Stability results for stochastic delayed recurrent neural networks with discrete and distributed delays

Guiling Chen, Onno van Gaans, Sjoerd Verduyn Lunel

Mathematical Institute, Leiden University, P.O.Box 9512, 2300 RA, Leiden, The Netherlands

Report MI-2013-08

Abstract

We present new conditions for asymptotic stability and exponential stability of a class of stochastic recurrent neural networks with discrete and distributed time varying delays. Our approaches are based on the method using fixed point theory and the method using an appropriate integral inequality, which do not resort to any Lyapunov function. Our results neither require the boundedness, monotonicity and differentiability of the activation functions nor differentiability of the time varying delays. In particular, a class of neural networks without stochastic perturbations is also considered by using the two approaches. Examples are given to illustrate our main results.

Keywords: Fixed point theory, asymptotic stability, exponential stability, stochastic recurrent neural networks, trivial solution, variable delays, Burkholder-Davis-Gundy inequality, Doob's inequality.

1. Introduction and main results

Neural networks have received an increasing interest in various areas [5, 8]. The stability of neural networks [6, 16, 37, 38] is critical for signal processing, especially in image processing and solving some classes of optimization problems. For the stochastic effects to the dynamical behaviors of neural networks, Liao and Mao [14, 15] initiated the study of stability and instability of stochastic neural networks.

Due to the finite switching speed of neurons and amplifiers, time delays which may lead to instability and bad performance in neural processing and signal transmission are commonly encountered in both biological and artificial neural networks. In addition, neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths [31]. Thus there will be a distribution of conduction velocities along these pathways and a distribution of propagation delays [39]. In these circumstances the signal propagation is not instantaneous and may not be suitably modeled with discrete delays. Therefore, a more appropriate way which incorporates continuously distributed delays in neural network models has been used. Further, due to random fluctuations and probabilistic causes in the network, noises do exist in a neural network. Thus, it is necessary and rewarding to study stochastic effects to the stability property of neural networks.

Many interesting articles [9, 10, 11, 28, 32] have considered some classes of the stochastic neural networks. Hu et al.[9] and Wan and Sun [32] studied a class of stochastic neural networks with the delays constant and discrete. The activation functions appearing in [9] are required to be bounded. Liao and Mao [17] investigated exponential stability of stochastic delay interval systems via Razumikihin-type theorems developed in [23], several exponential stability results were provided. However, the results are not only difficult to verify but also restrict to the case of the interval matrices $\tilde{A} = \tilde{B} = \tilde{C} = 0$. Sun and Cao [28] investigated the *pth* moment exponential stability of stochastic differential equations with discrete bounded delays by using the method of variation parameter, inequality technique and stochastic analysis. This method was firstly used in [32], which does not require the boundedness, monotonicity and differentiability of the activation functions. However, the stability criteria in [28] requires that the delay functions are bounded, differentiable and their derivatives are simultaneously required to be not greater than 1. This may impose a very strict constraint on model because time delays sometimes vary dramatically with time in real circuits (see [36]).

Huang et al. [10, 11] investigated the exponential stability of stochastic differential equations with discrete time-varying delays with the help of a Lyapunov function and Dini derivative. However, the use of their criteria depends very much on the choice of positive numbers k_{ij} etc. and a positive diagonal matrix M (see Theorem 3.3 in [10] and Theorem 3.3 in [11]).

Recently, Burton [2] has utilized the fixed point method to investigate the stability for deterministic systems, Luo [19] and Appleby [1] have applied this method to deal with the stability problems for stochastic delay differential equations, and afterwards, a great number of classes of stochastic delay differential equations are discussed by using fixed point method, see, for example, [5, 20, 21, 25, 26]. It turned out that fixed point method is a powerful technique in dealing with stability problems for differential equations with delays and stochastic differential equations with delays, and it can yield the existence, uniqueness and stability criteria of the considered system in one step by a fixed point argument, which is impossible when using the other methods. Chen [4, 3] has applied the method by using an appropriate integral inequality to study exponential stability of some classes of stochastic delay differential equations, and it turns out that it is a convenient way to discuss exponential stability of a system.

The aim of this paper is to study a general class of stochastic neural networks by using fixed point method and the method by employing an appropriate integral inequality. Indeed, we consider the following class of stochastic neural networks with varying discrete and distributed delays which is described by

$$\begin{aligned} dx_i(t) &= \left[-c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau(t))) + \sum_{j=1}^{n} l_{ij} \int_{t-r(t)}^{t} h_j(x_j(s))\, ds\right] dt \\ &\quad + \sum_{j=1}^{n} \sigma_{ij}(t, x_j(t), x_j(t-\tau(t)))\, d\omega_j(t), \end{aligned} \tag{1}$$

or

$$dx(t) = \left[-Cx(t) + Af(x(t)) + Bg(x(t-\tau(t))) + W\int_{t-r(t)}^{t} h(x(s))\, ds\right] dt + \sigma(t, x(t), x(t-\tau(t)))\, d\omega(t),$$

for $i = 1, 2, 3, \cdots, n$, where $x(t) = (x_1(t), x_2(t), \cdots x_n(t))^T \in \mathbb{R}^n$ is the state vector associated with the neurons; $C = \text{diag}(c_1, c_2, \cdots c_n) > 0$ where $c_i > 0$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and the external stochastic perturbations; $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ and $W = (l_{ij})_{n\times n}$ represent the connection weight matrix, delayed connection weight matrix and distributed delayed connection weight matrix, respectively; f_j, g_j, h_j are activation functions, $f(x(t)) = (f_1(x(t)), f_2(x(t)), \cdots f_n(x(t)))^T \in \mathbb{R}^n$, $g(x(t)) = (g_1(x(t)), g_2(x(t)), \cdots g_n(x(t)))^T \in \mathbb{R}^n$, $h(x(t)) = (h_1(x(t)), h_2(x(t)), \cdots h_n(x(t)))^T \in \mathbb{R}^n$, where $\tau(t)$ and $r(t)$ denote discrete time varying delay and distributed time varying delay, respectively. Moreover, $\omega(t) = (\omega_1(t), \omega_2(t), \cdots \omega_n(t))^T \in \mathbb{R}^n$ is an n-dimensional Brownion motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with natural complete filtration $\{\mathcal{F}_t\}_{t\geq 0}$ ($i.e. \mathcal{F}_t$ = completion of $\sigma\{\omega(s) : 0 \leq s \leq t\}$) and $\sigma : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $\sigma = (\sigma_{ij})_{n\times n}$ is the diffusion coefficient matrix. Denote $\vartheta = \inf_{t\geq 0}\{t-\tau(t), t-r(t)\}$.

The initial condition for the system (1) is given by

$$x(t) = \phi(t), \qquad t \in [\vartheta, 0], \tag{2}$$

where $\phi(t) = (\phi_1(t), \phi_2(t), \cdots \phi_n(t))^T \in C\left([\vartheta, 0], L^p_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)\right)$ with the norm defined as

$$\|\phi\|^p = \sup_{\vartheta\leq s\leq 0}\left[\mathbb{E}\sum_{i=1}^{n} |\phi_i(s)|^p\right],$$

where $\mathbb{E}$ denotes expectation with respect to the probability measure $\mathbb{P}$.

To obtain our main results, we suppose the following conditions are satisfied:

(A1) the delays $\tau(t), r(t)$ are continuous functions such that $t-\tau(t) \to \infty$ and $t-r(t) \to \infty$ as $t \to \infty$;

(A2) $f_j(x)$, $g_j(x)$, and $h_j(x)$ satisfy Lipschitz conditions. That is, for each $j = 1, 2, 3, \cdots n$, there exists constants α_j, β_j, γ_j such that for every $x, y \in \mathbb{R}^n$,

$$|f_j(x) - f_j(y)| \le \alpha_j |x - y|, \quad |g_j(x) - g_j(y)| \le \beta_j |x - y|, \quad |h_j(x) - h_j(y)| \le \gamma_j |x - y|;$$

(A3) Assume that $f(0) \equiv 0$, $g(0) \equiv 0$, $h(0) \equiv 0$, $\sigma(t, 0, 0) \equiv 0$;

(A4) $\sigma(t, x, y)$ satisfies a Lipschitz condition. That is, there are nonnegative constants μ_i and ν_i such that

$$\text{trace}\left[(\sigma(t,x,y) - \sigma(t,u,v))^T(\sigma(t,x,y) - \sigma(t,u,v))\right] \le \sum_{i=1}^{n}\left[\mu_i(x_i - u_i)^2 + \nu_i(y_i - v_i)^2\right].$$

It follows from [7, 22] that under the hypothesis $(A2)$, $(A3)$ and $(A4)$, system (1) with initial condition (2) has one unique global solution which is denoted by $x(t, 0, \phi)$ or $x(t)$, and $\mathbb{E}\sup_{0\le s\le t} \|x(s, 0, \phi)\|^p < \infty$ for $t > 0$. Clearly, system (1) admits the trivial solution $x(t, 0, 0) \equiv 0$.

Definition 1.1. *The trivial solution of system* (1) *is said to be stable in pth* ($p \ge 2$) *moment if for arbitrary given* $\epsilon > 0$, *there exists a* $\delta > 0$ *such that* $\mathbb{E}\|\phi\|^p < \delta$ *yields that*

$$\mathbb{E}\|x(t, 0, \phi)\|^p < \epsilon, \quad t \ge 0.$$

where $\phi(t) \in C\left([\vartheta, 0], L^p_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)\right)$. *In particular, when* $p = 2$, *the trivial solution is said to be mean square stable.*

Definition 1.2. *The trivial solution of system* (1) *is said to be asymptotically stable in pth* ($p \ge 2$) *moment if it is stable in pth* ($p \ge 2$) *moment and there exists a scalar* $\sigma > 0$, *such that* $E\|\phi\|^p < \sigma$ *implies*

$$\lim_{t\to\infty} \mathbb{E}\|x(t, 0, \phi)\|^p = 0.$$

where $\phi(t) \in C\left([\vartheta, 0], L^p_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)\right)$.

Definition 1.3. *The trivial solution of system* (1) *is said to be pth* ($p \ge 2$) *moment exponentially stable if there exists a pair of constants* $\lambda, C > 0$ *such that*

$$\mathbb{E}\|x(t, t_0, \phi)\|^p \le C\mathbb{E}\|\phi\|^p e^{-\lambda t}, \quad t \ge 0,$$

holds for $\phi(t) \in C\left([\vartheta, 0], L^p_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)\right)$. *Especially, when* $p = 2$, *we speak of exponentially stable in mean square.*

Different choices of norms are defined for space of stochastic processes. The norms we choose should be such that the space under consideration is complete and the equation yields a contraction with respect to the norm. For the system (1) with initial condition (2), we consider the following two different complete spaces which are defined by using two types of norms.

Define $\mathcal{S}_\phi$ the space of all $\mathcal{F}_t$-adapted processes $\varphi(t, \omega) : [\vartheta, \infty) \times \Omega \to \mathbb{R}^n$ such that $\varphi \in C\left([\vartheta, \infty), L^p_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)\right)$. Moreover, we set $\varphi(t, \omega) = \phi(t)$ for $t \in [\vartheta, 0]$ and $\mathbb{E}\sum_{i=1}^n |\varphi_i(t)|^p \to 0$ as $t \to \infty$, $i = 1, 2, \cdots n$. If we define the norm

$$\|\varphi\|^p := \sup_{t\ge\vartheta}\left(\mathbb{E}\sum_{i=1}^{n} |\varphi_i(t)|^p\right), \tag{3}$$

then $\mathcal{S}_\phi$ is a complete metric space. Using the contractive mapping defined on the space $\mathcal{S}_\phi$ and applying the contraction mapping principle, we obtain our first result.

Theorem 1.4. *Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the distributed delay* $r(t)$ *is bounded by a constant* r;

(ii)

$$\alpha \triangleq 5^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q} + 5^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q} \tag{4}$$

$$+5^{p-1}\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^p\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q} + 5^{p-1}n^{p-1}\sum_{i=1}^{n} c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right) < 1,$$

where $\mu = \max\{\mu_1, \mu_2, \cdots \mu_n\}$, $\nu = \max\{\nu_1, \nu_2, \cdots \nu_n\}$,

then the trivial solution of (1) *is pth moment asymptotically stable.*

Consider the case when both the discrete delay $\tau(t)$ and the distributed delay $r(t)$ are bounded by a constant τ. Define C_ϕ the space of all $\mathcal{F}_t$-adapted processes $\varphi(t,\omega) : [-\tau,\infty)\times\Omega \to \mathcal{R}^n$ such that $\varphi \in C\left([-\tau,\infty), L^p_{\mathcal{F}_0}(\Omega;\mathbb{R}^n)\right)$. Moreover, we set $\varphi(t,\omega) = \phi(t)$ for $t \in [-\tau, 0]$ and for $t \to \infty$, $\sum_{i=1}^n \mathbb{E}\sup_{t-\tau\le s\le t}|\varphi_i(s)|^p \to 0$. The norm on C_ϕ is defined as

$$\|\varphi\|^p = \sup_{t\ge 0}\left[\sum_{i=1}^{n}\mathbb{E}\left(\sup_{t-\tau\le s\le t}|\varphi_i(s)|^p\right)\right]. \tag{5}$$

then C_ϕ is a complete metric space. Using the contraction defined on the space C_ϕ and applying the contraction mapping principle, we obtain our second result.

Theorem 1.5. *Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the discrete delay* $\tau(t)$ *and the distributed delay* $r(t)$ *are bounded by a constant* τ*;*

(ii)

$$\alpha \triangleq 5^{p-1}e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q} + 5^{p-1}e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}$$

$$+5^{p-1}\tau^p e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q} + 5^{p-1}K_p n^p e^{pc\tau}q^p c^{1-p/2}(2c)^{-1}\left(\mu^{p/2}+\nu^{p/2}\right) < 1, \tag{6}$$

where $c = \min\{c_1, c_2, \cdots c_n\}$, $\mu = \max\{\mu_1, \mu_2, \cdots \mu_n\}$, $\nu = \max\{\nu_1, \nu_2, \cdots \nu_n\}$,

then the trivial solution of (1) *is pth moment asymptotically stable.*

Remark 1.6. *In Theorem* 1.5*, we obtain that*

$$\lim_{t\to\infty}\left\{\mathbb{E}\left[\sup_{t-\tau\le s\le t}\|x(s,0,\phi)\|^p\right]\right\} = 0,$$

that is, for any function $s \to x_t(s,0,\phi)$*, we have that* $\lim_{t\to\infty}\mathbb{E}\|x_t(\cdot,0,\phi)\|^p_{C[-\tau,0]} = 0$*, which implies*

$$\lim_{t\to\infty}\mathbb{E}\|x(t,0,\phi)\|^p = 0.$$

Remark 1.7. *In some papers, see, for example,* [18, 19, 33, 34]*, the norm for the space of stochastic process is defined as*

$$\|\varphi\|_{[0,t]} = \left\{\mathbb{E}\left(\sup_{s\in[0,t]}|\varphi(s,\omega)|^2\right)\right\}^{1/2}.$$

As in [19]*, in order to show* $P(\mathcal{S}) \subseteq \mathcal{S}$*, we need to estimate* $\mathbb{E}\sup_{s\in[0,t]}|I_5(s)|^2$*, where*

$$I_5(s) = \int_0^s e^{-\int_z^s h(u)\,du}\left[c(z)x(z) + e(z)x(z-\delta(z))\right]d\omega(z).$$

However, $I_5(s)$ *is not a local martingale* (*see Section* 8 *for its proof*)*. Hence, Burkholder-Davis-Gundy Inequality can not be applied directly.*

Using an appropriate integral inequality, we obtain sufficient conditions for exponential stability of (1) with initial condition (2), which is our third result.

Theorem 1.8. *Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the discrete delay* $\tau(t)$ *and distributed delay* $r(t)$ *are bounded by a constant* τ*;*

(ii)

$$5^{p-1}c^{-p}\sum_{i=1}^{n}\left[\sum_{j=1}^{n}|a_{ij}|^{q}|\alpha_{j}|^{q}\right]^{p/q}+5^{p-1}c^{-p}\sum_{i=1}^{n}\left[\sum_{j=1}^{n}|b_{ij}|^{q}|\beta_{j}|^{q}\right]^{p/q} \tag{7}$$

$$+5^{p-1}\left(\frac{\tau}{c}\right)^{p}\sum_{i=1}^{n}\left[\sum_{j=1}^{n}|l_{ij}|^{q}|\gamma_{j}|^{q}\right]^{p/q}+5^{p-1}n^{p}c^{-p/2}(\mu^{p/2}+\nu^{p/2})<1,$$

where $c=\min\{c_1,c_2,\cdots c_n\}$, $\mu=\max\{\mu_1,\mu_2,\cdots\mu_n\}$, $\nu=\max\{\nu_1,\nu_2,\cdots\nu_n\}$,

then the trivial solution of (1) *is exponentially stable in pth moment,*

Remark 1.9. *The stability criteria we provided in our main results are only in terms of the system parameters* c_i, a_{ij}, b_{ij}, l_{ij}, *etc. Hence, these criteria can usually be verified easily in applications.*

Remark 1.10. *Many articles, see, for example,* [27, 28] *have studied the case and special case of stochastic neural network* (1)*. However, the delays should satisfy the following condition:*

(H) *the discrete delay* $\tau(t)$ *is differentiable function and the distributed delay* $r(t)$ *is non-negative and bounded, that is, there exist constants* τ_M,ζ,r_M *such that*

$$0\le\tau(t)\le\tau_M,\quad \tau'(t)\le\zeta,\quad r(t)\le r_M, \tag{8}$$

In Sun and Cao [28]*, it seems that the constraint condition* (H) *on the discrete delays can be relaxed as* $\tau(t)$ *is bounded.*

As an example, we consider a two-dimensional stochastically perturbed Hopfield neural network with time-varying delays,

$$dx(t)=[-Cx(t)+Af(x(t))+Bg(x_\tau(t))]\,dt+\sigma(t,x(t),x_\tau(t))\,d\omega(t),$$

where $f(x)=\frac{1}{5}\arctan x$, $g(x)=\frac{1}{5}\tanh x=\frac{1}{5}(e^x-e^{-x})/(e^x+e^{-x})$, $\tau(t)=\frac{1}{2}|\sin t|+\frac{1}{2}$,

$$C=\begin{pmatrix}5&0\\0&4.5\end{pmatrix},\quad A=\begin{pmatrix}2&0.4\\0.6&1\end{pmatrix}\quad\text{and}\quad B=\begin{pmatrix}-0.8&2\\1&4\end{pmatrix}.$$

In this example, the discrete delay is bounded but not differentiable.

Consider the case when there are no stochastic effects on the system (1), which then comes down to the following neural network described by

$$\frac{dx_i(t)}{dt}=-c_ix_i(t)+\sum_{j=1}^{n}a_{ij}f_j(x_j(t))+\sum_{j=1}^{n}b_{ij}g_j(x_j(t-\tau(t)))+\sum_{j=1}^{n}d_{ij}\int_{t-r(t)}^{t}h_j(x_j(s))\,ds,\quad i=1,2,3,\cdots,n, \tag{9}$$

or

$$\frac{dx(t)}{dt}=-Cx(t)+Af(x(t))+Bg(x-\tau(t))+D\int_{t-r(t)}^{t}h(x(s))\,ds,$$

where $x(\cdot)=(x_1(\cdot),x_2(\cdot),\cdots,x_n(\cdot))^T$ is the neuron state vector of the transformed system (9).

The initial condition for the system (9) is

$$x(t)=\phi(t),\qquad t\in[\vartheta,0], \tag{10}$$

where ϕ is a continuous function with the norm defined by $\|\phi\|=\sup_{\vartheta\le s\le0}\sum_{i=1}^{n}|\phi_i(s)|$.

Assume that $(A1)-(A3)$ are satisfied, then (9) and (10) admit a trivial solution $x=0$. Denote by $x(t;s;\phi)=(x_1(t;s,\phi_1),\cdots,x_n(t;s,\phi_n))^T\in\mathbb{R}^n$ the solution of (9) with initial condition (10).

Definition 1.11. *For the system* (9) *with initial condition* (10), *we have that*

(i) *the trivial solution of* (9) *is said to be stable if for any* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for any initial condition* $\phi(s) \in C([\vartheta, 0], \mathbb{R}^n)$ *satisfying* $\|\phi\| < \delta$, *we have for the corresponding solution that* $\|x(t, s, \phi)\| < \epsilon$ *for* $t \geq 0$;

(ii) *the trivial solution of* (9) *is said to be asymptotically stable if it is stable and for any initial condition* $\phi(s) \in C([\vartheta, 0], \mathbb{R}^n)$ *we have for the corresponding solution that* $\lim_{t\to\infty} \|x(t, s, \phi)\| = 0$;

(iii) *the trivial solution of* (9) *is said to be globally exponentially stable if there exist scalars* $k > 0$ *and* $\alpha > 0$ *such that for any initial condition* $\phi(s) \in C([\vartheta, 0], \mathbb{R}^n)$, *we have for the corresponding solution that* $\|x(t, s, \phi)\| \leq \alpha e^{-kt}\|\phi\|$ *for* $t \geq 0$.

Define $\mathcal{H}_\phi = \mathcal{H}_{1\phi} \times \mathcal{H}_{2\phi} \times \cdots \mathcal{H}_{n\phi}$, where $\mathcal{H}_{i\phi}$ is the space consisting of continuous functions $\varphi_i(t) : \mathbb{R}^+ \to \mathbb{R}$ such that $\varphi_i(\theta) = \phi(\theta)$ for $\vartheta \leq \theta \leq 0$ and $\varphi_i(t) \to 0$ as $t \to \infty$, $i = 1, 2 \cdots n$. For any $\varphi(t) = (\varphi_1(t), \varphi_2(t), \cdots, \varphi_n(t)) \in \mathcal{H}_\phi$ and $\eta(t) = (\eta_1(t), \eta_2(t), \cdots, \eta_n(t)) \in \mathcal{H}_\phi$, if we define the metric as $d(\varphi, \eta) = \sup_{t\geq\vartheta} \sum_{i=1}^n |\varphi_i(t) - \eta_i(t)|$, then $\mathcal{H}_\phi$ becomes a complete metric space.

Using the contraction mapping defined on the space H_ϕ and applying the contraction mapping principle, we obtain our fourth result.

Theorem 1.12. *Suppose that the assumptions* (A1)-(A3) *hold. If the following conditions are satisfied,*

(i) *the distributed delay* $r(t)$ *is bounded by a constant* r;

(ii)

$$\alpha \triangleq \sum_{i=1}^n \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^n \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \sum_{i=1}^n \frac{r}{c_i} \max_{j=1,2,\cdots,n} |d_{ij}\gamma_j| < 1, \tag{11}$$

then the trivial solution of (9) *is asymptotically stable.*

By establishing an appropriate integral inequality, we obtain sufficient conditions for exponential stability of (9), which is our fifth result.

Theorem 1.13. *Suppose that the assumptions* (A1)-(A3) *hold. If the following conditions are satisfied,*

(i) *the discrete delay* $\tau(t)$ *and the distributed delay* $r(t)$ *are bounded by a constant* τ;

(ii)

$$\frac{1}{c} \sum_{i=1}^n \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \frac{1}{c} \sum_{i=1}^n \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \frac{1}{c} \sum_{i=1}^n \tau \max_{j=1,2,\cdots,n} |d_{ij}\gamma_j| < 1, \quad c = \min\{c_1, c_2, \cdots c_n\}, \tag{12}$$

then the trivial solution of (9) *with initial condition* (10) *is exponentially stable.*

Remark 1.14. *Several exponential stability results* [13, 29, 30] *were provided for the system* (9), *by constructing an appropriate Lyapunov functional and employing linear matrix inequality* (LMI) *method. However, the delays in those results should satisfy the following condition* (H). *From our main results, we need not know the particular form related to the delays, we only need to know that the delays are bounded. Furthermore, Theorem* 1.12 *is an extension and improvement of the result in Lai and Zhang* [12].

As an example, we consider a cellular neural network with time varying delays

$$\frac{dx(t)}{dt} = -Cx(t) + Ag(x(t)) + Bg(x - \tau(t)),$$

where

$$C = \begin{pmatrix} 2 & 0 \\ 0 & 4.5 \end{pmatrix}, \quad A = \begin{pmatrix} -0.2 & 0 \\ 2 & 0.1 \end{pmatrix}, \quad B = \begin{pmatrix} 0.1 & 0.01 \\ 0.2 & 0.1 \end{pmatrix}.$$

The activation function is described by $g(x) = \frac{|x+1|-|x-1|}{2}$. The time-varying delay is $\tau(t) = \frac{1}{5}|\cos t|$. It is clear that the discrete delay is bounded but not differentiable. Hence, the results in [13, 29, 30] are not applicable.

The rest of this paper is organized as follows. In Section 2, we present a proof of Theorem 1.4. The proof of Theorem 1.5 is presented in Section 3 and the proof of Theorem 1.8 is given in Section 4. we present the proofs of Theorem 1.12 and Theorem 1.13 in Section 5 and Section 6, respectively. Some examples are given to illustrate our main results in Section 7 and an appendix is given in Section 8.

2. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. We start with some preparations.

Lemma 2.1. ([32]) *If $\omega(t) = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is a n-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then we have the following formula*

$$\mathbb{E}\left(\int_0^t f_i(s)\,d\omega_i(s)\int_0^t f_j(s)\,d\omega_j(s)\right) = \mathbb{E}\int_0^t f_i(s) f_j(s)\,d\langle\omega_i, \omega_i\rangle_s,$$

where $\langle\omega_i, \omega_i\rangle_s = \delta_{ij}s$ are the cross-variations, δ_{ij} is correlation coefficient, $1 \le i, j \le n$.

If we multiply both sides of (1) by $e^{c_i t}$ and integrate from 0 to t, we obtain

$$\begin{aligned} x_i(t) &= e^{-c_i t}x_i(0) + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^n a_{ij} f_j(x_j(s))\,ds + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^n b_{ij}g_j(x_j(s-\tau(s)))\,ds \qquad (13)\\ &\quad + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^n l_{ij}\int_{s-r(s)}^s h_j(x_j(u))\,du\,ds + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^n \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\,d\omega_j(s). \end{aligned}$$

for $t \ge 0$, $i = 1, 2, 3, \cdots, n$.

Lemma 2.2. *Define an operator by $(Q\varphi)(t) = \phi(t)$ for $t \in [\vartheta, 0]$, and for $t \ge 0$, $i = 1, 2, 3, \cdots, n$,*

$$\begin{aligned} (Q\varphi_i)(t) &= e^{-c_i t}\varphi_i(0) + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^n a_{ij} f_j(\varphi_j(s))\,ds + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^n b_{ij}g_j(\varphi_j(s-\tau(s)))\,ds \qquad (14)\\ &\quad + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^n l_{ij}\int_{s-r(s)}^s h_j(\varphi_j(u))\,du\,ds + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^n \sigma_{ij}(s, \varphi_j(s), \varphi_j(s-\tau(s)))\,d\omega_j(s). \end{aligned}$$

Suppose that the assumption (A1)-(A4) *holds. If condition* (4) *holds, then $Q : \mathcal{S}_\phi \to \mathcal{S}_\phi$ and Q is a contraction mapping.*

Proof. Denote $(Q\varphi_i)(t) := J_{1i}(t) + J_{2i}(t) + J_{3i}(t) + J_{4i}(t) + J_{5i}(t)$, where

$$\begin{aligned} J_{1i}(t) &= e^{-c_i t}\varphi_i(0), \qquad J_{2i}(t) = \int_0^t e^{-c_i(t-s)}\sum_{j=1}^n a_{ij} f_j(\varphi_j(s))\,ds,\\ J_{3i}(t) &= \int_0^t e^{-c_i(t-s)}\sum_{j=1}^n b_{ij}g_j(\varphi_j(s-\tau(s)))\,ds,\\ J_{4i}(t) &= \int_0^t e^{-c_i(t-s)}\sum_{j=1}^n l_{ij}\int_{s-r(s)}^s h_j(\varphi_j(u))\,du\,ds,\\ J_{5i}(t) &= \int_0^t e^{-c_i(t-s)}\sum_{j=1}^n \sigma_{ij}(s, \varphi_j(s), \varphi_j(s-\tau(s)))\,d\omega_j(s). \end{aligned}$$

Step1. From the definition of Banach space $\mathcal{S}_\phi$, we have that $\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)|^p<\infty$, for all $t\geq 0$, $\varphi\in\mathcal{S}_\phi$.

Step2. We prove the continuity in *pth* moment of Q on $[0,\infty)$. Let $x\in\mathcal{S}_\phi$, $t_1\geq 0$, $|r|$ be sufficiently small and take the limit $r\to 0^+$. We have

$$\begin{aligned}\mathbb{E}\sum_{i=1}^{n}|J_{2i}(t_1+r)-J_{2i}(t_1)|^p &= \mathbb{E}\sum_{i=1}^{n}\bigg|\int_0^{t_1}\Big(e^{-c_i(t_1+r-s)}-e^{-c_i(t_1-s)}\Big)\sum_{j=1}^{n}a_{ij}f_j(x_j(s))\,ds\\ &\quad+\int_{t_1}^{t_1+r}e^{-c_i(t_1+r-s)}\sum_{j=1}^{n}a_{ij}f_j(x_j(s))\,ds\bigg|^p\to 0\quad\text{as}\quad r\to 0.\end{aligned}$$

Similarly, we have that

$$\mathbb{E}\sum_{i=1}^{n}|J_{3i}(t_1+r)-J_{3i}(t_1)|^p\to 0\quad\text{as}\quad r\to 0,\qquad \mathbb{E}\sum_{i=1}^{n}|J_{4i}(t_1+r)-J_{4i}(t_1)|^p\to 0\quad\text{as}\quad r\to 0.$$

In the following, we check the continuity of $J_{5i}(t)$.

$$\begin{aligned}&\mathbb{E}\sum_{i=1}^{n}|J_{5i}(t_1+r)-J_{5i}(t_1)|^p\\ &=\mathbb{E}\sum_{i=1}^{n}\bigg|\int_0^{t_1}\Big(e^{-c_i(t_1+r-s)}-e^{-c_i(t_1-s)}\Big)\sum_{j=1}^{n}\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,d\omega_j(s)\\ &\qquad+\int_{t_1}^{t_1+r}e^{-c_i(t_1+r-s)}\sum_{j=1}^{n}\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,d\omega_j(s)\bigg|^p\\ &\leq n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\bigg|\int_0^{t_1}\Big(e^{-c_i(t_1+r-s)}-e^{-c_i(t_1-s)}\Big)\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,d\omega_j(s)\\ &\qquad+\int_{t_1}^{t_1+r}e^{-c_i(t_1+r-s)}\sum_{j=1}^{n}\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,d\omega_j(s)\bigg|^p\\ &\leq(2n)^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\bigg|\int_0^{t_1}\Big|e^{-c_i(t_1+r-s)}-e^{-c_i(t_1-s)}\Big|\,|\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))|\,d\omega_j(s)\bigg|^p\\ &\qquad+(2n)^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\bigg|\int_{t_1}^{t_1+r}e^{-c_i(t_1+r-s)}|\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))|\,d\omega_j(s)\bigg|^p\\ &=(2n)^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\bigg\{\mathbb{E}\bigg[\int_0^{t_1}|e^{-c_i(t_1+r-s)}-e^{-c_i(t_1-s)}|^2\sigma_{ij}^2(s,x_j(s),x_j(s-\tau(s)))\,ds\bigg]^{p/2}\\ &\qquad+\mathbb{E}\bigg[\int_{t_1}^{t_1+r}e^{-2c_i(t_1+r-s)}\sigma_{ij}^2(s,x_j(s),x_j(s-\tau(s)))\,ds\bigg]^{p/2}\bigg\}\to 0\quad\text{as}\quad r\to 0.\end{aligned}$$

Thus, Q is indeed continuous in *pth* moment on $[0,\infty)$.

Step3. We prove that $Q(\mathcal{S}_\phi)\subseteq\mathcal{S}_\phi$.

$$\mathbb{E}\sum_{i=1}^{n}|Q\varphi_i(t)|^p=\mathbb{E}\sum_{i=1}^{n}\bigg|\sum_{j=1}^{5}J_{ji}(t)\bigg|^p\leq 5^{p-1}\sum_{j=1}^{5}\mathbb{E}\sum_{i=1}^{n}|J_{ji}(t)|^p. \tag{15}$$

Now, we estimate the terms on the right sides of the above inequality.

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^{n}|J_{2i}(t)|^{p} &= \sum_{i=1}^{n}\mathbb{E}\left|\int_{0}^{t}e^{-c_i(t-s)}\sum_{j=1}^{n}a_{ij}f_j(\varphi_j(s))\,ds\right|^{p}\\
&\le \sum_{i=1}^{n}\mathbb{E}\left[\int_{0}^{t}e^{-\frac{c_i(t-s)}{q}}e^{-\frac{c_i(t-s)}{p}}\sum_{j=1}^{n}|a_{ij}||f_j(\varphi_j(s))|\,ds\right]^{p}\\
&\le \sum_{i=1}^{n}\mathbb{E}\left\{\left[\int_{0}^{t}e^{-c_i(t-s)}\,ds\right]^{p/q}\int_{0}^{t}e^{-c_i(t-s)}\left[\sum_{j=1}^{n}|a_{ij}||f_j(\varphi_j(s))|\right]^{p}ds\right\}\\
&\le \sum_{i=1}^{n}c_i^{-p/q}\mathbb{E}\left\{\int_{0}^{t}e^{-c_i(t-s)}\left[\sum_{j=1}^{n}|a_{ij}||\alpha_j||\varphi_j(s)|\right]^{p}ds\right\}\\
&\le \sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|a_{ij}|^{q}|\alpha_j|^{q}\right)^{p/q}\int_{0}^{t}e^{-c_i(t-s)}\mathbb{E}\left[\sum_{j=1}^{n}|\varphi_j(s)|^{p}\right]ds. \qquad (16)
\end{aligned}$$

Since $\varphi(t)\in\mathcal{S}_\phi$, we have that $\lim_{t\to\infty}\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)|=0$. Thus for any $\epsilon>0$, there exists $T_1>0$ such that $t\ge T_1$ implies $\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)|<\epsilon$, combining with (16), we obtain that

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^{n}|J_{2i}(t)|^{p} &\le \sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|a_{ij}|^{q}|\alpha_j|^{q}\right)^{p/q}\int_{0}^{T_1}e^{-c_i(t-s)}\mathbb{E}\left[\sum_{j=1}^{n}|\varphi_j(s)|^{p}\right]ds\\
&\quad+\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|a_{ij}|^{q}|\alpha_j|^{q}\right)^{p/q}\int_{T_1}^{t}e^{-c_i(t-s)}\mathbb{E}\left[\sum_{j=1}^{n}|\varphi_j(s)|^{p}\right]ds\\
&< \sum_{i=1}^{n}c_i^{-p}e^{-c_it}(e^{c_iT_1}-1)\left(\sum_{j=1}^{n}|a_{ij}|^{q}|\alpha_j|^{q}\right)^{p/q}\sup_{0\le s\le T_1}\left\{\mathbb{E}\left[\sum_{j=1}^{n}|\varphi_j(s)|^{p}\right]\right\}+\epsilon\sum_{i=1}^{n}c_i^{-p}\left[\sum_{j=1}^{n}|a_{ij}|^{q}|\alpha_j|^{q}\right]^{p/q}
\end{aligned}$$

Hence, from the fact that $c_i>0$ $(i=1,2,\cdots,n)$, we obtain that $\mathbb{E}\sum_{i=1}^{n}|J_{2i}(t)|^{p}\to 0$ as $t\to\infty$.

With the similar computation as (16), we obtain that

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^{n}|J_{3i}(t)|^{p} &\le \sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|b_{ij}|^{q}|\beta_j|^{q}\right)^{p/q}\int_{0}^{t}e^{-c_i(t-s)}\mathbb{E}\left[\sum_{j=1}^{n}|\varphi_j(s-\tau(s)))|^{p}\right]ds\\
\mathbb{E}\sum_{i=1}^{n}|J_{4i}(t)|^{p} &\le \sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|l_{ij}|^{q}|\gamma_j|^{q}\right)^{p/q}\int_{0}^{t}e^{-c_i(t-s)}\mathbb{E}\left[\sum_{j=1}^{n}\left|\int_{s-r(s)}^{s}\varphi_j(u)\,du\right|^{p}\right]ds\\
&\le \sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^{q}|\gamma_j|^{q}\right)^{p/q}\int_{0}^{t}e^{-c_i(t-s)}\int_{s-r(s)}^{s}\mathbb{E}\left[\sum_{j=1}^{n}|\varphi_j(u)|^{p}\right]du\,ds. \qquad (17)
\end{aligned}$$

Using Lemma 2.1, we obtain that

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^{n}|J_{5i}(t)|^{p} &= \sum_{i=1}^{n}\mathbb{E}\left|\int_{0}^{t}e^{-c_i(t-s)}\sum_{j=1}^{n}\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,d\omega_j(s)\right|^{p} \qquad (18)\\
&\le n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left\{\left[\int_{0}^{t}e^{-c_i(t-s)}|\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))|\,d\omega_j(s)\right]^{2}\right\}^{p/2}\\
&= n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\int_{0}^{t}e^{-2c_i(t-s)}\sigma_{ij}^{2}(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,ds\right]^{p/2}
\end{aligned}$$

$$
\begin{aligned}
&\le n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\int_0^t e^{-2c_i(t-s)}\left(\mu_j\varphi_j^2(s)+\nu_j\varphi_j^2(s-\tau(s))\right)ds\right]^{p/2}\\
&\le n^{p-1}2^{p/2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\left(\int_0^t e^{-2c_i(t-s)}\mu_j\varphi_j^2(s)\,ds\right)^{p/2}+\left(\int_0^t e^{-2c_i(t-s)}\nu_j\varphi_j^2(s-\tau(s))\,ds\right)^{p/2}\right]\\
&\le n^{p-1}2^{p/2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\left(\int_0^t e^{-2c_i(t-s)}\,ds\right)^{p/2-1}\int_0^t e^{-2c_i(t-s)}\mu_j^{p/2}|\varphi_j(s)|^p\,ds\right]\\
&\quad+n^{p-1}2^{p/2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left\{\left(\int_0^t e^{-2c_i(t-s)}\,ds\right)^{p/2-1}\int_0^t e^{-2c_i(t-s)}\nu_j^{p/2}|\varphi_j(s-\tau(s))|^p\,ds\right\}\\
&\le n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left\{\mu^{p/2}\int_0^t e^{-2c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)ds+\nu^{p/2}\int_0^t e^{-2c_i(t-s)}E\left(\sum_{j=1}^{n}|\varphi_j(s-\tau(s))|^p\right)ds\right\}\\
&\le n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left\{\mu^{p/2}\int_0^t e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)ds+\nu^{p/2}\int_0^t e^{-c_i(t-s)}E\left[\sum_{j=1}^{n}|\varphi_j(s-\tau(s))|^p\right]ds\right\}.
\end{aligned}
$$

Since $\mathbb{E}\sum_{i=1}^n|\varphi_i(t)|\to 0$, $t-\tau(t)\to\infty$ and $t-r(t)\to\infty$ as $t\to\infty$, for each $\epsilon>0$, there exists $T_2>0$ such that $t\ge T_2$ implies $\mathbb{E}\sum_{i=1}^n|\varphi_i(t-\tau(s))|<\epsilon$ and $\mathbb{E}\sum_{i=1}^n|\varphi_i(t-r(t))|<\epsilon$. From (17), we obtain that

$$
\begin{aligned}
&\mathbb{E}\sum_{i=1}^{n}|J_{3i}(t)|^p\\
&\le\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\int_0^{T_2}e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s-\tau(s)))|^p\right)ds\\
&\quad+\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\int_{T_2}^{t}e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s-\tau(s)))|^p\right)ds\\
&<\sum_{i=1}^{n}\left(\frac{1}{c_i}\right)^{p/q}e^{-c_it}\int_0^{T_2}e^{c_is}\,ds\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\sup_{\vartheta\le s\le T_2}\left[\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s-\tau(s)))|^p\right)\right]+\epsilon\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\sum_{i=1}^{n}|J_{4i}(t)|^p\\
&\le\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^{T_2}e^{-c_i(t-s)}\int_{s-r(s)}^{s}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(u)|^p\right)du\,ds\\
&\quad+\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_{T_2}^{t}e^{-c_i(t-s)}\int_{s-r(s)}^{s}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(u)|^p\right)du\,ds\\
&<\sum_{i=1}^{n}re^{-c_it}\left(\frac{r}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\sup_{\vartheta\le u\le T_2}\left[\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(u)|^p\right)\right]\int_0^{T_2}e^{c_is}\,ds+\sum_{i=1}^{n}\frac{\epsilon r}{c_i}\left(\frac{r}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}.
\end{aligned}
$$

Further, from (18), we obtain

$$
\mathbb{E}\sum_{i=1}^{n}|J_{5i}(t)|^p
$$

$$
\begin{aligned}
&\le n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left[\mu^{p/2}\int_0^t e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)ds+\nu^{p/2}\int_0^t e^{-c_i(t-s)}E\left(\sum_{j=1}^{n}|\varphi_j(s-\tau(s))|^p\right)ds\right)\\
&\le n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left(\mu^{p/2}\int_0^{T_2} e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)ds+\nu^{p/2}\int_0^{T_2} e^{-c_i(t-s)}E\left(\sum_{j=1}^{n}|\varphi_j(s-\tau(s))|^p\right)ds\right]\\
&\quad+n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left\{\mu^{p/2}\int_{T_2}^t e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)ds+\nu^{p/2}\int_{T_2}^t e^{-c_i(t-s)}E\left(\sum_{j=1}^{n}|\varphi_j(s-\tau(s))|^p\right)ds\right\}\\
&< n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left\{\mu^{p/2}\sup_{0\le s\le T_2}\left[\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)\right]+\nu^{p/2}\sup_{\vartheta\le s\le T_2}\left[E\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)\right]\right\}\int_0^{T_2}e^{-c_i(t-s)}\,ds\\
&\quad+n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left(\frac{\epsilon(\mu^{p/2}+\nu^{p/2})}{c_i}\right).
\end{aligned}
$$

Hence, let $t\to\infty$, from the fact that $c_i>0$ ($i=1,2,\cdots,n$), we obtain that

$$
\mathbb{E}\sum_{i=1}^{n}|J_{3i}(t)|^p\to 0,\quad \mathbb{E}\sum_{i=1}^{n}|J_{4i}(t)|^p\to 0,\quad\text{and}\quad \mathbb{E}\sum_{i=1}^{n}|J_{5i}(t)|^p\to 0.
$$

Thus, combining with (15), we obtain that $\mathbb{E}\sum_{i=1}^{n}|Q\varphi_i(t)|^p\to 0$ as $\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)|^p\to 0$. Therefore, $Q:\mathcal{S}_\phi\to\mathcal{S}_\phi$.

Step4. We prove that Q is a contraction mapping. For any $\varphi,\psi\in\mathcal{S}_\phi$, from (16)-(18), we obtain

$$
\begin{aligned}
&\sup_{s\ge\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}|Q\varphi_i(s)-Q\psi_i(s)|^p\right\}\\
&\le 4^{p-1}\sup_{s\ge\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}a_{ij}\Big(f_j(x_j(u))-f_j(y_j(u))\Big)du\right|^p\right\}\\
&\quad+4^{p-1}\sup_{s\ge\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}b_{ij}\Big(g_j(x_j(u-\tau(u)))-g_j(y_j(u-\tau(u)))\Big)du\right|^p\right\}\\
&\quad+4^{p-1}\sup_{s\ge\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}l_{ij}\int_{s-r(s)}^{s}\Big(h_j(\varphi_j(v))-h_j(\psi_j(v))\Big)dv\,du\right|^p\right\}\\
&\quad+4^{p-1}\sup_{s\ge\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}\Big(\sigma_{ij}(s,x_j(s),x_j(u-\tau(u)))-\sigma_{ij}(s,y_j(s),y_j(s-\tau(u)))\Big)d\omega_j(u)\right|^p\right\}\\
&\le 4^{p-1}\sup_{s\ge\vartheta}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_0^s e^{-c_i(s-u)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(u)-\psi(u)|^p\right)du\\
&\quad+4^{p-1}\sup_{s\ge\vartheta}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\int_0^s e^{-c_i(s-u)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(u-\tau(u)))-\psi_j(u-\tau(u)))|^p\right)du\\
&\quad+4^{p-1}\sup_{s\ge\vartheta}\sum_{i=1}^{n}\left(\frac{\tau}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^s e^{-c_i(s-u)}\int_{u-r(u)}^{u}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(v)-\psi_j(v)|^p\right)dv\,du\\
&\quad+4^{p-1}n^{p-1}\sup_{s\ge\vartheta}\left\{\sum_{i=1}^{n}c_i^{1-p/2}\left[\mu^{p/2}\int_0^s e^{-c_i(s-u)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(u)-\psi_j(u)|^p\right)du\right.\right.
\end{aligned}
$$

$$
+\nu^{p/2}\int_0^s e^{-c_i(s-u)}E\left(\sum_{j=1}^n|\varphi_j(u-\tau(u))-\psi_j(u-\tau(u))|^p\right)du\Bigg]\Bigg\}
$$

$$
\begin{aligned}
&\le 4^{p-1}\Bigg\{\sum_{i=1}^n c_i^{-p}\left(\sum_{j=1}^n|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+\sum_{i=1}^n c_i^{-p}\left(\sum_{j=1}^n|b_{ij}|^q|\beta_j|^q\right)^{p/q}\\
&\quad+\sum_{i=1}^n\left(\frac{r}{c_i}\right)^p\left(\sum_{j=1}^n|l_{ij}|^q|\gamma_j|^q\right)^{p/q}+n^{p-1}\sum_{i=1}^n c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right)\Bigg\}\sup_{s\ge\vartheta}\left\{\mathbb{E}\sum_{j=1}^n|\varphi_j(s)-\psi_j(s)|^p\right\}\\
&=\alpha\sup_{s\ge\vartheta}\left\{\mathbb{E}\sum_{j=1}^n|\varphi_j(s)-\psi_j(s)|^p\right\}.
\end{aligned}
$$

From (4), we obtain that $Q:\mathcal{S}_\phi\to\mathcal{S}_\phi$ is a contraction mapping. □

We are now ready to prove Theorem 1.4.

Proof. From Lemma 2.2, by the contraction mapping principle, we obtain that Q has a unique fixed point $x(t)$, which is a solution of (1) with $x(t)=\phi(t)$ as $t\in[\vartheta,0]$ and $\mathbb{E}\sum_{i=1}^n|x_i(t)|^p\to 0$ as $t\to\infty$.

Now, we prove that the trivial solution of (1) is *pth* moment stable. Let $\epsilon>0$ be given and choose $\delta>0$ ($\delta<\epsilon$) such that $5^{p-1}\delta<(1-\alpha)\epsilon$.

If $x(t)=(x_1(t),x_2(t),\cdots x_n(t))^T$ is a solution of (1) with the initial condition (2) satisfying $\mathbb{E}\sum_{i=1}^n|\phi_i(t)|^p<\delta$, then $x(t)=(Qx)(t)$ defined in (14). We claim that $\mathbb{E}\sum_{i=1}^n|x_i(t)|^p<\epsilon$ for all $t\ge 0$. Notice that $\mathbb{E}\sum_{i=1}^n|x_i(t)|^p<\epsilon$ for $t\in[\vartheta,0]$, we suppose that there exists $t^*>0$ such that $\mathbb{E}\sum_{i=1}^n|x_i(t^*)|^p=\epsilon$ and $\mathbb{E}\sum_{i=1}^n|x_i(t)|^p<\epsilon$ for $\vartheta\le t<t^*$, then it follows from (4), we obtain that

$$
\begin{aligned}
\mathbb{E}\sum_{i=1}^n|x_i(t^*)|^p &\le 5^{p-1}\mathbb{E}\sum_{i=1}^n e^{-pc_it^*}|x_i(0)|^p+5^{p-1}\sum_{i=1}^n c_i^{-p/q}\left(\sum_{j=1}^n|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_0^{t^*}e^{-c_i(t^*-s)}\mathbb{E}\left(\sum_{j=1}^n|x_j(s)|^p\right)ds\\
&\quad+5^{p-1}\sum_{i=1}^n c_i^{-p/q}\left(\sum_{j=1}^n|b_{ij}|^q|\beta_j|^q\right)^{p/q}\int_0^{t^*}e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^n|x_j(s-\tau(s)))|^p\right)ds\\
&\quad+5^{p-1}\sum_{i=1}^n\left(\frac{r}{c_i}\right)^{p/q}\left[\sum_{j=1}^n|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^{t^*}e^{-c_i(t^*-s)}\int_{s-r(s)}^s\mathbb{E}\left(\sum_{j=1}^n|x_j(u)|^p\right)du\,ds\\
&\quad+5^{p-1}n^{p-1}\sum_{i=1}^n c_i^{1-p/2}\left[\mu^{p/2}\int_0^{t^*}e^{-c_i(t^*-s)}\mathbb{E}\left(\sum_{j=1}^n|x_j(s)|^p\right)ds\right.\\
&\quad\left.+\nu^{p/2}\int_0^{t^*}e^{-c_i(t-s)}E\left(\sum_{j=1}^n|x_j(s-\tau(s))|^p\right)ds\right]\\
&\le \Bigg[5^{p-1}\sum_{i=1}^n c_i^{-p}\left(\sum_{j=1}^n|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+5^{p-1}\sum_{i=1}^n c_i^{-p}\left(\sum_{j=1}^n|b_{ij}|^q|\beta_j|^q\right)^{p/q}\\
&\quad+5^{p-1}\sum_{i=1}^n\left(\frac{r}{c_i}\right)^p\left(\sum_{j=1}^n|l_{ij}|^q|\gamma_j|^q\right)^{p/q}+5^{p-1}n^{p-1}\sum_{i=1}^n c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right)\Bigg]\epsilon+5^{p-1}\delta\\
&< (1-\alpha)\epsilon+\alpha\epsilon=\epsilon.
\end{aligned}
$$

which is a contradiction. Therefore, the trivial solution of (1) is asymptotically stable in *pth* moment. □

Corollary 2.3. *Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the distributed delay $r(t)$ is bounded by a constant r;*

(ii)

$$5\sum_{i=1}^{n} c_i^{-2}\left(\sum_{j=1}^{n} a_{ij}^2\alpha_j^2\right) + 5\sum_{i=1}^{n} c_i^{-2}\left(\sum_{j=1}^{n} b_{ij}^2\beta_j^2\right) + 5\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^2\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right) + 20n\sum_{i=1}^{n} c_i^{-1}(\mu+\nu) \le \alpha,$$

where c, μ, ν are defined as in Theorem 1.4*,*

then the trivial solution of (1) *is asymptotically stable in mean square*

Consider the stochastic neural networks without distributed delays

$$dx_i(t) = \left[-c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau(t)))\right] dt + \sum_{j=1}^{n} \sigma_{ij}(t, x_j(t), x_j(t-\tau(t)))\, d\omega_j(t) \tag{19}$$

for $i = 1, 2, 3, \cdots, n$.

Corollary 2.4. *Suppose that the assumptions* (A1)-(A4) *hold. The trivial solution of* (19) *is asymptotically stable in pth moment if the following inequality holds,*

$$4^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q} + 4^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q} + 4^{p-1}n^{p-1}\sum_{i=1}^{n} c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right) \le \alpha, \tag{20}$$

where μ, ν are defined as in Theorem 1.4*.*

Remark 2.5. *Note that the discrete delay $\tau(t)$ in Corollary 2.4 can be unbounded.*

3. Proof of Theorem 1.5

In this section, we prove Theorem 1.5. We start with some preparations.

Lemma 3.1. *Define an operator by $(P\varphi)(t) = \phi(t)$ for $t \in [-\tau, 0]$, and for $t \ge 0$, $(P\varphi)(t)$ is defined as* (14)*, if there is $\alpha \in (0, 1)$ such that* (6) *holds, then $P : C_\phi \to C_\phi$ is a contraction mapping.*

Proof. From the proof of Theorem 1.4, we obtain that P is continuous in *pth* moment on $[0, \infty)$. Now, we prove that $P(C_\phi) \subseteq C_\phi$.

$$\sum_{i=1}^{n}\mathbb{E}\left[\sup_{t-\tau\le s\le t}|Q\varphi_i(s)|^p\right] = \sum_{i=1}^{n}\mathbb{E}\left[\sup_{t-\tau\le s\le t}\left|\sum_{j=1}^{5} J_{ji}(s)\right|^p\right] \le 5^{p-1}\sum_{j=1}^{5}\sum_{i=1}^{n}\mathbb{E}\left[\sup_{t-\tau\le s\le t}|J_{ji}(s)|^p\right].$$

We estimate the terms on the right-hand side of the above inequality. Let $c = \min\{c_1, c_2, c_3, \cdots, c_n\}$,

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\le s\le t}|J_{2i}(s)|^p\right] &= \mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\le s\le t}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n} a_{ij} f_j(\varphi_j(u))\, du\right|^p\right] \\
&\le c^{-p/q}\mathbb{E}\left\{\sum_{i=1}^{n}\sup_{t-\tau\le s\le t}\left[\int_0^s e^{-c(s-u)}\left(\sum_{j=1}^{n}|a_{ij}|\alpha_j|\varphi_j(u)|\right)^p du\right]\right\} \\
&\le c^{-p/q}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\int_0^s e^{-c(s-u)}\left(\sum_{j=1}^{n}|\varphi_j(u)|^p\right) du\right]\right\} \\
&\le c^{-p/q}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\sum_{j=1}^{n}\mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\int_0^s e^{-c(s-u)}|\varphi_j(u)|^p\, du\right]\right\}
\end{aligned}$$

$$\leq \quad c^{-p/q}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^{q}|\alpha_j|^{q}\right)^{p/q}\sum_{j=1}^{n}\mathbb{E}\left\{\sup_{t-\tau\leq s\leq t}\left[\int_0^s e^{-c(s-u)}\left(\sup_{u-\tau\leq v\leq u}|\varphi_j(v)|^p\right)du\right]\right\}$$

$$\leq \quad e^{c\tau}c^{-p/q}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^{q}|\alpha_j|^{q}\right)^{p/q}\sum_{j=1}^{n}\mathbb{E}\left[\int_0^t e^{-c(t-u)}\left(\sup_{u-\tau\leq v\leq u}|\varphi_j(v)|^p\right)du\right]$$

$$\leq \quad e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^{q}|\alpha_j|^{q}\right)^{p/q}\sum_{j=1}^{n}\mathbb{E}\left(\sup_{t-\tau\leq s\leq t}|x_j(s)|^p\right). \tag{21}$$

Since $\sum_{j=1}^{n}\mathbb{E}\sup_{t-\tau\leq s\leq t}|\varphi_j(s)|^p\to 0$ as $t\to\infty$, then for any $\epsilon>0$, there exists such that $t\geq T_1$ implies

$$\sum_{j=1}^{n}\mathbb{E}\left[\sup_{t-\tau\leq s\leq t}|\varphi_j(s)|^p\right]<\epsilon.$$

Then, combining with (21), we obatin that

$$\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\leq s\leq t}|J_{2i}(s)|^p\right]<\left[e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^{q}|\alpha_j|^{q}\right)^{p/q}\right]\epsilon.$$

Hence, we obtain that $\mathbb{E}\sum_{i=1}^{n}\sup_{t-\tau\leq s\leq t}|J_{2i}(s)|^p\to 0$ as $t\to\infty$. Similarly, we obtain that

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^{n}\sup_{t-\tau\leq s\leq t}|J_{3i}(s)|^p &\leq c^{-p/q}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^{q}|\beta_j|^{q}\right)^{p/q}\mathbb{E}\left\{\sup_{t-\tau\leq s\leq t}\left[\int_0^s e^{-c(s-u)}\left(\sum_{j=1}^{n}|\varphi_j(u-\tau(u))|^p\right)du\right]\right\}\\
&\leq c^{-p/q}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^{q}|\beta_j|^{q}\right)^{p/q}\sum_{j=1}^{n}\mathbb{E}\left\{\sup_{t-\tau\leq s\leq t}\left[\int_0^s e^{-c(s-u)}|\varphi_j(u-\tau(u))|^p\,du\right]\right\}\\
&\leq c^{-p/q}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^{q}|\beta_j|^{q}\right)^{p/q}\sum_{j=1}^{n}\mathbb{E}\left\{\sup_{t-\tau\leq s\leq t}\left[\int_0^s e^{-c(s-u)}\sup_{u-\tau\leq v\leq u}|\varphi_j(v)|^p\,du\right]\right\}\\
&\leq e^{c\tau}c^{-p/q}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^{q}|\beta_j|^{q}\right)^{p/q}\sum_{j=1}^{n}\mathbb{E}\left[\int_0^t e^{-c(t-u)}\sup_{u-\tau\leq v\leq u}|\varphi_j(v)|^p\,du\right]\\
&\leq e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^{q}|\beta_j|^{q}\right)^{p/q}\sum_{j=1}^{n}\mathbb{E}\left[\sup_{t-\tau\leq s\leq t}|\varphi_j(s)|^p\right].
\end{aligned}\tag{22}$$

Since $\sum_{j=1}^{n}\mathbb{E}\left[\sup_{t-\tau\leq s\leq t}|\varphi_j(s)|^p\right]\to 0$ as $t\to\infty$, that is, for any $\epsilon>0$, there exists such that $t\geq T_2$ implies $\sum_{j=1}^{n}\mathbb{E}\left[\sup_{t-\tau\leq s\leq t}|\varphi_j(s)|^p\right]<\epsilon$, so combining with (22) we have

$$\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\leq s\leq t}|J_{3i}(s)|^p\right]<\left[e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^{q}|\beta_j|^{q}\right)^{p/q}\right]\epsilon.$$

Hence, $\mathbb{E}\sum_{i=1}^{n}\sup_{t-\tau\leq s\leq t}|J_{3i}(s)|^p\to 0$ as $t\to\infty$.

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\leq s\leq t}|J_{4i}(s)|^p\right] &\leq c^{-p/q}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|l_{ij}|^{q}|\gamma_j|^{q}\right)^{p/q}\mathbb{E}\left\{\sup_{t-\tau\leq s\leq t}\left[\int_0^s e^{-c(s-u)}\sum_{j=1}^{n}\left|\int_{u-r(u)}^{u}\varphi_j(v)\,dv\right|^p du\right]\right\}\\
&\leq c^{-p/q}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|l_{ij}|^{q}|\gamma_j|^{q}\right)^{p/q}\sum_{j=1}^{n}\mathbb{E}\left\{\sup_{t-\tau\leq s\leq t}\left[\int_0^s e^{-c(s-u)}\left|\int_{u-r(u)}^{u}\varphi_j(v)\,dv\right|^p du\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&\le \tau^p c^{-p/q} \sum_{i=1}^n \left(\sum_{j=1}^n |l_{ij}|^q |\gamma_j|^q \right)^{p/q} \sum_{j=1}^n \mathbb{E}\left\{ \sup_{t-\tau\le s\le t} \left[\int_0^s e^{-c(s-u)} \sup_{u-\tau\le v\le u} |\varphi_j(v)|^p \, du \right] \right\} \\
&\le \tau^p e^{c\tau} c^{-p/q} \sum_{i=1}^n \left(\sum_{j=1}^n |l_{ij}|^q |\gamma_j|^q \right)^{p/q} \sum_{j=1}^n \mathbb{E}\left[\int_0^t e^{-c(t-u)} \sup_{u-\tau\le v\le u} |\varphi_j(v)|^p \, du \right] \\
&\le \tau^p e^{c\tau} c^{-p} \sum_{i=1}^n \left(\sum_{j=1}^n |l_{ij}|^q |\beta_j|^q \right)^{p/q} \sum_{j=1}^n \mathbb{E}\left[\sup_{t-\tau\le s\le t} |\varphi_j(s)|^p \right].
\end{aligned}
$$

Hence, we have that $\mathbb{E}\left[\sum_{i=1}^n \sup_{t-\tau\le s\le t} |J_{4i}(s)|^p\right] \to 0$ as $t\to\infty$. Let $\mu = \max\{\mu_1, \mu_2, \cdots, \mu_n\}$, $\nu = \max\{\nu_1, \nu_2, \cdots, \nu_n\}$,

$$
\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^n \sup_{t-\tau\le s\le t} |J_{5i}(s)|^p \right] &\le n^{p-1} \sum_{i=1}^n \sum_{j=1}^n \mathbb{E}\left[\sup_{t-\tau\le s\le t} \left| \int_0^s e^{-c_i(s-u)} \sigma_{ij}(u, \varphi_j(u), \varphi_j(u-\tau(u))) \, d\omega_j(u) \right|^p \right] \\
&\le n^{p-1} \sum_{i=1}^n \sum_{j=1}^n \mathbb{E}\left\{ \sup_{t-\tau\le s\le t} \left[\sup_{t-\tau\le r\le t} \left| \int_0^s e^{-c(r-u)} \sigma_{ij}(u, \varphi_j(u), \varphi_j(u-\tau(u))) \, d\omega_j(u) \right|^p \right] \right\} \\
&\le n^{p-1} \sum_{i=1}^n \sum_{j=1}^n q^p \sup_{t-\tau\le s\le t} \left\{ \mathbb{E}\left[\sup_{t-\tau\le r\le t} \left| \int_0^s e^{-c(r-u)} \sigma_{ij}(u, \varphi_j(u), \varphi_j(u-\tau(u))) \, d\omega_j(u) \right|^p \right] \right\} \\
&\le n^{p-1} e^{pc\tau} \sum_{i=1}^n \sum_{j=1}^n q^p \sup_{t-\tau\le s\le t} \left[\mathbb{E}\left| \int_0^s e^{-c(t-u)} \sigma_{ij}(u, \varphi_j(u), \varphi_j(u-\tau(u))) \, d\omega_j(u) \right|^p \right] \\
&\le K_p n^{p-1} e^{pc\tau} \sum_{i=1}^n \sum_{j=1}^n q^p \sup_{t-\tau\le s\le t} \left[\mathbb{E}\left(\int_0^s e^{-2c(t-u)} \sigma_{ij}^2(u, \varphi_j(u), \varphi_j(u-\tau(u))) \, du \right)^{p/2} \right] \\
&\le K_p n^{p-1} e^{pc\tau} \sum_{i=1}^n \sum_{j=1}^n q^p 2^{p/2-1} \sup_{t-\tau\le s\le t} \mathbb{E}\left[\left(\int_0^s e^{-2c(t-u)} \left(\mu_j \varphi_j^2(u) \right) du \right)^{p/2} \right] \\
&\quad + K_p n^{p-1} e^{pc\tau} \sum_{i=1}^n \sum_{j=1}^n q^p 2^{p/2-1} \sup_{t-\tau\le s\le t} \left[\mathbb{E}\left(\int_0^s e^{-2c(T-u)} \left(\nu_j \varphi_j^2(u-\tau(u)) \right) du \right)^{p/2} \right] \\
&\le K_p n^{p-1} e^{pc\tau} \sum_{i=1}^n \sum_{j=1}^n q^p 2^{p/2-1} \sup_{t-\tau\le s\le t} \left\{ \mathbb{E}\left[\left(\int_0^s e^{-2c(t-u)} \, du \right)^{p/2-1} \right.\right. \\
&\quad \left.\left. \times \left(\int_0^s e^{-2c(t-u)} \mu_j^{p/2} |\varphi_j(u)|^p \, du + \int_0^s e^{-2c(T-u)} \nu_j^{p/2} |\varphi_j(u-\tau(u))|^p \, du \right) \right] \right\} \\
&\le K_p n^p e^{pc\tau} q^p c^{1-p/2} (\mu^{p/2} + \nu^{p/2}) \int_0^t e^{-2c(T-u)} \sum_{j=1}^n \mathbb{E}\left[\sup_{u-\tau\le v\le u} |\varphi_j(v)|^p \, du \right] \\
&\le K_p n^p e^{pc\tau} q^p c^{1-p/2} (2c)^{-1} (\mu^{p/2} + \nu^{p/2}) \sum_{j=1}^n \mathbb{E}\left[\sup_{t-\tau\le s\le t} |\varphi_j(s)|^p \right]. \qquad (23)
\end{aligned}
$$

Since $\sum_{j=1}^n \mathbb{E}\left[\sup_{t-\tau\le s\le t} |\varphi_j(s)|^p\right] \to 0$ as $t\to\infty$, that is, for any $\epsilon > 0$, there exists $T_3 > 0$ such that $t \ge T_3$ implies $\sum_{j=1}^n \mathbb{E}\left[\sup_{t-\tau\le s\le t} |\varphi_j(s)|^p\right] < \epsilon$, so combining with (23) we have

$$
\mathbb{E}\left[\sum_{i=1}^n \sup_{t-\tau\le s\le t} |J_{5i}(s)|^p \right] < K_p n^p e^{pc\tau} q^p c^{1-p/2} (2c)^{-1} (\mu^{p/2} + \nu^{p/2}) \epsilon.
$$

Hence, $\mathbb{E}\left[\sum_{i=1}^n \sup_{t-\tau\le s\le t} |J_{5i}(s)|^p\right] \to 0$ as $t\to\infty$. Thus, $P(C_\phi) \subseteq C_\phi$.

Finally, we prove that Q is a contraction mapping. For any $\varphi, \psi \in C_\phi$, from (21)-(23), we obtain that

$$\begin{aligned}
&\sup_{t\geq 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\leq s\leq t}|Q\varphi_i(s)-Q\psi_i(s)|^p\right]\right\}\\
&\leq 4^{p-1}\sup_{t\geq 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\leq s\leq t}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}a_{ij}\Big(f_j(\varphi_j(u))-f_j(\psi_j(u))\Big)du\right|^p\right]\right\}\\
&\quad+4^{p-1}\sup_{t\geq 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\leq s\leq t}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}b_{ij}\Big(g_j(\varphi_j(u-\tau(u)))-g_j(\psi_j(u-\tau(u)))\Big)du\right|^p\right]\right\}\\
&\quad+4^{p-1}\sup_{t\geq 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\leq s\leq t}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}l_{ij}\int_{s-r(s)}^{s}\Big(h_j(\varphi_j(v))-h_j(\psi_j(v))\Big)dv\,du\right|^p\right]\right\}\\
&\quad+4^{p-1}\sup_{t\geq 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\leq s\leq t}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}[\sigma_{ij}(u,\varphi_j(u),\varphi_j(u-\tau(u)))-\sigma_{ij}(u,\psi_j(u),\psi_j(u-\tau(u)))]\,d\omega_j(u)\right|^p\right]\right\}\\
&\leq 4^{p-1}\Bigg\{e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}+\tau^p e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\\
&\quad+K_p n^p e^{pc\tau}q^p c^{1-p/2}(2c)^{-1}\left(\mu^{p/2}+\nu^{p/2}\right)\Bigg\}\sup_{t\geq 0}\sum_{j=1}^{n}\mathbb{E}\left[\sup_{t-\tau\leq s\leq t}|\varphi_j(s)-\psi_j(s)|^p\right]\\
&=\alpha\sup_{t\geq 0}\sum_{j=1}^{n}\mathbb{E}\left[\sup_{t-\tau\leq s\leq t}|\varphi_j(s)-\psi_j(s)|^p\right].
\end{aligned}$$

From (6), we obtain that $Q: C_\phi \to C_\phi$ is a contraction mapping. □

We are now ready to prove Theorem 1.5

Proof. From Lemma 3.1, by the contraction mapping principle, we obtain that P has a unique fixed point $x(t)$, which is a solution of (1) with $x(t) = \phi(t)$ as $t \in [-\tau, 0]$ and $\sum_{i=1}^{n}\mathbb{E}\left[\sup_{t-\tau\leq s\leq t}|x_i(s)|^p\right] \to 0$ as $t \to \infty$.

We prove that the trivial solution of (1) is *pth* moment stable. Let $\epsilon > 0$ be given and choose $\delta > 0$ ($\delta < \epsilon$) satisfying

$$5^{p-1}e^{-pct^*}\delta < (1-\alpha)\epsilon. \tag{24}$$

If $x(t) = (x_1(t), x_2(t), \cdots x_n(t))^T$ is a solution of (1) with the initial condtion satisfying $\|\phi\|^p < \delta$, then $x(t) = (Px)(t)$ defined in (14). We claim that $\|x\|^p < \epsilon$ for all $t \geq 0$. Notice that $\|\phi(t)\|^p < \epsilon$ for $t \in [-\tau, 0]$, we suppose that there exists $t^* > 0$ such that $\sum_{i=1}^{n}\mathbb{E}\left[\sup_{t^*-\tau\leq s\leq t^*}|x_i(s)|^p\right] = \epsilon$ and $\sum_{i=1}^{n}\mathbb{E}\left[\sup_{t-\tau\leq s\leq t}|x_i(s)|^p\right] < \epsilon$ for $-\tau \leq t < t^*$, then it follows from (4) and (24), we obtain that

$$\begin{aligned}
\sum_{i=1}^{n}\mathbb{E}\left[\sup_{t^*-\tau\leq s\leq t^*}|x_i(s)|^p\right] &\leq 5^{p-1}\sum_{j=1}^{5}\sum_{i=1}^{n}\mathbb{E}\left[\sup_{t^*-\tau\leq s\leq t^*}|J_{ji}(s)|^p\right]\\
&\leq 5^{p-1}e^{-pct^*}\delta+5^{p-1}\Bigg\{e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\\
&\qquad+\tau^p e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}+K_p n^p e^{pc\tau}q^p c^{1-p/2}(2c)^{-1}\left(\mu^{p/2}+\nu^{p/2}\right)\Bigg\}\epsilon\\
&< (1-\alpha)\epsilon+\alpha\epsilon=\epsilon.
\end{aligned}$$

which is a contradiction. Therefore, the trivial solution of (1) is asymptotically stable in *pth* moment. □

4. Proof of Theorem 1.8

In this section, we prove Theorem 1.8. We start with a lemma presenting an integral inequality.

Lemma 4.1. *Consider $\gamma > 0$, positive constants $\lambda_1, \lambda_2, \lambda_3$ and a function $y : [-\tau, \infty) \to [0, \infty)$. If $\lambda_1 + \lambda_2 + \tau\lambda_3 < c$ and the following inequality holds,*

$$y(t) \le \begin{cases} y_0 e^{-ct} + \lambda_1 \int_0^t e^{-c(t-s)} y(s)\,ds + \lambda_2 \int_0^t e^{-c(t-s)} y(s-\tau(s))\,ds + \lambda_3 \int_0^t e^{-c(t-s)} \int_{s-r(s)}^s y(u)\,du\,ds & t \ge 0, \\ y_0 e^{-ct}, \quad t \in [-\tau, 0], \end{cases} \tag{25}$$

then we have $y(t) \le y_0 e^{-\gamma t} (t \ge -\tau)$, where γ is a positive root of the algebraic equation $\frac{1}{c-\gamma}\left(\lambda_1 + e^{\gamma\tau}\lambda_2 + \frac{e^{\gamma\tau}-1}{\gamma}\lambda_3\right) = 1$.

Proof. Let $F(\gamma) = \frac{1}{c-\gamma}\left(\lambda_1 + e^{\gamma\tau}\lambda_2 + \frac{e^{\gamma\tau}-1}{\gamma}\lambda_3\right) - 1$. We have $F(0)F(c^-) < 0$, that is, there exists a positive constant $\gamma \in (0, c)$ such that $F(\gamma) = 0$. For any $\epsilon > 0$, let

$$C_\epsilon = \epsilon + y_0.$$

To prove the lemma, we claim that (25) implies

$$y(t) \le C_\epsilon e^{-\gamma t}, \quad t \ge -\tau. \tag{26}$$

It is easily shown that (26) holds for $t \in [-\tau, 0]$. Assume that there exists $t_1^* > 0$ such that

$$y(t) < C_\epsilon e^{-\gamma t}, \quad t \in [-\tau, t_1^*), \qquad y(t_1^*) = C_\epsilon e^{-\gamma t_1^*}. \tag{27}$$

Combining with (25), we have

$$\begin{aligned} y(t_1^*) &\le y_0 e^{-ct_1^*} + \lambda_1 \int_0^{t_1^*} e^{-c(t_1^*-s)} y(s)\,ds + \lambda_2 \int_0^{t_1^*} e^{-c(t_1^*-s)} y(s-\tau(s))\,ds + \lambda_3 \int_0^{t_1^*} e^{-c(t_1^*-s)} \int_{s-r(s)}^s y(u)\,du\,ds \\ &< y_0 e^{-ct_1^*} + C_\epsilon\lambda_1 \int_0^{t_1^*} e^{-c(t_1^*-s)} e^{-\gamma s}\,ds + C_\epsilon\lambda_2 \int_0^{t_1^*} e^{-c(t_1^*-s)} e^{-\gamma(s-\tau(s))}\,ds + C_\epsilon\lambda_3 \int_0^{t_1^*} e^{-c(t_1^*-s)} \int_{s-r(s)}^s e^{-\gamma u}\,du\,ds \\ &= \left[y_0 - \frac{C_\epsilon}{c-\gamma}\left(\lambda_1 + e^{\gamma\tau}\lambda_2 + \frac{e^{\gamma\tau}-1}{\gamma}\lambda_3\right)\right] e^{-ct_1^*} + \frac{C_\epsilon}{c-\gamma}\left(\lambda_1 + e^{\gamma\tau}\lambda_2 + \frac{e^{\gamma\tau}-1}{\gamma}\lambda_3\right) e^{-\gamma t_1^*}. \end{aligned}$$

From the definition of C_ϵ, we have

$$y_0 - \frac{C_\epsilon}{c-\gamma}\left(\lambda_1 + e^{\gamma\tau}\lambda_2 + \frac{e^{\gamma\tau}-1}{\gamma}\lambda_3\right) = y_0 - C_\epsilon < 0.$$

Then, together with the definition of γ, we obtain that $y(t_1^*) < C_\epsilon e^{-\gamma t_1^*}$, which contradicts (27), so (26) holds. As $\epsilon > 0$ is arbitrarily small, in view of (26), it follows that $y(t) \le y_0 e^{-\gamma t}$, for $t \ge -\tau$. □

Proof. For the representation (13), using (16)-(18), we obtain that

$$\begin{aligned} E\sum_{i=1}^n |x_i(t)|^p \le\ & 5^{p-1} e^{-ct} \sum_{i=1}^n \mathbb{E}|\phi_i(0)|^p + 5^{p-1} c^{-p/q} \sum_{i=1}^n \left(\sum_{j=1}^n |a_{ij}|^q |\alpha_j|^q\right)^{p/q} \int_0^t e^{-c(t-s)} \mathbb{E}\left(\sum_{j=1}^n |x_j(s)|^p\right) ds \\ & + 5^{p-1} c^{-p/q} \sum_{i=1}^n \left(\sum_{j=1}^n |b_{ij}|^q |\beta_j|^q\right)^{p/q} \int_0^t e^{-c(t-s)} \mathbb{E}\left[\sum_{j=1}^n |x_j(s-\tau(s)))|^p\right] ds \\ & + 5^{p-1} \left(\frac{\tau}{c}\right)^{p/q} \sum_{i=1}^n \left(\sum_{j=1}^n |l_{ij}|^q |\gamma_j|^q\right)^{p/q} \int_0^t e^{-c(t-s)} \int_{s-r(s)}^s \mathbb{E}\left[\sum_{j=1}^n |x_j(u)|^p\right] du\,ds \\ & + 5^{p-1} n^p c^{1-p/2} \left\{\mu^{p/2} \int_0^t e^{-c(t-s)} \mathbb{E}\left[\sum_{j=1}^n |x_j(s)|^p\right] ds + \nu^{p/2} \int_0^t e^{-c(t-s)} E\left[\sum_{j=1}^n |x_j(s-\tau(s))|^p\right] ds\right\}. \end{aligned}$$

Hence, by using Lemma 4.1 and (7), we obtain that the trivial solution of (1) is exponentially stable in *pth* moment. □

Corollary 4.2. *Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the discrete delay* $\tau(t)$ *and distributed delay* $r(t)$ *are bounded by a constant* τ*;*

(ii)

$$5c^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}^{2}\alpha_{j}^{2}+5c^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}^{2}\beta_{j}^{2}+5c^{-2}\tau^{2}\sum_{i=1}^{n}\sum_{j=1}^{n}l_{ij}^{2}\gamma_{j}^{2}+20n^{2}c^{-1}(\mu+\nu)<1,$$

where c,μ,ν *are defined as in Theorem* 1.4*,*

then the trivial solution of (1) *is exponentially stable in mean square,*

Corollary 4.3. *Let* $p\geq 2$*. Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the discrete delay* $\tau(t)$ *and distributed delay* $r(t)$ *are bounded by a constant* τ*;*

(ii)

$$4^{p-1}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^{q}|\alpha_{j}|^{q}\right)^{p/q}+4^{p-1}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^{q}|\beta_{j}|^{q}\right)^{p/q}+4^{p-1}n^{p}c^{-p/2}(\mu^{p/2}+\nu^{p/2})<1,$$

where c,μ,ν *are defined as in Theorem* 1.4*,*

then the trivial solution of (19) *is exponentially stable in pth moment.*

5. Proof of Theorem 1.12

In this section, we prove Theorem 1.12. We start with some preparations.

Multiply both sides of (9) by $e^{c_i t}$ and integrate from 0 to t, we obtain that for $t\geq 0$,

$$\begin{aligned} x_i(t) \;&=\; e^{-c_i t}x_i(0)+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}a_{ij}g_j(x_j(s))\,ds+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}b_{ij}g_j(x_j(s-\tau(s)))\,ds \\ &\quad+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}d_{ij}\int_{s-r(s)}^{s}g_j(x_j(u))\,du\,ds,\quad i=1,2,3,\cdots,n. \end{aligned} \tag{28}$$

Lemma 5.1. *Define an operator by* $(Px)(\theta)=\phi(\theta)$*, for* $\vartheta\leq\theta\leq 0$*, and for* $t\geq 0$*,*

$$\begin{aligned} Px_i(t) \;&=\; e^{-c_i t}x_i(0)+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}a_{ij}g_j(x_j(s))\,ds+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}b_{ij}g_j(x_j(s-\tau(s)))\,ds \\ &\quad+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}d_{ij}\int_{s-r(s)}^{s}g_j(x_j(u))\,du\,ds:=I_1(t)+I_2(t)+I_3(t)+I_4(t). \end{aligned} \tag{29}$$

If there exists $\alpha\in(0,1)$ *such that* (11) *holds, then* $P:\mathcal{H}_\phi\to\mathcal{H}_\phi$ *and P is a contraction mapping.*

Proof. First, we prove that $P\mathcal{H}_\phi\subseteq\mathcal{H}_\phi$. In view of (29), we have that, for fixed time $t_1\geq 0$, it is easy to check that $\lim_{r\to 0}[(Px_i)(t_1+r)-(Px_i)(t_1)]=0$. Thus, P is continuous on $[0,\infty)$. Note that $(Px_i)(\theta)=\phi(\theta)$ for $\vartheta\leq\theta\leq 0$, we obtain that P is indeed continuous on $[\vartheta,\infty)$.

Next, we prove that $\lim_{t\to\infty}(Px_i)(t) = 0$ for $x_i(t) \in \mathcal{H}_{i\phi}$. Since $x_i(t) \in \mathcal{H}_{i\phi}$, we have that $\lim_{t\to\infty} x_i(t) = 0$. Then for any $\epsilon > 0$, there exists $T_i > 0$ such that $s \geq T_i$ implies $|x_i(s)| < \epsilon$. Choose $T = \max_{i=1,2,\cdots,n}\{T_i\}$, combining with condition (A2),

$$\begin{aligned}
|I_2(t)| &= \left|\int_0^t e^{-c_i(t-s)} \sum_{j=1}^n a_{ij} f_j(x_j(s))\, ds\right| \\
&\leq \int_0^T e^{-c_i(t-s)} \sum_{j=1}^n |a_{ij}k_j||x_j(s)|\, ds + \int_T^t e^{-c_i(t-s)} \sum_{j=1}^n |a_{ij}\alpha_j||x_j(s)|\, ds \\
&\leq \sum_{j=1}^n |a_{ij}\alpha_j| \sup_{0\leq s\leq T} |x_j(s)| \int_0^T e^{-c_i(t-s)}\, ds + \epsilon \sum_{j=1}^n |a_{ij}\alpha_j| \int_T^t e^{-c_i(t-s)}\, ds \\
&\leq e^{-c_i t} \sum_{j=1}^n |a_{ij}\alpha_j| \sup_{0\leq s\leq T} |x_j(s)| \int_0^T e^{-c_i s}\, ds + \frac{\epsilon}{c_i} \sum_{j=1}^n |a_{ij}\alpha_j|. \qquad (30)
\end{aligned}$$

From the fact that $c_i > 0$ $(i = 1, 2, \cdots, n)$ and estimate (30), we have that $I_2(t) \to 0$ as $t \to \infty$.

Since $x_i(t) \to 0$ and $t - \tau(t) \to \infty$ as $t \to \infty$, for each $\epsilon > 0$, there exists $T_i' > 0$ such that $s \geq T_i'$ implies $|x_i(s - \tau(s))| < \epsilon$ for $i = 1, 2, \cdots, n$. Choose $T' = \max_{i=1,2,\cdots n}\{T_i'\}$, we obtain

$$\begin{aligned}
|I_3(t)| &= \left|\int_0^t e^{-c_i(t-s)} \sum_{j=1}^n b_{ij} g_j(x_j(s - \tau(s)))\, ds\right| \\
&\leq \int_0^{T'} e^{-c_i(t-s)} \sum_{j=1}^n |b_{ij}\beta_j||x_j(s - \tau(s))|\, ds + \int_{T'}^t e^{-c_i(t-s)} \sum_{j=1}^n |b_{ij}k_j||x_j(s - \tau(s))|\, ds \\
&\leq e^{-c_i t} \sum_{j=1}^n |b_{ij}\beta_j| \sup_{\vartheta\leq s\leq T'} |x_j(s)| \int_0^{T'} e^{c_i s}\, ds + \frac{\epsilon}{c_i} \sum_{j=1}^n |b_{ij}\beta_j|. \qquad (31)
\end{aligned}$$

From the fact that $c_i > 0$ $(i = 1, 2, \cdots, n)$ and estimate (31), we have that $I_3(t) \to 0$ as $t \to \infty$.

Since $x_i(t) \to 0$ and $t - r(t) \to \infty$ as $t \to \infty$, for each $\epsilon > 0$, there exists $T_i^* > 0$ such that $s \geq T_i^*$ implies $|x_i(s - r(s))| < \epsilon$ for $i = 1, 2, \cdots, n$. Choose $T^* = \max_{i=1,2,\cdots n}\{T_i^*\}$, we obtain

$$\begin{aligned}
|I_4(t)| &= \left|\int_0^t e^{-c_i(t-s)} \sum_{j=1}^n d_{ij} \int_{s-r(s)}^s h_j(x_j(u))\, du\, ds\right| \\
&\leq \int_0^{T^*} e^{-c_i(t-s)} \sum_{j=1}^n |d_{ij}\gamma_j| \int_{s-r(s)}^s |x_j(u)|\, du\, ds + \epsilon r \int_{T^*}^t e^{-c_i(t-s)} \sum_{j=1}^n |d_{ij}\gamma_j|\, ds \\
&\leq r \sum_{j=1}^n |d_{ij}\gamma_j| \sup_{\vartheta\leq u\leq T^*} |x_j(u)| \int_0^{T^*} e^{-c_i(t-s)}\, ds + \frac{\epsilon r}{c_i} \sum_{j=1}^n |d_{ij}\gamma_j|. \qquad (32)
\end{aligned}$$

From the fact that $c_i > 0 (i = 1, 2, \cdots, n)$ and estimate (32), we have that $I_4(t) \to 0$ as $t \to \infty$. From the above estimate, we conclude that $\lim_{t\to\infty}(Px_i)(t) = 0$ for $x_i(t) \in \mathcal{H}_{i\phi}$. Therefore, $P : \mathcal{H}_\phi \to \mathcal{H}_\phi$.

Now, we prove that P is a contraction mapping. For any $x(t), y(t) \in \mathcal{H}_\phi$, from (30) and (32), we obtain that

$$\begin{aligned}
\sum_{i=1}^n |(Px_i)(t) - (Py_i)(t)| \leq{}& \sum_{i=1}^n \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n |x_j(s) - y_j(s)|\, ds \\
&+ \sum_{i=1}^n \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n |x_j(s - \tau(s)) - y_j(s - \tau(s))|\, ds
\end{aligned}$$

$$+\sum_{i=1}^{n}\max_{j=1,2,\cdots,n}|d_{ij}\gamma_j|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}\int_{s-r(s)}^{s}|x_j(u)-y_j(u)|\,du\,ds$$

$$\leq \sum_{i=1}^{n}\left\{\frac{1}{c_i}\max_{j=1,2,\cdots,n}|a_{ij}\alpha_j|+\frac{1}{c_i}\max_{j=1,2,\cdots,n}|b_{ij}\beta_j|+\frac{r}{c_i}\max_{j=1,2,\cdots,n}|d_{ij}\gamma_j|\right\}\sup_{\vartheta\leq s\leq t}\sum_{j=1}^{n}|x_j(s)-y_j(s)|$$

$$= \alpha\sup_{\vartheta\leq s\leq t}\sum_{j=1}^{n}|x_j(s)-y_j(s)|.$$

From (11), we obtain that P is a contraction mapping. □

We are now ready to prove Theorem 1.12.

Proof. Let P be defined as in Lemma 5.1, by contraction mapping principle, P has a unique fixed point $x\in\mathcal{H}_\phi$ with $x(\theta)=\phi(\theta)$ on $\vartheta\leq\theta\leq 0$ and $x(t)\to 0$ as $t\to\infty$.

To obtain asymptotically stable, we need to prove that the trivial equilibrium $x=0$ of (9) is stable. For any $\epsilon>0$, choose $\sigma>0$ and $\sigma<\epsilon$ satisfying the condition $\sigma+\epsilon\alpha<\epsilon$.

If $x(t,s,\phi)=(x_1(t,s,\phi),x_2(t,s,\phi),\cdots,x_n(t,s,\phi))$ is the solution of (9) with the initial condition $\|\phi\|<\sigma$, the we claim that $\|x(t,s,\phi)\|<\epsilon$ for all $t\geq 0$. Indeed, we suppose that there exists $t^*>0$ such that

$$\sum_{i=1}^{n}|x_i(t^*;s,\phi)|=\epsilon,\qquad\text{and}\qquad\sum_{i=1}^{n}|x_i(t;s,\phi)|<\epsilon\quad\text{for}\quad 0\leq t<t^*. \tag{33}$$

From (11) and (28), we obtain

$$\begin{aligned}\sum_{i=1}^{n}|x_i(t^*;s,\phi)| \;\leq\; & \sum_{i=1}^{n}\Bigg[|e^{-c_it^*}x_i(0)|+\int_0^{t^*}e^{-c_i(t^*-s)}\sum_{j=1}^{n}|a_{ij}f_j(x_j(s))|\,ds+\int_0^{t^*}e^{-c_i(t^*-s)}\sum_{j=1}^{n}|b_{ij}g_j(x_j(s-\tau(s)))|\,ds\\ &+\int_0^{t^*}e^{-c_i(t^*-s)}\sum_{j=1}^{n}|d_{ij}\int_{s-r(s)}^{s}h_j(x_j(u))|\,du\,ds\Bigg]\\ <\;& \sigma+\epsilon\left(\sum_{i=1}^{n}\frac{1}{c_i}\max_{j=1,2,\cdots,n}|a_{ij}\alpha_j|+\sum_{i=1}^{n}\frac{1}{c_i}\max_{j=1,2,\cdots,n}|b_{ij}\beta_j|+\sum_{i=1}^{n}\frac{r}{c_i}\max_{j=1,2,\cdots,n}|d_{ij}\gamma_j|\right)\leq\sigma+\epsilon\alpha<\epsilon.\end{aligned}$$

which contradicts (33). Therefore, $\|x(t,s,\phi)\|<\epsilon$ for all $t\geq 0$. This completes the proof. □

Let $d_{ij}\equiv 0$ for $i=1,2,\cdots n,\ j=1,2,\cdots n$, the system is reduced to

$$\frac{dx_i(t)}{dt}=-c_ix_i(t)+\sum_{j=1}^{n}a_{ij}f_j(x_j(t))+\sum_{j=1}^{n}b_{ij}g_j(x_j(t-\tau(t))), \tag{34}$$

which is the description of cellular neural network with time-varying delays. Following the result of theorem 1.12, we have the following corollary.

Corollary 5.2. *Suppose that the assumptions* (A1)-(A3) *hold. If the following condition is satisfied,*

$$\sum_{i=1}^{n}\frac{1}{c_i}\max_{j=1,2,\cdots,n}|a_{ij}\alpha_j|+\sum_{i=1}^{n}\frac{1}{c_i}\max_{j=1,2,\cdots,n}|b_{ij}\beta_j|<1, \tag{35}$$

then the trivial solution of (34) *is asymptotically stable.*

Remark 5.3. *Note that the delay in Corollary* 5.2 *can be unbounded. Lai and Zhang* [12] *studied the asymptotic stability* (34) *as well. However, the additional condition*

$$\max_{i=1,2,\cdots n}\left[\frac{1}{c_i}\sum_{j=1}^{n}|a_{ij}k_j|+\frac{1}{c_i}\sum_{j=1}^{n}|b_{ij}k_j|\right]<\frac{1}{\sqrt{n}} \tag{36}$$

is needed in Theorem 4.1 *of* [12]. *It is clearly that Corollary* 5.2 *is an improvement of the result in* [12].

6. Proof of Theorem 1.13

Proof. From the represention (28), we obtain that

$$\begin{aligned}\sum_{i=1}^{n}|x_i(t)| \quad\le\quad & e^{-ct}\sum_{i=1}^{n}|x_i(0)|+\sum_{i=1}^{n}\max_{j=1,2,\cdots n}\{|a_{ij}k_j|\}\int_0^t e^{-c(t-s)}\sum_{j=1}^{n}|x_j(s)|\,ds\\ &+\sum_{i=1}^{n}\max_{j=1,2,\cdots n}\{|b_{ij}k_j|\}\int_0^t e^{-c(t-s)}\sum_{j=1}^{n}|x_j(s-\tau(s))|\,ds\\ &+\sum_{i=1}^{n}\max_{j=1,2,\cdots n}\{|d_{ij}k_j|\}\int_0^t e^{-c(t-s)}\sum_{j=1}^{n}\int_{s-r(s)}^{s}|x_j(u)|\,du\,ds.\end{aligned}$$

Combining with Lemma 4.1, we obtain that the trivial solution of (9) with initial condition (10) is exponentially stable. □

For the cellular neural network (34), we have the following result.

Corollary 6.1. *Suppose that the assumptions* (A1)-(A3) *hold. If the following conditions are satisfied,*

(i) *the discrete delay $\tau(t)$ and distributed delay $r(t)$ are bounded by a constant τ;*

(ii)

$$\sum_{i=1}^{n}\max_{j=1,2,\cdots,n}|a_{ij}k_j|+\sum_{i=1}^{n}\max_{j=1,2,\cdots,n}|b_{ij}k_j|<c,\qquad c=\min\{c_1,c_2,\cdots c_1\},$$

then the trivial solution of (34) *with initial condition* (10) *is exponentially stable.*

7. Examples

Example 7.1. *Consider the following two-dimensional cellular neural network*

$$\frac{dx(t)}{dt}=-Cx(t)+Ag(x(t))+Bg(x-\tau(t)),$$

where

$$C=\begin{pmatrix}c_1 & 0\\ 0 & c_2\end{pmatrix}=\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix},\quad A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}=\begin{pmatrix}6/7 & 3/7\\ -1/7 & -1/7\end{pmatrix},\quad B=\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}=\begin{pmatrix}6/7 & 2/7\\ 3/7 & 1/7\end{pmatrix}.$$

The activation function is described by $g_i(x)=\frac{|x+1|-|x-1|}{2}$, for $i=1,2$. The time-varying delay $\tau(t)$ is continuous and $|\tau(t)|\le\tau$, where τ is a constant.

It is clear that $\alpha_i=\beta_i=1$ for $i=1,2$. We check the condition (35) in Corollary 5.2,

$$\sum_{i=1}^{2}\frac{1}{c_i}\max_{j=1,2}|a_{ij}\alpha_j|+\sum_{i=1}^{2}\frac{1}{c_i}\max_{j=1,2}|b_{ij}\beta_j|\le\frac{1}{3}\times\left(\frac{6}{7}+\frac{1}{7}+\frac{6}{7}+\frac{3}{7}\right)=\frac{16}{21}<1.$$

Hence, by Corollary 5.2, the trivial equilibrium $x=0$ of this cellular neural network is asymptotically stable.

However, the condition (36) becomes

$$\max_{i=1,2}\left\{\frac{1}{c_i}\sum_{j=1}^{2}|a_{ij}\alpha_j|+\frac{1}{c_i}\sum_{j=1}^{2}|b_{ij}\beta_j|\right\}=\frac{17}{21}>\frac{1}{\sqrt{2}}.$$

Hence, Theorem 4.1 of [12] is not applicable.

Example 7.2. *Consider the two-dimensional stochastic recurrent neural network with time-varying delays*

$$\begin{aligned} dx(t) &= -\begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} dt + \begin{pmatrix} 2 & 0.4 \\ 0.6 & 1 \end{pmatrix}\begin{pmatrix} 0.2\tanh(x_1(t)) \\ 0.2\tanh(x_2(t)) \end{pmatrix} dt \\ &\quad + \begin{pmatrix} -0.8 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0.2\tanh(x_1(t-\tau_1(t))) \\ 0.2\tanh(x_2(t-\tau_2(t))) \end{pmatrix} dt + \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \int_{t-r(t)}^{t} 0.2\tanh(x_1(s))\,ds \\ \int_{t-r(t)}^{t} 0.2\tanh(x_2(s))\,ds) \end{pmatrix} dt \\ &\quad + \sigma(t, x(t), x(t-\tau(t)))\,dw(t), \end{aligned} \tag{37}$$

where $\tau(t), r(t)$ *are continuous functions such that* $t - \tau(t) \to \infty$ *as* $t \to \infty$ *and* $|r(t)| \le 1$, $\sigma : \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$ *satisfies*

$$trace\left[\sigma^T(t,x,y)\sigma(t,x,y)\right] \le 0.003(x_1^2 + x_2^2 + y_1^2 + y_2^2).$$

We suppose $p = 2$, and take $\mu_i = \nu_i = 0.003$ for $i = 1, 2$, by simple computation, we have $\alpha_i = 0.2$, for $i = 1, 2$, $c = \min\{c_1, c_2\} = 5$, $\mu = \nu = 0.003$. From Corollary 2.3, we have that

$$5\sum_{i=1}^{2} c_i^{-2}\left[\sum_{j=1}^{2} a_{ij}^2\alpha_j^2\right] + 5\sum_{i=1}^{2} c_i^{-2}\left[\sum_{j=1}^{2} b_{ij}^2\alpha_j^2\right] + 5\sum_{i=1}^{2}\left(\frac{\tau}{c_i}\right)^2\left[\sum_{j=1}^{2} l_{ij}^2\alpha_j^2\right] + 20\times 2\times\sum_{i=1}^{2} c_i^{-1}(\mu+\nu) < 0.256 < 1.$$

Then the trivial solution of (37) is mean square asymptotically stable.

If $\tau(t)$ is bounded, from Corollary 4.2, we obtain that

$$5c^{-2}\sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}^2\alpha_j^2 + 5c^{-2}\sum_{i=1}^{2}\sum_{j=1}^{n} b_{ij}^2\alpha_j^2 + 5c^{-2}\tau^2\sum_{i=1}^{2}\sum_{j=1}^{2} l_{ij}^2\alpha_j^2 + 20\times 4c^{-1}(\mu+\nu) < 0.298.$$

Hence, the trivial solution of (37) is mean square exponentially stable.

Example 7.3. *Consider a two-dimensional stochastically perturbed HNN with time-varying delays,*

$$dx(t) = [-Cx(t) + Af(x(t)) + Bg(x_\tau(t))]\,dt + \sigma(t, x(t), x_\tau(t))\,d\omega(t), \tag{38}$$

where $f(x) = \frac{1}{5}\arctan x$, $g(x) = \frac{1}{5}\tanh x = \frac{1}{5}(e^x - e^{-x})/(e^x + e^{-x})$, $\tau(t) = \frac{1}{2}\sin t + \frac{1}{2}$,

$$C = \begin{pmatrix} 5 & 0 \\ 0 & 4.5 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 0.4 \\ 0.6 & 1 \end{pmatrix} \quad and \quad B = \begin{pmatrix} -0.8 & 2 \\ 1 & 4 \end{pmatrix}.$$

In this example, let $p = 3$, *take* $\alpha_j = 0.2, \beta_j = 0.2$, $j = 1, 2$, $\sigma : \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$ *satisfies*

$$trace\left[\sigma^T(t,x,y)\sigma(t,x,y)\right] \le 0.01(x_1^2 + x_2^2 + y_1^2 + y_2^2).$$

Note that the exponential stability of (38) has been studied in Sun and Cao [28] by employing the method of variation parameter, inequality technique and stochastic analysis.

Now, we check the condition in Corollary 4.3,

$$4^{p-1}c^{-(1+p/q)}\sum_{i=1}^{2}\left[\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right]^{p/q} + 4^{p-1}c^{-(1+p/q)}\sum_{i=1}^{2}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q} + 4^{p-1}2^p c^{-p/2}(\mu^{p/2} + \nu^{p/2}) < 0.18 < 1.$$

From Corollary 4.3, the trivial solution of (38) is exponentially stable.

8. Appendix

In this section, we first show that $I_5(s)$ in [19] is not a local martingale and then we present some examples about Banach spaces.

Definition 8.1. *A real valued $\mathcal{F}_t$-adapted process $M = \{M(t) : t \geq 0\}$ is a martingale if $\mathbb{E}|M(t)| < \infty$ for all $t \geq 0$ and*

$$\mathbb{E}[M(t)|\mathcal{F}_s] = M(s), \qquad a.s. \quad \text{for} \quad \text{all} \quad 0 \leq \text{s} < \text{t} < \infty.$$

Lemma 8.2. *For continuous function $\sigma(t)$, $\int_0^t e^{-c(t-s)}\sigma(s)\,d\omega(s)$ is not a martingale.*

Proof. In fact, for $0 \leq u \leq t$,

$$\begin{aligned}\mathbb{E}\left[\int_0^t e^{-c(t-s)}\sigma(s)\,d\omega(s) \mid \mathcal{F}_u\right] &= \mathbb{E}\left[\int_0^u e^{-c(t-s)}\sigma(s)\,d\omega(s) \mid \mathcal{F}_u\right] + \mathbb{E}\left[\int_u^t e^{-c(t-s)}\sigma(s)\,d\omega(s) \mid \mathcal{F}_u\right] \\ &= \int_0^u e^{-c(t-s)}\sigma(s)\,d\omega(s) \neq \int_0^u e^{-c(u-s)}\sigma(s)\,d\omega(s). \end{aligned} \tag{39}$$

□

Lemma 8.3. ([24]) *If $M(t)$ is a local martingale and for every t, $\mathbb{E}\sup_{s\in[0,t]} |M(s)| < \infty$, then $M(t)$ is a martingale.*

Lemma 8.4. *For continuous function $\sigma(t)$, $\int_0^t e^{-c(t-s)}\sigma(s)\,d\omega(s)$ is not a local martingale.*

Proof. We suppose that $\int_0^t e^{-c(t-s)}\sigma(s)\,d\omega(s)$ is a local martingale. For every t, we have that

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s e^{-c(s-u)}\sigma(u)\,d\omega(u)\right| &= \mathbb{E}\sup_{s\in[0,t]} e^{-cs}\left|\int_0^s e^{cu}\sigma(u)\,d\omega(u)\right| \\ &\leq \mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s e^{cu}\sigma(u)\,d\omega(u)\right| \\ &\leq K_1\mathbb{E}\left(\int_0^t e^{2cu}\sigma^2(u)\,du\right)^{1/2} \leq K_1\left(\int_0^t e^{2cu}\mathbb{E}\sigma^2(u)\,du\right)^{1/2} < \infty.\end{aligned}$$

From Lemma 8.3, we obtain that M is a martingale. However, from Lemma 8.2, we know that $\int_0^t e^{-c(t-s)}\sigma(s)\,d\omega(s)$ is not a martingale, which is a contradiction. □

A normed linear space is a metric space with respect to the metric d derived from its norm, where $d(x, y) = \|x-y\|$.

Definition 8.5. *A Banach space is a normed linear space that is complete metric space with respect to the metric derived from its norm.*

Here are some examples of Banach spaces.

Example 8.6. *The space $C([a, b])$ of continuous, real-valued (or complex-valued) functions on $[a, b]$ with the sup-normed is a Banach space. More generally, we have the following examples.*

(i) *If X is a Banach space, the space $C([a, b]; X)$ of continuous, X-valued functions on $[a, b]$ equipped with the sup-norm is a Banach space.*

(ii) *If X is a Banach space, the space $BC([a, b]; X) := \{\varphi \mid \varphi \in C([a, b]; X),\ \|\varphi\| < \infty\}$ of bounded continuous, X-valued functions on $[a, b]$ equipped with the sup-norm is a Banach space.*

(iii) *If X is a Banach space, the space $\{\varphi \mid \varphi \in C([a, b]; X),\ \lim_{t\to\infty}\varphi(t) = 0\}$ and the sapce*

$$\left\{\varphi \mid \varphi \in C([a, b]; X), \|\varphi\| = \sup_{s\in[a,b]} |\varphi(s)| \text{ is bounded and } \lim_{t\to\infty}\varphi(t) = 0\right\}$$

are Banach spaces with respect to the sup-norm. Clearly, the space

$$C_0([a, b]; L^p(\Omega, \mathbb{R}^n)) := \left\{\varphi \mid \varphi \in C([a, b]; L^p(\Omega, \mathbb{R}^n)),\ \lim_{t\to\infty}\mathbb{E}|\varphi(t)|^p = 0\right\}$$

is a Banach spaces with respect to the norm defined by $\|\varphi\|^p := \sup_s [\mathbb{E}|\varphi(s)|^p]$.

The following lemma presents a Banach space that is used in this paper.

Lemma 8.7. *Suppose that $\mathcal{F}_t$ is complete, that is, contains all null sets. Then the space*

$$D := \{\varphi \in C_0([a,b]; L^p(\Omega, \mathbb{R}^n)),\ \varphi(t) \text{ is } \mathcal{F}_t - \text{measurable for all } t\}$$

is a closed subspace of $C_0([a,b]; L^p(\Omega, \mathbb{R}^n))$.

Proof. Let $\varphi(t), \psi(t) \in D$, then $\varphi(t)$ and $\psi(t)$ are $\mathcal{F}_t$-measurable for all t, so $\varphi(t) + \psi(t)$ and $\alpha\varphi(t)$ ($\alpha \in \mathbb{C}$) are $\mathcal{F}_t$-measurable for all t.

Suppose that the sequence $\varphi_1(t), \varphi_2(t), \cdots \varphi_n(t) \cdots \in D$, $\varphi(t) \in C_0([a,b]; L^p(\Omega, \mathbb{R}^n))$ and $\varphi_n(t) \to \varphi(t)$ as $n \to \infty$ for all t, we claim that $\varphi(t)$ is $\mathcal{F}_t$-measurable. In fact, since $\varphi_n(t) \to \varphi(t)$ as $n \to \infty$, then

$$\sup_{s\in\Omega} [\mathbb{E}|\varphi_n(s) - \varphi(s)|^p] \to 0 \quad \text{as} \quad n \to \infty.$$

So, for every t, we obtain that $\mathbb{E}|\varphi_n(s) - \varphi(s)|^p \to 0$ as $n \to \infty$, which implies that there exists a subsequence $(\varphi_{n_k}(t))_k$ such that $\varphi_{n_k}(t) \to \varphi(t)$ a.e. on Ω. On the other hand, $\mathcal{F}_t$ is complete. Hence, we obtain that $\varphi(t)$ is $\mathcal{F}_t$-measurable, which implies that D is a closed subspace of the space $C_0([a,b]; L^p(\Omega, \mathbb{R}^n))$. □

References

[1] J.A.D. Appleby, Fixed points, stability and harmless stochastic perturbations, Preprint, 2008.

[2] T.A. Burton, Stability by fixed point theory for functional differential equations. Dover Publication, New York, 2006.

[3] H.B. Chen, Integral inequality and exponential stability for neutral stochastic partial differential equations with delays, *Journal of inequalities and applications* 2009, Art. ID 297478, 15 pages.

[4] H.B. Chen, Impulsive-integral inequality and exponential stability for stochastic partial differential equations with delays, *Statistics and Probability Letters* **80** (2010), 50-56.

[5] A. Djoudi, R. Khemis, Fixed points techniques and stability for neutral nonlinear differential equations with unbounded delays, *Georgian Mathematical Journal* **13**(2006), No.1, 25-34.

[6] M. Forti, A. Tesi, New conditions for global stability of neural networks with application to linear and quadratic programming problems, *IEEE Trans. Circ. Syst. I*, **42** (1995), 354-366.

[7] A. Friedman, Stochastic differential equations and applications, Academic press, New York, 1976.

[8] S. Haykin Neural networks, Prentice-Hall, Englewood Cliffs, 1994.

[9] J. Hu, S. Zhong, L. Liang, Exponential stability analysis of stochastic delayed cellular neural network, *Chaos Soliton Fract.* **27** (2006), 1006-1010.

[10] C.X. Huang, Y.G. He, H.N. Wang, Mean square exponential stability of stochastic recurrent neural networks with time-varying delays, *Computers and Mathematics with Applications* **56** (2008), 1773-1778.

[11] C.X. Huang, Y.G. He, L.H. Huang, W.J. Zhu, *p*th moment stability analysis of stochastic recurrent neural networks with time-varying delays, *Information Science* **178** (2008), 2194-2203.

[12] X.H. Lai, Y.T. Zhang, Fixed point and asymptotic analysis of cellular neural networks, *Journal of Applied Mathematics* **2012** (2012), Article ID 689845, 12 pages doi:10.1155/2012/689845.

[13] T. Li, Q. Luo, C.Y. Sun, B.Y. Zhang, Exponential stability of recurrent neural networks with time-varying discrete and distributed delays, *Nonlinear Analysis: Real World Applications* **10** (2009), 2581-2589.

[14] X. Liao, X. Mao, Exponential stability and instability of stochastic neural networks, *Stochastic Analysis and Application* **14** (2) (1996), 165-185.

[15] X. Liao, X. Mao, Stability of stochastic neural networks, *Neural, Parallel and Scientific Computations* **4** (2) (1996), 205-224.

[16] X.X. Liao, J. Wang, Algebraic criteria for global exponential stability of cellular neural networks with multiple time delays, *IEEE TARANS. Circ. Syat. I.* **50** (2003), 268-275.

[17] X. Liao, X. Mao. Exponential stability of stochastic delay interval systems, *Syst. Control. Lett.*, 2000, 40: 171-181.

[18] D.Z. Liu, G.Y. Yang and W. Zhang, The stability of neutral stochastic delay differential equations with poisson jumps by fixed points, *Journal of Computational and Applied Mathematics* **235**(2011), 3115-3120.

[19] J.W. Luo, Fixed points and stability of neutral stochastic delay differential equations. *J. Math. Anal. Appl.*, **334**(2007), 431-440.

[20] J.W. Luo and T. Taniguchi, Fixed points and stability of stochastic neutral partial differential equations with infinite delays, *Stochastic Analysis and Applications* **27** (2009), 1163-1173.

[21] J.W. Luo, Fixed points and exponential stability of mild solutions of stochastic partial differential equations with delays, *J.Math.Anal.Appl.* **342** (2008), 753-760.

[22] S.E.A. Mohammed, Stochastic functional differential equations, *Longman Scientific and Technical*, 1986.

[23] X.R. Mao, Razumikihin-type theorems on exponential stability of stochastic functional differential equations, *Stochast. Proc. Appl.*, 1996, 65: 233-250.

[24] P.E. Protter, Stochastic Integration and Differential Equations, 2nd Edition, 2004.

[25] R. Sakthivel, J. Luo, Asymptotic stability of nonlinear impulsive stochastic differential equations, *Statist. Probab. Lett.* **79** (2009), 1219-1223.

[26] R. Sakthivel, J. Luo, Asymptotic stability of impulsive stochastic partial differential equations with infinite delays. *J. Math. Anal. Appl.* **356** (2009), 1-6.
[27] R. Sakthivel, R. Samidurai, S.M. Anthoni, Asymptotic stability of stochastic delayed recurrent neural networks with impulsive effects, *J Optim Theory Appl* **147** (2010), 583-596.
[28] Y.H. Sun, J.D. Cao, *pth* moment exponential stability of stochastic recurrent neural networks with time-varying delays, *Nonlinear Analysis: Real World Applications* **8** (2007) 1171-1185.
[29] J.K. Tian, S.M. Zhong, Y. Wang, Improved exponential stability criteria for neural networks with time-varying delays, *Neurocomputing* **97** (2012), 164-173.
[30] J.K. Tian, S.M. Zhong, New delay-dependent exponential stability criteria for neural networks with discrete and distributed time-varying delays, *Neurocomputing* **74** (2011), 3365-3375.
[31] B.D. Vries, J.C. Principle, The gamma model- a new neural model for temporal processing, *Neural Networks*, **5** (1992), 565-576.
[32] L. Wan, J. Sun, Mean square exponential stability of stochastic delayed Hopfield neural networks. *Phys. Lett.* A **343** (2005), 306-318.
[33] M. Wu, N.J. Huang and C.W. Zhao, Fixed points and stability in neutral stochastic differential equations with variable delays, *Fixed Point Theory and Applications*, Volume 2008, Article ID 407352, 11 pages. doi: 10.1155/2008/407352.
[34] M. Wu, N.J. Huang and C.W. Zhao, Stability of half-linear neutral stochastic differential equations with delays, *Bull.Aust.Math.Soc.* **80** (2009), 369-383.
[35] J.J. Yu, K.J. Zhang, S.M. Fei, T. Li, Simplified exponential stability analysis for recurrent neural networks with discrete and distributed time-varying delays, *Applied Mathematics and Computation* **205** (2008), 465-474.
[36] Z. Yuan, L. Yuan, L. Huang, Dynamics of periodic Cohen-Grossberg neural networks with varying delays, *Neurocomputing* **70** (2006), 164-172.
[37] Z.G. Zeng, D.S. Wang, Global stability of a general class of discrete-time recurrent neural networks, *Neural Computation*, **18** (2006), 848-870.
[38] Z.G. Zeng, J. Wang, Complete stability of cellular neural networks with time-varying delays, *IEEE Trans. Circ. Syst. I.*, **53** (2006), 944-955.
[39] H.Y. Zhao, Global asymptotic stability of Hopfield neural network involving distributed delays, *Neural Networks* **17** (2004), 47-53.