A GENERAL SMOOTHING INEQUALITY FOR DISORDERED POLYMERS

FRANCESCO CARAVENNA AND FRANK DEN HOLLANDER

Abstract. This note sharpens the smoothing inequality of Giacomin and Toninelli [7], [8] for disordered polymers. This inequality is shown to be valid for any disorder distribution with locally finite exponential moments, and to provide an asymptotically sharp constant for weak disorder. A key tool in the proof is an estimate that compares the effect on the free energy of tilting, respectively, shifting the disorder distribution. This estimate holds in large generality (way beyond disordered polymers) and is of independent interest.

1. Introduction and main results

Understanding the effect of disorder on phase transitions is a key topic in statistical physics. In a celebrated paper, Harris [9] proposed a criterion that predicts whether or not the addition of an arbitrarily small amount of quenched disorder is able to modify the critical behavior of a system close to a phase transition. The rigorous justification of this criterion for a class of *pinning models* has been an active direction of research in the mathematical literature (see Giacomin [6] for an overview). One of the key tools in this program is the *smoothing inequality* of Giacomin and Toninelli [7], [8]. It is the purpose of this note to generalize and sharpen this inequality.

Section 1.1 provides motivation, Section 1.2 states the necessary model assumptions, Section 1.3 defines the free energy, Section 1.4 states our main theorems, while Section 1.5 discusses the context of these theorems. Proofs are given in Sections 2–4.

1.1. **Motivation.** We begin by describing a class of models that motivates our main results in Section 1.4. We use the notation $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Consider a recurrent Markov chain $S := (S_n)_{n\in\mathbb{N}_0}$ on a countable set E, starting at a distinguished point denoted by 0, defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$, and let $\tau_1 := \inf\{n \in \mathbb{N} \colon \; S_n = 0\}$ be its first return time to 0. The key assumption is that for some $\alpha \in [0, \infty)$,

$$\mathrm{P}(\tau_1 > n) = n^{-\alpha+o(1)}, \qquad n \to \infty. \tag{1.1}$$

The case of a transient Markov chain, i.e., $\mathrm{P}(\tau_1 = \infty) > 0$, can be included as well, and requires that (1.1) holds conditionally on $\{\tau_1 < \infty\}$.

Given an $\mathbb{R}$-valued sequence $\omega := (\omega_n)_{n\in\mathbb{N}}$ (the *disorder* sequence), a function $\varphi\colon \mathsf{E} \to \mathbb{R}$ (the *potential*), and parameters $N \in \mathbb{N}$, $\beta \geq 0$, $h \in \mathbb{R}$ (the *system size*, the *disorder strength* and the *disorder shift*), we define the *partition function*

$$Z^{\omega,\varphi}_{N,\beta,h} = \mathrm{E}\left[e^{\sum_{n=1}^{N}(h+\beta\omega_n)\varphi(S_n)}\, 1_{\{S_N=0\}}\right] \in [0, \infty], \tag{1.2}$$

i.e., at each time n the Markov chain gets an exponential reward or penalty proportional to $h + \beta\omega_n$, modulated by a factor $\varphi(S_n)$. The sequence ω is to be thought of as a typical realization of a random process. Note that

Date: June 13, 2013.

- the choice $\varphi(x) := \mathbf{1}_{\{0\}}(x)$ corresponds to the *pinning model* (see Giacomin [5], [6], den Hollander [10]);
- when $\mathsf{E} = \mathbb{Z}$ and S is nearest-neighbor with symmetric excursions out of 0, the choice $\varphi(x) := \mathbf{1}_{(-\infty,0]}(x)$ corresponds to the *copolymer model* (see [5], [10]); †

Thus, the modulating potential φ allows us to interpolate between different classes of models. When S is simple random walk on $\mathbb{Z}^d$ and $\varphi(x) \approx |x|^{-\vartheta}$ as $|x| \to \infty$ for some $\vartheta \in (0, \infty)$, the model displays interesting features that are currently under investigation (Caravenna and den Hollander [4]).

1.2. **Assumptions.** Although our main focus will be on the model in (1.2), we list the assumptions that we actually need. We start with the disorder.

Assumption 1.1 (The disorder). *The disorder $\omega = (\omega_n)_{n\in\mathbb{N}}$ is an i.i.d. sequence of $\mathbb{R}$-valued random variables, defined on a probability space $(\Omega', \mathcal{F}', \mathbb{P})$, such that*

$$\exists\, t_0 \in (0, \infty] : \quad \mathrm{M}(t) := \mathbb{E}\big[e^{t\omega_1}\big] < \infty \quad \forall\, |t| < t_0, \qquad (1.3)$$
$$\mathbb{E}[\omega_n] = 0, \quad \mathbb{V}\mathrm{ar}(\omega_n) = 1.$$

The crucial assumption is that the disorder distribution has locally finite exponential moments. The choice of zero mean and unit variance is a convenient normalization only (since we can play with the parameters β and h).

For $\delta \in (-t_0, t_0)$, we denote by $\mathbb{P}_\delta$ the *tilted law* under which $\omega = (\omega_n)_{n\in\mathbb{N}}$ is i.i.d. with marginal distribution

$$\mathbb{P}_\delta(\omega_1 \in \mathrm{d}x) := e^{\delta x - \log \mathrm{M}(\delta)}\, \mathbb{P}(\omega_1 \in \mathrm{d}x). \qquad (1.4)$$

Next we state our assumptions on the partition function $Z_{N,\omega,\beta,h}$ we will be able to handle, defined for $N \in \mathbb{N}$, $\beta \geq 0$, $h \in \mathbb{R}$ and $\mathbb{P}$-a.e. $\omega \in \mathbb{R}^{\mathbb{N}}$ (keeping in mind (1.2) as a special case).

Assumption 1.2 (The partition function [I]). *$Z_{N,\omega,\beta,h}$ is a measurable function defined on $\mathbb{N} \times \mathbb{R}^{\mathbb{N}} \times [0, \infty) \times \mathbb{R}$, taking values in $[0, \infty)$ and satisfying the following conditions:*

(1) *$Z_{N,\omega,\beta,h}$ is a function of N and of $(h + \beta\omega_n)_{1\leq n\leq N}$.*

(2) *$Z_{N+M,\omega,\beta,h} \geq Z_{N,\omega,\beta,h}\, Z_{M,\vartheta^N\omega,\beta,h}$ for all $N, M \in \mathbb{N}$, where ϑ is the left-shift acting on ω, i.e., $(\vartheta^N\omega)_n := \omega_{N+n}$ for $N \in \mathbb{N}$.*

(3) *There exists a $\gamma \in (0, \infty)$ such that, for N in a subsequence of $\mathbb{N}$,*

$$Z_{N,\omega,\beta,h} \geq \frac{c_{\beta,h}(\omega)}{N^\gamma} \quad \textit{with} \quad \mathbb{E}_\delta[\log c_{\beta,h}(\omega)] > -\infty \quad \forall\, \delta \in (-t_0, t_0). \qquad (1.5)$$

Remark 1.3. Note that properties (1) and (2) are satisfied for the model in (1.2). For property (3) to be satisfied as well, we need to make additional assumptions on φ and/or S. For instance, for the pinning model property (3) holds with $\gamma = (1 + \alpha) + \varepsilon$, for any fixed $\varepsilon > 0$ (and for a suitable choice of $c_{\beta,h}(\omega) = c^{\varepsilon}_{\beta,h}(\omega)$), which follows from (1.1) after restricting the expectation in (1.2) to the event $\{\tau_1 = N\}$. Alternatively, when $\mathsf{E} = \mathbb{Z}^d$, if φ vanishes in a half-space and S is symmetric (as for the copolymer model), property (1.5) with $\gamma = (1 + \alpha) + \varepsilon$ again follows from (1.1).

†The standard copolymer model is defined through a *bond* interaction: $\varphi(S_n)$ is replaced by $\varphi(S_{n-1}, S_n) := \mathbf{1}_{(-\infty,0]}(\frac{1}{2}[S_{n-1} + S_n])$, and (β, h) by $(-2\lambda, -2\lambda h)$. This can be still cast in the framework of (1.2) by picking $\mathsf{E} = \mathbb{Z}^2$, taking the pair process (S_{n-1}, S_n) as the Markov chain, and $(0, 0)$ as 0.

As a matter of fact, properties (1) and (2) are rather mild: they are satisfied for many $(1+d)$-dimensional directed models (possibly after a minor modification of the partition function that does not change the free energy defined below). In contrast, property (3) is a more severe restriction. Roughly speaking, it says that the *disorder can be avoided at a cost that is only polynomial in the system size.*

1.3. **Free energy.** If Assumptions 1.1 and 1.2 are satisfied, then we can define the *free energy*

$$\textsc{f}(\beta, h; \delta) := \limsup_{N\to\infty} \frac{1}{N} \mathbb{E}_\delta \big[\log Z_{N,\omega,\beta,h} \big] \tag{1.6}$$

for $\beta \geq 0$, $h \in \mathbb{R}$, $\delta \in (-t_0, t_0)$ when ω is chosen according to $\mathbb{P}_\delta$.

Remark 1.4. (a) In the general framework of Assumption 1.2, it may happen that $\textsc{f}(\beta, h; \delta) = \infty$ for some values of the parameters. However, for the model in (1.2) we have $\textsc{f}(\beta, h; \delta) < \infty$ as soon as φ is bounded (see (3.4) below).
(b) By property (2) in Assumption 1.2, the lim sup in (1.6) may be replaced by sup, or by lim restricted to those values of N for which $\mathbb{E}_\delta \big[\log Z_{N,\omega,\beta,h} \big] > -\infty$, which by properties (2)–(3) form a sub-lattice $\textsc{t}\mathbb{N}$. By Kingman's super-additive ergodic theorem, we may also remove the expectation $\mathbb{E}_\delta$ in (1.6), because the limit as $N \to \infty$, $N \in \textsc{t}\mathbb{N}$, exists and is constant $\mathbb{P}_\delta$-a.s.

A direct consequence of (1.5) is the inequality $\textsc{f}(\beta, h; \delta) \geq 0$, which is a crucial feature of the class of models we consider. In many interesting cases, like for pinning and copolymer models, the free energy is zero in some closed region of the parameter space and strictly positive in its complement, with both regions non-empty. When this happens, the free energy is not an analytic function and the model is said to undergo a *phase transition.* It is then of physical and mathematical interest to study the regularity of the free energy close to the *critical curve* separating the two regions.

More concretely, consider the case when $h \mapsto Z_{N,\omega,\beta,h}$ is monotone (like for the model in (1.2) when φ has a sign), say non-decreasing, so that $h \mapsto \textsc{f}(\beta, h; \delta)$ is non-decreasing as well. Then for every $\beta \geq 0$ there exists a critical value $h_c(\beta) \in \mathbb{R} \cup \{\pm\infty\}$ such that $\textsc{f}(\beta, h; 0) = 0$ for $h < h_c(\beta)$ and $\textsc{f}(\beta, h; 0) > 0$ for $h > h_c(\beta)$ (we consider $\delta = 0$ for simplicity). If $h \mapsto \textsc{f}(\beta, h; 0)$ is continuous as well, as is typical, then $\textsc{f}(\beta, h_c(\beta); 0) = 0$ and it is interesting to understand how the free energy vanishes as $h \downarrow h_c(\beta)$. For homogeneous pinning models, i.e., when $\beta = 0$, it is known that

$$\textsc{f}(0, h_c(0) + t; 0) = t^{\max\{\frac{1}{\alpha}, 1\} + o(1)}, \qquad t \downarrow 0. \tag{1.7}$$

(See [5, Theorem 2.1] for more precise estimates.) On the other hand, as soon as disorder is present, i.e., when $\beta > 0$, it was shown by Giacomin and Toninelli [7], [8] that, under some mild restrictions on the disorder distribution,

$$\exists\, c \in (0, \infty): \quad 0 \leq \textsc{f}(\beta, h_c(\beta) + t; 0) \leq \frac{c}{\beta^2}\, t^2. \tag{1.8}$$

Comparing (1.7) and (1.8), we see that when $\alpha > \frac{1}{2}$ the addition of disorder has a *smoothing* effect on the way in which the free energy vanishes at the critical line.

1.4. **Main results.** The goal of this note is to generalize and sharpen (1.8), namely, to show that no assumption on the disorder distribution other than (1.3) is required, and to provide estimates on the constant c that are optimal in some sense (see below). We will stay in the general framework of Assumption 1.2, with no mention of "critical lines".

• *Tilting.* First we prove a smoothing inequality for $\mathrm{F}(\beta, h; \delta)$ with respect to the tilt parameter δ rather than the shift parameter h. Although both tilting and shifting are natural ways to control the disorder bias, the latter is often preferred in the literature because the free energy typically is a convex function of the shift parameter h (like for the model in (1.2)). However, for the purpose of the smoothing inequality the tilt parameter δ turns out to be more natural.

Theorem 1.5 (Smoothing inequality with respect to a disorder tilt)**.** *Subject to Assumptions* 1.1 *and* 1.2*, if* $\mathrm{F}(\bar\beta, \bar h; 0) = 0$ *for some* $\bar\beta > 0$ *and* $\bar h \in \mathbb{R}$*, then for all* $\delta \in (-t_0, t_0)$*,*

$$0 \le \mathrm{F}(\bar\beta, \bar h; \delta) \le \frac{\gamma}{2} B_\delta \, \delta^2 \tag{1.9}$$

where the constants t_0 *and* γ *are defined in* (1.3) *and* (1.5)*, while*

$$B_\delta := \frac{2}{\delta} \left| (\log \mathrm{M})'(\delta) - \frac{\log \mathrm{M}(\delta)}{\delta} \right| \in (0, \infty) \qquad \textit{satisfies} \qquad \lim_{\delta \to 0} B_\delta = 1. \tag{1.10}$$

Remark 1.6. For pinning and copolymer models satisfying (1.1), we can set $\gamma = 1 + \alpha$ in (1.9), by Remark 1.3.

Theorem 1.5 is proved in Section 2 through a direct translation of the argument developed in Giacomin and Toninelli [8]. The proof is based on the concept of *rare stretch strategy*, which has been a crucial tool in the study of disordered polymer models since the papers by Monthus [11], Bodineau and Giacomin [3].

• *Shifting.* Next we consider the effect of a disorder shift. In the *Gaussian case*, i.e., when $\mathbb{P}(\omega_1 \in \cdot) = N(0,1)$, tilting is the same as shifting: in fact $\mathbb{P}_\delta(\omega_1 \in \cdot) = N(\delta, 1)$ and so ω_n under $\mathbb{P}_\delta$ is distributed like $\omega_n + \delta$ under $\mathbb{P}$. Recalling property (1), we then get

$$\mathrm{F}(\beta, h; \delta) = \mathrm{F}(\beta, h + \beta\delta; 0) \tag{1.11}$$

and, since $\mathrm{M}(\delta) = e^{\delta^2/2}$, it follows from (1.9) that if $\mathrm{F}(\bar\beta, \bar h; 0) = 0$ with $\bar\beta > 0$, then

$$0 \le \mathrm{F}(\bar\beta, \bar h + t; 0) \le \frac{\gamma}{2\bar\beta^2} t^2 \qquad \forall\, t \in \mathbb{R}. \tag{1.12}$$

This is precisely the smoothing inequality with respect to a disorder shift in (1.8), with an explicit constant (see also Giacomin [5, Theorem 5.6 and Remark 5.7]).

For a general disorder distribution tilting is different from shifting. However, we may still hope that (1.11) holds approximately. This is what was shown in Giacomin and Toninelli [7], under additional restrictions on the disorder distribution and with non-optimal constants. The main result of this note, Theorem 1.8 below, shows that the effects on the free energy of tilting or shifting the disorder distribution are asymptotically equivalent, in large generality and with asymptotically optimal constants in the weak interaction limit. Since this result is unrelated to Theorem 1.5 and is of independent interest, we formulate it for a very general class of statistical physics models, way beyond disordered polymer models.

Assumption 1.7 (The partition function [II])**.** *The partition function is defined as*

$$Z_{N,\omega,\beta,h} := \mathrm{E}_N \left[e^{\sum_{n=1}^N (h + \beta\omega_n)\sigma_n} \right], \tag{1.13}$$

where, for fixed $N \in \mathbb{N}$*,* $(\sigma_i)_{1 \le i \le N}$ *are* $\mathbb{R}$*-valued measurable functions, defined on a finite measure space* $(\Omega_N, \mathcal{F}_N, \mathrm{P}_N)$*, that are uniformly bounded, have a sign, say*

$$\exists\, s_0 > 0\colon \qquad \mathrm{P}_N \left(\left\{ 0 \le \sigma_i \le s_0,\, \forall\, 1 \le i \le N \right\}^c \right) = 0 \qquad \forall\, N \in \mathbb{N}, \tag{1.14}$$

and satisfy $-\infty < \limsup_{N\to\infty} \frac{1}{N} \log \mathrm{P}_N(\Omega_N) < \infty$*.*

We emphasize that the σ_i's need not be independent, nor exchangeable. A more detailed discussion on Assumption 1.7 is given below.

We can now state the approximate version of (1.11). The free energy $\mathrm{F}(\beta, h; \delta)$ is again defined by (1.6).

Theorem 1.8 (Asymptotic equivalence of tilting and shifting)**.** *Subject to Assumptions* 1.1 *and* 1.7*, and with* $\varepsilon_0 := \min\{\frac{t_0}{2}, \frac{t_0}{2s_0}\}$ *(where* s_0, t_0 *are defined in* (1.14) *and* (1.3)*), for all* $\beta \in [0, \varepsilon_0)$ *and* $\delta \in (-\varepsilon_0, \varepsilon_0)$ *there exist* $0 < C^-_{\beta,\delta} \le C^+_{\beta,\delta} < \infty$ *such that*

$$\forall\, \delta \in [0, \varepsilon_0): \quad \mathrm{F}\big(\beta, h + C^-_{\beta,\delta}\, \beta\delta; 0\big) \le \mathrm{F}(\beta, h; \delta) \le \mathrm{F}\big(\beta, h + C^+_{\beta,\delta}\, \beta\delta; 0\big), \tag{1.15}$$

while for $\delta \in (-\varepsilon_0, 0]$ *the same relation holds with* $C^-_{\beta,\delta}$ *and* $C^+_{\beta,\delta}$ *interchanged. Moreover,* $(\beta, \delta) \mapsto C^\pm_{\beta,\delta}$ *is continuous with* $C^\pm_{0,0} = 1$*, and hence*

$$\lim_{(\beta,\delta)\to(0,0)} C^\pm_{\beta,\delta} = 1. \tag{1.16}$$

Furthermore, $\delta \mapsto C^\pm_{\beta,\delta}\, \delta$ *is strictly increasing.*

The proof of Theorem 1.8 is given in Section 3. The general strategy and consists in showing that the derivatives of $\mathrm{F}(\beta, h; \delta)$ with respect to δ and h are comparable. Compared to Giacomin and Toninelli [7], several estimates need to be sharpened considerably.

• *Smoothing.* Combining Theorems 1.5 and 1.8, we finally obtain our smoothing inequality with respect to a shift, with explicit control on the constant.

Theorem 1.9 (Smoothing inequality with respect to a disorder shift)**.** *Subject to Assumptions* 1.1, 1.2 *and* 1.7*, there is an* $\varepsilon_0' > 0$ *with the following property: if* $\mathrm{F}(\bar\beta, \bar h; 0) = 0$ *for some* $\bar\beta \in (0, \varepsilon_0')$ *and* $\bar h \in \mathbb{R}$*, then for* $t \in (-\bar\beta\varepsilon_0', \bar\beta\varepsilon_0')$*,*

$$0 \le \mathrm{F}(\bar\beta, \bar h + t; 0) \le \frac{\gamma}{2\bar\beta^2}\, A_{\bar\beta, \frac{t}{\bar\beta}}\, t^2, \tag{1.17}$$

where $(\beta, \delta) \mapsto A_{\beta,\delta}$ *is continuous from* $(0, \varepsilon_0') \times (\varepsilon_0', \varepsilon_0')$ *to* $(0, \infty)$*, and is such that*

$$\lim_{(\beta,\delta)\to(0,0)} A_{\beta,\delta} = 1. \tag{1.18}$$

1.5. **Discussion.** We comment on the results obtained in Section 1.4.

1. The version of the smoothing inequality in Theorem 1.9, *with the precision on the constant*, is picked up and used in Berger, Caravenna, Poisat, Sun and Zygouras [2] to obtain the sharp asymptotics of the critical curve $\beta \mapsto h_c(\beta)$ for pinning and copolymer models in the weak disorder regime $\beta \downarrow 0$, for the case $\alpha \in (1, \infty)$ (recall (1.1)).

2. The smoothing inequality in (1.17), *at the level of generality at which it is stated*, is optimal in the following sense.

- We cannot hope for an exponent strictly larger than 2 in the right-hand side of (1.17), because pinning models with $\mathrm{P}(\tau_1 = n) \sim (\log n)/n^{3/2}$ are in the "irrelevant disorder regime", and it is known that $\mathrm{F}(\beta, h_c(\beta) + t; 0) \sim \mathrm{F}(0, h_c(0) + t; 0) = t^{2+o(1)}$ as $t \downarrow 0$ for fixed $\beta > 0$ small enough (see Alexander [1, Theorem 1.2]).
- We cannot hope for an asymptotically smaller constant, i.e., $\lim_{(\beta,\delta)\to(0,0)} A_{\beta,\delta} < 1$, because the proof in Berger, Caravenna, Poisat, Sun and Zygouras [2] would yield a contradiction (the lower bound would be strictly larger than the upper bound).

Of course, for specific models the inequality (1.17) can sometimes be strengthened. For instance, pinning models satisfying (1.1) with $\alpha \in (0, \frac{1}{2})$ are such that $\textsc{f}(\beta, h_c(\beta) + t; 0) \sim \textsc{f}(0, h_c(0) + t; 0) = t^{1/\alpha + o(1)}$ as $t \downarrow 0$ (see (1.7)), again by Alexander [1, Theorem 1.2].

3. Compared with Assumption 1.2, Assumption 1.7 prescribes a specific form for the partition function $Z_{N,\omega,\beta,h}$ and therefore is more restrictive. On the other hand, in view of the minor constraints put on the σ_i's, (1.13) is so general that the absence of any restrictive conditions like (2) or (3) makes Assumption 1.7 effectively much weaker than Assumption 1.2. For instance, since (1.2) is a special case of (1.13), with $\mathrm{P}_N(\cdot) = \mathrm{P}(\,\cdot \cap \{S_N = 0\})$ (which, incidentally, explains why P_N is allowed to be a finite measure, and not necessarily a probability), the model in (1.2) satisfies Assumption 1.7 as soon as the function φ is bounded and has a sign, without the need for any requirement like (1.1).

We emphasize that many other (also non-directed) disordered models fall into Assumption 1.7. For instance, for $L \in \mathbb{N}$ set $\Lambda_L := \{-L, \ldots, +L\}^d$, $N := |\Lambda_L| = (2L+1)^d$, $\Omega_N := \{-1, +1\}^{\Lambda_L}$, and let $(\eta_i)_{i \in \Lambda_L}$ be the coordinate projections on Ω_N. If P_N is the standard Ising Gibbs measure on Ω_N, defined by $\mathrm{P}_N(\{\eta_i\}_{i \in \Lambda_L}) := (1/Z_N) \exp[J \sum_{i,j \in \Lambda_L, \, |i-j|=1} \eta_i \eta_j]$, then the random variables $\sigma_i := \frac{1}{2}(\eta_i + 1)$ satisfy Assumption 1.7.

4. It follows easily from (1.6) and (1.13) that (with obvious notation)

$$\textsc{f}_{(\sigma_n + c)_{n \in \mathbb{N}}}(\beta, h; \delta) = \textsc{f}_{(\sigma_n)_{n \in \mathbb{N}}}(\beta, h; \delta) + (\beta m_\delta + h) c. \tag{1.19}$$

Therefore, when the σ_n's are uniformly bounded but not necessarily non-negative, we can first perform a uniform translation to transform them into non-negative random variables, next apply (1.15), and finally use (1.19) to come back to the original σ_n's.

Still, the non-negativity assumption on the σ_n's in (1.14) cannot be dropped from Theorem 1.8. In fact, if $\textsc{f}(\beta, h; \delta)$ is differentiable in h and δ, then (1.15) implies that

$$\forall h \in \mathbb{R}: \qquad \frac{\partial \textsc{f}}{\partial \delta}(\beta, h; 0) = \big[1 + o(1)\big]\, \beta\, \frac{\partial \textsc{f}}{\partial h}(\beta, h; 0), \qquad \beta \downarrow 0. \tag{1.20}$$

This relation, which is a necessary condition for (1.15) when the free energy is differentiable, may be violated when the σ_n's take both signs. For instance, let $(\sigma_n)_{n \in \mathbb{N}}$ under $\mathrm{P}_N := \mathrm{P}$ be i.i.d. with $\mathrm{P}(\sigma_n = -1) = \mathrm{P}(\sigma_n = +1) = \frac{1}{2}$, and let the marginal distribution of the disorder be $\mathbb{P}(\omega_n = -a^{-1}) = a^2/(a^2+1)$, $\mathbb{P}(\omega_n = a) = 1/(a^2+1)$ with $a > 0$ (note that $\mathbb{E}(\omega_1) = 0$ and $\mathbb{V}\mathrm{ar}(\omega_1) = 1$, so that (1.3) is satisfied). The free energy is easily computed:

$$\textsc{f}(\beta, h; \delta) = \mathbb{E}_\delta[\cosh(h + \beta\omega_1)] = \frac{e^{a\delta}\cosh(h + a\beta) + a^2 e^{-a^{-1}\delta}\cosh(h - a^{-1}\beta)}{e^{a\delta} + a^2 e^{-a^{-1}\delta}}. \tag{1.21}$$

In particular,

$$\frac{\partial \textsc{f}}{\partial h}(\beta, 0; 0) = \frac{\sinh(a\beta) + a^2 \sinh(-a^{-1}\beta)}{1 + a^2} = \frac{a^2 - 1}{6a}\beta^3 + o(\beta^3), \tag{1.22}$$

$$\frac{\partial \textsc{f}}{\partial \delta}(\beta, 0; 0) = \frac{a\cosh(a\beta) - a\cosh(-a^{-1}\beta)}{1 + a^2} = \frac{a^2 - 1}{2a}\,\beta^2 \,+\, o(\beta^2), \tag{1.23}$$

and hence (1.20) does *not* hold for $a \neq 1$ (the left-hand side is $\approx \beta^2$, while the right-hand side is $\approx \beta^4$). Intuitively, such a discrepancy arises for values of h at which $\frac{\partial \textsc{f}}{\partial h}(0, h; 0) = 0$, which means that the average $\mathrm{E}_{N,\omega,0,h}(\frac{1}{N}\sum_{n=1}^N \sigma_n)$ tends to zero as $N \to \infty$, where $\mathrm{P}_{N,\omega,\beta,h}$ is the Gibbs law associated to the partition function $Z_{N,\omega,\beta,h}$ (see (3.2) below). When the σ_n's are non-negative, their individual variances under $\mathrm{P}_{N,\omega,0,h}$ must be small, but this is no longer true when the σ_n's can also take negative values. This is why one might have $\frac{\partial \textsc{f}}{\partial \delta}(\beta, h; 0) \gg \beta \frac{\partial \textsc{f}}{\partial h}(\beta, h; 0)$ for $\beta > 0$ small (compare (3.18) with (3.21)-(3.22) below).

2. Smoothing with respect to a tilt: proof of Theorem 1.5

2.1. **The $(\mathcal{G},\mathcal{C})$-rare stretch strategy.** Fix $\beta \ge 0$ and $h \in \mathbb{R}$. For $\ell \in \mathbb{N}$, let $\mathcal{A}_\ell \subseteq \mathbb{R}^\ell$ be a subset of "disorder stretches" such that there exist constants $\mathcal{G} \in [0,\infty)$ and $\mathcal{C} \in [0,\infty)$ with the following properties, along a diverging sequence of $\ell \in \mathbb{N}$:

- $\frac{1}{\ell}\log Z_{\ell,\omega,\beta,h} \ge \mathcal{G}$ for all $\omega = \omega_{(0,\ell]} := (\omega_1,\dots,\omega_\ell) \in \mathcal{A}_\ell$ (recall Assumption 1.2 (1));
- $\frac{1}{\ell}\log \mathbb{P}(\mathcal{A}_\ell) \ge -\mathcal{C}$.

The notation $(\mathcal{G},\mathcal{C})$ stands for *gain* versus *cost*. Recall that γ is the exponent in (1.5).

Lemma 2.1. *The following implication holds:*

$$\mathcal{G} - \gamma\,\mathcal{C} > 0 \quad \Longrightarrow \quad \textsc{f}(\beta,h;0) > 0. \tag{2.1}$$

Proof. Fix $\ell \in \mathbb{N}$ large enough so that the above conditions hold, and for $\omega \in \mathbb{R}^{\mathbb{N}}$ denote by $T_1(\omega), T_2(\omega), \dots$ the distances between the endpoints of the stretches in $\mathcal{A}_\ell$:

$$T_1(\omega) := \inf\big\{N \in \ell\mathbb{N}\colon\ \omega_{(N-\ell,N]} \in \mathcal{A}_\ell\big\}, \qquad T_{k+1}(\omega) := T_1(\vartheta^{T_1(\omega)+\dots+T_k(\omega)}(\omega)). \tag{2.2}$$

Note that $\{T_k\}_{k\in\mathbb{N}}$ is i.i.d. with marginal law given by $\ell\,\mathrm{GEO}(\mathbb{P}(\mathcal{A}_\ell))$. In particular,

$$\mathbb{E}(T_1) = \ell/\mathbb{P}(\mathcal{A}_\ell) \le \ell\, e^{\mathcal{C}\ell}. \tag{2.3}$$

Henceforth we suppress the subscripts β, h. Since $(\vartheta^{(T_1+\dots+T_i)-\ell}\omega)_{(0,\ell]} \in \mathcal{A}_\ell$ by construction, applying properties (2)-(3) in Assumption 1.2 and the definition of $\mathcal{G}$, we get

$$Z_{T_1+\dots+T_k,\omega} \ge \prod_{i=1}^k Z_{T_i-\ell,\vartheta^{(T_1+\dots+T_{i-1})}\omega}\, Z_{\ell,\vartheta^{(T_1+\dots+T_i)-\ell}\omega} \ge e^{k\mathcal{G}\ell}\prod_{i=1}^k \frac{c_{\beta,h}(\vartheta^{(T_1+\dots+T_{i-1})}\omega)}{(T_i)^\gamma}, \tag{2.4}$$

where we set $Z_0 := 1$ for convenience. Recalling (1.6) and Remark 1.4, for $\mathbb{P}$-a.e. ω we can write, by the strong law of large numbers and Jensen's inequality,

$$\begin{aligned} \textsc{f}(\beta,h;0) &= \lim_{k\to\infty} \frac{1}{T_1+\dots+T_k}\log Z_{T_1+\dots+T_k,\omega,\beta,h} \\ &\ge \frac{1}{\mathbb{E}(T_1(\omega))}\big\{\ell\mathcal{G} + \mathbb{E}[\log c_{\beta,h}(\omega)] - \gamma\,\mathbb{E}[\log(T_1)]\big\} \\ &\ge \frac{1}{\mathbb{E}(T_1(\omega))}\big\{\ell\mathcal{G} + \mathbb{E}[\log c_{\beta,h}(\omega)] - \gamma\,\log\mathbb{E}(T_1)\big\} \\ &= e^{-\mathcal{C}\ell}\bigg\{(\mathcal{G}-\gamma\mathcal{C}) + \frac{\mathbb{E}[\log c_{\beta,h}(\omega)]}{\ell} - \gamma\frac{\log\ell}{\ell}\bigg\}. \end{aligned} \tag{2.5}$$

If $\mathcal{G} - \gamma\mathcal{C} > 0$, then we can choose $\ell \in \mathbb{N}$ large enough (but finite!) such that the right-hand side is strictly positive. This proves (2.1). ☐

2.2. **Proof of Theorem 1.5.** We use Lemma 2.1. Fix $\beta > 0$, $h \in \mathbb{R}$, $\delta \in (-t_0, t_0)$ and $\varepsilon > 0$, and define the set of good atypical stretches as

$$\mathcal{A}_\ell := \bigg\{(\omega_1,\dots,\omega_\ell) \in \mathbb{R}^\ell\colon\ \frac{1}{\ell}\log Z_{\ell,\omega,\beta,h} \ge \textsc{f}(\beta,h;\delta) - \varepsilon\bigg\}, \tag{2.6}$$

so that $\mathcal{G} = \textsc{f}(\beta,h;\delta) - \varepsilon$ by construction. It remains to determine $\mathcal{C}$, for which we need to estimate the probability of $\mathbb{P}(\mathcal{A}_\ell)$ from below.

By the definition (1.6) of $\textsc{f}(\beta, h; \delta)$ together with Kingman's super-additive ergodic theorem (see Remark 1.4), the event $\mathcal{A}_\ell$ is typical for $\mathbb{P}_\delta$:

$$\lim_{\ell \to +\infty} \mathbb{P}_\delta(\mathcal{A}_\ell) = 1. \tag{2.7}$$

Denoting by $\mathbb{P}^\ell_\delta$ (resp. $\mathbb{P}^\ell$) the restriction of $\mathbb{P}_\delta$ (resp. $\mathbb{P}$) on $\sigma(\omega_1, \ldots, \omega_\ell)$, we have, by Jensen's inequality and (1.4),

$$\begin{aligned} \mathbb{P}(\mathcal{A}_\ell) &= \mathbb{P}_\delta(\mathcal{A}_\ell)\, \mathbb{E}_\delta\left(e^{-\log \frac{\mathrm{d}\mathbb{P}^\ell_\delta}{\mathrm{d}\mathbb{P}^\ell}} \middle| \mathcal{A}_\ell \right) \ge \mathbb{P}_\delta(\mathcal{A}_\ell)\, e^{-\mathbb{E}_\delta\left(\log \frac{\mathrm{d}\mathbb{P}^\ell_\delta}{\mathrm{d}\mathbb{P}^\ell} \middle| \mathcal{A}_\ell \right)} \\ &= \mathbb{P}_\delta(\mathcal{A}_\ell)\, e^{-\frac{1}{\mathbb{P}_\delta(\mathcal{A}_\ell)} \mathbb{E}_\delta\left[\left(\log \frac{\mathrm{d}\mathbb{P}^\ell_\delta}{\mathrm{d}\mathbb{P}^\ell} \right) 1_{\mathcal{A}_\ell} \right]} \\ &= \mathbb{P}_\delta(\mathcal{A}_\ell)\, e^{-\frac{\ell}{\mathbb{P}_\delta(\mathcal{A}_\ell)} \mathbb{E}_\delta\left[\left(\delta \frac{\omega_1 + \ldots + \omega_\ell}{\ell} - \log \mathrm{M}(\delta) \right) 1_{\mathcal{A}_\ell} \right]} . \end{aligned} \tag{2.8}$$

Recalling (1.4) and Assumption 1.1, we abbreviate

$$m_\delta := \mathbb{E}_\delta(\omega_1) = (\log \mathrm{M})'(\delta) = \delta + o(\delta), \qquad \delta \to 0. \tag{2.9}$$

By the strong law of large numbers, it follows from (2.7)-(2.8) that for every $\varepsilon > 0$ we have, for ℓ large enough,

$$\frac{1}{\ell} \log \mathbb{P}(\mathcal{A}_\ell) \ge -\big[\delta\, m_\delta - \log \mathrm{M}(\delta)\big] \,-\, \varepsilon =: -\mathcal{C}, \tag{2.10}$$

We can conclude. We know from (2.1) that $\textsc{f}(\beta, h; 0) > 0$ when

$$\mathcal{G} - \gamma \mathcal{C} = \textsc{f}(\beta, h; \delta) - \gamma\big[\delta\, m_\delta - \log \mathrm{M}(\delta)\big] - 2\varepsilon > 0. \tag{2.11}$$

If $\textsc{f}(\bar\beta, \bar h) = 0$, as in the assumptions of Theorem 1.5, it follows that $\mathcal{G} - \gamma\mathcal{C} \le 0$, i.e.,

$$\textsc{f}(\bar\beta, \bar h; \delta) \le \gamma\big[\delta\, m_\delta - \log \mathrm{M}(\delta)\big] + 2\varepsilon, \qquad \forall \delta \in (-t_0, t_0)\,. \tag{2.12}$$

Since this equality holds for every $\varepsilon > 0$, it must hold also for $\varepsilon = 0$, proving (1.9). □

3. Asymptotic equivalence of tilting and shifting: proof of Theorem 1.8

Throughout this section, we work under Assumptions 1.1 and 1.7.

3.1. **Notation.**

Denote the empirical average of the variables σ_i's by

$$\overline{\sigma}_N := \frac{1}{N} \sum_{i=1}^N \sigma_i. \tag{3.1}$$

The finite-volume Gibbs measure associated with the partition function in (1.13) is the *probability* on Ω_N defined, for $N \in \mathbb{N}$, $\omega \in \mathbb{R}^{\mathbb{N}}$, $\beta \ge 0$ and $h \in \mathbb{R}$, by

$$\begin{aligned} &\mathrm{P}_{N,\omega,\beta,h}(\,\cdot\,) := \frac{1}{Z_{N,\omega,\beta,h}}\, \mathrm{E}_N\left[e^{\sum_{n=1}^N (h + \beta\omega_n)\sigma_n}\, 1_{\{\cdot\}} \right], \\ &\text{where} \quad Z_{N,\omega,\beta,h} := \mathrm{E}_N\left[e^{\sum_{n=1}^N (h + \beta\omega_n)\sigma_n} \right]. \end{aligned} \tag{3.2}$$

Let us spell out the definition (1.6) of the free energy, recalling (1.4):

$$\begin{aligned} \textsc{f}(\beta, h; \delta) &:= \limsup_{N\to\infty} \textsc{f}_N(\beta, h; \delta) := \limsup_{N\to\infty} \frac{1}{N} \mathbb{E}_\delta\big[\log Z_{N,\omega,\beta,h}\big] \\ &= \limsup_{N\to\infty} \frac{1}{N} \mathbb{E}\big[e^{\sum_{n=1}^N [\delta\omega_n - \log \mathrm{M}(\delta)]} \log Z_{N,\omega,\beta,h}\big]. \end{aligned} \tag{3.3}$$

Note that, by (1.14),

$$\left|\sum_{n=1}^{N}(h+\beta\omega_n)\sigma_n\right| \le \sum_{n=1}^{N}(|h|+\beta|\omega_n|)\,|\sigma_n| \le s_0\sum_{n=1}^{N}(|h|+\beta|\omega_n|), \tag{3.4}$$

so that $|\mathrm{F}(\beta,h;\delta)| \le s_0(|h|+\beta\mathbb{E}_\delta(|\omega_1|)) + |\limsup_{N\to\infty}\frac{1}{N}\log \mathrm{P}_N(\Omega_N)| < \infty$.

3.2. **Preparation.** Before proving Theorem 1.8, we need some preparation. Recalling (3.1), we define for $[a,b]\subseteq\mathbb{R}$ with $a<b$ a restricted version of the partition function and the free energy, in which the empirical average $\overline{\sigma}_N$ is constrained to lie in $[a,b]$:

$$\begin{aligned} Z^{[a,b]}_{N,\omega,\beta,h} &:= \mathrm{E}_N\left[e^{\sum_{n=1}^{N}(h+\beta\omega_n)\sigma_n}\,\mathbb{1}_{\{\overline{\sigma}_N\in[a,b]\}}\right],\\ \mathrm{F}^{[a,b]}(\beta,h;\delta) &:= \limsup_{N\to\infty}\mathrm{F}^{[a,b]}_N(\beta,h;\delta) := \limsup_{N\to\infty}\frac{1}{N}\mathbb{E}_\delta\big[\log Z^{[a,b]}_{N,\omega,\beta,h}\big]. \end{aligned} \tag{3.5}$$

The corresponding restricted Gibbs measure is the probability defined by (recall (3.2))

$$\mathrm{P}^{[a,b]}_{N,\omega,\beta,h}(\,\cdot\,) := \mathrm{P}_{N,\omega,\beta,h}(\,\cdot\,|\,\overline{\sigma}_N\in[a,b]) = \frac{\mathrm{E}_N\left[e^{\sum_{n=1}^{N}(h+\beta\omega_n)\sigma_n}\,\mathbb{1}_{\{\overline{\sigma}_N\in[a,b]\}}\,\mathbb{1}_{\{\,\cdot\,\}}\right]}{Z^{[a,b]}_{N,\omega,\beta,h}}. \tag{3.6}$$

Note that $Z_{N,\omega,\beta,h} = Z^{[0,s_0]}_{N,\omega,\beta,h}$, by (1.14). Furthermore, $Z^{[a,b]}_{N,\omega,\beta,h} \le Z^{[c,d]}_{N,\omega,\beta,h}$ when $[a,b]\subseteq[c,d]$. Therefore

$$\mathrm{F}^{[a,b]}(\beta,h;\delta) \le \mathrm{F}^{[c,d]}(\beta,h;\delta) \le \mathrm{F}(\beta,h;\delta), \qquad [a,b]\subseteq[c,d]. \tag{3.7}$$

In particular, for $x\in\mathbb{R}$ we may define

$$\mathrm{F}^{\{x\}}(\beta,h;\delta) := \lim_{n\to\infty}\mathrm{F}^{[a_n,b_n]}(\beta,h;\delta) \in [-\infty,+\infty), \tag{3.8}$$

where $a_n\uparrow x$ and $b_n\downarrow x$ are arbitrary strictly monotone sequences (it is easily seen that the limit does not depend on the choice of these sequences).

Note that $\mathrm{F}^{\{x\}}(\beta,h;\delta) = -\infty$ when $x\notin[0,s_0]$, by (1.14). The following result is standard:

$$\mathrm{F}(\beta,h;\delta) = \sup_{x\in[0,s_0]}\mathrm{F}^{\{x\}}(\beta,h;\delta). \tag{3.9}$$

In fact, by (3.7) $\mathrm{F}^{[a,b]}(\beta,h;\delta) \le \mathrm{F}(\beta,h;\delta)$ for every $[a,b]\subseteq\mathbb{R}$, hence by (3.8) $\mathrm{F}(\beta,h;\delta) \ge \mathrm{F}^{\{x\}}(\beta,h;\delta)$ for every $x\in\mathbb{R}$. It follows that the inequality $\ge$ holds in (3.9). For the reverse inequality, note that if $a<b<c$, then $[a,c]\subseteq[a,b]\cup[b,c]$ and so

$$Z^{[a,c]}_{N,\omega,\beta,h} \le Z^{[a,b]}_{N,\omega,\beta,h} + Z^{[b,c]}_{N,\omega,\beta,h} \le 2\,\max\left\{Z^{[a,b]}_{N,\omega,\beta,h}, Z^{[b,c]}_{N,\omega,\beta,h}\right\}. \tag{3.10}$$

Recalling (3.5), we see that

$$\mathrm{F}^{[a,c]}(\beta,h;\delta) \le \max\left\{\mathrm{F}^{[a,b]}(\beta,h;\delta),\, \mathrm{F}^{[b,c]}(\beta,h;\delta)\right\}. \tag{3.11}$$

Since $\mathrm{F}(\beta,h;\delta) = \mathrm{F}^{[0,s_0]}(\beta,h;\delta)$, we can build a sequence of closed intervals $(I_n)_{n\in\mathbb{N}_0}$, where $I_0 = [0,s_0]$ and where I_{n+1} is either the first half or the second half of I_n, such that $\mathrm{F}^{I_n}(\beta,h;\delta) \le \mathrm{F}^{I_{n+1}}(\beta,h;\delta)$ for all $n\in\mathbb{N}$. In particular,

$$\mathrm{F}(\beta,h;\delta) \le \lim_{n\to\infty}\mathrm{F}^{I_n}(\beta,h;\delta). \tag{3.12}$$

By compactness, there exists an $\overline{x} \in [0, s_0]$ such that $I_n \downarrow \{\overline{x}\}$, i.e., $\bigcap_{n\in\mathbb{N}} I_n = \{\overline{x}\}$. If $I_n = [a_n, b_n]$, then we set $J_n := [a_n - \frac{1}{n}, b_n + \frac{1}{n}]$, so that we still have $J_n \downarrow \{\overline{x}\}$, and $\overline{x}$ lies in the interior of each J_n. Since $\mathrm{F}^{I_n}(\beta, h; \delta) \le \mathrm{F}^{J_n}(\beta, h; \delta)$, recalling (3.8) we obtain

$$\mathrm{F}(\beta, h; \delta) \le \lim_{n\to\infty} \mathrm{F}^{I_n}(\beta, h; \delta) \le \lim_{n\to\infty} \mathrm{F}^{J_n}(\beta, h; \delta) = \mathrm{F}^{\{\overline{x}\}}(\beta, h; \delta) \le \sup_{x\in[0,s_0]} \mathrm{F}^{\{x\}}(\beta, h; \delta), \tag{3.13}$$

and the proof of (3.9) is complete.

3.3. **Proof of Theorem 1.8.** By (3.9), it suffices to show that (1.15) is satisfied with $\mathrm{F}^{\{x\}}$ instead of F, for every fixed $x \in [0, s_0]$. It is of course important that the constants $C^{\pm}_{\beta,\delta}$ do not depend on x.

1. First we consider the case $x = 0$. We claim that

$$\mathrm{F}^{\{0\}}(\beta, h; \delta) = \lim_{\varepsilon\downarrow 0} \bigg(\limsup_{N\to\infty} \frac{1}{N} \log \mathrm{P}_N(0 \le \overline{\sigma}_N \le \varepsilon) \bigg). \tag{3.14}$$

Since the right-hand side of (3.14) is a constant that does not depend on $\beta \ge 0$, $\delta \in (-t_0, t_0)$ and $h \in \mathbb{R}$, (1.15) is trivially satisfied with $\mathrm{F}^{\{0\}}$ instead of F, whatever the definition of $C^{\pm}_{\beta,\delta}$ is. To prove (3.14) note that, by Cauchy-Schwarz,

$$\left| \sum_{n=1}^{N} (h + \beta\omega_n)\sigma_n \right| \le \sqrt{\sum_{n=1}^{N} (h + \beta\omega_n)^2} \sqrt{\sum_{n=1}^{N} |\sigma_n|^2} \le N\, s_0 \sqrt{\overline{\sigma}_N} \sqrt{\frac{1}{N}\sum_{n=1}^{N} (h + \beta\omega_n)^2}, \tag{3.15}$$

because $0 \le \sigma_n = |\sigma_n| \le s_0$ by (1.14). Recalling (3.5), for every $N \in \mathbb{N}$ we get

$$\begin{aligned} \left| \frac{1}{N}\mathbb{E}_\delta\big[\log Z^{[a,b]}_{N,\omega,\beta,h}\big] - \frac{1}{N}\log \mathrm{P}_N(a \le \overline{\sigma}_N \le b) \right| &\le s_0 \sqrt{b}\, \mathbb{E}_\delta \left[\sqrt{\frac{1}{N}\sum_{n=1}^{N} (h + \beta\omega_n)^2} \right] \\ &\le s_0 \sqrt{b} \sqrt{\mathbb{E}_\delta\big[(h + \beta\omega_1)^2\big]}, \end{aligned} \tag{3.16}$$

where we use Jensen. Note that the right-hand side is a finite constant. If $|a_N - b_N| \le c$ for all $N \in \mathbb{N}$, then $|\limsup_N a_N - \limsup_N b_N| \le c$, and so

$$\left| \mathrm{F}^{[a,b]}(\beta, h; \delta) - \bigg(\limsup_{N\to\infty} \frac{1}{N} \log \mathrm{P}_N(a \le \overline{\sigma}_N \le b) \bigg) \right| \le s_0 \sqrt{b} \sqrt{\mathbb{E}_\delta\big[(h + \beta\omega_1)^2\big]}. \tag{3.17}$$

Taking $[a, b] = [-\varepsilon, \varepsilon]$ and letting $\varepsilon \downarrow 0$, we get (3.14) from (3.8) .

2. Next we consider the case $x \in (0, s_0]$. Roughly speaking, the strategy of the proof is to show that the derivatives of the free energy with respect to δ and to h are comparable. Unless otherwise specified, we work with generic values of the parameters in the admissible range $\beta \ge 0$, $h \in \mathbb{R}$ and $\delta \in (-t_0, t_0)$. Henceforth we fix $0 < a < b < \infty$. Recalling (3.5) and (3.6), we see that the derivative with respect to h of the (restricted) finite-volume free energy $\mathrm{F}^{[a,b]}_N(\beta, h; \delta)$ can be expressed as

$$\frac{\partial}{\partial h}\mathrm{F}^{[a,b]}_N(\beta, h; \delta) = \frac{1}{N}\mathbb{E}_\delta\left[\frac{\partial}{\partial h} \log Z^{[a,b]}_{N,\omega,\beta,h}\right] = \mathbb{E}_\delta\big[\, \mathrm{E}^{[a,b]}_{N,\omega,\beta,h}\, [\overline{\sigma}_N]\big]. \tag{3.18}$$

3. The derivative with respect to δ requires some further estimates. Recalling (3.2)-(3.3), we have

$$\frac{\partial}{\partial\delta}\mathrm{F}_N^{[a,b]}(\beta,h;\delta) = \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}_\delta\Big[(\omega_n - m_\delta)\,\log Z^{[a,b]}_{N,\omega,\beta,h}\Big], \tag{3.19}$$

where $m_\delta := \mathbb{E}_\delta(\omega_n) = (\log \mathrm{M})'(\delta)$ by (2.9). Subtracting a centering term with zero mean, we get

$$\begin{aligned}\frac{\partial}{\partial\delta}\mathrm{F}_N^{[a,b]}(\beta,h;\delta) &= \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}_\delta\Big[(\omega_n - m_\delta)\Big(\log Z^{[a,b]}_{N,\omega,\beta,h} - \log Z^{[a,b]}_{N,\omega,\beta,h}|_{\omega_n=m_\delta}\Big)\Big] \\ &= \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}_\delta\bigg[(\omega_n - m_\delta)\int_{m_\delta}^{\omega_n}\bigg(\frac{\partial}{\partial\omega_n}\log Z^{[a,b]}_{N,\omega,\beta,h}\bigg)\bigg|_{\omega_n=y}\mathrm{d}y\bigg],\end{aligned} \tag{3.20}$$

where we agree that $\int_a^b(\ldots) := -\int_b^a(\ldots)$ when $a > b$. Abbreviate

$$f_n(\omega,y) := \frac{1}{\beta}\bigg(\frac{\partial}{\partial\omega_n}\log Z^{[a,b]}_{N,\omega,\beta,h}\bigg)\bigg|_{\omega_n=y} = \mathrm{E}^{[a,b]}_{N,\omega,\beta,h}\,|_{\omega_n=y}[\sigma_n], \tag{3.21}$$

where the second equality follows easily from (3.5) via (3.6). Note that $f_n(\omega,y)$ depends on the ω_i's for $i \neq n$, not on ω_n. Therefore (3.20) can be rewritten as

$$\frac{\partial}{\partial\delta}\mathrm{F}_N^{[a,b]}(\beta,h;\delta) = \frac{\beta}{N}\sum_{n=1}^{N}\mathbb{E}_\delta\Big[(\omega_n - m_\delta)^2\,\frac{1}{\omega_n - m_\delta}\int_{m_\delta}^{\omega_n} f_n(\omega,y)\,\mathrm{d}y\Big]. \tag{3.22}$$

4. By (3.21), the integral average in (3.22) should be close to $\mathrm{E}^{[a,b]}_{N,\omega,\beta,h}[\sigma_n]$. If we could factorize the expectation over $\mathbb{E}_\delta$, then the right-hand side in (3.22) would become $\approx \beta\,\mathbb{V}\mathrm{ar}_\delta(\omega_1)\,\mathbb{E}_\delta\big[\,\mathrm{E}^{[a,b]}_{N,\omega,\beta,h}[\overline{\sigma}_N]\big]$. Recalling (3.18), we see that this is precisely what we want, because $\mathbb{V}\mathrm{ar}_\delta(\omega_1) \approx 1$ for δ small. In order to turn these arguments into a proof, we need to estimate the dependence of $f_n(\omega,y)$ on y. To that end we note that

$$\begin{aligned}\frac{\partial}{\partial\omega_n}f_n(\omega,\omega_n) &= \frac{1}{\beta}\frac{\partial^2}{\partial\omega_n^2}\log Z^{[a,b]}_{N,\omega,\beta,h} \;=\; \beta\ \mathrm{Var}^{[a,b]}_{N,\omega,\beta,h}[\sigma_n] \\ &\le \beta\ \mathrm{E}^{[a,b]}_{N,\omega,\beta,h}[\sigma_n^2] \le s_0\,\beta\ \mathrm{E}^{[a,b]}_{N,\omega,\beta,h}[\sigma_n] = s_0\,\beta\,f_n(\omega,\omega_n)\end{aligned} \tag{3.23}$$

because $0 \le \sigma_n \le s_0$, by (1.14). Therefore

$$\frac{\partial}{\partial y}f_n(\omega,y) \ge 0, \qquad \frac{\partial}{\partial y}\big(e^{-s_0\,\beta\,y}\,f_n(\omega,y)\big) \le 0, \tag{3.24}$$

and integrating these relations we get

$$e^{-s_0\beta(y-y')^-}\,f_n(\omega,y') \le f_n(\omega,y) \le e^{s_0\beta(y-y')^+}\,f_n(\omega,y') \qquad \forall\, y,y' \in \mathbb{R}. \tag{3.25}$$

Introducing the function

$$g(x) := \begin{cases}\dfrac{e^x - 1}{x} & \text{if } x \neq 0,\\ 1 & \text{if } x = 0,\end{cases} \tag{3.26}$$

taking $y' = m_\delta$ in (3.25) and integrating over y, we easily obtain the bounds

$$\begin{aligned}&g\big(-\beta s_0(\omega_n - m_\delta)^-\big)\,f_n(\omega,m_\delta) \\ &\qquad \le \frac{1}{\omega_n - m_\delta}\int_{m_\delta}^{\omega_n} f_n(\omega,y)\,\mathrm{d}y \le g\big(\beta s_0(\omega_n - m_\delta)^+\big)\,f_n(\omega,m_\delta).\end{aligned} \tag{3.27}$$

5. Before inserting this estimate into (3.22), let us pause for a brief integrability interlude. The random variable $g(-\beta s_0(\omega_n - m_\delta)^-)$ is bounded, so there is no integrability concern. On the other hand, the random variable $g(\beta s_0(\omega_n - m_\delta)^+)$ is unbounded and a little care is required. Note that

$$g(\beta s_0(\omega_n - m_\delta)^+) \le A + B\, e^{\beta s_0 \omega_n} \tag{3.28}$$

for $A, B > 0$, and that $\mathbb{E}_\delta(e^{t\omega_1}) < \infty$ for $t + \delta \in (-t_0, +t_0)$, by (1.3) and (1.4). Therefore, when we integrate $g(\beta s_0(\omega_n - m_\delta)^+)$ (possibly times a polynomial of ω_n) over $\mathbb{P}_\delta$, to have a finite outcome we need to ensure that $\beta s_0 + \delta \in (-t_0, +t_0)$. This is simply achieved through the restrictions $\delta \in (-\varepsilon_0, \varepsilon_0)$ and $\beta \in [0, \varepsilon_0)$, where $\varepsilon_0 := \min\{\frac{t_0}{2}, \frac{t_0}{2s_0}\}$, as in the statement of Theorem 1.8. We make these restrictions henceforth.

6. Let us now substitute the estimate (3.27) into (3.22). Since $f_n(\omega, m_\delta)$ does not depend on ω_n, the expectation over $\mathbb{E}_\delta$ factorizes and we obtain

$$\begin{aligned} \mathbb{E}_\delta\Big[(\omega_1 - m_\delta)^2\, g\big(-\beta s_0(\omega_1 - m_\delta)^-\big)\Big] & \left(\frac{\beta}{N}\sum_{n=1}^N \mathbb{E}_\delta\Big[f_n(\omega, m_\delta)\Big]\right) \\ &\le \frac{\partial}{\partial\delta}\mathrm{F}_N^{[a,b]}(\beta, h; \delta) \\ &\le \mathbb{E}_\delta\Big[(\omega_1 - m_\delta)^2\, g\big(\beta s_0(\omega_1 - m_\delta)^+\big)\Big] \left(\frac{\beta}{N}\sum_{n=1}^N \mathbb{E}_\delta\Big[f_n(\omega, m_\delta)\Big]\right). \end{aligned} \tag{3.29}$$

We next want to replace $f_n(\omega, m_\delta)$ by $f_n(\omega, \omega_n) = \mathrm{E}_{N,\omega,\beta,h}^{[a,b]}\big[\sigma_n\big]$ (recall (3.21)). To this end, we again apply (3.25), this time with $y = \omega_n$ and $y' = m_\delta$. Since $f_n(\omega, m_\delta)$ does not depend on ω_n, we have

$$\frac{\mathbb{E}_\delta\big[f_n(\omega, \omega_n)\big]}{\mathbb{E}_\delta\big[e^{s_0\beta(\omega_1 - m_\delta)^+}\big]} \le \mathbb{E}_\delta\Big[f_n(\omega, m_\delta)\Big] \le \frac{\mathbb{E}_\delta\big[f_n(\omega, \omega_n)\big]}{\mathbb{E}_\delta\big[e^{-s_0\beta(\omega_1 - m_\delta)^-}\big]}. \tag{3.30}$$

We can now introduce the constants

$$\begin{aligned} c^+_{\beta,\delta} &:= \frac{\mathbb{E}_\delta\big[(\omega_1 - m_\delta)^2\, g\big(\beta s_0(\omega_1 - m_\delta)^+\big)\big]}{\mathbb{E}_\delta\big[e^{-s_0\beta(\omega_1 - m_\delta)^-}\big]}, \\ c^-_{\beta,\delta} &:= \frac{\mathbb{E}_\delta\big[(\omega_1 - m_\delta)^2\, g\big(-\beta s_0(\omega_1 - m_\delta)^-\big)\big]}{\mathbb{E}_\delta\big[e^{s_0\beta(\omega_1 - m_\delta)^+}\big]}, \end{aligned} \tag{3.31}$$

and note that $0 < c^-_{\beta,\delta} \le c^+_{\beta,\delta} < \infty$ for all $\delta \in (-\varepsilon_0, \varepsilon_0)$ and $\beta \in [0, \varepsilon_0)$. We have already observed that $f_n(\omega, \omega_n) = \mathrm{E}_{N,\omega,\beta,h}^{[a,b]}\big[\sigma_n\big]$ by (3.21), and so from (3.29)-(3.30) we obtain the following estimate: for every $\beta \in [0, \varepsilon_0)$, $h \in \mathbb{R}$, $\delta \in (-\varepsilon_0, \varepsilon_0)$ and $0 < a < b < \infty$

$$c^-_{\beta,\delta}\, \beta\, \mathbb{E}_\delta\big[\,\mathrm{E}_{N,\omega,\beta,h}^{[a,b]}\big[\overline{\sigma}_N\big]\big] \le \frac{\partial}{\partial\delta}\mathrm{F}_N^{[a,b]}(\beta, h; \delta) \le c^+_{\beta,\delta}\, \beta\, \mathbb{E}_\delta\big[\,\mathrm{E}_{N,\omega,\beta,h}^{[a,b]}\big[\overline{\sigma}_N\big]\big]. \tag{3.32}$$

Note the analogy with the expression in (3.18) for $\frac{\partial}{\partial h}\mathrm{F}_N^{[a,b]}(\beta, h; \delta)$.

7. We are close to the final conclusion. Since by (3.6) we have $a \le \mathrm{E}_{N,\omega,\beta,h}^{[a,b]}\big[\overline{\sigma}_N\big] \le b$, it follows from (3.32) that, for every $\delta \in [0, \varepsilon_0)$

$$C^-_{\beta,\delta}\, \beta\, a\, \delta \le \mathrm{F}^{[a,b]}(\beta, h; \delta) - \mathrm{F}^{[a,b]}(\beta, h; 0) \le C^+_{\beta,\delta}\, \beta\, b\, \delta, \tag{3.33}$$

where we set

$$C^{\pm}_{\beta,\delta} := \begin{cases} \frac{1}{\delta}\int_0^{\delta} c^{\pm}_{\beta,\delta'}\,\mathrm{d}\delta' & \text{if } \delta \in (-\varepsilon_0,\varepsilon_0)\setminus\{0\},\\ c^{\pm}_{\beta,0} & \text{if } \delta = 0. \end{cases} \tag{3.34}$$

Analogously to (3.33), from (3.18) we obtain, for every $\xi \geq 0$,

$$a\,\xi \leq \mathrm{F}^{[a,b]}(\beta, h+\xi; 0) - \mathrm{F}^{[a,b]}(\beta, h; 0) \leq b\,\xi. \tag{3.35}$$

Choosing $\xi = C^{+}_{\beta,\delta}\frac{b}{a}\beta\delta$ and $\xi = C^{-}_{\beta,\delta}\frac{a}{b}\beta\delta$, respectively, and combining (3.33)-(3.35), we finally get the following relation, which holds for all $\beta,\delta \in [0,\varepsilon_0)$, $h \in \mathbb{R}$ and $0 < a < b < \infty$:

$$\mathrm{F}^{[a,b]}\big(\beta, h + C^{-}_{\beta,\delta}\tfrac{a}{b}\beta\delta; 0\big) \leq \mathrm{F}^{[a,b]}(\beta, h|\delta) \leq \mathrm{F}^{[a,b]}\big(\beta, h + C^{+}_{\beta,\delta}\tfrac{b}{a}\beta\delta; 0\big). \tag{3.36}$$

Next, fix any $x > 0$ and $\eta > 0$. If $a_n \uparrow x$ and $b_n \downarrow x$, then $a_n/b_n \geq 1-\eta$ and $b_n/a_n \leq 1+\eta$ for large n. Since $h \mapsto \mathrm{F}^{[a,b]}(\beta,h;\delta)$ is non-decreasing (recall (3.18) and the fact that $\overline{\sigma}_N \geq 0$ by (1.14)), for n large enough we have

$$\mathrm{F}^{[a_n,b_n]}\big(\beta, h + C^{-}_{\beta,\delta}(1-\eta)\beta\delta; 0\big) \leq \mathrm{F}^{[a_n,b_n]}(\beta, h;\delta) \leq \mathrm{F}^{[a_n,b_n]}\big(\beta, h + C^{+}_{\beta,\delta}(1+\eta)\beta\delta; 0\big). \tag{3.37}$$

Recalling (3.8) and (3.9), we can let $n \to \infty$ to get that, for every $x > 0$,

$$\mathrm{F}^{\{x\}}\big(\beta, h + C^{-}_{\beta,\delta}(1-\eta)\beta\delta; 0\big) \leq \mathrm{F}^{\{x\}}(\beta, h;\delta) \leq \mathrm{F}^{\{x\}}\big(\beta, h + C^{+}_{\beta,\delta}(1+\eta)\beta\delta; 0\big). \tag{3.38}$$

This relation also holds for $x = 0$ because $\mathrm{F}^{\{0\}}(\beta,h;\delta)$ is a constant, as we showed in (3.14). Taking the supremum over $x \in [0, s_0]$, we have therefore shown that, for all $\beta,\delta \in [0,\varepsilon_0)$ and $h \in \mathbb{R}$,

$$\mathrm{F}\big(\beta, h + C^{-}_{\beta,\delta}(1-\eta)\beta\delta; 0\big) \leq \mathrm{F}(\beta, h;\delta) \leq \mathrm{F}\big(\beta, h + C^{+}_{\beta,\delta}(1+\eta)\beta\delta; 0\big). \tag{3.39}$$

Since $h \mapsto \mathrm{F}^{[a,b]}(\beta,h;\delta)$ is convex and finite, and hence continuous, we can let $\eta \downarrow 0$ to obtain (1.15) for $\delta \in [0,\varepsilon_0)$.

8. The case $\delta \in (-\varepsilon_0, 0]$ is analogous. The inequality in (3.33) is replaced by

$$C^{-}_{\beta,\delta}\,\beta\,a\,(-\delta) \leq \mathrm{F}^{[a,b]}(\beta,h;0) \;-\; \mathrm{F}^{[a,b]}(\beta,h;\delta) \leq C^{+}_{\beta,\delta}\,\beta\,b\,(-\delta), \tag{3.40}$$

while (3.35) for $\xi \leq 0$ becomes

$$a\,(-\xi) \leq \mathrm{F}^{[a,b]}(\beta,h;0) - \mathrm{F}^{[a,b]}(\beta,h+\xi;0) \leq b\,(-\xi). \tag{3.41}$$

Choosing $\xi = C^{+}_{\beta,\delta}\frac{b}{a}\beta\delta$ and $\xi = C^{-}_{\beta,\delta}\frac{a}{b}\beta\delta$, respectively, we get

$$\mathrm{F}^{[a,b]}\big(\beta, h + C^{+}_{\beta,\delta}\tfrac{b}{a}\beta\delta; 0\big) \leq \mathrm{F}^{[a,b]}(\beta, h;\delta) \leq \mathrm{F}^{[a,b]}\big(\beta, h + C^{-}_{\beta,\delta}\tfrac{a}{b}\beta\delta; 0\big). \tag{3.42}$$

Taking $a \uparrow x$, $b \downarrow x$ and afterwards taking the supremum over $x \in [0, s_0]$, we get (1.15) for $\delta \in (-\varepsilon_0, 0]$.

9. Finally, by (3.34), we have $0 < C^{-}_{\beta,\delta} \leq C^{+}_{\beta,\delta} < \infty$ for all $\beta \in [0,\varepsilon_0)$ and $\delta \in (-\varepsilon_0,\varepsilon_0)$. By dominated convergence, $(\beta,\delta) \mapsto c^{\pm}_{\beta,\delta}$ are continuous on $[0,\varepsilon_0)\times(-\varepsilon_0,\varepsilon_0)$, and hence also $(\beta,\delta) \mapsto C^{\pm}_{\beta,\delta}$ is continuous. Since $C^{\pm}_{0,0} = \mathbb{V}\mathrm{ar}(\omega_1) = 1$, the proof is complete. □

4. Smoothing with respect to a shift: proof of Theorem 1.9

Equations (1.13) and (1.14) imply that $h \mapsto \mathrm{F}(\beta, h; \delta)$ is non-decreasing. Since $\mathrm{F}(\beta, h; \delta) \geq 0$ under Assumption 1.2, by (1.5), if $\mathrm{F}(\bar\beta, \bar h; 0) = 0$, then $\mathrm{F}(\bar\beta, \bar h + t; 0) = 0$ for all $t \leq 0$, and (1.17) is trivially satisfied. Henceforth we assume $t > 0$.

Recalling the statement of Theorem 1.8, we set $F_\beta(\delta) := C^-_{\beta,\delta}\, \delta$. This is a continuous and strictly increasing function of δ, with $F_\beta(0) = 0$, and hence it maps the open interval $(0, \varepsilon_0)$ into $(0, \varepsilon_0')$, for some $\varepsilon_0' > 0$. Applying the first inequality in (1.15) for $t \in (0, \bar\beta \varepsilon_0')$, we can write

$$\mathrm{F}(\bar\beta, \bar h + t; 0) = \mathrm{F}\big(\bar\beta, \bar h + \bar\beta F_{\bar\beta}(F^{-1}_{\bar\beta}(\tfrac{t}{\bar\beta})); 0\big) \leq \mathrm{F}\big(\bar\beta, \bar h; F^{-1}_{\bar\beta}(\tfrac{t}{\bar\beta})\big). \tag{4.1}$$

Applying (1.9), we obtain

$$\mathrm{F}(\bar\beta, \bar h + t; 0) \leq \frac{\gamma}{2\bar\beta^2}\, A_{\bar\beta, \frac{t}{\bar\beta}}\, t^2, \tag{4.2}$$

where

$$A_{\beta,\delta} := B_{F^{-1}_\beta(\delta)} \left(\frac{F^{-1}_\beta(\delta)}{\delta} \right)^2. \tag{4.3}$$

It follows from (1.16) that

$$\lim_{(\beta,\delta)\to(0,0)} \frac{F^{-1}_\beta(\delta)}{\delta} = 1. \tag{4.4}$$

Since $\lim_{\delta\to 0} B_\delta = 1$, we obtain $\lim_{(\beta,\delta)\to(0,0)} A_{\beta,\delta} = 1$. □

References

[1] K. Alexander, The effect of disorder on polymer depinning transitions, Commun. Math. Phys. 279 (2008), 117–146.

[2] Q. Berger, F. Caravenna, J. Poisat, R. Sun, N. Zygouras, The critical curve of the random pinning and copolymer models at weak coupling, arXiv.org:1301.5308 [math.PR].

[3] T. Bodineau, G. Giacomin, On the localization transition of random copolymers near selective interfaces, J. Stat. Phys. 117 (2004) 801–818.

[4] F. Caravenna, F. den Hollander, work in progress.

[5] G. Giacomin, *Random Polymer Models*, Imperial College Press, 2007.

[6] G. Giacomin, *Disorder and Critical Phenomena Through Basic Probability Models*, École d'Été de Probabilités de Saint-Flour XL–2010, Lecture Notes in Mathematics 2025, Springer, 2011.

[7] G. Giacomin, F.L. Toninelli, Smoothing effect of quenched disorder on polymer depinning transitions, Commun. Math. Phys. 266 (2006) 1–16.

[8] G. Giacomin, F.L. Toninelli, Smoothing of depinning transitions for directed polymers with quenched disorder, Phys. Rev. Lett. 96 (2006) 070602.

[9] A.B. Harris, Effects of random defects on the critical behaviour of Ising models, J. Phys. C 7 (1974) 1671–1692.

[10] F. den Hollander, *Random Polymers*, École d'Été de Probabilités de Saint-Flour XXXVII–2007, Lecture Notes in Mathematics 1974, Springer, 2009.

[11] C. Monthus, On the localization of random heteropolymers at the interface between two selective solvents, Eur. Phys. J. B13 (2000) 111–130.