Stability results for nonlinear functional differential equations using fixed point methods

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Abstract

We present new conditions for stability of the zero solution for three distinct classes of scalar nonlinear delay differential equations. Our approach is based on fixed point methods and has the advantage that our conditions neither require boundedness of delays nor fixed sign conditions on the coefficient functions. Our work extends and improves a number of recent stability results for nonlinear functional differential equations in a unified framework. A number of examples are given to illustrate our main results.

Keywords: Fixed point theory, asymptotic stability, contraction mapping principle, (neutral) integro-differential equation, variable delay.

1. Introduction and main results

Lyapunov's direct method provides simple geometric theorems for deciding the stability or instability of an equilibrium point of a differential equation. However, in the context of functional differential equations, Lyapunov's direct method is not always as effective, in particular if the delay is unbounded or if the differential equation has unbounded terms. Therefore, it was recently proposed by Burton [6] and co-workers to use fixed point methods as an alternative. While Lyapunov's direct method usually requires pointwise conditions, fixed point methods need conditions of an averaging nature, and, therefore, can handle various delays or unbounded terms more easily.

A typical stability result based on fixed point theory arguments follows a number of standard arguments adapted to the special structure of the equation under consideration. This leads to many different results in the literature for different classes of equations, for example, with time dependent delays, distributed delays, neutral terms, and certain nonlinearities, see [2-14]. The aim of this paper is to study the approach using fixed point theory in a systematic way and to unify recent results in the literature by considering three general classes of equations. For each of these classes of equations, we combine different techniques to prove new stability theorems. In addition, we present a number of examples to illustrate our results.

The first class consists of scalar neutral integro-differential equations of the form

$$x'(t) - c(t)x'(t - r_1(t)) = -a(t)x(t - r_2(t)) + \int_{t-r_3(t)}^{t} g(t, x(s))\, d\mu(t, s), \quad t \geq 0, \tag{1}$$

where for $j = 1, 2, 3$, the delays $r_j(t) : [0, \infty) \to [0, \infty)$ are continuous functions, the coefficients $a, c : [0, \infty) \to \mathbb{R}$ are continuous, where $r_0 = \min[\inf_{t\geq 0}\{t - r_1(t)\}, \inf_{t\geq 0}\{t - r_2(t)\}, \inf_{t\geq 0}\{t - r_3(t)\}]$. The kernel $\mu(t, s)$ is of bounded variation for each t and $g : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and for each t, $xg(t, x) > 0$ if $x \neq 0$ is sufficient small. We assume that g satisfies:

(G) $g(t, 0) = 0$, there exists an $l > 0$ such that g satisfies a Lipschitz condition with respect to x on $[0, \infty) \times [-l, l]$, that is, there exists a constant $L > 0$, such that

$$|g(t, x) - g(t, y)| \leq L|x - y|, \quad \text{for} \quad t \geq 0 \quad \text{and} \quad x, y \in [-l, l].$$

A standard fix point argument shows that the differential equation (1) provided with an initial condition

$$x(t) = \phi(t) \quad t \in [r_0, 0], \tag{2}$$

where $\phi(s) \in C([r_0, 0], \mathbb{R})$ defines a well-posed initial-value problem and we denote by $x(t) := x(t, \phi)$ the solution of (1) with initial condition (2).

Definition 1.1. *The zero solution of* (1) *is said to be stable if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every initial function $\phi : [r_0, 0] \to (-\delta, \delta)$, we have that the corresponding solution satisfies $|x(t)| < \epsilon$ for $t \geq 0$.*

Definition 1.2. *The zero solution of* (1) *is said to be asymptotically stable if it is stable and there exists a $\delta > 0$ such that for every initial function $\phi : [r_0, 0] \to (-\delta, \delta)$, the corresponding solution $x(t)$ tends to zero as $t \to \infty$.*

In our first result we obtain sufficient and necessary conditions for the asymptotic stability of (1) by introducing two auxiliiary continuous functions $h_1(t)$ and $h_2(t)$ which will be used to define an appropriate map defined on a complete metric space so that we can apply a fix point argument.

Theorem 1.3. *Consider the neutral integro-differential equation* (1) *an suppose the following conditions are satisfied*

(i) *the delay $r_2(t)$ is differentiable, the delay $r_1(t)$ is twice differentiable with $r_1'(t) \neq 1$, and $t - r_j(t) \to \infty$ as $t \to \infty$, $j = 1, 2, 3$;*

(ii) *there exists a constant $\alpha \in (0, 1)$ and continuous functions $h_j : [r_0, \infty) \to \mathbb{R}$ ($j = 1, 2$) such that*

$$\begin{aligned}
&\left|\frac{c(t)}{1 - r_1'(t)}\right| + \sum_{j=1}^{2} \int_{t-r_j(t)}^{t} |h_j(s)|\, ds + \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\, du} \left(V_{[s-r_3(s),s]}(\mu(s,\cdot))\right) ds \\
&+ \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\, du} |h_2(s - r_2(s))(1 - r_2'(s)) - a(s)|\, ds \\
&+ \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\, du} |h_1(s - r_1(s))(1 - r_1'(s)) - k(s)|\, ds \\
&+ \sum_{j=1}^{2} \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\, du}\, |h_1(s) + h_2(s)| \left(\int_{s-r_j(s)}^{s} |h_j(u)|\, du \right) ds \leq \alpha,
\end{aligned}$$

where $k(s) = \left((1 - r_1'(s))^2\right)^{-1} \left([c(s)(h_1(u) + h_2(u)) + c'(s)](1 - r_1'(s)) + c(s) r_1''(s) \right)$ and $V_{[s-r_3(s),s]}(\mu(s,\cdot))$ denotes the total variation of $\mu(s,\cdot)$ on $[s - r_3(s), s]$;

(iii) *and such that*

$$\liminf_{t\to\infty} \int_0^t (h_1(s) + h_2(s))\, ds > -\infty.$$

Then the zero solution of (1) *is asymptotically stable if and only if*

(iv)

$$\int_0^t (h_1(s) + h_2(s))\, ds \to \infty \quad as \quad t \to \infty.$$

Remark 1.4. *Theorem* 1.3 *contains all the stability results for* (1) *discussed in* [2, 3, 4, 5, 6, 8, 9, 11]. *Note that, in addition, in our result the delays can be unbounded and that the coefficients can change sign. See Example* 2.6 *and Example* 2.8.

A simple illustrative example is the scalar equation

$$x'(t) - c(t)x'(t - r(t)) = -a(t)x(t) + \int_{t-r(t)}^{t} k(t, s)x(s)\, ds, \quad t \geq 0,$$

where $r(t)$ is a variable delay, $a, c : [0,\infty) \to \mathbb{R}$ are continuous functions, $k(t,s)$ is continuous with respect to both arguments.

The second class of functional differential equations that we will study in this paper is of the form

$$x'(t) = -\int_{t-r(t)}^{t} a(t,s)g(s,x(s))\,ds, \tag{3}$$

where $r(t) : [0,\infty) \to [0,\infty)$, $a(t,s) : [0,\infty)\times[r_0,\infty) \to \mathbb{R}$ are continuous functions, g is a continuous function that satisfies a Lipschitz condition with respect to x on $[r_0,\infty)\times[-l,l]$, where $r_0 = \inf_{t\geq 0}\{t - r(t)\}$.

Theorem 1.5. *Consider the functional differential equation* (3) *and suppose that the following conditions are satisfied*

(i) $g(s,-x) = -g(s,x)$;

(ii) *there exists an* $l > 0$ *such that* g *satisfies a Lipschitz condition with respect to* x *on* $[r_0,\infty)\times[-l,l]$*, that is, there exists a constant* $L > 0$*, such that*

$$|g(s,x) - g(s,y)| \leq L|x-y| \quad for \quad s \geq r_0 \quad and \quad x, y \in [-l,l];$$

(iii) *there are functions* w *and* W *that are continuous, odd and strictly increasing on* $[-l,l]$ *such that* $w(x) \leq g(s,x) \leq W(x)$ *for* $x \in [0,l]$;

(iv) $x - w(x)$ *is non-decreasing on* $[0,l]$;

(v) $|x - g(s,x)| \leq l - w(l)$ *for* $x \in [-l,l]$;

(vi) $v : [r_0,\infty) \to \mathbb{R}$ *is a continuous function,* $v(t) \geq 0$ *for* $t \geq 0$;

(vii) *there exists a continuous function* q *such that*

$$\left|\int_t^u a(s,u)\,ds\right| \leq q(u), \quad \text{for} \quad t - r(t) \leq u \leq t;$$

(viii) *a positive number* $\alpha < w(l)[W(l)]^{-1}$ *exists such that*

$$\int_{t-r(t)}^{t}\left|v(u) + \int_t^u a(s,u)\,ds\right|du + \int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-r(s)}^{s}\left|v(u) + \int_s^u a(s,u)\,ds\right|du\,ds$$
$$+ \int_0^t e^{-\int_s^t v(u)\,du}\left|v(s-r(s)) + \int_s^{s-r(s)} a(u, s-r(s))\,du\right||1 - r'(s)|\,ds \leq \alpha.$$

Then there exists a $\delta \in (0,l)$ *such that, for each continuous initial function* $\phi : [r_0, 0] \to (-\delta,\delta)$*, there is a unique solution* $x : [0,\infty) \to \mathbb{R}$ *with* $x(t) = \phi(t)$ *on* $[r_0,0]$ *of* (3) *such that* $|x(t)|$ *is bounded by* l *on* $[r_0,\infty)$*. This implies that the zero solution of* (3) *is stable.*

Remark 1.6. *The proof is based on a generaliztion of some ideas of Jin and Luo [12] who discussed the case when* $g(s,x) = g(x)$*. We eliminate the condition that* $t - r(t)$ *is strictly increasing and obtain weaker conditions in Theorem* 1.5 *than those obtained in Theorem* 4.1 *of Becker and Burton* [9]*. See Example* 2.8.

A simple example is the scalar equation

$$x'(t) = -\int_{t-r(t)}^{t} a(s)x(s)\,ds, \quad t \geq 0,$$

where $r(t)$ is a variable delay, $r_0 = \inf_{t\geq 0}\{t - r(t)\}$, $a : [r_0,\infty)$ is a continuous function.

The third class consists of nonlinear delay differential equations of the form

$$x'(t) = -a(t)f(x(t - r_1(t))) + b(t)g(x(t - r_2(t))), \quad t \geq 0, \tag{4}$$

where $r_1, r_2 : [0,\infty) \to [0,\infty)$ are continuous functions, $r_0 = \min\{\inf_{t\geq 0}\{t - r_1(t)\}, \inf_{t\geq 0}\{t - r_2(t)\}\}$. The coefficients $a, b : [0,\infty) \to \mathbb{R}$ and $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Suppose, in addition, that $r_1(t)$ is differentiable, $t - r_j(t) \to \infty$ as $t \to \infty$, $j = 1,2$, and that there exists a continuous function $\tilde{a} : [0,\infty) \to \mathbb{R}$ such that $a(t) = \tilde{a}(t)(1 - r_1'(t))$ and, finally, that the inverse function $h(t)$ of $t - r_1(t)$ exists. We then have the following result.

Theorem 1.7. *Consider the nonlinear delay differential equation* (4) *and suppose that*

(i) $v(t) : [r_0, \infty) \to \mathbb{R}$ *is a continuous function,* $v(t) \geq 0$ *as* $t \geq 0$*;*
(ii) *there exists a constant* $l > 0$ *such that* $f(x)$, $x - f(x)$, *and* $g(x)$ *satisfy a Lipschitz condition with constant* $L > 0$ *on the interval* $[-l, l]$*;*
(iii) *the functions* f *and* g *are odd, increasing on* $[0, l]$, $x - f(x)$ *is nondecreasing on* $[0, l]$*;*
(iv) *there exists an* $\alpha \in (0, 1)$ *with* $\alpha g(l) < (1 - \alpha) f(l)$ *such that for* $t \geq 0$,

$$\int_0^t e^{-\int_s^t v(u)\,du} |\tilde{a}(h(s))|\,ds + \int_0^t e^{-\int_s^t v(u)\,du} |b(s)|\,ds + \int_0^t e^{-\int_s^t v(u)\,du} |v(s - r_1(s))(1 - r_1'(s))|\,ds$$
$$+ \int_{t-r_1(t)}^t |\tilde{a}(h(s)) + v(s)|\,ds + \int_0^t e^{-\int_s^t v(u)\,du} v(s) \int_{s-r_1(s)}^s |\tilde{a}(h(u)) + v(u)|\,du\,ds \leq \alpha.$$

Then the zero solution of (4) *is stable.*

Remark 1.8. *Burton* [6] *studied the special case when* $b(t) \equiv 0$ *and* r_1 *is a constant. Following the technique developed by Burton* [5], *Ding and Li* [3] *studied stability properties of* (4) *as well. However, condition* (iv) *in Ding and Li* [3] *is restrictive. By introducing a continuous function* $v(t)$ *for constructing a fixed point mapping argument, the alternative condition* (iv) *in Theorem* 1.7 *is obtained. Note that the condition that the functions* $t - r_1(t)$ *and* $t - r_2(t)$ *are strictly increasing is not needed in Theorem* 1.7.

A simple example is the scalar equation

$$x'(t) = -a(t)x(t - r_1(t)) + b(t)x(t - r_2(t)), \quad t \geq 0,$$

where $r_j(t)$, $j = 1, 2$, are variable delays, $a, b : [0, \infty) \to \mathbb{R}$ are continuous functions.

The rest of this paper is organized as follows. In Section 2 we present a proof of Theorem 1.3. The proof of Theorem 1.5 is presented in Section 3 and the proof of Theorem 1.7 is given in Section 4.

2. Proof of Theorem 1.3

In this section, we will prove Theorem 1.3. We start with some preparations. First define

$$S_\phi^l = \left\{ x \mid x \in C([r_0, \infty), \mathbb{R}),\ \|x\| = \sup_{t \geq r_0} |x(t)| \leq l,\ x(t) = \phi(t) \text{ for } t \in [r_0, 0], \text{ and } x(t) \to 0 \text{ as } t \to \infty \right\}.$$

If we define the metric $\rho(x, y) = \|x - y\| = \sup_{t \geq r_0}\{|x(t) - y(t)|\}$, then S_ϕ^l becomes a complete metric space.

If we multiply both sides of equation (1) by $e^{\int_0^t (h_1(s)+h_2(s))\,ds}$, integrate from 0 to t, and perform an integration by parts, we obtain

$$\begin{aligned} x(t) &= \left\{ \phi(0) - \frac{c(0)}{1 - r_1'(0)} \phi(-r_1(0)) - \sum_{j=1}^2 \int_{-r_j(0)}^0 h_j(s)\phi(s)\,ds \right\} e^{-\int_0^t (h_1(s)+h_2(s))\,ds} + \frac{c(t)}{1 - r_1'(t)} x(t - r_1(t)) \\ &\quad + \sum_{j=1}^2 \int_{t-r_j(t)}^t h_j(s)x(s)\,ds + \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du} [h_2(s - r_2(s))(1 - r_2'(s)) - a(s)] x(s - r_2(s))\,ds \\ &\quad + \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du} [h_1(s - r_1(s))(1 - r_1'(s)) - k(s)] x(s - r_1(s))\,ds \\ &\quad + \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du} \int_{s-r_3(s)}^s g(s, x(u))\,d\mu(s, u)\,ds \\ &\quad - \sum_{j=1}^2 \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du} (h_1(s) + h_2(s)) \int_{s-r_j(s)}^s h_j(u)x(u)\,du\,ds. \end{aligned}$$

Lemma 2.1. *Let $\varphi \in S_\phi^l$ and define an operator P by $(P\varphi)(t) = \phi(t)$ for $t \in [r_0, 0]$ and, for $t \geq 0$,*

$$\begin{aligned}(P\varphi)(t) &= \left\{\phi(0) - \frac{c(0)}{1-r_1'(0)}\phi(-r_1(0)) - \sum_{j=1}^{2}\int_{-r_j(0)}^{0} h_j(s)\phi(s)\,ds\right\} e^{-\int_0^t (h_1(s)+h_2(s))\,ds} + \frac{c(t)}{1-r_1'(t)}\varphi(t-r_1(t)) \\ &+ \sum_{j=1}^{2}\int_{t-r_j(t)}^{t} h_j(s)\varphi(s)\,ds + \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}[h_2(s-r_2(s))(1-r_2'(s)) - a(s)]\varphi(s-r_2(s))\,ds \\ &+ \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}[h_1(s-r_1(s))(1-r_1'(s)) - k(s)]\varphi(s-r_1(s))\,ds \\ &+ \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}\int_{s-r_3(s)}^{s} g(s,\varphi(u))\,d\mu(s,u)\,ds \\ &- \sum_{j=1}^{2}\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}(h_1(s)+h_2(s))\int_{s-r_j(s)}^{s} h_j(u)\varphi(u)\,du\,ds. \qquad (5)\end{aligned}$$

If conditions (i)-(iv) *in Theorem* 1.3 *are satisfied, then there exists $\delta > 0$ such that for any $\phi : [r_0, 0] \to (-\delta, \delta)$, we have that $P : S_\phi^l \to S_\phi^l$ and P is a contraction with respect to the metric defined on S_ϕ^l.*

Proof. Let $J = \sup_{t\geq 0}\left\{e^{-\int_0^t (h_1(s)+h_2(s))\,ds}\right\}$, by (iv), J is well defined. Suppose that (iv) holds.

It is clear that $P\varphi \in C([r_0,\infty),\mathbb{R})$. Hence, by (ii) and condition (G), we have

$$\begin{aligned}|(P\varphi)(t)| &\leq \|\phi\|\left(1 + \left|\frac{c(0)}{1-r_1'(0)}\right| + \sum_{j=1}^{2}\int_{-r_j(0)}^{0}|h_j(s)|\,ds\right)e^{-\int_0^t (h_1(s)+h_2(s))\,ds} \\ &\quad + l\left\{\left|\frac{c(t)}{1-r_1'(t)}\right| + \sum_{j=1}^{2}\int_{t-r_j(t)}^{t}|h_j(s)|\,ds + \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}\,(V_{[s-r_3(s),s]}(\mu(s,\cdot)))\,ds\right. \\ &\quad + \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}|h_2(s-r_2(s))(1-r_2'(s)) - a(s)|\,ds \\ &\quad + \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}|h_1(s-r_1(s))(1-r_1'(s)) - k(s)|\,ds \\ &\quad \left. + \sum_{j=1}^{2}\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}|h_1(s)+h_2(s)|\int_{s-r_j(s)}^{t}|h_j(u)|\,du\,ds\right\} \\ &\leq \|\phi\|\left(1 + \left|\frac{c(0)}{1-r_1'(0)}\right| + \sum_{j=1}^{2}\int_{-r_j(0)}^{0}|h_j(s)|\,ds\right)J + l\alpha.\end{aligned}$$

From this estimate, it follows that if

$$\delta := \frac{(1-\alpha)l}{\left(1 + \frac{|c(0)|}{|1-r_1'(0)|} + \sum_{j=1}^{2}\int_{-r_j(0)}^{0}|h_j(s)|\,ds\right)J},$$

then $\|\phi\| \leq \delta$ implies that $|(P\varphi)(t)| \leq l$.

Next, we show that $(P\varphi)(t) \to 0$ as $t \to \infty$. Since $\varphi(t) \to 0$ and $t - r_j(t) \to \infty$ as $t \to \infty$, for each $\epsilon > 0$, there exists a $T_1 > 0$ such that $t > T_1$ implies $|\varphi(t - r_j(t))| < \epsilon$, for $j = 1, 2$. Thus for $t \geq T_1$,

$$|I_2| := \left|\sum_{j=1}^{2}\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}(h_1(s)+h_2(s))\left(\int_{s-r_j(s)}^{s} h_j(u)\varphi(u)\,du\right)ds\right|$$

$$\begin{aligned}
&\le \sum_{j=1}^{2}\int_0^{T_1} e^{-\int_s^t (h_1(u)+h_2(u))\,du}|h_1(s)+h_2(s)|\left(\int_{s-r_j(s)}^{s}|h_j(u)||\varphi(u)|\,du\right)ds\\
&\quad+\sum_{j=1}^{2}\int_{T_1}^{t} e^{-\int_s^t (h_1(u)+h_2(u))\,du}|h_1(s)+h_2(s)|\left(\int_{s-r_j(s)}^{s}|h_j(u)||\varphi(u)|\,du\right)ds\\
&\le l\sum_{j=1}^{2}\int_0^{T_1} e^{-\int_s^t (h_1(u)+h_2(u))\,du}|h_1(s)+h_2(s)|\left(\int_{s-r_j(s)}^{s}|h_j(u)|\,du\right)ds\\
&\quad+\epsilon\sum_{j=1}^{2}\int_{T_1}^{t} e^{-\int_s^t (h_1(u)+h_2(u))\,du}|h_1(s)+h_2(s)|\left(\int_{s-r_j(s)}^{s}|h_j(u)|\,du\right)ds. \qquad (6)
\end{aligned}$$

By condition (iv), there exists $T_2 > T_1$ such that $t > T_2$ implies

$$l\sum_{j=1}^{2}\int_0^{T_1} e^{-\int_s^t (h_1(u)+h_2(u))\,du}|h_1(s)+h_2(s)|\left(\int_{s-r_j(s)}^{s}|h_j(u)|\,du\right)ds < \epsilon.$$

Applying (ii), we have $|I_2| \to 0$ as $t \to \infty$.

Since $\varphi(t) \to 0$ and $t - r_3(t) \to \infty$ as $t \to \infty$, for each $\epsilon > 0$, there exists a $T_3 > 0$ such that $t > T_3$ implies $|\varphi(t - r_3(t))| < \epsilon$. Thus for $t \ge T_3$,

$$\begin{aligned}
|I_3| &:= \left|\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}\int_{s-r_3(s)}^{s} g(s,\varphi(u))\,d\mu(s,u)\,ds\right|\\
&\le l\int_0^{T_3} e^{-\int_s^t (h_1(u)+h_2(u))\,du}\left(V_{[s-r_3(s),s]}(\mu(s,\cdot))\right)ds+\epsilon\int_{T_3}^{t} e^{-\int_s^t (h_1(u)+h_2(u))\,du}\left(V_{[s-r_3(s),s]}(\mu(s,\cdot))\right)ds. \qquad (7)
\end{aligned}$$

By condition (iv), there exists $T_4 > T_3$ such that $t > T_4$ implies

$$l\int_0^{T_3} e^{-\int_s^t (h_1(u)+h_2(u))\,du}\left(V_{[s-r_3(s),s]}(\mu(s,\cdot))\right)ds < \epsilon.$$

Applying (ii), we have $|I_3| \to 0$ as $t \to \infty$.

Similarly, we can show that the rest terms in (5) approach zero as $t \to \infty$, which yields $(P\varphi)(t) \to 0$ as $t \to \infty$.

Finally, we show that P is a contraction mapping with contraction constant α. In fact, for $\varphi, \eta \in S_\phi^l$,

$$\begin{aligned}
|(P\varphi)(t)-(P\eta)(t)| &\le \Bigg\{\left|\frac{c(t)}{1-r_1'(t)}\right|+\sum_{j=1}^{2}\int_{t-r_j(t)}^{t}|h_j(s)|\,ds+\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}\left(V_{[s-r_3(s),s]}(\mu(s,\cdot))\right)ds\\
&\quad+\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}|h_2(s-r_2(s))(1-r_2'(s))-a(s)|\,ds\\
&\quad+\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}|h_1(s-r_1(s))(1-r_1'(s))-k(s)|\,ds\\
&\quad+\sum_{j=1}^{2}\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}|(h_1(s)+h_2(s))|\int_{s-r_j(s)}^{t}|h_j(u)|\,du\,ds\Bigg\}\|\varphi-\eta\| \le \alpha\|\varphi-\eta\|.
\end{aligned}$$

Thus, $P : S_\phi^l \to S_\phi^l$ and P is a contraction. □

We are now ready to prove Theorem 1.3.

Proof. Let P be defined as in Lemma 2.1. By the contraction mapping principle, P has a unique fixed point x in S_ϕ^l which is by construction a solution of (1) with $x(t) = \phi(t)$ on $[r_0, 0]$ and $x(t) \to 0$ as $t \to \infty$.

Let $\epsilon > 0$ be given, then we choose $m > 0$ so that $m < \min\{l, \epsilon\}$. By considering S_ϕ^m, we obtain that there is a $\delta > 0$ such that $\|\phi\| < \delta$ implies that the unique solution of (1) with $x(t) = \phi(t)$ on $[r_0, 0]$ satifies $|x(t)| \le m < \epsilon$ for all $t \ge r_0$. This shows that the zero solution of (1) is asymptotically stable if (iv) holds.

Conversely, we suppose that condition (iv) fails. Then by (iii), there exists a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \int_0^{t_n} (h_1(s) + h_2(s))\, ds = v$ for some $v \in \mathbb{R}$. We may choose a positive constant M such that

$$-M \le \int_0^{t_n} (h_1(s) + h_2(s))\, ds \le M, \quad \text{for} \quad \text{all} \quad n \ge 1. \tag{8}$$

To simplify our expressions, we define

$$\begin{aligned} w(s) \quad := \quad & |h_2(s - r_2(s))(1 - r_2'(s)) - a(s)| + |h_1(s - r_1(s))(1 - r_1'(s)) - k(s)| \\ & + V_{[s-r_3(s),s]}(\mu(s,\cdot)) + |(h_1(s) + h_2(s))| \sum_{j=1}^{2} \int_{s-r_j(s)}^{t} |h_j(u)|\, du, \quad s \ge 0. \end{aligned}$$

By (ii) we have

$$\int_0^{t_n} e^{-\int_s^{t_n} (h_1(u)+h_2(u))\, du} w(s)\, ds \le \alpha, \quad \text{for} \quad \text{all} \quad n \ge 1. \tag{9}$$

Combining (8) and (9), we have

$$\int_0^{t_n} e^{\int_0^s (h_1(u)+h_2(u))\, du} w(s)\, ds \le \alpha e^{\int_0^{t_n} (h_1(u)+h_2(u))\, du} \le \alpha e^M, \quad \text{for} \quad \text{all} \quad n \ge 1,$$

which yields that the sequence $\int_0^{t_n} e^{\int_0^s (h_1(u)+h_2(u))\, du} w(s)\, ds$ is bounded. Therefore, there exists a convergent subsequence and without loss of generality, we can assume that

$$\lim_{k\to\infty} \int_0^{t_{n_k}} e^{\int_0^s (h_1(u)+h_2(u))\, du} w(s)\, ds = \gamma, \quad \text{for} \quad \text{some} \quad \gamma \in \mathbb{R}^+.$$

We choose a positive integer $\overline{k}$ so large that

$$\lim_{k\to\infty} \int_{t_{n_{\overline{k}}}}^{t_{n_k}} e^{\int_0^s (h_1(u)+h_2(u))\, du} w(s)\, ds \le \frac{\delta_0}{4J},$$

for all $n_k > n_{\overline{k}}$, where $\delta_0 > 0$ satisfies $2\delta_0 J e^M + \alpha < 1$.

Now, we consider the solution $x(t) = x(t, t_{n_{\overline{k}}}, \phi)$ of (1) with $x(t_{n_{\overline{k}}}) = \delta_0$ and $x(s) \le \delta_0$ for $t_{n_{\overline{k}}} - r_0 \le s \le t_{n_{\overline{k}}}$, and we may choose ϕ such that $|x(t)| \le 1$ for $t \ge t_{n_{\overline{k}}}$ and

$$x(t_{n_{\overline{k}}}) - \frac{c(t_{n_{\overline{k}}})}{1 - r_1'(t_{n_{\overline{k}}})} x(t_{n_{\overline{k}}} - r_1(t_{n_{\overline{k}}})) - \sum_{j=1}^{2} \int_{t_{n_{\overline{k}}} - r_j(t_{n_{\overline{k}}})}^{t_{n_{\overline{k}}}} h_j(s) x(s)\, ds \ge \frac{1}{2}\delta_0. \tag{10}$$

So, it follows from (10) with $x(t) = (Px)(t)$ that for $k \ge \overline{k}$,

$$\begin{aligned} & \left| x(t_{n_k}) - \frac{c(t_{n_k})}{1 - r_1'(t_{n_k})} x(t_{n_k} - r_1(t_{n_k})) - \sum_{j=1}^{2} \int_{t_{n_k} - r_j(t_{n_k})}^{t_{n_k}} h_j(s) x(s)\, ds \right| \\ & \ge \frac{1}{2}\delta_0 e^{-\int_{t_{n_{\overline{k}}}}^{t_{n_k}} (h_1(u)+h_2(u))\, du} - \int_{t_{n_{\overline{k}}}}^{t_{n_k}} e^{-\int_s^{t_{n_k}} (h_1(u)+h_2(u))\, du} w(s)\, ds \\ & = e^{-\int_{t_{n_{\overline{k}}}}^{t_{n_k}} (h_1(u)+h_2(u))\, du} \left(\frac{1}{2}\delta_0 - e^{-\int_0^{t_{n_{\overline{k}}}} (h_1(u)+h_2(u))\, du} \int_{t_{n_{\overline{k}}}}^{t_{n_k}} e^{\int_0^s (h_1(u)+h_2(u))\, du} w(s)\, ds \right) \\ & \ge e^{-\int_{t_{n_{\overline{k}}}}^{t_{n_k}} (h_1(u)+h_2(u))\, du} \left(\frac{1}{2}\delta_0 - J \int_{t_{n_{\overline{k}}}}^{t_{n_k}} e^{\int_0^s (h_1(u)+h_2(u))\, du} w(s)\, ds \right) \\ & \ge \frac{1}{4}\delta_0 e^{-\int_{t_{n_{\overline{k}}}}^{t_{n_k}} (h_1(u)+h_2(u))\, du} \ge \frac{1}{4}\delta_0 e^{-2M} > 0. \end{aligned} \tag{11}$$

On the other hand, suppose that $x(t) = x(t, t_{n_{\bar{k}}}, \phi) \to 0$ as $t \to \infty$. Since $t_{n_k} - r_j(t_{n_k}) \to \infty$ as $k \to \infty$, $j = 1, 2$ and (ii) holds, this would imply that

$$x(t_{n_k}) - \frac{c(t_{n_k})}{1 - r_1'(t_{n_k})} x(t_{n_k} - r_1(t_{n_k})) - \sum_{j=1}^{2} \int_{t_{n_k} - r_j(t_{n_k})}^{t_{n_k}} h_j(s)x(s)\, ds \to 0 \quad as \quad k \to \infty,$$

which contradicts the estimate. Hence condition (iv) is necessary for the asymptotic stability of the zero solution of (1). This completes the proof of Theorem 1.3. □

Corollary 2.2. *Consider the equation*

$$x'(t) - c(t)x'(t - r(t)) = -a(t)x(t) + \int_{t-r(t)}^{t} g(t, x(s))\, d\mu(t, s). \tag{12}$$

Assume that $r(t)$ is twice differentiable, $r'(t) \neq 1$, $t - r(t) \to \infty$ as $t \to \infty$, g satisfies condition (G). Suppose that there exists a constant $\alpha \in (0, 1)$ and a continuous function $v : [r_0, \infty) \to \mathbb{R}$ such that $\liminf_{t\to\infty} \int_0^t v(s)\, ds > -\infty$ and

$$\int_{t-r(t)}^{t} |v(s) - a(s)|\, ds + \int_0^t e^{-\int_s^t v(u)\, du} |v(s)| \int_{s-r(s)}^{s} |v(u) - a(u)|\, du\, ds + \left| \frac{c(t)}{1 - r'(t)} \right|$$
$$+ \int_0^t e^{-\int_s^t v(u)\, du} \left| [v(s - r(s)) - a(s - r(s))](1 - r'(s)) - k(s) \right| ds + \int_0^t e^{-\int_s^t v(u)\, du} V_{[s-r(s), s]}(\mu(s, \cdot))\, ds \leq \alpha,$$

where

$$k(s) = \frac{[c(s)v(s) + c'(s)](1 - r'(s)) + c(s)r''(s)}{(1 - r'(s))^2}. \tag{13}$$

Then the zero solution of (12) is asymptotically stable if and only if $\int_0^t v(s)\, ds \to \infty$ as $t \to \infty$.

Corollary 2.3. *Consider the equation*

$$x'(t) - c(t)x'(t - r_1(t)) = -a(t)x(t - r_2(t)) + b(t)g(t, x(t - r_3(t))). \tag{14}$$

Assume that $r_2(t)$ is differentiable, $r_1(t)$ is twice differentiable, $r_1'(t) \neq 1$, $t - r_j(t) \to \infty$ as $t \to \infty$, $j = 1, 2, 3$, g satisfies condition (G). Suppose that there exists a constant $\alpha \in (0, 1)$ and continuous functions $h_j : [r_0, \infty) \to \mathbb{R}$, $(j = 1, 2)$, such that $\liminf_{t\to\infty} \int_0^t (h_1(s) + h_2(s))\, ds > -\infty$ and

$$\left| \frac{c(t)}{1 - r_1'(t)} \right| + \sum_{j=1}^{2} \int_{t-r_j(t)}^{t} |h_j(s)|\, ds + \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\, du} |h_2(s - r_2(s))(1 - r_2'(s)) - a(s)|\, ds$$
$$+ \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\, du} \left[|h_1(s - r_1(s))(1 - r_1'(s)) - k(s)| + |b(s)| \right] ds$$
$$+ \sum_{j=1}^{2} \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\, du} |h_1(s) + h_2(s)| \int_{s-r_j(s)}^{s} |h_j(u)|\, du\, ds \leq \alpha,$$

where $k(s)$ and $h_j(s)$, $(j = 1, 2)$, are defined as in Theorem 1.3. Then the zero solution of (14) is asymptotically stable if and only if $\int_0^t (h_1(s) + h_2(s))\, ds \to \infty \quad as \quad t \to \infty$.

Corollary 2.4. *Consider the equation*

$$x'(t) - c(t)x'(t - r(t)) = -a(t)x(t) + b(t)g(x(t - r(t))). \tag{15}$$

Assume that $r(t)$ is twice differentiable, $r'(t) \neq 1$, $t - r(t) \to \infty$ as $t \to \infty$, g satisfies condition (G). Suppose that there exists a constant $\alpha \in (0,1)$ and a continuous function $v : [r_0, \infty) \to \mathbb{R}$ such that $\liminf_{t\to\infty} \int_0^t v(s)\,ds > -\infty$ and

$$\left|\frac{c(t)}{1-r'(t)}\right| + \int_{t-r(t)}^{t} |v(s)-a(s)|\,ds + \int_0^t e^{-\int_s^t v(u)\,du}|v(s)| \int_{s-r(s)}^{s} |v(u)-a(u)|\,du\,ds$$
$$+ \int_0^t e^{-\int_s^t v(u)\,du}\Big\{ \Big|[v(s-r(s)) - a(s-r(s))](1-r'(s)) - k(s)\Big| + |b(s)|\Big\}\,ds \le \alpha,$$

where $k(s)$ is defined as in (13). Then the zero solution of (15) is asymptotically stable if and only if $\int_0^t v(s)\,ds \to \infty$ as $t \to \infty$.

Example 2.5. *Consider the neutral differential equation*

$$x'(t) = -\frac{1}{t+1}x(t) + \frac{1}{2t+2}x(t-0.05t) + 0.05x'(t-0.05t), \tag{16}$$

Define $a(t) = \frac{1}{t+1}$, $b(t) = \frac{1}{2t+2}$, $c(t) = 0.05$, $r(t) = 0.05t$ and $v(t) = \frac{2}{t+1}$. Then

$$\frac{|c(t)|}{|1-r'(t)|} = \frac{0.05}{1-0.05} = \frac{1}{19} \approx 0.0526.$$

Since $|v(s)-a(s)| = \frac{1}{s+1}$, $k(s) = \frac{2}{19(s+1)}$, we have

$$\int_{t-r(t)}^{t} |v(s)-a(s)|\,ds = \int_{0.95t}^{t} \frac{1}{s+1}\,ds \quad < \quad 0.0513,$$
$$\int_0^t e^{-\int_s^t v(u)\,du}|v(s)| \int_{s-r(s)}^{s} |v(u)-a(u)|\,du\,ds \quad < \quad 0.0513,$$

$$\int_0^t e^{-\int_s^t v(u)\,du}\Big\{ \Big|[v(s-r(s)) - a(s-r(s))](1-r'(s)) - k(s) + b(s)\Big|\Big\}\,ds$$
$$= \int_0^t e^{-\int_s^t \frac{2}{u+1}\,du}\left|\frac{0.95}{0.95s+1} - \frac{2}{19(s+1)} + \frac{1}{2(s+1)}\right|\,ds \le \frac{17}{2\times 19} + \frac{1}{4} < 0.697.$$

Hence, we have

$$\left|\frac{c(t)}{1-r'(t)}\right| + \int_{t-r(t)}^{t} |v(s)-a(s)|\,ds + \int_0^t e^{-\int_s^t v(u)\,du}|v(s)| \int_{s-r(s)}^{s} |v(u)-a(u)|\,du\,ds$$
$$+ \int_0^t e^{-\int_s^t v(u)\,du}\Big\{ \Big|[v(s-r(s)) - a(s-r(s))](1-r'(s)) - k(s) + b(s)\Big|\Big\}\,ds \approx 0.852 < 1,$$

and since $\int_0^t v(s)\,ds = \int_0^t \frac{2}{s+1}\,ds = 2\ln(t+1)$, the conditions of Corollary 2.4 are satisfied. Therefore, the zero solution of (16) is asymptotically stable.

Example 2.6. *Consider the following differential equation*

$$x'(t) = -\frac{1}{32}\left(\frac{1}{4} - \frac{1}{3}\sin t + \epsilon_1(t)\right)x\left(t - \left(1 - \frac{1}{3}\cos t + \epsilon_2(t)\right)\right) + \frac{\cos t}{256}g(x(t-r_3(t))), \tag{17}$$

where $|\epsilon_j(t)| < \epsilon < \frac{2}{51}$, $|\epsilon_j'(t)| < \epsilon < \frac{2}{51}$, $j = 1,2$, $r_3(t) \in C(\mathbb{R}^+, \mathbb{R}^+)$ is an arbitrary continuous function which satisfies $t - r_3(t) \to \infty$ as $t \to \infty$, g satisfies condition (G). Define $a(t) = \frac{1}{32}\left(\frac{1}{4} - \frac{1}{3}\sin t + \epsilon_1(t)\right)$, $b(t) = \frac{\cos t}{256}$, $r_2(t) = 1 - \frac{1}{3}\cos t + \epsilon_2(t)$, and $v(t) = \frac{1}{32}$. Then

$$\int_{t-r_2(t)}^{t} |v(s)|\,ds = \int_{t-1+\frac{1}{3}\cos t - \epsilon_2(t)}^{t} \frac{1}{32}\,ds = \frac{1}{32}(1 - \frac{1}{3}\cos t + \epsilon_2(t)) \quad \le \quad \frac{1}{24} + \frac{\epsilon}{32},$$
$$\int_0^t e^{-\int_s^t \frac{1}{32}\,du}\frac{1}{32}\int_{s-1+\frac{1}{3}\cos s + \epsilon_2(s)}^{s} \frac{1}{32}\,du\,ds \quad \le \quad \frac{1}{24} + \frac{\epsilon}{32},$$

$$\begin{aligned}\int_0^t e^{-\int_s^t v(u)\,du}\left(|v(s-r_2(s))(1-r_2'(s))-a(s)|+|b(s)|\right)\,ds &= \int_0^t e^{-\int_s^t \frac{1}{32}\,du}\left(\frac{1}{32}\times\frac{3}{4}+\frac{\epsilon}{32}+\frac{1}{32}\times\frac{1}{8}\right)ds\\ &\le \frac{7}{8}+\epsilon.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{t-r_2(t)}^t |v(s)|\,ds+\int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-r_2(s)}^s |v(u)|\,du\,ds+\int_0^t e^{-\int_s^t v(u)\,du}\left(|v(s-r_2(s))(1-r_2'(s))-a(s)|+|b(s)|\right)\,ds\\ &\le \frac{1}{24}+\frac{\epsilon}{32}+\frac{1}{24}+\frac{\epsilon}{32}+\frac{7}{8}+\epsilon=\frac{23}{24}+\frac{17\epsilon}{16}<1,\end{aligned}$$

and since $\int_0^t v(s)\,ds=\int_0^t \frac{1}{32}\,ds=\frac{1}{32}t$, the conditions of Corollary 2.3 are satisfied. Therefore, the zero solution of (17) is asymptotically stable.

Example 2.7. *Consider the following differential equation*

$$x'(t)=-a(t)x\left(t-1-\frac{1}{3}\cos t\right)+b(t)g(x(t-r_3(t))),\tag{18}$$

where $0<m_1\le a(t)$, $|b(t)|\le M_2$, $r_3(t)\in C(\mathbb{R}^+,\mathbb{R}^+)$ is an arbitrary continuous function which satisfies $t-r_3(t)\to\infty$ as $t\to\infty$, g satisfies condition (G).

Define $r_2(t)=1-\frac{1}{3}\cos t$, if we choose $v(t)=v$ is a constant satisfying $v>\frac{4m_1}{5}$, we have

$$\begin{aligned}\int_{t-r_2(t)}^t |v(s)|\,ds=\int_{t-1+\frac{1}{3}\cos t}^t v\,ds=v(1-\frac{1}{3}\cos t) &\le \frac{4}{3}v,\\ \int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-r_2(s)}^s |v(u)|\,du\,ds &\le \frac{4}{3}v,\\ \int_0^t e^{-\int_s^t v(u)\,du}\left(|v(s-r_2(s))(1-r_2'(s))-a(s)|+|b(s)|\right)\,ds &\le \int_0^t e^{-(t-s)v}(\frac{5v}{4}-m_1+M_2)\,ds\le\frac{5}{4}-\frac{m_1-M_2}{v}.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{t-r_2(t)}^t |v(s)|\,ds+\int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-r_2(s)}^s |v(u)|\,du\,ds\\ &\quad+\int_0^t e^{-\int_s^t v(u)\,du}\left(|v(s-r_2(s))(1-r_2'(s))-a(s)|+|b(s)|\right)\,ds\le\frac{8}{3}v+\frac{5}{4}-\frac{m_1-M_2}{v}.\end{aligned}$$

Next, choose v such that $\frac{8}{3}v+\frac{5}{4}-\frac{m_1-M_2}{v}<1$, and since $\int_0^t v(s)\,ds=\int_0^t v\,ds=vt$, then the conditions of Corollary 2.3 are satisfied. Therefore, the zero solution of (18) is asymptotically stable.

For instance, if we choose $v=\frac{1}{32}$, $m_1=\frac{1}{32}$, $M_2=\frac{1}{64}$, we have $\frac{8}{3}v+\frac{5}{4}-\frac{m_1-M_2}{v}=\frac{5}{6}<1$.

Example 2.8. *Consider the following differential equation*

$$x'(t)=-\frac{1}{128}\left(1-2\sin\frac{5t}{16}\right)x(t-1),\tag{19}$$

Define $a(t)=\frac{1}{128}\left(1-2\sin\frac{5t}{16}\right)$, $v(t)=\frac{1}{32}$, we obtain that

$$\int_{t-1}^t |v(s)|\,ds=\frac{1}{32},\qquad \int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-1}^s |v(u)|\,du\,ds\le\frac{1}{32},$$

$$\int_0^t e^{-\int_s^t v(u)\,du}\,|v(s-1)-a(s)|\,ds=\frac{3}{128}\int_0^t e^{-v(t-s)}\,ds+\frac{1}{64}\int_0^t e^{-v(t-s)}\sin\left(\frac{5}{16}s\right)ds<0.104,$$

Hence,

$$\int_{t-1}^{t}|v(s)|\,ds+\int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-1}^{s}|v(u)|\,du\,ds+\int_0^t e^{-\int_s^t v(u)\,du}\,|v(s-1)-a(s)|\;ds<0.9165.$$

Since $\int_0^t v(s)\,ds=\frac{t}{32}\to\infty$ *as* $t\to\infty$, *the conditions of Corollary 2.3 are satisfied. Therefore, the zero solution of (19) is asymptotically stable.*

Remark 2.9. *Zhao* [13] *investigated the case for which* $\left|\frac{c(t)}{1-\tau'(t)}\right|<1$ *does not hold by considering the following neutral differential equation*

$$x'(t)=-b(t)x(t-\tau(t))+c(t)x'(t-\tau(t)),$$

and presented new criteria for asymptotic stability of the zero solution by employing an auxiliary function $p(t)$*, but there seems to be a mistake in his computations on page 6. We obtain that* (8) *in* [13] *actually should be*

$$\begin{aligned} z'(t) &= -\frac{p'(t)}{p(t)}z(t)-\frac{b(t)p(t-\tau(t))-c(t)p'(t-\tau(t))}{p(t)}z(t-\tau(t))\\ &\quad+\frac{c(t)p(t-\tau(t))}{p(t)}z'(t-\tau(t)), \end{aligned}$$

which is a special form of (15)*. By using the condition in* Corollary 2.3*, the correct condition* (iii) *in* Theorem 3.1 *on page 6 of Zhao* [13] *should be*

$$\begin{aligned} &\left|\frac{p(t-\tau(t))c(t)}{p(t)(1-\tau'(t))}\right|+\int_{t-\tau(t)}^{t}\left|v(s)-\frac{p'(s)}{p(s)}\right|\,ds+\int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-\tau(s)}^{s}\left|v(u)-\frac{p'(u)}{p(u)}\right|\,du\,ds\\ &+\int_0^t e^{-\int_s^t v(u)\,du}\left\{\left|-\beta(s)+\left(v(s-\tau(s))-\frac{p'(s-\tau(s))}{p(s-\tau(s))}\right)(1-\tau'(s))-k(s)\right|\right\}\,ds\le\alpha, \end{aligned}$$

where

$$\beta(s)=\frac{b(s)p(s-\tau(s))-c(s)p'(s-\tau(s))}{p(s)},$$

and

$$k(s)=\frac{[C(s)v(s)+C'(s)](1-\tau'(s))+C(s)\tau''(s)}{(1-\tau'(s))^2},\quad C(s)=\frac{c(s)p(s-\tau(s))}{p(s)}.$$

3. Proof of Theorem 1.5

In this section, we will prove Theorem 1.5. We start with some preparations. First we write (3) in the following form

$$x'(t)=B(t,t-r(t))(1-r'(t))g(t-r(t),x(t-r(t)))+\frac{d}{dt}\int_{t-r(t)}^{t}B(t,s)g(s,x(s))\,ds,\tag{20}$$

where

$$B(t,s):=\int_t^s a(u,s)\,du.\tag{21}$$

If we multiply both sides of (20) by $e^{\int_0^t v(s)\,ds}$, integrate from 0 to t, and perform an integration by parts, then we obtain

$$\begin{aligned} x(t) &= \left\{\phi(0)-\int_{-r(0)}^{0}[v(s)+B(0,s)]g(s,\phi(s))\,ds\right\}e^{-\int_0^t v(s)\,ds}+\int_{t-r(t)}^{t}[v(s)+B(t,s)]g(s,\phi(s))ds\\ &\quad-\int_0^t e^{-\int_s^t v(u)\,du}v(s)\int_{s-r(s)}^{s}[v(u)+B(s,u)]g(u,x(u))\,du\,ds+\int_0^t e^{-\int_s^t v(u)\,du}v(s)[x(s)-g(s,x(s))]\,ds\\ &\quad+\int_0^t e^{-\int_s^t v(u)\,du}[v(s-r(s))+B(s,s-r(s))](1-r'(s))g(s-r(s),x(s-r(s)))\,ds. \end{aligned}$$

By (ii), we choose a common Lipschitz constant L for $g(s,x)$ and $x-g(s,x)$ on $[-l,l]$. For $t\in[r_0,\infty)$ and a constant $k>4$, we define

$$h(t)=kL\int_0^t[v(u)+q(u)+p(u)]\,du, \tag{22}$$

where q is as defined in (iv) of Theorem 1.5 and

$$p(u)=[v(u-r(u))+B(u,u-r(u))](1-r'(u)).$$

Now, let C be the space of all continuous functions $\varphi:[r_0,\infty)\to\mathbb{R}$ such that

$$|\varphi|_h:=\sup\left\{|\varphi(t)|e^{-h(t)}:t\in[r_0,\infty)\right\}<\infty,$$

where h is given by (22). Then $(C,|\cdot|_h)$ is a Banach space, which can be verified by Cauchy's criterion for uniform convergence. Thus (C,d) is a complete metric space, where d denotes the induced metric: $d(\varphi,\eta)=|\varphi-\eta|_h$ for $\varphi,\eta\in C$. Define

$$C_\phi^l=\left\{\varphi\mid\varphi\in C,\ \|\varphi\|=\sup_{t\geq r_0}|\varphi(t)|\leq l,\ \varphi(t)=\phi(t)\text{ for }t\in[r_0,0]\right\},$$

where $\phi:[r_0,0]\to[-l,l]$ is a given continuous initial function. Then C_ϕ^l is a closed subspace of C and hence a complete metric space as well with the metric inherited from C.

Lemma 3.1. *Let $\varphi\in C_\phi^l$ and define the operator P by $(P\varphi)(t)=\phi(t)$ for $t\in[r_0,0]$ and, for $t\geq 0$,*

$$\begin{aligned}(P\varphi)(t)\ =\ &\left\{\phi(0)-\int_{-r(0)}^0[v(s)+B(0,s)]g(s,\phi(s))\,ds\right\}e^{-\int_0^t v(s)\,ds}+\int_{t-r(t)}^t[v(s)+B(t,s)]g(s,\phi(s))ds\\&-\int_0^t e^{-\int_s^t v(u)\,du}v(s)\int_{s-r(s)}^s[v(u)+B(s,u)]g(u,x(u))\,du\,ds+\int_0^t e^{-\int_s^t v(u)\,du}v(s)[x(s)-g(s,x(s))]\,ds\\&+\int_0^t e^{-\int_s^t v(u)\,du}[v(s-r(s))+B(s,s-r(s))](1-r'(s))g(s-r(s),x(s-r(s)))\,ds.\end{aligned}\tag{23}$$

If the conditions (i)-(viii) *in Theorem* 1.5 *are satisfied, then there exists $\delta>0$ such that for any $\phi:[r_0,0]\to(-\delta,\delta)$, we have that $P:C_\phi^l\to C_\phi^l$ and P is a contraction.*

Proof. First of all, given $\varphi\in C_\phi^l$ we show that $P\varphi\in C_\phi^l$. Let $\phi:[r_0,0]\to(-\delta,\delta)$ be a continuous function, where $\delta>0$ satisfies

$$\delta+W(\delta)\int_{-r(0)}^0|v(u)+B(0,u)|\,du\leq w(l)-\alpha W(l). \tag{24}$$

Such δ exists, since $W(0)=0$ and W is continuous on $[0,l]$. Note that $w(l)-\alpha W(l)>0$ by (iii) and (viii). By (iv), $w(l)\leq l$. For any $\varphi\in C_\phi^l$, by (24), we have $|(P\varphi)(t)|=|\phi(t)|<l$ for $t\in[r_0,0]$. Now we consider $(P\varphi)(t)$ for $t>0$. By (i) and (iii), $|g(s,x)|\leq W(l)$ for $x\in[-l,l]$ and $t\geq r_0$, thus using (iii) and (v), we obtain

$$\begin{aligned}|P(\varphi)(t)|\ &\leq\ \delta+W(\delta)\int_{-r(0)}^0|v(u)+B(0,u)|\,du\\&\leq\ w(l)-\alpha W(l)+W(l)\int_{t-r(t)}^t|v(u)+B(t,u)|\,du+W(l)\int_0^t e^{-\int_s^t v(u)\,du}v(s)\int_{s-r(s)}^s|v(u)+B(s,u)|\,du\,ds\\&\quad+W(l)\int_0^t e^{-\int_s^t v(u)\,du}|v(s-r(s))+B(s,s-r(s))||1-r'(s)|\,ds+(l-w(l))\int_0^t e^{-\int_s^t v(u)\,du}v(s)\,ds\\&\leq\ w(l)-\alpha W(l)+\alpha W(l)+l-w(l)=l.\end{aligned}\tag{25}$$

So $|P(\varphi)(t)| \le l$ for $t \in [r_0, \infty)$. Therefore, $P\varphi \in C^l_\phi$.

Next, we show that P is a contraction mapping on C^l_ϕ. Suppose that $\varphi, \eta \in C^l_\phi$,

$$\begin{aligned}|P\varphi(t) - P\eta(t)|e^{-h(t)} \le & \int_{t-r(t)}^t e^{-kL\int_u^t [v(s)+q(s)]\,ds}|v(u) + B(t,u)|L|\varphi(u) - \eta(u)|e^{-h(u)}\,du \\ & + \int_0^t e^{-\int_s^t v(u)\,du} v(s) \int_{s-r(s)}^s e^{-kL\int_u^s [v(\theta)+q(\theta)]\,d\theta}|v(u) + B(s,u)| \cdot L|\varphi(u) - \eta(u)|e^{-h(u)}\,du\,ds \\ & + \int_0^t e^{-kL\int_s^t p(u)\,du} p(s)L|\varphi(s - r(s)) - \eta(s - r(s))|e^{-h(s-r(s))}\,ds \\ & + \int_0^t e^{-(kL+1)\int_s^t v(u)\,du} v(s)L|\varphi(s) - \eta(s)|e^{-h(s)}\,ds,\end{aligned}$$

since $|v(u) + B(t,u)| \le v(u) + q(u)$, for $t - r(t) \le u \le t$, we have

$$|P\varphi - P\eta|_h \le \left(\frac{1}{kL} + \frac{1}{kL} + \frac{1}{kL} + \frac{1}{kL}\right)L|\varphi - \eta|_h = \frac{4}{k}|\varphi - \eta|_h < |\varphi - \eta|_h,$$

since $k > 4$. This shows that P is indeed a contraction mapping. □

We are now ready to prove Theorem 1.5.

Proof. By the contraction mapping principle, P has a unique fixed point $x \in C^l_\phi$, which is by construction a solution of (3) on $[0, \infty)$ and $|x(t)| \le l$ for $t \ge r_0$. Hence $x(t)$ is the only continuous function satisfying (3) for $t \ge 0$ with $x(t) = \phi(t)$ on $[r_0, 0]$.

Let $\epsilon > 0$ be given and choose $m > 0$, such that $m < \min\{\epsilon, l\}$, replacing l with m in (25), we see that there is $\delta > 0$ such that $|\varphi| \le m < \epsilon$ for $t \ge r_0$. Hence, the zero solution of (3) is stable. This completes the proof of Theorem 1.5. □

Example 3.2. *Consider the following integro-differential equation*

$$x'(t) = -\int_{0.4635t}^t \frac{0.9}{s^2+1}\left(\frac{4}{5} + \frac{1}{10}\sin^2 s\right)x^3(s)\,ds. \tag{26}$$

We check the condition (vii) of Theorem 4.1 in Becker and Burton [9], $f(t) = \frac{t}{0.4635}$, *then*

$$G(t,s) = \int_t^{s/0.4635} \frac{0.9}{s^2+1}\,du = \frac{0.9(s/0.4635 - t)}{s^2+1},$$

for $t \ge 0$ *and* $0.4635t \le s \le t$. *Consequently,*

$$\lim_{t\ge 0}\left\{2\int_{0.4635t}^t |G(t,u)|\,du\right\} = 0.9 \times 2\left(-\frac{\ln 0.4635 + 1}{0.4635} + 1\right) = 0.9027.$$

Then there exists some $t_0 > 0$ *such that for* $t \ge t_0$, *we have*

$$2\int_{0.4635t}^t |G(t,u)|\,du > 0.9020.$$

Since $\frac{w(\frac{1}{2})}{W(\frac{1}{2})} = \frac{8}{9} = 0.8889 < 0.9020$. *This implies that condition (vii) of Theorem 4.1 in Becker and Burton [9] does not hold. Thus Theorem 4.1 of Becker and Burton [9] can not be applied to equation (26). However, by (21),*

$$B(t,s) = \int_t^s \frac{0.9}{s^2+1}\,du = \frac{0.9(s-t)}{s^2+1}.$$

Choosing $v(t) = \frac{0.9t}{t^2+1}$,

$$\int_{t-r(t)}^{t} |v(u) + B(t,u)|\,du = 0.9 \times \int_{0.4635t}^{t} |\frac{2u-t}{u^2+1}|\,du = 0.9 \times \int_{0.4635t}^{0.5t} \frac{t-2u}{u^2+1}\,du + 0.9 \times \int_{0.5t}^{t} \frac{2u-t}{u^2+1}\,du$$
$$= 0.9 \times \left[t\,(2\arctan 0.5t - \arctan t - \arctan 0.4635t) + \ln(t^2+1) + \ln(0.4635^2 t^2 + 1) - 2\ln(0.25t^2+1)\right]$$
$$:= w(t).$$

Since the function $w(t)$ *is increasing on* $[0,\infty)$ *and*

$$\lim_{t\to\infty} w(t) = 0.9 \times (1/0.4635 - 3 + 2\ln 2 + 2\ln 0.927) = 0.3530,$$

we have

$$\int_{t-r(t)}^{t} |v(u) + B(t,u)|\,du < 0.3530,$$

$$\begin{aligned}\int_0^t e^{-\int_s^t v(u)\,du} |v(s-r(s)) + B(s, s-r(s))||1-r'(s)|\,ds &= (1/0.4635 - 2)\int_0^t e^{-\int_s^t \frac{0.9u}{u^2+1}\,du} \frac{0.9s}{s^2 + 1/0.4635^2}\,ds \\ &< 1/0.4635 - 2 = 0.1575,\end{aligned}$$

and $\int_0^t e^{-\int_s^t v(u)\,du} v(s) \int_{s-r(s)}^{s} |v(u) + B(s,u)|\,du\,ds < 0.3530$. *Hence, we have*

$$\int_{t-r(t)}^{t} |v(u) + B(t,u)|\,du + \int_0^t e^{-\int_s^t v(u)\,du} v(s) \int_{s-r(s)}^{s} |v(u) + B(s,u)|\,du\,ds$$
$$+ \int_0^t e^{-\int_s^t v(u)\,du} |v(s-r(s)) + B(s,s-r(s))||1-r'(s)|\,ds < 0.8635 < \frac{w(\frac{1}{2})}{W(\frac{1}{2})} = \frac{8}{9} = 0.8889.$$

By Theorem 1.5, the zero solution of (26) is stable. Together with Remark 1.6, this shows that our result extends the result of Becker and Burton [9].

4. Proof of Theorem 1.7

In this section, we will prove Theorem 1.7. We start with some preparations. Equation (4) can be written in the following equivalent form

$$x'(t) = -\tilde{a}(h(t))f(x(t)) + \frac{d}{dt}\int_{t-r_1(t)}^{t} \tilde{a}(h(s))f(x(s))\,ds + b(t)g(x(t-r_2(t))). \tag{27}$$

If we multiply both sides of (27) by $e^{\int_0^t v(s)\,ds}$, integrate from 0 to t, and perform an integration by parts, then we obtain

$$\begin{aligned} x(t) &= \left\{\phi(0) - \int_{-r_1(0)}^{0} [\tilde{a}(h(s)) + v(s)]f(\phi(s))\,ds\right\} e^{-\int_0^t v(s)\,ds} + \int_0^t e^{-\int_s^t v(u)\,du} v(s)[x(s) - f(x(s))]\,ds \\ &- \int_0^t e^{-\int_s^t v(u)\,du} \tilde{a}(h(s)) f(x(s))\,ds - \int_0^t e^{-\int_s^t v(u)\,du} v(s) \int_{s-r_1(s)}^{s} [\tilde{a}(h(u)) + v(u)] f(x(u))\,du\,ds \\ &+ \int_0^t e^{-\int_s^t v(u)\,du} v(s-r_1(s))(1-r_1'(s)) f(x(s-r_1(s)))\,ds + \int_{t-r_1(t)}^{t} [\tilde{a}(h(s)) + v(s)] f(x(s))\,ds \\ &+ \int_0^t e^{-\int_s^t v(u)\,du} b(s) g(x(s-r_2(s)))\,ds. \end{aligned}$$

Let C be the weighted space of all continuous functions $\varphi : [r_0, \infty) \to \mathbb{R}$ with

$$|\varphi|_q := \sup\{|\varphi(t)|e^{-q(t)} : t \in [r_0, \infty)\} < \infty.$$

Here the weight function $q : [r_0, \infty) \to \mathbb{R}$ is defined as follows

$$q(t) = \begin{cases} 1 & \text{for} \quad t \in [r_0, 0], \\ dl \int_0^t [v(s) + |\tilde{a}(h(s))| + |b(s)| + w(s)]\, ds & \text{for} \quad t \in [0, \infty), \end{cases}$$

where $d > 5$ is a constant, and

$$w(s) = \begin{cases} 0 & \text{for} \quad s \in [r_0, 0], \\ |v(s - r_1(s))(1 - r_1'(s))| & \text{for} \quad s \in [0, \infty). \end{cases}$$

The space $(C, |\cdot|_q)$ becomes a Banach space, which can be verified by Cauchy's criterion for uniform convergence. Define

$$C_\phi^l = \left\{\varphi : [r_0, \infty) \to \mathbb{R} | \varphi \in C, \quad \|\varphi\| = \sup_{t \ge r_0} |\varphi(t)| \le l, \quad \varphi(t) = \phi(t) \quad \text{for} \quad \mathrm{t} \in [\mathrm{r}_0, 0]\right\}$$

where $\phi : [r_0, 0] \to [-l, l]$ is a given continuous initial function. Then C_ϕ^l is a closed subset of $(C, |\cdot|_q)$ and hence a complete metric space with the metric inherited from C.

Lemma 4.1. *Let $\varphi \in C_\phi^l$. Define the operator P by $(P\varphi)(t) = \phi(t), t \in [r_0, 0]$, and, for $t \ge 0$,*

$$\begin{aligned}(P\varphi)(t) \quad = \quad & \left\{\phi(0) - \int_{-r_1(0)}^0 [\tilde{a}(h(s)) + v(s)] f(\phi(s))\, ds\right\} e^{-\int_0^t v(s)\, ds} + \int_0^t e^{-\int_s^t v(u)\, du} v(s)[\varphi(s) - f(\varphi(s))]\, ds \\ & - \int_0^t e^{-\int_s^t v(u)\, du} \tilde{a}(h(s)) f(\varphi(s))\, ds - \int_0^t e^{-\int_s^t v(u)\, du} v(s) \int_{s - r_1(s)}^s [\tilde{a}(h(u)) + v(u)] f(\varphi(u))\, du\, ds \\ & + \int_0^t e^{-\int_s^t v(u)\, du} v(s - r_1(s))(1 - r_1'(s)) f(\varphi(s - r_1(s)))\, ds + \int_{t - r_1(t)}^t [\tilde{a}(h(s)) + v(s)] f(\varphi(s))\, ds \\ & + \int_0^t e^{-\int_s^t v(u)\, du} b(s) g(\varphi(s - r_2(s)))\, ds.\end{aligned}$$

If the conditions (i)-(iv) *in Theorem* 1.7 *are satisfied, then there exists $\delta > 0$ such that for any $\phi : [r_0, 0] \to (-\delta, \delta)$, we have that $P : C_\phi^l \to C_\phi^l$ and P is a contraction.*

Proof. Since f is odd and satisfies a Lipschitz condition on $[-l, l]$, and $f(0) = 0$, we can choose a $\delta < l$ that satisfies

$$\delta + f(\delta) \int_{-r_1(0)}^0 |\tilde{a}(h(s)) + v(s)|\, ds \le (1 - \alpha) f(l) - \alpha g(l).$$

Let $\phi : [r_0, 0] \to (-\delta, \delta)$ be a continuous function. Thus $|\phi(t)| \le l$ for $t \in [r_0, 0]$. Now we show for such ϕ, $P : C_\phi^l \to C_\phi^l$. In fact, for arbitrary $\varphi \in C_\phi^l$, it follows from the conditions in Theorem 1.7 that we have for $t > 0$,

$$\begin{aligned}|(P\varphi)(t)| \quad \le \quad & \delta + f(\delta) \int_{-r_1(0)}^0 |\tilde{a}(h(s)) + v(s)|\, ds + f(l) \left\{ \int_0^t e^{-\int_s^t v(u)\, du} |\tilde{a}(h(s))|\, ds + \int_{t - r_1(t)}^t |\tilde{a}(h(s)) + v(s)|\, ds \right. \\ & \left. + \int_0^t e^{-\int_s^t v(u)\, du} |v(s)| \int_{s - r_1(s)}^s |\tilde{a}(h(u)) + v(u)|\, du\, ds + \int_0^t e^{-\int_s^t v(u)\, du} |v(s - r_1(s))(1 - r_1'(s))|\, ds \right\} \\ & + g(l) \int_0^t e^{-\int_s^t v(u)\, du} |b(s)|\, ds + (l - f(l)) \int_0^t e^{-\int_s^t v(u)\, du} v(s)\, ds \\ \le \quad & (1 - \alpha) f(l) - \alpha g(l) + \alpha f(l) + \alpha g(l) + l - f(l) = l.\end{aligned}$$

Hence $|(P\varphi)(t)| \le l$ for $t \in [r_0, \infty)$. Therefore $P\varphi \in C^l_\phi$.

Next, we will show that P is a contraction mapping in C^l_ϕ. For $\varphi, \eta \in C^l_\phi$,

$$\begin{aligned}
&|(P\varphi)(t) - (P\eta)(t)|\, e^{-q(t)} \\
&\le \int_0^t e^{-dL\int_s^t [v(u)+|\tilde{a}(h(u))|]\,du}\,[v(s) + |\tilde{a}(h(s))|]\, L|\varphi(s) - \eta(s)|e^{-q(s)}\,ds \\
&\quad + \int_0^t e^{-\int_s^t v(u)\,du} v(s) \int_{s-r_1(s)}^s e^{-dL\int_s^t [v(s)+|\tilde{a}(h(s))|]\,ds}[|\tilde{a}(h(u))| + v(u)] \times L|\varphi(u) - \eta(u)|e^{-q(u)}\,du\,ds \\
&\quad + \int_0^t e^{-dL\int_s^t w(u)\,du} w(s)L|\varphi(s - r_1(s)) - \eta(s - r_1(s))|e^{-q(s-r_1(s))}\,ds \\
&\quad + \int_{t-r_1(t)}^t e^{-dL\int_s^t [v(u)+|\tilde{a}(h(u))|]\,du}[|\tilde{a}(h(s))| + v(s)]L|\varphi(s) - \eta(s)|e^{-q(s)}\,ds \\
&\quad + \int_0^t e^{-dL\int_s^t |b(u)|\,du}|b(s)|L|\varphi(s - r_2(s)) - \eta(s - r_2(s))|e^{-q(s-r_2(s))}\,ds.
\end{aligned}$$

So, we have

$$|(P\varphi)(t) - (P\eta)(t)|e^{-q(t)} \quad \le \quad \left(\frac{1}{dL} + \frac{1}{dL} + \frac{1}{dL} + \frac{1}{dL} + \frac{1}{dL}\right) L|\varphi - \eta|_q \le \frac{5}{d}|\varphi - \eta|_q,$$

for all $t > 0$. Thus $|P\varphi - P\eta|_q \le \frac{5}{d}|\varphi - \eta|_q$. Since $d > 5$, we conclude that P is a contraction on $(C^l_\phi, |\cdot|_q)$. □

We are now ready to prove Theorem 1.7.

Proof. By the contraction mapping principle, P has a unique fixed point x in C^l_ϕ, which is by construction a solution of (4) with $x(t) = \phi(t)$ on $[r_0, 0]$ and $|x(t)| \le l$.

Let $\epsilon > 0$ be given. Then, we choose $m > 0$ so that $m < min\{l, \epsilon\}$. By considering C^m_ϕ, we obtain existence of a $\delta > 0$ such that $\|\phi\| < \delta$ implies that the unique solution of (4) with $x(t) = \phi(t)$ on $[r_0, 0]$ satifies $|x(t)| \le m < \epsilon$ for all $t \ge r_0$. This shows that the zero solution of (4) is stable. This completes the proof of Theorem 1.7. □

Remark 4.2. *It is an open problem whether the zero solution of* (4) *is asymptotically stable. Our method of proof can not be used to solve this problem. The reason is that if we would add the condition to C^l_ϕ that $\varphi(t) \to 0$ as $t \to \infty$, then C^l_ϕ would no longer be complete under the weighted metric.*

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