A DYNAMICAL SYSTEM PERTURBED BY STOCHASTIC INTERVENTIONS-2D CASE

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Abstract. Deterministic dynamical systems that are subject to stochastic interventions are studied. Unlike previous works, the deterministic dynamical systems are perturbed at fixed time points by random jumps in state space, where the laws of these jumps can depend on the system's state just before intervention. As a conceivable example connected to modeling population dynamics in mathematical biology, we consider regular random harvesting from a deterministically growing population. The deterministic part of the model is given by *The Rosenzweig-MacArthur Model (RMM)*, and the stochastic interventions are controlled by means of a Markov operator. Under suitable parameter conditions, we show existence of a closed invariant ball for this model and show that the Markov operator restricted to that ball has a unique ergodic measure. Moreover, we show uniform convergence to the ergodic measure. The two main tools of our analysis are a recent improvement of the general lower bound technique, and a re-norming process. The first is used to show the asymptotic stability of the unique ergodic measure, the second is shown to ensure the invariance of the ball. We also give a full characterization of the support of the ergodic measure.

1. Introduction

In this article, we investigate deterministic dynamical systems that are stochastically perturbed by stochastic interventions where the *shape* of the distribution of the jumps (interventions) is state dependent, i.e. depends on the system's state

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just before the intervention. As for the time of the next jump, in general, two types of interventions may occur; random or fixed. In this work we consider the latter, where these times are equidistant. That is, we study the long term behaviour of a system, a population of individuals for instance, in which the state of the population is changing in time, described by some deterministic dynamical system, while at discrete fixed points in time quick interventions change its state after which it continues to evolve deterministically till the next time of intervention.

Modeling the dynamics of a population in biology one naturally encounters instantaneous perturbations as described above. Even when these perturbations are actually realized by short-term process, it is often convenient to neglect the duration of these perturbations and consider them to be instantaneous. This leads, in many cases, to a dynamical system of continuous trajectories with stochastic jump perturbations. Apparently, based on intervention times, we distinguish two extreme types of such systems. One with random jumps, i.e. the time of next intervention is random, with law dependent on the population state just after the last intervention, but the change in state is deterministic, as in models for cell division (cf. Lasota & Mackey [7]). In the other one, intervention times may be fixed, but the jumps are random with state-dependent distribution, as in a growing population of bacteria where samples are drawn regularly from the system in an experiment, or a lagoon from which fish and possibly shark are caught daily using a net as we consider here. Both type of stochastic interventions may be combined too.

It appears that such type of systems lacked mathematical attention over the past years, while a different type of deterministic systems have been studied extensively; for instance, stochastic differential equations. However, in such equations only the amplitude of the stochastic perturbations is allowed to be state dependent. The current state of the theory cannot accommodate for models in which also the shape of the distribution of the jumps may depend on state.

Following the approach for the one dimensional model in [1], we make a further step towards analyzing such models in higher dimensions and developing related tools. To that end we focus on a particular example, which is linked to application in biological population dynamics. Namely, we consider a population of predators and preys, fish and shark say, that reproduce, grow and die, and that is subject to occasional external intervention ('fishing') at a discrete set of time points in which part of the population is removed. The 'catch size' is random, but limited to a maximal amount. The distribution Q_x for the catch size depends on the numbers of individuals x present in the population at time just before the intervention. Population behavior in-between interventions is modeled deterministically using the Rosenzweig-MacArthur Model (RMM) (c.f. [5, 10]), which is biologically more realistic model than the well-known Lotka-Volterra predator-prey model, because it takes satiation and prey handling into account. Under suitable assumptions on Q_x, the catch size distribution, we establish a formula for the transition probabilities at intervention times, which indeed yield a Markov operator. Under conditions on the parameters of the model, we show existence of a closed invariant ball for this model and show that the Markov operator restricted to that ball has an invariant distribution for the population state, which is actually a unique ergodic measure. Moreover, we show uniform convergence to the ergodic measure from any distribution supported on the ball.

In our analysis, we employ equicontinuity conditions on the Markov operator (the 'e-property') and a recently improved form of the general lower bound technique, which could be found in [1, 11, 12, 14], to establish asymptotic stability of the ergodic measure (distribution). The invariant ball is actually a ball for an other metric than the Euclidean metric. We use a re-norming process tailored to the problem, which ensures the invariance of the ball. The major steps of the proof of the asymptotic stability in [1] concern the support of the distribution of the population just after the intervention times. In the one dimensional case considered there, the supports are intervals and their dynamics reduce to the dynamics of the end points. We show that these arguments can be replaced by investigations of the dynamics of a suitable map acting on the collection of closed bounded subsets. We use tools like the hyperspace of closed bounded subsets, the Hausdorff metric, and the Portmanteau theorem to give a characterization of the support of the ergodic measure. Below, we shall denote by $B(x, r)$ the open ball centered at x of radius $r > 0$ and $\overline{B}(x, r)$ the closed ball.

2. Model Description

Essentially, we will consider a deterministic dynamical system with stochastic interventions at fixed times with **equal length time intervals**. That is, a system which is subject to interventions at fixed, equally spaced discrete times $0 < t_1 < t_2 \ldots < t_n < \ldots$ at which the state is changed to a new state randomly. Put $\Delta t := t_{i+1} - t_i$. The law describing this new state depends on the state just before the intervention.

2.1. **The Rosenzweig-MacArthur Model (RMM).** The original version of Lotka-Volterra predator-prey model was built according to the assumption that the responses of the populations would be proportional to the product of their densities so that

$$\begin{aligned} dN/dt =& aN - bNP, \\ dP/dt =& cNP - dP, \end{aligned}$$

where N and P are the densities of prey and predator, respectively, a and d are their per-capita rate of change in the absence of each other, and b and c are their respective rates of change due to interaction.

The Rosenzweig-MacArthur Model (RMM) (c.f. [10]), sometimes called a Kolmogorov-type predator-prey model see for instance [5], includes logarithmic growth of the prey population and satiation of predation, modeled with a Holling type II functional response:

$$\dot{v} = rv\left(1 - \frac{v}{K}\right) - \frac{av}{b+v} \cdot p, \tag{1}$$

$$\dot{p} = -dp + h \cdot \frac{av}{b+v} \cdot p. \tag{2}$$

Where v and p stands for the population size of the victim (prey) and the predator, respectively. The biologically relevant parameter settings for this model are $a, b, d, h, r, K > 0$. Let us assume that these conditions hold. Moreover, we focus our attention on solutions in $\mathbb{R}_+^2$, since only these are biologically interpretable.

For the proof of Lemma 3 below, we will make use of the following Remark.

Remark 1. *(Gronwall's Lemma*, the integrable form, cf. [13]) Let the real function $\phi(t)$ be continuous for $0 \le t \le a$, and let $\phi(t) \le \alpha + \beta \int_0^t \phi(\tau)\, d\tau$ in $[0, a]$ with $\beta > 0$. Then $\phi(t) \le \alpha e^{\beta t}$ in $[0, a]$.

Sometimes the differentiable form of above lemma will be more useful.

Remark 2. *(Gronwall's Lemma*, the differentiable form) Let I denote any interval of the form $[a, b]$, $[a, b)$ or $[a, \infty)$. Let $u : I \longrightarrow \mathbb{R}$ be differentiable with $u(a) > 0$ and $f : I \longrightarrow \mathbb{R}$ be continuous. If $u' \le f(t)\, u(t)$, for all $t \in \text{interior}(I)$, then $u(t) \le u(a)\, e^{\int_a^t f(s) ds}$, for all $t \in I$.

Lemma 3. *Let* $\gamma := \frac{d}{ha-d}$. *For the RMM described by the equations* (1) *and* (2), *we have the following properties:*

(1) *Any solution that starts in* $\mathbb{R}_+^2$ *will remain in* $\mathbb{R}_+^2$ *for all time that this solution exists.*

(2) *Solutions to* (1) *and* (2) *that start in* $(v_0, p_0) \in \mathbb{R}_+^2$ *will satisfy*

$$p(t) \le p_0 e^{(ha-d)t}, \qquad v(t) \le v_0 e^{rt} \tag{3}$$

for all time t *for which the solution exists. Moreover, from the defining equations* (1)- (2) *it follows that the system is well posed.*

(3) *The steady states of* (1)- (2) *are* $(0,0), (K,0), \left(b\gamma, \frac{rb}{a}(1+\gamma)\left(1-\frac{b}{K}\gamma\right)\right)$. *Moreover, There is a steady state in* $\mathrm{Int}\left(\mathbb{R}_+^2\right)$ *if and only if* $a > \left(1+\frac{b}{K}\right)\cdot\frac{d}{h}$.

(4) *The steady state* $(K,0)$ *is stable if and only if* $a < \left(1+\frac{b}{K}\right)\cdot\frac{d}{h}$.

(5) *The trivial steady state* $(0,0)$ *is unstable for all parameter settings with* $a,b,d,h,r,K>0$.

(6) *If* $a<\frac{d}{h}$, *then any solution starting in* $\mathbb{R}_+^2\setminus\{0\}$ *will eventually converge to* $(K,0)$.

(7) *Suppose the condition mentioned in part 4 above holds, then the unique steady state in* $\mathrm{Int}\left(\mathbb{R}_+^2\right)$ *is stable if and only if* $\gamma<\frac{K}{b}<1+2\gamma$.

Proof. The proof of the first part of the lemma will be presented later. To prove the second statement we first note that the quantity $\frac{av}{b+v}\cdot p$ in equation (1) is a positive quantity and therefore $\frac{dv}{dt}\le rv\left(1-\frac{v}{K}\right)$. Also, $\left(1-\frac{v}{K}\right)\le 1$ for a positive solution. Thus, $\frac{dv}{dt}\le rv$. Thus, $v(t)\le v_0+r\int_0^t v(\tau)\,d\tau$. By Gronwall's lemma, $v(t)\le v_0\mathrm{e}^{rt}$. Furthermore, Equation 2 could be written as $\frac{dp}{dt}=\left(-d+ha\cdot\frac{v(t)}{b+v(t)}\right)p(t)$. Since $\frac{v(t)}{b+v(t)}\le 1$, $\frac{dp}{dt}\le(-d+ha)\,p(t)$. By Gronwall's lemma (differentiable form), $p(t)\le p_0\mathrm{e}^{(ha-d)t}$. Note that $(ha-d)$ may be less than zero and that is why we need the differentiable form of Gronwall's lemma. To compute the steady states of equations (1)- (2), we put $\frac{dv}{dt}=0$ and $\frac{dp}{dt}=0$. So we get:

$$v\left(r-\frac{rv}{K}-\frac{ap}{b+v}\right)=0 \tag{4}$$

$$p\left(-d+\frac{hav}{b+v}\right)=0 \tag{5}$$

Solving these equations yield three steady states; the trivial one $(v,p)=(0,0)$, $(K,0)$ and for the third one $-d+\frac{hav}{b+v}=0$ gives $v=\frac{bd}{ha-d}$, also $r-\frac{rv}{K}-\frac{ap}{b+v}=0$ gives $ap=(b+v)(r-\frac{rv}{K})$, or equivalently, $p=\frac{r}{a}(b+v)(1-\frac{v}{K})$. substituting $v=\frac{db}{ha-d}$ we get $p=\frac{r}{a}\left(b+\frac{db}{ha-d}\right)\left(1-\frac{db}{(ha-d)K}\right)$ which is the same thing as $p=\frac{rb}{a}\left(1+\frac{d}{ha-d}\right)\left(1-\frac{db}{(ha-d)K}\right)$. A steady state in $\mathrm{Int}\left(\mathbb{R}_+^2\right)$ occurs when $v>0$ and $p>0$. Thus, $v=\frac{db}{ha-d}>0$ which implies $ha-d>0$ or $a>\frac{d}{h}$. Also, $p=\frac{rb}{a}\left(1+\frac{d}{ha-d}\right)\left(1-\frac{db}{(ha-d)K}\right)>0$. Since $\frac{rb}{a}>0$ and $\left(1+\frac{d}{ha-d}\right)>0$,

$\left(1-\frac{db}{(ha-d)K}\right)>0$ which implies $\frac{haK-dK-db}{(ha-d)K}>0$. since $(ha-d)>0$, $haK>d(K+b)$ or equivalently $a>\left(1+\frac{b}{K}\right)\cdot\frac{d}{h}$. Moreover, if $a>\left(1+\frac{b}{K}\right)\cdot\frac{d}{h}$, which implies $a>\frac{d}{h}$, then $v>0$ and $p>0$. That is, the steady state is in Int $\left(\mathbb{R}_+^2\right)$. For the fourth statement, we need the Jacobian matrix

$$J=\begin{bmatrix}\frac{\partial F_1}{\partial v} & \frac{\partial F_1}{\partial p}\\ \frac{\partial F_2}{\partial v} & \frac{\partial F_2}{\partial p}\end{bmatrix}=\begin{bmatrix}r\left(1-\frac{2v}{K}\right)-\frac{abp}{(b+v)^2} & \frac{-av}{b+v}\\ \frac{abhp}{(b+v)^2} & -d+\frac{ahv}{b+v}\end{bmatrix}.$$

Assume that the steady state $(K,0)$ is stable. One has

$$J\left(K,0\right)=\begin{bmatrix}-r & \frac{aK}{b+K}\\ 0 & \frac{-db-dK+haK}{b+K}\end{bmatrix}.$$

The characteristic equation is

$$\mid J\left(K,0\right)-\lambda I\mid=\begin{vmatrix}-r-\lambda & \frac{aK}{b+K}\\ 0 & \frac{-db-dK+haK}{b+K}-\lambda\end{vmatrix}=0.$$

So, $\lambda_1=-r<0$. $\lambda_2=\frac{-db-dK+haK}{b+K}$. Since the steady state $(K,0)$ is stable, λ_2 should be less than zero. Hence, $\frac{-d(b+K)+haK}{b+K}<0$. Since $b+K>0$, $-d(b+K)+haK<0$. Consequently, $a<\left(1+\frac{b}{K}\right)\cdot\frac{d}{h}$. Moreover, if we start by assuming that $a<\left(1+\frac{b}{K}\right)\cdot\frac{d}{h}$, then the Jacobian matrix at $(K,0)$ is given by the above formula with eigenvalues $\lambda_1=-r<0$, $\lambda_2=-d+\frac{haK}{b+K}$. Since $a<\left(1+\frac{b}{K}\right)\cdot\frac{d}{h}$, $\lambda_2<0$. i.e $(K,0)$is stable. For the fifth part, we evaluate the Jacobian matrix at $(0,0)$ which is

$$J(0,0)=\begin{bmatrix}r & 0\\ 0 & -d\end{bmatrix}.$$

Thus, the eigenvalues are $\lambda_1=r$, $\lambda_2=-d$. Since $r,d>0$, the sign of the eigenvalues will always differ. Hence, the fixed point (steady state) is a saddle point. therefore, it is unstable. For the sixth part, we solve explicitly the initial value problem that corresponds with equation (1) and then we evaluate the limit as t tends to infinity. Since $\frac{dv}{dt}\leq rv\left(1-\frac{v}{K}\right)$, we consider the initial value problem $\frac{dv}{dt}=rv\left(1-\frac{v}{K}\right)$ in

$(0,\infty)$ with $v(0)=v_0$. Using the separation of variables technique we get $\frac{v(t)}{1-\frac{v(t)}{K}} = C\mathrm{e}^{rt}$, where C is a constant. Or $v(t) = \frac{1}{\frac{1}{C}\mathrm{e}^{-rt}+\frac{1}{K}}$. with $t=0$, $v(0)=v_0$ we get $\frac{1}{C} = \frac{1}{v_0} - \frac{1}{K}$. Thus, $v(t) = \frac{Kv_0\mathrm{e}^{rt}}{K-v_0+v_0\mathrm{e}^{rt}}$, or simply

$$v(t) = \left(\frac{1}{K} + \left(\frac{1}{v_0} - \frac{1}{K}\right)\mathrm{e}^{-rt}\right)^{-1}.$$

Hence, $\lim_{t\to\infty} v(t) = K$. For the other convergence we use what we show above, $p(t) \le p_0\mathrm{e}^{(ha-d)t}$. If $a < \frac{d}{h}$, then $ha-d<0$. Thus, $\lim_{t\to\infty} p(t) \le \lim_{t\to\infty} \frac{p_0}{\mathrm{e}^{(ha-d)t}} = 0$. As a result of these calculations we conclude that any solution starting in $\mathbb{R}^2_+\setminus\{0\}$ will eventually converge to $(K,0)$, on condition $a<\frac{d}{h}$. To prove the last part of the lemma, we firstly evaluate the Jacobian matrix at the unique steady state in the interior of $\left(\mathbb{R}^2_+\right)$, $(v^*,p^*) = \left(\frac{bd}{ha-d}, \frac{rb}{a}\left(1+\frac{d}{ha-d}\right)\left(1-\frac{bd}{(ha-d)K}\right)\right)$, which is given by

$$J(v^*,p^*) = \begin{bmatrix} r(1-2\frac{v^*}{K}) - \frac{abp^*}{(b+v^*)^2} & \frac{-av^*}{b+v^*} \\ \frac{abhp^*}{(b+v^*)^2} & -d+\frac{ahv^*}{b+v^*} \end{bmatrix}.$$

Since $\frac{b}{b+v^*} = \frac{ha-d}{ha} = 1-\frac{d}{ha}$, $\frac{bv^*}{b+v^*} = \frac{ha-d}{ha}\cdot\frac{bd}{ha-d} = \frac{bd}{ha}$ and $\frac{v^*}{b+v^*} = \frac{d}{ha}$. By 4, $\frac{ap^*}{b+v^*} = r\left(1-\frac{v^*}{K}\right)$. Thus, the above matrix becomes

$$J(v^*,p^*) = \begin{bmatrix} \frac{rd}{ha}\left(1-\frac{v^*}{K}\right) - \frac{rv^*}{K} & \frac{-d}{h} \\ r\left(h-\frac{d}{a}\right)\left(1-\frac{v^*}{K}\right) & 0 \end{bmatrix}. \tag{6}$$

Therefore, the determinant of $J(v^*,p^*) - \lambda I$ is given by

$$\begin{aligned} \mid J(v^*,p^*) - \lambda I \mid &= \begin{vmatrix} \frac{rd}{ha}\left(1-\frac{v^*}{K}\right) - \frac{rv^*}{K} - \lambda & \frac{-d}{h} \\ r\left(h-\frac{d}{a}\right)\left(1-\frac{v^*}{K}\right) & -\lambda \end{vmatrix} \\ &= \lambda^2 - \lambda\left(\frac{rd}{ha}\left(1-\frac{v^*}{K}\right) - \frac{rv^*}{K}\right) + \frac{rd}{h}\left(h-\frac{d}{a}\right)\left(1-\frac{v^*}{K}\right). \end{aligned}$$

For simplicity, let $C = \frac{rd}{ha}\left(1-\frac{v^*}{K}\right)$. Then $-\frac{rv^*}{K} = \frac{haC}{d} - r$. Hence, $\mid J(v^*,p^*) - \lambda I \mid = \lambda^2 - \lambda\left(C - r + \frac{haC}{d}\right) + C(ah-d)$. The discriminant of the quadratic equation $\lambda^2 - \lambda\left(C-r+\frac{haC}{d}\right) + C(ah-d) = 0$ is $D = \left(C-r+\frac{haC}{d}\right)^2 - 4C(ah-d)$. [If λ_1, λ_2 are the solutions of the equation, then $\lambda_i = \frac{\left(C-r+\frac{haC}{d}\right)\pm\sqrt{D}}{2}$. Moreover, the equation could be written as $\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2]$. If $D \geq 0$, then we have two real roots λ_1, λ_2 satisfying $\lambda_1\lambda_2 = C(ah-d)$. So, if $C(ah-d) < 0$, then either λ_1 or λ_2 is greater than zero and, therefore, we have an unstable steady state. While if $C(ah-d) > 0$, then $\lambda_1$and λ_2 have the same sign, which would be negative if the sum $\lambda_1 + \lambda_2 = \left(C - r + \frac{haC}{d}\right)$ is negative and vice versa. This indicates that the interior steady state is stable if $\left\{D \geq 0,\, C(ah-d) > 0, \text{ and } \left(C - r + \frac{haC}{d}\right) < 0\right\}$. Since $a > \left(1 + \frac{b}{K}\right) \cdot \frac{d}{h}$, $a > \frac{d}{h}$. So the condition $C(ah-d) > 0$ yields $C > 0$. On the other hand, if $D < 0$, then the quadratic equation has two complex roots $\lambda_1 = \overline{\lambda_2}$. Thus, $\mathrm{Re}\lambda_1 = \mathrm{Re}\lambda_2$ (i.e they have the same sign which is the sign of the term $\left(C - r + \frac{haC}{d}\right)$). This indicates that the interior steady state is stable if $\left\{D < 0, \text{ and } \left(C - r + \frac{haC}{d}\right) < 0\right\}$. $D < 0$ implies $C(ah-d) > \frac{1}{4}\left(C - r + \frac{haC}{d}\right)^2 \geq 0$. Again, since $(ah-d) > 0$, $C > 0$ .Consequently, what ever the sign of D is, there exist a stable steady state if and only if $C > 0$ and $\left(C - r + \frac{haC}{d}\right) < 0$ (in addition to the pre-mentioned condition $a > \frac{d}{h}$). $C > 0$ gives $\frac{d}{ha-d} < \frac{K}{b}$ and $\left(C - r + \frac{haC}{d}\right) < 0$ gives $\frac{rd}{ha}\left(1 - \frac{v^*}{K}\right) - \frac{rv^*}{K} < 0$, or $\frac{d}{ha} < \left(1 + \frac{d}{ha}\right)\frac{bd}{(ha-d)K}$, i.e $\frac{K}{b} < \frac{ha+d}{ha-d}$. Combining the conditions together we get $\frac{d}{ha-d} < \frac{K}{b} < \frac{ha+d}{ha-d}$. □

We would prove the existence and uniqueness of a solution to the RMM with the initial value (v_0, p_0) at $t = 0$.

Lemma 4. *The (semilinear) initial value problem representing the RMM*

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R}_+, \quad u(0) = u_0 \tag{7}$$

where $u(t) = \begin{bmatrix} v(t) \\ p(t) \end{bmatrix}$, $A = \begin{bmatrix} r & 0 \\ 0 & -d \end{bmatrix}$, $f : [0, \infty) \times \mathbb{R}_+^2 \longrightarrow \mathbb{R}^2$,

$$f(t, u(t)) = \begin{cases} \frac{-rv^2}{K} - \frac{av}{b+v}p \\ h \cdot \frac{av}{b+v} \cdot p \end{cases}$$

has a unique solution.

Proof. We will make use of the theory of operator semigroups. First note that the initial value problem (IVP) (7) does not necessarily have a solution of any kind. However, if it has (a classical or strong) solution then this solution u satisfies the integral equation

$$u(t) = T(t)\,u_0 + \int_0^t T(t-s)\,f(s,u(s))\,ds, \tag{8}$$

but not conversely since a solution of (8) is not necessarily differentiable. (Where $(T(t))_{t\geq 0}$ is a C_0-semigroup on a Banach space X generated by A). Furthermore, the continuous solution of (8) will be called a *mild* solution of the IVP. A mild solution is thus a kind of generalized solution. Moreover, u is a mild solution of IVP if and only if u is a fixed point of an operator S, defined as follows:

$$(Su)(t) = T(t)\,u_0 + \int_0^t T(t-s)\,f(s,u(s))\,ds.$$

Moreover, if $f : [t_0, T] \times X \longrightarrow X$ is continuously differentiable from $[t_0, T] \times X$ into X, then the mild solution of IVP with $u_0 \in D(A)$ is a classical solution of IVP. A theorem (c.f. A. Pazy [9]) states that if $f : [0, \infty) \times X \longrightarrow X$ is continuous in t for $t \geq 0$ and locally Lipschitz continuous in u, uniformly in t on bounded intervals and A is the infinitesimal generator of C_0-semigroup $T(t)$ on X then for every $u_0 \in X$ there is a $t_{\max} \leq \infty$ such that the initial value problem

$$u'(t) = Au(t) + f(t, u(t)), \qquad t \geq 0, \quad u(0) = u_0$$

has a unique mild solution u on $[0, t_{\max})$. Moreover, (c.f. Engel and Nagel [3]) for any $A \in M_n(\mathbb{C})$ and $t \geq 0$, $T = \left(\mathrm{e}^{tA}\right)_{t\geq 0}$ is the (uniformly continuous) one parameter semigroup generated by A. In addition, if A is a diagonal matrix, i.e $A = \mathrm{diag}(a_1, ..., a_n)$, then $\mathrm{e}^{tA} = \mathrm{diag}(\mathrm{e}^{ta_1}, ..., \mathrm{e}^{ta_n})$. In our case, A and f satisfies the required conditions in the (above stated) theorem. Thus, there exists a unique solution. □

Coming back to the first assertion of Lemma 3, the uniqueness of the solutions discussed above implies that solutions cannot intersect unless they are equal. Also, P and V axes consists of solutions. This implies that solutions that start in $\mathbb{R}_+^2$ will remain in $\mathbb{R}_+^2$ for all time that this solution exists. Otherwise, if a solution starts in $\mathbb{R}_+^2$ and part of the trajectory of the solution is outside $\mathbb{R}_+^2$, then the solution should intersect at least one of the axes, i.e this solution will intersect with another solution on the axes, and this is impossible.

We denote by ϕ_t the semi-flow (solution operator) associated to (1) and (2):

$$\phi_t\left(v_0, p_0\right) := \left(v\left(t\right), p\left(t\right)\right).$$

There is no explicit expression for ϕ_t.

2.2. **Stochastic Interventions.** We will consider stochastic interventions at fixed equally spaced times t. The distribution for the jump to the new state after intervention will be state dependent. We suppose that at the intervention times the number of individuals of each type is diminished by a random amount, the *catch size*. We need the following crucial assumptions on the law Q_x for the catch size, where $x := (v, p) \in \text{Int}\left(\mathbb{R}_+^2\right)$ is the population size just before the intervention. We call Q_x the *catch size distribution.* Let $C \subseteq \mathbb{R}_+^2$ be a non-empty compact set and $C = \overline{\text{Int}\,(C)}$ with $0 \in C$ and let $S = \left\{x : x - C \subseteq \text{Int}\left(\mathbb{R}_+^2\right)\right\}$.

A1): The support of Q_x is C, for each $x \in S$.

A2): Q_x has density q_x w.r.t. Lebesgue measure on $\mathbb{R}^2$, for each $x \in S$.

A3): The map $x \mapsto q_x\left(\cdot\right) : S \to \mathrm{L}^1\left(\mathbb{R}^2\right)$ is continuous.

2.3. **Introduction of the transition probability P.** The evolution of the system between the interventions is given by the RMM, so the population just before the next intervention will be $x' = \phi_{\Delta t}\left(x\right)$ where $x = (v, p)$ is the state just after the previous intervention. Then the population y just after the intervention will be in a Borel set $A \subseteq [0, \infty) \times [0, \infty)$ with probability $Q_{x'}\left(x' - A\right)$. In fact, we end in

A if and only if the subtracted amount is $x' - y$ with $y \in A$. This happens with probability $Q_{x'}(x' - A)$. Thus, the distribution of the population size just after the next intervention is $Q_{\phi_{\Delta t}(x)}(\phi_{\Delta t}(x) - \cdot)$. Furthermore, if the population size just after the n-th intervention would have distribution μ, then the distribution just after the $(n+1)$- intervention equals $\int_{x \in \mathbb{R}_+^2} Q_{\phi_{\Delta t}(x)}(\phi_{\Delta t}(x) - A)\, d\mu(x)$. We denote this transition by P:

$$(P\mu)(A) = \int_{\mathbb{R}_+^2} Q_{\phi_{\Delta t}(x)}(\phi_{\Delta t}(x) - A)\, d\mu(x)\,, \tag{9}$$

or simply

$$(P\mu)(A) = \int_S p(x, A)\, d\mu(x)\,, \quad \mu \in \mathcal{M}^+(S)$$

where $p(x, A) = Q_{\phi_{\Delta t}(x)}(\phi_{\Delta t}(x) - A)$, with $A \subseteq \mathbb{R}_+^2$ Borel. Observe that $P : \mathcal{M}^+(S) \to \mathcal{M}^+(S)$ is a Markov operator with transition kernel[1] $p(x, \cdot) = P\delta_x = Q_{\phi_{\Delta t}(x)}(\phi_{\Delta t}(x) - \cdot)$. So,

$$(P\mu)(A) = \int_S (P\delta_x)(A)\, d\mu(x)\,.$$

Therefore it suffices to understand the iterates $(P^n\delta_x)_n$ in order to determine the dynamics of P.

3. Existence and stability of a non-trivial ergodic measure

We start with the following simple observation:

Lemma 5. *If $g : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 near 0, and $Dg(0) = 0$, then g is locally Lipschitz near 0, and for all $\epsilon > 0$, there exists R such that for any x, y satisfying $\| x \|, \| y \| \leq R$, then*

$$\| g(x) - g(y) \| \leq \epsilon \| x - y \|\,.$$

[1]A function $p : S \times \mathcal{B}(S) \longrightarrow [0, 1]$, defined as $p(x, E) = P\delta_x(E)$ for $x \in S$ and $E \in \mathcal{B}(S)$ is called the *transition function* (transition kernel). (c.f. Ethier and Kurtz, [4]).

Proof. As g is C^1, on a ball centered at 0 with radius R,

$$\| g(x) - g(y) \| \leq \sup_{\|z\| \leq R} \| Dg(z) \| \| x - y \|$$

for $\| x \|, \| y \| \leq R$. Since $Dg(0) = 0$ and $z \mapsto Dg(z)$ is continuous, for $\epsilon > 0$ there exists R such that $\| Dg(z) \| < \epsilon$ for all z with $\| z \| \leq R$. □

The dynamics of a differential equation of the form

$$\begin{cases} z'(s) = & Az(s) + g(z(s)) \\ z(0) = & x \end{cases}, \tag{10}$$

where A is an $n \times n$ matrix and g a smooth function with $g(0) = 0$ and $Dg(0) = 0$ need not be contractive near 0, even if the eigenvalues of A have negative real parts. For instance, it could be that a solution $t \mapsto z(t, x)$ follows an elliptic spiral orbit so that its distance to zero may increase before converging to zero. We costruct a norm on $\mathbb{R}^n$, depending on A, for which the dynamics are contracting. We start with a simple Lemma.

Lemma 6. *Let A be an $n \times n$ matrix, let $g : \mathbb{R}^n \to \mathbb{R}^n$ be C^1 near 0 with $g(0) = 0$ and $Dg(0) = 0$, let $z(s, x)$ be the solution of (10). Then*

$$\| x \|_A := \sup_{t \geq 0} \| e^{At} x \|, \quad x \in \mathbb{R}^n$$

defines a norm on $\mathbb{R}^n$ such that for every $\epsilon > 0$ there exists an $R_\epsilon > 0$ such that $\| g(x) - g(y) \|_A \leq \epsilon \| x - y \|_A$ for all $x, y \in \overline{B_A}(0, R_\epsilon)$, (i.e. $x, y \in \mathbb{R}^n$ with $\| x \|_A \leq R_\epsilon$, $\| y \|_A \leq R_\epsilon$). Then

$$\| z(t, x_1) - z(t, x_0) \|_A \leq \| x_1 - x_0 \|_A \, e^{(\epsilon + w)t},$$

for all $t \geq 0$, and $x_i \in \overline{B_A}(0, R_\epsilon)$ where w is the maximum of the real parts of the eigenvalues of A.

Proof. According to Pazy, for each $x \in \mathbb{R}^n$ there exists a mild solution of (10) of the form (8),

$$z(t,x) = \mathrm{e}^{At}x + \int_0^t \mathrm{e}^{A(t-s)} g(z(s,x))\,ds.$$

Note that $\|\cdot\|_A$ is a norm (c.f. [2], Lemma II.3.10, p78) and $\| \mathrm{e}^{At}x \|_A \le \mathrm{e}^{wt} \| x \|$ where w is the maximum of the real parts of the eigenvalues of A (c.f. [2], Corollary IV.3.12, P281). Let $\epsilon > 0$ and take $R_0 > 0$ such that $\| g(x) - g(y) \|_A \le \epsilon \| x - y \|_A$ for all x, y with $\| x \|, \| y \| \le R_0$. If $\| x \|, \| y \| \le \frac{1}{2}R_0$ then there exists $T > 0$ such that $\| z(t,x) \| \le R_0$ and $\| z(t,y) \| \le R_0$ for all $t \in [0,T]$. Choose $R \in (0, \frac{1}{2}R_0]$ such that $\{x : \| x \|_A \le R\} \subseteq \left\{x : \| x \| \le \frac{1}{2}R_0\right\}$. For x and y such that $\| x \|_A \le R$ and $\| y \|_A \le R$ $t \in [0,T]$ we have

$$z(t,x_1) - z(t,x_0) = \mathrm{e}^{At}(x_1 - x_0) + \int_0^t \mathrm{e}^{A(t-s)} \left(g(z(s,x_1)) - g(z(s,x_0))\right) ds,$$

so

$$\begin{aligned}
\| z(t,x_1) - z(t,x_0) \|_A \le & \| \mathrm{e}^{At}(x_1 - x_0) \|_A \\
& + \int_0^t \| \mathrm{e}^{A(t-s)} \|_A \| g(z(s,x_1)) - g(z(s,x_0)) \|_A \,ds \\
& \le \mathrm{e}^{wt} \| x_1 - x_0 \|_A + \int_0^t \mathrm{e}^{w(t-s)} \epsilon \| z(s,x_1) - z(s,x_0) \|_A \,ds.
\end{aligned}$$

Thus, multiplying by $\mathrm{e}^{(-w)t}$,

$$\mathrm{e}^{(-w)t} \| z(t,x_1) - z(t,x_0) \|_A \le \| x_1 - x_0 \|_A + \epsilon \int_0^t \mathrm{e}^{(-w)t} \mathrm{e}^{w(t-s)} \| z(s,x_1) - z(s,x_0) \|_A \,ds,$$

or simply

$$\mathrm{e}^{(-w)t} \| z(t,x_1) - z(t,x_0) \|_A \le \| x_1 - x_0 \|_A + \epsilon \int_0^t \mathrm{e}^{-ws} \| z(s,x_1) - z(s,x_0) \|_A \,ds.$$

By Gronwall's Lemma, $\mathrm{e}^{(-w)t} \parallel z(t,x_1) - z(t,x_0) \parallel_A \leq \parallel x_1 - x_0 \parallel_A \mathrm{e}^{\epsilon t}$ hence,

$$\parallel z(t,x_1) - z(t,x_0) \parallel_A \leq \parallel x_1 - x_0 \parallel_A \mathrm{e}^{(\epsilon+w)t}.$$

□

Corollary 7. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous such that*

$$x'(t) = f(x(t)) \tag{11}$$

has a steady state x^*. Let $\Delta t > 0$. If f is C^1 near x^* and all eigenvalues of $A = (Df)(x^*)$ have negative real part, then there exists a norm $\parallel \cdot \parallel_A$ on $\mathbb{R}^n$, a $\theta \in [0,1)$, and an $R^* > 0$ such that the closed $\parallel \cdot \parallel_A$-ball $\overline{B}_A(x^*, R^*)$ is invariant under $\phi_{\Delta t}$ and

$$\parallel \phi_{\Delta t}(x) - \phi_{\Delta t}(y) \parallel_A \leq \theta \parallel x - y \parallel_A \tag{12}$$

for all $x, y \in \overline{B}_A(x^*, R^*)$, where $\phi_{\Delta t}(x)$ is the solution of (11) at time Δt, starting at x at time 0.

Proof. The deviation from equilibruim, $z(t) = x(t) - x^*$, satisfies (10) with $A = Df(x^*)$ and $g(z) = f(z) - Df(x^*)$. Let w be the spectral bound of A, choose $\epsilon > 0$ so small that $\theta := \mathrm{e}^{(\epsilon+w)t} < 1$. By shifting x^* to 0, we infer from Lemma 6 that there exists an $R^* > 0$ such that (12) holds for all $x, y \in \overline{B}_A(x^*, R^*)$. Since $\phi_{\Delta t}(x^*) = x^*$, this ball is invariant under $\phi_{\Delta t}$ by (12). Note that $\epsilon + w < 0$. Taking $\theta := \mathrm{e}^{(\epsilon+w)\Delta t} < 1$, for $t = \Delta t$. Hence, $\phi_{\Delta t}$ is a strict contraction on sufficiently small A-ball around x^*. □

Remark 8. With suitable adaptions, the arguments above also hold in a Banach space setting:

Let $(X, \parallel \cdot \parallel)$ be a Banach space, A the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on X with growth bound $w < 0$, let $g : X \to X$ be such that for every $\epsilon > 0$ there exists $R > 0$ such that for $\parallel x \parallel \leq R$, then $\parallel g(x) \parallel \leq \epsilon \parallel x \parallel$. If

$t \mapsto z(t,x)$ is the mild solution of

$$\begin{cases} z'(t) = & Az(t) + g(z(t)) \\ z(0) = & x \end{cases}, \tag{13}$$

then there exists an equivalent norm $\| \cdot \|_A$, an $\alpha > 0$ and an $R_0 \geq 0$ such that for all $x \in X$ with $\| x \|_A \leq R_0$, the mild solution $z(t,x)$ to (13) exists for all time and satisfies $\| z(t,x) \|_A \leq \| x \|_A \, \mathrm{e}^{-\alpha t}$.

We are going to apply the next Lemma to $\Omega = B(x^*, R)$, where $x^* = (v,p)$ is the steady state in $\mathrm{Int}\left(\mathbb{R}^2_+\right)$, $R > 0$. Recall that the diameter of a subset C of a metric space (S,d) is defined by $\mathrm{diam}(C) := \sup\{d(x,y) : x, y \in S\}$.

If $(X, \| \cdot \|)$ is a Banach space, $\phi : X \to X$, and $\psi(E) := \phi(E) - C$, then ψ is monotonic. That is, if $A \subset B$, then $\psi(A) \subset \psi(B)$. Remark that $\psi(A) = \phi_{\Delta t}(A) - C \subset \phi_{\Delta t}(B) - C = \psi(B)$. Also, if ϕ is a contraction, then so is ψ, as we will see in Proposition 18.

The next lemma applies to the the situation of Corollary 7.

Lemma 9. *Consider a normed space $(S, \| \cdot \|)$, a bounded subset $C \subseteq S$ with $0 \in C$, $x^* \in S$, and $R > 0$. Let $\Omega = B(x^*, R)$ be invariant under a map $\phi : \Omega \to \Omega$ and such that for some $0 \leq \theta < 1$, $d(\phi(x), x^*) \leq \theta d(x, x^*)$, for all $x \in \Omega$, where d is the metric induced by $\| \cdot \|$. If $\mathrm{diam}(C) < (1-\theta) R$, then $\phi(x) - C \subseteq B(x^*, R)$ for all $x \in \Omega$. Thus, if $E \subset B(x^*, R)$, then $\psi(E) \subset B(x^*, R)$.*

Proof. For $A \subseteq S$, define

$$\delta(A, x^*) := \sup_{x \in A} d(x, x^*),$$

We will show that $\delta(\phi(x) - C, x^*) \leq R$ for all $x \in \Omega$. For this purpose

$$\begin{aligned}\delta(\phi(x) - C, x^*) &= \sup_{y \in C} \delta(\phi(x) - y, x^*) \\ &\leq \sup_{y \in C} \{\delta(\phi(x) - y, \phi(x)) + \delta(\phi(x), x^*)\} \\ &= \delta(\phi(x), x^*) + \sup_{y \in C} \delta(\phi(x) - y, \phi(x)) \\ &\leq \delta(\phi(x), x^*) + \operatorname{diam} C \leq \theta d(x, x^*) + \operatorname{diam} C\end{aligned}$$

If $\operatorname{diam}(C) \leq (1-\theta) R$ and $x \in \Omega = B(x^*, R)$, then $d(\phi(x) - C, x^*) < \theta R + (1-\theta) R = R$. Thus, $\phi_{\Delta t}(x) - C \subset B(x^*, R)$. □

Remark 10. (The portmanteau Theorem): If $P^n \delta_{x_0} \to \mu^*$, then for U open,

$$\liminf_{n \to \infty} P^n \delta_{x_0}(U) \geq \mu^*(U),$$

and for C closed,

$$\limsup_{n \to \infty} P^n \delta_{x_0}(C) \leq \mu^*(C).$$

Moreover, if E is Borel measurable, such that $\mu^*\left(\overline{E} \setminus E\right) = 0$, then $P^n \delta_{x_0}(E) \to \mu^*(E)$. ([8], Theorem 6.1, Ch. II, p.40).

Consider the situation of Corollary 7. Let C be a set with $\operatorname{diam} C < (1-\theta) R^*$, and $0 \in C$. Let us take x_0 as a starting point, let $\overline{B}_A(x^*, R^*)$ be the invariant closed ball (in the new norm) such that $x_0 \in \overline{B}_A(x^*, R^*)$. Let $x_n := \phi_{n\Delta t}(x_0)$. Then $\psi(\{x_0\}) := \phi_{\Delta t}(x_0) - C \subset \overline{B}_A(x^*, R^*)$ by construction–see Lemma 9. Since $0 \in C$, $x_1 := \phi_{\Delta t}(x_0) \in \phi_{\Delta t}(x_0) - C$ (if $C = \{0\}$ then $\phi_{\Delta t}(x_0) \in \phi_{\Delta t}(x_0)$). Moreover, $\phi_{\Delta t}(x_1) \in \phi_{\Delta t}(\phi_{\Delta t}(x_0) - C)$ and also $\phi_{\Delta t}(x_1) \in \phi_{\Delta t}(\phi_{\Delta t}(x_0) - C) - C$ and so on. Thus, the map ψ, defined as

$$\psi : S \supset E \mapsto \phi_{\Delta t}(E) - C,$$

is monotonic whenever C is compact and non-empty, where $\psi(E)$ is the possible population compositions after the first fishing when starting with population in

E. Based on this notation, starting at x_0, after n-fishing events, the population composition is in $\psi^n(\{x_0\})$. Note that $\psi^n(\{x_0\})$ contains x_n.

Let us return to the RMM with intervention distribution Q_x and Markov operator P as given in Section 2.

Lemma 11. *For all* $x_0 \in S$*,* $\operatorname{supp}(P\delta_{x_0}) = \phi_{\Delta t}(x_0) - C =: \psi(\{x_0\})$.

Proof. By definition of ψ, $\psi(\{x\}) = \phi_{\Delta t}(x) - C$, and by definition of P,

$$\begin{aligned}(P\delta_{x_0})(A) =& \int\limits_{\mathbb{R}^2_+} Q_{\phi_{\Delta t}(x)}(\phi_{\Delta t}(x) - A)\, d\delta_{x_0}(x) \\ =& Q_{\phi_{\Delta t}(x_0)}(\phi_{\Delta t}(x_0) - A).\end{aligned}$$

Let $z \in \phi_{\Delta t}(x_0) - C$, then $z = \phi_{\Delta t}(x_0) - c$ for some $c \in C$. Let $r > 0$.

$$\begin{aligned}P\delta_{x_0}(B(z,r)) =& Q_{\phi_{\Delta t}(x_0)}(\phi_{\Delta t}(x_0) - B(z,r)) \\ =& Q_{\phi_{\Delta t}(x_0)}(B(\phi_{\Delta t}(x_0) - z, r)) \\ =& Q_{\phi_{\Delta t}(x_0)}(B(c,r)) > 0,\end{aligned}$$

the last step is due to the fact that $c \in C = \operatorname{supp}\left(Q_{\phi_{\Delta t}(x_0)}\right)$. So, $z \in \operatorname{supp}(P\delta_{x_0})$. Hence, $\phi_{\Delta t}(x_0) - C \subseteq \operatorname{supp}(P\delta_{x_0})$. For the other inclusion, let $z \notin \phi_{\Delta t}(x_0) - C$. Then, as C is closed, there exists $r > 0$ such that

$$B(z,r) \cap (\phi_{\Delta t}(x_0) - C) = \phi.$$

That is $B(z,r) \subseteq (\phi_{\Delta t}(x_0) - C)^c$. Hence, if $x \in B(\phi_{\Delta t}(x_0) - z, r)$, then

$$\phi_{\Delta t}(x_0) - x \in \phi_{\Delta t}(x_0) - B(\phi_{\Delta t}(x_0) - z, r) = B(z,r) \subseteq (\phi_{\Delta t}(x_0) - C)^c.$$

So $x \notin C$. Hence, $B(\phi_{\Delta t}(x_0) - z, r) \subseteq C^c$. Thus, $Q_{\phi_{\Delta t}(x_0)}(B(\phi_{\Delta t}(x_0) - z, r)) = 0$, and therefore $P\delta_{x_0}(B(z,r)) = 0$. Thus, $z \notin \operatorname{supp}(P\delta_{x_0})$. Combining the two inclusions we get the result. □

Lemma 12. *For every ball B, in the Euclidean norm or $\|\cdot\|_A$, the map*

$$y \mapsto P\delta_y(B) : S \to [0,1]$$

is continuous.

Proof. Let $y_0 \in S$ and let $(y_n) \in S$ be such that $y_n \to y_0$. Then $\delta_{y_n} \to \delta_{y_0}$ in $\mathcal{M}(S)_{BL}$. Since P is Markov-Feller, $P\delta_{y_n} \to P\delta_{y_0}$. Having $P\delta_{y_0}\left(\overline{B} \setminus B\right) = 0$, since $Q_{\phi_{\Delta t}(y_0)}$ has density with respect to the Lebesgue measure on $\mathbb{R}^n$, the Portmanteau Theorem (Remark (10)) yields $P\delta_{y_n}(B) \to P\delta_{y_0}(B)$. Hence, the map $y \mapsto P\delta_y(B)$ is continuous at y_0. □

Using induction we get the following Lemma

Lemma 13. *For each $n \in \mathbb{N}$, one has the following:*

(1) $\operatorname{supp}(P^n\delta_{x_0}) = \psi^n(\{x_0\})$.
(2) If $x_n = \phi^n_{\Delta t}(x_0)$, then $x_n \in \operatorname{supp}(P^n\delta_{x_0})$.

Proof. (1) Let $A_n := \operatorname{supp}(P^n\delta_{x_0})$. Let $y_0 \in A_n$, $r > 0$ and $z_0 \in \psi(\{y_0\})$. Put $B = B(z_0, r)$. Since $\operatorname{supp}(P\delta_{y_0}) = \phi_{\Delta t}(y_0) - C$, $P\delta_{y_0}(B) > 0$. By the continuity of the map $y \mapsto P\delta_y(B)$ at y_0, Lemma (12), there exists U_0 an open neighbourhood of y_0 such that $P\delta_y(B) \geq \alpha > 0$ for all $y \in U_0$. Then

$$\begin{aligned}
P^{n+1}\delta_{x_0}(B) =& P(P^n\delta_{x_0})(B) = P\left(\int_{\mathbb{R}^2_+} \delta_y[P^n\delta_{x_0}](dy)\right)(B) \\
&= \left(\int_{A_n} P\delta_y[P^n\delta_{x_0}](dy)\right)(B) \\
&= \int_{A_n} P\delta_y(B)\cdot[P^n\delta_{x_0}](dy) \\
&\geq \int_{U_0} P\delta_y(B)\cdot[P^n\delta_{x_0}](dy) \geq \alpha\cdot[P^n\delta_{x_0}](U_0) > 0.
\end{aligned}$$

The last step is due to having $y_0 \in U_0$ and $y_0 \in \operatorname{supp}(P^n \delta_{x_0})$. Since r was arbitrary, $z_0 \in \operatorname{supp}\left(P^{n+1}\delta_{x_0}\right)$. Thus,

$$\bigcup_{y_0 \in A_n} \psi\left(\{y_0\}\right) \subset \operatorname{supp}\left(P^{n+1}\delta_{x_0}\right),$$

and since $\operatorname{supp}\left(P^{n+1}\delta_{x_0}\right)$ is closed,

$$\overline{\bigcup_{y_0 \in A_n} \psi\left(\{y_0\}\right)} \subset \operatorname{supp}\left(P^{n+1}\delta_{x_0}\right).$$

For the other inclusion, we will use a contradiction argument. Let $z \notin \overline{\bigcup_{y_0 \in A_n} \psi\left(\{y_0\}\right)}$. Then there exists $r > 0$ such that $B := B(z, r)$ and $\overline{\bigcup_{y_0 \in A_n} \psi\left(\{y_0\}\right)}$ are disjoint. We use again that

$$P^{n+1}\delta_{x_0}(B) = \int_{A_n} P\delta_y(B) \cdot \left[P^n \delta_{x_0}\right](dy).$$

Since $y \in A_n$ implies $B \cap \psi\left(\{y\}\right) = \phi$, and $P\delta_y$ has support $\psi\left(\{y\}\right)$, $P\delta_y(B) = 0$. Thus the above integral equals zero. So, $z \notin \operatorname{supp}\left(P^{n+1}\delta_{x_0}\right)$. Hence,

$$\operatorname{supp}\left(P^{n+1}\delta_{x_0}\right) \subset \overline{\bigcup_{y_0 \in A_n} \psi\left(\{y_0\}\right)}.$$

Combining both inclusions yields

$$\operatorname{supp}\left(P^{n+1}\delta_{x_0}\right) = \overline{\bigcup_{y_0 \in A_n} \psi\left(\{y_0\}\right)}.$$

By definition, $\psi(A_n) = \bigcup_{y_0 \in A_n} \psi\left(\{y_0\}\right)$, and $\psi(E) = \phi_{\Delta t}(E) - C$ is compact whenever E is compact, since $\phi_{\Delta t}$ is continuous and C is compact. Hence, by Lemma 11, A_1 is compact and therefore by the identity above, $A_2 = \psi(A_1)$, and then inductively $A_n = \psi^n\left(\{x_0\}\right)$. As for (2), since $0 \in C$, $x_n \in \psi^n\left(\{x_0\}\right) = \operatorname{supp}\left(P^n \delta_{x_0}\right)$. $\square$

Let us apply the above results to the RMM and stochastic interventions as in A1)-A3). The RMM has a steady state x^* in the interior of $\mathbb{R}_+^2$ under the parameter settings of Lemma 3 (7). Application of Corollary 7 has the following consequence.

Corollary 14. *Let $\Delta t > 0$. Under the parameter conditions of (7) of Lemma 3, there exist $\theta \in [0, 1)$ and $R^* > 0$, as in Corollary 7, such that the Markov operator P given by* (9) *maps the set*

$$\mathcal{P}_A^* := \left\{\mu \in \mathcal{P}\left(\mathbb{R}_+^2\right) : \operatorname{supp}(\mu) \subset \overline{B}_A(x^*, R^*)\right\}$$

into itself, where $A = J(v^*, p^*)$ given by (6).

Proof. Due to Lemma 3, the linearization A of the RMM at its steady state x^* in $\mathbb{R}_0^+$ has two eigenvalues, both with negative real part. Hence, Corollary (7) yields existence of an $R^* > 0$ and $\theta \in [0, 1)$ such that $d_A(\phi_{\Delta t}(x), x^*) \leq \theta d(x, x^*)$ for all $x \in \overline{B}_A(x^*, R)$. Since $\operatorname{diam}(C) < (1 - \theta) R$, Lemma 9 yields that $\psi(\{x\}) \subseteq \overline{B}_A(x^*, R)$ for every $x \in \overline{B}_A(x^*, R)$. Application of Lemma 13 completes the proof. □

Theorem 15. *Let $\Delta t > 0$, $\theta \in [0, 1)$ and R^* as in Corollary 7. If* $\operatorname{diam}(C) < (1 - \theta) R^*$, *then the Markov operator P defined by Q_x satisfying A1)-A3) according to* (9) *is ultra-Feller on the invariant ball $\overline{B}_A(x^*, R^*)$.*

Proof. Take $U = \overline{B}_A(x^*, R^*)$. The claim is to show that the map $x \mapsto P\delta_x : U \to \mathcal{M}(S)_{TV}$ is continuous, that is, fix x_0, to show $\| P\delta_x - P\delta_{x_0} \|_{TV} \longrightarrow 0$ as $x \longrightarrow x_0$ that is $\| p(x, \cdot) - p(x_0, \cdot) \|_{TV} \longrightarrow 0$ or $\| H(x, .) - H(x_0, .) \|_{L^1(U,dy)} \longrightarrow 0$ as $x \longrightarrow x_0$, where $H(x, y) = q_{\phi_{\Delta t}(x)}(\phi_{\Delta t}(x) - y)$, with q as in A3). Let us write $\hat{x} = \phi_{\Delta t}(x)$. Consider

$$f : (x, z) \mapsto q(\hat{x}, \hat{z} - \cdot) = T_{\hat{z}}[q(\hat{x}, \cdot)] : U \times U \to \mathrm{L}^1\left(\mathbb{R}^2\right),$$

which is separately continuous. Since translation $T_{\hat{z}}$ is strongly continuous on $\mathrm{L}^1\left(\mathbb{R}^2\right)$, the map $(z, f) \mapsto T_{\hat{z}} f$ is jointly continuous on $U \times K$, for any $K \subset \mathrm{L}^1\left(\mathbb{R}^2\right)$ compact (e.g. [3], Lemma 1.5.2.). Now let $x_0 \in U$ and $(x_n)_n$ be a sequence in U such that $x_n \to x_0$, then $(q(\hat{x_n}, \cdot))_{n=0}^{\infty}$ are contained in a compact subset of $\mathrm{L}^1(\mathbb{R})$, according to assumption A3). So,

$$f(x_n, x_n) \to f(x_0, x_0),$$

as desired. □

Theorem 16. *Assume the conditions of Theorem 15. The Markov operator P restricted to $\mathcal{P}_A^*$ has a unique invariant (ergodic) measure μ^* with* $\operatorname{supp}(\mu^*) \subset \overline{B}_A(x^*, R^*)$.

Proof. Due to the previous Corollary, $\overline{B}_A(x^*, R)$ is invariant under P. Since P is Markov-Feller (!) and $\overline{B}_A(x^*, R)$ is compact, the Krylov-Bogoliubov method yields existence of an invariant measure μ^* with $\operatorname{supp}(\mu^*) \subset \overline{B}_A(x^*, R)$. In order to obtain uniqueness, we show that there exists $z \in \overline{B}_A(x^*, R)$ such that for all $r > 0$ and $x, y \in \overline{B}_A(x^*, R)$, there exist n_x and n_y such that

$$P^{n_x}\delta_x(B_A(z, r)) > 0 \quad \text{and} \quad P^{n_y}\delta_y(B_A(z, r)) > 0,$$

and apply Theorem 7.4.6. p.156 in [14], gives the uniqueness of μ^*.

For any $x \in \overline{B}_A(x^*, R)$, we have that $x_n := \phi_{\Delta t}^n(x)$ converges to x^*. Since $\operatorname{supp}(Q_y) = C$ contains 0 for all y, we have $x_n \in \psi^n(\{x\}) = \operatorname{supp}(P^n\delta_x)$. Let $r > 0$. Then there exists $n_x \in \mathbb{N}$ such that $d_A(x_n, x^*) \leq r$ for $n \geq n_x$ and then $x_n \in B_A(x^*, r)$, so that $P^n\delta_x(B(x^*, r)) > 0$ for $n \geq n_x$. Ergodicity of μ^* follows from ultra-Feller property of P. □

Theorem 17. *Assume the conditions of Theorem 15. The unique ergodic measure μ^* is asymptotically stable on $\overline{B}_A(x^*, R^*)$. (That is, for any $x \in \overline{B}_A(x^*, R^*)$, $P^n\delta_x \to \mu^*$).*

Proof. Let $x_0 \in \overline{B}_A(x^*, R^*)$ and denote $x_n = \phi_{\Delta t}^n(x_0)$. By iteration, Corollary 7 yields

$$\| x_n - x^* \|_A \leq \theta^n \| x_0 - x^* \|_A .$$

Let $r > 0$ and $N_r := \min\{n|\, R^*\theta^n < r\}$. Then for every $n \geq N_r$ and every $x_0 \in \overline{B}_A(x^*, R^*)$ we have $\| x_n - x^* \|_A \leq \theta^n \| x_0 - x^* \|_A \leq \theta^n R^* < r$. Since $0 \in C$, we have $x_n \in \psi^n(\{x_0\}) = \operatorname{supp} P^n\delta_{x_0}$. So for $n \geq N_r$, the ball $B_A(x^*, r)$ contains x_n, so $B_A(x^*, r)$ is an open set containing x_n. Hence, $P^n\delta_{x_0}(B_A(x^*, r)) > 0$, for all $n \geq N_r$ and all $x_0 \in B_A(x^*, r)$. Thus, by Lemma 18 in [1], one has

$$\liminf_{n\to\infty} P^n\delta_x(B(z, r)) > 0,$$

i.e. the unique ergodic measure μ^* is asymptotically stable. □

4. The Support of the Invariant Measure

In this section we give a characterization of the support of the ergodic measure μ^*. Recall that the metric space $\mathcal{H}\left(\overline{B}_A(x^*, R^*)\right)$ of compact subsets of $\overline{B}_A$ endowed with the Hausdorff metric d_H is called the *hyperspace* of $\overline{B}_A$.

Proposition 18. *The map ψ is a contraction on the hyperspace $\mathcal{H}\left(\overline{B}_A(x^*, R^*)\right)$.*

Proof. Remark that ψ leaves $\overline{B}_A(x^*, R^*)$ invariant: Lemma 9. Consider the semidistance

$$\begin{aligned}
\delta(\psi(E), \psi(F)) &= \sup\{d(e, \psi(F))\,|e \in \psi(E)\} \\
&= \sup\{d(\phi_{\Delta t}(x) - y, \psi(F))\,|x \in E,\, y \in C\} \\
&= \sup_{x\in E, y\in C} \inf_{a\in F, z\in C} \| \phi_{\Delta t}(x) - y - (\phi_{\Delta t}(a) - z) \|_A \\
&\leq \sup_{x\in E, y\in C} \inf_{a\in F, z\in C} \| \phi_{\Delta t}(x) - \phi_{\Delta t}(a) \|_A + \| z - y \|_A \\
&=^{\text{choose } z=y} \sup_{x\in E,} \inf_{a\in F} \| \phi_{\Delta t}(x) - \phi_{\Delta t}(a) \|_A \\
&\leq \sup_{x\in E,} \inf_{a\in F} \mathrm{e}^{(\epsilon+w)\Delta t} \| x - a \|_A \\
&= \mathrm{e}^{(\epsilon+w)\Delta t}\delta(E, F)
\end{aligned}$$

provided that $E, F \subset \overline{B}_A\left(x^*, \hat{R^*}\right)$, then $\epsilon + w < 0$. Hence, ψ is a contraction in $\mathcal{H}\left(\overline{B}_A\right)$. Or simply, ψ is a contraction since ϕ is, this is because

$$\mid \psi \mid_{L,\overline{B}_A} \leq \mid \phi \mid_{L,\overline{B}_A}.$$

□

Remark 19. Recall that if T is a strict contraction and for some x one has $T^n x \to z$, then $T^n y \to z$ for all y. This is because

$$\begin{aligned} d\left(T^n y, z\right) \leq& d\left(T^n y, T^n x\right) + d\left(T^n x, z\right) \\ \leq& \theta^n d\left(y, x\right) + d\left(T^n x, z\right), \end{aligned}$$

the first term goes to zero since $\theta < 1$, and the second goes to zero since $T^n x \to z$.

Following [6], Theorem 10.1.6. p. 297, one has:

Corollary 20. *If S is complete, then so is the hyperspace $\mathcal{H}(S)$, for the Hausdorff metric.*

Remark 21. Since $\overline{B}_A\left(x^*, R^*\right)$ is complete, the hyperspace $\mathcal{H}\left(\overline{B}_A\left(x^*, R^*\right)\right)$ is complete as well.

A direct consequence of the Banach Fixed Point Theorem, Proposition (18), and the completeness of $\overline{B}_A\left(x^*, R^*\right)$ is

Corollary 22. *Since ψ is a contraction, ψ has a unique fixed point E^* in $\mathcal{H}\left(\overline{B}_A\right)$ such that $\psi^n(E) \to E^*$ as $n \to \infty$ for all $E \in \mathcal{H}$ (for any closed subset E of $\overline{B}_A$).*

Lemma 23. *$\psi\left(supp\left(\mu^*\right)\right) \subset supp\left(\mu^*\right)$.*

Proof. Let $z \in \psi\left(\text{supp}\left(\mu^*\right)\right)$, and $r > 0$. Then there exists $x \in \text{supp}\left(\mu^*\right)$, and $c \in C$ such that $z = \phi_{\Delta t}(x) - c$, and $Q_{\phi_{\Delta t}(x)}\left(\phi_{\Delta t}(x) - B(z, r)\right) > 0$. This is because, $\phi_{\Delta t}(x) - z = c$, so

$$\phi_{\Delta t}(x) - B(z, r) = B\left(\phi_{\Delta t}(x) - z, r\right) = B(c, r).$$

Now since $\phi_{\Delta t}(x) - B(z,r)$ is an open set, *and* $C = \overline{\mathrm{Int}(C)}$, it follows that

$$\phi_{\Delta t}(x) - B(z,r) \cap \mathrm{int}\, C$$

is non-empty, this holds whenever $\mathrm{int}(C) \neq \phi$. Thus, there exists $\delta > 0$ such that for all $y \in B(\phi_{\Delta t}(x), \delta)$ one has $y - B(z,r) \bigcap \mathrm{int}\, C \neq \phi$. As $\phi_{\Delta t}$ is continuous, there is $\gamma > 0$, such that $\phi_{\Delta t}(u) \in B(\phi_{\Delta t}(x), \delta)$ for all $u \in B(x, \gamma)$. Hence, for $u \in B(x,\gamma)$ one has, $Q_{\phi_{\Delta t}(u)}(\phi_{\Delta t}(u) - B(z,r)) > 0$. Therefore,

$$\int\limits_{\mathrm{supp}(\mu^*)} Q_{\phi_{\Delta t}(x)}(\phi_{\Delta t}(x) - B(z,r))\, d\mu^* > 0.$$

Consequently, since $\int_S (\cdot)\, d\mu^* = \int_{\mathrm{supp}(\mu^*)} (\cdot)\, d\mu^*$,

$$\mu^*(B(z,r)) = P\mu^*(B(z,r)) = \int\limits_S Q_{\phi_{\Delta t}(x)}(\phi_{\Delta t}(x) - B(z,r))\, d\mu^* > 0.$$

Hence, $z \in \mathrm{supp}(\mu^*)$. □

Lemma 24. *Let* E^* *be as such in Corollary* (22). *As* $\psi(supp(\mu^*)) \subset supp(\mu^*)$, *then* $E^* \subset \mathrm{supp}(\mu^*)$.

Proof. The proof is divided into two steps. (1) We show that if $\psi(\mathrm{supp}(\mu^*)) \subset \mathrm{supp}(\mu^*)$, then $\psi^{(n)}(\mathrm{supp}(\mu^*)) \subset \mathrm{supp}(\mu^*)$. Since ψ is monotonic, if $A \subseteq B$, then $\psi(A) \subseteq \psi(B)$. So, if $\psi(\mathrm{supp}(\mu^*)) \subset \mathrm{supp}(\mu^*)$, then $\psi[\psi(\mathrm{supp}(\mu^*))] \subset \psi[\mathrm{supp}(\mu^*)]$. Hence,

$$\begin{aligned}\psi^2(\mathrm{supp}(\mu^*)) &\subset \psi[\mathrm{supp}(\mu^*)] \\ &\subset \mathrm{supp}(\mu^*).\end{aligned}$$

That is, using induction, if $\psi(\mathrm{supp}(\mu^*)) \subset \mathrm{supp}(\mu^*)$, then $\psi^{(n)}(\mathrm{supp}(\mu^*)) \subset \mathrm{supp}(\mu^*)$. (2) We show that if $\psi^{(n)}(\mathrm{supp}(\mu^*)) \subset \mathrm{supp}(\mu^*)$, then $E^* \subset \mathrm{supp}(\mu^*)$. Note that E^* is such that $d_H(E_n, E^*) \to 0$ as $n \to \infty$, where $E_n = \psi^{(n)}(\mathrm{supp}(\mu^*))$. Thus, E^* has the form (construct a Cauchy sequence then the limit will have this form)

$$E^* := \bigcap_{k=1}^{\infty} \overline{\bigcup_{n\geq k} E_n}.$$

Since each E_n is closed and $E_n \subset \operatorname{supp}(\mu^*)$ for all n, $\overline{\bigcup_{n\geq k} E_n} \subset \operatorname{supp}(\mu^*)$. Hence, $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n\geq k} E_n} \subset \operatorname{supp}(\mu^*)$, i.e. $E^* \subset \operatorname{supp}(\mu^*)$. □

Theorem 25. *Let E^* be the unique fixed point of ψ in $\mathcal{H}\left(\overline{B}_A(x^*, R^*)\right)$. Then*

$$\operatorname{supp}(\mu^*) = E^*.$$

Proof. We have $E^* \subset \operatorname{supp}(\mu^*)$ (by Lemma 24). For the other way around, since $E_n = \psi^n(\{x_0\}) = \operatorname{supp}(P^n\delta_{x_0})$ and $P^n\delta_x \to \mu^*$ (see Theorem 17), and by the Portmanteau theorem, we have, for all k,

$$\begin{aligned}\mu^*\left(\overline{\bigcup_{n\geq k} E_n}\right) &\geq \limsup_{m\to\infty} P^m\delta_{x_0}\left(\overline{\bigcup_{n\geq k} E_n}\right)\\ &\geq \limsup_{m\to\infty} P^m\delta_{x_0}(E_m) = 1,\end{aligned}$$

also, $\overline{\bigcup_{n\geq k+1} E_n} \subseteq \overline{\bigcup_{n\geq k} E_n}$. So,

$$\mu^*\left(\bigcap_{k=1}^{\infty}\overline{\bigcup_{n\geq k} E_n}\right) = \lim_{k\to\infty}\mu^*\left(\overline{\bigcup_{n\geq k} E_n}\right) = 1.$$

□

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