A CHARACTERISATION OF C^*-ALGEBRAS THROUGH POSITIVITY OF FUNCTIONALS

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ABSTRACT. We show that a unital involutive Banach algebra, with identity of norm one and continuous involution, is a C^*-algebra, with the given involution and norm, if every continuous linear functional attaining its norm at the identity is positive.

If $\mathcal{A}$ is an involutive Banach algebra, then a linear map $\omega : \mathcal{A} \to \mathbb{C}$ is called positive if $\omega(a^*a) \geq 0$, for all $a \in \mathcal{A}$. If the involution is isometric, and $\mathcal{A}$ has an identity 1 of norm one, then ω is automatically continuous, and $\|\omega\| = \omega(1)$, see [4, Lemma I.9.9].[1] For a unital C^*-algebra $\mathcal{A}$, there is a converse: if $\omega : \mathcal{A} \to \mathbb{C}$ is continuous, and $\omega(1) = \|\omega\|$, then ω is positive (cf. [4, Lemma III.3.2]). Thus the positive continuous linear functionals on a unital C^*-algebra are precisely the continuous linear functionals attaining their norm at the identity. Consequently, any Hahn-Banach extension of a positive linear functional, defined on a unital C^*-subalgebra, is automatically positive again. As is well known, this is a basic characteristic of C^*-algebras that makes the theory of states on such algebras a success.

If $\mathcal{A}$ is a unital involutive Banach algebra with identity of norm one, but not a C^*-algebra, then this converse, as valid for unital C^*-algebras, need not hold: even when the involution is isometric, there can exist continuous linear functionals on $\mathcal{A}$ that attain their norm at the identity, but which fail to be positive. For example, for $H^\infty(\mathbb{D})$, the algebra of bounded holomorphic functions on the open unit disk, supplied with the supremum norm and involution $f^*(z) = \overline{f(\bar{z})}$ ($z \in \mathbb{D}$, $f \in H^\infty(\mathbb{D})$), all point evaluations attain their norm at the identity, but only the evaluation in points in $(-1, 1)$ are positive. As another example, consider $\ell^1(\mathbb{Z})$, the group algebra of the integers. Then its dual can be identified with $\ell^\infty(\mathbb{Z})$, and the continuous linear functionals attaining their norm at the identity are then the bounded maps $\omega : \mathbb{Z} \to \mathbb{C}$, such that $\omega(0) = \|\omega\|_\infty$. Not all such continuous linear functionals are positive.[2] For example, if $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$, then $\omega_\lambda \in \ell^\infty(\mathbb{Z})$, defined by $\omega(0) = 1$, $\omega(1) = \lambda$, and $\omega(n) = 0$ if $n \neq 0, 1$, attains its norm at the identity of $\ell^1(\mathbb{Z})$. However, if we define $\ell_0 : \mathbb{Z} \to \mathbb{C}$ by $\ell_0(0) = 1$, $\ell_0(1) = 1$, and $\ell_0(n) = 0$ if $n \neq 0, 1$, then $\ell_0 \in \ell^1(\mathbb{Z})$, but $\omega_0(\ell_0^*\ell_0) = 2 + \lambda$ need not even be real.

It is the aim of this note to show that the existence of examples as above is no coincidence: there necessarily exist continuous linear functionals that attain their norm at the identity, yet are not positive, *because* the algebra in question has a continuous involution, but is not a C^*-algebra. This is the main content of the result below which, with a rather elementary proof, follows from the far less

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[1] More generally, cf. [3, Theorem 11.31]: even if the involution is not continuous, a positive linear functional is always continuous. If the involution is continuous, and $\|a^*\| \leq \beta\|a\|$, for all $a \in \mathcal{A}$, then $\|\omega\| \leq \sqrt{\beta}\,\omega(1)$.

[2] Of course, Bochner's theorem describes the ones that *are* positive.

elementary Vidav-Palmer theorem [1, Theorem 38.14]. We formulate the latter first for convenience.

Theorem (Vidav-Palmer). *Let $\mathcal{A}$ be a unital Banach algebra with identity of norm one. Let $\mathcal{A}_S$ be the real linear subspace of all $a \in A$ such that $\omega(a)$ is real, for every continuous linear functional ω on $\mathcal{A}$ such that $\|\omega\| = \omega(1)$. If $\mathcal{A} = \mathcal{A}_S + i\,\mathcal{A}_S$, then this is automatically a direct sum of real linear subspaces, and the well defined map $(a_1 + ia_2) \mapsto (a_1 - ia_2)$ $(a_1, a_2 \in \mathcal{A}_S)$ is an involution on $\mathcal{A}$ which, together with the given norm, makes $\mathcal{A}$ into a C^*-algebra.*

As further preparation let us note that, if $\mathcal{A}$ is a unital involutive Banach algebra with identity of norm one and a continuous involution, and if $a \in \mathcal{A}$ is self-adjoint with spectral radius less than 1, then there exists a self-adjoint element $b \in \mathcal{A}$ such that $1 - a = b^2$. Indeed, using the continuity of the involution, the proof as usually given for a unital Banach algebra with isometric involution, cf. [4, Lemma I.9.8], which is based on the fact that the coefficients of the power series around 0 of the principal branch of $\sqrt{1-z}$ on $\mathbb{D}$ are all real, goes through unchanged.

Theorem. *Let $\mathcal{A}$ be a unital involutive Banach algebra with identity 1 of norm one. Then the following are equivalent:*

(1) *The involution is continuous, and, if ω is a continuous linear functional on $\mathcal{A}$ such that $\|\omega\| = \omega(1)$, then ω is positive;*
(2) *The involution is continuous, and, if ω is a continuous linear functional on $\mathcal{A}$ such that $\|\omega\| = \omega(1)$, and $a \in \mathcal{A}$ is self-adjoint, then $\omega(a^2)$ is real;*
(3) *$\mathcal{A}$ is a C^*-algebra with the given norm and involution.*

Proof. We need only prove that (2) implies (3). Suppose that $a \in \mathcal{A}$ is self-adjoint and that $\|a\| < 1$. Then, as remarked preceding the theorem, there exists a self-adjoint $b \in \mathcal{A}$ such that $1 - a = b^2$. If ω is a continuous linear functional on $\mathcal{A}$ such that $\|\omega\| = \omega(1)$, then the assumption in (2) implies that $1 - \omega(a) = \omega(1-a) = \omega(b^2)$ is real. Hence $\omega(a)$ is real. This implies that $\omega(a)$ is real, for all self-adjoint $a \in \mathcal{A}$, and for all continuous linear functionals ω on $\mathcal{A}$ such that $\|\omega\| = \omega(1)$. Since certainly every element of $\mathcal{A}$ can be written as $a_1 + ia_2$, for self-adjoint $a_1, a_2 \in \mathcal{A}$, this shows that $\mathcal{A} = \mathcal{A}_S + i\,\mathcal{A}_S$. Then the Vidav-Palmer theorem yields that the involution in that theorem, which agrees with the given one, together with the given norm, makes $\mathcal{A}$ into a C^*-algebra. □

In [2, Theorem 11.2.5], a number of equivalent criteria are given for a unital involutive Banach algebra—with a possibly discontinuous involution—to be a C^*-algebra, but positivity of certain continuous linear functionals is not among them. The proof above of such a criterion is made possible by the extra condition of the continuity of the involution. Although, given the Vidav-Palmer theorem, the proof is quite straightforward, we are not aware of a reference for this characterisation of C^*-algebras through positivity of linear functionals. Since the result seems to have a certain appeal, we thought it worthwhile to make it explicit.

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