SHARPNESS VERSUS ROBUSTNESS OF THE PERCOLATION TRANSITION IN 2D CONTACT PROCESSES

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Abstract. We study versions of the contact process with two or three states, and with infections occurring at a rate depending on the infection density. Motivated by a model for vegetation patterns in arid landscapes, we focus on percolation under invariant measures of such processes. We prove that the percolation transition is *sharp* (for one of the three-state processes we consider this requires a reasonable assumption). This is shown to contradict a form of 'robust critical behaviour' (with power law cluster size distribution for a range of parameter values) suggested in [13].

1. Introduction and background

Percolative systems with weak dependencies, such as the contact process and its variants, are in the spotlight of recent mathematical research. The present article studies versions of the two-dimensional contact process that are motivated by models for vegetation patterns in arid landscapes, as put forward by biologists and agricultural researchers [13]. The central question we address is whether or not the percolation transition for these modified contact processes is *sharp*.

The models and questions we consider are defined precisely in the following subsections. Briefly speaking, however, we consider two main modifications of the 'standard' two-dimensional contact process. Firstly, rather than two states 0 and 1, we allow three states: -1, 0 and 1. Secondly, the transition rates are allowed to vary with the overall density of 1's in the process itself. Contact processes with three states have been considered by several authors before, e.g. [22, 23]. We consider two different types of 3-state contact processes, one of which is closely related to the process in [23], the other of which has not previously appeared in the mathematical literature. The second modification has,

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to the best of our knowledge, not been considered previously in the mathematical literature; we call such processes 'density-driven' (see Definition 1.1) and prove their existence in Section 6.

Our main focus is the question of *percolation* in such processes: whether or not, under an invariant distribution of the process, there can be an unbounded connected set of 1's. For certain parameter values an unbounded connected set of 1's occurs with positive probability and for others not. As the parameters are varied, one obtains in this sense a phase transition which we refer to as the *percolation transition*. In [13] it is suggested, based on numerical simulation, that the type of model we consider may exhibit a form of 'robust critical behaviour', different from the usual 'sharp' phase transition in standard percolation models. We critically discuss this suggestion, based on rigorous results about the percolation transition. Our main results, on sharpness of the transition and lack of 'robustness', are stated in Theorems 2.2, 2.3, 2.4, 2.7 and 2.9 below.

The contact process is one of the most-studied interacting particle systems, see e.g. [18] and references therein. Several multi-type variants have been studied; most of them have been proposed as models in theoretical biology, and focus has typically been on the survival versus extinction of species. See, for example, Cox–Schinazi [7], Durrett–Neuhauser [8], Durrett–Swindle [9], Konno–Schinazi–Tanemura [15], Kuczek [16], Neuhauser [22]. The question of percolation under invariant distributions of the contact process was first studied by Liggett and Steif [19], and a sharpness result for percolation under such distributions was first proved by van den Berg [2].

We begin by describing the type of model we consider in more detail.

1.1. Contact processes with two or three states.

The ordinary contact process on $\mathbb{Z}^d$ is a Markov process with state space $\{0,1\}^{\mathbb{Z}^d}$. Elements $x \in \mathbb{Z}^d$ are sometimes called 'sites' (and sometimes interpreted as 'individuals'). An element of $\{0,1\}^{\mathbb{Z}^d}$ is typically denoted by $\eta = (\eta_x : x \in \mathbb{Z}^d)$ or ξ, and those $x \in \mathbb{Z}^d$ for which $\eta_x = 1$ are typically called 'infected'. Sometimes η is identified with the collection of infected sites.

The dynamics of the contact process are as follows. Infected individuals recover at rate κ, independently of each other (often κ is set to 1). Alternatively, a healthy site can become infected by an infected neighbour at rate λ. This occurs independently for different sites and independently of the recoveries. For $\xi \in \{0,1\}^{\mathbb{Z}^d}$ we write P^ξ for the probability measure governing the contact process $X = (X_x(t) : x \in \mathbb{Z}^d, t \geq 0)$ with initial state ξ. If ν is a probability measure on $\{0,1\}^{\mathbb{Z}^d}$ we write P^ν for the probability measure governing the contact process X with initial state sampled from ν.

The main facts about the contact process are the following. Firstly, for each $\kappa > 0$ there is a critical value $\lambda_c = \lambda_c(\kappa) \in (0, \infty)$ such that if $\lambda \le \lambda_c$ then the process 'dies out'. This means that, for each initial state ξ and each $x \in \mathbb{Z}^d$, we have that $X_x(t) \to 0$ almost surely under P^ξ. In this regime the only stationary distribution of the process is the point mass $\delta_\varnothing$ on the configuration consisting of all zeros. For any $\lambda \ge 0$ there exists an *upper invariant measure* $\overline{\nu}$ on $\{0,1\}^{\mathbb{Z}^d}$ which can be obtained as the limiting distribution when initally *all* sites are infected (this follows from standard monotonicity arguments [18]). In view of the foregoing, for $\lambda \le \lambda_c$ we have $\overline{\nu} = \delta_\varnothing$. On the other hand, if $\lambda > \lambda_c$ then there is positive chance that infection is transmitted indefinitely, and hence $\overline{\nu} \ne \delta_\varnothing$. In this regime, there is more than one invariant distribution, each invariant distribution being a convex combination of $\delta_\varnothing$ and $\overline{\nu} \ne \delta_\varnothing$. (The latter result is known as *complete convergence.*)

We now describe the three variations of the ordinary contact process on which this paper focuses.

1.1.1. *The 2-state contact process.* Apart from recoveries and neighbour-infections, as described above, we also include 'spontaneous infections' at rate $h \ge 0$. This means that any site switches from state 0 to 1 at rate h, in addition to the neighbour-rate given above. As soon as $h > 0$ the contact process has a unique stationary distribution, which we also denote by $\overline{\nu}$. As proved in [1], $\overline{\nu}$ is continuous at $h = 0$ in the sense that the limit of $\overline{\nu}$ as $h \downarrow 0$ equals the upper invariant measure $\overline{\nu}$ as defined previously. We refer to the contact process described above as the '2-state contact process' since each site can be in one of two states: 0 or 1. By 'ordinary contact process' we refer to the special case $h = 0$.

1.1.2. *The 3-state contact process.* We also consider the following 3-state contact process. The state space is now $\{-1,0,1\}^{\mathbb{Z}^d}$, and the parameters are $\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}$ and $h, \tilde{h}$. The state of a site may change spontaneously from 1 to 0, from 0 to -1, from -1 to 0 or from 0 to 1, at rates $\kappa, \tilde{\kappa}, \tilde{h}, h$ respectively. Alternatively, a site which is in state -1 or 0 may change to state 0 or 1, respectively, at a rate proportional to the number of nearest neighbours which are in state 1, the constants of proportionality being given by $\tilde{\lambda}$ and λ, respectively. These transition rates are informally summarized in the following table:

Spontaneous rates	Neighbour rates
$1 \to 0$ rate κ	$0 \to 1$ rate $\lambda\cdot\#$(type 1 nbrs)
$0 \to -1$ rate $\tilde{\kappa}$	$-1 \to 0$ rate $\tilde{\lambda}\cdot\#$(type 1 nbrs)
$0 \to 1$ rate h	
$-1 \to 0$ rate $\tilde{h}$	

If $\tilde{\kappa} = \tilde{\lambda} = \tilde{h} = 0$ we thus essentially recover the 2-state contact process.

This 3-state process is inspired by a model proposed to study the desertification of arid regions in [13]. The intuition is that 0 represents a 'vacant' patch of 'good' soil, 1 represents a vegetated patch, and -1 represents a 'bad' patch of soil which must first be improved (to state 0) before vegetation can grow there. Type 1 patches can influence the states of neighbouring patches either by spreading seeds ($0 \to 1$) or improving the soil ($-1 \to 0$), for example by binding the soil better with roots. Much less is known about the 3-state process than about the 2-state process. To a large extent this is because the notion of 'path' along which infection spreads is no longer sufficient. In particular, we do not know if there is a unique stationary distribution if all the parameters are strictly positive, as is the case for the 2-state process. However, most of our results on the 3-state process are conditional on the assumption that there is a unique stationary distribution $\overline{\nu}$ in this situation. A more precise formulation of the assumption is stated in Assumption 4.5.

1.1.3. *Contact process in a random environment.* Finally, we consider also the following contact process which also has the three states -1, 0 and 1. This process is essentially a version of the 2-state process in a 'random environment' (the state -1 in taking the role of the 'environment') and is closely related to a process studied by Remenik [23]. It is defined as follows. The parameters are $\kappa, \kappa^\star, \lambda, h, \tilde{h}$, and the transitions are summarized in the following table.

Spontaneous rates	Neighbour rates
$1 \to 0$ rate κ	$0 \to 1$ rate $\lambda \cdot \#(\text{type 1 nbrs})$
$(0 \text{ or } 1) \to -1$ rate $\kappa^\star$	
$0 \to 1$ rate h	
$-1 \to 0$ rate $\tilde{h}$	

Thus a site changes state to -1 at rate $\kappa^\star$ *regardless of* the current state, and transitions out of the state -1 occur at a rate independent of the number of type 1 neighbours. We call this process the RE-process, to emphasise the interpretation in terms of a random environment. Note that, although it is a process with three states, we reserve the name '3-state process' for the previous process.

In the case $h = 0$ this process is the one studied in [23]. Remenik puts it forward as a model for the spread of vegetation, with a slightly different interpretation of transitions to state -1 than in the 3-state process. In Remenik's model the interpretation is that "if a site becomes uninhabitable, the particles living there will soon die" (quote from [23]). In the 3-state process transitions to -1 only occur for uninhabited sites (in state 0) with the motivation that they "may undergo further degradation, for example, by processes such as erosion and soil-crust formation" (quote from [13]).

The RE-process is considerably easier to study than the 3-state process. Indeed, for the case $h = 0$, Remenik interpreted the model as a hidden Markov chain and, building on results by Broman [6], proved strong results such as complete convergence. For the case $h > 0$, exponential convergence to a unique invariant distribution is stated in Lemma 4.4 below.

1.2. **Density-driven contact processes.** It is straightforward to generalize the definitions of the contact processes we consider to allow time-varying infection rates $\lambda(t)$ and $h(t)$ (see Section 4.1 for more on this). Furthermore, in the context of vegetation spread it seems natural to allow the rates governing transitions from state 0 to 1 (λ and h) to depend on the overall density of 1's in the process itself. For example, one may imagine that seeds can be blown over large distances to spread vegetation, and that whether a seed which has landed on a vacant piece of soil indeed becomes a plant may depend on the overall competition of the other plants. Indeed the model proposed in [13] includes such a mechanism. The model there is defined in discrete time and in a finite region, and it is not immediately obvious that it is possible to define such a process in continuous time and on the infinite graph $\mathbb{Z}^d$. However, in Section 6 we prove the existence of the following class of processes.

Let $X(t)$ be a translation invariant 2- or 3-state process or RE-process, and write $\rho(t) = P(X_0(t) = 1)$ for the *density* of the process.

DEFINITION 1.1 (DDCP). *Let the functions* $\Lambda, H\colon [0,1] \to [0,\infty)$ *be given, and let* $X(t)$ *be a translation-invariant 2-state contact process with parameters* κ, $\lambda(\cdot)$ *and* $h(\cdot)$, *or a translation-invariant 3-state contact process with parameters* κ, $\tilde{\kappa}$, $\tilde{\lambda}$, $\tilde{h}$ *and* $\lambda(\cdot)$, $h(\cdot)$, *or a translation-invariant* RE*-process with parameters* κ, $\kappa^\star$, $\tilde{h}$, *and* $\lambda(\cdot)$, $h(\cdot)$. *This process is called* a density-driven contact process specified by Λ and H *if* λ, h *satisfy* $\lambda(t) = \Lambda(\rho(t))$ *and* $h(t) = H(\rho(t))$ *for all* $t \geq 0$.

We use the abbreviation DDCP for 'density-driven contact process'. In words, a DDCP has infection rates $\lambda(t)$, $h(t)$ which vary *instantaneously* with the density $\rho(t)$ *of the process* $X(t)$ *itself.* Thus the process constantly updates its infection rates based on the current prevalence of 1's.

1.3. **Outline.** In Section 2 we state our main results, which concern on the one hand 'sharpness' and on the other 'lack of robustness'. In Section 3 we prove our results on lack of robustness, deferring the proofs of our sharpness results to Section 5. In Section 4 we describe methods and results from the literature which are needed for the proofs of our main sharpness results. Here we also formulate our standing assumption on the 3-state process, Assumption 4.5. Section 5 contains the proofs of our sharpness results (Theorems 2.4, 2.7, and 2.9). In the final Section 6, we prove in general the existence of density-driven processes.

2. Main results

2.1. **Percolation and the question of robustness.**

For any configuration $\eta \in \{0,1\}^{\mathbb{Z}^d}$ or $\eta \in \{-1,0,1\}^{\mathbb{Z}^d}$, consider the subgraph induced by sites in state 1 and the nearest-neighbor relation. Let C_0 denote the connected component of the subgraph of 1's containing the origin 0, and write $|C_0|$ for the number of sites in C_0. If ν is a probability measure on $\{0,1\}^{\mathbb{Z}^d}$ or $\{-1,0,1\}^{\mathbb{Z}^d}$, we say that *percolation occurs* under ν if $\nu(|C_0| = \infty) > 0$. For the rest of this section we fix $d = 2$.

A major focus of this article is to study the phenomenon of percolation when ν is an invariant measure of a 2- or 3-state or RE-contact process, possibly density-driven. Indeed, one of the main motivations is an intriguing suggestion in [13] concerning a specific version of the density-driven 3-state model (with explicitly given forms of the functions $h(t)$ and $\lambda(t)$, involving certain parameters). In our context (where the medium is the *infinite* lattice and time is continuous) that version is given by the functions in (2.1) below. The suggestion in [13] is that this model has a form of 'robust critical behaviour': that there is a non-negligible set of parameter values for which the model has an invariant measure under which the size of an occupied cluster has a power-law distribution.

As the authors of [13] remark, such behaviour is different "from classical critical systems, where power laws only occur at the transition point". Further, the authors suggest that this uncommon behaviour may be explained by strong local positive interactions. (The latter means that the transitions from -1 to 0 and the transitions from 0 to 1 are 'enhanced' by the presence of occupied sites in the neighborhood). Later in their paper they argue that an important aspect to explain their 'observed' robust critical behaviour would be that 'disturbances' (transitions to the -1 state) do not affect directly the occupied sites: they first have to change to the 0 state, which "constrains the spatial propagation of the disturbance". In later life-sciences papers the robust criticality is debated [12, 14, 20, 21].

The arguments in [13] and those in the articles mentioned above lack mathematical rigour. Our aim is to contribute by lifting the discussion to a rigorous mathematical level, and by proving mathematical theorems that are relevant for the above mentioned discussion. Our following result, Theorem 2.2, shows that in our formulation of the model in [13], criticality is rare, in a strong and well-defined sense. We also show a more general, but weaker, statement of a similar form (Theorem 2.3).

DEFINITION 2.1. *We call a distribution ν on $\{0,1\}^{\mathbb{Z}^2}$ or $\{-1,0,1\}^{\mathbb{Z}^2}$* critical *(for percolation) if $\nu(|C_0| \geq n)$ converges to zero subexponentially; that is, $\nu(|C_0| \geq n) \to 0$ as $n \to \infty$, but*

$$\liminf_{n\to\infty} \frac{-\log \nu(|C_0| \geq n)}{n} = 0.$$

Calling such a distribution ν 'critical' may be somewhat imprecise, partly as it seems to ignore the possibility of a discontinuous phase transition. However, the name is meant to capture the idea that power law cluster sizes are associated with critical behaviour.

The precise form of the DDCP corresponding to the model in [13] is given by

$$\begin{aligned} \Lambda(\rho) &= \beta\,\frac{1-\delta}{4}\,(\varepsilon - g\rho), \\ H(\rho) &= \beta\delta\,\rho\,(\varepsilon - g\rho), \end{aligned} \tag{2.1}$$

where β, ε and g are positive parameters and $\delta \in (0,1)$. This choice of functions is motivated in the Methods supplement to [13]. Briefly, β represents the seed production rate, δ the fraction of seeds that are spread over long distances, ε the establishment probability of a seed not subject to competition, and g a competitive effect due to the presence of other plants.

For the DDCP where λ and h are density-dependent and given by (2.1) we have the following result:

THEOREM 2.2 (Lack of robustness). *Let $d = 2$ and recall Definition 2.1.*

1. *Consider the 2-state* DDCP *with parameters $\kappa > 0$, and $\Lambda(\cdot)$, $H(\cdot)$ given by (2.1). For almost all κ, β, δ, ε and g, this process does not have a critical invariant measure.*
2. *Similarly, for almost all $\kappa, \kappa^\star, \tilde{h}$, β, δ, ε and g, the* RE-DDCP *with $\Lambda(\cdot)$ and $H(\cdot)$ given by (2.1) does not have a critical invariant measure.*
3. *Similarly, for almost all $\kappa, \tilde{\kappa}, \tilde{\lambda}, \tilde{h}$, β, δ, ε and g, under Assumption 4.5, the 3-state* DDCP *with $\Lambda(\cdot)$ and $H(\cdot)$ given by (2.1) does not have a critical invariant measure.*

We also have the following result, which on the one hand holds for much more general Λ, H, but on the other hand has a weaker conclusion. We say that two functions $f, g\colon [0,1] \to \mathbb{R}$ differ at most ε if $|f(r) - g(r)| < \varepsilon$ for all $r \in [0,1]$. The result is formulated and proved for the 3-state model but straightforwardly extends to the 2-state and RE-cases as well.

THEOREM 2.3. *Let Λ, H be continuous, strictly positive functions, and suppose the 3-state* DDCP *with parameters $\kappa, \tilde{\kappa}, \tilde{\lambda}$, $\tilde{h} > 0$ and*

$\Lambda, H > 0$ has a critical invariant distribution. Then, under Assumption 4.5, for every $\varepsilon > 0$ there are parameters $\kappa', \tilde{\kappa}', \tilde{\lambda}'$, $\tilde{h}'$ and Λ', H' which each differ at most ε from the original parameters, and for which the corresponding DDCP *has no critical invariant measure.*

Theorems 2.2 and 2.3 are proved in Section 3. These results cast considerable doubt on the suggestions in [13] discussed in the beginning of this section.

2.2. **Sharpness of percolation transitions.** The main step in proving Theorems 2.2 and 2.3 is to establish *sharpness results* for percolation under the invariant measures of contact processes, which we state in this section. Such results are also of independent interest. Given these sharpness results, the proofs of Theorems 2.2 and 2.3 are relatively elementary. For $x, y \in \mathbb{R}^k$ we use the notation $x \prec y$ to indicate that each coordinate of x is strictly smaller than the corresponding coordinate of y.

2-state process. Starting with the ordinary 2-state process (with $h = 0$), recall that $\overline{\nu}$ denotes the upper invariant measure and that λ_{c} denotes the critical point (see Section 1.1). For κ fixed, define $\lambda_{\mathrm{p}} := \inf\big\{\lambda \geq 0 : \overline{\nu}(|C_0| = \infty) > 0\big\}$. We see that

$$\overline{\nu}(|C_0| = \infty) \begin{cases} > 0 & \text{if } \lambda > \lambda_{\mathrm{p}}, \\ = 0 & \text{if } \lambda < \lambda_{\mathrm{p}}. \end{cases}$$

Clearly, $\lambda_{\mathrm{p}} \geq \lambda_{\mathrm{c}}$. Moreover, Liggett and Steif [19] prove that $\lambda_{\mathrm{p}} < \infty$ for the ordinary contact process in dimension $d \geq 2$. It is not known, though widely believed, that $\lambda_{\mathrm{p}} > \lambda_{\mathrm{c}}$ (see open question 2 in Section 6 of [19]). For $d = 2$ it was further proved by van den Berg [2] that the transition is *sharp* in that for each $\lambda < \lambda_{\mathrm{p}}$ there is $c > 0$ such that $\overline{\nu}(|C_0| \geq n) \leq e^{-cn}$.

This article establishes similar results for the case $h > 0$. First note that, by monotonicity, the above mentioned result that $\lambda_{\mathrm{p}} < \infty$, immediately carries over to the general 2-state model (with $h > 0$ and λ_{p} replaced by $\lambda_{\mathrm{p}}(h)$). In Section 5 we prove the following sharp-transition result.

THEOREM 2.4 (Sharpness for 2-state contact process). *Let $d = 2$ and fix $\kappa > 0$. If the parameters $\lambda, h > 0$ are such that $\overline{\nu}(|C_0| = \infty) = 0$ then for all $(\lambda', h') \prec (\lambda, h)$ there is $c > 0$ such that $\overline{\nu}(|C_0| \geq n) \leq e^{-cn}$ for all $n \geq 1$.*

Theorem 2.4 has the following consequence. For fixed $\kappa > 0$ define

$$h_{\mathrm{p}} = h_{\mathrm{p}}(\lambda) := \inf\big\{h \geq 0 : \overline{\nu}(|C_0| = \infty) > 0\big\}.$$

From comparison with Bernoulli percolation it follows that $h_{\mathrm{p}} < \infty$. It is easy to see that $h_{\mathrm{p}}(\lambda)$ is non-increasing in λ, and we have that $h_{\mathrm{p}}(\lambda) > 0$ if $\lambda < \lambda_{\mathrm{p}}$ (see Remark 2.6).

COROLLARY 2.5. *Consider the 2-state contact process with* $d = 2$.

(1) *For each fixed* λ, *if* $h < h_{\mathrm{p}}(\lambda)$ *then there is* $c > 0$ *such that* $\overline{\nu}(|C_0| \geq n) \leq e^{-cn}$ *for all* $n \geq 1$.
(2) *For all but at most countably many* $h > 0$ *we have that if* $\lambda < \lambda_{\mathrm{p}}(h)$ *then there is* $c > 0$, *such that* $\overline{\nu}(|C_0| \geq n) \leq e^{-cn}$ *for all* $n \geq 1$.

The proof appears in Section 5.3. The reason we obtain a stronger statement for the transition in h is that $h_{\mathrm{p}}(\lambda)$ is easily seen to be continuous, see (4.1). We have not been able to establish continuity of $\lambda_{\mathrm{p}}(h)$.

REMARK 2.6. One can show that $h_{\mathrm{p}}(\lambda) > 0$ if $\lambda < \lambda_{\mathrm{p}}$. Here is a very brief outline of the argument. We may assume that $\lambda \neq \lambda_{\mathrm{c}}$, and hence that the contact process dynamics converge exponentially fast to equilibrium both for $h > 0$ and $h = 0$ (see Lemma 4.4 and [2, Theorem 2.1]). Thus we can approximate $\overline{\nu}$ using a contact process realization defined in a bounded space–time region. By [2, Theorem 1.1] the probability of a vertical crossing of a large enough rectangle (of any aspect ratio) can be made arbitrarily small under such a finite-volume measure for $h = 0$. By continuity of such probabilities in h the same holds for small enough $h > 0$, and hence also under $\overline{\nu}$ for small enough $h > 0$. The result then follows from the finite-size criterion, Lemma 4.6. On the other hand, clearly $h_{\mathrm{p}} = 0$ if $\lambda > \lambda_{\mathrm{p}}$.

3-state process. For the 3-state process all our results are conditional on the uniqueness of the invariant measure $\overline{\nu}$ when all parameters are positive, as well as a certain assumption on convergence of the dynamics (Assumption 4.5). First, again note that by comparison with Bernoulli percolation it follows immediately that $\overline{\nu}(|C_0| = \infty) > 0$ provided $h, \tilde{h}$ are large enough, or $\tilde{h} > 0$ and h is large enough. In Section 5 we prove the following result analogous to Theorem 2.4.

THEOREM 2.7 (Sharpness for 3-state contact process). *Consider the 3-state contact process with* $d = 2$ *and* $\kappa, \tilde{\kappa} > 0$ *fixed. Under Assumption 4.5 we have the following. If the parameters* $\lambda, \tilde{\lambda}, h, \tilde{h} > 0$ *are such that* $\overline{\nu}(|C_0| = \infty) = 0$, *then whenever* $(\lambda', \tilde{\lambda}', h', \tilde{h}') \prec (\lambda, \tilde{\lambda}, h, \tilde{h})$, *there is* $c > 0$ *such that* $\overline{\nu}(|C_0| \geq n) \leq e^{-cn}$ *for all* $n \geq 1$.

For the 3-state process we have the following analog of Corollary 2.5. For each choice of $\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}, \tilde{h} > 0$ define h_{p} in the same way as for the 2-state model.

Corollary 2.8. *Consider the 3-state contact process with* $d = 2$. *Under Assumption 4.5, for all* $\kappa, \tilde{\kappa} > 0$ *and almost all* $\tilde{\lambda}, \tilde{h} > 0$ *the following holds: for all but countably many* $\lambda > 0$, *if* $h < h_{\mathrm{p}}(\lambda)$ *then* $\overline{\nu}(|C_0| \geq n) \leq e^{-cn}$ *for some* $c > 0$ *and all* $n \geq 1$.

re-*process.* For the re-process, uniqueness of, and exponential convergence to, the invariant distribution $\overline{\nu}$ holds whenever $h > 0$ (see Lemma 4.4). As before, $\overline{\nu}(|C_0| = \infty) > 0$ provided $h, \tilde{h}$ are large enough. We have the following.

Theorem 2.9 (Sharpness for re-process). *Consider the* re-*process with* $d = 2$ *and* $\kappa, \kappa^\star > 0$ *fixed. If the parameters* $\lambda, h, \tilde{h} > 0$ *are such that* $\overline{\nu}(|C_0| = \infty) = 0$, *then whenever* $(\lambda', h', \tilde{h}') \prec (\lambda, h, \tilde{h})$, *there is* $c > 0$ *such that* $\overline{\nu}(|C_0| \geq n) \leq e^{-cn}$ *for all* $n \geq 1$.

Corollary 2.10. *Consider the* re-*process with* $d = 2$. *For all* $\kappa, \kappa^\star > 0$ *and almost all* $\tilde{h} > 0$ *the following holds: for all but countably many* $\lambda > 0$, *if* $h < h_{\mathrm{p}}(\lambda)$ *then* $\overline{\nu}(|C_0| \geq n) \leq e^{-cn}$ *for some* $c > 0$ *and all* $n \geq 1$.

3. Proofs of nonrobustness for density-driven processes

In this section we prove Theorems 2.2 and 2.3 on lack of robustness in the model proposed in [13] assuming the sharpness results (Corollaries 2.5, 2.8 and 2.10). We begin with a discussion about the stationary distributions of ddcp.

3.1. Stationary distributions for density-driven processes.

We first consider the 2-state case. Let $\kappa > 0$ be fixed. Let $X(t), t \geq 0$, be a ddcp for the parameters κ, Λ and H. Suppose that X is stationary, i.e. the distribution of $X(t)$ is constant in t. Denote this distribution by ν. By stationarity, the occupation density $\rho(t)$ of $X(t)$ is constant, say $\rho(t) \equiv \rho$. Writing $\lambda = \Lambda(\rho)$ and $h = H(\rho)$ we thus find that ν is a stationary distribution for the contact process with constant parameters κ, λ and h.

There are two possibilities. If $h = f(\rho) = 0$ then ν is a convex combination $\nu = \theta\overline{\nu} + (1-\theta)\delta_\varnothing$, where $\overline{\nu}$ is the upper invariant measure introduced in Section 1.1. We are mainly concerned with the other case when $h = f(\rho) > 0$. Since for $h > 0$ there is only one stationary distribution $\overline{\nu}$ for the 2-state contact process, it follows that $\nu = \overline{\nu}$. Let $\overline{\rho}(\lambda, h) = \overline{\nu}(\{\eta : \eta_0 = 1\})$ denote the density of $\overline{\nu}$. It follows that λ and h satisfy the fixed point equations $\lambda = \Lambda(\overline{\rho}(\lambda, h))$ and $h = H(\overline{\rho}(\lambda, h))$, respectively. Conversely, if $h > 0$ and λ satisfy these fixed point equations, then $\overline{\nu}$ is stationary for the 2-state ddcp defined by Λ, H. We summarize these findings in the following proposition:

Proposition 3.1. *Suppose $H(\rho) > 0$ for all $\rho \in [0,1]$. Then the stationary distributions of the 2-state* DDCP *specified by Λ, H are precisely the upper invariant measures $\overline{\nu}$ for λ, h satisfying $\lambda = \Lambda(\overline{\rho}(\lambda, h))$ and $h = H(\overline{\rho}(\lambda, h))$.*

Under Assumption 4.5, very similar arguments apply to the 3-state case, and we obtain the following result, where $\overline{\rho}(\lambda, h)$ denotes the occupation density of the upper invariant measure $\overline{\nu}$ for the contact process with parameters κ, $\tilde{\kappa}$, λ, $\tilde{\lambda}$, h, $\tilde{h}$ (which under Assumption 4.5 is the unique stationary distribution).

Proposition 3.2. *Let $\kappa, \tilde{\kappa}, \tilde{\lambda}, \tilde{h} > 0$ be fixed. Suppose $\Lambda(\rho), H(\rho) > 0$ for all $\rho \in [0,1]$. Then, under Assumption 4.5, the stationary distributions of the 3-state* DDCP *specified by Λ, H are precisely the measures $\overline{\nu}$ for λ, h satisfying $\lambda = \Lambda(\overline{\rho}(\lambda, h))$ and $h = H(\overline{\rho}(\lambda, h))$.*

Similar results hold for the RE-process, which we leave to the reader to formulate explicitly.

3.2. **Proof of Theorem 2.2.** Writing $\gamma_1 = \beta(1-\delta)/4$ and $\gamma_2 = \beta\delta$, it is sufficient to prove the following claim.

Claim: For almost all κ, γ_1, γ_2, ε and g, the DDCP with $\Lambda(\rho) = \gamma_1(\varepsilon - g\rho)$ and $H(\rho) = \gamma_2\rho(\varepsilon - g\rho)$ does not have a critical invariant measure.

The proof of the claim is simpler for the 2-state process so we begin by treating that case.

2-state case. Suppose the 2-state DDCP does have a critical invariant measure, which we denote by ν. Since ν is invariant we have that the density $\rho(t) = \rho$ is constant. Since ν is critical we have $\rho > 0$ and hence $\lambda, h > 0$. As noted above ν is the upper invariant measure for the 2-state contact process with constant parameters $\lambda = \Lambda(\rho)$ and $h = H(\rho)$. Moreover, since ν is critical, h must be equal to $h_{\mathrm{p}}(\lambda)$ by Corollary 2.5. Hence the following two equations hold, where $\rho_{\mathrm{p}}(\lambda) = \rho(\lambda, h_{\mathrm{p}}(\lambda))$ denotes the density of the upper invariant measure for the 2-state contact process with parameters λ and $h = h_{\mathrm{p}}(\lambda)$:

$$\lambda = \gamma_1(\varepsilon - g\rho_{\mathrm{p}}(\lambda)), \tag{3.1}$$

$$h = h_{\mathrm{p}}(\lambda) = \gamma_2\rho_{\mathrm{p}}(\lambda)(\varepsilon - g\rho_{\mathrm{p}}(\lambda)). \tag{3.2}$$

To prove the claim, fix κ, g and γ_1. With these parameters fixed, it is clear that for each λ there is at most one ε such that (3.1) holds. Hence, for almost all ε the set $L = L(\varepsilon) = L(\varepsilon, \kappa, g, \gamma_1)$ of those λ for which (3.1) holds has Lebesgue measure 0. (Note that $\rho_{\mathrm{p}}(\lambda)$ is measurable since $\rho(\lambda, h)$ and $h_{\mathrm{p}}(\lambda)$ are measurable.) Now also fix (besides the above mentioned parameters which were already fixed) the parameter ε such that L indeed has measure 0. Note that for each λ there is at most one γ_2 such that (3.2) holds. Let $L' \subset L$ be the set of those $\lambda \in L$ for which there is indeed such a γ_2. Rearranging (3.2) we can write this γ_2 as a function of $\lambda \in L'$:

$$\gamma_2 = \frac{h_{\mathrm{p}}(\lambda)}{\rho_{\mathrm{p}}(\lambda)(\varepsilon - g\rho_{\mathrm{p}}(\lambda))}.$$

Since (3.1) also holds, we can 'eliminate' $\rho_{\mathrm{p}}(\lambda)$ from the above expression for γ_2 and get

$$\gamma_2 = \frac{h_{\mathrm{p}}(\lambda) g \gamma_1^2}{\lambda(\varepsilon\gamma_1 - \lambda)}, \quad \lambda \in L'. \tag{3.3}$$

Write the right hand side of (3.3) as a function

$$F(\lambda) := \frac{h_{\mathrm{p}}(\lambda) g \gamma_1^2}{\lambda(\varepsilon\gamma_1 - \lambda)}.$$

We want to show that $F(L')$ has measure 0. To do this, we use the fact that $h_{\mathrm{p}}(\lambda)$ is uniformly Lipschitz continuous in λ. (This is not hard to see using that 'spontaneous infection has a stronger effect that neighbour infection'; see (4.1).) Hence the numerator in the definition of the function F above is locally Lipschitz. It follows that F is locally Lipschitz outside the point $\lambda = \varepsilon\gamma_1$. Hence, since L' has measure 0, $F(L')$ also has measure 0 (any cover of L' with 'small' intervals is mapped under F to a cover of $F(L')$ with comparably small intervals).

Summarizing, for all κ, g and γ_1 we have that for almost all ε the set of those γ_2 for which the DDCP with parameters κ, g, ε, γ_1 and γ_2 has a critical invariant measure, has Lebesgue measure 0. This is (slightly stronger than) the Claim, and completes the proof of Theorem 2.2 in the 2-state case.

3-state case. The proof for the 3-state case is more complicated since we do not have sharpness in h for *all* values of the other parameters. However, by Corollary 2.8 for all $\kappa, \tilde{\kappa}$ and almost all $\tilde{\lambda}, \tilde{h}$ the set of $\lambda > 0$ for which sharpness in h fails is at most countable. We call λ 'bad' if sharpness in h fails, and we henceforth assume that $\kappa, \tilde{\kappa}, \tilde{\lambda}, \tilde{h}$ are fixed and chosen so that the set of bad λ is at most countable.

Suppose the DDCP has a critical invariant measure ν. As in the 2-state case we have that ρ, λ, h do not vary with t, and that $\rho, \lambda, h > 0$. We now consider the two cases, λ 'bad' or not. If λ is *not* bad then, as for the 2-state case, h must equal $h_{\mathrm{p}}(\lambda)$. In this case the argument is the same as for the 2-state case. The only points that need to be checked are, firstly, that ρ is measurable as a function of λ and h, and, secondly, that h_{p} still satisfies the Lipschitz condition (4.1) in λ. Both these things can be seen in the same way as for the 2-state model.

We thus consider the case when λ is bad. For a fixed bad λ, we write $\rho = \rho(h)$. We can no longer conclude that $h = h_{\mathrm{p}}(\lambda)$, but we still have the relations

$$\lambda = \gamma_1(\varepsilon - g\rho(h)), \quad \text{and} \quad h = \gamma_2\rho(h)(\varepsilon - g\rho(h)). \tag{3.4}$$

We aim to show that, for each fixed bad λ, the set of choices of the parameters $\gamma_1, \gamma_2, \varepsilon, g$ such that (3.4) holds has measure zero. This concludes the proof since a countable union of measure zero sets has measure zero.

If (3.4) holds, we may rearrange to obtain the relations

$$\rho(h) = \frac{\gamma_1}{\lambda\gamma_2}h, \quad \text{and} \quad \varepsilon = \frac{\lambda}{\gamma_1} + g\rho(h) = \frac{\lambda}{\gamma_1} + g\frac{\gamma_1}{\lambda\gamma_2}h. \tag{3.5}$$

We now fix arbitrary $\gamma_2, g > 0$. Then for almost all γ_1, the first relation in (3.5) can only hold for h in a set of measure zero. This follows from Fubini's theorem (using polar coordinates) and the fact that the set $\{(h, \rho(h)) : h > 0\}$ has two-dimensional Lebesgue measure zero. We now fix γ_1 such that the first relation in (3.5) only holds for h in a set of measure zero. It follows that the second relation in (3.5) can only hold for ε in a set of measure zero. This concludes the proof in the 3-state case.

RE-*case.* Finally, the proof for the RE-case is similar to that for the 3-state case, using Corollary 2.10 in place of Corollary 2.8. □

3.3. **Proof of Theorem 2.3.** Consider the 3-state contact process with parameters κ, $\tilde{\kappa}$, λ, $\tilde{\lambda}$, h, $\tilde{h}$, all strictly positive. Under Assumption 4.5, there is a unique invariant distribution $\overline{\nu}$ for this process, and we have (as stated precisely in the same Assumption) exponentially fast convergence to that distribution, from any starting distribution. Recall that $\overline{\rho}$ denotes the density of $\overline{\nu}$ (i.e. the probability under $\overline{\nu}$ that a given vertex has value 1).

LEMMA 3.3. *$\overline{\rho}$ is continuous from the right in each of the parameters λ, $\tilde{\lambda}$, h and $\tilde{h}$.*

Proof. Let μ_t denote the distribution at time t if we start the process with all vertices in state 1. By uniqueness, $\overline{\nu}$ is the limit, as $t \to \infty$, of μ_t. From Lemma 4.2 we have that μ_t is stochastically increasing in each of the parameters h, $\tilde{h}$, λ and $\tilde{\lambda}$ (and stochastically decreasing in κ and $\tilde{\kappa}$). Also by obvious monotonicity, μ_t is stochastically decreasing in t. For each $t \geq 0$ the density $\rho(t)$ under μ_t is continuous in each of the parameters λ, $\tilde{\lambda}$, h, $\tilde{h}$, and by the above we have that $\rho(t) \searrow \overline{\rho}$. The result follows since we can interchange the order of any two decreasing limits. □

Proof of Theorem 2.3. Let ν denote the critical invariant measure mentioned in the statement of the theorem. Let ρ denote its density, and let $\lambda = \Lambda(\rho)$ and $h = H(\rho)$. Then, as in Proposition 3.2, ν is invariant under the 3-state contact process dynamics with parameters κ, $\tilde{\kappa}$, λ,

$\tilde{\lambda}$, h, $\tilde{h}$. Hence, under Assumption 4.5, ν is the unique measure $\overline{\nu}$ for these parameters. So we have

$$\lambda = \Lambda(\rho(\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}, h, \tilde{h})), \text{ and}$$
$$h = H(\rho(\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}, h, \tilde{h})).$$

Now we increase each of the 'good' parameters $\lambda, \tilde{\lambda}, h$ and $\tilde{h}$ by an amount $\in (0, \varepsilon/2)$ so small that

$$\Lambda(\rho(\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}, h, \tilde{h})) \text{ and } H(\rho(\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}, h, \tilde{h}))$$

change by at most $\varepsilon/2$. This is possible by the continuity of Λ and H and Lemma 3.3. Denote the new parameters by $\kappa' = \kappa$, $\tilde{\kappa}' = \tilde{\kappa}$, $\lambda', \tilde{\lambda}', h', \tilde{h}'$. From the above it follows that there are continuous functions Λ' and H' which differ at most ε from Λ and H respectively, such that

$$\lambda' = \Lambda'(\rho(\kappa', \tilde{\kappa}', \lambda', \tilde{\lambda}', h', \tilde{h}')), \text{ and}$$
$$h' = H'(\rho(\kappa', \tilde{\kappa}', \lambda', \tilde{\lambda}', h', \tilde{h}'))$$

(For example, one may take $\Lambda'(r) = \Lambda(r) + \lambda' - \Lambda(\rho(\kappa', \tilde{\kappa}', \lambda', \tilde{\lambda}', h', \tilde{h}'))$, and take H' in a similar way). Let ν' be the invariant measure for the contact process with fixed parameters $\kappa', \tilde{\kappa}', \lambda', \tilde{\lambda}', h', \tilde{h}'$. From the above we conclude that ν' is invariant under the DDCP dynamics with parameters $\kappa', \tilde{\kappa}', \tilde{\lambda}', \tilde{h}'$, Λ' and H', and each of these 'new' parameters differs at most ε from the corresponding 'old' one. Moreover, by Theorem 2.7, this ν' is not critical. This completes the proof of Theorem 2.3. □

4. Ingredients from the literature

In this section we discuss a number of results and methods needed for the proofs of our main sharpness results, Theorems 2.4, 2.7 and 2.9. First we discuss graphical representations and convergence to equilibrium for contact processes, then some methods from percolation theory as well as influence results.

4.1. **Graphical representations.** Central to the study of the 2-state contact process is the following graphical representation. Let $D = (D_x : x \in \mathbb{Z}^d)$ denote a Poisson process of intensity κ on $\mathbb{Z}^d \times [0, \infty)$; that is to say, the D_x are independent Poisson processes of rate κ on $[0, \infty)$. Next, for each ordered pair (x, y) of nearest-neighbour sites, let A_{xy} be a Poisson process on $[0, \infty)$ of rate λ. The A_{xy} are taken independent for different pairs of sites, and indepentent of D; we write A for the collection of A_{xy}. We think of D as the process of 'recovery points' and A as a process of 'arrows'. Finally let $U = (U_x : x \in \mathbb{Z}^d)$ be a Poisson process with intensity $h \geq 0$; the elements of U are called 'spontaneous infection points'. Given an initial state ξ, a realization of the contact process X may be obtained from the triple (D, A, U)

as follows. At a point $(x,t) \in D$ the site x changes from state 1 to 0 (alternatively, if it already was in state 0, it remains in state 0). If at time t the state of x is 1 and $(xy,t) \in A$ then (y,t) changes form state 0 to state 1 (alternatively, remains in state 1). At a point $(x,t) \in U$ the site x changes from state 0 to 1 (alternatively, remains in state 1).

We consider also the 2-state contact process with time-varying parameters κ, λ, h. Such a process is easily defined by way of its graphical representation, as follows. Let $\kappa(\cdot)$, $\lambda(\cdot)$ and $h(\cdot)$ be nonnegative integrable functions, and let the D_x, A_{xy} and U_x be independent Poisson processes of rates $\kappa(\cdot)$, $\lambda(\cdot)$ and $h(\cdot)$ respectively. The contact process with these recovery-, neighbour infection- and spontaneous infection-processes is defined as above.

The 2-state contact process has the following monotonicity in the initial state and graphical representation. Suppose we have two initial states $\xi, \xi' \in \{0,1\}^{\mathbb{Z}^d}$ such that $\xi'_x \geq \xi_x$ for all $x \in \mathbb{Z}^d$ (we write $\xi' \geq \xi$ for this), as well as two arrow-processes A, A' satisfying $A \subseteq A'$, two recovery processes D, D' satisfying $D' \subseteq D$, and two spontaneous infection processes U, U' satisfying $U \subseteq U'$. Then the corresponding contact processes X, X' satisfy $X'(t) \geq X(t)$ for all $t \geq 0$. Using basic properties of Poisson processes we deduce the following:

LEMMA 4.1. *Let ξ, ξ' be as above and let $\lambda(\cdot)$, $\lambda'(\cdot)$, $\kappa(\cdot)$, $\kappa'(\cdot)$ and $h(\cdot)$, $h'(\cdot)$ be such that $\lambda(t) \leq \lambda'(t)$, $\kappa(t) \geq \kappa'(t)$ and $h(t) \leq h'(t)$ for all $t \geq 0$. There is a coupling P of the processes X and X' with these parameters and initial conditions such that*

$$P(\forall t \geq 0 : X'(t) \geq X(t)) = 1.$$

The graphical representation allows us to easily show that $h_{\mathrm{p}}(\lambda)$, as defined in Section 2.2, is continuous. Indeed, for each $\alpha \geq 0$, the 2-state contact process with parameters κ, $\lambda + \alpha$ and h is stochastically dominated by the contact process with parameters κ, λ and $h + 4\alpha$. This can be seen by comparing the space–time diagrams, in particular comparing the effects of incoming arrows versus spontaneous infection points. So we have

$$h_{\mathrm{p}}(\lambda) \geq h_{\mathrm{p}}(\lambda + \alpha) \geq h_{\mathrm{p}}(\lambda) - 4\alpha, \tag{4.1}$$

which implies that $h_{\mathrm{p}}(\lambda)$ is (uniformly Lipschitz) continuous.

Graphical representation for the 3-state process. The 3-state process also has a natural graphical representation in terms of Poisson processes of 'marks' and 'arrows'. We write D_1 and D_2 for independent Poisson processes of intensities κ and $\tilde{\kappa}$, which we think of as the processes of 'down' marks ($1 \to 0$ and $0 \to -1$, respectively). Independently of these and of each other we define Poisson processes U_1 and U_2 of 'up' marks ($0 \to 1$ and $-1 \to 0$) of respective intensities h and $\tilde{h}$. Finally, independently of all these and of each other we define Poisson processes

A_1 and A_2 of arrows ($0 \to 1$ and $-1 \to 0$, respectively) with respective intensities λ and $\tilde{\lambda}$. The rates of these processes are summarized in the following table:

Spontaneous transitions (on $\{x\} \times [0, \infty)$)	Neighbour transitions (on $\{xy\} \times [0, \infty)$)
D_1 rate κ	A_1 rate λ
D_2 rate $\tilde{\kappa}$	A_2 rate $\tilde{\lambda}$
U_1 rate h	
U_2 rate $\tilde{h}$	

As for the 2-state process, the rates are all allowed to be time-varying. We note that the 3-state process shares with the 2-state process the following monotonicity in the initial condition and in the graphical representation. Let X denote the 3-state process with initial state $\xi \in \{-1, 0, 1\}^{\mathbb{Z}^d}$ and graphical representation D_1, D_2, U_1, U_2, A_1 and A_2, and let X' denote the process with initial condition $\xi' \in \{-1, 0, 1\}^{\mathbb{Z}^d}$ and graphical representation D_1', D_2', U_1', U_2', A_1' and A_2'. If the following hold, then $X'(t) \geq X(t)$ for all $t \geq 0$: $\xi' \geq \xi$, $D_1' \subseteq D_1$, $D_2' \subseteq D_2$, $U_1 \subseteq U_1'$, $U_2 \subseteq U_2'$, $A_1 \subseteq A_1'$ and $A_2 \subseteq A_2'$. Using the natural couplings of Poisson processes we deduce the following:

LEMMA 4.2. *The 3-state process is (stochastically) increasing in the initial state and the parameters $\lambda, \tilde{\lambda}, h, \tilde{h}$ and decreasing in $\kappa, \tilde{\kappa}$.*

Graphical representation for the RE*-process.* There is also a graphical representation for the RE-process, which is almost identical to that of the 3-state process. The only changes are that the Poisson process D_2 has intensity $\kappa^\star$, and represents transition to state -1 irrespective of the previous state. Further, there is no process A_2 for the RE-process. The monotonicity statement follows as before:

LEMMA 4.3. *The* RE*-process is (stochastically) increasing in the initial state and the parameters $\lambda, h, \tilde{h}$ and decreasing in $\kappa, \kappa^\star$.*

For all three processes we consider (2- and 3-state and RE) the monotonicity properties described above imply that if the initial condition ξ consists of only 1's then the distribution of the process at time t is stochastically decreasing in t. Standard arguments then imply that the process decreases (stochastically) to a limiting distribution, which in all three cases we denote by $\overline{\nu}$ and call the *upper invariant measure.*

4.2. **Convergence to equilibrium.** Consider the contact process (2- or 3-state or RE) with all parameters constant. We are mainly concerned with the case when all parameters are positive, so we assume for the 2-state process that $\kappa, \lambda, h > 0$, for the 3-state process that $\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}, h, \tilde{h} > 0$, and for the RE-process that $\kappa, \kappa^\star, \lambda, h, \tilde{h} > 0$.

It is well-known (and can be easily proved by a standard coupling argument using the graphical representation) that the assumption $h > 0$ implies exponentially fast convergence to equilibrium in the 2-state process. Moreover, almost exactly the same argument extends to the RE-process. For any $\xi \in \{0,1\}^{\mathbb{Z}^d}$ let us write μ_t^ξ for the law of the contact process with initial state ξ. Further, for a finite set $\Lambda \subseteq \mathbb{Z}^d$ let $\mu_{t;\Lambda}^\xi$ denote the restriction of μ_t^ξ to Λ. Similarly, let $\overline{\nu}_\Lambda$ denote the restriction of the upper invariant measure $\overline{\nu}$ to Λ (ie, marginal of μ_t^ξ on $\{0,1\}^\Lambda$). We use the same notation for the RE-process (with $\{0,1\}$ replaced by $\{-1,0,1\}$).

LEMMA 4.4. *For the 2-state and* RE*-processes with $h > 0$ and any initial state ξ we have that*

$$d_{\mathrm{tv}}(\mu_{t;\Lambda}^\xi, \overline{\nu}_\Lambda) \le |\Lambda| e^{-ht}.$$

For the 3-state process we have not been able to establish exponential convergence to equilibrium along the lines of Lemma 4.4. However, it is natural to suppose that such a result should hold when all parameters $\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}, h, \tilde{h} > 0$. Our results for the 3-state process in Section 5 rely on such a convergence result: they are formulated conditional on the following assumption.

ASSUMPTION 4.5 (Exponential convergence to equilibrium). *For the 3-state process with strictly positive parameters*

(1) *there is a unique stationary distribution $\overline{\nu}$, and*
(2) *there are constants $C_1, C_2 > 0$ such that for all finite $\Lambda \subseteq \mathbb{Z}^d$ and all initial configurations ξ we have*

$$d_{\mathrm{tv}}(\mu_{t;\Lambda}^\xi, \overline{\nu}_\Lambda) \le C_1 |\Lambda| e^{-C_2 t}.$$

Here are some heuristic arguments supporting the assumption. Compared to the 2-state process, the extra state -1 in the 3-state process introduces 'delays' during which particles are insensitive to infection attempts. The delay periods are of random length but with exponential tails, and hence we do not expect the qualitiative properties of convergence speed to equilibrium to be different from the 2-state case. Also, by standard general arguments (see [17, Theorem 4.1]) Assumption 4.5 holds for a certain parameter range, namely when the 'spontaneous rates' are sufficiently large compared with the 'neighbour rates'. As already stated, the exponential convergence holds for the RE-process, and intuitively the 3-state and RE-processes should not differ substantially as far as speed of convergence etc. is concerned.

4.3. **Finite-size criterion.** Next we present a so-called finite-size criterion for percolation. Its analog for Bernoulli percolation is a well-known classical result which, as pointed out in [2] (see Lemma 2.3 in that paper), can be generalized to the case where the configurations come from the supercritical ordinary contact process. The same arguments as in [2] yield our Lemma 4.6 below.

Let $d = 2$ and let $H(m,n)$ denote the event that there is a left-right crossing of the rectangle $[0,m] \times [0,n]$ (ie, that the subgraph of $[0,m] \times [0,n]$ spanned by sites in state 1 contains a path from some $(0,x)$ to some (m,y) where $0 \leq x,y \leq n$). Let $V(m,n)$ denote the event that there is an up-down crossing of the rectangle $[0,m] \times [0,n]$.

LEMMA 4.6. *(For the 3-state case we assume Assumption 4.5 here). There is a (universal) constant $\hat{\varepsilon} > 0$ such that the following holds. For all choices of strictly positive parameters for the 2- or 3-state or* RE-*contact process, there is $\hat{n}$ (depending on the parameters) such that*

(1) *If for some $n \geq \hat{n}$ we have $\overline{\nu}(V(3n,n)) < \hat{\varepsilon}$, then there is $c > 0$ such that $\overline{\nu}(|C_0| \geq k) \leq e^{-ck}$ for all $k \geq 0$.*
(2) *If for some $n \geq \hat{n}$ we have $\overline{\nu}(H(3n,n)) > 1 - \hat{\varepsilon}$, then $\overline{\nu}(|C_0| = \infty) > 0$.*

4.4. **An influence result.** We further need the following combination of the Margulis-Russo formula and an influence result which essentially comes from Talagrand's paper [24] (which in turn is closely related to [11]), where all the p_i's in the description below are equal. For our particular situation we straightforwardly generalized the form (with two different p_i's) in [2, Corollary 2.9]. See also e.g. [10] and [4, 5].

Let $X = (X_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n)$ be a collection of independent $\{0,1\}$-valued random variables such that

$$P(X_{i,j} = 1) = p_i \quad \text{for all } j \in \{1, \ldots, n\}.$$

For fixed i,j, let $X^{(i,j)}$ denote the random vector obtained from X by replacing $X_{i,j}$ with $1 - X_{i,j}$ (and keeping all other $X_{i',j'}$ the same). For an event A, define the *influence* of $X_{i,j}$ on A as

$$I_{i,j}(A) = P(\{X \in A\} \vartriangle \{X^{(i,j)} \in A\}),$$

where $\vartriangle$ denotes symmetric difference.

LEMMA 4.7. *Fix $k \in \{1, \ldots, m\}$ and suppose that H is an event which is increasing in the $X_{i,j}$ for $i \leq k$, and decreasing in the $X_{i,j}$ for $i \geq k+1$. Let N denote the number of indices (i,j) such that $I_{i,j}(H)$ is maximal. There is an absolute constant K such that*

$$\sum_{i=1}^{k} \frac{\partial}{\partial p_i} P(H) - \sum_{i=k+1}^{m} \frac{\partial}{\partial p_i} P(H) \geq \frac{P(H)(1-P(H))}{K \max_i p_i \log(2/\min_i p_i)} \log N.$$

For our application, m represents the number of different types of symbols. We apply it with $m = 3$ and $k = 2$ for the 2-state process, $m = 6$ and $k = 4$ for the 3-state process, and $m = 5$ and $k = 3$ for the RE-process.

4.5. **RSW-result.** The following result is usually referred to as a RSW-type result as this type of result was pioneered, for Bernoulli percolation, in papers by Russo, Seymour and Welsh. A highly non-trivial extension of (a weak version of) the original RSW-result to a dependent percolation model, namely on the random Voronoi model, was proved by Bollobás and Riordan [4] (and modified in [3] to a form which is closer to Lemma 4.8 below). As pointed out in [4], the result holds under quite mild geometric, positive-association and spatial mixing conditions. In [2] (see Proposition 2.4 in that paper) it is explained that these conditions hold for the supercritical ordinary contact process. The same arguments hold for our models.

LEMMA 4.8. *Consider the upper invariant measure $\overline{\nu}$ of the 2-state or* RE*-process with $h > 0$, or the 3-state process under Assumption 4.5. If for some $\alpha > 0$ we have $\limsup_{n\to\infty} \overline{\nu}(H(\alpha n, n)) > 0$ then for all $\alpha > 0$ we have $\limsup_{n\to\infty} \overline{\nu}(H(\alpha n, n)) > 0$.*

5. Proofs of sharpness results

In this section we prove Theorems 2.4, 2.7 and 2.9 and Corollaries 2.5, 2.8 and 2.10. Here is an outline of the argument that follows. Suppose $\overline{\nu}$ is an invariant measure for which the cluster size $|C_0|$ does *not* have exponential tails. The first part of Lemma 4.6 together with Lemma 4.8 imply that certain crossing probabilities then have uniformly positive probability under $\overline{\nu}$. We want to apply Lemma 4.7 to show that, with an arbitrarily small increase of the relevant parameters, we can 'boost' this to get crossing probabilities close to 1. The second part of Lemma 4.6 then tells us that $|C_0|$ is now infinite with positive probability.

One of the main technical obstacles with carrying out this argument is that Lemma 4.7 applies to events which are defined in terms of a finite number of Bernoulli variables, whereas contact processes are defined in terms of 'continuous' objects (Poisson processes). The first step is therefore a *stability coupling*, a type of coupling which was also used in [4] for the Voronoi model, and later in [2] for the ordinary contact process. It tells us that if we increase the 'good' parameters then we can encode the contact process sufficiently well in terms of Bernoulli variables. This is the topic of Section 5.1. For the 3-state process we give a detailed proof of this part of the argument because it is considerably more complicated than the corresponding one in [2, Lemma 3.2] for the ordinary contact process. (For the 2-state and RE-processes we give an outline.) The subsequent parts of the proof appear in Section 5.2. Recall that we are only considering the case $d = 2$.

We start by pointing out that the monotonicity lemmas (Lemmas 4.1, 4.2, and 4.3) imply that Theorems 2.4, 2.7, and 2.9 follow once we establish the following claim: Fix any $\kappa, \lambda, h > 0$ and consider the parameterization $\{(\kappa, r\lambda, rh) : r \geq 0\}$ for the 2-state contact process. Similarly, let $\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}, h, \tilde{h} > 0$ be fixed, and consider the parameterization $\{(\kappa, \tilde{\kappa}, r\lambda, r\tilde{\lambda}, rh, r\tilde{h}) : r \geq 0\}$ for the 3-state process; and let $\kappa, \kappa^\star, \lambda, h, \tilde{h} > 0$ be fixed, and consider the parameterization $\{(\kappa, \kappa^\star, r\lambda, rh, r\tilde{h}) : r \geq 0\}$ for the RE-process. Define

$$r_{\mathrm{p}} = \inf\{r \geq 0 : \overline{\nu}(|C_0| = \infty) > 0\} \tag{5.1}$$

as a function of the other parameters. Then the percolation transition is sharp in r, in that if $r < r_{\mathrm{p}}$ then $\overline{\nu}(|C_0| \geq n) \leq e^{-cn}$ for some $c > 0$.

It is convenient to rescale time so that the total rate of 'events per line' is 1. That is, we assume

$$\begin{aligned} &\kappa + h + 4\lambda = 1, && \text{(2-state process)} \\ &\kappa + \tilde{\kappa} + 4\lambda + 4\tilde{\lambda} + h + \tilde{h} = 1 && \text{(3-state process)}, \\ &\kappa + \kappa^\star + 4\lambda + h + \tilde{h} = 1 && \text{(RE-process)}. \end{aligned} \tag{5.2}$$

This clearly leaves the invariant measure $\overline{\nu}$ unchanged. Write

$$q = \begin{cases} 4\lambda + h, & \text{(2-state)}, \\ 4\lambda + 4\tilde{\lambda} + h + \tilde{h} & \text{(3-state)}, \\ 4\lambda + h + \tilde{h} & \text{(RE)}. \end{cases}$$

Thus q equals the part of the sum in (5.2) which is decreased in the sharpness theorems. The symbols whose rates are thereby increased are called 'up' symbols, and the remaining 'down' symbols. Thus for the 2-state process the 'up' symbols are the elements of U and A, whereas for the 3-state and RE-process the 'down' symbols are D_1 and D_2 and the remaining are 'up'. We vary the parameter $q \in [0, 1]$, and leave the ratio between the rates of any two *up* or any two *down* symbols constant. We are then required to prove, under the relevant assumptions, that if q is such that $\overline{\nu}(|C_0| \geq n)$ goes to 0 as $n \to \infty$, but slower than exponentially, then for any $q'' > q$,

$$\overline{\nu}(|C_0| = \infty) > 0.$$

The objective of the stability coupling is the following. We wish to discretize time into intervals of length $\delta = n^{-\alpha}$ (for a certain $\alpha > 0$), and then apply the influence bound of Lemma 4.7. However, even if we choose n very large, we cannot avoid that there are intervals with more than one symbol. The solution is an intermediate step: for $q' \in (q, q'')$ we couple the processes for values q and q' in such a way that 'essential' symbols have distance at least δ. The existence of such a coupling is stated in Lemma 5.1 below. Subsequently, we use the influence bound of Lemma 4.7 to conclude that when q' is further increased to q'', then the criterion for percolation in Lemma 4.6 is satisfied. A similar program has been carried out in [2] for the ordinary contact process. The current situation is, particularly for the 3-state case, considerably more complicated than in [2].

5.1. **Stability coupling.** The 3-state processes with different values of q can be coupled in a natural way, and a similar statement holds for the other two processes. Since this coupling procedure serves as a 'starting point' for the more complicated coupling in Lemma 5.1 below, we give a brief sketch here. Let Π be a Poisson point process with unit density on $\mathbb{Z}^2 \times \mathbb{R}$, and write $[\Pi]$ for the support of Π (i.e. the set of points in a realization of this point process). We interpret $(x,t) \in [\Pi]$ as a *symbol* in the graphical representation. In a second step we decide the *type* of the symbol. Types are from the set

$$\mathrm{T} = \begin{cases} \{D, U, A^{\uparrow}, A^{\downarrow}, A^{\leftarrow}, A^{\rightarrow}\} & \text{(2-state)},\\ \{D_1, D_2, U_1, U_2, A_1^{\uparrow}, A_1^{\downarrow}, A_1^{\leftarrow}, A_1^{\rightarrow}, A_2^{\uparrow}, A_2^{\downarrow}, A_2^{\leftarrow}, A_2^{\rightarrow}\} & \text{(3-state)},\\ \{D_1, D_2, U_1, U_2, A_1^{\uparrow}, A_1^{\downarrow}, A_1^{\leftarrow}, A_1^{\rightarrow}\} & (\textsc{re}), \end{cases}$$

corresponding to the notation in Section 4.1, arrow superscripts indicating the direction of (incoming) arrows. For each symbol $(x,t) \in [\Pi]$, we consider three independent random variables drawn uniformly from the unit interval, denoted $Q_{(x,t)}, B_{(x,t)}$, and $G_{(x,t)}$. These are independent also of all other random variables used. We assign an *up* symbol whenever $Q_{(x,t)} \le q$ and a *down* symbol whenever $Q_{(x,t)} > q$. In the 2-state process an up symbol is assigned type U if $G_{(x,t)} \le h/(h+4\lambda)$, and one the types $A^{\uparrow}$, $A^{\downarrow}$, $A^{\leftarrow}$, $A^{\rightarrow}$ (to be decided in the natural way) if $G_{(x,t)} > h/(h+4\lambda)$. In the 2-state process there is only one down symbol, and the value of $B_{(x,t)}$ will not be used here. For the 3-state process we assign type D_1 if $Q_{(x,t)} > q$ and $B_{(x,t)} \le \kappa/(\kappa+\tilde{\kappa})$ and we assign type D_2 if $Q_{(x,t)} > q$ and $B_{(x,t)} > \kappa/(\kappa+\tilde{\kappa})$. Similarly, whenever $Q_{(x,t)} \le q$, we assign an *up* symbol based on the outcome of $G_{(x,t)}$, in such a way that the marginal distributions for the ten different *up* symbols (four different arrows of type A_1, another four of type A_2, and the two types U_1 and U_2) have the desired form. A very similar construction holds for the RE-process (without the A_2 symbols). We write H^q for the graphical representation thus obtained. So H^q consists of the processes D, A, U for the 2-state case, $D_1, D_2, A_1, A_2, U_1, U_2$ for the 3-state case, and D_1, D_2, A_1, U_1, U_2 for the RE-case, as in Section 4.1. (Of course, H^q depends not only on q but also on the remaining parameters κ, λ, $\tilde{\kappa}, \tilde{\lambda}$ etc; however, we suppress this from the notation.) The reader may convince her-/himself that the marginal distributions coincide with the definition of Section 4.1.

We write $\mathbb{P}$ for the probability measure governing Π, $Q_{(x,t)}$, $B_{(x,t)}$ and $G_{(x,t)}$ as above, and $\mathbb{P}_q$ for the probability measure governing the resulting graphical representation H^q. Thus $\mathbb{P}$ is a coupling of all the $\mathbb{P}_q$'s, $0 \le q \le 1$. Let $R_n = [0,6n] \times [0,3n] \subseteq \mathbb{Z}^2$. For $x \in R_n$ and $q \in [0,1]$, we define the random variable $\eta_x^{(q,n)}$ as the indicator function of the event that the graphical representation H^q inside the space-time box $\{y \in \mathbb{Z}^2 \colon d(x,y) \le \sqrt{n}\} \times [-\sqrt{n}, 0]$ is such that $(x,0)$ is in state 1, subject to the boundary condition assigning state 1 to any point (y,t)

with $d(x,y) = \sqrt{n}$ or $t = -\sqrt{n}$. (Here $d(x,y)$ denotes the usual graph distance.) Recall the length $\delta = n^{-\alpha}$ introduced in the paragraph preceding this section. For $v \in R_n$ and $k \in \mathbb{N}$, $0 \le k \le n/\delta$, and type $\tau \in \mathrm{T}$, we introduce the indicator functions

(5.3)
$$X_\tau^{(q,k,\delta)}(v) := \mathbb{1}\big\{\exists \text{ symbol of type } \tau \text{ in } \{v\} \times (-k\delta, (-k+1)\delta]\big\}.$$

For each δ-interval, these X variables only indicate whether there are symbols of a certain type in the interval, but do not tell us their precise locations or order. However, this information is often enough to conclude the state at $(x,0)$: We define $\eta_x^{(q,n,\delta)}(t)$ as the maximal $m \in \{-1,0,1\}$ for which the values of the elements of

$$\big\{X_\tau^{(q,k,\delta)}(v) \colon \tau \in T,\ v \in R_n \text{ and } k \in \mathbb{N},\ 0 \le k \le n/\delta\big\}$$

imply that $\eta_x^{(q,n)}(t) \ge m$. Clearly $\eta_x^{(q,n,\delta)} \le \eta_x^{(q,n)}$ for all $\delta > 0$. The following result holds for all three processes (subject to the correct interpretation); note that for the 3-state process we do not require Assumption 4.5 for this result.

LEMMA 5.1 (Stability coupling). *Let $\alpha > 0$ and, for each n, let $\delta = \delta_n = n^{-\alpha}$. For any $0 < q < q' < 1$, there is a coupling $\tilde{\mathbb{P}} = \tilde{\mathbb{P}}_{q,q',n}$ of $\mathbb{P}_q$ and $\mathbb{P}_{q'}$ such that $\tilde{\mathbb{P}}\big(\forall x \in R_n : \eta_x^{(q,n)} \le \eta_x^{(q',n,\delta)}\big) \to 1$ as $n \to \infty$.*

We give full details for the 3-state case, which is considerably more complicated than the proof of the corresponding result for the ordinary contact process, Lemma 3.2 in [2]. The 2-state and RE-case are easier and more similar to [2], so for these cases we outline the argument at the end of this section.

Proof for 3-state process. We let $\delta_1 = \sqrt{\delta} = n^{-\alpha/2}$, and throughout the proof consider intervals I of the form $\{x\} \times [-(k+1)\delta_1, -k\delta_1]$ whose intersection with $\mathrm{ST}_n = R_n \times [-n, 0]$ is nonempty ($k \in \mathbb{N}_0$). We call such an interval I *occupied* whenever $I \cap [\Pi] \ne \varnothing$. Moreover, we call two intervals $I^{(x,k)}$ and $I^{(y,\ell)}$ *neighbors* if either $x = y$ and $|k - \ell| = 1$, or $d(x,y) = 1$ and $k = \ell$. This neighborhood relation determines the notion of *clusters of neighboring occupied intervals*, henceforth called *clusters*. We use the notation $\mathcal{C}$ for clusters, and define the *size* $|\mathcal{C}|$ to be the number of *symbols* in $\mathcal{C}$ (not the number of intervals). More precisely, $|\mathcal{C}| = \sum_{I \in \mathcal{C}} |I \cap [\Pi]|$, where $|I \cap [\Pi]|$ is the cardinality of $I \cap [\Pi]$. Note that the clusters depend only on the Poisson process Π, hence the law of the clusters is independent of any of the parameters $\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}, h, \tilde{h}$, and the occupation of the intervals is pairwise independent.

Let S_n^α be the event that each occupied cluster in ST_n has size smaller than $\lceil 9/\alpha \rceil$. It follows in an elementary way from properties of the

Poisson process (see [2, equation (18)]) that

$$\lim_{n\to\infty} \mathbb{P}\big(S_n^\alpha\big) = 1. \tag{5.4}$$

Before proceeding with the argument, here are the main ideas. Ideally, we would like to take α so large that the size of the largest cluster shrinks to 1. However, later on (just before (5.13)) we need to take α rather close to 0; hence the existence of clusters of size ≥ 2 cannot be ruled out, and a result of the form (5.4) is essentially the best bound we get. (The precise value $9/\alpha$ is not important). We solve the issue by factorizing our probability space $\Omega = \Omega_1 \times \Omega_2$, where Ω_1 determines the clusters of intervals, the relative order (w.r.t. the time coordinates) of the symbols within each cluster, as well as some other information, and Ω_2 is responsible for the 'fine-tuning' (including the precise location of the symbols). We then first sample from Ω_1, which in particular fixes the clusters. For each cluster, when sampling from Ω_2, we use a 'crossover', which sacrifices an unnecessarily 'good' event in order to avoid a 'bad' event (where 'bad' means lack of δ-stability). 'Crossover' techniques have been used earlier for Voronoi percolation in [4, Theorem 6.1], and for percolation in the (ordinary) contact process in [2]. However, it turns out that the model we consider requires a considerably more subtle 'crossover recipe' than in [2].

We now give a detailed description (still for the 3-state case). Outcomes $\omega_1 \in \Omega_1$ contain the following partial information about Π: First of all, for any interval I, ω_1 determines the number of elements of $[\Pi] \cap I$. This identifies the clusters of ST_n. We call an interval $I^{(x,k)} = \{x\} \times [-(k+1)\delta_1, -k\delta_1]$ *vertically isolated* whenever $I^{(x,k)}$ contains precisely one symbol and both $I^{(x,k-1)}$ and $I^{(x,k+1)}$ are not occupied. Further, for any cluster $\mathcal{C}$, we let ω_1 also determine the relative order of symbols in $\mathcal{C}$ (w.r.t. the time coordinates of the symbols), and the value of $G_{(x,t)}$ for all symbols in $\mathcal{C}$. (For the ease of description we here name symbols by (x,t) although the precise time t is not yet determined. Further, recall that $G_{(x,t)}$ tells *which* good type a symbol has *if* its type is good). Finally, we also let ω_1 determine the value of $B_{(x,t)}$ for symbols in *vertically isolated intervals only.*

Outcomes $\omega_2 \in \Omega_2$ determine the precise location of symbols $(x,t) \in \Pi$, as well as the value $Q_{(x,t)}$ for *all* $(x,t) \in [\Pi]$, and the value of $B_{(x,t)}$ for every (x,t) that is *not contained in a vertically isolated interval.* Write $\mathcal{F}_1$ and $\mathcal{F}_2$ for the corresponding σ-algebras on Ω_1 and Ω_2 respectively.

Following the discussion at the beginning of the section, we can obtain the graphical representation H_n^q of the 3-state process on the space-time box ST_n as function of ω_1, ω_2, and q:

$$H_n^q = H_n^q(\omega_1, \omega_2). \tag{5.5}$$

(Actually, H_n^q depends also on the remaining parameters κ, λ, h etc but we suppress this dependence.) Since $\eta_x^{(q,n)}$ is itself a monotone (in q) function of H^q_n, $\mathbb{P}$ gives a coupling of $\eta_x^{(q,n)}$ and $\eta_x^{(q',n)}$ such that $\eta_x^{(q,n)} \geq \eta_x^{(q',n)}$ for any $q < q'$. The restriction of H^q_n to a cluster $\mathcal{C}$ is denoted $H^q_{n,\mathcal{C}} = H^q_{n,\mathcal{C}}(\omega_1, \omega_2)$.

The 'crossover' referred to above is a mapping $\Omega_2 \to \Omega_2$ which depends on the outcome of $\omega_1 \in \Omega_1$. To this end, fix an instance $\omega_1 \in \Omega_1$ (which, as mentioned above, particularly fixes the clusters). Since $\mathbb{P}(\,\cdot \mid \mathcal{F}_1)$ acts independently on the different clusters, we can write $\mathbb{P}(\,\cdot \mid \mathcal{F}_1) = \prod_{\mathcal{C}} \mathbb{P}_{\mathcal{C}}(\,\cdot \mid \mathcal{F}_1)$. Fix a cluster $\mathcal{C}$ and write $\mathbb{P}^{\omega_1}_{\mathcal{C}}(\,\cdot) = \mathbb{P}_{\mathcal{C}}(\,\cdot \mid \mathcal{F}_1)(\omega_1)$. In light of (5.4), we proceed under the assumption that $|\mathcal{C}| < \lceil 9/\alpha \rceil$. On the probability space Ω_2 we now define two events. (The definition of these events involves a cluster $\mathcal{C}$ and hence also ω_1; however, recall that we consider ω_1 as fixed here). The first event is

> $\mathcal{B}$: in $H^q_{n,\mathcal{C}}$ there are two symbols whose time coordinates differ by less than $\delta = n^{-\alpha}$.

The probability of $\mathcal{B}$ is maximized when all $|\mathcal{C}|$ symbols are in one single interval, so that

$$\mathbb{P}^{\omega_1}_{\mathcal{C}}(\mathcal{B}) \leq |\mathcal{C}|^2 \frac{2\delta}{\delta_1} \leq 2\lceil 9/\alpha \rceil^2 n^{-\alpha/2}, \quad \text{if } |\mathcal{C}| < \lceil 9/\alpha \rceil, \tag{5.6}$$

which goes to 0 as $n \to \infty$.

Before we state the other event, we need the following notion: A *maximal connected vertical chain* is a union of occupied δ-intervals $I^{(x,k)}$, $I^{(x,k+1)}, \dots, I^{(x,k+m-1)}$, with $k \geq 0$, $m \geq 1$, and where $I^{(x,k+m+1)}$ and (in case $k \geq 1$) $I^{(x,k-1)}$ are vacant. We call m the length of the chain. Note that a vertically isolated interval (defined earlier) is a maximal connected vertical chain of length 1.

We now define the event $\mathcal{G}$ that:

(1) in $H^q_{n,\mathcal{C}}$ all symbols are *down* symbols (i.e., all symbols in $\mathcal{C}$ have Q-value larger than q),
(2) in $H^q_{n,\mathcal{C}}$, each maximal connected vertical chain of length ≥ 2 has lowest symbol of type D_1 and all other symbols of type D_2, and
(3) in $H^{q'}_{n,\mathcal{C}}$ all symbols are *up* symbols (i.e., all symbols in $\mathcal{C}$ have Q-value smaller than q).

From the above definitions it follows straighforwardly that

$$\mathbb{P}^{\omega_1}_{\mathcal{C}}(\mathcal{G}) \geq (q'-q)^{|\mathcal{C}|} \big(\min\{\kappa/(\kappa+\tilde{\kappa}), \tilde{\kappa}/(\kappa+\tilde{\kappa})\}\big)^{|\mathcal{C}|}. \tag{5.7}$$

By this and (5.6) we thus may choose n sufficiently large (not depending on $\mathcal{C}$ or otherwise on ω_1) such that

$$\mathbb{P}^{\omega_1}_{\mathcal{C}}(\mathcal{G}) \geq \mathbb{P}^{\omega_1}_{\mathcal{C}}(\mathcal{B}) \qquad \text{if } |\mathcal{C}| < \lceil 9/\alpha \rceil. \tag{5.8}$$

Write $\mathcal{B}' = \mathcal{B} \setminus \mathcal{G}$. From the above we get, for n sufficiently large, that if $|\mathcal{C}| < \lceil 9/\alpha \rceil$ then there exists a measurable subset $\mathcal{G}' \subset \mathcal{G} \setminus \mathcal{B}$ and a measure-preserving 1-1 map $\psi_{\mathcal{C}}$ on Ω_2 such that

- $\psi_{\mathcal{C}}(\mathcal{B}') = \mathcal{G}'$,
- $\psi_{\mathcal{C}}(\mathcal{G}') = \mathcal{B}'$, and
- $\psi_{\mathcal{C}}(\omega_2) = \omega_2$ whenever $\omega_2 \notin \mathcal{B}' \cup \mathcal{G}'$.

If, on the other hand, $|\mathcal{C}| \geq \lceil 9/\alpha \rceil$, then we let $\psi_{\mathcal{C}}$ be the identity on Ω_2.

The map $\psi_{\mathcal{C}}$ is the crossover mentioned before. Since $\psi_{\mathcal{C}}$ is measure-preserving on Ω_2, we obtain a new coupling of the graphical representations on the cluster $\mathcal{C}$ by considering the graphical representation

$$\tilde{H}^{q'}_{n,\mathcal{C}} := H^{q'}_{n,\mathcal{C}}(\omega_1, \psi_{\mathcal{C}}(\omega_2)),$$

cf. (5.5). The 'overall coupling' is then obtained by constructing $\tilde{H}^{q'}_{n,\mathcal{C}}$ for each cluster $\mathcal{C}$ independently. The resulting graphical representation is denoted $\tilde{H}^{q'}_n$. We construct $\eta^{(q,n)}_x$ from the graphical representation $H^q_n = H^q_n(\omega_1, \omega_2)$, and $\eta^{(q',n,\delta)}_x$ from $\tilde{H}^{q'}_n$.

Finally, we check that this coupling has the desired properties. For a given ω_1, let $\mathcal{C}$ be one of the clusters. Recall that $H^{q'}_{n,\mathcal{C}} \geq H^q_{n,\mathcal{C}}$. We have to study $\tilde{H}^{q'}_{n,\mathcal{C}}$ and compare it with $H^q_{n,\mathcal{C}}$. These objects depend on ω_2. There are three cases:

(i) Case $\omega_2 \notin (\mathcal{B} \cup \mathcal{G})$. This case is simple: by the definition of the coupling procedure, we have $\psi_{\mathcal{C}}(\omega_2) = \omega_2$, and $\tilde{H}^{q'}_{n,\mathcal{C}} = H^{q'}_{n,\mathcal{C}}$, which (as we recalled above) dominates $H^q_{n,\mathcal{C}}$. Moreover, since $\omega_2 \notin \mathcal{B}$ we know that $H^q_{n,\mathcal{C}}$, and hence also $\tilde{H}^{q'}_{n,\mathcal{C}}$, does not have two symbols of which the time coordinates differ less than δ. This settles case (i).

(ii) Case $\omega_2 \in \mathcal{B}'$. Then, by the definition of the coupling procedure, $\psi_{\mathcal{C}}(\omega_2) \in \mathcal{G}' \subseteq \mathcal{G} \setminus \mathcal{B}$. By the definition of $\mathcal{G}$, this implies that all symbols in $\tilde{H}^{q'}_{n,\mathcal{C}}$ are up symbols. Since the precise type of an up symbol is determined by ω_1, we get that each symbol in $\tilde{H}^{q'}_{n,\mathcal{C}}$ 'dominates' the corresponding symbol in $H^q_{n,\mathcal{C}}$. Moreover, since $\psi_{\mathcal{C}}(\omega_2)$ is not in $\mathcal{B}$, there are no symbols in $\tilde{H}^{q'}_{n,\mathcal{C}}$ of which the time coordinates differ less than δ. Finally, the *order* (w.r.t. time) of the symbols in $\tilde{H}^{q'}_{n,\mathcal{C}}$ is the same as for $H^q_{n,\mathcal{C}}$ (recall that the order is determined by ω_1). This settles case (ii).

(iii) Case $\omega_2 \in \mathcal{G}$. Then, by the definition of $\mathcal{G}$, the types in $H^q_{n,\mathcal{C}}$ on the maximal connected vertical chains that are not single vertically isolated intervals, are as 'unfavourable' as possible: Consider such a chain and let $I^{(x,k)}$ be its 'highest' (i.e., with largest time index) interval. Since the symbol with smallest time coordinate on the chain has type D_1 and the others D_2, and since there are no incoming arrows, it follows that $\eta^{(q,n)}_x(-k\delta) = -1$. Further, each single vertically isolated interval has (by the definition of $\mathcal{G}$) in $H^q_{n,\mathcal{C}}$ a 'down' symbol. Since the precise type of this down symbol is determined by ω_1, it follows that the corresponding symbol in $\tilde{H}^{q'}_{n,\mathcal{C}}$ is either the same type of down symbol, or an up symbol. From these considerations it follows that, no matter how the symbols in $\tilde{H}^{q'}_{n,\mathcal{C}}$ are located precisely, we have that, if $\mathcal{C}$ would be the only cluster, then each space-time point (x,t) which is the 'highest point' of a maximal connected vertical chain of $\mathcal{C}$, satisfies

$$\eta_x^{(q',n,\delta)}(t) \geq \eta_x^{(q,n)}(t). \tag{5.9}$$

This settles the last case.

At the end of case (iii) we stated that (5.9) would hold if $\mathcal{C}$ is the only cluster. In fact, by combining this statement with the conclusions concerning case (i) and (ii), and the monotonicity of the contact process dynamics, it follows that (5.9) also holds (for such (x,t)) if there are other clusters (as long as all clusters have size $\leq 9/\alpha$). This completes the proof of Lemma 5.1 for the 3-state case. □

Sketch proof for 2-state and RE*-process.* The argument for the 2-state and RE-case has the same structure as the 3-state case, but the details are considerably simpler. One difference is that now the only information represented by ω_1 is the number of symbols in each interval (which in turn defines the clusters) and the values of $U_{(x,t)}$. All other information (the precise locations of the symbols and the $Q_{(x,t)}$- and $B_{(x,t)}$-values) are represented by ω_2. Another difference is that we modify the definition of the event $\mathcal{G}$ to the following:

(1) in $H^q_{n,\mathcal{C}}$ all symbols are of type D (2-state) / D_2 (RE-process);
(2) in $H^{q'}_{n,\mathcal{C}}$ all symbols are *up* symbols.

This implies $Q_{(x,t)} \in (q,q')$ for all symbols in $H^q_{n,\mathcal{C}}$. In particular, no special 'treatment' of maximal connected vertical chains is needed anymore. Equation (5.7) becomes

$$\mathbb{P}^{\omega_1}_{\mathcal{C}}(\mathcal{G}) \geq (q'-q)^{|\mathcal{C}|}\big(\kappa/(\kappa+\kappa^\star)\big)^{|\mathcal{C}|},$$

(with $\kappa^\star = 0$ in the 2-state process) so that (5.8) still holds for large enough n. Thus we may define crossover maps $\psi_{\mathcal{C}}$ and the modified graphical representation $\tilde{H}^q_n$ as before. To check that this coupling has the required properties, we distinguish again the three cases (i)–(iii) as for the 3-state model. Indeed, the arguments for cases (i) and (ii) apply verbatim as in the 3-state case. Case (iii) is now considerably simpler than before, because $\omega_2 \in \mathcal{G}$ implies that$H^q_{n,\mathcal{C}}$ has only D/D_2 symbols and hence $\tilde{H}^{q'}_{n,\mathcal{C}}$ is always 'at least as good'. (In fact, because of the simpler structure of the dynamics in the 2-state case, one gets a stronger result, namely that (if all clusters have size at most $\lceil 9/\alpha \rceil$) for all x and all t that are multiples of $-\delta_1$ we have $\eta_x^{(q',n,\delta)}(t) \geq \eta_x^{(q,n)}(t)$.) □

5.2. **Proofs of Theorems 2.4, 2.7 and 2.9.** Let $\overline{\nu}_q$ denote the upper invariant measure for the (2- or 3-state) contact process defined as above with parameter value q. Let $0 < q_1 < 1$ be such that under $\overline{\nu}_{q_1}$ the cluster size $|C_0|$ does *not* have exponential tails. Let $q_2 > q_1$. We will deduce that $\overline{\nu}_{q_2}(|C_0| = \infty) > 0$. This immediately implies Theorems 2.4, 2.7 and 2.9.

By the first part of Lemma 4.6 and Lemma 4.8 we have that there exists $\varepsilon_1 > 0$ and a sequence $n_i \to \infty$ such that

$$\overline{\nu}_{q_1}(H(4n_i, n_i)) \geq \varepsilon_1 \quad \text{for all } i \geq 1.$$

Let L_n denote the specific $4n$-by-n rectangle $[n, 5n] \times [n, 2n]$, and write $H_i = H(L_{n_i})$. By monotonicity (cf. Lemmas 4.1 and 4.2 and the discussion around there) the realization of $\overline{\nu}_q$ is dominated by $\eta^{(q,n)}$ for all $n \geq 1$. We deduce that

$$\mathbb{P}(\eta^{(q_1,n_i)} \in H_i) \geq \varepsilon_1 \quad \text{for all } i \geq 1. \tag{5.10}$$

Fix $q' \in (q, q_2)$. By Lemma 5.1 and (5.10),

$$\mathbb{P}(\eta^{(q',n_i,\delta)} \in H_i) \geq \varepsilon_2 := \varepsilon_1/2$$

for all large enough $i \geq 1$. The latter probability is defined in terms of the Bernoulli variables X of (5.3), so in principle Lemma 4.7 could now be applied. However, we have no good way of bounding the number N of variables with maximal influence. To get around this, we consider a 'symmetrized' version of the event H_i. A similar method was used in e.g. [4] and [2] and is standard in this type of argument; here we use the 'truncation' implicit in the definition of the $\eta^{(n,\delta)}$ and hence, ultimately, the fast convergence of the dynamics (Lemma 4.4 and Assumption 4.5).

Recall that R_n is the box $[0, 6n] \times [0, 3n]$, and consider the 'periodic' set R_n^{per} obtained from R_n by identifying the left and right sides; that is, identifying points $(6n, y)$ and $(0, y)$. We can consider L_n as a subset of R_n^{per} rather than $\mathbb{Z}^2$. Since the variables $\eta^{(q',n_i,\delta)}$ are 'truncated' at distance $\sqrt{n}$ the probability that $\eta^{(q',n_i,\delta)} \in H_i$ is (for large enough i) unchanged under this change of geometry. Let A_i be the event that there is a horizontal crossing of 1's of at least one of the $6n_i - 1$ horizontal translates of L_{n_i} in $R_{n_i}^{\text{per}}$. Thus

$$\pi_i(q') := \mathbb{P}(\eta^{(q',n_i,\delta)} \in A_i) \geq \mathbb{P}(\eta^{(q',n_i,\delta)} \in H_i) \geq \varepsilon_2, \tag{5.11}$$

for all sufficiently large i.

We apply Lemma 4.7 to the event A_i. By symmetry, all $6n_i - 1$ horizontal translates of $X_\tau^{(q',k,\delta)}(v)$ have the same influence, so the number N of Lemma 4.7 satisfies $N \geq 6n_i - 1 \geq n_i$. The number m of that lemma corresponds to the number of different types τ (*not* distinguishing between different directions of arrows) and thus $m = 3$ for the 2-state process, $m = 6$ for the 3-state process, and $m = 5$ for the RE-process. (The number n of variables of each type which appears in that lemma does not figure in the conclusion, so it is irrelevant for us.)

For the next step of the argument, we consider the three processes one by one.

2-state case. We start by computing (in terms of q) the probabilities p_1, p_2, p_3 that appear in Lemma 4.7. Here p_1 equals the probability that a given $X_\tau^{(q,k,\delta)}(v)$ equals 1 for $\tau = U$, whereas p_2 and p_3 are the corresponding probabilities for $\tau \in \{A^\uparrow, A^\downarrow, A^\leftarrow, A^\rightarrow\}$ and $\tau = D$, respectively. Recall that we have rescaled time so that $h + 4\lambda + \kappa = 1$, and that we vary the parameter $q = h + 4\lambda$ while keeping the ratio λ/h fixed. So $\kappa = 1 - q$ and there are constants $r_1, r_2 > 0$ (satisfying $r_1 + 4r_2 = 1$) such that $h = r_1 q$ and $\lambda = r_2 q$. We thus have $p_1 = 1 - e^{-r_1\delta q}$, $p_2 = 1 - e^{-r_2\delta q}$ and $p_3 = 1 - e^{-(1-q)\delta}$, and hence

$$
\begin{aligned}
\frac{d\pi_i}{dq} &\geq \delta C \left(\frac{\partial \pi_i}{\partial p_1} + \frac{\partial \pi_i}{\partial p_2} - \frac{\partial \pi_i}{\partial p_3} \right) \\
&\geq \delta C \log N \frac{\pi_i(1-\pi_i)}{K'\delta \log(2/\delta)},
\end{aligned}
\tag{5.12}
$$

for some constants C, K', where we used Lemma 4.7.

Let $\varepsilon_3 > 0$ and suppose that $\pi_i(q'') < 1 - \varepsilon_3$ for all $q'' \in (q', q_2)$. Using that $N \geq n_i$ and $\delta = n_i^{-\alpha}$ we deduce from (5.11) and (5.12) that $\pi_i(q_2) \geq C(q_2 - q')\varepsilon_2\varepsilon_3/\alpha$. Choosing α suffiently small we reach the following conclusion:

$$
\forall \varepsilon^* > 0\, \exists \alpha > 0 : \text{ for large enough } i,\ \pi_i(q_2) \geq 1 - \varepsilon^*. \tag{5.13}
$$

Before concluding the proof of sharpness, we show that we can reach the same conclusion (5.13) for the 3-state and RE-processes.

3-state and RE-*case.* Consider the 3-state case first. We now let $p_1, \dots, p_6$ denote the probabilities that $X_\tau^{(q',k,\delta)}(v)$ equals 1 for $\tau = U_1$, $\tau = U_2$, $\tau \in A_2$, $\tau \in A_1$, $\tau = D_1$, and $\tau = D_2$, respectively. Recall that $\kappa + \tilde\kappa + 4\lambda + 4\tilde\lambda + h + \tilde h = 1$ and that we increase $q = 4\lambda + 4\tilde\lambda + h + \tilde h$ while keeping $\kappa/\tilde\kappa$ and the ratios between any two of $\tilde\lambda, \lambda, \tilde h, h$ fixed. This implies that there are constants $r_1, \dots, r_6 \in (0,1)$ such that $h = r_1 q$, $\tilde h = r_2 q$, $\lambda = r_3 q$, $\tilde\lambda = r_4 q$, $\kappa = r_5(1-q)$, and $\tilde\kappa = r_6(1-q)$. Hence p_j equals $1 - e^{-r_j q\delta}$ for $1 \leq j \leq 4$, and $1 - e^{-r_j(1-q)\delta}$ for $j = 5, 6$. It follows that for i large enough

$$
\begin{aligned}
\frac{d\pi_i}{dq} &= \sum_{j=1}^{4} \delta r_j e^{-r_j\delta q} \frac{\partial \pi_i}{\partial p_j} - \sum_{j=5}^{6} \delta r_j e^{-r_j\delta(1-q)} \frac{\partial \pi_i}{\partial p_j} \\
&\geq \delta C \Big[\sum_{j=1}^{4} \frac{\partial \pi_i}{\partial p_j} - \sum_{j=5}^{6} \frac{\partial \pi_i}{\partial p_j} \Big] \\
&\geq \delta C \log N \frac{\pi_i(1-\pi_i)}{K'\delta \log(2/\delta)},
\end{aligned}
\tag{5.14}
$$

for some constants C, K', where the last inequality comes from Lemma 4.7. In the same way as for (5.13), we deduce that

$$\forall \varepsilon^* > 0 \, \exists \alpha > 0 : \text{ for large enough } i, \; \pi_i(q_2) \geq 1 - \varepsilon^*. \tag{5.15}$$

For the RE-case, we obtain (5.15) in literally the same way, except that $\tilde{\lambda} = r_4 = p_4 = 0$ (because there are no A_2 symbols) and $\tilde{\kappa} = \kappa^\star$.

Conclusion of proof. This final argument is the same for all three processes. Note that the event A_i implies that there is a horizontal crossing of at least one of the following rectangles (regarded as subsets of R_n^{per}):

$$[jn_i, (j+3)n_i \,(\text{mod } 6n_i))] \times [n_i, 2n_i] \quad 0 \leq j \leq 5.$$

Thus (by using the FKG-inequality) for $\hat{\varepsilon} > 0$ as in Lemma 4.6,

$$\mathbb{P}(\eta^{(q_2,n_i,\delta)} \in H(3n_i, n_i)) \geq 1 - (1 - \mathbb{P}(\eta^{(q_2,n_i,\delta)} \in A_i))^{1/6} \geq 1 - \hat{\varepsilon}/2$$

for all sufficiently large i, where the last inequality comes from (5.13) for the 2-state and from (5.15) for the other processes. The family of random variables $(\eta_x^{(q_2,n_i,\delta)} : x \in [n,4n] \times [n,2n])$ is clearly stochastically dominated by the family $(\eta_x^{(q_2,n_i)} : x \in [n,4n] \times [n,2n])$, and the law of this latter family has (by Lemma 4.4 and Assumption 4.5) total variation distance at most

$$C_1' \cdot 3n_i^2 \exp\big(-C_2'\sqrt{n_i}\big)$$

from $\overline{\nu}_{q_2}$. Hence $\overline{\nu}_{q_2}(H(3n_i, n_i)) \geq 1 - \hat{\varepsilon}$ for large enough i, which by Lemma 4.6 implies that $\overline{\nu}_{q_2}(|C_0| = \infty) > 0$. This completes the proof of Theorems 2.4, 2.7 and 2.9. ☐

5.3. **Proofs of Corollaries 2.5, 2.8 and 2.10.** In this section we show that Corollary 2.5 follows easily from Theorem 2.4, and we deduce Corollaries 2.8 and 2.10 from Theorems 2.7 and 2.9, respecitvely. Recall that for $x, y \in \mathbb{R}^k$ we write $x \prec y$ if each coordinate of x is strictly smaller than the corresponding coordinate of y.

Proof of Corollary 2.5. We fix the parameter $\kappa > 0$. Starting with assertion (2) of the corollary, recall that $\lambda_{\text{p}}(h)$ is non-increasing. Let $\mathcal{C}$ denote the set of $h > 0$ at which $\lambda_{\text{p}}(h)$ is right-continuous, and fix $h \in \mathcal{C}$. Let $\delta > 0$. By right-contiunuity there is $h' > h$ such that $\lambda_{\text{p}}(h') > \lambda_{\text{p}}(h) - \delta$. Hence $(h, \lambda_{\text{p}}(h) - \delta) \prec (h', \lambda_{\text{p}}(h'))$. The result now follows from Theorem 2.4 and the fact that $[0,\infty) \setminus \mathcal{C}$ is at most countable (since $\lambda_{\text{p}}(h)$ is monotone).

The argument for (1) is similar, noting first that by (4.1), the function $h_{\text{p}}(\lambda)$ is continuous on the entire interval $[0,\infty)$. ☐

We now turn to Corollary 2.8. We write $\mathbb{R}_+ = (0,\infty)$. In proving Corollary 2.8 we make use of the following fact.

LEMMA 5.2. *Let $n \geq 1$ be fixed and let $B \subseteq \mathbb{R}_+^3$ be a measurable set with the following property:*

$$\text{(5.16)} \quad \textit{if } a_1, a_2, \ldots, a_m \in B \textit{ satisfy } a_1 \prec a_2 \prec \cdots \prec a_m \textit{ then } m \leq n.$$

For $y \in \mathbb{R}_+^2$ write $B(y) = (\mathbb{R}_+ \times \{y\}) \cap B$ and let Γ denote the set of $y \in \mathbb{R}_+^2$ such that $B(y)$ is uncountable. Then Γ has measure zero.

Proof. The partially ordered set $(B, \prec)$ has height at most n, so by (the dual version of) Dilworth's theorem, B can be partitioned into n antichains. An antichain in this situation is a set satisfying (5.16) with $n = 1$, so it suffices to consider that case.

For $n = 1$, note that if $x < x'$ and both (x, y) and (x', y) belong to B then property (5.16) is preserved if the interval $[x, x'] \times \{y\}$ is added to B. Thus we may assume that B is maximal in the sense that it includes all such intervals. With each $y \in \Gamma$ we may thus associate a rational number $q(y)$ such that $(q(y), y)$ lies in an interval of $B(y)$. We now write $y \in \mathbb{R}_+^2$ in polar coordinates (θ, r) with $\theta \in (0, \pi/2)$ and $r > 0$. Fix θ and $r < r'$ and write $y = (\theta, r)$ and $y' = (\theta, r')$. If $(x, y) \in B(y)$ and $(x', y') \in B(y')$ then $x' \leq x$, by (5.16) with $n = 1$. Thus, if $y, y' \in \Gamma$ then we may choose $q(y') < q(y)$. It follows that for each $\theta \in (0, \pi/2)$ the set of $r > 0$ such that $(\theta, r) \in \Gamma$ is at most countable. By Fubini's theorem (using polar coordinates) it follows that Γ has measure zero. □

Proof of Corollary 2.8. Fix arbitrary $\kappa, \tilde{\kappa} > 0$. Recall that h_p is decreasing in each of the parameters $\lambda, \tilde{\lambda}, \tilde{h}$. Since $h_\mathrm{p}(\lambda, \tilde{\lambda}, \tilde{h}) < \infty$ for $\tilde{h} > 0$ we may restrict $(\lambda, \tilde{\lambda}, \tilde{h})$ to one of the (countably many) sets where h_p is bounded above by a fixed integer K. We call a triple $(\lambda, \tilde{\lambda}, \tilde{h})$ 'bad' if there is some $\delta > 0$ such that

$$\text{(5.17)} \qquad h_\mathrm{p}(\lambda', \tilde{\lambda}', \tilde{h}') \leq h_\mathrm{p}(\lambda, \tilde{\lambda}, \tilde{h}) - \delta \text{ for all } (\lambda', \tilde{\lambda}', \tilde{h}') \succ (\lambda, \tilde{\lambda}, \tilde{h}).$$

If B denotes the set of 'bad' points, then we may write $B = \cup_{n \geq 1} B_n$, where B_n is the set of points such that (5.17) holds with $\delta = K/n$. The set B_n satisfies (5.16), so it follows from Lemma 5.2 that for almost all pairs $(\tilde{\lambda}, \tilde{h})$ the set of $\lambda > 0$ such that $(\lambda, \tilde{\lambda}, \tilde{h})$ is bad is countable.

We are therefore done if we show that if $(\lambda, \tilde{\lambda}, \tilde{h})$ is *not* bad, and $h < h_\mathrm{p}(\lambda, \tilde{\lambda}, \tilde{h})$, then the cluster size decays exponentially for the parameter values $\kappa, \tilde{\kappa}, \lambda, \tilde{\lambda}, h, \tilde{h}$. Writing $\delta = h_\mathrm{p}(\lambda, \tilde{\lambda}, \tilde{h}) - h$ we have that there exists $(\lambda', \tilde{\lambda}', \tilde{h}') \succ (\lambda, \tilde{\lambda}, \tilde{h})$ such that

$$h_\mathrm{p}(\lambda', \tilde{\lambda}', \tilde{h}') > h_\mathrm{p}(\lambda, \tilde{\lambda}, \tilde{h}) - \delta = h.$$

The result now follows from Theorem 2.7. □

Proof of Corollary 2.10. Fix $\kappa, \kappa^\star > 0$. The argument for the RE-case is very similar to the one already given for the 3-state case, using the analog of Lemma 5.2 with $B \subseteq \mathbb{R}_+^2$ (in which case we can actually show that Γ is at most countable). One small difference is that we may now have $h_\mathrm{p} = \infty$ for some $(\lambda, \tilde{h})$. We now say that $(\lambda, \tilde{h})$ is *bad* if $h_\mathrm{p}(\lambda, \tilde{h}) < \infty$ and there exists $\delta > 0$ such that

$$h_\mathrm{p}(\lambda', \tilde{h}') \leq h_\mathrm{p}(\lambda, \tilde{h}) - \delta \text{ for all } (\lambda', \tilde{h}') \succ (\lambda, \tilde{h}), \tag{5.18}$$

and that $(\lambda, \tilde{h})$ is *terrible* if $h_\mathrm{p}(\lambda, \tilde{h}) = \infty$ but $h_\mathrm{p}(\lambda, \tilde{h}) < \infty$ for all $(\lambda', \tilde{h}') \succ (\lambda, \tilde{h})$. The set B of bad points may be written as

$$B = \cup_{K \geq 1} \cup_{n \geq 1} B_n^{(K)},$$

where $B_n^{(K)}$ is the set of points $(\lambda, \tilde{h})$ such that $h_\mathrm{p}(\lambda, \tilde{h}) \leq K$ and (5.18) holds with $\delta = K/n$. Thus $B_n^{(K)}$ satisfies (5.16). The set T of terrible points satisfies (5.16) with $n = 1$. It follows that for almost all $\tilde{h} > 0$, the set of λ such that $(\lambda, \tilde{h}) \in B \cup T$ is at most countable. If $(\lambda, \tilde{h}) \notin B \cup T$ then the result follows in the same way as for Corollary 2.8. □

6. Existence of density-driven processes

In this section we prove the existence of DDCP (Definition 1.1) using a fixed-point argument. Although this result is strictly speaking not needed for our main results on sharpness and lack of robustness (since, as discussed in Section 3.1, *stationary* DDCP are simply contact processes with constant parameters), we find it interesting in itself.

We consider 2-state processes with constant κ and 3-state processes with constant $\kappa, \tilde{\kappa}, \tilde{\lambda}, \tilde{h}$. Recall that we write $\rho(t) = P(X_0(t) = 1)$ for the density of the process. We let L_b^∞ denote the set of measurable $h : [0, \infty) \to [0, \infty)$ which are bounded on each compact subinterval.

We prove the following existence result for the 2- and 3-state processes; a completely analogous result holds for the RE-process.

Theorem 6.1. *Let $\Lambda, H \colon [0,1] \to [0, \infty)$ be uniformly Lipschitz continuous.*

(a) *For each $\kappa \geq 0$ and each translation-invariant probability measure ν on $\{0,1\}^{\mathbb{Z}^d}$, there is a unique pair $(\lambda, h) \in L_b^\infty \times L_b^\infty$ such that the 2-state contact process with initial distribution ν and parameters κ, $\lambda(\cdot)$ and $h(\cdot)$ satisfies $\lambda(t) = \Lambda(\rho(t))$ and $h(t) = H(\rho(t))$ for all $t \geq 0$.*

(b) *For all $\kappa, \tilde{\kappa}, \tilde{\lambda}, \tilde{h} \geq 0$ and each translation-invariant probability measure ν on $\{-1,0,1\}^{\mathbb{Z}^d}$, there is a unique pair $(\lambda, h) \in L_b^\infty \times L_b^\infty$ such that the 3-state contact process with initial distribution ν and parameters $\kappa, \tilde{\kappa}, \lambda(\cdot), \tilde{\lambda}, h(\cdot), \tilde{h}$ satisfies $\lambda(t) = \Lambda(\rho(t))$ and $h(t) = H(\rho(t))$ for all $t \geq 0$.*

Proof. Since part (a) is a special case of (b), we only prove (b).

Let $h, h', \lambda, \lambda' \in L^\infty([0,\infty), [0,\infty))$, and let D_1, D_2, U_2 and A_2 be as in Section 4.1. The intensities of these processes are kept fixed. Let $\underline{U}_1$ be a Poisson process of intensity $h(t) \wedge h'(t)$. Let $\tilde{U}_1^{(h)}$ and $\tilde{U}_1^{(h')}$ denote independent Poisson processes (independent also of $\underline{U}_1$) with intensities $h(t)-(h(t)\wedge h'(t))$ and $h'(t)-(h(t)\wedge h'(t))$, respectively. Write $U_1^{(h)} = \underline{U}_1 \cup \tilde{U}_1^{(h)}$, $U_1^{(h')} = \underline{U}_1 \cup \tilde{U}_1^{(h')}$ and $\overline{U}_1 = \underline{U}_1 \cup \tilde{U}_1^{(h)} \cup \tilde{U}_1^{(h')}$. In the same way (and independently of the Poisson processes above) we define $\underline{A}_1$, $A_1^{(\lambda)}$, $A_1^{(\lambda')}$ and $\overline{A}_1$. Furthermore, let $m := 1 \vee \sup\{\lambda(t) \vee \lambda'(t) : t \geq 0\}$, and let $A_1^{(m)}$ be obtained from $\overline{A}_1$ by appending another independent Poisson process of intensity $m-(\lambda(t)\vee\lambda'(t))$. Note that an element of $A_1^{(m)}$ (at time coordinate t) belongs to $\overline{A}_1 \setminus \underline{A}_1$ with probability $|\lambda'(t)-\lambda(t)|/m \leq \|\lambda'-\lambda\|_\infty$.

Let $\underline{X}$ be the contact process with $0 \to 1$ transitions given by $\underline{U}_1$ and $\underline{A}_1$ (and remaining transitions given by D_1, D_2, U_2, A_2). Similarly, $X^{(h,\lambda)}$, $X^{(h',\lambda')}$, and $\overline{X}$ denote the contact processes with $0 \to 1$ transitions given by $U_1^{(h)}$ and $A_1^{(\lambda)}$, with $U_1^{(h')}$ and $A_1^{(\lambda')}$, and with $\overline{U}_1$ and $\overline{A}_1$, respectively. The construction above is done such that $\underline{X} \leq X^{(h,\lambda)}, X^{(h',\lambda')} \leq \overline{X}$ holds and $\overline{U}_1 \setminus \underline{U}_1$ has rate $|h(t)-h'(t)|$.

Now consider the set $Z_t^{\leftarrow}$, which is defined as the set of space-time points (x,s), $0 \leq s \leq t$, such that there is a space-time path from (x,s) to $(0,t)$ using arrows from $A_1^{(m)} \cup A_2$. Let B_t be the event that none of the arrows in the definition of $Z_t^{\leftarrow}$ belongs to $\overline{A}_1 \setminus \underline{A}_1$. If B_t occurs and $(\overline{U}_1 \setminus \underline{U}_1) \cap Z_t^{\leftarrow} = \varnothing$, then $\overline{X}_0(t) = \underline{X}_0(t)$ and hence $X_0^{(h,\lambda)}(t) = X_0^{(h',\lambda')}(t)$.

Equip $L^\infty([0,\infty),[0,\infty))^2$ with the norm $\|(\lambda,h)\| = \|\lambda\|_\infty + \|h\|_\infty$, and consider the mapping R from this space to $L^\infty([0,\infty),[0,1])$ given by letting $R(\lambda,h)(t) = P(X_0(t)=1)$ where X is the 3-state contact process with rates $\kappa, \tilde{\kappa}, \lambda(\cdot), \tilde{\lambda}, h(\cdot), \tilde{h}$. By the above, for all $\alpha \geq 0$ and all $0 \leq t \leq \alpha$ we have that

$$\begin{aligned}|R(\lambda,h)(t)-R(\lambda',h')(t)| &\leq P(B_t^c \text{ occurs or } (\overline{U}_1 \setminus \underline{U}_1) \cap Z_t^{\leftarrow} \neq \varnothing) \\ &\leq E|Z_\alpha^{\leftarrow}|(\|\lambda-\lambda'\|_\infty + \|h-h'\|_\infty),\end{aligned}$$

where $|Z_\alpha^{\leftarrow}|$ is defined as the sum of the total Lebesgue measure of all intervals constituting $Z_\alpha^{\leftarrow}$, plus the number of arrows in $Z_\alpha^{\leftarrow}$. Let K_1, K_2 be uniform Lipschitz constants for Λ, H. It follows that

$$\begin{aligned}&\|\Lambda(R(\lambda,h)) - \Lambda(R(\lambda',h'))\|_\infty + \|H(R(\lambda,h)) - H(R(\lambda',h'))\|_\infty \\ &\qquad \leq (K_1+K_2)E|Z_\alpha^{\leftarrow}|(\|\lambda-\lambda'\|_\infty + \|h-h'\|_\infty).\end{aligned}$$

By standard comparison with a branching process, it is easy to see that $E|Z_\alpha^{\leftarrow}|$ is finite for α sufficiently small, and goes to 0 as $\alpha \to 0$. Hence there is an $\alpha_0 > 0$ such that the mapping $\Gamma = \Gamma_{\alpha_0}^{\nu} : (\lambda,h) \mapsto (\Lambda(R(\lambda,h)), H(R(\lambda,h)))$ is a contraction of $L^\infty([0,\alpha_0],[0,\infty))^2$. By Banach's fixed point theorem, this gives the desired result for the time interval $[0,\alpha_0]$. By repeating ('concatenating') this result, it can be extended to $[0,2\alpha_0]$, $[0,3\alpha_0]$, etcetera, which completes the proof. □

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