Exponential stability of impulsive neutral stochastic partial differential equations with variable delays and Poisson jumps

Guiling Chen, Onno van Gaans, Sjoerd Verduyn Lunel

Mathematical Institute, Leiden University, P.O.Box 9512, 2300 RA, Leiden, The Netherlands

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Abstract

We study existence and exponential stability in *pth* moment of mild solution to a class of impulsive neutral stochastic partial differential equations with variable delays and Poisson jumps. Sufficient conditions ensuring exponential stability in *pth* moment of mild solution to such class of differential equations are obtained by using fixed point theory.

1. Introduction

Many real world problems in science and engineering can be modelled by stochastic delay differential equations. The qualitative behavior of stochastic delay differential equations, regarding the stability, oscillation, boundedness and the existence of periodic solutions, has been studied by many investigators, e.g. [3, 4, 5, 7, 11, 17, 19]. In particular, the exponential stability of mild solutions of various stochastic delay differential equations has been established [1, 6, 9, 10, 12, 13, 15, 18].

The classical technique applied in the study of stability of stochastic delay differential equations is based on a stochastic version of Lyapunov's direct method. However, it may be difficult to apply Lyapunov's direct method to specific problems on exponential stability of solutions in stochastic delay differential equations. To overcome this difficulty, Luo [9] firstly applied fixed point theory to study the exponential stability of mild solutions of stochastic delay differential equations. The conditions ensuring the exponential stability given by [9] neither require the monotone decreasing behavior of delays, nor ask for a fixed sign of the coefficient functions.

On the other hand, besides delay effects, impulsive effects likewise exist in a great variety of evolutionary processes in which states are changed abruptly at certain moments of time, involving such fields are medicine, biology, economics, mechanics, electronics and telecommunications, etc. Many interesting results about impulsive effects to stochastic delay differential equations have been obtained by many authors, see, for example, Chen [6] has used an appropriate impulsive-integral inequality to establish the sufficient conditions for exponential stability of impulsive stochastic partial differential equations with variable delays, and it turns out that it is a convenient way to study asymptotic stability and exponential stability of mild solution of impulsive stochastic partial differential equations with bounded delays. Sakthivel and Luo [16, 17] have discussed the asymptotic stability of mild solution of impulsive stochastic partial differential equations with infinite delays by using fixed point method. This powerful method is also an effective tool to deal with exponential stability for mild solution to stochastic partial differential equations with variable delays, see, for example, [9] and Cui et al. [2]. To the best of our knowledge, there is no article discussing impulsive effects to stochastic partial differential equations with variable delays and poisson jumps.

The aim of paper is to study the existence and exponential stability of a class of impulsive stochastic partial differential equations with variable delays and poisson jumps by using fixed point method.

This paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, we study exponential stability of mild solution of impulsive neutral stochastic delay differential equations by using fixed point method.

2. Preliminaries

Let $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\}$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. right continuous and $\mathcal{F}_0$ containing all P-null sets). Let X, Y be two real separable Hilbert spaces which are both equipped with a norm denoted by $\|\cdot\|$, let $\mathcal{L}(Y, X)$ denote the space of all bounded linear operators from Y into X.

Suppose $\{p(t), t \geq 0\}$ is a σ-finite stationary $\mathcal{F}_t$-adapted Possion point process taking values in measurable space $(U, \mathcal{B}(U))$. The random measure N_p defined by $N_p((0,t] \times \Lambda) := \sum_{s\in(0,t]} 1_\Lambda(p(s))$ for $\Lambda \in \mathcal{B}(U)$ is called the Poisson random measure induced by $p(\cdot)$, thus, we can define the measure $\tilde{N}$ by $\tilde{N}(dt, dy) = N_p(dt, dy) - \nu(dy)dt$, where ν is the characteristic measure of N_p, which is called the compensated Poisson random measure. Let $w = (w_t)_{t\geq 0}$, independent of the Poisson point process, be a Y-valued Wiener process defined on $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\}$ with covariance operator Q, that is

$$E\langle w(t), x\rangle_Y \langle w(s), y\rangle_Y = (t \wedge s)\langle Qx, y\rangle_Y, \quad x, y \in Y,$$

where Q is a positive, self-adjoint, trace class operator on Y. Furthermore, $\mathcal{L}_2^0(Y, X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y to X with the norm

$$\|\xi\|^2_{\mathcal{L}_2^0} := tr(\xi Q \xi^*) < \infty, \quad \xi \in \mathcal{L}_2^0(Y, X).$$

For the construction of a stochastic integral in a Hilbert space, see Da Prato and Zabczyk [8].

For Borel set $Z \in \mathcal{B}(U \setminus \{0\})$, we consider the following impulsive neutral stochastic delay differential equation with poisson jumps

$$\begin{cases} d[x(t) + u(t, x(t-\tau(t)))] = [Ax(t)dt + f(t, x(t-\delta(t)))]dt + g(t, x(t-\rho(t))dw(t) \\ \qquad\qquad + \int_Z h(t, x(t-\sigma(t)), y)\, \tilde{N}(dt, dy), \qquad t \geq 0, \quad t \neq t_k, \\ \Delta x(t_k) = I_k x(t_k^-), \quad t = t_k, \quad k = 1, 2, \cdots, \\ x_0(\vartheta) = \phi, \quad \vartheta \in [-\tau, 0], \quad a.s. \end{cases} \tag{1}$$

where $\phi \in PC$ and the functions $\tau(t), \delta(t), \rho(t), \sigma(t) : [0, \infty) \to [0, \tau]$ $(\tau > 0)$ are continuous functions. Let $PC \equiv PC([-\tau, 0]; X)$ be the space of all almost surely bounded $\mathcal{F}_0$-measurable functions from $[-\tau, 0]$ into X that are continuous everywhere except for a finite number of points s at which the left and right limits of $\xi(s-)$ and $\xi(s+)$ exist and $\xi(s^+) = \xi(s)$ as usual, equipped with the supremum norm $\|\phi\|_0 = \mathrm{essup}_{\omega\in\Omega} \sup_{t\in[-r,0]} \|\phi(t)(\omega)\|$; A is the infinitesimal generator of a strongly continuous semigroup of linear operators $S(t)(t \geq 0)$ in X, refer to [14] for detailed information. Moreover, the fixed moments of time t_k satisfy $0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = \infty$; $x(t_k^-)$ and $x(t_k^+)$ represent the left and right limits of $x(t)$ at time $t = t_k$, $k = 1, 2, \cdots$, respectively. $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ denotes the jump in the state x at time t_k with $I_k(\cdot) : X \to X (k = 1, 2, \cdots)$ determining the size of the jump; $u, f : [0, \infty) \times X \to X$, $g : [0, \infty) \times X \to \mathcal{L}_2^0(Y, X)$, $h : [0, \infty) \times X \times U \to X$ are given functions to be specified later.

Definition 2.1. *An X-valued stochastic process $x(t)$, $t \in [0, +\infty)$, is called the mild solution of (1) if*

(a) *$x(t)$ is adapted to $\mathcal{F}_t$, $t \geq 0$.*
(b) *$x(t)$ has* càdlàg *paths on $[0, +\infty)$ almost surely, and for $t \in [0, +\infty)$, $x(t)$ satisfies the following integral equation*

$$\begin{aligned} x(t) \;=\;& S(t)(\phi(0) + u(0, \phi)) - u(t, x(t-\tau(t))) - \int_0^t AS(t-s)u(s, x(s-\tau(s)))\, ds \\ &+ \int_0^t S(t-s)f(s, x(s-\delta(s)))\, ds + \int_0^t S(t-s)g(s, x(s-\rho(s)))\, dw(s) \\ &+ \int_0^t \int_Z S(t-s)h(s, x(s-\sigma(s)), y)\, \tilde{N}(ds, dy) + \sum_{0<t_k<t} S(t-t_k)I_k x(t_k^-) \end{aligned} \tag{2}$$

and

$$x_0(\cdot) = \phi \in PC, \quad a.s.$$

Definition 2.2. *Let $p \geq 1$ be an integer. Equation* (1) *is said to be exponentially stable in pth moment, if for any initial value ϕ, there exists a pair of positive constants $\lambda > 0$ and C such that*

$$\mathbb{E}\|x(t)\|^p \leq C\|\phi\|_0^p e^{-\lambda t}, \quad t \geq 0.$$

where $\mathbb{E}$ denotes expectation with respect to the probability measure $\mathbb{P}$ and x is the mild solution of (1).

To obtain our main results, we impose the following assumptions:

(A1) A is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ in X such that $0 \in \rho(-A)$, the resolvent set of $-A$, and $S(t)$ is uniformly bounded,

$$\|S(t)\| \leq Me^{-\gamma t}, \quad t \geq 0,$$

for some constants $\gamma, M > 0$.

(A2) The mappings $f(t,\cdot), \sigma(t,\cdot)$ and $h(t,\cdot)$ satisfy global Lipschitz conditions, that is, there exist $L_1, L_2, L_3 > 0$ such that for any $x, y \in H$ and $t \geq 0$,

$$\begin{aligned}
\|f(t,x) - f(t,y)\| &\leq L_1\|x-y\|, \quad L_1 > 0,\\
\|g(t,x) - g(t,y)\| &\leq L_2\|x-y\|, \quad L_2 > 0,\\
\int_Z \|h(t,x,z) - h(t,y,z)\|^2 \nu(dz) &\leq L_3^2\|x-y\|^2, \quad L_3 > 0,
\end{aligned}$$

we further assume that $f(t,0) = g(t,0) = h(t,0,z) = 0$ for all $t \geq 0$, $z \in Z$. Then Equation (1) has a trivial solution $x = 0$ when $\phi = 0$.

(A3) The mapping $(-A)^\alpha u(t,\cdot)$ satisfies a uniformly Lipschitz condition: there exists a positive constant $K > 0$, such that for any $x, y \in X$

$$\|(-A)^\alpha u(t,x) - (-A)^\alpha u(t,y)\| \leq K\|x-y\|, \quad u(t,0) = 0, \quad t \geq 0,$$

for $\alpha \in (1/p, 1]$ (for some $p \geq 2$) and $u(t,\cdot) \in D((-A)^\alpha)$.

(A4) $I_k \in C(X,X)$ and there exists a positive constant q_k such that $\|I_k(x) - I_k(y)\| \leq q_k\|x-y\|$ and $I_k(0) = 0, k = 1,2,3,\cdots$, for each $x, y \in X$.

Lemma 2.3. (Theorem 6.13, [14]) *Suppose that the assumption* (A1) *holds, then for any $\beta \in (0,1]$, we have that*

(i) for each $x \in \mathcal{D}((-A)^\beta)$,

$$S(t)(-A)^\beta x = (-A)^\beta S(t)x,$$

(ii) there exists a positive constant $M_\beta > 0$ such that

$$\left\|(-A)^\beta S(t)\right\| \leq M_\beta t^{-\beta} e^{-\gamma t}, \quad t > 0.$$

Lemma 2.4. (Da Prato and Zabczyk[8]) *For any $p \geq 2$ and for an arbitrary $\mathcal{L}_2^0$-valued predictable process $\Phi(\cdot)$,*

$$\sup_{s\in[0,t]} E\left\|\int_0^s \Phi(u)dw(u)\right\|^p \leq c_p\left(\int_0^t \left(E\|\Phi(s)\|_{\mathcal{L}_2^0}^p\right)^{2/p} ds\right)^{p/2}, \quad \text{where} \quad c_p = (p(p-1)/2)^{p/2}.$$

3. Main results

In this section, we study the existence and exponential stability in the *pth* moment of mild solutions of the system (1) by means of fixed point method.

Denote by $\mathcal{S}_\phi$ the space of all $\mathcal{F}$-adapted processes: $\varphi(t,\omega) : [-\tau,\infty)\times\Omega \to X$ which is almost surely continuous in $t \neq t_k$ ($k = 1,2,\cdots$) for fixed $\omega \in \Omega$, $\lim_{t\to t_k^-}\varphi(t)$ and $\lim_{t\to t_k^+}\varphi(t)$ exist, and $\lim_{t\to t_k^-}\varphi(t) = \varphi(t_k)$. Moreover, $\phi(s,\omega) = \phi(s)$ for $s \in [-\tau,0]$ and $e^{\alpha t}\mathbb{E}\|\varphi(t)\|^p \to 0$ as $t \to \infty$, where $0 < \alpha < \gamma$. If we define the metric as

$$\|\varphi\|_{\mathcal{S}_\phi} := \sup_{s\geq 0}\mathbb{E}\|\varphi(s)\|^p, \tag{3}$$

then $\mathcal{S}_\phi$ is a complete metric space with respect to (3). Using the contraction mapping defined on the space $\mathcal{S}_\phi$ and applying the contraction mapping principle, we obtain the following result.

Theorem 3.1. *Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *there exists a constant* $\tilde{q}$ *such that* $q_k \leq \tilde{q}(t_k - t_{k-1})$, $k = 1, 2, \cdots$,

(ii)

$$6^{p-1}\|(-A)^{-\alpha}\|^p K^p + 6^{p-1} M_{1-\alpha}^p K^p \gamma^{-p\alpha}(\Gamma(1 + p(\alpha - 1)/(p - 1)))^{p-1} + 6^{p-1} M^p L_1^p \gamma^{-p}$$
$$+6^{p-1} c_p M^p L_2^p \left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2} \gamma^{-1} + 6^{p-1} c_p M^p L_3^p \left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2} \gamma^{-1} + 6^{p-1} M^p \tilde{q}^p \gamma^{-p} < 1,$$

where $\Gamma(\cdot)$ *is the Gamma function,* $M_{1-\alpha}$ *is the corresponding constant as in Lemma* 2.3,

then the mild solution of the system (1) *is exponentially stable.*

Proof. Define an operator by $(\pi\varphi)(t) = \phi(t)$ for $t \in [-\tau, 0]$, and for $t \geq 0$,

$$\begin{aligned}(\pi\varphi)(t) &= S(t)(\phi(0) + u(0, \phi)) - u(t, \varphi(t - \tau(t))) - \int_0^t AS(t - s)u(s, \varphi(s - \tau(s)))\, ds \\ &\quad + \int_0^t S(t - s) f(s, \varphi(s - \delta(s)))\, ds + \int_0^t S(t - s) g(s, \varphi(s - \rho(s)))\, dw(s) \\ &\quad + \int_0^t \int_Z S(t - s) h(s, \varphi(s - \sigma(s)), y)\, \tilde{N}(ds, dy) + \sum_{0<t_k<t} S(t - t_k) I_k(\varphi(t_k^-)) := \sum_{i=1}^{7} J_i(t). \end{aligned} \tag{4}$$

First, we prove the continuity in *pth* moment of π on $[0, \infty)$. Let $\varphi \in \mathcal{S}_\phi$, $t_1 \geq 0$, and $|r|$ be sufficiently small, from (4), we have

$$\mathbb{E}\|(\pi\varphi)(t_1 + r) - (\pi\varphi)(t_1)\|^p \leq 7^{p-1} \sum_{i=1}^{7} \mathbb{E}\|J_i(t_1 + r) - J_i(t_1)\|^p. \tag{5}$$

It is easily to check that $\mathbb{E}\|J_i(t_1 + r) - J_i(t_1)\|^p \to 0$ as $r \to 0$, $i = 1, 2, 3, 4, 7$. Further, by using Hölder inequality and Lemma 2.4, we obtain

$$\begin{aligned}\mathbb{E}\|J_5(t_1 + r) - J_5(t_1)\|^p &\leq 2^{p-1}\mathbb{E}\left\|\int_0^{t_1} (S(t_1 + r - s) - S(t_1)) g(s, \varphi(s - \rho(s)))\, dw(s)\right\|^p \\ &\quad +2^{p-1}\mathbb{E}\left\|\int_{t_1}^{t} S(t_1 + r - s) g(s, \varphi(s - \rho(s)))\, dw(s)\right\|^p \\ &\leq 2^{p-1} c_p \left(\int_0^{t_1} (\mathbb{E}\|(S(t_1 + r - s) - S(t_1)) g(s, \varphi(s - \rho(s)))\|^p)^{2/p}\ ds\right)^{p/2} \\ &\quad +2^{p-1} c_p \left(\int_{t_1}^{t} (\mathbb{E}\|S(t_1 + r - s) g(s, \varphi(s - \rho(s)))\|^p)^{2/p}\ ds\right)^{p/2} \to 0 \quad \text{as} \quad r \to 0.\end{aligned}$$

Similarly, we can verify that $\mathbb{E}\|J_6(t_1 + r) - J_6(t_1)\|^p \to 0$ as $r \to 0$. Thus, π is indeed continuous in *pth* moment on $[0, \infty)$.

Next, We show that $\pi(\mathcal{S}_\phi) \subseteq \mathcal{S}_\phi$. It follows form (4) that

$$\begin{aligned}e^{\alpha t}\mathbb{E}\|(\pi\varphi)(t)\|^p &\leq 7^{p-1} e^{\alpha t}\mathbb{E}\|S(t)(\phi(0) + u(0, \phi))\|^p + 7^{p-1} e^{\alpha t}\mathbb{E}\|u(t, \varphi(t - \tau(t)))\|^p \\ &\quad +7^{p-1} e^{\alpha t}\mathbb{E}\left\|\int_0^t AS(t - s)u(s, \varphi(s - \tau(s)))\, ds\right\|^p \\ &\quad +7^{p-1} e^{\alpha t}\mathbb{E}\left\|\int_0^t S(t - s) f(s, \varphi(s - \delta(s)))\, ds\right\|^p + 7^{p-1} e^{\alpha t}\mathbb{E}\left\|\int_0^t S(t - s) g(s, \varphi(s - \rho(s)))\, dw(s)\right\|^p \\ &\quad +7^{p-1} e^{\alpha t}\mathbb{E}\left\|\int_0^t \int_Z S(t - s) h(s, \varphi(s - \sigma(s)), y)\, \tilde{N}(ds, dy)\right\|^p + 7^{p-1} e^{\alpha t}\mathbb{E}\left\|\sum_{0<t_k<t} S(t - t_k) I_k \varphi(t_k^-)\right\|^p .\end{aligned} \tag{6}$$

Now, we estimate the terms on the right-hand side of (6). By the condition (A1) and (A3), we obtain

$$7^{p-1}e^{\alpha t}\mathbb{E}\|S(t)(\phi(0)+u(0,\phi))\|^p \le M^p e^{(\alpha-p\gamma)t}\mathbb{E}\|\phi(0)+u(0,\phi)\|^p \to 0 \quad \text{as} \quad t\to\infty \tag{7}$$

and

$$\begin{aligned} 7^{p-1}e^{\alpha t}\mathbb{E}\|u(t,\varphi(t-\tau(t)))\|^p &\le 7^{p-1}e^{\alpha t}\|(-A)^{-\alpha}\|^p E\|(-A)^{\alpha}u(t,\varphi(t-\tau(t)))\|_H^p \\ &\le 7^{p-1}e^{\alpha\tau}K^p\|(-A)^{-\alpha}\|^p e^{\alpha(t-\tau(t))}E\|\varphi(t-\tau(t))\|^p \to 0 \quad \text{as} \quad t\to\infty. \end{aligned} \tag{8}$$

Using Hölder inequality, we obtain

$$\begin{aligned} &7^{p-1}e^{\alpha t}\mathbb{E}\left\|\int_0^t AS(t-s)u(s,\varphi(s-\tau(s)))\,ds\right\|^p \\ &\le 7^{p-1}e^{\alpha t}M_{1-\alpha}^p K^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}\int_0^t e^{-\gamma(t-s)}E\|\varphi(s-\tau(s))\|^p\,ds \\ &\le 7^{p-1}M_{1-\alpha}^p K^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}e^{\alpha\tau}\int_0^t e^{-(\gamma-\alpha)(t-s)}e^{\alpha(s-\tau(s))}E\|\varphi(s-\tau(s))\|^p\,ds. \end{aligned} \tag{9}$$

For any $\varphi\in\mathcal{S}_\phi$ and $\epsilon>0$, there exists $t_1>0$ such that $e^{\alpha(s-\tau(s))}E\|\varphi(s-\tau(s))\|^p<\epsilon$ for $t\ge t_1$. Thus, from (9),

$$\begin{aligned} &7^{p-1}e^{\alpha t}\mathbb{E}\left\|\int_0^t AS(t-s)u(s,\varphi(s-\tau(s)))\,ds\right\|^p \\ &\le 7^{p-1}M_{1-\alpha}^p K^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}e^{\alpha\tau}\int_0^{t_1} e^{-(\gamma-\alpha)(t-s)}e^{\alpha(s-\tau(s))}E\|\varphi(s-\tau(s))\|^p\,ds \\ &\quad +7^{p-1}M_{1-\alpha}^p K^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}e^{\alpha\tau}\int_{t_1}^t e^{-(\gamma-\alpha)(t-s)}e^{\alpha(s-\tau(s))}E\|\varphi(s-\tau(s))\|^p\,ds \\ &\le 7^{p-1}M_{1-\alpha}^p K^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}e^{\alpha\tau}\int_0^{t_1} e^{-(\gamma-\alpha)(t-s)}e^{\alpha(s-\tau(s))}E\|\varphi(s-\tau(s))\|^p\,ds \\ &\quad +7^{p-1}M_{1-\alpha}^p K^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}\,(e^{\alpha\tau}/(\gamma-\alpha))\,\epsilon. \end{aligned} \tag{10}$$

Since $e^{-(\gamma-\alpha)t}\to 0$ as $t\to\infty$, then there exists $t_2\ge t_1$ such that for $t\ge t_2$, we have

$$\begin{aligned} &7^{p-1}M_{1-\alpha}^p K^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}e^{\alpha\tau}\int_0^{t_1} e^{-(\gamma-\alpha)(t-s)}e^{\alpha(s-\tau(s))}E\|\varphi(s-\tau(s))\|^p\,ds \\ &\quad\le \epsilon-7^{p-1}M_{1-\alpha}^p K^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}\,(e^{\alpha\tau}/(\gamma-\alpha))\,\epsilon. \end{aligned} \tag{11}$$

Hence, from (10)and (11), we obtain that

$$7^{p-1}e^{\alpha t}\mathbb{E}\left\|\int_0^t AS(t-s)u(s,\varphi(s-\tau(s)))\,ds\right\|^p \to 0 \quad \text{as} \quad t\to\infty.$$

As for the fourth term on the right-hand side of (6), for any $\varphi\in\mathcal{S}_\phi$, we have that for $p>2$,

$$\begin{aligned} 7^{p-1}e^{\alpha t}\mathbb{E}\left\|\int_0^t S(t-s)f(s,\varphi(s-\delta(s)))\,ds\right\|^p &\le 7^{p-1}e^{\alpha t}E\left[\int_0^t Me^{-\gamma(t-s)}\|f(s,\varphi(s-\delta(s)))\|\,ds\right]^p \qquad (12)\\ &\le 7^{p-1}e^{\alpha t}M^pL_1^pE\left[\int_0^t e^{-\gamma(t-s)}\|\varphi(s-\delta(s))\|\,ds\right]^p \\ &\le 7^{p-1}e^{\alpha t}M^pL_1^p\left[\int_0^t e^{-\gamma(t-s)}\,ds\right]^{p-1}\int_0^t e^{-\gamma(t-s)}E\|\varphi(s-\delta(s))\|^p\,ds \\ &\le 7^{p-1}e^{\alpha\tau}M^pL_1^p\gamma^{1-p}\int_0^t e^{-(\gamma-\alpha)(t-s)}e^{\alpha(s-\delta(s))}E\|\varphi(s-\delta(s))\|^p\,ds. \end{aligned}$$

From Lemma 2.4 and Hölder inequality, we obtain

$$\begin{aligned}
&7^{p-1}e^{\alpha t}\mathbb{E}\left\|\int_0^t S(t-s)g(s,\varphi(s-\rho(s)))\,dw(s)\right\|^p \\
&\le 7^{p-1}e^{\alpha t}c_pM^p\left[\int_0^t \left(e^{-\gamma p(t-s)}\mathbb{E}\|g(s,\varphi(s-\rho(s)))\|^p\right)^{2/p}\,ds\right]^{p/2} \\
&\le 7^{p-1}e^{\alpha t}c_pM^pL_2^p\left[\int_0^t \left(e^{-\gamma p(t-s)}\mathbb{E}\|\varphi(s-\rho(s))\|^p\right)^{2/p}\,ds\right]^{p/2} \\
&\le 7^{p-1}e^{\alpha t}c_pM^pL_2^p\left[\int_0^t e^{-\left(\frac{2(p-1)}{p-2}\right)\gamma(t-s)}\,ds\right]^{p-1}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|\varphi(s-\rho(s))\|^p\,ds \\
&\le 7^{p-1}c_pM^pL_2^p\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}e^{\alpha\tau}\int_0^t e^{-(\gamma-\alpha)(t-s)}e^{\alpha(s-\rho(s))}\mathbb{E}\|\varphi(s-\rho(s))\|^p\,ds \qquad (13)
\end{aligned}$$

and

$$\begin{aligned}
&7^{p-1}e^{\alpha t}\mathbb{E}\left\|\int_0^t\int_Z S(t-s)h(s,\varphi(s-\sigma(s)),z)\,\tilde{N}(ds,dz)\right\|^p \\
&\le 7^{p-1}e^{\alpha t}c_p\mathbb{E}\left[\int_0^t\int_Z \|S(t-s)h(s,\varphi(s-\sigma(s)),z)\|^2\,ds\nu(dz)\right]^{p/2} \\
&\le 7^{p-1}e^{\alpha t}c_pM^p\mathbb{E}\left[\int_0^t\int_Z e^{-2\gamma(t-s)}\|h(s,\varphi(s-\sigma(s)),z)\|^2\,ds\nu(dz)\right]^{p/2} \\
&\le 7^{p-1}e^{\alpha t}c_pM^p\mathbb{E}\left[\int_0^t e^{-2\gamma(t-s)}\int_Z \|h(s,\varphi(s-\sigma(s)),z)\|^2\nu(dz)\,ds\right]^{p/2} \\
&\le 7^{p-1}e^{\alpha t}c_pM^pL_3^p\left[\int_0^t e^{-2\gamma(t-s)}\mathbb{E}\|\varphi(s-\sigma(s))\|^2\,ds\right]^{p/2} \\
&\le 7^{p-1}e^{\alpha t}c_pM^pL_3^p\left(\int_0^t e^{-\frac{2(p-1)}{p}\cdot\frac{p}{p-2}\gamma(t-s)}\,ds\right)^{(p-2)/2}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|\varphi(s-\sigma(s))\|^p\,ds \\
&\le 7^{p-1}c_pM^pL_3^p\left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}e^{\alpha\tau}\int_0^t e^{-(\gamma-\alpha)(t-s)}e^{\alpha(s-\sigma(s))}\mathbb{E}\|\varphi(s-\sigma(s))\|^p\,ds, \qquad (14)
\end{aligned}$$

where $c_p = (p(p-1)/2)^{p/2}$. We remark that if $p = 2$, the inequality (13) also holds with $0^0 := 1$. Similar to the proof of (9), from estimate (12), (13) and (14), we obtain that

$$\begin{aligned}
&7^{p-1}e^{\alpha t}\mathbb{E}\left\|\int_0^t S(t-s)f(s,\varphi(s-\delta(s)))\,ds\right\|^p \to 0 \quad \text{as} \quad t\to\infty, \\
&7^{p-1}e^{\alpha t}\mathbb{E}\left\|\int_0^t S(t-s)g(s,\varphi(s-\rho(s)))\,dw(s)\right\|^p \to 0 \quad \text{as} \quad t\to\infty, \\
&7^{p-1}e^{\alpha t}\mathbb{E}\left\|\int_0^t\int_Z S(t-s)h(s,\varphi(s-\sigma(s)),z)\,\tilde{N}(ds,dz)\right\|^p \to 0 \quad \text{as} \quad t\to\infty. \qquad (15)
\end{aligned}$$

Now, we estimate the impulsive term, from the condition (i), we obtain

$$\begin{aligned}
7^{p-1}e^{\alpha t}\mathbb{E}\left\|\sum_{0<t_k<t} S(t-t_k)I_k\varphi(t_k^-)\right\|^p &\le 7^{p-1}e^{\alpha t}\mathbb{E}\left(\sum_{0<t_k<t} Me^{-\gamma(t-t_k)}q_k\|\varphi(t_k^-)\|\right)^p \\
&\le 7^{p-1}e^{\alpha t}\mathbb{E}\left(\sum_{0<t_k<t} Me^{-\gamma(t-t_k)}\tilde{q}\|\varphi(t_k^-)(t_k-t_{k-1})\|\right)^p
\end{aligned}$$

$$\begin{aligned}
&\le 7^{p-1}e^{\alpha t}\mathbb{E}\left(\int_0^t Me^{-\gamma(t-s)}\tilde{q}\|\varphi(s)\|\,ds\right)^p \\
&\le 7^{p-1}M^p\tilde{q}^p\left(\int_0^t e^{-\gamma(t-s)}\,ds\right)^{p-1}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|\varphi(s)\|^p\,ds \\
&\le 7^{p-1}M^p\tilde{q}^p\gamma^{1-p}\int_0^t e^{-(\gamma-\alpha)(t-s)}e^{\alpha s}\mathbb{E}\|\varphi(s)\|^p\,ds. \qquad (16)
\end{aligned}$$

From (16), we obtain

$$7^{p-1}e^{\alpha t}\mathbb{E}\left\|\sum_{0<t_k<t}S(t-t_k)I_k\varphi(t_k^-)\right\|^p \to 0 \quad \text{as} \quad t\to\infty. \qquad (17)$$

Hence, from (7), (8), (15) and (17), we obtain that $e^{\alpha t}\mathbb{E}\|(\pi\varphi)(t)\|^p \to 0 \quad \text{as} \quad t\to\infty$. Therefore, $\pi(\mathcal{S}_\phi)\subseteq\mathcal{S}_\phi$.

Finally, we show that π is a contraction mapping. For any $\varphi,\psi\in\mathcal{S}_\phi$, we obtain

$$\begin{aligned}
\sup_{t\ge0}\mathbb{E}\|(\pi\varphi)(t)-(\pi\psi)(t)\|^p &\le 6^{p-1}\sup_{t\ge0}\mathbb{E}\|u(t,\varphi(t-\tau(t)))-u(t,\psi(t-\tau(t)))\|^p \\
&\quad +6^{p-1}\sup_{t\ge0}\mathbb{E}\left\|\int_0^t AS(t-s)(u(s,\varphi(s-\tau(s)))-u(s,\psi(s-\tau(s))))\,ds\right\|^p \\
&\quad +6^{p-1}\sup_{t\ge0}\mathbb{E}\left\|\int_0^t S(t-s)(f(s,\varphi(s-\delta(s)))-f(s,\psi(s-\delta(s))))\,ds\right\|^p \\
&\quad +6^{p-1}\sup_{t\ge0}\mathbb{E}\left\|\int_0^t S(t-s)(g(s,\varphi(s-\rho(s)))-g(s,\psi(s-\rho(s)))))\,dw(s)\right\|^p \\
&\quad +6^{p-1}\sup_{t\ge0}\mathbb{E}\left\|\int_0^t\int_Z S(t-s)(h(s,\varphi(s-\sigma(s)),y)-h(s,\psi(s-\sigma(s)),y))\,\tilde{N}(ds,dy)\right\|^p \\
&\quad +6^{p-1}\sup_{t\ge0}\mathbb{E}\left\|\sum_{0<t_k<t}S(t-t_k)(I_k(\varphi(t_k^-))-I_k(\psi(t_k^-)))\right\|^p \\
&\le \Bigg[6^{p-1}\|(-A)^{-\alpha}\|^pK^p+6^{p-1}M_{1-\alpha}^pK^p\gamma^{-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1} \\
&\quad +6^{p-1}M^pL_1^p\gamma^{-p}+6^{p-1}c_pM^pL_2^p\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\gamma^{-1} \\
&\quad +6^{p-1}c_pM^pL_3^p\left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}\gamma^{-1}+6^{p-1}M^p\tilde{q}^p\gamma^{-p}\Bigg]\sup_{t\ge0}\mathbb{E}\|\varphi(t)-\psi(t)\|^p.
\end{aligned}$$

Thus, by the condition (ii), we know that π is a contraction mapping. Hence, by the contraction mapping principle, π has a unique fixed point $x(t)$ in $\mathcal{S}_\phi$, which is a solution of the system (1) with $x(s)=\phi(s)$ on $[-\tau,0]$ and $e^{\alpha t}\mathbb{E}\|x(t)\|^p\to0$ as $t\to\infty$. □

Remark 3.2. *In Theorem* 3.1, *We do not require the monotone decreasing behavior of the delays, i.e.* $\tau'(t)\le0$, $\delta'(t)\le0$, $\rho'(t)\le0$, $\sigma'(t)\le0$ *for* $t\ge0$.

Remark 3.3. *Sakthivel and Luo* [16, 17] *and Jiang and Shen* [3] *studied asymptotic stability of special cases of the system* (1) *by using fixed point theory. In* [3, 16, 17], *the estimate of the impulsive term is*

$$\sup_{t\in[0,T]}\mathbb{E}\left\|\sum_{0<t_k<t}S(t-t_k)(I_k(x(t_k^-))-I_k(y(t_k^-)))\right\|^p \le M^pe^{-\gamma pT}\mathbb{E}\left(\sum_{k=1}^m q_k^p\right)\sup_{t\in[0,T]}\mathbb{E}\|x(t)-y(t)\|^p$$

which seems to be a mistake. It should be

$$\sup_{t\ge0}\mathbb{E}\left\|\sum_{0<t_k<t}S(t-t_k)(I_k(x(t_k^-))-I_k(y(t_k^-)))\right\|^p \le M^p\sup_{t\ge0}\mathbb{E}\left[\sum_{0<t_k<t}e^{-\gamma(t-t_k)}q_k\|x(t_k^-)-y(t_k^-)\|\right]^p.$$

To estimate the impulsive effects to the system (1), *we consider the case with which the impulses satisfy condition* (i) *in Theorem* 3.1. *Note that k in condition* (i) *can be equal to infinity.*

If the impulsive effects $I_k(\cdot) \equiv 0$ ($k = 1, 2, \cdots$), system (1) reduces to the following neutral stochastic partial differential equations with delays

$$\begin{cases} d[x(t) + u(t, x(t-\tau(t)))] = [Ax(t)dt + f(t, x(t-\delta(t)))]dt + g(t, x(t-\rho(t))d\omega(t) \\ \qquad\qquad + \int_z h(t, x(t-\theta(t)), y)\,\tilde{N}(dt, dy), \quad t \geq 0 \\ x_0(\cdot) = \phi \in PC. \end{cases} \tag{18}$$

Corollary 3.4. *Consider the stochastic partial differential equation* (18) *and suppose that the conditions* (A1)-(A4) *are satisfied. Then the mild solution of* (18) *is exponential stability in pth moment, if the following inequality*

$$5^{p-1}\|(-A)^{-\alpha}\|^p K^p + 5^{p-1} M_{1-\alpha}^p K^p \gamma^{-p\alpha}(\Gamma(1 + p(\alpha-1)/(p-1)))^{p-1}$$
$$+5^{p-1}M^p L_1^p \gamma^{-p} + 5^{p-1}c_p M^p L_2^p \left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2} \gamma^{-1} + 5^{p-1}c_p M^p L_3^p \left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2} \gamma^{-1} < 1$$

holds, where $c_p = (p(p-1)/2)^{p/2}$.

Remark 3.5. *Cui et al.* [2] *have studied exponential stability in mean square of mild solution to* (18) *by using fixed point theorem under the assumption that* $\|S(t)\| \leq e^{-\gamma t}$ *for* $t \geq 0$.

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