\left[\tau_i\,1_{\{\tau_i<L\}}+L\,1_{\{\tau_i\geq L\}}\right]. \tag{B.6}$$

For any $-\infty<q<0<p<1$ with $p^{-1}+q^{-1}=1$ and $g<0$, it follows from (3.13) that

$$\begin{aligned}e^{\bar hN}F_N^{\hat\beta,\hat h,\bar\beta,\bar h,\omega}(g)&=E_0^*\left(\exp\left[-gk_N+N\left(\Phi_{\hat\beta,\hat h}(R_N^\omega)+\bar\beta\,\Phi(R_N^\omega)\right)\right]\right),\\&\geq\left(\tfrac12\right)^N\,E_0^*\left(\exp\left[-gk_N(L)+N\bar\beta\,\Phi(R_N^\omega)\right]\right)\\&\geq\left(\tfrac12\right)^N\,E_0^*\left(e^{-gpk_N(L)}\right)^{1/p}E_0^*\left(e^{Nq\bar\beta\,\Phi(R_N^\omega)}\right)^{1/q}\\&=\left(\tfrac12\right)^N\,\mathcal{N}_L(pg)^{N/p}E_0^*\left(e^{Nq\bar\beta\,\Phi(R_N^\omega)}\right)^{1/q},\end{aligned} \tag{B.7}$$

where

$$\mathcal{N}_L(g) = \sum_{n=1}^{L-1} \rho(n) e^{-ng} + e^{-Lg} \sum_{n \geq L} \rho(n). \tag{B.8}$$

The first inequality in (B.7) uses that $\Phi_{\hat{\beta},\hat{h}}(Q) \geq -\log 2$, $k_N \geq k_N(L)$ and $g < 0$. The second inequality follows from the reverse Hölder inequality with the above choice of $p$ and $q$. Note that $\mathcal{N}_L(g)$ is finite for $g \in \mathbb{R}$ and $\lim_{L\to\infty} \mathcal{N}_L(g) = \mathcal{N}(g)$. It therefore follows from (3.16), (B.4) and (B.7) that

$$\begin{aligned} s^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta};g) &= \limsup_{N\to\infty} \frac{1}{N} \log\left(e^{N\bar{h}} F_N^{\hat{\beta},\hat{h},\bar{\beta},\bar{h},\omega}(g)\right) \\ &\geq -\log 2 + \frac{1}{p} \log \mathcal{N}_L(pg) + \frac{1}{q} \bar{M}(-\bar{\beta} q). \end{aligned} \tag{B.9}$$

Letting $L \to \infty$, we get from (2.6) that $s^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta};g) = \infty$, since $\mathcal{N}(pg) = \infty$ for $g \in (-\infty, 0)$.

**Step 2.** In this step, which is divided into 2 substeps, we consider the case $g > 0$.

**2a. Lower bound:** For $M > 0$, define

$$\Phi^{-M}(Q) = \int_{\bar{E}} (\bar{\pi}_{1,1} Q)(d\bar{\omega}_1)[\bar{\omega}_1 \vee (-M)]. \tag{B.10}$$

Note that $\Phi^{-M}$ is lower semi-continuous and that

$$\bar{\beta}\Phi^{-M}(Q) + \Phi_{\hat{\beta},\hat{h}}(Q) \leq \bar{\beta}\Phi(Q) + \Phi_{\hat{\beta},\hat{h}}(Q) - \bar{\beta} \int_{\bar{\omega}_1 < -M} \bar{\omega}_1 \, (\bar{\pi}_{1,1}Q)(d\bar{\omega}_1). \tag{B.11}$$

Therefore, for any $p, q > 1$ with $1/p + 1/q = 1$, it follows from the Hölder inequality that

$$\begin{aligned} &\frac{1}{N} \log E_g^* \left( e^{N\left(\Phi_{\hat{\beta},\hat{h}}(R_N^\omega) + \bar{\beta}\,\Phi^{-M}(R_N^\omega)\right)} \right) \\ &\leq \frac{1}{pN} \log E_g^* \left( e^{pN\left[\Phi_{\hat{\beta},\hat{h}}(R_N^\omega) + \bar{\beta}\,\Phi(R_N^\omega)\right]} \right) + \frac{1}{qN} \log E_g^* \left( e^{-q\,\bar{\beta} \sum_{i=1}^N \bar{\omega}_{k_i} 1_{\{\bar{\omega}_{k_i} < -M\}}} \right). \end{aligned} \tag{B.12}$$

The rest of the proof consists of taking the appropriate limits and showing that the left-hand side of (B.12) is bounded from below by $S^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta};g)$, while the second term in the right-hand side tends to zero and the first term tends to $s^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta};g)$.

Let us start with the second term in the right-hand side of (B.12). Note from (2.6) that

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{qN} \log E_g^* \left( e^{-q\,\bar{\beta} \sum_{i=1}^N \bar{\omega}_{k_i} 1_{\{\bar{\omega}_{k_i} < -M\}}} \right) \\ &= -\frac{1}{q} \log \mathcal{N}(g) + \limsup_{N\to\infty} \frac{1}{qN} \log E_0^* \left( e^{-gk_N - q\,\bar{\beta} \sum_{i=1}^N \bar{\omega}_{k_i} 1_{\{\bar{\omega}_{k_i} < -M\}}} \right) \\ &\leq -\frac{1}{q} \log \mathcal{N}(g) + \frac{1}{2q} \log \mathcal{N}(2g) + \limsup_{N\to\infty} \frac{1}{2qN} \log E_0^* \left( e^{-2q\,\bar{\beta} \sum_{i=1}^N \bar{\omega}_{k_i} 1_{\{\bar{\omega}_{k_i} < -M\}}} \right) \\ &\leq -\frac{1}{q} \log \mathcal{N}(g) + \frac{1}{2q} \log \mathcal{N}(2g) + \frac{1}{2q} \log \int_{\bar{E}} e^{-2q\bar{\beta}\bar{\omega}_1 1_{\{\bar{\omega}_1 < -M\}}} \bar{\mu}(d\bar{\omega}_1). \end{aligned} \tag{B.13}$$

The first inequality uses the Cauchy-Schwarz inequality, the second inequality uses (B.4). Note from (1.3) that the above bound tends to zero upon when $M \to \infty$ followed by $q \to \infty$.

For the first term in the right-hand side of (B.12) we proceed as follows. Note from Lemma A.2 that $\hat{\omega}$-a.s.

$$-gk_N + pN\Phi_{\hat{\beta},\hat{h}}(R_N^x) \le NK(\hat{\omega}, p, \hat{\beta}, \hat{h}, g), \tag{B.14}$$

where we use that

$$-gk_N + pN\Phi_{\hat{\beta},\hat{h}}(R_N^{\omega}) = p\left(-\frac{g}{p}k_N + N\Phi_{\hat{\beta},\hat{h}}(R_N^{\omega})\right). \tag{B.15}$$

Therefore, for any $1 < p < \infty$, it follows from (B.4) and (B.14) that $\hat{\omega}$-a.s.

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{pN} \log E_g^*\left(e^{Np\left[\Phi_{\hat{\beta},\hat{h}}(R_N^{\omega})+\bar{\beta}\,\Phi(R_N^{\omega})\right]}\right)\\
&\qquad = -\frac{1}{p}\log\mathcal{N}(g) + \limsup_{N\to\infty}\frac{1}{pN}\log E_0^*\left(e^{-gk_N+Np\left[\Phi_{\hat{\beta},\hat{h}}(R_N^{\omega})+\bar{\beta}\,\Phi(R_N^{\omega})\right]}\right)\\
&\qquad \le \frac{1}{p}\left(K(p,\hat{\beta},\hat{h},g) + \bar{M}(-p\bar{\beta}) - \log\mathcal{N}(g)\right) < \infty.
\end{aligned} \tag{B.16}$$

Next, for $-\infty < r < 0 < s < 1$ with $r^{-1} + s^{-1} = 1$, it follows from the argument leading to (B.9) that

$$\begin{aligned}
&\limsup_{N\to\infty} \frac{1}{pN} \log E_g^*\left(e^{Np\left[\Phi_{\hat{\beta},\hat{h}}(R_N^{\omega})+\bar{\beta}\,\Phi(R_N^{\omega})\right]}\right)\\
&\qquad \ge \log\frac{1}{2} + \frac{1}{p}\mathcal{N}(g) + \frac{1}{sp}\mathcal{N}(sg) + \frac{1}{pr}\bar{M}(-pr\bar{\beta}) > -\infty,
\end{aligned} \tag{B.17}$$

since $g \in (0,\infty)$. Define

$$S(p) = \limsup_{N\to\infty}\frac{1}{N}\log E_g^*\left(e^{Np\left[\Phi_{\hat{\beta},\hat{h}}(R_N^{\omega})+\bar{\beta}\,\Phi(R_N^{\omega})\right]}\right). \tag{B.18}$$

By (B.16–B.17), the map $p \mapsto S(p)$ is convex and finite on $(0,\infty)$, and hence continuous on $(0,\infty)$. It therefore follows from (3.15) that the left-hand side of (B.16) converges to $s^{\text{que}}(\hat{\beta},\hat{h},\bar{\beta};g) - \log\mathcal{N}(g)$ as $p \downarrow 1$. It follows from (B.16–B.17) that this limit is finite, which proves the finiteness of the map $g \mapsto s^{\text{que}}(\hat{\beta},\hat{h},\bar{\beta};g)$ on $(0,\infty)$.

Finally, we turn to the left-hand side of the inequality in (B.12). For any $\epsilon > 0$ and $Q \in \mathcal{C}^{\text{fin}} \cap \mathcal{R}$, note from the lower semi-continuity of the map $Q \mapsto \bar{\beta}\Phi^{-M}(Q) + \Phi_{\hat{\beta},\hat{h}}(Q)$ that the set

$$A_\epsilon(Q) = \left\{Q' \in \mathcal{P}^{\text{inv}}(\widetilde{E}^{\mathbb{N}}):\ \bar{\beta}\Phi^{-M}(Q') + \Phi_{\hat{\beta},\hat{h}}(Q') \ge \bar{\beta}\Phi^{-M}(Q) + \Phi_{\hat{\beta},\hat{h}}(Q) - \epsilon\right\} \tag{B.19}$$

is open. This implies that

$$\begin{aligned}
&\limsup_{N\to\infty}\frac{1}{N}\log E_g^*\left(e^{N\left[\Phi_{\hat{\beta},\hat{h}}(R_N^{\omega})+\bar{\beta}\,\Phi^{-M}(R_N^{\omega})\right]}\right)\\
&\qquad \ge \liminf_{N\to\infty}\frac{1}{N}\log E_g^*\left(e^{N\left[\Phi_{\hat{\beta},\hat{h}}(R_N^{\omega})+\bar{\beta}\,\Phi^{-M}(R_N^{\omega})\right]}1_{A_\epsilon(Q)}(R_N^{\omega})\right)\\
&\qquad \ge \bar{\beta}\Phi^{-M}(Q) + \Phi_{\hat{\beta},\hat{h}}(Q) - \epsilon - \inf_{Q'\in A_\epsilon(Q)} I_g^{\text{que}}(Q')\\
&\qquad \ge \bar{\beta}\Phi^{-M}(Q) + \Phi_{\hat{\beta},\hat{h}}(Q) - I_g^{\text{que}}(Q) - \epsilon\\
&\qquad \ge \bar{\beta}\Phi(Q) + \Phi_{\hat{\beta},\hat{h}}(Q) - I_g^{\text{que}}(Q) - \epsilon\\
&\qquad = \bar{\beta}\Phi(Q) + \Phi_{\hat{\beta},\hat{h}}(Q) - gm_Q - I^{\text{ann}}(Q) - \log\mathcal{N}(g) - \epsilon.
\end{aligned} \tag{B.20}$$

The second inequality uses Theorem 2.3, the third inequality uses that $Q \in A_\epsilon(Q)$, the fourth inequality follows from the fact that $\Phi \le \Phi^{-M}$, while the equality follows from Lemma 2.9. It therefore follows from (B.12–B.13), (B.20) and the comment below (B.18) that, after taking the supremum over $\mathcal{C}^{\mathrm{fin}} \cap \mathcal{R}$ followed by $M \to \infty$, $\epsilon \to 0$ and $p \downarrow 1$,

$$s^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g) \ge \sup_{Q\in\mathcal{C}^{\mathrm{fin}}\cap\mathcal{R}} \Big[\bar\beta\Phi(Q) + \Phi_{\hat\beta,\hat h}(Q) - gm_Q - I^{\mathrm{ann}}(Q)\Big] = S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g). \tag{B.21}$$

**2b. Upper bound:** Let $\chi(\hat\omega_{I_i}) = \log\phi_{\hat\beta,\hat h}(\hat\omega_{I_i})$. For $M>0$, define

$$\begin{aligned} \Phi^M(Q) &= \int_{\bar E} (\bar\pi_{1,1}Q)(d\bar\omega_1)(\bar\omega_1 \wedge M),\\ \Phi^M_{\hat\beta,\hat h}(Q) &= \int_{\tilde{\hat E}} (\hat\pi_1 Q)(d\hat\omega_{I_1})(\chi(\hat\omega_{I_1})\wedge M). \end{aligned} \tag{B.22}$$

Note that $\Phi^M$ and $\Phi^M_{\hat\beta,\hat h}$ are upper semi-continuous and

$$\bar\beta\Phi(Q) + \Phi_{\hat\beta,\hat h}(Q) - \int_{\bar\omega_1\ge M} \bar\omega_1\,(\bar\pi_{1,1}Q)(d\bar\omega_1) - \int_{\chi\ge M} (\hat\pi_1 Q)(d\hat\omega_{I_1})\chi(\hat\omega_{I_1}) \le \bar\beta\Phi^M(Q) + \Phi^M_{\hat\beta,\hat h}(Q). \tag{B.23}$$

Therefore, for any $-\infty < q < 0 < p < 1$ with $q^{-1} + p^{-1} = 1$, the reverse Hölder inequality gives

$$\begin{aligned} &\frac{1}{N}\log E^*_g\left(e^{N\left[\bar\beta\Phi^M(R^\omega_N) + \Phi^M_{\hat\beta,\hat h}(R^\omega_N)\right]}\right)\\ &\quad\ge \frac{1}{qN}\log E^*_g\left(e^{-q\left[\bar\beta\sum_{i=1}^N \bar\omega_{k_i}1_{\{\bar\omega_{k_i}\ge M\}} + \sum_{i=1}^N \chi(\hat\omega_{I_i})1_{\{\chi(\hat\omega_{I_i})\ge M\}}\right]}\right)\\ &\qquad + \frac{1}{pN}\log E^*_g\left(e^{pN\left[\bar\beta\Phi(R^\omega_N) + \Phi_{\hat\beta,\hat h}(R^\omega_N)\right]}\right). \end{aligned} \tag{B.24}$$

The rest of the proof for the upper bound follows after showing that the left-hand side of (B.24) gives rise to the desired upper bound, while the right-hand side gives rise to $s^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g)$ after taking appropriate limits.

It follows from [3], Step 2 in the proof of Lemma B.1, that

$$\frac{g}{q}k_N + \sum_{i=1}^N \chi(\hat\omega_{I_i})1_{\{\chi(\hat\omega_{I_i})\ge M\}} \le 0 \tag{B.25}$$

for $M$ large enough. Hence, for $M$ large enough, it follows from (B.4), (B.25) and $q<0$ that

$$\begin{aligned} &\limsup_{N\to\infty}\frac{1}{qN}\log E^*_g\left(e^{-q\left[\bar\beta\sum_{i=1}^N \bar\omega_{k_i}1_{\{\bar\omega_{k_i}\ge M\}} + \sum_{i=1}^N \chi(\hat\omega_{I_i})1_{\{\chi(\hat\omega_{I_i})\ge M\}}\right]}\right)\\ &= -\frac{1}{q}\log\mathcal{N}(g) + \limsup_{N\to\infty}\frac{1}{qN}\log E^*_0\left(e^{-q\left[\frac{g}{q}k_N + \sum_{i=1}^N \chi(\hat\omega_{I_i})1_{\{\chi(\hat\omega_{I_i})\ge M\}} + \bar\beta\sum_{i=1}^N \bar\omega_{k_i}1_{\{\bar\omega_{k_i}\ge M\}}\right]}\right)\\ &\ge \frac{1}{q}\left(\log\int_{\mathbb{R}} e^{-q\bar\beta\bar\omega_1 1_{\{\bar\omega_1\ge M\}}}\,\bar\mu(d\bar\omega_1) - \log\mathcal{N}(g)\right), \end{aligned} \tag{B.26}$$

which tends to zero as $M\to\infty$ followed by $q\to-\infty$. Furthermore, it follows from (B.16–B.18) and the remark below (B.18) that

$$s^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g) - \log\mathcal{N}(g) = \lim_{p\uparrow1}\limsup_{N\to\infty}\frac{1}{pN}\log E^*_g\left(e^{pN\left[\bar\beta\Phi(R^\omega_N) + \Phi_{\hat\beta,\hat h}(R^\omega_N)\right]}\right). \tag{B.27}$$

Since $\bar{\beta}\Phi^M + \Phi^M_{\hat{\beta},\hat{h}}$ is upper semi-continuous, it follows from Dembo and Zeitouni [6], Lemma 4.3.6, and Theorem 2.3 that

$$\begin{aligned}\limsup_{N\to\infty} \log E_g^*\left(e^{N\left[\bar{\beta}\Phi^M(R_N^\omega)+\Phi^M_{\hat{\beta},\hat{h}}(R_N^\omega)\right]}\right) &\leq \sup_{Q\in\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\otimes\mathbb{N}})}\left[\bar{\beta}\Phi^M(Q)+\Phi^M_{\hat{\beta},\hat{h}}(Q)-I_g^{\mathrm{que}}(Q)\right]\\ &= \sup_{Q\in\mathcal{R}}\left[\bar{\beta}\Phi^M(Q)+\Phi^M_{\hat{\beta},\hat{h}}(Q)-I_g^{\mathrm{que}}(Q)\right]\\ &= \sup_{Q\in\mathcal{C}^{\mathrm{fin}}\cap\mathcal{R}}\left[\bar{\beta}\Phi^M(Q)+\Phi^M_{\hat{\beta},\hat{h}}(Q)-gm_Q-I^{\mathrm{ann}}(Q)\right]-\log\mathcal{N}(g)\\ &\leq \sup_{Q\in\mathcal{C}^{\mathrm{fin}}\cap\mathcal{R}}\left[\bar{\beta}\Phi(Q)+\Phi_{\hat{\beta},\hat{h}}(Q)-gm_Q-I^{\mathrm{ann}}(Q)\right]-\log\mathcal{N}(g).\end{aligned} \tag{B.28}$$

The first equality uses that $I_g^{\mathrm{que}}(Q)=\infty$ for $Q\notin\mathcal{R}$ (recall (2.14)) and the fact that $\hat{\beta}\Phi^M+\Phi^M_{\hat{\beta},\hat{h}}\leq M(1+\bar{\beta})$, the second equality uses (2.9) and the fact that $I_g^{\mathrm{que}}=I_g^{\mathrm{ann}}$ on $\mathcal{R}$. (The removal of $Q$'s with $m_Q=I^{\mathrm{ann}}(Q)=\infty$ again follows from $\hat{\beta}\Phi^M+\Phi^M_{\hat{\beta},\hat{h}}\leq M(1+\bar{\beta})$), the last inequality uses that $\Phi^M(Q)\leq\Phi(Q)$ and $\Phi^M_{\hat{\beta},\hat{h}}(Q)\leq\Phi_{\hat{\beta},\hat{h}}(Q)$. Therefore, combining (B.24–B.28) and letting $M\to\infty$ and $p\uparrow 1$ in the appropriate order, we conclude the proof of the upper bound.

**Step 3.** For $g=0$ we show that

$$s^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta};0)=S^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta};0)=S_*^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta}) \tag{B.29}$$

(recall (3.8)). The proof comes in steps 3a and 3b below, which show that

$$\begin{aligned}\lim_{g\downarrow 0} S^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta};g)&=S^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta};0)=S_*^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta}),\\ \lim_{g\downarrow 0} s^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta};g)&=s^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta};0)=S_*^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta}).\end{aligned} \tag{B.30}$$

We need the following lemma, whose proof is deferred to Section B.1.

**Lemma B.2** *Fix $\hat{\beta}$, $\hat{h}>0$ and $\bar{\beta}\geq 0$. Then*

$$\lim_{g\downarrow 0} s^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta};g)\geq S_*^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta}). \tag{B.31}$$

**3a.** Since the map $g\mapsto S^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta};g)$ is non-increasing on $[0,\infty)$, we have

$$S^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta};0+)=\lim_{g\downarrow 0} S^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta};g)\leq S^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta};0). \tag{B.32}$$

However, it follows from (3.4) that, for any $Q'\in\mathcal{C}^{\mathrm{fin}}\cap\mathcal{R}$,

$$\begin{aligned}S^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta};g)&=\sup_{Q\in\mathcal{C}^{\mathrm{fin}}\cap\mathcal{R}}\left[\bar{\beta}\Phi(Q)+\Phi_{\hat{\beta},\hat{h}}(Q)-gm_Q-I^{\mathrm{ann}}(Q)\right]\\ &\geq \bar{\beta}\Phi(Q')+\Phi_{\hat{\beta},\hat{h}}(Q')-gm_{Q'}-I^{\mathrm{ann}}(Q').\end{aligned} \tag{B.33}$$

Therefore, taking the limit $g\downarrow 0$ and taking the supremum over $Q'\in\mathcal{C}^{\mathrm{fin}}\cap\mathcal{R}$, we get

$$S^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta};0+)\geq \sup_{Q\in\mathcal{C}^{\mathrm{fin}}\cap\mathcal{R}}\left[\bar{\beta}\Phi(Q)+\Phi_{\hat{\beta},\hat{h}}(Q)-I^{\mathrm{ann}}(Q)\right]=S^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta};0). \tag{B.34}$$

Thus, we have

$$S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0+) = S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0). \tag{B.35}$$

Next, note from (3.4) and (3.8) that

$$\begin{aligned} S^{\mathrm{que}}_*(\hat\beta,\hat h,\bar\beta) &= \sup_{Q\in\mathcal{C}^{\mathrm{fin}}}\left[\bar\beta\Phi(Q)+\Phi_{\hat\beta,\hat h}(Q)-I^{\mathrm{que}}(Q)\right]\\ &\geq \sup_{Q\in\mathcal{C}^{\mathrm{fin}}\cap\mathcal{R}}\left[\bar\beta\Phi(Q)+\Phi_{\hat\beta,\hat h}(Q)-I^{\mathrm{que}}(Q)\right]\\ &= S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0), \end{aligned} \tag{B.36}$$

where we use that $I^{\mathrm{que}} = I^{\mathrm{ann}}$ on $\mathcal{R}$. It follows from Steps 1 and 2 that $S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g) = s^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g)$ for $g>0$. It therefore follows from Lemma B.2 that

$$\lim_{g\downarrow 0} s^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g) = \lim_{g\downarrow 0} S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g) = S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0) \geq S^{\mathrm{que}}_*(\hat\beta,\hat h,\bar\beta). \tag{B.37}$$

**3b.** Recall from (3.15) that

$$s^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g) = \limsup_{N\to\infty}\frac{1}{N}\log E_0^*\left(\exp\left[N\left(-g m_{R_N^\omega}+\Phi_{\hat\beta,\hat h}(R_N^\omega)+\bar\beta\,\Phi(R_N^\omega)\right)\right]\right). \tag{B.38}$$

Hence the map $g\mapsto s^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g)$ is convex and non-increasing. Therefore, when $s^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0)<\infty$, it follows from convexity and finiteness (implying continuity) of the map $g\mapsto s^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g)$ on $[0,\infty)$ that

$$s^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0) = \lim_{g\downarrow 0} s^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g) = S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0+) = S^{\mathrm{que}}_*(\hat\beta,\hat h,\bar\beta). \tag{B.39}$$

It follows from Lemma B.2 that

$$s^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0) \geq \lim_{g\downarrow 0} s^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g) \geq S^{\mathrm{que}}_*(\hat\beta,\hat h,\bar\beta). \tag{B.40}$$

Therefore

$$s^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0) = S^{\mathrm{que}}_*(\hat\beta,\hat h,\bar\beta) \quad\text{whenever}\quad S^{\mathrm{que}}_*(\hat\beta,\hat h,\bar\beta)=\infty. \tag{B.41}$$

Now, for $\hat\beta>0$ and $\bar\beta\geq 0$, it follows from the proof of Lemma 5.2 that $S^{\mathrm{que}}_*(\hat\beta,\hat h,\bar\beta)=\infty$ when $\hat h<\hat h_c^{\mathrm{ann}}(\hat\beta/\alpha)$ and $s^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0)<\infty$ when $\hat h>\hat h_c^{\mathrm{ann}}(\hat\beta/\alpha)$. Thus, from (B.39) and the remark above it and (B.41), we get for $\hat h\neq\hat h_c^{\mathrm{ann}}(\hat\beta/\alpha)$ that

$$s^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0) = s^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0+) = S^{\mathrm{que}}_*(\hat\beta,\hat h,\bar\beta). \tag{B.42}$$

This equality holds at $\hat h=\hat h_c^{\mathrm{ann}}(\hat\beta/\alpha)$ when $s^{\mathrm{que}}(\hat\beta,\hat h_c^{\mathrm{ann}}(\hat\beta/\alpha),\bar\beta;0)<\infty$. ∎

## B.1 Proof of Lemma B.2

*Proof.* For $L>0$, $M\in\mathbb{N}$, $\epsilon>0$ and $Q'\in\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})$ with $I^{\mathrm{que}}(Q')<\infty$ and $m_{Q'}<\infty$, let

$$\begin{aligned} \Phi^{M,-L}(Q) &= \int_{\bar E}(\bar\pi_{1,1}Q)(d\bar\omega_1)[\bar\omega_1\wedge M\vee(-L)],\\ \bar\Phi^{-L}(Q) &= \int_{\bar E}(\bar\pi_{1,1}Q)(d\bar\omega_1)\bar\omega_1 1_{\{\bar\omega_1<-L\}},\\ \mathcal{C}^{\mathrm{fin}}_* &= \left\{Q\in\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})\colon\ I^{\mathrm{que}}(Q)<\infty,\, m_Q<\infty\right\},\\ A_{Q'} &= \left\{Q\in\mathcal{P}^{\mathrm{inv}}(\widetilde{E}^{\mathbb{N}})\colon\ \bar\beta\Phi^{M,-L}(Q)+\Phi^{M}_{\hat\beta,\hat h}(Q) > \bar\beta\Phi^{M,-L}(Q')+\Phi^{M}_{\hat\beta,\hat h}(Q')-\epsilon\right\}. \end{aligned} \tag{B.43}$$

The map $Q \mapsto \bar\beta\Phi^{M,-L}(Q) + \Phi^M_{\hat\beta,\hat h}(Q)$ is lower semi-continuous, and so the set $A_{Q'}$ is open. Moreover, $\Phi^M \geq \Phi^{M,-L} + \bar\Phi^{-L}$, $\Phi^M \leq \Phi^{M,-L}$, $\Phi_{\hat\beta,\hat h} \geq \Phi^M_{\hat\beta,\hat h}$ and $\Phi \geq \Phi^M$ (recall (B.10) and (B.22)).

Define

$$s_M^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g) = \limsup_{N\to\infty} \frac{1}{N} \log E_0^*\left(\exp\left[N\left(-g m_{R_N^\omega} + \Phi^M_{\hat\beta,\hat h}(R_N^\omega) + \bar\beta\,\Phi^M(R_N^\omega)\right)\right]\right). \tag{B.44}$$

Note that $s_M^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g) < \infty$ for $g \geq 0$. It therefore follows from the convexity of the map $g \mapsto s_M^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g)$ that

$$\lim_{g\downarrow 0} s_M^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g) = s_M^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0), \tag{B.45}$$

and from (B.38) that

$$\lim_{g\downarrow 0} s^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g) \geq \lim_{g\downarrow 0} s_M^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;g) = s_M^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0). \tag{B.46}$$

Now, for $-\infty < q < 0 < p < 1$ with $p^{-1} + q^{-1} = 1$, note from (B.44) that

$$\begin{aligned}
s_M^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0) &= \limsup_{N\to\infty} \frac{1}{N}\log E_0^*\left(e^{N\left(\Phi^M_{\hat\beta,\hat h}(R_N^\omega) + \bar\beta\,\Phi^M(R_N^\omega)\right)}\right)\\
&\geq \limsup_{N\to\infty} \frac{1}{N}\log E_0^*\left(e^{N\left(\Phi^M_{\hat\beta,\hat h}(R_N^\omega) + \bar\beta\,\Phi^{M,-L}(R_N^\omega) + \bar\beta\,\bar\Phi^{-L}(R_N^\omega)\right)}\right)\\
&\geq \frac{1}{p}\limsup_{N\to\infty} \frac{1}{N}\log E_0^*\left(e^{pN\left(\Phi^M_{\hat\beta,\hat h}(R_N^\omega) + \bar\beta\,\Phi^{M,-L}(R_N^\omega)\right)}\right)\\
&+ \frac{1}{q}\limsup_{N\to\infty} \frac{1}{N}\log E_0^*\left(e^{qN\bar\beta\bar\Phi^{-L}(R_N^\omega)}\right)\\
&\geq \frac{1}{p}\liminf_{N\to\infty} \frac{1}{N}\log E_0^*\left(e^{pN\left(\Phi^M_{\hat\beta,\hat h}(R_N^\omega) + \bar\beta\,\Phi^{M,-L}(R_N^\omega)\right)} 1_{A_{Q'}}(R_N^\omega)\right)\\
&+ \frac{1}{q}\log\int_{\bar E} e^{q\bar\beta\bar\omega_1 1_{\{\bar\omega_1<-L\}}}\,\bar\mu(d\bar\omega_1)\\
&\geq \bar\beta\Phi^{M,-L}(Q') + \Phi^M_{\hat\beta,\hat h}(Q') - \frac{1}{p}\,I^{\mathrm{que}}(Q') + \frac{1}{q}\log\int_{\bar E} e^{q\bar\beta\bar\omega_1 1_{\{\bar\omega_1<-L\}}}\,\bar\mu(d\bar\omega_1) - \epsilon\\
&\geq \bar\beta\Phi^{M}(Q') + \Phi^M_{\hat\beta,\hat h}(Q') - \frac{1}{p}\,I^{\mathrm{que}}(Q') + \frac{1}{q}\log\int_{\bar E} e^{q\bar\beta\bar\omega_1 1_{\{\bar\omega_1<-L\}}}\,\bar\mu(d\bar\omega_1) - \epsilon.
\end{aligned} \tag{B.47}$$

The second inequality uses the reverse Hölder inequality, the third inequality uses (B.4) and $q < 0$, the fourth inequality uses the definition of $A_{Q'}$, Theorem 2.3, (2.15) and the fact that $I^{\mathrm{que}}(Q') \geq \inf_{Q\in A_{Q'}} I^{\mathrm{que}}(Q)$, the last inequality uses that $\Phi^{M,-L} \geq \Phi^M$.

Letting $M \to \infty$ and applying Fatou's lemma to $\bar\beta\Phi^M(Q') + \Phi^M_{\hat\beta,\hat h}(Q')$, followed by $L \to \infty$, $p \uparrow 1$ and $\epsilon \downarrow 0$, we get

$$\liminf_{M\to\infty} s_M^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0) \geq \bar\beta\Phi(Q') + \Phi_{\hat\beta,\hat h}(Q') - I^{\mathrm{que}}(Q'). \tag{B.48}$$

Consequently, taking the supremum over $Q' \in \mathcal{C}_*^{\mathrm{fin}}$, we get

$$\begin{aligned}
\liminf_{M\to\infty} s_M^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0) &\geq \sup_{Q\in\mathcal{C}_*^{\mathrm{fin}}}\left[\bar\beta\Phi(Q') + \Phi_{\hat\beta,\hat h}(Q') - I^{\mathrm{que}}(Q')\right]\\
&= \sup_{Q\in\mathcal{C}^{\mathrm{fin}}}\left[\bar\beta\Phi(Q') + \Phi_{\hat\beta,\hat h}(Q') - I^{\mathrm{que}}(Q')\right] = S_*^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta).
\end{aligned} \tag{B.49}$$

The first equality uses that those $Q \in C^{\rm fin}$ for which $I^{\rm que}(Q) = \infty$ do not contribute to the supremum over $C^{\rm fin}$, because $\Phi < \infty$ and $\Phi_{\hat\beta,\hat h} < \infty$ on $C^{\rm fin}$. It therefore follows from (B.45–B.46) and (B.49) that

$$\lim_{g\downarrow 0} s^{\rm que}(\hat\beta,\hat h,\bar\beta;g) \geq S^{\rm que}_*(\hat\beta,\hat h,\bar\beta). \tag{B.50}$$

∎

# C Proof of Lemma 5.3

In this Appendix we prove Lemma 5.3. To do so we need another lemma, which we state and prove in Section C.1. In Section C.2 we use this lemma to prove Lemma 5.3.

## C.1 A preparatory lemma

**Lemma C.1** *For every $\hat\beta > 0$, $\bar\beta \geq 0$ and $\hat h \geq \hat h_c^{\rm ann}(\hat\beta)$,*

$$S^{\rm que}(\hat\beta,\hat h,\bar\beta;0) \leq \sup_{\bar Q \in \bar C^{\rm fin}} \Big[\bar\beta\Phi(\bar Q) + \Xi_{\hat\beta,\hat h}(r_{\bar Q}) - \bar I^{\rm que}(\bar Q)\Big], \tag{C.1}$$

*where*

$$\Xi_{\hat\beta,\hat h}(r_{\bar Q}) = \sum_{n\in\mathbb{N}} r_{\bar Q}(n) \log\left(\tfrac12\left[1+e^{n[\hat M(2\hat\beta)-2\hat\beta\hat h]}\right]\right) \tag{C.2}$$

*and $r_{\bar Q}$ is the word length distribution under $\bar Q$.*

*Proof.* Throughout the proof, $\hat\beta > 0$, $\bar\beta \geq 0$ and $\hat h \geq \hat h_c^{\rm ann}(\hat\beta)$ are fixed. Put

$$S_N^\omega = E_0^*\left(e^{N\left[\bar\beta\Phi(R_N^\omega)+\Phi_{\hat\beta,\hat h}(R_N^\omega)\right]}\right) = E_0^*\left(e^{N\left[\bar\beta\Phi(R_N^{\bar\omega})+\Phi_{\hat\beta,\hat h}(R_N^{\hat\omega})\right]}\right). \tag{C.3}$$

Note from (B.5) and the Borel-Cantelli lemma that, for every $\epsilon > 0$ and $\bar\omega$-a.s., there exists an $N_0 = N_0(\bar\omega,\epsilon) < \infty$ such that

$$E_0^*\left(e^{N\bar\beta\Phi(R_N^\omega)}\right) = E_0^*\left(e^{N\bar\beta\Phi(R_N^{\bar\omega})}\right) \leq e^{N[\bar M(-\bar\beta)+\epsilon]} \qquad \forall\, N \geq N_0. \tag{C.4}$$

Therefore, $\bar\omega$-a.s. and for all $N \geq N_0$,

$$\begin{aligned}
\mathbb{E}_{\hat\omega}\left(S_N^\omega\right) &= \sum_{0=k_0<k_1<\ldots<k_N<\infty} \prod_{i=1}^N \rho(k_i-k_{i-1})\, e^{\bar\beta\bar\omega_{k_i}}\, \tfrac12\left[1+\mathbb{E}_{\hat\omega}\left(e^{-2\hat\beta\sum_{k=k_{i-1}+1}^{k_i}(\hat\omega_k+\hat h)}\right)\right]\\
&= \sum_{0=k_0<k_1<\ldots<k_N<\infty} \prod_{i=1}^N \rho(k_i-k_{i-1})\, e^{\bar\beta\bar\omega_{k_i}}\, \tfrac12\left(1+e^{(k_i-k_{i-1})[\hat M(2\hat\beta)-2\hat\beta\hat h]}\right)\\
&= \sum_{0=k_0<k_1<\ldots<k_N<\infty} \left(\prod_{i=1}^N \rho(k_i-k_{i-1})\right)\left(e^{N[\bar\beta\Phi(R_N^\omega)+\Xi_{\hat\beta,\hat h}(r_{R_N^\omega})]}\right)\\
&= E_0^*\left(e^{N[\bar\beta\Phi(R_N^\omega)+\Xi_{\hat\beta,\hat h}(r_{R_N^\omega})]}\right) < \infty.
\end{aligned} \tag{C.5}$$

Finiteness follows from (C.4) and the fact that $\Xi_{\hat\beta,\hat h} \leq 0$ if $\hat h \geq \hat h_c^{\mathrm{ann}}(\hat\beta)$. Therefore, for every $\delta > 0$, $\bar\omega$-a.s. and $N \geq N_0$, we have

$$\mathbb{P}_{\hat\omega}\left(\frac{1}{N}\log S_N^\omega \geq \frac{1}{N}\log \mathbb{E}_{\hat\omega}(S_N^\omega) + \delta\right) = \mathbb{P}_{\hat\omega}\left(S_N^\omega \geq \mathbb{E}_{\hat\omega}(S_N^\omega)\, e^{N\delta}\right) \leq e^{-N\delta}. \tag{C.6}$$

From the Borel-Cantelli lemma we therefore obtain that $\omega$-a.s.

$$\begin{aligned} s^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0) &= \limsup_{N\to\infty}\frac{1}{N}\log S_N^\omega \leq \limsup_{N\to\infty}\frac{1}{N}\log \mathbb{E}_{\hat\omega}(S_N^\omega) + \delta \\ &= \sup_{\bar Q \in \bar{\mathcal{C}}^{\mathrm{fin}}}\left[\bar\beta\Phi(\bar Q) + \Xi_{\hat\beta,\hat h}(r_{\bar Q}) - \bar I^{\mathrm{que}}(\bar Q)\right] + \delta. \end{aligned} \tag{C.7}$$

The equality uses Steps 1 and 2 in the proof of Lemma B.1 and the observation that $\Xi_{\hat\beta,\hat h}$ is independent of $\omega$ (i.e., only pinning disorder is present), and $-\log 2 \leq \Xi_{\hat\beta,\hat h} \leq 0$ for $\hat h \geq \hat h_c^{\mathrm{ann}}(\hat\beta)$, where we use (2.15) instead of (2.14). Finally, let $\delta \downarrow 0$. ∎

## C.2 Proof of Lemma 5.3

*Proof.* Throughout the proof, $\bar\beta \geq 0$ and $\hat\beta > 0$ are fixed and $\hat h \geq \hat h_c^{\mathrm{ann}}(\hat\beta)$. Note from (3.6) and (3.7) that $\Phi_{\hat\beta,\infty} \equiv -\log 2$. Therefore, replacing $\bar\beta\bar\Phi + \Phi_{\hat\beta,\hat h}$ by $\bar\beta\bar\Phi - \log 2$ in (3.8), we get

$$\begin{aligned} S^{\mathrm{que}}(\hat\beta,\infty,\bar\beta;0) &= S_*^{\mathrm{que}}(\hat\beta,\infty,\bar\beta) = s^{\mathrm{que}}(\hat\beta,\infty,\bar\beta;0) \\ &= \limsup_{N\to\infty}\frac{1}{N}\log E_0^*\left(e^{N[\log\frac{1}{2} + \bar\beta\Phi(R_N^{\bar\omega})]}\right) \\ &= -\log 2 + \sup_{\bar Q\in\bar{\mathcal{C}}^{\mathrm{fin}}}\left[\bar\beta\Phi(\bar Q) - \bar I^{\mathrm{que}}(\bar Q)\right] \\ &= -\log 2 + \bar h_c^{\mathrm{que}}(\bar\beta). \end{aligned} \tag{C.8}$$

The fourth equality follows from the proof of Lemma B.1, while the last equality uses [5], Theorem 1.3. Next, note that

$$S^{\mathrm{que}}(\hat\beta,\infty^-,\bar\beta;0) = \lim_{\hat h\uparrow\infty} S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0) \geq S^{\mathrm{que}}(\hat\beta,\infty,\bar\beta;0), \tag{C.9}$$

since the map $\hat h \mapsto S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0)$ is non-increasing. For $\hat h \geq \hat h_c^{\mathrm{ann}}(\hat\beta)$ it follows from (C.1) that

$$\begin{aligned} S^{\mathrm{que}}(\hat\beta,\hat h,\bar\beta;0) &\leq \sup_{\bar Q\in\bar{\mathcal{C}}^{\mathrm{fin}}}\left[\bar\beta\Phi(\bar Q) + \Xi_{\hat\beta,\hat h}(r_{\bar Q}) - \bar I^{\mathrm{que}}(\bar Q)\right] \\ &= \sup_{\substack{r\in\mathcal{P}(\mathbb{N});\\ m_r<\infty}}\ \sup_{\substack{\bar Q\in\bar{\mathcal{C}}^{\mathrm{fin}};\\ r_{\bar Q}=r}}\left[\bar\beta\Phi(\bar Q) + \Xi_{\hat\beta,\hat h}(r) - \bar I^{\mathrm{que}}(\bar Q)\right] \\ &\leq \sup_{\substack{r\in\mathcal{P}(\mathbb{N});\\ m_r<\infty}}\left(\Xi_{\hat\beta,\hat h}(r) + \sup_{\bar Q\in\bar{\mathcal{C}}^{\mathrm{fin}}}\left[\bar\beta\Phi(\bar Q) - \bar I^{\mathrm{que}}(\bar Q)\right]\right) \\ &\leq \log\left[\tfrac{1}{2}\left(1 + e^{[\hat M(2\hat\beta) - 2\hat\beta\hat h]}\right)\right] + \sup_{\bar Q\in\bar{\mathcal{C}}^{\mathrm{fin}}}\left[\bar\beta\Phi(\bar Q) - \bar I^{\mathrm{que}}(\bar Q)\right] \\ &= \log\left[\tfrac{1}{2}\left(1 + e^{[\hat M(2\hat\beta) - 2\hat\beta\hat h]}\right)\right] + \bar h_c^{\mathrm{que}}(\bar\beta). \end{aligned} \tag{C.10}$$

The third inequality uses that, for $\hat{h} \geq \hat{h}_c^{\mathrm{ann}}(\hat{\beta})$, $\Xi_{\hat{\beta},\hat{h}}(r) \leq \log\left[\frac{1}{2}\left(1+e^{[\hat{M}(2\hat{\beta})-2\hat{\beta}\hat{h}]}\right)\right]$ for all $r \in \mathcal{P}(\mathbb{N})$. Therefore

$$\lim_{\hat{h}\uparrow\infty} S^{\mathrm{que}}(\hat{\beta},\hat{h},\bar{\beta};0) \leq -\log 2 + \bar{h}_c^{\mathrm{que}}(\bar{\beta}) = S^{\mathrm{que}}(\hat{\beta},\infty,\bar{\beta};0). \tag{C.11}$$

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