ERGODICITY AND STABILITY OF A DYNAMICAL SYSTEM PERTURBED BY IMPULSIVE RANDOM INTERVENTIONS

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ABSTRACT. We determine all ergodic measures and their stability properties of a Markov operator that is associated to a Markov chain which ensues from impulsive random interventions in a one-dimensional deterministic dynamical system at equally spaced time points. This setting is inspired by a biological application in population dynamics, where samples ('catches') are drawn regularly from a growing population or part of a bacterial population is eradicated, e.g. through antibiotics. On the way, we formulate and prove a version of Orey's convergence Theorem and exponential ergodicity using essentially Banach lattice arguments and Banach's Fixed Point Theorem, valid in the generality of a Polish state space. We use the Krylov-Bogoliubov-Beboutov-Yosida decomposition to show that we found all ergodic measures. Finally, we prove that the extinction probability is a continuous function of the initial population size that is strictly positive on part of the state space.

1. INTRODUCTION

This paper considers the long-term behavior of a system that results from a particular type of stochastic perturbation of a deterministic dynamical system. That is, the dynamics of the latter is interrupted by a fast intervention through which the state of the system is changed randomly to a new position in state space according to a law that depends on the state just before this intervention. The times of intervention may either be equally spaced and predetermined – as we consider here – or

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be random themselves, with the distribution for the duration of the time intervals between interventions dependent on the state at the start too.

This type of dynamics occurs naturally for example in the modeling of particular biological systems. One can think of a growing bacterial colony, modeled deterministically, from which samples of random size are drawn regularly in an experiment, a marine ecosystem from which fish are harvested with a net or a simple model for the development of a bacterial infection that is treated with antibiotics that do not eradicate all (at once). One may further consider the modeling and analysis of stochasticity in gene regulatory networks with feedback through a cell's signaling system [15, 20], or a more quantitative study of developmental heterogeneity in populations of genetically identical bacteria when exposed to a randomly changing environment [4]. See [17] for application to cell growth and division.

The associated mathematical models deserve further attention than they obtained over the past years, in particular in view of the applications. The vast amount of results on stochastic differential equations (SDEs) with either a Wiener or a jump process as random perturbation (cf. [21, 23]) seems inadequate for these settings. In the current SDE approach only the amplitude of the perturbations can depend on the state of the system, not the *shape* of its distribution, nor can the random time between interventions in a jump process. This is an essential modeling ingredient in these applications. Ji, *et. al.* [16] considers an SDE version of the system that we study here.

In this paper we consider as deterministic system the solution to the logistic or Verhulst equation

$$\frac{dv}{dt} = rv\big(1 - \frac{v}{K}\big), \tag{1}$$

which may model autonomous growth of bacteria in a colony or a fish population. We take a one-dimensional system in order to focus on mathematical issues in proving existence, (non)uniqueness and stability of invariant measures for the

Markov chain associated to (1) that is obtained by diminishing the solution at times $t_n := n\Delta t$ ($\Delta t > 0$), with a random amount, the *catch size*, that is drawn from the fixed interval $[0, m_c]$ according to a law $Q_{v'}$. Essential for the applications is the dependence of this law on the state v' just before intervention. The *maximal catch size* m_c is fixed. Any negative outcome as a result of the random perturbation is reduced directly to the value $v = 0$ ('*extinction*'). After intervention the deterministic system (1) continues with the new state as initial condition.

We explore an analytic approach towards existence and stability of ergodic measures, motivated by our background in functional analysis and operator theory. That is, we associate a Markov operator P on finite Borel measures on $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ to the mentioned Markov chain (see Section 2) and subsequently study and employ the functional analytic properties of this operator, e.g. that P is ultra-Feller. Essentially, we use Banach lattice techniques to show that P is a contraction on measures supported on a suitable invariant interval in $\mathbb{R}_+$, for the total variation norm that is naturally associated to the lattice structure. In this manner exponential ergodicity (cf. [19]) follows from Banach's Fixed Point Theorem. Our proof is inspired by that of Orey's Convergence Theorem (cf. [22], or [19], Theorem 18.1.2), but takes full advantage of the Banach lattice formulation. Results on exponential ergodicity with a similar spirit of reducing the problem to a fixed point argument can be found in [12].

In preparation of these existence and stability results we 'trace' the supports of iterates of the transition kernel associated to P, i.e. $P^n\delta_x$, where δ_x is the Dirac measure at x (Section 3). It allows us to compute the support of the ergodic measure that was shown to exist (Section 5). This is particularly interesting for the applications, as it provides at least a range in which the population size will be on the long term. Determining the precise density of this invariant measure was not feasible, given the generality in which we studied the problem.

Finally, we show that the system has *precisely two ergodic measures*, namely the trivial measure δ_0 ('population extinct') and the non-trivial ergodic measure μ^*

mentioned above. We do so (in Section 6) by employing the so-called Krylov-Bogoliubov-Beboutov-Yosida (KBBY) ergodic decomposition of the state space (cf. [31] Chapter XIII, [33] Chapter 2 or [30]). It provides *a priori* a description of all ergodic measures. By means of the results obtained so far, we can then show that this description can yield only the two measures δ_0 and μ^*. As a further result of the KBBY decomposition we obtain that the *extinction probability* is continuous as a function of the population size $v(0)$ at the start and is zero if $v(0)$ is beyond a computable threshold v_-^*, given the parameters r, K and m_c.

We are aware that the part on existence and stability of the ergodic measure μ^* can also be obtained through the theory on T-chains (and e-chains) and results on exponential ergodicity as presented in for example [19] (e.g. Theorem 16.0.2 or 18.0.2). The main difficulty in application of the characterizations in [19] is of course in verifying suitable conditions, e.g. aperiodicity. We discuss in Section 7 how this approach could proceed based upon the wealth of (sometimes technical) probabilistic results for this purpose in [19]. In these arguments we need to use, too, the information on supports of iterates of the transition kernel of Section 3 that we used in our analytic approach. It may depend on background and scope of the reader which approach he enjoys most.

1.1. **Preliminaries: Notation.** Before proceeding, let us introduce the following notational conventions. Unless otherwise mentioned, (S, d) will denote a complete separable metric space, viewed as a measurable space with respect to its Borel σ-algebra $\mathcal{B}(S)$. We write $\mathcal{M}(S)$ to denote the real vector space of all signed finite Borel measures on S, containing $\mathcal{M}^+(S)$, the cone of positive measures. $\mathcal{P}(S)$ is the set of probability measures in $\mathcal{M}^+(S)$. We denote the total variation norm on $\mathcal{M}(S)$ by $\|\cdot\|_{TV}$ and write $\mathcal{M}(S)_{TV}$ for the Banach space consisting of $\mathcal{M}(S)$ endowed with this norm. We denote by $\mathrm{BM}(S)$ the real vector space of all bounded Borel measurable functions from S to $\mathbb{R}$ (equipped with the supremum norm) and by $\mathrm{BL}(S)$ the Banach space of bounded Lipschitz functions from S to $\mathbb{R}$, with the norm $\| f \|_{\mathrm{BL}} = | f |_{\mathrm{Lip}} + \| f \|_\infty$, where $| f |_{\mathrm{Lip}}$ denotes the Lipschitz constant of f. We write 1_E for the indicator function of $E \subset S$. For $f : S \to \mathbb{R}$ measurable and $\mu \in \mathcal{M}(S)$, we write $\langle f, \mu \rangle$ for $\int_S f \, d\mu$. $C_b(S)$ denotes the Banach space of bounded continuous functions from S to $\mathbb{R}$, endowed with the supremum norm $\| \cdot \|_\infty$. For $x \in S$ and $r > 0$, $B(x, r)$ denotes the open ball around x with radius r, and δ_x denotes the Dirac measure (point measure) at x.

1.2. **Preliminaries: Markov Operators and Semigroups.** Let us firstly recall some facts on Markov operators. An operator $P : \mathcal{M}^+(S) \to \mathcal{M}^+(S)$ is called a *Markov operator* if $P\mu(S) = \mu(S)$, for all $\mu \in \mathcal{M}^+(S)$, and

$$P(\lambda_1 \mu_1 + \lambda_2 \mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2$$

for $\lambda_1, \lambda_2 \geq 0$ and $\mu_1, \mu_2 \in \mathcal{M}^+(S)$, (cf. [27, 29]).

A Markov operator P is said to be *regular* if there exists an operator U which maps $\mathrm{BM}(S)$ into itself such that $\langle Uf, \mu \rangle = \langle f, P\mu \rangle$, for all $f \in \mathrm{BM}(S)$ and $\mu \in \mathcal{M}^+(S)$. In that case U is unique and is called the *dual* of P. A regular Markov operator on S is *Markov-Feller* if its dual maps $C_b(S)$ into itself. It is *strong Feller* if its dual maps $\mathrm{BM}(S)$ into $C_b(S)$. Finally, a Markov operator P is *ultra-Feller* if the map $x \to P\delta_x$ from S to $\mathcal{M}(S)_{TV}$ is continuous (c.f. [26, 29]).

Remark 1. If P and Q are two Markov operators on $\mathcal{M}^+(S)$ that are both strong Feller, then the product PQ is ultra-Feller. In particular, if P is strong Feller, then P^2 is ultra Feller ([5], Thoreme IX.18, or [26]). Note that there exist strong Feller operators which are not ultra Feller, even if S is compact.

A measure $\mu \in \mathcal{M}^+(S)$ is called *invariant* under P if $P\mu = \mu$. A Borel set E is P-*invariant* if for all $x \in E$, the measure $P\delta_x$ is concentrated on E (i.e. $P\delta_x(E) = 1$). In the literature many different definitions of the concept of ergodic measure exist, see e.g. [24]. We follow [14], [32], p18: $\mu \in \mathcal{P}(S)$ is ergodic with respect to P if $\mu(E) = 0$ or $\mu(E) = 1$ for every P-invariant set E. The ergodic measures (in this sense) are the extreme points of the convex set of invariant probability measures. In case that the set of ergodic measures is finite, a measure is invariant if and only if it is a convex combination of ergodic measures (cf. [1], Theorem 19.25).

The following notion is defined by Szarek in [28]: A Markov operator P *overlaps supports* if for every $x, y \in S$ there exists $n_0 \in \mathbb{N}$, such that

$$\operatorname{supp}(P^{n_0}\delta_x) \cap \operatorname{supp}(P^{n_0}\delta_y) \neq \phi.$$

We say that P *strictly overlaps supports* if for every $x, y \in S$ there exists $n_0 \in \mathbb{N}$, such that the intersection contains a non-empty open set, say U, i.e.

$$\operatorname{supp}(P^{n_0}\delta_x) \cap \operatorname{supp}(P^{n_0}\delta_y) \supseteq U.$$

1.3. **Preliminaries: The Banach lattice structure of the space of finite measures.** Let S be a set with a σ-algebra Σ. First recall that $\mathcal{M}(S)$ is a vector lattice, $\|\cdot\|_{TV}$ is vector lattice norm on it, and $(\mathcal{M}(S), \|\cdot\|_{TV})$ is a Banach lattice. If $\mu_1, \mu_2 \in \mathcal{M}(S)$, then (cf. Bogachev, [2], p. 176 and 279)

$$(\mu_1 \wedge \mu_2)(A) = \inf\{\mu_1(B) + \mu_2(A \setminus B) : B \subset A \text{ measurable}\}. \tag{2}$$

The total variation norm has different expressions. We shall use

$$\| \mu \|_{TV} = | \mu | (S) = \sup_{B \in \mathcal{B}(S)} \mu(B) - \inf_{B \in \mathcal{B}(S)} \mu(B) = \sup_{B \in \mathcal{B}(S)} | \mu(B) | . \tag{3}$$

Also, if $\mu_1, \mu_2 \in \mathcal{M}^+(S)$, then $\| \mu_1 + \mu_2 \|_{TV} = \| \mu_1 \|_{TV} + \| \mu_1 \|_{TV}$, and μ_1 and μ_2 are mutually singular if and only if $\mu_1 \wedge \mu_2 = 0$. The Jordan decomposition $\mu = \mu^+ - \mu^-$ relates to the lattice structure: $\mu^+ = \mu \vee 0$ and $\mu^- = -(\mu \wedge 0)$.

Lemma 2. *For any $\mu, \nu \in \mathcal{M}^+(S)$ there exist mutually singular measures $\mu', \nu' \in \mathcal{M}^+(S)$ such that $\mu - \nu = \mu' - \nu'$, $\| \mu' \|_{TV} = \| \nu' \|_{TV} = \| \mu - \nu \|_{TV} /2$.*

Proof. Take $\mu' = (\mu - \nu)^+$ and $\nu' = (\mu - \nu)^-$. Then $\mu' - \nu' = \mu - \nu$ and $\mu' + \nu' = | \mu - \nu |$. Also, $\mu' \wedge \nu' = 0$. Now $\mu'(S) - \nu'(S) = \mu(S) - \nu(S) = 0$, so

$\mu'(S) = \nu'(S)$, that is $\| \mu' \|_{TV} = \| \nu' \|_{TV}$. Moreover, $\| \mu' + \nu' \|_{TV} = \|| \mu - \nu |\|_{TV}$ and $\| \mu' + \nu' \|_{TV} = \| \mu' \|_{TV} + \| \nu' \|_{TV}$, so $\| \mu' \|_{TV} + \| \nu' \|_{TV} = \|| \mu - \nu |\|_{TV}$. Hence, $\| \mu' \|_{TV} = \| \nu' \|_{TV} = \| \mu - \nu \|_{TV} /2$. □

2. Model Description

Fix $\Delta t > 0$ and put $t_n := n\Delta t$, $n \in \mathbb{N}_0$. The sequence $(t_n)_{n=1}^{\infty}$ should be viewed as the times of random interventions in the deterministic system that we shall describe now.

2.1. **The Deterministic Population Model.** In this paper we consider the *logistic equation,* which is the simplest, though realistic, model for a developing population, given by (1), where $v(t)$ is the expected number of individuals in the population at time t (often called 'victims' in a predator prey model), $r > 0$ is its maximum per-capita rate of change, and $K > 0$ is the so-called *carrying capacity* of the environment. The proof of the following lemma is straightforward.

Lemma 3. *The unique solution to* (1) *with* $v(0) = v_0$ *is explicitly given by*

$$v(t) = \phi_t(v_0) := \left(\frac{1}{K} + \left(\frac{1}{v_0} - \frac{1}{K} \right) \mathrm{e}^{-rt} \right)^{-1}. \tag{4}$$

The steady states are $v = 0$ *and* $v = K$, *where* K *is stable, while* 0 *is unstable. Moreover, the intervals* $[0, K]$ *and* , $[K, \infty)$ *are positively invariant.*

2.2. **Stochastic Interventions.** We will consider stochastic interventions at times t_n, $n \in \mathbb{N}$. The distribution for the jump to the new state after intervention will be state dependent. We suppose that at the intervention times the number of individuals is diminished by a random amount, the *catch size*, which is between 0 and a fixed maximum m_c. We need the following crucial assumptions on the law Q_x for the catch size, where x is the population size just before the intervention.

We call Q_x the *catch size distribution.* We assume that for every $x \in [0,\infty)$, Q_x is a probability measure on $[0,m_c]$ (viewed as measure on $\mathbb{R}$) such that

A1): $\operatorname{supp}(Q_x) = [0,m_c]$ for $x > 0$.

A2): Q_x has density function $y \mapsto q(x,y)$ with respect to Lebesgue measure on $\mathbb{R}$ for each $x > 0$.

A3): The map $x \mapsto q(x,\cdot) : (0,\infty) \to \mathrm{L}^1(\mathbb{R})$ is continuous.

A4): $Q_x([0,x)) \to 0$ as $x \downarrow 0$.

Assumption A4) expresses, essentially, that the fewer individuals there are, the more likely it will be to catch all.

Lemma 4. *If $x \mapsto q(x,y)$ is continuous on $[0,\infty)$ for almost every $y \in \mathbb{R}$, then q satisfies A3).*

Proof. Since $q(x,y)$ and $q(x_0,y)$ are densities, due to the Vitali-Scheff Theorem (Scheff [25], Bogachev [2]) a sufficient condition that

$$\lim_{x\to x_0} \int_{\mathbb{R}} \mid q(x,y) - q(x_0,y) \mid dy = 0$$

in $[0,\infty)$ is that $\lim_{x\to x_0} q(x,y) = q(x_0,y)$ for almost every y, which we have by assumption. □

2.3. **Introduction of the Markov operator P.** If the population size just before an intervention equals x, then the population y just after the intervention will be in a Borel set $A \subseteq [0,\infty)$ with probability $Q_x(x-A)$. So, if the population size at time 0 equals x, then the population size just before the first intervention at time Δt equals $\phi_{\Delta t}(x)$, thus, the distribution of the population size just after the first intervention is $Q_{\phi_{\Delta t}(x)}(\phi_{\Delta t}(x) - \cdot)$. Furthermore, if the population size just after the n-th intervention would have distribution μ, then the distribution just after the $(n+1)$- intervention equals $P\mu$ where P is defined as follows. First put

$$\left(\widetilde{P}\mu\right)(A) = \int\limits_{[0,\infty)} Q_{\phi_{\Delta t}(x)}\left(\phi_{\Delta t}(x) - A\right) d\mu(x)$$

for any $\mu \in \mathcal{M}^+([0,\infty))$. Note that $\widetilde{P} : \mathcal{M}^+([0,\infty)) \to \mathcal{M}^+(\mathbb{R})$ is a total mass conserving positively linear map. It has kernel[1] $\widetilde{p}(x,\cdot) := \widetilde{P}\delta_x = Q_{\phi_{\Delta t}(x)}\left(\phi_{\Delta t}(x) - \cdot\right)$. Finally, define P as the push-forward of $\widetilde{P}$ under $F(x) = x^+ := \max(x, 0)$:

$$P\mu := F^\star \widetilde{P}\mu = \left(\widetilde{P}\mu\right) \circ F^{-1}. \tag{5}$$

Note that $P\delta_0(\{0\}) = \widetilde{P}\delta_0((-\infty, 0]) = Q_0([0,\infty)) = 1$, irrespective of the choice of Q_0. Thus, $P\delta_0 = \delta_0$. We call δ_0 the trivial invariant measure of P.

The population size just after the n-th intervention is a random variable X_n. Then $(X_n)_n$ is a Markov chain and P is its corresponding Markov operator.

The following property of our model is crucial in our analysis:

Theorem 5. *The Markov operator P defined by* (5) *with Q_x satisfying A1)-A4) is ultra-Feller on* $[0,\infty)$.

Proof. We need to show that the map $x \mapsto P\delta_x : [0,\infty) \to \mathcal{M}(\mathbb{R})_{TV}$ is continuous. First we show right continuity at 0. For $x > 0$, such that $\phi_{\Delta t}(x) < m_c$, one has:

$$d_x := P\delta_x(\{0\}) = Q_{\phi_{\Delta t}(x)}\left([\phi_{\Delta t}(x), m_c]\right) = 1 - Q_{\phi_{\Delta t}(x)}\left([0, \phi_{\Delta t}(x)]\right) > 0.$$

Then $v_x := P\delta_x - d_x\delta_0$ is a positive measure on $\mathbb{R}_+$ with $\| v_x \| = 1 - d_x$. Consequently, $\| P\delta_0 - P\delta_x \| = \| (1 - d_x)\delta_0 - v_x \|_{TV} \leq 2(1 - d_x)$. The latter converges to 0 as $x \downarrow 0$, according to A4). Now consider continuity on $(0,\infty)$. Fix x_0 in $(0,\infty)$. We may write $\widetilde{P}\delta_x$ as $T_{\hat{x}}[q(\hat{x},\cdot)]\, dm$ where T_z denotes the translation-reflection map $T_z g(y) := g(z - y)$ on $\mathrm{L}^1(\mathbb{R})$, $\hat{x} := \phi_{\Delta t}(x)$, and m is Lebesgue measure. Consider

$$f : (x, z) \mapsto q(\hat{x}, \hat{z} - \cdot) = T_{\hat{z}}[q(\hat{x},\cdot)] : (0,\infty) \times (0,\infty) \to \mathrm{L}^1(\mathbb{R}).$$

[1] A function $p : S \times \mathcal{B}(S) \to [0,1]$, defined as $p(x,E) = P\delta_x(E)$ for $x \in S$ and $E \in \mathcal{B}(S)$ is called the *transition function* (transition kernel) (c.f. Ethier and Kurtz, [9]).

Since $(T_z)_{z\geq 0}$ is strongly continuous semigroup on $\mathrm{L}^1(\mathbb{R})$, the map $(z,g)\mapsto T_z g$ is jointly continuous on $[0,\infty)\times K$, for any $K\subset \mathrm{L}^1(\mathbb{R})$ compact (e.g. [8], Lemma I.5.2). Now let $(x_n)_n$ be a sequence in $(0,\infty)$ such that $x_n\to x_0$, then $(q(\hat{x_n},\cdot))_{n=0}^{\infty}$ is contained in a compact subset of $\mathrm{L}^1(\mathbb{R})$, according to assumption A3). So, $f(x_n,x_n)\to f(x_0,x_0)$, i.e. $x\mapsto \widetilde{P}\delta_x$ is continuous for $\|\cdot\|_{TV}$ at x_0. Note that the push-forward under a continuous map is continuous for $\|\cdot\|_{TV}$. Hence, $x\mapsto P\delta_x = F^\star\widetilde{P}\delta_x$ is continuous for $\|\cdot\|_{TV}$. □

2.4. **An Example of a distribution that satisfies A1)-A4).** If there are many individuals a catch may be expected to be large, if there are few, the catch will be small. A distribution which captures these properties and satisfies assumptions A1)-A4) is a suitably scaled Beta distribution;

$$Q_x \sim \mathrm{Beta}\left(0, m_c, \beta, \beta\frac{x}{x^*}\right), \tag{6}$$

for some positive parameters β and x^*. The meaning of x^* is that it is the population size just before intervention at which the distribution Q_{x^*} has expectation $\frac{1}{2}m_c$.

Recall that the Beta distribution $\mathrm{Beta}(a,b,\alpha,\beta)$ is defined on $[a,b]$ and given by

$$\frac{1}{B(\alpha,\beta)}(x-a)^{\alpha-1}\frac{(b-x)^{\beta-1}}{(b-a)^{\alpha+\beta-1}},\quad a<x<b,$$

where $B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ is the Beta function.

Note that the density function q associated to Q_x is continuous on $[0,\infty)\times\mathbb{R}\setminus\{0,m_c\}$, but may have singularities for y at 0 and m_c. Thus, q satisfies Lemma 4, hence condition A3). Clearly Q_x satisfies A1) and A2). Note that Q_x also satisfies A4).

In this example $P\delta_x$ is given by

$$P\delta_x = \begin{cases} \mathrm{F}_x(y)\,dy & \text{for } x : \phi_{\Delta t}(x) \geq m_c \\ \mathrm{F}_x(y)\,dy + d_x\delta_0 & \text{for } x : \phi_{\Delta t}(x) < m_c \end{cases},$$

where

$$\mathrm{F}_x(y) = \begin{cases} \mathrm{Beta}\left(\phi_{\Delta t}(x) - m_c, \phi_{\Delta t}(x), \beta\frac{\phi_{\Delta t}(x)}{x^*}, \beta\right)(y) & \text{for } 0 \leq y \leq \phi_{\Delta t}(x) \\ 0 & \text{elsewhere} \end{cases},$$

and

$$d_x = 1 - \int_0^{\phi_{\Delta t}(x)} \mathrm{F}_x(y)\,.$$

3. Tracing Supports

We shall determine all ergodic measures for the Markov operator P on $[0,\infty)$ in Sections 5 and 6 that follow. For applications it is interesting to know the supports of these measures. Zaharopol [32] provides expressions for their supports, in terms of the sets $\mathrm{supp}\,(P^n\delta_x)$, $x \in [0,\infty)$, $n \in \mathbb{N}$. In this section we determine the latter sets explicitly. This will prove useful both in establishing existence, uniqueness and stability of the non-trivial ergodic measure and the computations of its support.

At this point we first would like to analyze what happens in the extreme deterministic cases: nothing is caught at times t_n or more interestingly, if it is 'always' the maximum catch size m_c. In the first case, if the initial population size is x, then the population size at $t = t_n$ equals $a_n^+(x) := \phi_{n\Delta t}(x)$. If at each time the maximum size m_c is caught, then the population size at time t_n will be $a_n^-(x) := \psi^n(x)$, where

$$\psi : [0,\infty) \to [0,\infty) : x \mapsto (\phi_{\Delta t}(x) - m_c)^+\,.$$

It will turn out that $\mathrm{supp}\,(P^n\delta_x) = [a_n^-(x), a_n^+(x)]$, see Proposition 8.

For the latter case we have to analyze the iterates of the map ψ.

In order to formulate our results conveniently, define

$$m_c^* := K \cdot \frac{\left(e^{\frac{1}{2}r\Delta t} - 1\right)}{\left(e^{\frac{1}{2}r\Delta t} + 1\right)},$$

the *critical catch size*, and for m_c satisfying $(K - m_c)^2 \geq \frac{4Km_c}{e^{r\Delta t}-1}$,

$$v_\pm^* := \frac{1}{2}(K - m_c) \pm \frac{1}{2}\sqrt{(K - m_c)^2 - \frac{4Km_c}{e^{r\Delta t} - 1}}, \tag{7}$$

which is a real number. Note that $m_c^* < K$. Moreover, if $m_c \leq K$, then

$$(K - m_c)^2 \geq \frac{4Km_c}{e^{r\Delta t} - 1} \quad \text{if and ony if} \quad m_c \leq m_c^*.$$

Proposition 6. *(Fixed points of ψ) We distinguish the following cases:*

(1) *If $m_c^* < m_c$, then ψ has one fixed point, which is 0.*
(2) *If $0 < m_c < m_c^*$, then ψ has three distinct fixed points in $[0, \infty)$ which are $0, v_-^*, v_+^*$ given by equation (7).*
(3) *If $m_c = m_c^*$, then v_-^* and v_+^* coincide and ψ has two fixed points in $[0, \infty)$; 0 and $v_\pm^*$.*

Proof. One can solve the fixed point equation $\phi_{\Delta t}(v^*) - m_c = v^*$ explicitly for v^* using expression (4) and obtain the stated result by straightforward calculation. □

Proposition 7. *(Stability of fixed points of ψ) Suppose that $0 < m_c < m_c^*$. Then one has the following cases:*

(1) If $0 < x < v_-^*$, then $a_n^-(x) \downarrow 0$ as $n \to \infty$ and there exists n_0 such that $a_n^-(x) = 0$ for all $n \geq n_0$.
(2) If $v_-^* < x < v_+^*$, then $a_n^-(x) \uparrow v_+^*$ as $n \to \infty$.
(3) If $x \geq v_+^*$, then $a_n^-(x) \downarrow v_+^*$.
(4) If $x \in (0, \infty)$, then $a_n^+(x) \to K$.

Proof. The results follow easily from the interpretation of the graph of ψ. □

We have the following result on the support of $P^n \delta_x$:

Proposition 8. *For each $x \in [0, \infty)$, and $n \in \mathbb{N}$,* $\operatorname{supp}(P^n \delta_x) = [a_n^-(x), a_n^+(x)]$.

Proof. According to assumption A1), we see that $P\delta_x$ has as support the interval

$$[\psi(x), \phi_{\Delta t}(x)] = \left[a_1^-(x), a_1^+(x)\right].$$

If $P^n \delta_x$ is supported on the interval $I_n = [\alpha, \beta]$, then

$$P^{n+1}\delta_x = P(P^n\delta_x) = P\left(\int_{[0,\infty)} \delta_y [P^n\delta_x](dy)\right) = \int_{I_n} P\delta_y [P^n\delta_x](dy),$$

which, by monotonicity of ψ and $\phi_{\Delta t}$, is supported on

$$\overline{\bigcup_{y \in I_n} [\psi(y), \phi_{\Delta t}(y)]} = [\psi(\alpha), \phi_{\Delta t}(\beta)].$$

Induction yields the result. $\square$

Corollary 9. *If $0 < m_c < m_c^*$, then the sets $\left[v_-^*, K\right]$, $\left[v_+^*, K\right]$, $[v_-^*, \infty)$ and $[v_+^*, \infty)$ are P-invariant. In particular, P leaves $\mathcal{M}^+\left(\left[v_+^*, K\right]\right)$, $\mathcal{M}^+\left(\left[v_-^*, K\right]\right)$, $\mathcal{M}^+\left([v_+^*, \infty)\right)$ and $\mathcal{M}^+\left([v_-^*, \infty)\right)$ invariant.*

Proof. The first part follows from Propositions 7 and 8. For the second part, take a measure μ such that $\operatorname{supp}\mu \subset [v_-^*, \infty)$. If $y < v_-^*$ and $r > 0$ such that $y + r < v_-^*$, then, with $E = (y - r, y + r)$, $P\mu(E) = \int_{[v_-^*, \infty)} P\delta_x(E)\, d\mu(x) = 0$, according to Propositions 7 and 8. Hence, $y \notin \operatorname{supp}(P\mu)$ and we conclude $\operatorname{supp}(P\mu) \subset [v_-^*, \infty)$. The other cases are shown by similar reasoning. $\square$

4. A Banach lattice approach to exponential ergodicity

We now consider the case with random catch size. We begin by an abstract result, Theorem 13, which holds for an arbitrary Polish space S. It yields convergence of the n- step transition probabilities, as time goes to ∞, to a unique invariant probability measure in the total variation topology. The proof of Theorem 13 is partially inspired by Orey, [22], but takes a Banach lattice formulation. Note that P is a positive contraction in $(\mathcal{M}(S), \|\cdot\|_{TV})$. We present conditions under which it is also a *strict* contraction on the probability measures, so that, for each probability measure μ, $P^n\mu$ converges to a unique fixed point within the probability measures. We use essentially Banach lattice arguments. The stability results in total variation norm and exponential ergodicity as found in [19] e.g. Theorem 18.0.2 thus boil down to a Banach fixed point argument. In this sense our approach is similar to [12]. We first discuss the measurability of the map $(x, y) \mapsto \| P^n\delta_x - P^n\delta_y \|_{TV}$.

Lemma 10. *Let $n \in \mathbb{N}$, and let P be a regular Markov operator.*

(1) *For every $A \in \mathcal{B}(S)$ the map $(x, y) \mapsto (P^n\delta_x - P^n\delta_y)(A) : S \mapsto \mathbb{R}$ is measurable.*
(2) *The map $(x, y) \mapsto \| P^n\delta_x - P^n\delta_y \|_{TV} : S \mapsto [0, \infty)$ is measurable.*

Proof. (1) As P is regular, there exists a bounded linear map $U : BM(S) \to BM(S)$ which is dual to P. Since $x \mapsto \mathtt{1}_A(x)$ is measurable, the map $x \mapsto (U\mathtt{1}_A)(x)$ is measurable and therefore $x \mapsto P\delta_x(A) =< P\delta_x, \mathtt{1}_A >=< \delta_x, U\mathtt{1}_A >= U\mathtt{1}_A(x)$ is measurable.

(2) Since (S, d) is separable, $\mathcal{B} := \mathcal{B}(S)$ is countably generated (separable). Thus, there exists a countable subset $\mathcal{A}$ of $\mathcal{B}$ such that $\mathcal{A}$ is an algebra and a generator of $\mathcal{B}$ and such that for each μ and each $B \in \mathcal{B}$, and for any $\epsilon > 0$, there exists $A \in \mathcal{A}$, such that $| \mu(A) - \mu(B) | < \epsilon$ (see Bogachev [2]). Hence, $\sup_{B\in\mathcal{B}} \mu(B) = \sup_{A\in\mathcal{A}} \mu(A)$ and $\inf_{B\in\mathcal{B}} \mu(B) = \inf_{A\in\mathcal{A}} \mu(A)$. So, for $x, y \in S$ and $n \in \mathbb{N}$, we have by (3)

$$\| P^n\delta_x - P^n\delta_y \|_{TV} = \sup_{A\in\mathcal{A}} (P^n\delta_x - P^n\delta_y)(A) - \inf_{A\in\mathcal{A}} (P^n\delta_x - P^n\delta_y)(A).$$

For each $A \in \mathcal{A}$ the map $(x, y) \mapsto (P^n\delta_x - P^n\delta_y)(A) : S \mapsto \mathbb{R}$ is measurable. Because $\mathcal{A}$ is countable, it follows that $(x, y) \mapsto \| P^n\delta_x - P^n\delta_y \|_{TV}$ is measurable. $\square$

Lemma 11. *Let P be a regular Markov operator on S. For every $\mu, \nu \in \mathcal{M}^+(S)$ and $n \in \mathbb{N}$,*

$$\| \iint (P^n (\delta_x - \delta_y)) \, \mu(dx) \, \nu(dy) \|_{TV} \leq \iint \| P^n (\delta_x - \delta_y) \|_{TV} \, \mu(dx) \, \nu(dy) \,. \tag{8}$$

Proof. The difficulty is that the integral at the left hand side is not a Bochner integral in $\mathcal{M}(S)_{TV}$. Recall that according to Lemma 10 (1), the map $(x, y) \mapsto (P^n \delta_x - P^n \delta_y)(E) : S \mapsto \mathbb{R}$ is measurable for all $n \geq 1$ and for all $E \in \mathcal{B}(S)$. So, the integral at the left hand side is well-defined when viewed as a set-wise integral. According to Lemma 10 (2), the map $(x, y) \mapsto \| P^n \delta_x - P^n \delta_y \|_{TV} : S \mapsto [0, \infty)$ is measurable for all $n \geq 1$, and therefore, the integral at right hand side is well-defined. For $B \in \mathcal{B}(S)$,

$$\begin{aligned} | \iint (P^n (\delta_x - \delta_y))(B) \, \mu(dx) \, \nu(dy) | &\leq \iint | (P^n (\delta_x - \delta_y))(B) | \, \mu(dx) \, \nu(dy) \\ &\leq \iint \| P^n (\delta_x - \delta_y) \|_{TV} \, \mu(dx) \, \nu(dy) \end{aligned}$$

and (8) follows. □

Remark 12. The Markov operator P extends to a positive linear operator on $\mathcal{M}(S)_{TV}$ given by $P\mu := P\mu^+ - P\mu^-$. Since $\| P\mu \|_{TV} = \| \mu \|_{TV}$ for all $\mu \in \mathcal{M}^+(S)$, this extension is bounded with operator norm equal 1. Hence, for $\mu, \nu \in \mathcal{M}^+(S)$, the map $n \mapsto \| P^n (\mu - \nu) \|_{TV}$ is non-increasing.

Theorem 13. *Let P be a regular Markov operator on S such that there exists $n_1 \in \mathbb{N}$ for which $\alpha := \inf_{x,y \in S} \| P^{n_1} \delta_x \wedge P^{n_1} \delta_y \|_{TV} > 0$. Then for all $n \geq n_1$ one has*

$$\| P^n (\mu - \nu) \|_{TV} \leq \theta^n \| \mu - \nu \|_{TV},$$

where $\theta = (1 - \alpha)^{\frac{1}{n_1}} < 1$. Consequently, there exists a unique ergodic measure μ^* and

$$\sup_{\mu \in \mathcal{P}(S)} \| P^n \mu - \mu^* \|_{TV} \leq 2\theta^n \to 0$$

as $n \to \infty$.

Proof. Let $\mu, \nu \in \mathcal{P}(S)$ be two probability measures. Recall that $\| P^n (\mu - \nu) \|_{TV}$ is non-increasing as a function of n (see Remark 12). We should show that there exists an n_1 and $\rho_1 < 1$, independent of μ and ν such that

$$\| P^{n_1} (\mu - \nu) \|_{TV} \leq \rho_1 \| \mu - \nu \|_{TV}, \tag{9}$$

then iteration gives $\| P^{kn_1} (\mu - \nu) \|_{TV} < \rho_1^k \| \mu - \nu \|_{TV}$, $k \in \mathbb{N}$ and this combined with the monotonicity of $\| P^n (\mu - \nu) \|_{TV}$ implies the ergodicity of P. Note that inequality (9) indicates that P^{n_1} is a strict contraction with norm $\| P^{n_1} \|_{TV} = \rho < 1$. Due to Lemma 2 there exist $\mu', \nu' \in \mathcal{M}^+(S)$, such that $\mu - \nu = \mu' - \nu'$, $\| \mu' \|_{TV} = \| \nu' \|_{TV} = \| \mu - \nu \|_{TV} /2$, and $\mu' \wedge \nu' = 0$. The following argument will show that it suffices to prove (9) for the case $\mu = \delta_x$, $\nu = \delta_y$, $x \neq y$. Observe that according to Lemma 11

$$\begin{aligned}
\| P^n (\mu - \nu) \|_{TV} &= \| P^n (\mu' - \nu') \|_{TV} \\
&= \left(\| \mu' \|_{TV}\right)^{-1} \left\| \iint \left(P^n (\delta_x - \delta_y)\right) \mu'(dx) \, \nu'(dy) \right\| \\
&\leq \left(\| \mu' \|_{TV}\right)^{-1} \iint \| P^n (\delta_x - \delta_y) \| \, \mu'(dx) \, \nu'(dy) .
\end{aligned}$$

Consequently, once we prove that $\| P^n (\delta_x - \delta_y) \|_{TV} \leq 2\rho$ for any $x, y \in S$, we will get for any two probability measures μ, ν

$$\begin{aligned}
\| P^n (\mu - \nu) \|_{TV} &\leq 2\rho \left(\| \mu' \|_{TV}\right)^{-1} \| \mu' \|_{TV} \| \nu' \|_{TV} \\
&\leq 2\rho \frac{1}{2} \| \mu - \nu \|_{TV} = \rho \| \mu - \nu \|_{TV} .
\end{aligned}$$

Now using the vector lattice identity $| \mu - \nu | = \mu + \nu - 2 (\mu \wedge \nu)$ we obtain

$$\begin{aligned}
\| P^n \delta_x - P^n \delta_y \|_{TV} &= \| P^n \delta_x \|_{TV} + \| P^n \delta_y \|_{TV} - 2 \| P^n \delta_x \wedge P^n \delta_y \|_{TV} \\
&\leq 2 \left(1 - \| P^n \delta_x \wedge P^n \delta_y \|_{TV}\right) = 2 (1 - \alpha) .
\end{aligned}$$

By taking $\rho := 1 - \alpha < 1$, the result follows. $\square$

5. Unique ergodicity and total variation stability on $[v_-^*, \infty)$

Consider again the Markov operator P given by (5). Theorem 13 yields:

Theorem 14. *If $0 < m_c < m_c^*$, and A1)-A4) hold, then the restriction of the Markov operator P defined by (5) is uniformly exponentially ergodic on $\mathcal{P}([r, R])$, for any $v_-^* \leq r \leq v_+^*$ and any $R \geq K$. That is,*

$$\sup_{\mu \in \mathcal{P}([r,R])} \| P^n\mu - \mu^* \|_{TV} \to 0$$

at an exponential rate as $n \to \infty$.

Proof. We need to show that $\| P^n\delta_x \wedge P^n\delta_y \|_{TV} > 0$. For this purpose we first show that P *strictly* overlaps supports. Recall that for $x \in [r, R]$, $\operatorname{supp}(P^n\delta_x) = [a_n^-(x), a_n^+(x)]$, $a_n^-(x) \to v_+^*$ and $a_n^+(x) \to K$. So, one can take n_0 large such that for all $x, y \in [r, R]$, $(a_n^-(x), a_n^+(x)) \cap (a_n^-(y), a_n^+(y)) \neq \phi$, because $a_n^+(x) > a_n^-(y)$ for all $n \geq n_0$. Then the intersection of the supports of $P^n\delta_x$ and $P^n\delta_y$ contains a non-empty open interval. This yields that $P^n\delta_x \wedge P^n\delta_y \neq 0$. Hence, $\| P^n\delta_x \wedge P^n\delta_y \|_{TV} > 0$. Next, since $x \mapsto P^n\delta_x : [r, R] \to \mathcal{M}(S)_{TV}$ is continuous and $(\mathcal{M}(S), \| \cdot \|_{TV})$ is a Banach lattice, the map $(x, y) \mapsto P^n\delta_x \wedge P^n\delta_y$ is continuous. Since the norm is a continuous mapping, $(x, y) \mapsto \| P^n\delta_x \wedge P^n\delta_y \|_{TV}$ is continuous. Since $[r, R]$ is compact, there exists $\alpha > 0$ such that for all $x, y \in [r, R]$ one has

$$\| P^n\delta_x \wedge P^n\delta_y \|_{TV} \geq \alpha > 0.$$

The result now follows due to Theorem 13. □

The non-compact case $[v_-^*, \infty)$ is captured in the following:

Theorem 15. *For every $\mu \in \mathcal{P}\left([v_-^*, \infty)\right)$ one has $\| P^n\mu - \mu^* \|_{TV} \to 0$ as $n \to \infty$. Hence, P is uniquely ergodic[2] on $[v_-^*, \infty)$. The convergence is not uniform.*

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[2]We call P *uniquely ergodic* if P has exactly one invariant probability measure, which is then an ergodic measure (cf. Zaharopol [32], p18).

Proof. Let $\mu \in \mathcal{P}\left([v_-^*, \infty)\right)$ be such that supp$\mu \subseteq [v_-^*, \infty)$. Fix $\epsilon > 0$. Take $\mu_\epsilon \in \mathcal{P}\left([v_-^*, \infty)\right)$ such that $\| \mu - \mu_\epsilon \|_{TV} < \epsilon/2$ and such that suppμ_ϵ is compact. Then $\| P^n \mu_\epsilon - \mu^* \|_{TV} \to 0$ as $n \to \infty$ by Corollary 14. Take n_0 such that

$$\| P^n \mu_\epsilon - \mu^* \|_{TV} < \epsilon/2$$

for all $n \geq n_0$. Then

$$\begin{aligned} \| P^n \mu - \mu^* \|_{TV} &\leq \| P^n \mu_\epsilon - \mu^* \|_{TV} + \| P^n \left(\mu_\epsilon - \mu\right) \|_{TV} \\ &\leq \| P^n \mu_\epsilon - \mu^* \|_{TV} + \| \mu_\epsilon - \mu \|_{TV} \leq \epsilon \end{aligned}$$

for $n \geq n_0$. Thus, $P^n \mu \to \mu^*$ in $\| \cdot \|_{TV}$. To show that the convergence is not uniform in δ_x for $x \in [v_-^*, \infty)$, take $y \in [v_-^*, \infty)$, $y > x$ such that $a_{n_1}^- (y) > a_{n_1}^+ (x)$ for some n_1 (see Proposition 7). Then there is no overlapping in the supports (Proposition 8), i.e. $P^n \delta_x \wedge P^n \delta_y = 0$, hence $\| P^n \delta_x - P^n \delta_y \|_{TV} = 2$. Suppose the convergence is uniform, then $\sup_{x \in [v_-^*, \infty)} \| P^n \delta_x - \mu^* \|_{TV} \to 0$. So, there exists n_1 such that $\| P^n \delta_x - \mu^* \|_{TV} < 1$ for all $x \in [v_-^*, \infty)$. Hence,

$$\| P^n \delta_x - P^n \delta_y \|_{TV} \leq \| P^n \delta_x - \mu^* \|_{TV} + \| \mu^* - P^n \delta_y \|_{TV} < 2,$$

which is a contradiction. $\square$

Moreover, one has

Theorem 16. *If $0 < m_c < m_c^*$, and A1)-A4) hold, then the ergodic measure μ^* for the Markov operator P defined by (5) has support $\left[v_+^*, K\right]$. Hence, P is strictly ergodic*[3] *on $\left[v_+^*, K\right]$.*

Proof. In order to prove that $\mathrm{supp}\left(\mu^*\right) = \left[v_+^*, K\right]$, first note that for any $x \in \left[v_-^*, K\right]$, $a_n(x)$ converges to v_+^* (see Proposition 7) and $\phi_{n\Delta t}(x)$ to K. For any

[3]If P is uniquely ergodic and the support of the unique invariant probability is the entire space S, then P is called *strictly ergodic* (cf. Zaharopol [32], p18).

probability measure μ on $\left[v_-^*, K\right]$ and $0 \le k \le n$ one has

$$(P^n \mu)(A) = \int\limits_{\left[v_-^*, K\right]} \left(P^{n-k}\delta_x\right)(A)\, d\left[P^k \mu\right](x),$$

for any $A \subseteq \left[v_-^*, K\right]$ Borel. In particular, $P^n \mu^* = \int_{\left[v_-^*, K\right]} P^n \delta_x \, d\mu^*(x)$. Hence,

$$\operatorname{supp}(P^n \mu^*) \subseteq \overline{\bigcup_{x \in \left[v_-^*, K\right]} \operatorname{supp}(P^n \delta_x)}.$$

Since

$$\operatorname{supp}(P^n \delta_x) \subseteq \left[a_n^-(x), \phi_{n\Delta t}(x)\right] \subseteq \left[a_n^-\left(v_-^*\right), \phi_{n\Delta t}(K)\right] = \left[a_n^-\left(v_-^*\right), K\right]$$

for all $x \in \left[v_-^*, K\right]$, $\operatorname{supp}(P^n \mu^*) \subseteq \left[a_n^-\left(v_-^*\right), K\right]$. Since $P^n \mu^* = \mu^*$ for all n, we obtain

$$\operatorname{supp}(\mu^*) = \operatorname{supp}(P^n \mu^*) \subseteq \bigcap_{n=1}^{\infty} \left[a_n^-\left(v_-^*\right), K\right] = \left[v_+^*, K\right].$$

For the other inclusion, let $x \in \left(v_+^*, K\right)$ and $U \subseteq \left[v_-^*, K\right]$ open with $x \in U$. Take $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq U$ and its closure $[x-\delta, x+\delta] \subseteq \left(v_+^*, K\right)$. Take n so large that $a_n^-(K) < x - \delta$ and $\phi_{n\Delta t}\left(v_+^*\right) > x + \delta$. Then for every $y \in \left[v_+^*, K\right]$,

$$\operatorname{supp}(P^n \delta_y) = \left[a_n^-(y), \phi_{n\Delta t}(y)\right] \supseteq \left[a_n^-(K), \phi_{n\Delta t}\left(v_+^*\right)\right] \supseteq [x-\delta, x+\delta],$$

so $P^n \delta_y((x-\delta, x+\delta)) > 0$. Hence, using invariance of μ^*,

$$\begin{aligned} \mu^*((x-\delta, x+\delta)) &= \int\limits_{\left[v_-^*, K\right]} P^n \delta_y((x-\delta, x+\delta))\, d\mu^*(y) \\ &\ge \int\limits_{\left[v_+^*, K\right]} P^n \delta_y((x-\delta, x+\delta))\, d\mu^*(y) > 0, \end{aligned}$$

so that $\mu^*(U) > 0$. Therefore, $x \in \operatorname{supp}(\mu^*)$. It follows that $\left[v_+^*, K\right] \subseteq \operatorname{supp}(\mu^*)$. Hence, $\operatorname{supp}(\mu^*) = \left[v_+^*, K\right]$. □

6. Dynamics on the Full State Space $[0,\infty)$

If we consider the perturbed dynamical system of Section 2 on the entire state space $[0,\infty)$, there will be more than one invariant measure. Since $\{0\}$ is an invariant set, the point measure δ_0 will be an invariant measure of P, in addition to μ^* of Section 5. The main objective of this section is to prove that the ergodic measures μ^* and δ_0 are the only ergodic measures of P on $[0,\infty)$. Moreover, we examine the dynamics of P on this space. In particular, we investigate the *extinction probability* of the population, which is $\lim_{n\to\infty} P^n\delta_x(\{0\})$. We shall show that this limit exists and that it is a continuous function of x.

Based on [29] we define on $\mathcal{M}(S)$ the dual bounded Lipschitz norm, $\|\cdot\|_{BL}^*$, as follows:

$$\| \mu \|_{BL}^* := \sup_{f\in BL,\, \|f\|_{BL}\le 1} \left| \int f d\mu \right| = \sup_{f\in BL\setminus\{0\}} \frac{|\langle f,\mu\rangle|}{\| f \|_{BL}}.$$

Thus, $|\langle f,\mu\rangle| \le \| \mu \|_{BL}^* \cdot \| f \|_{BL}$. One says that (μ_n), $\mu_n\in\mathcal{M}(S)$, converges weakly to $\mu\in\mathcal{M}(S)$ if $\lim_{n\to\infty}\langle f,\mu_n\rangle = \langle f,\mu\rangle$ for every continuous bounded function $f:\mathbb{R}\to\mathbb{R}$. The convergence $\lim_{n\to\infty}\| \mu_n-\mu \|_{BL}^* = 0$ for $\mu_n,\mu\in\mathcal{P}(S)$ is equivalent to the weak convergence of $(\mu_n)_{n\ge 1}$ to μ (see Dudley, [6, 7]). We denote by $\mathcal{S}_{\mathrm{BL}}$ the closure of D in $\mathrm{BL}(S)^*$, where $D := \mathrm{span}\{\delta_x : x\in S\}$.

In order to show that there cannot be ergodic measures other than μ^* and δ_0, we employ the so-called Krylov-Bogoliubov-Beboutov-Yosida (KBBY) ergodic decomposition of the state space (cf. Ch.2 in [32] or Ch XIII in Yoshida [31] and the references found there for the case of locally compact state spaces and Markov-Feller operators. For more general Markov operators see Zaharopol [33] and Worm [30] for the general case of a Polish state space). Central in the KBBY decomposition is the consideration of Cesaro averages of a Markov operator P and associated limits in $\mathcal{S}_{BL}$:

$$P^{(n)} = \frac{1}{n}\sum_{k=0}^{n-1} P^k, \quad \epsilon_x := \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} P^k\delta_x,$$

whenever the limit exists. In our setting, this is the case for any $x \in [0, \infty)$, ϵ_x is an invariant measure necessarily, since P is Markov-Feller. This follows from the Krylov-Bogoliubov Theorem for x in the P-invariant compact set $[0, K]$, and from Theorem 15 for $x > K$. The KBBY decomposition yields a subset $\Gamma_{\text{cpie}} \subset [0, \infty)$ such that ϵ_x is an ergodic measure for each $x \in \Gamma_{\text{cpie}}$. Moreover, each ergodic measure equals ϵ_x for some $x \in \Gamma_{cpie}$ (cf. [32], Theorem 2.1.3). Note that if a sequence converges, then its Cesaro average converges, with the same limit. This yields that if $x \in [v_-^*, \infty)$, then $\epsilon_x = \mu^*$ since $P^k \delta_x \to \mu^*$.

Corollary 14 and Theorem 15 yield that $\epsilon_x = \mu^*$ for all $x \in [v_-^*, \infty)$. Clearly, $\epsilon_0 = \delta_0$.

Proposition 17. *If $x \in \left(0, v_-^*\right)$, then ϵ_x is not ergodic.*

Proof. According to Proposition 8, $\operatorname{supp}\left(P^{n_0}\delta_x\right) = \left[a_n^-(x), a_n^+(x)\right]$ and, according to Proposition 7 (1), there exists n_0 such that $a_n^-(x) = 0$ for all $n \geq n_0$. Take $w^* \in \left(0, v_-^*\right)$, such that $\phi_{\Delta t}(w^*) < m_c$. Then, for $n \geq n_0$, $(0, w^*) \cap (0, a_n^+(x))$ is a non-empty open set in the support of $P^n \delta_x$, so $P^n \delta_x\left((0, w^*)\right) > 0$. For $y \in (0, w^*)$ one has

$$P\delta_y\left(\{0\}\right) = F^\star \widetilde{P}\delta_y\left(\{0\}\right) = \widetilde{P}\delta_y\left(F^{-1}\{0\}\right) = \widetilde{P}\delta_y\left((-\infty, 0]\right),$$

because $F(x) = x^+$. Hence

$$\begin{aligned} P\delta_y\left(\{0\}\right) = &\int_{[0,\infty)} Q_{\phi_{\Delta t}(u)}\left(\phi_{\Delta t}(u) - (-\infty, 0]\right) d\delta_y(u) \\ =& Q_{\phi_{\Delta t}(y)}\left(\left[\phi_{\Delta t}(y), \infty\right)\right) > 0, \end{aligned}$$

since $\phi_{\Delta t}(y) < \phi_{\Delta t}(w^*) < m_c$ and $\operatorname{supp}\left(Q_{\phi_{\Delta t}(y)}\right) = [0, m_c]$. Therefore,

$$\begin{aligned} c := \left(P^{n_0+1}\delta_x\right)\left(\{0\}\right) &= \int_{[0,\infty)} P\delta_y\left(\{0\}\right) dP^{n_0}\delta_x(y) \\ &\geq \int_{[0,w^*)} P\delta_y\left(\{0\}\right) dP^{n_0}\delta_x(y) > 0. \end{aligned}$$

Then $c \in (0,1]$, and $P^{n_0+1}\delta_x \geq c\delta_0$. Hence, for all $n \geq n_0+1$ one has $P^n\delta_x \geq c\delta_0$. Recall that

$$\begin{aligned} P^{(n)}\delta_x = \frac{1}{n}\sum_{k=0}^{n-1} P^k\delta_x =& \frac{1}{n}\sum_{k=0}^{n_0} P^k\delta_x + \frac{1}{n}\sum_{k=n_0+1}^{n-1} P^k\delta_x \\ =& \frac{1}{n}\left(\sum_{k=0}^{n_0} P^k\delta_x\right) + \frac{n-n_0-1}{n}\cdot\frac{1}{n-n_0-1}\sum_{k=n_0+1}^{n-1} P^k\delta_x \\ \geq& \frac{1}{n}\alpha + \left(1-\frac{n_0+1}{n}\right)c\delta_0 \\ \geq& \left(1-\frac{n_0+1}{n}\right)c\delta_0, \ \ n \geq n_0+1. \end{aligned}$$

So, $P^{(n)}\delta_x - \left(1-\frac{n_0+1}{n}\right)c\delta_0 \geq 0$. Since $\mathcal{M}^+([0,\infty))$ is closed in $\mathcal{S}_{BL}$, the limit is positive as well. That is, $\epsilon_x - c\delta_0 \geq 0$. Hence, $\epsilon_x \geq c\delta_0$. Similarly, for any $z^* \in \left(v_-^*, K\right)$, there exists $m_0 > 0$ such that $z^* \in \operatorname{supp}(P^{m_0}\delta_x)$. Thus, $d := P^{m_0}\delta_x\left(\left[v_-^*, K\right]\right) > 0$. Since $\left[v_-^*, K\right]$ is P-invariant, $P^{k+1}\delta_x\left(\left[v_-^*, K\right]\right) \geq d$ for all $k \geq m_0$. Then for $n \geq m_0$,

$$\begin{aligned} P^{(n)}\delta_x\left(\left[v_-^*, K\right]\right) =& \frac{1}{n}\sum_{k=0}^{m_0} P^k\delta_x\left(\left[v_-^*, K\right]\right) + \frac{1}{n}\sum_{k=m_0+1}^{n-1} P^k\delta_x\left(\left[v_-^*, K\right]\right) \\ \geq& \frac{n-m_0-1}{n(n-m_0-1)}\cdot\sum_{k=n_0+1}^{n-1} P^k\delta_x\left(\left[v_-^*, K\right]\right) \\ \geq& \frac{1}{n}(n-m_0-1)d \\ =& \left(1-\frac{m_0+1}{n}\right)d, \ \ n \geq m_0+1. \end{aligned}$$

The Portmanteau Theorem then yields that

$$\epsilon_x\left(\left[v_-^*, K\right]\right) \geq \limsup_{n\to\infty} P^{(n)}\delta_x\left(\left[v_-^*, K\right]\right) \geq d.$$

So, one has $\epsilon_x \neq c\delta_0$. Also, $\epsilon_x \neq \mu^*$ because $\epsilon_x(\{0\}) \geq c\delta_0(\{0\}) = c$ and $\mu^*(\{0\}) = 0$ because $0 \notin \operatorname{supp}\mu^*$. Thus,

$$\epsilon_x = (\epsilon_x - c\delta_0) + c\delta_0 = (1-c)\left(\frac{1}{1-c}(\epsilon_x - c\delta_0)\right) + c\delta_0, \ \ c \in (0,1).$$

That is, ϵ_x is a non-trivial convex combination of two invariant probability measures and therefore ϵ_x is not an extreme point of the set of invariant probability measures. Thus, ϵ_x is not ergodic. $\square$

Thus, from the KBBY decomposition we can deduce that δ_0 and μ^* are all (and the only) ergodic measures.

Theorem 18. *Under the conditions as in Theorem 16, the Markov operator P defined by (5) on $[0,\infty)$ has only two ergodic invariant measures: δ_0 and μ^*.*

Proof. As we discussed above, the set of ergodic measures equals $\{\epsilon_x : x \in [0,\infty)\}$. Clearly $\epsilon_0 = \delta_0$ is ergodic. For $x \in \left(0, v_-^*\right)$, Proposition 17 yields that ϵ_x is not ergodic. Also, $\epsilon_x = \mu^*$ for $x \in [v_-^*, \infty)$. Hence, the only ergodic measures are δ_0 and μ^*. $\square$

Corollary 19. *If μ is invariant, then there exists $p \in [0,1]$ such that*

$$\mu = p\delta_0 + (1-p)\,\mu^*.$$

The following Lemma could be found (in a more general situation) in [29], Proposition 7.3.3. p. 148.

Proposition 20. *The map $x \mapsto \epsilon_x : S \to \mathcal{S}_{BL}$ is continuous.*

Proof. The map $x \mapsto \epsilon_x : S \to \mathcal{S}_{BL}$ is continuous if and only if the map

$$x \mapsto < \epsilon_x, f >: S \to \mathbb{R}$$

is continuous for all $f \in C_b(S)$. One has

$$\begin{aligned} |< \epsilon_x - \epsilon_y, f >| &= \lim_{n\to\infty} |< \frac{1}{n}\sum_{k=0}^{n-1} P^k\delta_x - P^k\delta_y, f >| \\ &\leq \sup_n \frac{1}{n}\sum_{k=0}^{n-1} \| P^k\delta_x - P^k\delta_y \|_{TV} \| f \|_\infty \ . \end{aligned}$$

Since P is ultra Feller, there exists an open neighborhood U_x of x such that for $y \in U_x$ and for all k,

$$\| P^k \delta_x - P^k \delta_y \|_{TV} \leq \| P\delta_x - P\delta_y \|_{TV} < \frac{\epsilon}{1 + \| f \|_\infty}.$$

Hence, $|< \epsilon_x - \epsilon_y, f >| < \epsilon$ for $y \in U_x$. □

Proposition 21. *There exists a continuous function $p : [0, \infty) \to [0, 1]$ such that for every $x \in [0, \infty)$, $\lim_{n\to\infty} P^{(n)} \delta_x = p(x) \delta_0 + (1 - p(x)) \mu^*$ in $\mathcal{M}([0, \infty))^+_{BL}$. Moreover, $p(x) = 0$ for $x \in [v^*_-, \infty)$, $p(0) = 1$, and $0 < p(x) < 1$ for $x \in \left(0, v^*_-\right)$.*

Proof. We argued above, that according to the KBBY decomposition, ϵ_x is an invariant probability measure for each $x \in [0, \infty)$. Thus, ϵ_x is a unique convex combination of δ_0 and μ^*, because of Corollary 19. Let $p(x) \in [0, 1]$ be the unique number such that $\epsilon_x = p(x) \delta_0 + (1 - p(x)) \mu^*$. Now let $0 < \delta < v^*_-$, and define $f(x) = \left[1 - \frac{1}{\delta} x\right]^+$. Then $f \in BL([0, \infty))$. Since $x \mapsto \epsilon_x : [0, \infty) \to \mathcal{M}([0, \infty))^+_{BL}$ is continuous, $x \mapsto < \epsilon_x, f >= p(x)$ is continuous. Since $f = 0$ on $\mathrm{supp}(\mu^*)$, we have $< \mu^*, f >= 0$, so $< \epsilon_x, f >= p(x) < \delta_0, f >= p(x) f(0) = p(x)$. Clearly, $p(0) = 1$. For $x \in [v^*_-, \infty)$, $p(x) = 0$ follows from Proposition 14 and Theorem 15. For $x \in \left(0, v^*_-\right)$, ϵ_x is not ergodic (Proposition 17), so $0 < p(x) < 1$ for $x \in \left(0, v^*_-\right)$. □

Recall that for $x \in [v^*_-, \infty)$, we have a stronger convergence than stated in Proposition 21: according to Theorem 15, $P^n \delta_x \to \mu^*$ for $\| \cdot \|_{TV}$.

In the remainder of this section we show that $P^n \delta_x (\{0\}) \to p(x)$ and

$$P^n \delta_x \left([v^*_-, \infty)\right) \to 1 - p(x)$$

as $n \to \infty$. Thus, $p(x)$ may be interpreted as extinction probability. This proceeds through the following series of lemmas.

Lemma 22. *For every $\gamma \in (0, 1)$, there exists $\delta > 0$ such that for all $y \in [0, \delta)$ one has $P\delta_y (\{0\}) \geq 1 - \gamma$.*

Proof. Due to Theorem 5, for every Borel set A the evaluation map $x \mapsto P\delta_x(A) : [0,\infty) \to [0,1]$ is continuous. Hence, $x \mapsto P\delta_x(\{0\})$ is continuous. At $x = 0$ one has $P\delta_0(\{0\}) = 1$. Thus, by the continuity at 0, there exists $\delta > 0$ such that $P\delta_x(\{0\}) \geq 1 - \gamma$ for all $y \in [0,\delta)$. $\square$

Lemma 23. *Let $\gamma \in (0,1)$ and $\epsilon_n \geq 0$ with $\epsilon_n \to 0$. If $\alpha_n \geq 0$, $n \in \mathbb{N}$, satisfy $\alpha_{n+1} \leq \gamma\alpha_n + \epsilon_n$, for every $n \in \mathbb{N}$, then $\alpha_n \to 0$ as $n \to \infty$.*

Proof. Define $\beta_n := \frac{1}{\gamma^n}\alpha_n$. Then $\beta_{n+1} = \frac{1}{\gamma^{n+1}}\alpha_{n+1} \leq \frac{1}{\gamma^{n+1}}\gamma\alpha_n + \frac{1}{\gamma^{n+1}}\epsilon_n = \beta_n + \frac{1}{\gamma^{n+1}}\epsilon_n$. By induction, $\beta_{n+1} = \beta_0 + \sum_{k=0}^n \frac{1}{\gamma^{k+1}}\epsilon_k$. So, $\alpha_{n+1} = \gamma^{n+1}\alpha_0 + \sum_{k=0}^n \gamma^{n-k}\epsilon_k$. For $n \geq m$ we have

$$\begin{aligned}\sum_{k=0}^n \gamma^{n-k}\epsilon_k &= \sum_{k=0}^{m-1} \gamma^{n-k}\epsilon_k + \sum_{k=m}^n \gamma^{n-k}\epsilon_k \\ &\leq \sum_{k=0}^{m-1} \gamma^{n-k} \sup_{j\geq 0}\epsilon_j + \sum_{k=m}^n \gamma^{n-k} \sup_{j\geq m}\epsilon_j \\ &= \gamma^n \sum_{k=0}^{m-1} \gamma^{-k} \sup_{j\geq 0}\epsilon_j + \frac{1}{1-\gamma}\sup_{j\geq m}\epsilon_j.\end{aligned}$$

So, $\limsup_{n\to\infty} \sum_{k=0}^n \gamma^{n-k}\epsilon_k \leq \frac{1}{1-\gamma}\sup_{j\geq m}\epsilon_j$ for all m. Thus, $\alpha_{n+1} \to 0$. $\square$

Lemma 24. *For every $x \in [0, v_-^*)$ and every closed $K \subset \left(0, v_-^*\right)$, $P^{(n)}\delta_x(K) \to 0$ as $n \to \infty$.*

Proof. Note that by Portmanteau, since $P^{(n)}\delta_x \to \epsilon_x$, for the closed set K one has

$$\limsup_{n\to\infty} P^{(n)}\delta_x(K) \leq \epsilon_x(K).$$

Note that (from Proposition 21) $\epsilon_x = p(x)\delta_0 + (1-p(x))\mu^*$. Since $K \subset \left(0, v_-^*\right)$, $\delta_0\left(0, v_-^*\right) = 0$ and $\mu^*\left(0, v_-^*\right) = 0$, we get $\epsilon_x(K) = 0$. So, $\limsup_{n\to\infty} P^{(n)}\delta_x(K) \leq 0$, hence $\lim_{n\to\infty} P^{(n)}\delta_x(K) = 0$. $\square$

Lemma 25. *For every $x \in [0,\infty)$, $P^{(n)}\delta_x\left(\left(0, v_-^*\right)\right) \to 0$ as $n \to \infty$.*

Proof. Let $\gamma \in (0,1)$ and take $\delta > 0$ as in Lemma 22. For $y \in [0,\delta)$,

$$P\delta_y\left((0,v_-^*]\right) \leq P\delta_y\left((0,\infty)\right) \leq P\delta_y\left([0,\infty)\right) - P\delta_y\left(\{0\}\right) \leq 1+\gamma-1=\gamma.$$

Note also that $P\delta_y\left((0,v_-^*]\right) \leq 1$. Hence,

$$\begin{aligned}
P^{n+1}\delta_x\left((0,v_-^*]\right) &= \int\limits_{[0,\infty)} P\delta_y\left((0,v_-^*]\right) dP^n\delta_x(y) \\
&= \int\limits_{(0,v_-^*]} P\delta_y\left((0,v_-^*]\right) dP^n\delta_x(y) \\
&= \int\limits_{(0,\delta)} P\delta_y\left((0,v_-^*]\right) dP^n\delta_x(y) + \int\limits_{[\delta,v_-^*]} P\delta_y\left((0,v_-^*]\right) dP^n\delta_x(y) \\
&\leq \gamma\, P^n\delta_x\left((0,\delta)\right) + P^n\delta_x\left([\delta,v_-^*]\right).
\end{aligned}$$

Hence,

$$\sum_{k=1}^{n} P^k\delta_x\left((0,v_-^*]\right) \leq \gamma\sum_{k=0}^{n-1} P^k\delta_x\left((0,\delta)\right) + \sum_{k=0}^{n-1} P^k\delta_x\left([\delta,v_-^*]\right).$$

Thus,

$$\begin{aligned}
P^{(n+1)}\delta_x\left((0,v_-^*]\right) =& \frac{1}{n+1}\sum_{k=0}^{n} P^k\delta_x\left((0,v_-^*]\right) \\
=& \frac{1}{n+1}\delta_x\left((0,v_-^*]\right) + \frac{1}{n+1}\sum_{k=1}^{n} P^k\delta_x\left((0,v_-^*]\right) \\
\leq& \frac{1}{n+1} + \frac{1}{n+1}\gamma\sum_{k=0}^{n-1} P^k\delta_x\left((0,\delta)\right) + \frac{1}{n+1}\sum_{k=0}^{n-1} P^k\delta_x\left([\delta,v_-^*]\right) \\
=& \frac{1}{n+1} + \gamma\frac{n}{n+1}P^{(n)}\delta_x\left((0,\delta)\right) + \frac{n}{n+1}P^{(n)}\delta_x\left([\delta,v_-^*]\right) \\
=& \frac{1}{n+1} + \gamma\left(1-\frac{1}{n+1}\right)P^{(n)}\delta_x\left((0,\delta)\right) \\
&+ \left(1-\frac{1}{n+1}\right)P^{(n)}\delta_x\left([\delta,v_-^*]\right).
\end{aligned}$$

Hence, $P^{(n+1)}\delta_x\left((0,\delta)\right) \leq \gamma\, P^{(n)}\delta_x\left((0,\delta)\right) + \epsilon_n$, where

$$\epsilon_n = \frac{1}{n+1} - \frac{\gamma}{n+1}P^{(n)}\delta_x\left((0,\delta)\right) + \left(1-\frac{1}{n+1}\right)P^{(n)}\delta_x\left([\delta,v_-^*]\right) - P^{(n+1)}\delta_x\left([\delta,v_-^*]\right).$$

Put $\alpha_n := P^{(n)}\delta_x\left((0,\delta)\right)$. Then, one has $\alpha_{n+1} \le \gamma\alpha_n + \epsilon_n$. Also, by Lemma 24 $\epsilon_n \to 0$. Hence, using Lemma 23, $\alpha_n \to 0$ as $n \to \infty$. $\square$

Lemma 26. *For all $x \in [0,\infty)$, $P^n\delta_x\left(\left(0, v_-^*\right)\right) \to 0$ as $n \to \infty$.*

Proof. Note that for all $x \in [0,\infty)$, the map $n \mapsto P^n\delta_x\left(\left(0, v_-^*\right)\right)$ is decreasing. As $\left(P^n\delta_x\left(\left(0, v_-^*\right)\right)\right)_n$ is a decreasing sequence bounded from below by 0, it is convergent and

$$\lim_{n\to\infty} P^n\delta_x\left(\left(0, v_-^*\right)\right) = \lim_{n\to\infty} P^{(n)}\delta_x\left(\left(0, v_-^*\right)\right) = 0.$$

$\square$

The extinction probability of the population starting at x equals $\lim_{n\to\infty} P^n\delta_x\left(\{0\}\right)$ and is given precisely by the following proposition.

Proposition 27. *One has:*

$$\lim_{n\to\infty} P^n\delta_x\left(\{0\}\right) = p\left(x\right), \quad \lim_{n\to\infty} P^n\delta_x\left(\left[v_-^*, \infty\right)\right) = 1 - p\left(x\right).$$

Proof. Recall that, since $P^n\delta_x$ is a probability measure on $[0,\infty)$ for all n, one has $P^n\delta_x\left([0,\infty)\right) = 1$ for all n. So,

$$\lim_{n\to\infty} \left\{P^n\delta_x\left(\{0\}\right) + P^n\delta_x\left(\left(0, v_-^*\right)\right) + P^n\delta_x\left(\left[v_-^*, \infty\right)\right)\right\} = 1.$$

Lemma 26 says that $P^n\delta_x\left(\left(0, v_-^*\right)\right) \to 0$. By the Portmanteau Theorem, one has $\limsup_{n\to\infty} P^{(n)}\delta_x\left(\{0\}\right) \le \epsilon_x\left(\{0\}\right) \le p\left(x\right)$. If we put $L_0 = \lim_{n\to\infty} P^n\delta_x\left(\{0\}\right) = \limsup_{n\to\infty} P^{(n)}\delta_x\left(\{0\}\right)$, then $L_0 \le p\left(x\right)$. Also,

$$\limsup_{n\to\infty} P^{(n)}\delta_x\left(\left[v_-^*, \infty\right)\right) \le \epsilon_x\left(\left[v_-^*, \infty\right)\right) \le 1 - p\left(x\right).$$

If we put $L_K = \lim_{n\to\infty} P^n\delta_x\left(\left[v_-^*, \infty\right)\right) = \limsup_{n\to\infty} P^{(n)}\delta_x\left(\left[v_-^*, \infty\right)\right)$, then $L_K \le 1 - p\left(x\right)$. Thus, $1 = L_0 + L_K \le p\left(x\right) + L_K \le p\left(x\right) + 1 - p\left(x\right) = 1$. Hence, $\lim_{n\to\infty} P^n\delta_x\left(\{0\}\right) = p\left(x\right)$ and $\lim_{n\to\infty} P^n\delta_x\left(\left[v_-^*, \infty\right)\right) = 1 - p\left(x\right)$. $\square$

7. An alternative approach to exponential ergodicity

One could roughly say that a version of the Feller Property combined with some type of irreducibility yields some kind of ergodicity. Many results of this type are known and go back to a classical result often derived from theorems of Khas'minski and Doob, which states that topological irreducibility together with the strong Feller property implies uniqueness of the invariant measure. For many purposes the strong Feller property is too restrictive, and there are different ways known in which it can be relaxed to a weaker property that is still stronger than the Feller property. Meyn and Tweedie [19] use T-chains, combined with (ψ-)irreducibility and aperiodicity to get exponential ergodicity (Theorem 16.0.2). More recently, Hairer and Mattingly [10], have devised an *asymptotic strong Feller* property, which is weaker than the strong Feller property and which in combination with a weak type of irreducibility yields ergodicity.

In our situation the Markov operator P given by (5) has the ultra Feller property, which is stronger than the strong Feller property. Therefore, we obtained ergodicity on $\left[v_-^*, K\right]$ by only using a fairly weak irreducibility condition, which we established by tracing supports (see Theorems 13 and 14).

Instead, the theory of Meyn and Tweedie could have been applied. In the remaining part we shall sketch how this could proceed. It seems that proving aperiodicity also requires the result on supports of iterates of the kernel that are essential in our approach (Section 3). We use terminology from [19].

Proposition 28. *The Markov operator P defined by* (5)*, and restricted to* $\left[v_-^*, K\right]$ *yields a ψ-irreducible T-chain, Φ, with $\psi = P\delta_K$.*

Proof. That P induces a T-chain follows from Theorem 6.2.9 of [19] and the support of $P\delta_K$ having non-empty interior (Proposition 8). For ψ-irreducibility use [19] Theorem 6.1.5, which requires Φ to be a strong Feller chain. This follows from Theorem 5. It also requires the state space S to have a reachable point. According to

Proposition 8, the point $x^* = K$ is reachable, since for every open set U containing K, and for every $x \in \left[v_-^*, K\right]$, there exists n such that $\operatorname{supp}(P^n \delta_x) \cap U \neq \phi$. $\square$

In order to apply the Meyn and Tweedie $\|\cdot\|_{TV}$-stability results (for instance, Theorem 16.0.2.) we need to prove aperiodicity.

Proposition 29. *The ψ-irreducible T-chain induced by P is aperiodic.*

Proof. If the chain were periodic, there would be a d-cycle with $d \geq 2$. That is, there exist $D_i \subset \left[v_-^*, K\right]$, $i = 1, 2, \ldots, d$ pairwise disjoint such that for each $x \in D_i$, $P\delta_x\left(D_{i+1(\mathrm{mod}\, d)}\right) = 1$ and $\left(\cup_{i=1}^d D_i\right)^c$ is ψ-null (see [19], Theorem 5.4.4). Let $x \in D_1$, then $P^n\delta_x\left(D_{n+1(\mathrm{mod}\, d)}\right) = 1$ for all n. According to proposition 8, $P^n\delta_x\left(\left[a_n^-(x), a_n^+(x)\right] \setminus D_{n+1(\mathrm{mod}\, d)}\right) = 0$. Since $P^n\delta_x$ has strictly positive density (a.e.) with respect to Lebesgue measure λ, we find that $\lambda\left(\left[a_n^-(x), a_n^+(x)\right] \setminus D_{n+1}\right) = 0$ for all n. Now take n so large that $\left(a_n^-(x), a_n^+(x)\right) \cap \left(a_{n+1}^-(x), a_{n+1}^+(x)\right) \neq \phi$. Then, taking indices of D modulo d,

$$
\begin{aligned}
0 \neq & \lambda\left(\left[a_n^-(x), a_n^+(x)\right] \cap \left[a_{n+1}^-(x), a_{n+1}^+(x)\right]\right) \\
\leq & \lambda\left((D_{n+1} \cap D_{n+2})\right) \cup \left(\left[a_n^-(x), a_n^+(x)\right] \setminus D_{n+1}\right) \cup \left(\left[a_{n+1}^-(x), a_{n+1}^+(x)\right] \setminus D_{n+2}\right) \\
= & 0
\end{aligned}
$$

because $D_{n+1} \cap D_{n+2} = 0$, and we arrive at a contradiction. $\square$

Theorem 6.0.1. (ii) of [19] yields that any compact set is petite, so $\left[v_-^*, K\right]$ is petite. Theorem 5.5.7. of [19] yields that $\left[v_-^*, K\right]$ is small. Finally, Theorem 16.0.2 of [19] yields uniform exponential ergodicity from part (v). So we have

Corollary 30. *The Markov operator P defined by* (5)*, when restricted to $\left[v_-^*, K\right]$, is uniformly exponentially ergodic.*

A similar argument applies to restriction of P to $[r, R]$, as in Theorem 14.

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