New stability results for impulsive stochastic delayed recurrent neural networks by using fixed point theory

Guiling Chen, Onno van Gaans, Sjoerd Verduyn Lunel

Mathematical Institute, Leiden University, P.O.Box 9512, 2300 RA, Leiden, The Netherlands

Report MI-2013-09

Abstract

New sufficient conditions for asymptotic stability and exponential stability of a class of impulsive stochastic delayed recurrent neural networks are presented by using fixed point method. Our results neither require the boundedness, monotonicity and differentiability of the activation functions nor differentiability of the time varying delays, and the abrupt changes can be linear and nonlinear. In particular, a class of impulsive delayed neural networks without stochastic perturbations is also considered. Two examples are given to illustrate our main results.

Keywords: Fixed point theory, contraction mapping principle, asymptotic stability, exponential stability, stochastic delayed recurrent neural networks, trivial solution, variable delays, impulsive effects.

1. Introduction and main results

Neural networks have received an increasing interest in various areas such as optimization, signal processing, pattern recognition and image processing [3, 5, 6] in the past few decades.

Due to the finite switching speed of neurons and amplifiers in the implementation of neural networks, time delays which may lead to instability and bad performance in neural processing and signal transmission are commonly encountered. In addition, neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths [18]. Thus there will be a distribution of conduction velocities along these pathways and a distribution of propagation delays [20]. In these circumstances the signal propagation is not instantaneous and can not be modeled with discrete delays. Therefore, a more appropriate way which incorporates continuously distributed delays in neural network models is proposed. Further, due to random fluctuations and probabilistic causes in the network, noises do exist in a neural network. On the other hand, besides delay and stochastic effects, impulsive effects are also likely to exist in the neural networks systems, which could be stabilized or destabilized the systems. Therefore, it is necessary to take delay effects, stochastic effects and impulsive effects into account on dynamical behaviors of neural networks.

Lyapunov's direct method has long been viewed the main classical method of studying stability problems in many areas of stochastic delay differential equations. However, there is a large set of problems for which it has been ineffective when we use it for studying some classes of stochastic delay differential equations. For instance, it is ineffective if the equation has unbounded terms or if the delay is unbounded, and it is ineffective when dealing with some real world problems which naturally involve average properties. Therefore, an alternative may be explored to overcome such difficulties.

Recently, Burton [2] has utilized the fixed point theory to investigate the stability for deterministic systems, which shows that some of the difficulties posed by Lyapunov's direct method vanish when applying fixed point theory. Luo [13] and Appleby [1] have applied this powerful method to deal with the stability problems for stochastic delay differential equations, and afterwards, a great number of classes of stochastic delay differential equations are discussed by using fixed point method, see, for example, [4, 11, 12, 15, 16]. It turned out that fixed point method is a powerful technique in dealing with stability problems for differential equations with delays and stochastic differential equations with delays.

In this paper, we apply fixed point method to study asymptotic stability and exponential stability of a class of stochastic delayed neural networks with impulsive effects, which is described by

$$\begin{cases} dx_i(t) = \left[-c_i x_i(t) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) + \sum_{j=1}^n b_{ij} g_j(x_j(t-\tau(t))) + \sum_{j=1}^n l_{ij} \int_{t-r(t)}^t h_j(x_j(s))\, ds\right] dt \\ \qquad\qquad + \sum_{j=1}^n \sigma_{ij}(t, x_j(t), x_j(t-\tau(t)))\, d\omega_j(t), \quad t \neq t_k \\ \Delta x_i(t_k) = I_{ik}(x_i(t_k)), \quad t = t_k, \quad k = 1, 2, 3, \cdots \end{cases} \tag{1}$$

or

$$\begin{cases} dx(t) = \left[-Cx(t) + Af(x(t)) + Bg(x(t-\tau(t))) + W \int_{t-r(t)}^t h(x(s))\, ds\right] dt + \sigma(t, x(t), x(t-\tau(t)))\, d\omega(t), \quad t \neq t_k \\ \Delta x(t_k) = I_k(x(t_k)), \quad t = t_k, \quad k = 1, 2, 3, \cdots \end{cases}$$

$i = 1, 2, 3, \cdots, n$, where $x(t) = (x_1(t), x_2(t), \cdots x_n(t))^T \in \mathbb{R}^n$ is the state vector associated with the neurons; $C = \mathrm{diag}(c_1, c_2, \cdots c_n) > 0$ where $c_i > 0$ represents the rate with which the ith unit will reset its ponential to the resting state in isolation when disconnected from the network and the external stochastic perturbations; $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ and $W = (l_{ij})_{n\times n}$ represent the connection weight matrix, delayed connection weight matrix and distributed delayed connection weight matrix, respectively; f_j, g_j, h_j are activation functions, $f(x(t)) = (f_1(x(t)), f_2(x(t)), \cdots f_n(x(t)))^T \in \mathbb{R}^n$, $g(x(t)) = (g_1(x(t)), g_2(x(t)), \cdots g_n(x(t)))^T \in \mathbb{R}^n$, $h(x(t)) = (h_1(x(t)), h_2(x(t)), \cdots h_n(x(t)))^T \in \mathbb{R}^n$. Moreover, $\omega(t) = (\omega_1(t), \omega_2(t), \cdots \omega_n(t))^T \in \mathbb{R}^n$ is an n-dimensional Brownion motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with natural complete filtration $\{\mathcal{F}_t\}_{t\geq 0}$ $(i.e.\mathcal{F}_t$ = completion of $\sigma\{\omega(s) : 0 \leq s \leq t\})$ and $\sigma : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $\sigma = (\sigma_{ij})_{n\times n}$ is the diffusion coefficient matrix. $\Delta x_i(t_k) = I_{ik}(x_i(t_k)) = x_i(t_k^+) - x_i(t_k^-)$ is the impulse at moment t_k, and $t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim_{k\to\infty} t_k = +\infty$, $x_i(t_k^+)$ and $x_i(t_k^-)$ stand for the right-hand and left-hand limit of $x_i(t)$ at $t = t_k$, respectively. $I_{ik}(x_i(t_k))$ shows the abrupt change of $x_i(t)$ at the impulsive moment t_k and $I_{ik}(\cdot) \in C\left(L^p_{\mathcal{F}_t}(\Omega; \mathbb{R}^n), L^p_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)\right)$. $\tau(t)$ and $r(t)$ denote discrete time varying delay and distributed time varying delay, respectively. Denote that $\vartheta = \inf_{t\geq 0}\{t-\tau(t), t-r(t)\}$.

The initial condition for the system (1) is given by

$$x(t) = \phi(t), \qquad t \in [\vartheta, 0], \tag{2}$$

where $\phi(t) = (\phi_1(t), \phi_2(t), \cdots \phi_n(t))^T \in C\left([\vartheta, 0], L^p_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)\right)$ with the norm is defined as

$$||\phi||^p = \sup_{\vartheta\leq s\leq 0}\left[\mathbb{E}\sum_{i=1}^n |\phi_i(s)|^p\right],$$

where $\mathbb{E}$ denotes expectation with respect to the probability measure $\mathbb{P}$.

To obtain our main results, we suppose the following conditions are satisfied:

(A1) the delays $\tau(t), r(t)$ are continuous functions such that $t - \tau(t) \to \infty$ and $t - r(t) \to \infty$ as $t \to \infty$;

(A2) $f_j(x)$, $g_j(x)$, and $h_j(x)$ satisfy Lipschitz condition. That is, for each $j = 1, 2, 3, \cdots n$, there exists constants α_j, β_j, γ_j such that for every $x, y \in \mathbb{R}^n$,

$$|f_j(x) - f_j(y)| \leq \alpha_j|x-y|, \quad |g_j(x) - g_j(y)| \leq \beta_j|x-y|, \quad |h_j(x) - h_j(y)| \leq \gamma_j|x-y|;$$

(A3) There exists nonegative constants p_{ik} such that for any $x, y \in \mathbb{R}^n$,

$$|I_{ik}(x) - I_{ik}(y)| \leq p_{ik}|x-y|, \ i = 1, 2, \cdots, n, \ k = 1, 2, 3, \cdots;$$

(A4) Assume that $f(0) \equiv 0$, $g(0) \equiv 0$, $h(0) \equiv 0$, $\sigma(t, 0, 0) \equiv 0$, $I_{ik}(0) \equiv 0$;

(A5) $\sigma(t, x, y)$ satisfies a Lipschitz condition. That is, there are nonnegative constants μ_i and ν_i such that

$$\mathrm{trace}\left[(\sigma(t,x,y) - \sigma(t,u,v))^T(\sigma(t,x,y) - \sigma(t,u,v))\right] \leq \sum_{i=1}^n \left[\mu_i(x_i - u_i)^2 + \nu_i(y_i - v_i)^2\right].$$

The solution $x(t) := x(t; s, \phi)$ of the system (1) is, for the time t, a piecewise continuous vector-valued function with the first kind discontinuity at the points t_k ($k = 1, 2, \cdots$), where it is left continuous, i.e.,

$$x_i(t_k^-) = x_i(t_k), \qquad x_i(t_k^+) = x_i(t_k) + I_{ik}(x_i(t_k)), \qquad i = 1, 2, \cdots n, \qquad k = 1, 2, \cdots.$$

Definition 1.1. *The trivial solution of system* (1) *is said to be stable in pth* ($p \geq 2$) *moment if for arbitrary given* $\epsilon > 0$, *there exists a* $\delta > 0$ *such that* $\mathbb{E}\|\phi\|^p < \delta$ *guarantees that*

$$\mathbb{E}\|x(t, 0, \phi)\|^p < \epsilon, \quad t \geq 0.$$

where $\phi(t) \in C\left([\vartheta, 0], L^p_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)\right)$. *In particular, when* $p = 2$, *we say it is mean square stable.*

Definition 1.2. *The trivial solution of system* (1) *is said to be asymptotically stable in pth* ($p \geq 2$) *moment if it is stable in pth* ($p \geq 2$) *moment and there exists a scalar* $\sigma > 0$, *such that* $E\|\phi\|^p < \sigma$ *implies*

$$\lim_{t\to\infty} \mathbb{E}\|x(t, 0, \phi)\|^p = 0.$$

where $\phi(t) \in C\left([\vartheta, 0], L^p_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)\right)$.

Definition 1.3. *The trivial solution of system* (1) *is said to be pth* ($p \geq 2$) *moment exponentially stable if there exists a pair of positive constants* λ *and* C *such that*

$$\mathbb{E}\|x(t, t_0, \phi)\|^p \leq C\mathbb{E}\|\phi\|^p e^{-\lambda t}, \quad t \geq 0,$$

holds for $\phi(t) \in C\left([\vartheta, 0], L^p_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)\right)$. *Especially, when* $p = 2$, *it is called to be exponentially stable in mean square.*

Define $\mathcal{S}_\phi$ the space of all $\mathcal{F}_t$-adapted processes $\varphi(t, \omega) : [\vartheta, \infty) \times \Omega \to \mathbb{R}^n$ such that $\varphi \in C\left([\vartheta, \infty), L^p_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)\right)$, $\varphi(t, \omega)$ is continuous on $t \neq t_k$ ($k = 1, 2, \cdots$), $\lim_{t\to t_k^-} \varphi(t, \omega)$ and $\lim_{t\to t_k^+} \varphi(t, \omega)$ exist, and $\lim_{t\to t_k^-} \varphi(t, \omega) = \varphi(t_k, \omega)$ for $k = 1, 2, \cdots$. Moreover, we set $\varphi(t, \omega) = \phi(t)$ for $t \in [\vartheta, 0]$ and $\mathbb{E}\left(\sum_{i=1}^n |\varphi_i(t)|^p\right) \to 0$ as $t \to \infty$, $i = 1, 2, \cdots n$. If we define the metric as the form

$$\|\varphi\|^p := \sup_{t\geq\vartheta} \left(\mathbb{E} \sum_{i=1}^n |\varphi_i(t)|^p\right), \tag{3}$$

then $\mathcal{S}_\phi$ is a complete metric space with respect to the norm (3). Using the contraction mapping defined on the space $\mathcal{S}_\phi$ and applying the contraction mapping principle, we obtain our first main result.

Theorem 1.4. *Suppose that the assumptions* (A1)-(A5) *hold. Assume that the following conditions are satisfied,*

(i) *the distributed delay* $r(t)$ *is bounded by a constant* r;
(ii) *there exist constants* p_i ($i = 1, 2, \cdots n$) *such that* $p_{ik} \leq p_i(t_k - t_{k-1})$, $k = 1, 2, \cdots$;
(iii)

$$\begin{aligned}\alpha \quad \triangleq \quad & 6^{p-1} \sum_{i=1}^n c_i^{-p} \left(\sum_{j=1}^n |a_{ij}|^q |\alpha_j|^q\right)^{p/q} + 6^{p-1} \sum_{i=1}^n c_i^{-p} \left(\sum_{j=1}^n |b_{ij}|^q |\beta_j|^q\right)^{p/q} + 6^{p-1} \sum_{i=1}^n \left(\frac{r}{c_i}\right)^p \left(\sum_{j=1}^n |l_{ij}|^q |\gamma_j|^q\right)^{p/q} \\ & + 6^{p-1} n^{p-1} \sum_{i=1}^n c_i^{-p/2} \left(\mu^{p/2} + \nu^{p/2}\right) + 6^{p-1} c^{-1} \max_{i=1,2,\cdots,n} \left\{\frac{p_i^p}{c_i^{p-1}}\right\} < 1,\end{aligned} \tag{4}$$

where $c = \min\{c_1, c_2, \cdots c_n\}$, $\mu = \max\{\mu_1, \mu_2, \cdots \mu_n\}$, $\nu = \max\{\nu_1, \nu_2, \cdots \nu_n\}$.

Then the trivial solution of (1) *is pth moment asymptotically stable.*

Define C_ϕ the space of all $\mathcal{F}_t$-adapted processes $\varphi(t,\omega) : [\vartheta,\infty)\times\Omega \to \mathbb{R}^n$ such that $\varphi \in C\left([\vartheta,\infty), L^p_{\mathcal{F}_0}(\Omega;\mathbb{R}^n)\right)$, $\varphi(t,\omega)$ is continuous on $t \neq t_k$ $(k = 1,2,\cdots)$, $\lim_{t\to t_k^-}\varphi(t,\omega)$ and $\lim_{t\to t_k^+}\varphi(t,\omega)$ exist, and $\lim_{t\to t_k^-}\varphi(t,\omega) = \varphi(t_k,\omega)$ for $k = 1,2,\cdots$. Moreover, we set $\varphi(t,\omega) = \phi(t)$ for $t \in [\vartheta, 0]$ and $e^{\lambda t}\mathbb{E}\left(\sum_{i=1}^n |\varphi_i(t)|^p\right) \to 0$ as $t \to \infty$, $\lambda < \min\{c_1, c_2, \cdots c_n\}$, $i = 1,2,\cdots n$. Then C_ϕ is a complete metric space with respect to the norm (3). Using the contraction mapping defined on the space C_ϕ and applying the contraction mapping principle, we obtain our second main result.

Theorem 1.5. *Suppose that the assumptions* (A1)-(A5) *hold. Assume that the following conditions are satisfied,*

(i) *the discrete delay $\tau(t)$ and distributed delay $r(t)$ are bounded by a constant τ;*
(ii) *there exist constants p_i ($i = 1,2,\cdots n$) such that $p_{ik} \le p_i(t_k - t_{k-1})$, $k = 1,2,\cdots$;*
(iii)

$$\begin{aligned}\alpha \quad &\triangleq \quad 6^{p-1}\sum_{i=1}^n c_i^{-p}\left(\sum_{j=1}^n |a_{ij}|^q|\alpha_j|^q\right)^{p/q} + 6^{p-1}\sum_{i=1}^n c_i^{-p}\left(\sum_{j=1}^n |b_{ij}|^q|\beta_j|^q\right)^{p/q} + 6^{p-1}\sum_{i=1}^n\left(\frac{\tau}{c_i}\right)^p\left(\sum_{j=1}^n |l_{ij}|^q|\gamma_j|^q\right)^{p/q}\\ &+6^{p-1}n^{p-1}\sum_{i=1}^n c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right) + 6^{p-1}c^{-1}\max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\} < 1, \end{aligned}\tag{5}$$

where $c = \min\{c_1, c_2, \cdots c_n\}$, $\mu = \max\{\mu_1, \mu_2, \cdots \mu_n\}$, $\nu = \max\{\nu_1, \nu_2, \cdots \nu_n\}$.

Then the trivial solution of (1) *is pth moment exponentially stable.*

Remark 1.6. *Theorem* 1.5 *and Theorem* 1.4 *present sufficient conditions for exponential stability and asymptotic stability of the system* (1)*, respectively. In Theorem* 1.5*, both the discrete delay $\tau(t)$ and distributed delay $r(t)$ are required to be bounded, while the discrete delay $\tau(t)$ in Theorem* 1.4 *can be unbounded.*

Remark 1.7. *The system* (1) *is quite general and it includes several well-known neural network models as its special cases, see, for example, the models in* [7, 8, 9, 10, 14, 17, 19, 21]. *Sakthivel et al.* [14] *has considered asymptotic stability in mean square of the system* (1) *with linear impulsive effects, by employing Lyapunov functional method and using linear matrix inequality optimization approach. However, the time varying delays in* [14] *should satisfy*

$$0 \le h_1 \le \tau(t) \le h_2, \quad \tau'(t) \le \mu,$$

where h_1, h_2 are constants, the distributed delay $r(t)$ is bounded, $0 \le r(t) \le \bar{r}$, $\bar{r}$ is a constant. It is clearly that our main results do not require the differentiability of delays. On the other hand, condition (A2) *implies that the activation functions discussed in this paper may be unbounded, non-monotonic and / or non-differentiable.*

Remark 1.8. *In this paper, our approach is based on the method by using fixed point theory, and in one step, a fixed point argument can yield the existence, unqiueness and stability criteria of the considered system. However, when using Lyapunov's direct method, one must independently verify that a solution exists. The stability criteria we provided in our main results are only in terms of the system parameters c_i, a_{ij}, b_{ij}, l_{ij}, p_i etc. Hence, these criteria can be verified easily in applications.*

Consider the case when there is no stochastic perturbations on the system (1), the stochastic neural networks become usual neural network which can be described as

$$\begin{cases} \frac{dx_i(t)}{dt} = -c_ix_i(t) + \sum_{j=1}^n a_{ij}f_j(x_j(t)) + \sum_{j=1}^n b_{ij}g_j(x_j(t-\tau(t))) + \sum_{j=1}^n l_{ij}\int_{t-r(t)}^t h_j(x_j(s))\,ds, \quad t \neq t_k\\ \\ \Delta x_i(t_k) = I_{ik}(x_i(t_k)), \quad t = t_k, \quad k = 1,2,3,\cdots \end{cases}\tag{6}$$

or

$$\begin{cases} \frac{dx(t)}{dt} = -Cx(t) + Af(x(t)) + Bg(x-\tau(t)) + D\int_{t-r(t)}^t h(x(s))\,ds, \quad t \neq t_k\\ \\ \Delta x(t_k) = I_k(x(t_k)), \quad t = t_k, \quad k = 1,2,3,\cdots \end{cases}$$

for $i = 1, 2, 3, \cdots, n$, where $x(\cdot) = (x_1(\cdot), x_2(\cdot), \cdots, x_n(\cdot))^T$ is the neuron state vector of the transformed system (6).

The initial condition for the system (6) is

$$x(t) = \phi(t), \qquad t \in [\vartheta, 0], \tag{7}$$

where ϕ is a continuous function with the norm defined by $\|\phi\| = \sum_{i=1}^{n} \sup_{\vartheta \le s \le 0} |\phi_i(s)|$.

Definition 1.9. *The trivial solution of* (6) *is said to be stable if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any initial condition $\phi(s) \in C([\vartheta, 0], \mathbb{R}^n)$ satisfying $\|\phi\| < \delta$, we have that the corresponding solution $\|x(t, s, \phi)\| < \epsilon$ for $t \ge 0$.*

Definition 1.10. *The trivial solution of* (6) *is said to be asymptotically stable if it is stable and for any initial condition $\phi(s) \in C([\vartheta, 0], \mathbb{R}^n)$ we have that the corresponding solution* $\lim_{t\to\infty} \|x(t, s, \phi)\| = 0$.

Definition 1.11. *The trivial solution of* (6) *is said to be globally exponentially stable if there exist scalars $\lambda > 0$ and $\alpha > 0$ such that for any initial condition $\phi(s) \in C([\vartheta, 0], \mathbb{R}^n)$, we have that the corresponding solution $\|x(t, s, \phi)\| \le \alpha e^{-\lambda t}\|\phi\|$ for $t \ge 0$.*

Define $\mathcal{H}_\phi = \mathcal{H}_{1\phi} \times \mathcal{H}_{2\phi} \times \cdots \mathcal{H}_{n\phi}$, where $\mathcal{H}_{i\phi}$ is the space consisting of continuous functions $\varphi_i(t) : \mathbb{R}^+ \to \mathbb{R}$ such that $\varphi_i(t)$ is continuous on $t \neq t_k$ $(k = 1, 2, \cdots)$, $\lim_{t\to t_k^-} \varphi_i(t)$ and $\lim_{t\to t_k^+} \varphi_i(t)$ exist, and $\lim_{t\to t_k^-} \varphi_i(t) = \varphi_i(t_k)$. Moreover, we set $\varphi_i(\theta) = \phi(\theta)$ for $\vartheta \le \theta \le 0$ and $\varphi_i(t) \to 0$ as $t \to \infty$, $i = 1, 2 \cdots n$. For any $\varphi(t), \eta(t) \in \mathcal{H}_\phi$, if we define the metric as

$$d(\varphi, \eta) = \sum_{i=1}^{n} \sup_{t \ge \vartheta} |\varphi_i(t) - \eta_i(t)|, \tag{8}$$

then $\mathcal{H}_\phi$ is a complete metric space with respect to the norm (8). Using the contraction mapping defined on the space H_ϕ and applying the contraction mapping principle, we obtain our third main result.

Theorem 1.12. *Suppose that the assumptions* (A1)-(A4) *hold. Assume that the following conditions are satisfied,*

(i) *the distributed delay $r(t)$ is bounded by a constant r;*

(ii) *there exist constants p_i $(i = 1, 2, \cdots n)$ such that $p_{ik} \le p_i(t_k - t_{k-1})$, $k = 1, 2, \cdots$;*

(iii)

$$\alpha \triangleq \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \sum_{i=1}^{n} \frac{r}{c_i} \max_{j=1,2,\cdots,n} |l_{ij}\gamma_j| + \max_{i=1,2,\cdots,n} \left\{\frac{p_i}{c_i}\right\} < 1. \tag{9}$$

Then the trivial solution of (6) *is asymptotically stable.*

Define $\mathcal{B}_\phi = \mathcal{B}_{1\phi} \times \mathcal{B}_{2\phi} \times \cdots \mathcal{B}_{n\phi}$, where $\mathcal{B}_{i\phi}$ is the space consisting of continuous functions $\varphi_i(t) : \mathbb{R}^+ \to \mathbb{R}$ such that $\varphi_i(t)$ is continuous on $t \neq t_k$ $(k = 1, 2, \cdots)$, $\lim_{t\to t_k^-} \varphi_i(t)$ and $\lim_{t\to t_k^+} \varphi_i(t)$ exist, and $\lim_{t\to t_k^-} \varphi_i(t) = \varphi_i(t_k)$. Moreover, we set $\varphi_i(\theta) = \phi(\theta)$ for $\vartheta \le \theta \le 0$ and $e^{\lambda t}\varphi_i(t) \to 0$ as $t \to \infty$, where $\lambda < \min\{c_1, c_2, \cdots c_n\}$, $i = 1, 2 \cdots n$. Then $\mathcal{B}_\phi$ is a complete metric space with respect to the metric (8). Using the contraction mapping defined on the space $\mathcal{B}_\phi$ and applying the contraction mapping principle, we obtain our fourth main result.

Theorem 1.13. *Suppose that the assumptions* (A1)-(A4) *hold. Assume that the following conditions are satisfied,*

(i) *the discrete delay $\tau(t)$ and distributed delay $r(t)$ are bounded by a constant τ;*

(ii) *there exist constants p_i $(i = 1, 2, \cdots n)$ such that $p_{ik} \le p_i(t_k - t_{k-1})$, $k = 1, 2, \cdots$;*

(iii)

$$\alpha \triangleq \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \sum_{i=1}^{n} \frac{\tau}{c_i} \max_{j=1,2,\cdots,n} |l_{ij}\gamma_j| + \max_{i=1,2,\cdots,n} \left\{\frac{p_i}{c_i}\right\} < 1. \tag{10}$$

Then the trivial solution of (6) *is exponentially stable.*

Remark 1.14. *Zhang et al.* [21, 22] *have investigated exponential stability and asymptotic stability of a class of impulsive cellular neural networks by using fixed point method, which is a special case of the system* (6)*. Our results in Theorem* 1.12 *and Theorem* 1.13 *improve and extend the results in* [21, 22] (*see Remark* 4.3 *and Remark* 5.2 *for more information*).

The rest of this paper is organized as follows. The proofs of Theorem 1.4 and Theorem 1.5 are presented in Section 2 and Section 3, respectively. The proofs of Theorem 1.12 and Theorem 1.13 are provided in Section 4 and Section 5, respectively. Some examples are given to illustrate our main results in Section 6.

2. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. We start with some preparations.

Lemma 2.1. ([19]) *Set $\omega(t) = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is a n-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we have the following formula*

$$\mathbb{E}\left(\int_0^t f_i(s)\, d\omega_i(s) \int_0^t f_j(s)\, d\omega_j(s)\right) = \mathbb{E}\int_0^t f_i(s) f_j(s)\, d\langle\omega_i, \omega_i\rangle_s,$$

where $\langle\omega_i, \omega_i\rangle_s = \delta_{ij} s$, is cross-variations, δ_{ij} is correlation coefficient, $1 \le i, j \le n$.

Multiply both sides of (1) by $e^{c_i t}$, we obtain that for $t \neq t_k$, $i = 1, 2, 3, \cdots, n$,

$$\begin{aligned} de^{c_i t} x_i(t) &= e^{c_i t}\left[\sum_{j=1}^n a_{ij} f_j(x_j(t)) + \sum_{j=1}^n b_{ij} g_j(x_j(t-\tau(t))) + \sum_{j=1}^n l_{ij} \int_{t-r(t)}^t h_j(x_j(u))\, du\right] dt \\ &\quad + e^{c_i t} \sum_{j=1}^n \sigma_{ij}(t, x_j(t), x_j(t-\tau(t)))\, d\omega_j(t). \end{aligned} \tag{11}$$

Integrate (11) from $t_{k-1} + \epsilon$ ($\epsilon > 0$) to $t \in (t_{k-1}, t_k)$ ($k = 1, 2, \cdots$), we obtain that

$$\begin{aligned} e^{c_i t} x_i(t) &= e^{c_i (t_{k-1}+\epsilon)} x_i(t_{k-1}+\epsilon) + \int_{t_{k-1}+\epsilon}^t e^{c_i s} \sum_{j=1}^n \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, d\omega_j(s) \\ &\quad + \int_{t_{k-1}+\epsilon}^t e^{c_i s}\left[\sum_{j=1}^n a_{ij} f_j(x_j(s)) + \sum_{j=1}^n b_{ij} g_j(x_j(s-\tau(s))) + \sum_{j=1}^n l_{ij} \int_{s-r(s)}^s h_j(x_j(u))\, du\right] ds. \end{aligned} \tag{12}$$

Let $\epsilon \to 0$ in (12), for $t \in (t_{k-1}, t_k)$ ($k = 1, 2, \cdots$), we obtain that

$$\begin{aligned} e^{c_i t} x_i(t) &= e^{c_i t_{k-1}} x_i(t_{k-1}^+) + \int_{t_{k-1}}^t e^{c_i s} \sum_{j=1}^n \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, d\omega_j(s) \\ &\quad + \int_{t_{k-1}}^t e^{c_i s}\left[\sum_{j=1}^n a_{ij} f_j(x_j(s)) + \sum_{j=1}^n b_{ij} g_j(x_j(s-\tau(s))) + \sum_{j=1}^n l_{ij} \int_{s-r(s)}^s h_j(x_j(u))\, du\right] ds. \end{aligned} \tag{13}$$

Set $t = t_k - \epsilon$ ($\epsilon > 0$) in (13), we obtain that

$$\begin{aligned} e^{c_i (t_k-\epsilon)} x_i(t_k-\epsilon) &= e^{c_i t_{k-1}} x_i(t_{k-1}^+) + \int_{t_{k-1}}^{t_k-\epsilon} e^{c_i s} \sum_{j=1}^n \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, d\omega_j(s) \qquad (14) \\ &\quad + \int_{t_{k-1}}^{t_k-\epsilon} e^{c_i s}\left[\sum_{j=1}^n a_{ij} f_j(x_j(s)) + \sum_{j=1}^n b_{ij} g_j(x_j(s-\tau(s))) + \sum_{j=1}^n l_{ij} \int_{s-r(s)}^s h_j(x_j(u))\, du\right] ds. \end{aligned}$$

Let $\epsilon \to 0$ in (14), we obtain that

$$\begin{aligned} e^{c_i t_k} x_i(t_k^-) &= e^{c_i t_{k-1}} x_i(t_{k-1}^+) + \int_{t_{k-1}}^{t_k} e^{c_i s} \sum_{j=1}^n \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, d\omega_j(s) \\ &\quad + \int_{t_{k-1}}^{t_k} e^{c_i s} \left[\sum_{j=1}^n a_{ij} f_j(x_j(s)) + \sum_{j=1}^n b_{ij} g_j(x_j(s-\tau(s))) + \sum_{j=1}^n l_{ij} \int_{s-r(s)}^s h_j(x_j(u))\, du \right] ds. \qquad (15) \end{aligned}$$

Note that $x_i(t_k) = x_i(t_k^-)$, from (13) and (15), we obtain that for $t \in (t_{k-1}, t_k]$ $(k = 1, 2, \cdots)$,

$$\begin{aligned} e^{c_i t} x_i(t) &= e^{c_i t_{k-1}} x_i(t_{k-1}^+) + \int_{t_{k-1}}^t e^{c_i s} \left[\sum_{j=1}^n a_{ij} f_j(x_j(s)) + \sum_{j=1}^n b_{ij} g_j(x_j(s-\tau(s))) + \sum_{j=1}^n l_{ij} \int_{s-r(s)}^s h_j(x_j(u))\, du \right] ds \\ &\quad + \int_{t_{k-1}}^t e^{c_i s} \sum_{j=1}^n \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, d\omega_j(s) \\ &= e^{c_i t_{k-1}} x_i(t_{k-1}) + \int_{t_{k-1}}^t e^{c_i s} \left[\sum_{j=1}^n a_{ij} f_j(x_j(s)) + \sum_{j=1}^n b_{ij} g_j(x_j(s-\tau(s))) + \sum_{j=1}^n l_{ij} \int_{s-r(s)}^s h_j(x_j(u))\, du \right] ds \\ &\quad + \int_{t_{k-1}}^t e^{c_i s} \sum_{j=1}^n \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, d\omega_j(s) + e^{c_i t_{k-1}} I_{i(k-1)}(x_i(t_{k-1})). \end{aligned}$$

Hence,

$$\begin{aligned} e^{c_i t_{k-1}} x_i(t_{k-1}) &= e^{c_i t_{k-2}} x_i(t_{k-1}) + \int_{t_{k-2}}^{t_{k-1}} e^{c_i s} \left[\sum_{j=1}^n a_{ij} f_j(x_j(s)) + \sum_{j=1}^n b_{ij} g_j(x_j(s-\tau(s))) + \sum_{j=1}^n l_{ij} \int_{s-r(s)}^s h_j(x_j(u))\, du \right] ds \\ &\quad + \int_{t_{k-2}}^{t_{k-1}} e^{c_i s} \sum_{j=1}^n \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, d\omega_j(s) + e^{c_i t_{k-2}} I_{i(k-2)}(x_i(t_{k-2})) \\ &\vdots \\ e^{c_i t_2} x_i(t_2) &= e^{c_i t_1} x_i(t_1) + \int_{t_1}^{t_2} e^{c_i s} \left[\sum_{j=1}^n a_{ij} f_j(x_j(s)) + \sum_{j=1}^n b_{ij} g_j(x_j(s-\tau(s))) + \sum_{j=1}^n l_{ij} \int_{s-r(s)}^s h_j(x_j(u))\, du \right] ds \\ &\quad + \int_{t_1}^{t_2} e^{c_i s} \sum_{j=1}^n \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, d\omega_j(s) + e^{c_i t_1} I_{i1}(x_i(t_1)) \\ e^{c_i t_1} x_i(t_1) &= \phi_i(0) + \int_0^{t_1} e^{c_i s} \left[\sum_{j=1}^n a_{ij} f_j(x_j(s)) + \sum_{j=1}^n b_{ij} g_j(x_j(s-\tau(s))) + \sum_{j=1}^n l_{ij} \int_{s-r(s)}^s h_j(x_j(u))\, du \right] ds \\ &\quad + \int_0^{t_1} e^{c_i s} \sum_{j=1}^n \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, d\omega_j(s), \end{aligned}$$

which yields that for $t > 0$,

$$\begin{aligned} x_i(t) &= e^{-c_i t} \phi_i(0) + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n a_{ij} f_j(x_j(s))\, ds + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n b_{ij} g_j(x_j(s-\tau(s)))\, ds \\ &\quad + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n l_{ij} \int_{s-r(s)}^s h_j(x_j(u))\, du\, ds \\ &\quad + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, d\omega_j(s) + \sum_{0<t_k<t} e^{-c_i(t-t_k)} I_{ik}(x_i(t_k)). \end{aligned}$$

Lemma 2.2. *Define an operator by* $(Q\varphi)(t)=\phi(t)$ *for* $t\in[\vartheta,0]$, *and for* $t\geq 0$, $i=1,2,3,\cdots,n$,

$$\begin{aligned}(Q\varphi_i)(t) &= e^{-c_it}\phi_i(0)+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n a_{ij}f_j(\varphi_j(s))\,ds+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n b_{ij}g_j(\varphi_j(s-\tau(s)))\,ds\\ &\quad+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n l_{ij}\int_{s-r(s)}^s h_j(\varphi_j(u))\,du\,ds\\ &\quad+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n \sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,d\omega_j(s)+\sum_{0<t_k<t}e^{-c_i(t-t_k)}I_{ik}(\varphi_i(t_k)).\end{aligned}\tag{16}$$

Suppose that the assumptions (A1)-(A5) *hold, and assume that the conditions* (i)-(iii) *in Theorem* 1.4 *are satisfied, then* $Q:\mathcal{S}_\phi\to\mathcal{S}_\phi$ *and* Q *is a contraction mapping.*

Proof. Denote $(Q\varphi_i)(t):=J_{1i}(t)+J_{2i}(t)+J_{3i}(t)+J_{4i}(t)+J_{5i}(t)+J_{6i}(t)$, where

$$\begin{aligned}J_{1i}(t) &= e^{-c_it}\varphi_i(0),\qquad J_{2i}(t)=\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n a_{ij}f_j(\varphi_j(s))\,ds,\\ J_{3i}(t) &= \int_0^t e^{-c_i(t-s)}\sum_{j=1}^n b_{ij}g_j(\varphi_j(s-\tau(s)))\,ds,\\ J_{4i}(t) &= \int_0^t e^{-c_i(t-s)}\sum_{j=1}^n l_{ij}\int_{s-r(s)}^s h_j(\varphi_j(u))\,du\,ds,\\ J_{5i}(t) &= \int_0^t e^{-c_i(t-s)}\sum_{j=1}^n \sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,d\omega_j(s)\\ J_{6i}(t) &= \sum_{0<t_k<t}e^{-c_i(t-t_k)}P_{ik}(x_i(t_k)).\end{aligned}$$

Step1. From the definition of Banach space $\mathcal{S}_\phi$, we have that $\mathbb{E}\sum_{i=1}^n|\varphi_i(t)|^p<\infty$, for all $t\geq 0$, $\varphi\in\mathcal{S}_\phi$.

Step2. We prove the continuity in *pth* moment of Q on $[0,\infty)$. It is clearly that $(Q\varphi_i)(t)$ is continuous on $[\vartheta,0]$. For a fixed time $t>0$, it is easy to check that $J_{1i}(t),J_{2i}(t),J_{3i}(t),J_{4i}(t),J_{6i}(t)$ are continuous in *pth* moment on the fixed time $t\neq t_k$ $(k=1,2,\cdots)$. In the following, we check the *pth* moment continuity of $J_{5i}(t)$ on the fixed time $t\neq t_k$ $(k=1,2,\cdots)$. Let $\varphi\in\mathcal{S}_\phi$, $t\geq 0$, and $|r|$ be sufficiently small, take the limit $r\to 0$, we obatain that

$$\begin{aligned}\mathbb{E}\sum_{i=1}^n|J_{5i}(t+r)-J_{5i}(t)|^p &= \mathbb{E}\sum_{i=1}^n\Bigg|\int_0^t\left(e^{-c_i(t+r-s)}-e^{-c_i(t-s)}\right)\sum_{j=1}^n\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,d\omega_j(s)\\ &\qquad+\int_t^{t+r}e^{-c_i(t+r-s)}\sum_{j=1}^n\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,d\omega_j(s)\Bigg|^p\\ &\leq n^{p-1}\sum_{i=1}^n\sum_{j=1}^n\mathbb{E}\Bigg|\int_0^t\left(e^{-c_i(t+r-s)}-e^{-c_i(t-s)}\right)\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,d\omega_j(s)\\ &\qquad+\int_t^{t+r}e^{-c_i(t+r-s)}\sum_{j=1}^n\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,d\omega_j(s)\Bigg|^p\\ &\leq (2n)^{p-1}\sum_{i=1}^n\sum_{j=1}^n\mathbb{E}\left|\int_0^t\left|e^{-c_i(t_1+r-s)}-e^{-c_i(t-s)}\right||\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))|\,d\omega_j(s)\right|^p\\ &\quad+(2n)^{p-1}\sum_{i=1}^n\sum_{j=1}^n\mathbb{E}\left|\int_{t_1}^{t_1+r}e^{-c_i(t+r-s)}|\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))|\,d\omega_j(s)\right|^p\end{aligned}$$

$$= \quad (2n)^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\Bigg\{\mathbb{E}\left[\int_0^t |e^{-c_i(t+r-s)}-e^{-c_i(t-s)}|^2\sigma_{ij}^2(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,ds\right]^{p/2}$$
$$+\mathbb{E}\left[\int_t^{t+r} e^{-2c_i(t+r-s)}\sigma_{ij}^2(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,ds\right]^{p/2}\Bigg\}\to 0 \quad \text{as} \quad r\to 0.$$

Hence, $(Q\varphi_i)(t)$ is continuous in *pth* moment on the fixed time $t\neq t_k$ $(k=1,2,\cdots)$. On the other hand, as $t=t_k$, it is easy to check that $J_{1i}(t)$, $J_{2i}(t)$, $J_{3i}(t)$, $J_{4i}(t)$, $J_{5i}(t)$ are continuous in *pth* moment on the fixed time $t=t_k$ $(k=1,2,\cdots)$. In the following, we check *pth* moment continuity of $J_{6i}(t)$ on $t=t_k$ $(k=1,2,\cdots)$. Let $r<0$ be small enough,

$$\begin{aligned}\mathbb{E}\sum_{i=1}^{n}|J_{6i}(t_k+r)-J_{6i}(t_k)|^p &= \mathbb{E}\sum_{i=1}^{n}\left|\sum_{0<t_m<t_k+r}e^{-c_i(t_k+r-t_m)}I_{im}(\varphi_i(t_m))-\sum_{0<t_m<t_k}e^{-c_i(t_k-t_m)}I_{im}(\varphi_i(t_m))\right|^p\\ &\leq \mathbb{E}\sum_{i=1}^{n}\left|\left(e^{-c_i(t_k+r)}-e^{-c_it_k}\right)\sum_{0<t_m<t_k}e^{c_it_m}I_{im}(\varphi_i(t_m))\right|^p,\end{aligned}$$

which implies that $\lim_{r\to 0^-}\mathbb{E}\sum_{i=1}^{n}|J_{6i}(t_k+r)-J_{6i}(t_k)|^p=0$. Let $r>0$ be small enough,

$$\begin{aligned}\mathbb{E}\sum_{i=1}^{n}|J_{6i}(t_k+r)-J_{6i}(t_k)|^p &= \mathbb{E}\sum_{i=1}^{n}\left|\sum_{0<t_m<t_k+r}e^{-c_i(t_k+r-t_m)}I_{im}(\varphi_i(t_m))-\sum_{0<t_m<t_k}e^{-c_i(t_k-t_m)}I_{im}(\varphi_i(t_m))\right|^p\\ &= \mathbb{E}\sum_{i=1}^{n}\left|e^{-c_i(t_k+r)}\left[\sum_{0<t_m<t_k}e^{c_it_m}I_{im}(\varphi_i(t_m))+e^{c_it_k}I_{ik}(\varphi_i(t_k))\right]-e^{-c_it_k}\sum_{0<t_m<t_k}e^{c_it_m}I_{im}(\varphi_i(t_m))\right|^p\\ &= \mathbb{E}\sum_{i=1}^{n}\left|\left(e^{-c_i(t_k+r)}-e^{-c_it_k}\right)\sum_{0<t_m<t_k}e^{c_it_m}I_{im}(\varphi_i(t_m))+e^{-c_ir}I_{ik}(\varphi_i(t_k))\right|^p,\end{aligned}$$

which implies that $\lim_{r\to 0^+}\mathbb{E}\sum_{i=1}^{n}|J_{6i}(t_k+r)-J_{6i}(t_k)|^p=\mathbb{E}\sum_{i=1}^{n}|I_{ik}(\varphi_i(t_k))|^p$.

Based on the above discussion, we obtain that $(Q\varphi_i)(t):[\vartheta,\infty)\to L^p_{\mathcal{F}_0}(\Omega;\mathbb{R}^n)$ is continuous in *pth* moment on $t\neq t_k$ $(k=1,2,\cdots)$, and for $t=t_k$ $(k=1,2,\cdots)$, $\lim_{t\to t_k^+}(Q\varphi_i)(t)$ and $\lim_{t\to t_k^-}(Q\varphi_i)(t)$ exist. Furthermore, we also obtain that $\lim_{t\to t_k^-}(Q\varphi_i)(t)=(Q\varphi_i)(t_k)\neq\lim_{t\to t_k^+}(Q\varphi_i)(t)$.

Step3. We prove that $Q(\mathcal{S}_\phi)\subseteq\mathcal{S}_\phi$. From (16),

$$\mathbb{E}\sum_{i=1}^{n}|Q\varphi_i(t)|^p=\mathbb{E}\sum_{i=1}^{n}\left|\sum_{j=1}^{6}J_{ji}(t)\right|^p\leq 6^{p-1}\sum_{j=1}^{6}\mathbb{E}\left(\sum_{i=1}^{n}|J_{ji}(t)|^p\right). \tag{17}$$

Now, we estimate the right-hand terms of (17). By Hölder inequality,

$$\begin{aligned}\mathbb{E}\sum_{i=1}^{n}|J_{2i}(t)|^p &= \sum_{i=1}^{n}\mathbb{E}\left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}a_{ij}f_j(\varphi_j(s))\,ds\right|^p\\ &\leq \sum_{i=1}^{n}\mathbb{E}\left[\int_0^t e^{-\frac{c_i(t-s)}{q}}e^{-\frac{c_i(t-s)}{p}}\sum_{j=1}^{n}|a_{ij}||f_j(\varphi_j(s))|\,ds\right]^p\\ &\leq \sum_{i=1}^{n}\mathbb{E}\left\{\left[\int_0^t e^{-c_i(t-s)}\,ds\right]^{p/q}\int_0^t e^{-c_i(t-s)}\left[\sum_{j=1}^{n}|a_{ij}||f_j(\varphi_j(s))|\right]^p\,ds\right\}\\ &\leq \sum_{i=1}^{n}c_i^{-p/q}\mathbb{E}\left\{\int_0^t e^{-c_i(t-s)}\left[\sum_{j=1}^{n}|a_{ij}||\alpha_j||\varphi_j(s)|\right]^p\,ds\right\}\\ &\leq \sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_0^t e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)ds.\end{aligned} \tag{18}$$

With the similar computation as (18), we obtain that

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^{n}|J_{3i}(t)|^p &\le \sum_{i=1}^{n} c_i^{-p/q}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\int_0^t e^{-c_i(t-s)}\mathbb{E}\left[\sum_{j=1}^{n}|\varphi_j(s-\tau(s)))|^p\right]ds.\\
\mathbb{E}\sum_{i=1}^{n}|J_{4i}(t)|^p &\le \sum_{i=1}^{n} c_i^{-p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^t e^{-c_i(t-s)}\mathbb{E}\left[\sum_{j=1}^{n}\left|\int_{s-r(s)}^{s}\varphi_j(u)\,du\right|^p\right]ds\\
&\le \sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^t e^{-c_i(t-s)}\int_{s-r(s)}^{s}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(u)|^p\right)du\,ds. \qquad (19)
\end{aligned}$$

Using Lemma 2.1 and Hölder inequality, we obtain that

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^{n}|J_{5i}(t)|^p &= \sum_{i=1}^{n}\mathbb{E}\left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,d\omega_j(s)\right|^p \qquad (20)\\
&\le n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left\{\left[\int_0^t e^{-c_i(t-s)}|\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))|\,d\omega_j(s)\right]^2\right\}^{p/2}\\
&= n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\int_0^t e^{-2c_i(t-s)}\sigma_{ij}^2(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,ds\right]^{p/2}\\
&\le n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\int_0^t e^{-2c_i(t-s)}\left(\mu_j\varphi_j^2(s)+\nu_j\varphi_j^2(s-\tau(s))\right)ds\right]^{p/2}\\
&\le n^{p-1}2^{p/2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\left(\int_0^t e^{-2c_i(t-s)}\mu_j\varphi_j^2(s)\,ds\right)^{p/2}+\left(\int_0^t e^{-2c_i(t-s)}\nu_j\varphi_j^2(s-\tau(s))\,ds\right)^{p/2}\right]\\
&\le n^{p-1}2^{p/2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\left(\int_0^t e^{-2c_i(t-s)}\,ds\right)^{p/2-1}\int_0^t e^{-2c_i(t-s)}\mu_j^{p/2}|\varphi_j(s)|^p\,ds\right]\\
&\quad +n^{p-1}2^{p/2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left\{\left(\int_0^t e^{-2c_i(t-s)}\,ds\right)^{p/2-1}\int_0^t e^{-2c_i(t-s)}\nu_j^{p/2}|\varphi_j(s-\tau(s))|^p\,ds\right\}\\
&\le n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left\{\mu^{p/2}\int_0^t e^{-2c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)ds+\nu^{p/2}\int_0^t e^{-2c_i(t-s)}E\left(\sum_{j=1}^{n}|\varphi_j(s-\tau(s))|^p\right)ds\right\}\\
&\le n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left\{\mu^{p/2}\int_0^t e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)ds+\nu^{p/2}\int_0^t e^{-c_i(t-s)}E\left[\sum_{j=1}^{n}|\varphi_j(s-\tau(s))|^p\right]ds\right\}.
\end{aligned}$$

Further, from (A3), we know that $|I_{ik}(x_i(t_k))| \le p_{ik}|x_i(t_k)|$, combining with the condition (ii), we obtain that

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^{n}|J_{6i}(t)|^p &\le \mathbb{E}\sum_{i=1}^{n}\left[\sum_{0<t_k<t}e^{-c_i(t-t_k)}p_{ik}|\varphi_i(t_k)|\right]^p\\
&\le \mathbb{E}\sum_{i=1}^{n}\left[p_i\sum_{0<t_k<t}e^{-c_i(t-t_k)}|\varphi_i(t_k)|(t_k-t_{k-1})\right]^p\\
&\le \mathbb{E}\sum_{i=1}^{n}\left[p_i\int_0^t e^{-c_i(t-s)}|\varphi_i(s)|\,ds\right]^p\\
&\le \mathbb{E}\sum_{i=1}^{n}p_i^p\left(\int_0^t e^{-c_i(t-s)}\,ds\right)^{p/q}\int_0^t e^{-c_i(t-s)}|\varphi_i(s)|^p\,ds
\end{aligned}$$

$$\leq \max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\}\int_0^t e^{-c(t-s)}\mathbb{E}\left(\sum_{i=1}^n|\varphi_i(s)|^p\right)ds. \tag{21}$$

Since $\mathbb{E}\sum_{i=1}^n|\varphi_i(t)| \to 0$, $t-\tau(t)\to\infty$ and $t-r(t)\to\infty$ as $t\to\infty$. Thus, from (17) to (21), we obtain that $\mathbb{E}\sum_{i=1}^n|Q\varphi_i(t)|^p\to 0$ as $\mathbb{E}\sum_{i=1}^n|\varphi_i(t)|^p\to 0$. Therefore, $Q:\mathcal{S}_\phi\to\mathcal{S}_\phi$.

Step4. We prove that Q is a contraction mapping. For any $\varphi,\psi\in\mathcal{S}_\phi$, from (17) to (21), we obtain

$$\begin{aligned}
&\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{i=1}^n|Q\varphi_i(s)-Q\psi_i(s)|^p\right\}\\
&\leq 5^{p-1}\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{i=1}^n\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^n a_{ij}\Big(f_j(x_j(u))-f_j(y_j(u))\Big)du\right|^p\right\}\\
&\quad+5^{p-1}\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{i=1}^n\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^n b_{ij}\Big(g_j(x_j(u-\tau(u)))-g_j(y_j(u-\tau(u)))\Big)du\right|^p\right\}\\
&\quad+5^{p-1}\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{i=1}^n\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^n l_{ij}\int_{s-r(s)}^s\Big(h_j(\varphi_j(v))-h_j(\psi_j(v))\Big)dv\,du\right|^p\right\}\\
&\quad+5^{p-1}\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{i=1}^n\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^n\Big(\sigma_{ij}(s,x_j(s),x_j(u-\tau(u)))-\sigma_{ij}(s,y_j(s),y_j(s-\tau(u)))\Big)d\omega_j(u)\right|^p\right\}\\
&\quad+5^{p-1}\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{i=1}^n\left|\sum_{0<t_k<t}e^{-c_i(t-t_k)}\left(I_{ik}(\varphi_i(t_k))-I_{ik}(\psi_i(t_k))\right)\right|^p\right\}\\
&\leq 5^{p-1}\sup_{s\geq\vartheta}\sum_{i=1}^n c_i^{-p/q}\left(\sum_{j=1}^n|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_0^s e^{-c_i(s-u)}\mathbb{E}\left(\sum_{j=1}^n|\varphi_j(u)-\psi(u)|^p\right)du\\
&\quad+5^{p-1}\sup_{s\geq\vartheta}\sum_{i=1}^n c_i^{-p/q}\left(\sum_{j=1}^n|b_{ij}|^q|\beta_j|^q\right)^{p/q}\int_0^s e^{-c_i(s-u)}\mathbb{E}\left(\sum_{j=1}^n|\varphi_j(u-\tau(u)))-\psi_j(u-\tau(u)))|^p\right)du\\
&\quad+5^{p-1}\sup_{s\geq\vartheta}\sum_{i=1}^n\left(\frac{r}{c_i}\right)^{p/q}\left(\sum_{j=1}^n|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^s e^{-c_i(s-u)}\int_{u-r(u)}^u\mathbb{E}\left(\sum_{j=1}^n|\varphi_j(v)-\psi_j(v)|^p\right)dv\,du\\
&\quad+5^{p-1}n^{p-1}\sup_{s\geq\vartheta}\left\{\sum_{i=1}^n c_i^{1-p/2}\left[\mu^{p/2}\int_0^s e^{-c_i(s-u)}\mathbb{E}\left(\sum_{j=1}^n|\varphi_j(u)-\psi_j(u)|^p\right)du\right.\right.\\
&\qquad\left.\left.+\nu^{p/2}\int_0^s e^{-c_i(s-u)}E\left(\sum_{j=1}^n|\varphi_j(u-\tau(u))-\psi_j(u-\tau(u))|^p\right)du\right]\right\}\\
&\quad+5^{p-1}\max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\}\left\{\sup_{s\geq\vartheta}\int_0^s e^{-c(s-u)}\mathbb{E}\left(\sum_{i=1}^n|\varphi_i(u)-\psi(u)|^p\right)du\right\}\\
&\leq 5^{p-1}\left\{\sum_{i=1}^n c_i^{-p}\left(\sum_{j=1}^n|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+\sum_{i=1}^n c_i^{-p}\left(\sum_{j=1}^n|b_{ij}|^q|\beta_j|^q\right)^{p/q}+\sum_{i=1}^n\left(\frac{r}{c_i}\right)^p\left(\sum_{j=1}^n|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\right.\\
&\qquad\left.+n^{p-1}\sum_{i=1}^n c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right)+\frac{1}{c}\max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\}\right\}\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{j=1}^n|\varphi_j(s)-\psi_j(s)|^p\right\}
\end{aligned}$$

$$= \alpha \sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{j=1}^{n}|\varphi_j(s)-\psi_j(s)|^p\right\}.$$

From (4), we obtain that $Q:\mathcal{S}_\phi\to\mathcal{S}_\phi$ is a contraction mapping. $\square$

We are now ready to prove Theorem 1.4.

Proof. From Lemma 2.2, by the contraction mapping principle, we obtain that Q has a unique fixed point $x(t)$, which is a solution of (1) with $x(t)=\phi(t)$ as $t\in[\vartheta,0]$ and $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p\to 0$ as $t\to\infty$.

Now, we prove that the trivial solution of (1) is *pth* moment stable. From (4), For any $\epsilon>0$, we choose $\delta>0$ ($\delta<\epsilon$) such that $6^{p-1}\delta<(1-\alpha)\epsilon$.

If $x(t)=(x_1(t),x_2(t),\cdots x_n(t))^T$ is a solution of (1) with the initial condtion satisfying $\mathbb{E}\sum_{i=1}^{n}|\phi_i(t)|^p<\delta$, then $x(t)=(Qx)(t)$ defined in (16). We claim that $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p<\epsilon$ for all $t\geq 0$. Notice that $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p<\epsilon$ for $t\in[\vartheta,0]$, we suppose that there exists $t^*>0$ such that $\mathbb{E}\sum_{i=1}^{n}|x_i(t^*)|^p=\epsilon$ and $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p<\epsilon$ for $-\tau\leq t<t^*$, then it follows from (4), we obtain that

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^{n}|x_i(t^*)|^p \leq\ & 6^{p-1}\mathbb{E}\sum_{i=1}^{n}e^{-pc_it^*}|x_i(0)|^p+5^{p-1}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_0^{t^*}e^{-c_i(t^*-s)}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(s)|^p\right)ds\\
&+6^{p-1}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\int_0^{t^*}e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(s-\tau(s)))|^p\right)ds\\
&+6^{p-1}\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^{p/q}\left[\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right]^{p/q}\int_0^{t^*}e^{-c_i(t^*-s)}\int_{s-r(s)}^{s}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(u)|^p\right)du\,ds\\
&+6^{p-1}n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left[\mu^{p/2}\int_0^{t^*}e^{-c_i(t^*-s)}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(s)|^p\right)ds\right.\\
&\left.+\nu^{p/2}\int_0^{t^*}e^{-c_i(t-s)}E\left(\sum_{j=1}^{n}|x_j(s-\tau(s))|^p\right)ds\right]\\
&+6^{p-1}\max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\}\int_0^{t^*}e^{-c(t^*-s)}\mathbb{E}\left(\sum_{i=1}^{n}|x_i(s)|^p\right)ds\\
\leq\ & 6^{p-1}\delta+\left[6^{p-1}\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+6^{p-1}\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\right.\\
&\left.+6^{p-1}\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^{p}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}+6^{p-1}n^{p-1}\sum_{i=1}^{n}c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right)+\frac{1}{c}\max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\}\right]\epsilon\\
<\ & (1-\alpha)\epsilon+\alpha\epsilon=\epsilon,
\end{aligned}$$

which is a contradiction. Therefore, the trivial solution of (1) is asymptotically stable in *pth* moment. $\square$

Let $l_{ij}\equiv 0$, we come to the following stochastically perturbed hopfield neural networks with time-varying delays,

$$\begin{cases} dx(t)=[-Cx(t)+Af(x(t))+Bg(x(t-\tau(t)))]\,dt+\sigma(t,x(t),x(t-\tau(t)))\,d\omega(t), & t\neq t_k\\ \Delta x(t_k)=I_k(x(t_k)), \quad t=t_k, \quad k=1,2,3,\cdots. \end{cases} \tag{22}$$

Corollary 2.3. *Suppose that the assumptions* (A1)-(A5) *hold. Assume that*

(i) *there exist constants p_i ($i=1,2,\cdots n$) such that $p_{ik}\leq p_i(t_k-t_{k-1})$, $k=1,2,\cdots$,*

(ii)

$$5^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+5^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}$$
$$+5^{p-1}n^{p-1}\sum_{i=1}^{n} c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right)+5^{p-1}c^{-1}\max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\}<1,$$

where $c=\min\{c_1,c_2,\cdots c_n\}$, $\mu=\max\{\mu_1,\mu_2,\cdots\mu_n\}$, $\nu=\max\{\nu_1,\nu_2,\cdots\nu_n\}$.

Then the trivial solution of (22) *is pth moment asymptotically stable.*

Remark 2.4. *Note that the delay* $\tau(t)$ *in Corollary* 2.3 *can be unbounded.*

3. Proof of Theorem 1.5

Define the operator Q as (16). Following the proof of Theorem 1.4, we find that to show Theorem 1.5, we only need to prove that $e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|Q\varphi_i(t)|^p\to 0$ as $t\to\infty$. It follows from (16) that

$$e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|Q\varphi_i(t)|^p=e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}\left|\sum_{j=1}^{6}J_{ji}(t)\right|^p\le 6^{p-1}e^{\lambda t}\sum_{j=1}^{6}\mathbb{E}\left(\sum_{i=1}^{n}|J_{ji}(t)|^p\right).\tag{23}$$

Now, we estimate the right-hand terms of (23). First, by using Hölder inequality,

$$\begin{aligned}
e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|J_{2i}(t)|^p &= e^{\lambda t}\sum_{i=1}^{n}\mathbb{E}\left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}a_{ij}f_j(\varphi_j(s))\,ds\right|^p\\
&\le e^{\lambda t}e^{\lambda t}\sum_{i=1}^{n}\mathbb{E}\left[\int_0^t e^{-\frac{c_i(t-s)}{q}}e^{-\frac{c_i(t-s)}{p}}\sum_{j=1}^{n}|a_{ij}||f_j(\varphi_j(s))|\,ds\right]^p\\
&\le e^{\lambda t}\sum_{i=1}^{n}\mathbb{E}\left\{\left[\int_0^t e^{-c_i(t-s)}\,ds\right]^{p/q}\int_0^t e^{-c_i(t-s)}\left[\sum_{j=1}^{n}|a_{ij}||f_j(\varphi_j(s))|\right]^p ds\right\}\\
&\le e^{\lambda t}\sum_{i=1}^{n}c_i^{-p/q}\mathbb{E}\left\{\int_0^t e^{-c_i(t-s)}\left[\sum_{j=1}^{n}|a_{ij}||\alpha_j||\varphi_j(s)|\right]^p ds\right\}\\
&\le e^{\lambda t}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_0^t e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)ds\\
&= \sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_0^t e^{(\lambda-c_i)(t-s)}e^{\lambda s}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)ds.
\end{aligned}\tag{24}$$

With the similar computation as (24), we obtain that

$$\begin{aligned}
e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|J_{3i}(t)|^p &\le e^{\lambda t}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\int_0^t e^{-c_i(t-s)}\mathbb{E}\left[\sum_{j=1}^{n}|\varphi_j(s-\tau(s)))|^p\right]ds.\\
&\le e^{\lambda\tau}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\int_0^t e^{-(c_i-\lambda)(t-s)}e^{\lambda(s-\tau(s))}\mathbb{E}\left[\sum_{j=1}^{n}|\varphi_j(s-\tau(s)))|^p\right]ds.\\
e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|J_{4i}(t)|^p &\le e^{\lambda t}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^t e^{-c_i(t-s)}\mathbb{E}\left[\sum_{j=1}^{n}\left|\int_{s-r(s)}^{s}\varphi_j(u)\,du\right|^p\right]ds
\end{aligned}$$

$$
\begin{aligned}
&\leq e^{\lambda t}\sum_{i=1}^{n}\left(\frac{\tau}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^t e^{-c_i(t-s)}\int_{s-r(s)}^{s}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(u)|^p\right)du\,ds\\
&\leq e^{\lambda\tau}\sum_{i=1}^{n}\left(\frac{\tau}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^t e^{-(c_i-\lambda)(t-s)}\int_{s-r(s)}^{s}e^{\lambda u}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(u)|^p\right)du\,ds
\end{aligned}
\tag{25}
$$

Using Lemma 2.1 and Hölder inequality, we obtain that

$$
\begin{aligned}
e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|J_{5i}(t)|^p &= e^{\lambda t}\sum_{i=1}^{n}\mathbb{E}\left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,d\omega_j(s)\right|^p\\
&\leq e^{\lambda t}n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left\{\left[\int_0^t e^{-c_i(t-s)}|\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))|\,d\omega_j(s)\right]^2\right\}^{p/2}\\
&= e^{\lambda t}n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\int_0^t e^{-2c_i(t-s)}\sigma_{ij}^2(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,ds\right]^{p/2}\\
&\leq e^{\lambda t}n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\int_0^t e^{-2c_i(t-s)}\left(\mu_j\varphi_j^2(s)+\nu_j\varphi_j^2(s-\tau(s))\right)ds\right]^{p/2}\\
&\leq e^{\lambda t}n^{p-1}2^{p/2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\left(\int_0^t e^{-2c_i(t-s)}\mu_j\varphi_j^2(s)\,ds\right)^{p/2}+\left(\int_0^t e^{-2c_i(t-s)}\nu_j\varphi_j^2(s-\tau(s))\,ds\right)^{p/2}\right]\\
&\leq e^{\lambda t}n^{p-1}2^{p/2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\left(\int_0^t e^{-2c_i(t-s)}\,ds\right)^{p/2-1}\int_0^t e^{-2c_i(t-s)}\mu_j^{p/2}|\varphi_j(s)|^p\,ds\right]\\
&\quad+e^{\lambda t}n^{p-1}2^{p/2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left\{\left(\int_0^t e^{-2c_i(t-s)}\,ds\right)^{p/2-1}\int_0^t e^{-2c_i(t-s)}\nu_j^{p/2}|\varphi_j(s-\tau(s))|^p\,ds\right\}\\
&\leq e^{\lambda t}n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left\{\mu^{p/2}\int_0^t e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)ds\right\}\\
&\quad+e^{\lambda t}n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left\{\nu^{p/2}\int_0^t e^{-c_i(t-s)}E\left[\sum_{j=1}^{n}|\varphi_j(s-\tau(s))|^p\right]ds\right\}\\
&\leq n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left\{\mu^{p/2}\int_0^t e^{-(c_i-\lambda)(t-s)}e^{\lambda s}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)ds\right\}\\
&\quad+e^{\lambda\tau}n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left\{\nu^{p/2}\int_0^t e^{-(c_i-\lambda)(t-s)}e^{\lambda(s-\tau(s))}E\left[\sum_{j=1}^{n}|\varphi_j(s-\tau(s))|^p\right]ds\right\}
\end{aligned}
\tag{26}
$$

Further, from (A3), we know that $|I_{ik}(x_i(t_k))| \leq p_{ik}|x_i(t_k)|$, combining with the condition that $p_{ik} \leq p_i(t_k - t_{k-1})$, we obtain that

$$
\begin{aligned}
e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|J_{6i}(t)|^p &\leq e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}\left[\sum_{0<t_k<t}e^{-c_i(t-t_k)}p_{ik}|\varphi_i(t_k)|\right]^p\\
&\leq e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}\left[p_i\sum_{0<t_k<t}e^{-c_i(t-t_k)}|\varphi_i(t_k)|(t_k-t_{k-1})\right]^p
\end{aligned}
$$

$$
\begin{aligned}
&\le e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}\left[p_i\int_0^t e^{-c_i(t-s)}|\varphi_i(s)|\,ds\right]^p\\
&\le e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}p_i^p\left(\int_0^t e^{-c_i(t-s)}\,ds\right)^{p/q}\int_0^t e^{-c_i(t-s)}|\varphi_i(s)|^p\,ds\\
&\le \max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\}\int_0^t e^{-(c-\lambda)(t-s)}e^{\lambda s}\mathbb{E}\left(\sum_{i=1}^{n}|\varphi_i(s)|^p\right)ds. \qquad (27)
\end{aligned}
$$

Since $\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)| \to 0$, $t-\tau(t)\to\infty$ and $t-r(t)\to\infty$ as $t\to\infty$. Thus, from (23) to (27), we obtain that $e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|Q\varphi_i(t)|^p \to 0$ as $e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)|^p \to 0$. Hence, combining the proof of Theorem 1.4, there exists a unique fixed point $\varphi(\cdot)$ of Q in C_ϕ, which is a solution of the system (1) such that $e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)|^p \to 0$ as $t\to\infty$. This completes the proof.

Corollary 3.1. *Suppose that the assumptions* (A1)-(A5) *hold. Assume that*

(i) *the discrete delay $\tau(t)$ is bounded by a constant τ;*

(ii) *there exist constants p_i ($i=1,2,\cdots n$) such that $p_{ik}\le p_i(t_k-t_{k-1})$, $k=1,2,\cdots$,*

(iii)

$$
\begin{aligned}
&5^{p-1}\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+5^{p-1}\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\\
&+5^{p-1}n^{p-1}\sum_{i=1}^{n}c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right)+5^{p-1}c^{-1}\max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\}<1,
\end{aligned}
$$

where $c=\min\{c_1,c_2,\cdots c_n\}$, $\mu=\max\{\mu_1,\mu_2,\cdots\mu_n\}$, $\nu=\max\{\nu_1,\nu_2,\cdots\nu_n\}$.

Then the trivial solution of (22) *is pth moment exponentially stable.*

4. Proof of Theorem 1.12

In this section, we prove Theorem 1.12. We start with some preparations.

Using similar computations as in Section 2, we obtain that for $t\ge 0$, the system (6) is equivalent to

$$
\begin{aligned}
x_i(t) &= e^{-c_it}x_i(0)+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}a_{ij}g_j(x_j(s))\,ds+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}b_{ij}g_j(x_j(s-\tau(s)))\,ds\\
&\quad+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}l_{ij}\int_{s-r(s)}^{s}g_j(x_j(u))\,du\,ds+\sum_{0<t_k<t}e^{-c_i(t-t_k)}I_{ik}(x_i(t_k)),\quad i=1,2,3,\cdots,n,\quad k=1,2,\cdots.
\end{aligned}
$$

Lemma 4.1. *Define an operator by $(P\varphi)(\theta)=\phi(\theta)$, for $-\tau\le\theta\le 0$, and for $t\ge 0$,*

$$
\begin{aligned}
P\varphi_i(t) &= e^{-c_it}\varphi_i(0)+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}a_{ij}g_j(\varphi_j(s))\,ds+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}b_{ij}g_j(\varphi_j(s-\tau(s)))\,ds\\
&\quad+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}l_{ij}\int_{s-r(s)}^{s}g_j(\varphi_j(u))\,du\,ds+\sum_{0<t_k<t}e^{-c_i(t-t_k)}I_{ik}(x_i(t_k))\\
&:= I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t). \qquad (28)
\end{aligned}
$$

If condition (9) *holds, then $P:\mathcal{S}_\phi\to\mathcal{S}_\phi$ and P is a contraction mapping.*

Proof. First, we prove that $P\mathcal{S}_\phi \subseteq \mathcal{S}_\phi$. In view of (28), it is easy to check that $(Px_i)(t)$ is continuous on fixed time $t \neq t_k$ $(k = 1, 2, \cdots)$. On the other hand, as $t = t_k$ $(k = 1, 2, \cdots)$, it is not difficult to show that $I_1(t), I_2(t), I_3(t), I_4(t)$ is continuous on fixed time $t = t_k$ $(k = 1, 2, \cdots)$. Let $r < 0$ be small enough, we obtain that

$$\begin{aligned}
|J_{5i}(t_k + r) - J_{5i}(t_k)| &= \left|\sum_{0<t_m<t_k+r} e^{-c_i(t_k+r-t_m)} I_{im}(\varphi_i(t_m)) - \sum_{0<t_m<t_k} e^{-c_i(t_k-t_m)} I_{im}(\varphi_i(t_m))\right| \\
&\leq \left|\left(e^{-c_i(t_k+r)} - e^{-c_i t_k}\right) \sum_{0<t_m<t_k} e^{c_i t_m} I_{im}(\varphi_i(t_m))\right|,
\end{aligned}$$

which implies that $\lim_{r\to 0^-} |J_{5i}(t_k + r) - J_{5i}(t_k)| = 0$. Let $r > 0$ be small enough, we obtain that

$$\begin{aligned}
|J_{5i}(t_k + r) - J_{5i}(t_k)| &= \left|\sum_{0<t_m<t_k+r} e^{-c_i(t_k+r-t_m)} I_{im}(\varphi_i(t_m)) - \sum_{0<t_m<t_k} e^{-c_i(t_k-t_m)} I_{im}(\varphi_i(t_m))\right| \\
&= \left|e^{-c_i(t_k+r)}\left[\sum_{0<t_m<t_k} e^{c_i t_m} I_{im}(\varphi_i(t_m)) + e^{c_i t_k} I_{ik}(\varphi_i(t_k))\right] - e^{-c_i t_k} \sum_{0<t_m<t_k} e^{c_i t_m} I_{im}(\varphi_i(t_m))\right| \\
&= \left|\left(e^{-c_i(t_k+r)} - e^{-c_i t_k}\right) \sum_{0<t_m<t_k} e^{c_i t_m} I_{im}(\varphi_i(t_m)) + e^{-c_i r} I_{ik}(\varphi_i(t_k))\right|,
\end{aligned}$$

which implies that $\lim_{r\to 0^+} |J_{5i}(t_k + r) - J_{5i}(t_k)| = |I_{ik}(\varphi_i(t_k))|$.

Based on the above discussion, we obtain that $(P\varphi_i)(t) : [\vartheta, \infty) \to \mathbb{R}^n$ is continuous on $t \neq t_k$ $(k = 1, 2, \cdots)$, and for $t = t_k$ $(k = 1, 2, \cdots)$, $\lim_{t\to t_k^+}(P\varphi_i)(t)$ and $\lim_{t\to t_k^-}(P\varphi_i)(t)$ exist. Furthermore, we also obtain that $\lim_{t\to t_k^-}(P\varphi_i)(t) = (P\varphi_i)(t_k) \neq \lim_{t\to t_k^+}(P\varphi_i)(t)$.

Next, we prove that $\lim_{t\to\infty}(P\varphi_i)(t) = 0$ for $\varphi_i(t) \in \mathcal{S}_{i\phi}$. We estimate the right-hand terms of (28).

$$\begin{aligned}
|I_2(t)| = \left|\int_0^t e^{-c_i(t-s)} \sum_{j=1}^n a_{ij} f_j(\varphi_j(s))\, ds\right| &\leq \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n |a_{ij}\alpha_j||\varphi_j(s)|\, ds \\
&\leq \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n |\varphi_j(s)|\, ds \qquad (29)
\end{aligned}$$

$$\begin{aligned}
|I_3(t)| = \left|\int_0^t e^{-c_i(t-s)} \sum_{j=1}^n b_{ij} g_j(\varphi_j(s - \tau(s)))\, ds\right| &\leq \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n |b_{ij}\beta_j||\varphi_j(s - \tau(s))|\, ds \\
&\leq \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n |\varphi_j(s - \tau(s))|\, ds \qquad (30)
\end{aligned}$$

$$\begin{aligned}
|I_4(t)| = \left|\int_0^t e^{-c_i(t-s)} \sum_{j=1}^n l_{ij} \int_{s-r(s)}^s h_j(\varphi_j(u))\, du\, ds\right| &\leq \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n |l_{ij}\gamma_j| \int_{s-r(s)}^s |\varphi_j(u)|\, du\, ds \\
&\leq \max_{j=1,2,\cdots,n} |l_{ij}\gamma_j| \int_0^t e^{-c_i(t-s)} \int_{s-r(s)}^s \sum_{j=1}^n |\varphi_j(s)|\, ds \qquad (31)
\end{aligned}$$

$$|I_5(t)| = \left|\sum_{0<t_k<t} e^{-c_i(t-t_k)} I_{ik}(x_i(t_k))\right| \leq \left|\sum_{0<t_k<t} e^{-c_i(t-t_k)} p_i(t_k - t_{k-1}) x_i(t_k)\right| \leq p_i \int_0^t e^{-c_i(t-s)} |x_i(s)|\, ds \qquad (32)$$

From the fact that $c_i > 0$ ($i = 1, 2, \cdots, n$) and the estimate (29)-(32), we conclude that $\lim_{t\to\infty}(Px_i)(t) = 0$ for $x_i(t) \in \mathcal{S}_{i\phi}$. Therefore, $P : \mathcal{S}_\phi \to \mathcal{S}_\phi$.

Now, we prove that P is a contraction mapping. For any $x(t), y(t) \in \mathcal{S}_\phi$, from (29) to (32), we obtain that

$$\begin{aligned}
&\sum_{i=1}^{n} |(Px_i)(t) - (Py_i)(t)| \\
&\quad \le \sum_{i=1}^{n} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^{n} |x_j(s) - y_j(s)|\, ds \\
&\qquad + \sum_{i=1}^{n} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^{n} |x_j(s-\tau(s)) - y_j(s-\tau(s))|\, ds \\
&\qquad + \sum_{i=1}^{n} \max_{j=1,2,\cdots,n} |l_{ij}\gamma_j| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^{n} \int_{s-r(s)}^{s} |x_j(u) - y_j(u)|\, du\, ds + \sum_{i=1}^{n} p_i \int_0^t e^{-c_i(t-s)} |x_i(s) - y_i(s)|\, ds \\
&\quad \le \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| \sum_{j=1}^{n} \left\{ \sup_{\vartheta \le s \le t} |x_j(s) - y_j(s)| \right\} + \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| \sum_{j=1}^{n} \left\{ \sup_{\vartheta \le s \le t} |x_j(s) - y_j(s)| \right\} \\
&\qquad + \sum_{i=1}^{n} \frac{\tau}{c_i} \max_{j=1,2,\cdots,n} |l_{ij}\gamma_j| \sum_{j=1}^{n} \left\{ \sup_{\vartheta \le s \le t} |x_j(s) - y_j(s)| \right\} + \max_{j=1,2,\cdots,n} \left\{ \frac{p_i}{c_i} \right\} \sum_{j=1}^{n} \left\{ \sup_{\vartheta \le s \le t} |x_j(s) - y_j(s)| \right\}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\sum_{i=1}^{n} \sup_{-\tau \le s \le t} |(Px_i)(s) - (Py_i)(s)| \\
&\quad \le \left\{ \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \sum_{i=1}^{n} \frac{\tau}{c_i} \max_{j=1,2,\cdots,n} |l_{ij}\gamma_j| + \max_{j=1,2,\cdots,n} \left\{ \frac{p_i}{c_i} \right\} \right\} \\
&\qquad \times \sum_{j=1}^{n} \left\{ \sup_{\vartheta \le s \le t} |x_j(s) - y_j(s)| \right\} = \alpha \sum_{j=1}^{n} \left\{ \sup_{\vartheta \le s \le t} |x_j(s) - y_j(s)| \right\}.
\end{aligned}$$

From (9), we obtain that P is a contraction mapping. □

We are now ready to prove Theorem 1.12.

Proof. Let P be defined as in Lemma 4.1, by contraction mapping principle, P has a unique fixed point $x \in \mathcal{S}_\phi$ with $x(\theta) = \phi(\theta)$ on $-\tau \le \theta \le 0$ and $x(t) \to 0$ as $t \to \infty$.

To obtain asymptotically stable, we need to prove that the trivial equilibrium $x = 0$ of (6) is stable. From (9), For any $\epsilon > 0$, choose $\sigma > 0$ and $\sigma < \epsilon$ satisfying the condition $\sigma + \epsilon\alpha < \epsilon$.

If $x(t, s, \phi) = (x_1(t, s, \phi), x_2(t, s, \phi), \cdots, x_n(t, s, \phi))$ is the solution of (6) with the initial condition $\|\phi\| < \sigma$, the we claim that $\|x(t, s, \phi)\| < \epsilon$ for all $t \ge 0$. Indeed, we suppose that there exists $t^* > 0$ such that

$$\sum_{i=1}^{n} |x_i(t^*; s, \phi)| = \epsilon, \qquad \text{and} \qquad \sum_{i=1}^{n} |x_i(t; s, \phi)| < \epsilon \quad \text{for} \quad 0 \le t < t^*. \tag{33}$$

From (9), we obtain

$$\begin{aligned}
\sum_{i=1}^{n} |x_i(t^*; s, \phi)| \quad \le \quad & \sum_{i=1}^{n} \Bigg[|e^{-c_i t^*} x_i(0)| + \int_0^{t^*} e^{-c_i(t^*-s)} \sum_{j=1}^{n} |a_{ij} f_j(x_j(s))|\, ds + \int_0^{t^*} e^{-c_i(t^*-s)} \sum_{j=1}^{n} |b_{ij} g_j(x_j(s-\tau(s)))|\, ds \\
& + \int_0^{t^*} e^{-c_i(t^*-s)} \sum_{j=1}^{n} |l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))|\, du\, ds + p_i \int_0^{t^*} e^{-c_i(t^*-s)} |x_i(s)|\, ds \Bigg]
\end{aligned}$$

$$
\begin{aligned}
&< \quad \sigma + \epsilon \left[\sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \sum_{i=1}^{n} \frac{\tau}{c_i} \max_{j=1,2,\cdots,n} |l_{ij}\gamma_j| + \max_{j=1,2,\cdots,n} \left\{ \frac{p_i}{c_i} \right\} \right] \\
&\leq \quad \sigma + \epsilon\alpha < \epsilon.
\end{aligned}
$$

which contradicts (33). Therefore, $\|x(t, s, \phi)\| < \epsilon$ for all $t \geq 0$. This completes the proof. □

Let $l_{ij} \equiv 0$ for $i = 1, 2, \cdots n,\ j = 1, 2, \cdots n$, the system is reduced to

$$
\begin{cases}
\frac{dx_i(t)}{dt} = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t - \tau(t))), \quad t \neq t_k \\
\\
\Delta x_i(t_k) = I_{ik} x_i(t_k), \quad t = t_k, \quad k = 1, 2, 3, \cdots,
\end{cases}
\tag{34}
$$

which is the description of cellular neural network with time-varying delays. Following the result of Theorem 1.12, we have the following corollary. Note that the delay in Corollary 4.2 can be unbounded.

Corollary 4.2. *Suppose that the conditions* (A1)-(A4) *hold, assume that the following conditions are satisfied,*

(i) *there exist constants p_i ($i = 1, 2, \cdots n$) such that $p_{ik} \leq p_i(t_k - t_{k-1})$, $k = 1, 2, \cdots$;*

(ii)

$$
\alpha \triangleq \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \max_{i=1,2,\cdots,n} \left\{ \frac{p_i}{c_i} \right\} < 1. \tag{35}
$$

Then the trivial solution of (34) *is asymptotically stable.*

Remark 4.3. *Zhang and Guan* [22] *has studied asymptotic stability of* (34) *by using fixed point theory. The conditions in* [22] *are as follows*

(i) *there exists a constant μ such that $\inf_{k=1,2,\cdots}\{t_k - t_{k-1}\} \geq \mu$;*

(ii) *there exist constants p_i ($i = 1, 2, \cdots n$) such that $p_{ik} \leq p_i\mu$, $k = 1, 2, \cdots$;*

(iii)

$$
\lambda^* \triangleq \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \max_{i=1,2,\cdots,n} \left\{ \frac{p_i}{c_i} + p_i\mu \right\} < 1.
$$

(iv)

$$
\max_{i=1,2,\cdots n} \{\lambda_i\} < \frac{1}{\sqrt{n}}, \quad \text{where } \lambda_i = \frac{1}{c_i} \sum_{j=1}^{n} |a_{ij}\alpha_j| + \frac{1}{c_i} \sum_{j=1}^{n} |b_{ij}\beta_j| + \left(\frac{p_i}{c_i} + p_i\mu \right).
$$

It is clearly that Corollary 5.1 *is an improvement of the result in* [22].

5. Proof of Theorem 1.13

Define the operator P as in Section 4. Following the proof of Theorem 1.12, we find that to show Theorem 1.13, we only need to prove that $e^{\lambda t}(P\varphi_i)(t) \to 0$ as $t \to \infty$. We estimate the right-hand terms of (28), we obtain that

$$
\begin{aligned}
e^{\lambda t}|I_2(t)| = \left| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^{n} a_{ij} f_j(\varphi_j(s))\, ds \right| &\leq e^{\lambda t} \int_0^t e^{-c_i(t-s)} \sum_{j=1}^{n} |a_{ij}\alpha_j||\varphi_j(s)|\, ds \\
&\leq \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| \int_0^t e^{-(c_i-\lambda)(t-s)} e^{\lambda s} \sum_{j=1}^{n} |\varphi_j(s)|\, ds,
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
e^{\lambda t}|I_3(t)| &= e^{\lambda t}\left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n b_{ij}g_j(\varphi_j(s-\tau(s)))\,ds\right| \\
&\le e^{\lambda t}\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n |b_{ij}\beta_j||\varphi_j(s-\tau(s))|\,ds \\
&\le e^{\lambda\tau}\max_{j=1,2,\cdots,n}|b_{ij}\beta_j|\int_0^t e^{-(c_i-\lambda)(t-s)}e^{\lambda(s-\tau(s))}\sum_{j=1}^n|\varphi_j(s-\tau(s))|\,ds, \qquad (37)
\end{aligned}
$$

$$
\begin{aligned}
e^{\lambda t}|I_4(t)| &= e^{\lambda t}\left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n l_{ij}\int_{s-r(s)}^s h_j(\varphi_j(u))\,du\,ds\right| \\
&\le e^{\lambda t}\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n |l_{ij}\gamma_j|\int_{s-r(s)}^s|\varphi_j(u)|\,du\,ds \\
&\le e^{\lambda\tau}\max_{j=1,2,\cdots,n}|l_{ij}\gamma_j|\int_0^t e^{-(c_i-\lambda)(t-s)}\int_{s-r(s)}^s e^{\lambda u}\sum_{j=1}^n|\varphi_j(u)|\,du\,ds, \qquad (38)
\end{aligned}
$$

$$
e^{\lambda t}|I_5(t)| = e^{\lambda t}\left|\sum_{0<t_k<t} e^{-c_i(t-t_k)}I_{ik}(x_i(t_k))\right| \le e^{\lambda t}\left|\sum_{0<t_k<t} e^{-c_i(t-t_k)}p_i(t_k-t_{k-1})x_i(t_k)\right| \le p_i\int_0^t e^{-(c_i-\lambda)(t-s)}e^{\lambda s}|x_i(s)|\,ds. \qquad (39)
$$

From the fact that $\lambda < \min\{c_1, c_2, \cdots c_n\}$, $c_i > 0$ $(i = 1, 2, \cdots, n)$ and the above estimate, we obtain that $e^{\lambda t}P\varphi_i)(t) \to 0$ as $t \to \infty$.

Corollary 5.1. *Suppose that the conditions* (A1)-(A4) *hold, assume that the following conditions are satisfied,*

(i) *the delay $\tau(t)$ is bounded by a constant τ;*
(ii) *there exist constants p_i ($i = 1, 2, \cdots n$) such that $p_{ik} \le p_i(t_k - t_{k-1})$, $k = 1, 2, \cdots$;*
(iii)

$$
\alpha \triangleq \sum_{i=1}^n \frac{1}{c_i}\max_{j=1,2,\cdots,n}|a_{ij}\alpha_j| + \sum_{i=1}^n\frac{1}{c_i}\max_{j=1,2,\cdots,n}|b_{ij}\beta_j| + \max_{j=1,2,\cdots,n}\left\{\frac{p_i}{c_i}\right\} < 1. \qquad (40)
$$

Then the trivial solution of (34) *is exponentially stable.*

Remark 5.2. *Zhang and Luo* [21] *has studied exponential stability of* (34) *by using fixed point theory. The conditions in* [21] *are as follows*

(i) *the delay $\tau(t)$ is bounded by a constant τ;*
(ii) *there exists a constant μ such that* $\inf_{k=1,2,\cdots}\{t_k - t_{k-1}\} \ge \mu$;
(iii) *there exist constants p_i ($i = 1, 2, \cdots n$) such that $p_{ik} \le p_i\mu$, $k = 1, 2, \cdots$;*
(iv)

$$
\alpha \triangleq \sum_{i=1}^n \frac{1}{c_i}\max_{j=1,2,\cdots,n}|a_{ij}\alpha_j| + \sum_{i=1}^n\frac{1}{c_i}\max_{j=1,2,\cdots,n}|b_{ij}\beta_j| + \max_{i=1,2,\cdots,n}\left\{\frac{p_i}{c_i} + p_i\mu\right\} < 1.
$$

It is clearly that Corollary 5.1 *is an improvement of the result in* [21].

6. Examples

Example 6.1. *Consider the following two-dimensional cellular neural network*

$$\begin{cases} \frac{dx_i(t)}{dt} = -c_i x_i(t) + \sum_{j=1}^{2} a_{ij} g_j(x_j(t)) + \sum_{j=1}^{2} b_{ij} g_j(x_j(t-\tau(t))) \quad i = 1,2, \quad t \neq t_k \\ \\ \Delta x_i(t_k) = I_{ik} x_i(t_k), \quad t = t_k, \quad k = 1,2,3,\cdots \end{cases} \tag{41}$$

with the initial conditions $x_1(s) = \cos(s)$, $x_2(s) = \sin(s)$ *on* $-\frac{1}{2} \leq s \leq 0$, *where* $c_1 = c_2 = 3$, $a_{11} = 6/7$, $a_{12} = 3/7$, $a_{21} = -1/7$, $a_{22} = -1/7$, $b_{11} = 6/7$, $b_{12} = 2/7$, $b_{21} = 3/7$, $b_{22} = 1/7$, *the activation function is described by* $g_i(x) = \frac{|x+1|-|x-1|}{2}$, $\tau(t) = 0.4t + 1$. $I_{ik}(x_i(t_k)) = \arctan(0.4x_i(t_k))$, $t_k = t_{k-1} + 0.5k$, $i = 1,2$ *and* $k = 1,2,\cdots$.

It is clearly that $\alpha_i = \beta_i = 1$, $p_{ik} = 0.4$ for $i = 1,2$, $k = 1,2,\cdots$, we select $p_i = 0.8$, then

$$\sum_{i=1}^{2} \frac{1}{c_i} \max_{j=1,2} |a_{ij}\alpha_j| + \sum_{i=1}^{2} \frac{1}{c_i} \max_{j=1,2} |b_{ij}\beta_j| + \max_{j=1,2,\cdots,n} \left\{ \frac{p_i}{c_i} \right\} \leq \frac{1}{3} \times \left(\frac{6}{7} + \frac{1}{7} + \frac{6}{7} + \frac{3}{7} \right) + \frac{4}{35} = \frac{16}{21} + \frac{4}{35} < 0.88 < 1.$$

Hence, by Corollary 4.2, the trivial solution of (41) is asymptotically stable. However,

$$\sum_{i=1}^{2} \frac{1}{c_i} \max_{j=1,2} |a_{ij}\alpha_j| + \sum_{i=1}^{2} \frac{1}{c_i} \max_{j=1,2} |b_{ij}\beta_j| + \max_{i=1,2,\cdots,n} \left\{ \frac{p_i}{c_i} + p_i\mu \right\} > 1,$$

which implies that the result in [22] is not applicable.

Example 6.2. *Consider a two-dimensional stochastically perturbed hopfield neural network with time-varying delays,*

$$\begin{cases} dx(t) = [-Cx(t) + Af(x(t)) + Bg(x_\tau(t))]\,dt + \sigma(t, x(t), x_\tau(t))\,d\omega(t), \quad t \neq t_k \\ \\ \Delta x(t_k) = I_k(x(t_k)), \quad t = t_k, \quad k = 1,2,3,\cdots \end{cases} \tag{42}$$

where $f(x) = \frac{1}{5}\arctan x$, $g(x) = \frac{1}{5}\tanh x = \frac{1}{5}(e^x - e^{-x})/(e^x + e^{-x})$, $\tau(t) = \frac{1}{2}\sin t + \frac{1}{2}$,

$$C = \begin{pmatrix} 5 & 0 \\ 0 & 4.5 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 0.4 \\ 0.6 & 1 \end{pmatrix} \quad and \quad B = \begin{pmatrix} -0.8 & 2 \\ 1 & 4 \end{pmatrix}.$$

In this example, let $p = 3$, *take* $\alpha_j = 0.2, \beta_j = 0.2$, $j = 1,2$, $\sigma : \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$ *satisfies*

$$trace\left[\sigma^T(t,x,y)\sigma(t,x,y)\right] \leq 0.01\left(x_1^2 + x_2^2 + y_1^2 + y_2^2\right).$$

$I_{ik}(x_i(t_k)) = 0.1x_i(t_k)$, $t_k = t_{k-1} + 0.5$, $i = 1,2$ *and* $k = 1,2,\cdots$.

It is clearly that $p_{ik} = 0.1$, we choose $p_i = 0.2$, let $p = 2$, we check the condition in Corollary 2.3,

$$\begin{aligned} &5^{p-1} \sum_{i=1}^{n} c_i^{-p} \left(\sum_{j=1}^{n} |a_{ij}|^q |\alpha_j|^q \right)^{p/q} + 5^{p-1} \sum_{i=1}^{n} c_i^{-p} \left(\sum_{j=1}^{n} |b_{ij}|^q |\beta_j|^q \right)^{p/q} \\ &+5^{p-1} n^{p-1} \sum_{i=1}^{n} c_i^{-p/2} \left(\mu^{p/2} + \nu^{p/2} \right) + 5^{p-1} c^{-1} \max_{i=1,2} \left\{ \frac{p_i^p}{c_i^{p-1}} \right\} < 0.53 < 1. \end{aligned}$$

From Corollary 2.3, the trivial solution of (42) is asymptotically stable. On the other hand, since $|\tau(t)| = \left|\frac{1}{2}\sin t + \frac{1}{2}\right| \leq 1$, from Corollary 3.1, the trivial solution of (42) is exponentially stable.

References

[1] J.A.D. Appleby, Fixed points, stability and harmless stochastic perturbations, Preprint, 2008.

[2] T.A. Burton, Stability by fixed point theory for functional differential equations. Dover Publication, New York, 2006.

[3] A. Cichocki, R. Unbehauen, Neural networks for optimization and signal processing, Wiley, Chichester, 1993.

[4] A. Djoudi, R. Khemis, Fixed points techniques and stability for neutral nonlinear differential equations with unbounded delays. *Georgian Mathematical Journal* **13** (2006), No.1, 25-34.

[5] A. Friedman, Stochastic differential equations and applications, Academic press, New York, 1976.

[6] S. Haykin, Neural networks, Prentice-Hall, Englewood Cliffs, 1994.

[7] J. Hu, S. Zhong, L. Liang, Exponential stability analysis of stochastic delayed cellular neural network, *Chaos Soliton Fract.* **27** (2006), 1006-1010.

[8] X.H. Lai, Y.T. Zhang, Fixed point and asymptotic analysis of cellular neural networks, *Journal of Applied Mathematics* **2012** (2012), Article ID 689845, 12 pages doi:10.1155/2012/689845.

[9] X.A. Li, J.Z. Zou, E.W. Zhu, The *pth* moment exponential stability of stochastic cellular neural networks with impulses, *Advances in Difference Equations* **2013**, 2013: 6.

[10] B. Lisena, Exponential stability of Hopfield neural networks with impulses *Nonlinear Analysis: Real World Applications* **12** (2011), 1923-1930.

[11] J.W. Luo and T. Taniguchi, Fixed points and stability of stochastic neutral partial differential equations with infinite delays, *Stochastic Analysis and Applications* **27** (2009), 1163-1173.

[12] J.W. Luo, Fixed points and exponential stability of mild solutions of stochastic partial differential equations with delays, *J.Math.Anal.Appl.* **342** (2008), 753-760.

[13] Luo, J.W., Fixed points and stability of neutral stochastic delay differential equations. *J. Math. Anal. Appl.*, **334** (2007) 431-440.

[14] R. Sakthivel, R. Samidurai, S.M.Anthoni, Asymptotic stability of stochastic delayed recurrent neural networks with impulsive effects, *J Optim Theory Appl* **147** (2010), 583-596.

[15] R. Sakthivel, J. Luo, Asymptotic stability of nonlinear impulsive stochastic differential equations, *Statist. Probab. Lett.* **79** (2009), 1219-1223.

[16] R. Sakthivel, J. Luo, Asymptotic stability of impulsive stochastic partial differential equations with infinite delays. *J. Math. Anal. Appl.* **356** (2009), 1-6.

[17] Y.H. Sun, J.D. Cao, *pth* moment exponential stability of stochastic recurrent neural networks with time-varying delays. *Nonlinear Analysis: Real World Applications* **8** (2007), 1171-1185.

[18] B.D. Vries, J.C. Principle, The gamma model- a new neural model for temporal processing, *Neural Networks*, **5**(1992), 565-576.

[19] L. Wan, J. Sun, Mean square exponential stability of stochastic delayed Hopfield neural networks, *Phys. Lett.* A **343** (2005), 306-318.

[20] H.Y. Zhao, Global asymptotic stability of Hopfield neural network involving distributed delays, *Neural Networks* **17** (2004), 47-53.

[21] Y.T. Zhang, Q. Luo, Global exponential stability of impulsive cellular neural networks with time-varying delays via fixed point theory, *Advances in Difference Equations* 2013, 2013:23.

[22] Y.T. Zhang, Y. H. Guan, Asymptotic Stability of Impulsive Cellular Neural Networks with Infinite Delays via Fixed Point Theory, *Abstract and Applied Analysis* **2013** (2013), Article ID 427827, 10 pages.