BACKBONE SCALING LIMIT OF THE HIGH-DIMENSIONAL IIC

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ABSTRACT. We identify the scaling limit of the backbone of the high-dimensional incipient infinite cluster (IIC), both in the long- as well as in the finite-range setting. In the finite-range setting, this scaling limit is Brownian motion, in the long-range setting, it is a stable motion. The proof relies on a novel lace expansion for percolation which resembles the original expansion for self-avoiding walks by Brydges and Spencer in 1985. This expansion is interesting in its own right.

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1. INTRODUCTION

A challenging task in probability theory is to understand critical percolation, and in particular, to understand the scaling limits of critical percolation. In this paper we identify the scaling limit of the backbone of large critical percolation clusters and of the incipient infinite cluster on $\mathbb{Z}^d$. We prove that the backbone scales to a Brownian motion (if the model is finite range, and the dimension is sufficiently high).

Our results apply to the common *nearest-neighbor* bond percolation model in sufficiently high dimension. Moreover, they apply to spread-out percolation in dimension $d > 6$, and also to *long-range* percolation (where, in the latter case, the scaling limit is not always Brownian motion).

Bond percolation on $\mathbb{Z}^d$, a generalized setup. We start by introducing the percolation models. Our models are defined in terms of a (weight) function $D\colon \mathbb{Z}^d \to [0,1]$. We assume that D is invariant under lattice symmetries and rotations by 90°, that $\sum_{x\in\mathbb{Z}^d} D(x) = 1$, and that $D(0) = 0$. Let $p \ge 0$ be a parameter chosen such that $pD(x) \le 1$ for all x. For arbitrary lattice sites $u, v \in \mathbb{Z}^d$, we declare the bond $\{u, v\}$ *occupied* with probability $pD(v-u)$ and *vacant* otherwise. The occupation statuses of the bonds are independent random variables. Mind that p is *not* supposed to denote the probability of an event, and might be greater than 1.

Our results hold for a broad range of models. We state the precise assumptions on D below in Assumptions D and E, but first we describe three 'standard' models that satisfy these assumptions as examples to keep in mind.

The first 'standard' model is also the best known model, namely the *nearest neighbor model.* Here $D(x) = 1/(2d)$ for all x with $|x| = 1$, and $D(x) = 0$ for all other x. Here, $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$. Note that this definition implies that each nearest neighbor bond is occupied with probability $p/(2d)$, not with probability p.

The second 'standard' model is a *finite-range spread-out model,* where, for a given $L \in \mathbb{N}$, bonds of length up to L are occupied with equal probability, and longer bonds are always vacant, i.e.,

$$pD(x) = \frac{p}{(2L+1)^d - 1}\,\mathbb{1}_{\{0<\|x\|_\infty \le L\}}. \tag{1.1}$$

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The parameter L serves to spread out the connections and is typically fixed at a large value.

The third 'standard' model is a *long-range spread-out model*, where the occupation probabilities decay as a power of the length of the edge. Indeed, for $\alpha \in (0,\infty)$ we define

$$pD(x) = p\frac{\mathcal{N}_L}{\max\{|x|/L,1\}^{d+\alpha}}, \tag{1.2}$$

where $\mathcal{N}_L$ is a normalizing constant. The parameter α is known as the *power-law exponent*.

Our results apply in much greater generality than the above three models describe: all that we require is that the Fourier transform of D obeys certain bounds. To make this precise, we need some definitions. For $f:\mathbb{Z}^d \mapsto \mathbb{R}$, let $\hat{f}$ denote the Fourier transform of f, i.e.,

$$\hat{f}(k) = \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{ik\cdot x} f(x), \qquad k \in \mathbb{R}^d. \tag{1.3}$$

We write α for the power-law exponent if the model is long-range, and we use the convention that $\alpha = \infty$ for models that do not depend on α (e.g. finite-range models). We write $(2\wedge\alpha) = \min\{\alpha,2\}$. We write $f(k) \sim g(k)$ if the ratio of $f(k)/g(k) \to 1$ when $k \to 0$.

The results in this paper hold for models where D satisfies the following two assumptions:

Assumption D [Bounds on $\hat{D}$]. *Consider a d-dimensional percolation model. Let $L = 1$ for nearest-neighbor models. Otherwise, let L be the spread-out parameter. The model satisfies the following bounds: There exist constants $c_1, c_2 > 0$ and $\alpha \in (0,\infty)$ such that*

$$1 - \hat{D}(k) \geq c_1 L^{(2\wedge\alpha)}|k|^{(2\wedge\alpha)} \qquad \text{if} \quad \|k\|_\infty \leq L^{-1}; \tag{1.4}$$

$$1 - \hat{D}(k) > c_2 \qquad \text{if} \quad \|k\|_\infty \geq L^{-1}. \tag{1.5}$$

Furthermore, there exists a constant w with $0 < w = O(L^{(2\wedge\alpha)})$ such that, for $\varepsilon > 0$ sufficiently small,

$$1 - \hat{D}(k) \leq w|k|^{(2\wedge\alpha)} \qquad \text{if} \quad |k| \leq \varepsilon. \tag{1.6}$$

Assumption E [Convergence of $\hat{D}$]. *Consider a d-dimensional percolation model. There exists a constant $0 < v_\alpha < \infty$ such that, as $k \to 0$,*

$$1 - \hat{D}(k) \sim \begin{cases} v_\alpha |k|^{(2\wedge\alpha)} & \text{if } \alpha \neq 2, \\ v_2 |k|^2 \log(1/|k|) & \text{if } \alpha = 2. \end{cases} \tag{1.7}$$

Assumption D is also made in the companions to this paper, [20] and [21]. We do not use Assumption D explicitly in this paper, but it is a common assumption for high-dimensional percolation that we need to use results from, for instance, [20] and [23]. A detailed description of three families of models that satisfy Assumption D is given in the introduction of [20]. Assumption E is not always needed in high-dimensional percolation papers, but we do need it here, so we state it as a separate assumption. Assumption E is a natural assumption to make for a scaling limit result. In fact, Assumption E is even needed to show the scaling limit of a simple random walk with step distribution D. The same assumption is also made in [8, (1.1)] and [19, Lemma 1.1]. See [19] for an in-depth discussion of the asymptotics in (1.7).

Define the spatial variance

$$\sigma^2 \equiv \sum_{x\in\mathbb{Z}^d} |x|^2 D(x). \tag{1.8}$$

We say a model has finite variance if D is such that $\sigma^2 < \infty$ and we say the model has infinite variance otherwise. Of course, σ^2 is finite for any finite-range model. The variance of a long-range spread-out model is finite when $\alpha > 2$, but it is infinite when $\alpha \leq 2$. Models with finite variance behave very differently from models with infinite variance, as can be seen below.

We write $\mathbb{P}_p$ for the law of configurations of occupied bonds, and we write $\mathbb{E}_p$ for the corresponding expectation. Given a configuration, we say that x is connected to y, and write $x \leftrightarrow y$, if there is a path of occupied bonds from x to y (or if $x = y$). Let $\mathbb{B} = \mathbb{Z}^d \times \mathbb{Z}^d$, so that $(\mathbb{Z}^d, \mathbb{B})$ is a complete graph. We write $\mathcal{C}(x)$ for the subgraph of the occupied percolation cluster that contains x (and at times we abuse notation by also writing $\mathcal{C}(x)$ for either just the sites or the occupied bonds of this subgraph).

We usually work at (or just below) the percolation threshold $p = p_c$ where p_c is the critical value of p, i.e., $p_c \equiv \inf\{p \mid \mathbb{P}_p(|\mathcal{C}(0)| = \infty) > 0\}$. In the parametrization of this paper, p_c satisfies $p_c > 1$, and p_c tends to 1 as either $d \to \infty$ or $L \to \infty$; see [15, 23].

The incipient infinite cluster. It is common knowledge for two-dimensional models and for models with sufficiently high dimension that there are no infinite clusters at the critical point. Nevertheless, we can think of the critical point as the point where the infinite clusters are at the verge of appearing. So it is reasonable to believe that we can construct infinite clusters at the critical point via reasonable conditioning and limiting schemes. Indeed, Kesten [29] showed that such a scheme can be used to construct an infinite cluster at criticality in two dimensions. The resulting cluster is known as the *incipient infinite cluster* (IIC). Later, limit scheme constructions of the IIC were given for high-dimensional models as well. The IIC for spread-out *oriented* percolation above $4+1$ dimensions was constructed in [25], for finite-range spread-out percolation above 6 dimensions in [27], and in the general setting discussed above in [20].

The simplest known construction goes as follows. Define, for every event F that depends on the occupation status of *finitely many bonds*,

$$\mathbb{P}_{\text{IIC}}(F) \equiv \lim_{p \nearrow p_c} \frac{\sum_{x \in \mathbb{Z}^d} \mathbb{P}_p(F \cap \{0 \leftrightarrow x\})}{\sum_{x \in \mathbb{Z}^d} \mathbb{P}_p(0 \leftrightarrow x)}. \tag{1.9}$$

where we define the *susceptibility* $\chi(p) \equiv \sum_x \mathbb{P}_p(0 \leftrightarrow x)$. Because of the appearance of the factor $\chi(p)$ on the right-hand side of (1.9), we call this limit the susceptibility limit. It is proved in [20] that this construction works in the generalized setting of high-dimensional percolation (i.e., if the dimension is sufficiently high and if Assumption D holds). It is also shown in [20] that several related and natural constructions lead to the *same* limit. This indicates that the IIC is a natural and robust object.

The IIC in high-dimensional percolation has attracted considerable attention. For instance, it has been observed that random walk on the IIC is strongly subdiffusive. This phenomenon has been studied extensively in recent years (cf. [2, 21, 30]).

Another aspect of the IIC that has been studied is its scaling limit. It is widely believed that the scaling limit of very large critical percolation clusters is super-Brownian motion (SBM). There is plenty of supporting evidence for this conjecture, much of it coming from studies of the IIC. Indeed, the asymptotics of the r-point functions of the oriented percolation IIC have been identified as those for the canonical measure of super-Brownian motion conditioned to survive forever (ICSBM) [24]. The ICSBM measure was introduced by Evans [13]. It consists of a single infinite Brownian motion path (the immortal particle) together with super-Brownian motions branching off this path. The ICSBM can be viewed as an SBM conditioned to have infinite mass.

In this paper we take a step towards identifying the scaling limit of the IIC of high-dimensional percolation by identifying the scaling limit of a subgraph of the IIC that corresponds to the trace of the immortal particle of ICSBM. Indeed, the IIC contains an (essentially) unique infinite path, in the sense that there exists a random infinite family of bonds, called *backbone pivotal bonds*, that are contained in every unbounded path in the IIC. The union of all simple infinite paths starting from the origin is called the IIC *backbone*. This object should play a similar role as the immortal particle for ICSBM. We show that the scaling limit of the IIC backbone is Brownian motion for finite-variance models, while it is a stable motion for infinite-variance models. This is consistent with the conjecture that the IIC is super-Brownian or -stable motion conditioned to survive forever, and it might bring its proof significantly forward.

1.1. Main Results

The backbone-pivotal bonds are ordered by their appearance as $(e_i)_{i=1}^{\infty}$, in such a way that every infinite self-avoiding walk (SAW) started at the origin passes through e_i before it passes through e_{i+1}. Also, we can think of the backbone-pivotal bonds as being *directed* as $b = (x, y)$, where the direction is such that $\{0 \leftrightarrow x\}$ uses different bonds than $\{y \leftrightarrow \infty\}$. For a directed bond $b = (x, y)$, we write $\underline{b} = x$ for its *bottom*, and $\overline{b} = y$ for its *top*. Then we write

$$S_n = \overline{e}_n \tag{1.10}$$

for the lattice position of the top of the nth backbone pivotal bond e_n, with the convention that $S_0 = \overline{e}_0 = 0$. The process $(S_n)_{n=0}^{\infty}$ is a stochastic process, and we will study its scaling limit in high dimensions, where geometry tends to trivialize. That geometry is trivial in high dimensions can be understood by noting that the displacement $S_n - S_{n-1} = \overline{e}_n - \overline{e}_{n-1}$ is the displacement between two subsequent backbone pivotals, and, in high dimensions, these displacements should be weakly dependent. Therefore, we expect that the scaling limit of $(S_n)_{n=0}^{\infty}$ is the same as the scaling limit of a random walk with independent and identically distributed steps. This suggests that the scaling limit is either a Brownian motion or a stable motion, depending on the number of existing (spatial) moments of $\overline{e}_n - \overline{e}_{n-1}$. We define the scaling function as

$$f_\alpha(n) = \begin{cases} (v_\alpha n)^{-1/(2\wedge\alpha)} & \text{if } \alpha \neq 2, \\ (v_2 n \log n)^{-1/2} & \text{if } \alpha = 2. \end{cases} \tag{1.11}$$

Furthermore, we define the stochastic process X_n as

$$X_n(t) = f_\alpha(n)\, S_{\lceil nt \rceil}, \qquad t \geq 0. \tag{1.12}$$

Our results apply to the IIC but also to critical percolation. We need a few definitions to state these results for critical percolation. In the context of critical percolation, whenever we say that a bond b is *pivotal* we mean that it is pivotal for the event $\{x \leftrightarrow y\}$, i.e., if x is connected to y on the (possibly modified) configuration where b is made occupied, while x is not connected to y on the (possibly modified) configuration where b is made vacant. For every $n \geq 1$, we define a probability measure on *marked configurations*, i.e., on the set of pairs (ω, x) (where ω is a percolation configuration and $x \in \mathbb{Z}^d$), by

$$\mathbb{E}^*_{p_c,n}[F] = \frac{\mathbb{E}_{p_c}\Big[\sum_{x\in\mathbb{Z}^d} F(\cdot, x)\, \mathbb{1}_{\{0\leftrightarrow x \text{ with } n-1 \text{ occupied pivotal bonds}\}}\Big]}{\mathbb{E}_{p_c}\Big[\sum_{x\in\mathbb{Z}^d} \mathbb{1}_{\{0\leftrightarrow x \text{ with } n-1 \text{ occupied pivotal bonds}\}}\Big]} \tag{1.13}$$

for every non-negative measurable function $F = F(\omega, x)$. We write x_* for the distinguished vertex under $\mathbb{P}^*_{p_c,n}$ (but note that x_* is a random variable with respect to $\mathbb{P}^*_{p_c,n}$). Under this measure, we let $S'_0 = 0$, and $S'_1, S'_2, \ldots, S'_{n-1}$ be the top of the ith pivotal bond for the event $\{0 \leftrightarrow x_*\}$, and $S'_n = x_*$. This defines the random process $\big(S'_i\big)_{i=1}^{n}$.

Let Y_n be the rescaled version of $\big(S'_i\big)_{i=1}^{n}$,

$$Y_n(t) = f_\alpha(n)\, S'_{\lceil nt \rceil}, \qquad t \in [0,1]. \tag{1.14}$$

In the following theorem we let $(B^{(2)}_t, t \geq 0)$ be a standard d-dimensional Brownian motion, and for $\alpha \in (0,2)$ we let $(B^{(\alpha)}_t, t \geq 0)$ be a symmetric stable process with $\mathbb{E}[\exp(\lambda B^{(\alpha)}_1)] = \exp\{-|\lambda|^\alpha\}$.

Theorem 1.1 [Backbone scaling limit]. *Consider a nearest-neighbor model in sufficiently high dimension, or a finite-range spread-out model in dimension $d > 6$ with a sufficiently large spread-out parameter L, or a power-law spread-out model in dimension $d > 3(2 \wedge \alpha)$ with L sufficiently large. Then, there exist $K_\alpha > 0$ (depending on d, α, L) such that the following convergences in distribution hold as $n \to \infty$ in the space of right-continuous functions with left limits, respectively*

$\mathbb{D}([0,\infty),\mathbb{R}^d)$ *and* $\mathbb{D}([0,1],\mathbb{R}^d)$*, endowed with the Skorokhod* J_1 *topology:*

$$X_n \Rightarrow (K_\alpha)^{\frac{1}{2\wedge\alpha}} B^{(2\wedge\alpha)} \quad \textit{and} \quad Y_n \Rightarrow (K_\alpha)^{\frac{1}{2\wedge\alpha}} B^{(2\wedge\alpha)}. \tag{1.15}$$

The proof is similar in spirit to the proof of [19, Theorem 1.5], where it is shown that long-range self-avoiding walk has a similar scaling limit for $d > 2(2\wedge\alpha)$. In particular, both proofs use a lace expansion. The lace expansion in this paper takes the displacement along the backbone-pivotals into account. This approach makes it possible to analyse the resulting equations in a similar way as was done for self-avoiding walk.

The condition of Theorem 1.1 is that d or L is "sufficiently large", but how large is sufficient? It turns out that we can associate a parameter to every distribution D. We call this the *mean-field parameter* β. For nearest neighbor models we let $\beta = O(d^{-1})$ and for spread-out models we let $\beta = O(L^{-d})$. The mean-field parameter β can be made arbitrarily small by increasing d or L. In Section 4 we use the lace expansion to establish bounds on certain quantities in terms of power series in β. A small β ensures that these series converge.

The mean-field parameter β also appears in the bound on the triangle diagram,

$$\sum_{x,y} \mathbb{P}_{p_c}(0\leftrightarrow x)\mathbb{P}_{p_c}(x\leftrightarrow y)\mathbb{P}_{p_c}(y\leftrightarrow 0) = 1 + O(\beta). \tag{1.16}$$

This bound is known as the *strong triangle condition*, and it holds under the conditions of Theorem 1.1, [15, 23].

Theorem 1.1 shows that the set of pivotal bonds of the IIC backbone is close to the image of a Brownian or α-stable motion when properly renormalized. It is natural to ask whether the geometry of the entire backbone is well-captured by the pivotal bonds. Let $\mathcal{B}_n$ be the set of vertices of $\mathbb{Z}^d$ that are in the backbone, and that are separated from the origin by at most n backbone pivotal bonds. With this definition, the increasing union $\mathcal{B}_\infty$ of all the sets $\mathcal{B}_n, n \geq 0$ is the set of vertices of the backbone. For $n \geq 1$, we also let $\mathcal{S}_n = \mathcal{B}_n \setminus \mathcal{B}_{n-1}$, and let $\mathcal{S}_0 = \mathcal{B}_0$. The set $\mathcal{S}_n$ can be viewed as a the vertex set of the subgraph that is induced by the backbone. It is a doubly-connected graph (in the sense that removing any one of its edges cannot disconnect the graph). We call $\mathcal{S}_n$ the nth "sausage" of the backbone. Note that $\mathcal{S}_n$ could be just a single vertex.

We let $\mathcal{K}$ be the set of non-empty compact subsets of $\mathbb{R}^d$. The *Hausdorff distance* d_H between two subsets $A, A' \subseteq \mathbb{R}^d$ is given by

$$d_H(A,A') = \sup_{x\in A} d(x,A') \vee \sup_{x'\in A'} d(x',A),$$

where $d(x,A') = \inf_{x'\in A'} |x-x'|$. The space $(\mathcal{K}, d_H)$ is a complete, locally compact metric space [7].

In the remainder of the introduction, we are going to work under an extra hypothesis. We believe that this hypothesis is true in general, but we have only been able to prove it in certain cases (finite-range models are one such case). Under this hypothesis we can prove that the sausages are uniformly small in the scale $f_\alpha(n)$, so that compact subsets of the backbone are close – in the Hausdorff sense – to the set of pivotals. Recall that for events A, B, the event $A \circ B$ is the event of disjoint occurrence of A and B (see [14] for a definition).

Hypothesis H. *There exists a finite constant* $C > 0$ *such that for every* $m \geq 1$*,*

$$\max_{|x|\leq m} \mathbb{P}_{p_c}(\exists y \in \mathbb{Z}^d : |y| > 2m, \{0\leftrightarrow y\}\circ\{x\leftrightarrow y\}) \leq \frac{C}{m^{2(2\wedge\alpha)}}. \tag{1.17}$$

While we do not have a proof that shows that Hypothesis H holds under the general assumptions used in this paper, we prove it under slightly stronger assumptions:

Proposition 1.2 [Verification of Hypothesis H]. *Hypothesis H holds for*
(i) finite-range percolation under the strong triangle condition (1.16)*;*
(ii) long-range percolation under the assumption that $d > 4(2\wedge\alpha)$.

Theorem 1.3 [Hausdorff convergence of the IIC backbone]. *Assume Hypothesis H. Under the conditions of Theorem 1.1, and under the probability measure $\mathbb{P}_{\text{IIC}}$, for every $T \geq 0$, the following convergence in distribution holds in the space $(\mathcal{K}, d_H)$:*

$$(K_\alpha)^{(2\wedge\alpha)} f_\alpha(n)\, \mathcal{B}_{\lfloor nT \rfloor} \Rightarrow \{B_t^{(2\wedge\alpha)} : 0 \leq t \leq T\}^{\text{cl}},$$

where A^{cl} is the closure of $A \subseteq \mathbb{R}^d$.

Several variants of this result can be stated. For instance, we can view $(\mathcal{S}_{\lfloor nt \rfloor}, t \geq 0)$ and $(\mathcal{B}_{\lfloor nt \rfloor}, t \geq 0)$ as stochastic processes in the Skorokhod space $\mathbb{D}([0,\infty), \mathcal{K})$ endowed with the inherited Skorokhod J_1 topology [12].

Proposition 1.4. *Assume Hypothesis H. Under the conditions of Theorem 1.1, and under the probability measure $\mathbb{P}_{\text{IIC}}$,*

$$\begin{aligned} \left((K_\alpha)^{(2\wedge\alpha)} f_\alpha(n)\, \mathcal{S}_{\lfloor nt \rfloor}, t \geq 0\right) \quad &\Rightarrow \quad \left(\{B_t^{(2\wedge\alpha)}\}, t \geq 0\right), \\ \left((K_\alpha)^{(2\wedge\alpha)} f_\alpha(n)\, \mathcal{B}_{\lfloor nt \rfloor}, t \geq 0\right) \quad &\Rightarrow \quad \left(\{B_s^{(2\wedge\alpha)} : 0 \leq s \leq t\}^{\text{cl}}, t \geq 0\right), \end{aligned}$$

in distribution in $\mathbb{D}([0,\infty), \mathcal{K})$.

This result is proved in Section 8. Theorem 1.3 follows as a corollary.

1.2. Further results

In this section, we state some results that are used in the proof of Theorem 1.1 and that are interesting in their own right as well. The first such result is that the fixed time marginal distribution of the process X_n converges.

For $x \in \mathbb{Z}^d$, we define the IIC backbone two-point function by

$$\varrho_n(x) = \mathbb{Q}_{\text{IIC}}(S_n = x), \tag{1.18}$$

i.e., ϱ_n is the probability mass function for the position of the top of the nth backbone-pivotal. We study the two-point function with a *fixed* number of pivotal bonds,

$$\tau_n(x) = \mathbb{P}_{p_c}(0 \leftrightarrow x \text{ with } n \text{ pivotal bonds}). \tag{1.19}$$

For ϱ_n and τ_n we prove the following:

Theorem 1.5 [Weak convergence of end-to-end displacement]. *Let $k_n = f_\alpha(n)k$. Under the assumptions of Theorem 1.1, there exist positive constants $K_\alpha, A \in (0,\infty)$ (all of these constants depend on d and D, and α) such that the Fourier transforms of the IIC backbone two-point function ϱ_n and the two-point function with a fixed number of pivotals τ_n satisfy*

$$\hat{\varrho}_n(k_n) \to \exp\{-K_\alpha |k|^{2\wedge\alpha}\}, \quad \hat{\tau}_n(k_n) \to A \exp\{-K_\alpha |k|^{2\wedge\alpha}\} \quad \text{as } n \to \infty, \text{ uniformly in } k \in \mathbb{R}^d. \tag{1.20}$$

The constants that appear in this theorem are defined in terms of the lace-expansion coefficients: K_α is defined in (2.32), and A in (2.38) below.

The *mean-r displacement* along the IIC backbone is defined as the rth spatial moment of $\varrho_n(x)$. We write $f \asymp g$ if there are uniform positive constants c, C such that $cg \leq f \leq Cg$.

Theorem 1.6 [Mean-r displacement]. *Under the assumptions of Theorem 1.1, for any $r < (2 \wedge \alpha)$,*

$$\left(\sum_{x \in \mathbb{Z}^d} |x|^r \varrho_n(x)\right)^{1/r} \asymp \left(\sum_{x \in \mathbb{Z}^d} |x|^r \tau_n(x)\right)^{1/r} \asymp \begin{cases} n^{1/(2\wedge\alpha)}, & \text{if } \alpha \neq 2, \\ (n \log n)^{1/2}, & \text{if } \alpha = 2, \end{cases} \tag{1.21}$$

as $n \to \infty$.

Recently, Chen and Sakai [9] have shown that the same bounds as in (1.21) hold for *all* $r \in (0, \alpha)$ for the mean-r displacement of long-range self-avoiding walk and long-range oriented percolation. In their work, they also identify leading order constants and give bounds on the error terms.

The weaker statement in Theorem 1.6 lets us prove tightness of the sequences X_n and Y_n, and thus suffices for our purposes.

1.3. Discussion

The scaling limit of the IIC backbone is an important ingredient in the study of random walk on high-dimensional incipient infinite cluster in [21] (in particular, Assumption (S) therein). Indeed, Theorem 1.1 is used to estimate the number of backbone pivotals between the origin and the boundary of a large Euclidean ball. This in turn gets a lower bound on the effective resistance between the origin and the boundary of the ball. Effective resistances are key quantities when studying random walk properties (cf. [11]).

It would be interesting to see if we can use this result to identify the scaling limit of the complete IIC. Indeed, the IIC can be viewed as a backbone with 'critical clusters' connected to it by a single bond, as is the case for the IIC on the tree. But on the tree the critical clusters attached to the backbone are *independent and identically distributed*, whereas for the IIC on $\mathbb{Z}^d$, they are *mutually avoiding*. Perhaps the results for critical clusters, as in [17, 18], can be combined with the scaling limit of the backbone to get the scaling limit for the full IIC. This approach may be easier for *oriented percolation*, where we our knowledge the scaling limit of large critical clusters is more precise [28].

1.4. Overview

The rest of this paper is organized as follows. In Section 2 we give an outline of the proof of our main result. In Section 3 we get the lace expansion for $\varrho_m(x)$ and $\tau_m(x)$, establishing (2.7) and (2.9). In Section 4 we get moment estimates for the lace expansion coefficients $\pi_m(x)$ and $\psi_m(x)$ (formulated in Propositions 4.1–4.4). These are the key estimates for the remainder of the paper. In Section 5 we use these moment estimates to prove Propositions 2.3 and 2.4, thus completing the proof of Theorem 1.5. In Section 6 we prove Theorem 1.6. In Section 7 we complete the proof of Theorem 1.1 by proving Proposition 2.1 and Proposition 2.2. In Section 8 we discuss convergence in path space and prove Theorem 1.3 and Proposition 1.2. In the supplementary material to this paper [22] we prove of Propositions 4.1, 4.3 and Lemma 4.5.

2. Outline of the proof of Theorem 1.1

In this section we give an overview of the proof of our main result, Theorem 1.1. The claim that the backbones of the IIC and of large critical clusters converge in distribution as a process is proved if we manage to prove that the process has the following two properties: (1) its finite-dimensional distributions converge, and (2) the families $\{Y_n\}_{n=0}^{\infty}$ and $\{X_n\}_{n=0}^{\infty}$ are tight. To prove that X_n converges in distribution in $\mathbb{D}([0,\infty),\mathbb{R}^d)$, it suffices if we prove that the restriction of X_n to the interval $[0,T]$ converges in distribution (in $\mathbb{D}([0,T],\mathbb{R}^d)$) for every $T>0$. And by a simple scaling argument this is equivalent to proving the case where $T=1$. Therefore, we only consider the restriction of the process X_n to $[0,1]$ from now on.

Convergence of finite-dimensional distributions. By convergence of finite-dimensional distributions we mean that for every $N=1,2,3,\dots$, any $0<t_1<\cdots<t_N\le 1$, and any bounded continuous function $g\colon (\mathbb{R}^d)^N\to\mathbb{R}$,

$$\lim_{n\to\infty}\mathbb{E}_{\mathrm{IIC}}\big[g\big(X_n(t_1),\dots,X_n(t_N)\big)\big]=\mathbb{E}\big[g\big(B^{(2\wedge\alpha)}(t_1),\dots,B^{(2\wedge\alpha)}(t_N)\big)\big]. \tag{2.1}$$

If we have convergence of the characterstic functions, then convergence in distribution follows, so it suffices if we only consider functions g of the form

$$g(x_1,\dots,x_N)=\exp\{i\,\mathsf{k}\cdot(x_1,\dots,x_N)\}, \tag{2.2}$$

where $\mathsf{k} = \big(k^{(1)}, \dots, k^{(N)}\big) \in \mathbb{R}^{dN}$ and $x_i \in \mathbb{R}^d$, $i = 1, \dots, N$. The problem becomes easier still if we use the equivalent form

$$g(x_1, \dots, x_N) = \exp\{i\,\mathsf{k} \cdot (x_1, x_2 - x_1, \dots, x_N - x_{N-1})\}. \tag{2.3}$$

For $\mathsf{n} = (n^{(1)}, \dots, n^{(N)}) \in \mathbb{N}^N$ with $n^{(1)} < \dots < n^{(N)}$, we define

$$\underline{\hat\varrho}^{(N)}_{\mathsf{n}}(\mathsf{k}) = \mathbb{E}_{\mathsf{IIC}}\Big[\exp\Big\{i \sum_{j=1}^{N} k^{(j)} \cdot \big(S_{n^{(j)}} - S_{n^{(j-1)}}\big)\Big\}\Big] \tag{2.4}$$

as the characteristic function of the increments of $(S_n)_{n\ge 0}$, with $n^{(0)} = 0$. The quantity $\underline{\hat\tau}^{(N)}_{\mathsf{n}}(\mathsf{k})$ is defined accordingly, with S_n replaced by S'_n, and $\mathbb{E}_{\mathsf{IIC}}$ in (2.1) replaced by $\mathbb{E}^*_{p_c,n}$.

Proposition 2.1 [Finite-dimensional distributions]. *Let N be a positive integer, $k^{(1)}, \dots, k^{(N)} \in \mathbb{R}^d$, $0 = t^{(0)} < t^{(1)} < \dots < t^{(N)} < 1$. Denote*

$$\mathsf{k}_n = \big(k^{(1)}_n, \dots, k^{(N)}_n\big) = f_\alpha(n)\big(k^{(1)}, \dots, k^{(N)}\big), \qquad n\mathsf{t} = \big(\lfloor n t^{(1)}\rfloor, \dots, \lfloor n t^{(N-1)}\rfloor, \lfloor n t^{(N)}\rfloor\big). \tag{2.5}$$

Under the conditions of Theorem 1.1,

$$\lim_{n\to\infty} \underline{\hat\varrho}^{(N)}_{n\mathsf{t}}(\mathsf{k}_n) = \lim_{n\to\infty} \frac{\underline{\hat\tau}^{(N)}_{n\mathsf{t}}(\mathsf{k}_n)}{\hat\tau_{n t^{(N)}}(0)} = \exp\Big\{-K_\alpha \sum_{j=1}^{N} |k^{(j)}|^{2\wedge\alpha}\,(t^{(j)} - t^{(j-1)})\Big\}. \tag{2.6}$$

We conclude that the finite-dimensional distributions of the finite-range and long-range IIC backbone converge to those of Brownian motion or of α-stable Lévy motion, respectively. This also proves that Brownian or α-stable motion is the only possible scaling limit for the backbone process.

Tightness. To prove that the sequences X_n and Y_n converge in the space of càdlàg-functions, as we claimed in Theorem 1.1, we need to show that their finite-dimensional distributions converge (i.e., the previous proposition), but we also need that X_n and Y_n are *tight*. Using the moment estimates in Theorem 1.6 and a self-repellence property of X_n and Y_n (see Lemmas 3.3 and 3.6 below) we can easily prove tightness. We prove the following statement in Section 7.2.

Proposition 2.2. *The sequences X_n in* (1.12) *and Y_n in* (1.14) *are tight in $\mathbb{D}\big([0,1],\mathbb{R}^d\big)$.*

Lace expansion. Our proofs use a *lace expansion* that gets an expansion of the form

$$\varrho_{n+1}(x) = \psi_{n+1}(x) + \sum_{m=0}^{n} \big(\pi_m * p_c D * \varrho_{n-m}\big)(x) \tag{2.7}$$

for certain lace expansion coefficients $\pi_m(x), \psi_n(x)$. Define the *two-point function*

$$\tau_p(x) = \mathbb{P}_p(0 \leftrightarrow x). \tag{2.8}$$

The best-known lace expansion gives an expansion for $\tau_p(x)$ (cf. [5, 15]). So far, almost all results for high-dimensional have been proved with this lace expansion. We derive (2.7) in Section 3. We also derive a lace expansion for the critical two-point function with a fixed number of pivotals in Section 3, i.e., for τ_n as defined in (1.19). This expansion reads

$$\tau_{n+1}(x) = \pi_{n+1}(x) + \sum_{m=0}^{n} \big(\pi_m * p_c D * \tau_{n-m}\big)(x). \tag{2.9}$$

The coefficients $\pi_m(x)$ are the same as the ones appearing in (2.7).

We can use expansion (2.7) to prove Theorem 1.5. If we multiply (2.7) by z^{n+1} and sum over $n \ge 0$ we get

$$\mathrm{P}_z(x) = \Psi_z(x) + (z p_c D * \mathrm{P}_z * \Pi_z)(x), \qquad \mathrm{T}_z(x) = \Pi_z(x) + (z p_c D * \mathrm{T}_z * \Pi_z)(x) \tag{2.10}$$

where we define

$$\mathrm{P}_z(x) = \sum_{n=0}^{\infty} \varrho_n(x) z^n, \qquad \mathrm{T}_z(x) = \sum_{n=0}^{\infty} \tau_n(x) z^n. \tag{2.11}$$

and

$$\Pi_z(x) = \sum_{n=0}^{\infty} \pi_n(x) z^n, \qquad \Psi_z(x) = \sum_{n=0}^{\infty} \psi_n(x) z^n \tag{2.12}$$

The generating functions $\mathrm{P}_z(x)$ and $\mathrm{T}_z(x)$ are power-series in z. Since $\sum_x \varrho_n(x) = 1$ for all $n \geq 0$,

$$\sum_x \mathrm{P}_z(x) = \frac{1}{1-z}. \tag{2.13}$$

It follows that the radius of convergence z_c of the power-series $\sum_x \mathrm{P}_z(x)$ equals $z_c = 1$. Furthermore, we can use (2.7) to identify

$$\sum_x \mathrm{P}_z(x) = \frac{\sum_x \Psi_z(x)}{1 - zp_c \sum_x \Pi_z(x)}, \tag{2.14}$$

and for T_z,

$$\sum_x \mathrm{T}_z(x) = \frac{\sum_x \Pi_z(x)}{1 - zp_c \sum_x \Pi_z(x)}. \tag{2.15}$$

When we compare (2.14) and (2.15) we can see that $z_c = 1$ is also the radius of convergence of $\sum_x \mathrm{T}_z(x)$. Note that this only holds if the Fourier transform $\hat{\Psi}_z(k)$ is uniformly bounded in $z \leq 1$ and $k \in \mathbb{R}^d$. This turns out to be a simple consequence of bounds that we prove later (see Proposition 4.4), but until the end of this outline we simply assume boundedness.

Analyzing the expansion for P_z. We proceed by proving Theorem 1.5 subject to certain bounds on the lace-expansion coefficients $\pi_n(x)$. We will state these bounds below. Our argument is roughly the same as the arguments of [8] and [19, Section 2.1]. The main difference is that we have to deal with the term $\hat{\Psi}_z(k)$ in the numerator of (2.17), and complicates the analysis.

Taking the Fourier transformation of (2.10) gets

$$\hat{\mathrm{P}}_z(k) = \hat{\Psi}_z(k) + zp_c \hat{D}(k) \hat{\Pi}_z(k) \hat{\mathrm{P}}_z(k) \qquad k \in [-\pi,\pi)^d, \tag{2.16}$$

and this can be solved for $\hat{\mathrm{P}}_z(k)$ as

$$\hat{\mathrm{P}}_z(k) = \frac{\hat{\Psi}_z(k)}{1 - zp_c \hat{D}(k)\hat{\Pi}_z(k)} = \frac{\hat{\Psi}_1(0)}{1 - zp_c \hat{D}(k)\hat{\Pi}_z(k)} - \frac{\hat{\Psi}_1(0) - \hat{\Psi}_z(k)}{1 - zp_c \hat{D}(k)\hat{\Pi}_z(k)}, \qquad k \in [-\pi,\pi)^d. \tag{2.17}$$

Note that if we set $z = 1$ and $k = 0$ and we compare (2.17) with (2.12) and (2.13), we get

$$1 = p_c \hat{\Pi}_1(0) = p_c \sum_{x \in \mathbb{Z}^d} \sum_{n \in \mathbb{N}} \pi_n(x). \tag{2.18}$$

We rewrite the first denominator on the right-hand side of (2.17) as

$$1 - zp_c \hat{D}(k)\hat{\Pi}_z(k) = (1-z)\hat{A}(k) + \hat{B}(k) + \hat{E}_z(k) \tag{2.19}$$

with

$$\begin{aligned}
\hat{A}(k) &\equiv p_c \hat{D}(k) \left(\hat{\Pi}_1(k) + \partial_z \hat{\Pi}_z(k)|_{z=1} \right), && (2.20)\\
\hat{B}(k) &\equiv [1 - \hat{D}(k)] + p_c \hat{D}(k) \left(\hat{\Pi}_1(0) - \hat{\Pi}_1(k) \right), && (2.21)\\
\hat{E}_z(k) &\equiv zp_c \hat{D}(k) \left(\hat{\Pi}_1(k) - \hat{\Pi}_z(k) \right) - (1-z)\, p_c \hat{D}(k)\, \partial_z \hat{\Pi}_z(k)|_{z=1}, && (2.22)
\end{aligned}$$

so that

$$\hat{\mathrm{P}}_z(k) = \frac{\hat{\Psi}_1(0)}{(1-z)\hat{A}(k) + \hat{B}(k)} - \hat{\Theta}_z(k), \tag{2.23}$$

where

$$\hat{\Theta}_z(k) = \frac{\hat{\Psi}_1(0)\, \hat{E}_z(k)}{((1-z)\hat{A}(k) + \hat{B}(k))^2} + \frac{\hat{\Psi}_1(0) - \hat{\Psi}_z(k)}{1 - zp_c \hat{D}(k)\hat{\Pi}_z(k)}. \tag{2.24}$$

For the first term in (2.23) we write

$$\frac{\hat{\Psi}_1(0)}{(1-z)\hat{A}(k) + \hat{B}(k)} = \frac{\hat{\Psi}_1(0)}{\hat{A}(k) + \hat{B}(k)} \sum_{n=0}^{\infty} z^n \left(\frac{\hat{A}(k)}{\hat{A}(k) + \hat{B}(k)} \right)^n. \tag{2.25}$$

The geometric sum converges whenever $z < (\hat{A}(k)+\hat{B}(k))/\hat{A}(k)$ and the right-hand side approaches 1 as $|k| \to 0$. For $z < 1$, we can write $\hat{\Theta}_z(k)$ as a power series in z as well, i.e.,

$$\hat{\Theta}_z(k) = \sum_{n=0}^{\infty} \hat{\theta}_n(k)\, z^n. \tag{2.26}$$

In Section 5 we prove the following bound on the error term $\hat{\theta}_n$:

Proposition 2.3 [Error bound for percolation lace expansion]. *Under the conditions of Theorem 1.1, there exists an* $0 < \varepsilon < 1$ *and an* $N = N(\varepsilon)$ *such that, for* $n \geq N$, $|\hat{\theta}_n(k)| \leq O(n^{-\varepsilon} + |k|^{\varepsilon} \log n)$ *for all* $k \in [-\pi, \pi)^d$.

We prove this proposition in Section 5.

Recall that $\hat{\mathrm{P}}_z(k) = \sum_{n=0}^{\infty} \hat{\varrho}_n(k) z^n$, so Proposition 2.3 implies that there exists $0 < \varepsilon < 1$ and $N = N(\varepsilon) \geq 0$ such that for all $n \geq N$ and $k \in \mathbb{R}^d$ that satisfy $f_\alpha(n) k \equiv k_n \in [-\pi, \pi)^d$, we have

$$\hat{\varrho}_n(k_n) = \frac{\hat{\Psi}_1(0)}{\hat{A}(k_n) + \hat{B}(k_n)} \left(\frac{\hat{A}(k_n)}{\hat{A}(k_n) + \hat{B}(k_n)} \right)^n + O\big(n^{-\varepsilon} + |k|^{\varepsilon} \log n\big). \tag{2.27}$$

To determine the values of these coefficients, we study the small-k behaviour of the lace expansion coefficients $\hat{\Psi}_1(k)$ and $\hat{\Pi}_1(k)$.

Proposition 2.4. *Under the conditions of Theorem 1.1,*

$$\lim_{|k| \to 0} \frac{\hat{\Pi}_1(0) - \hat{\Pi}_1(k)}{1 - \hat{D}(k)} = \begin{cases} 1, & \text{if } \alpha \leq 2; \\ (2 d v_\alpha)^{-1} \sum_{x \in \mathbb{Z}^d} |x|^2\, \Pi_1(x), & \text{if } \alpha > 2. \end{cases} \tag{2.28}$$

This proposition is proved in Section 5. By Proposition 2.4,

$$\lim_{|k| \to 0} \frac{\hat{B}(k)}{1 - \hat{D}(k)} = \begin{cases} 1 + p_c, & \text{if } \alpha \leq 2; \\ 1 + p_c (2 d v_\alpha)^{-1} \sum_{x \in \mathbb{Z}^d} |x|^2\, \Pi_1(x), & \text{if } \alpha > 2. \end{cases} \tag{2.29}$$

Observe that k_n has been chosen such that

$$\lim_{n \to \infty} n[1 - \hat{D}(k_n)] = |k|^{2 \wedge \alpha}. \tag{2.30}$$

If a sequence h_n converges to a limit h, then the sequence $(1 - h_n/n)^n$ converges to e^{-h}. We apply this to (2.27) with

$$h_n = \frac{n \hat{B}(k_n)}{\hat{A}(k_n) + \hat{B}(k_n)} = n[1 - \hat{D}(k_n)]\, \frac{\hat{B}(k_n)}{1 - \hat{D}(k_n)}\, \frac{1}{\hat{A}(k_n) + \hat{B}(k_n)} \quad \overset{n \to \infty}{\longrightarrow} \quad K_\alpha |k|^{2 \wedge \alpha}, \tag{2.31}$$

where

$$K_\alpha \equiv \lim_{n \to \infty} \left(\frac{\hat{B}(k_n)}{1 - \hat{D}(k_n)} \cdot \frac{1}{\hat{A}(k_n) + \hat{B}(k_n)} \right) = \begin{cases} \dfrac{1 + p_c}{1 + p_c \sum_{x \in \mathbb{Z}^d} \sum_{n \in \mathbb{N}} n \pi_n(x)}, & \text{if } \alpha \leq 2; \\[2ex] \dfrac{1 + p_c (2 d v_\alpha)^{-1} \sum_{x \in \mathbb{Z}^d} \sum_{n \in \mathbb{N}} |x|^2 \pi_n(x)}{1 + p_c \sum_{x \in \mathbb{Z}^d} \sum_{n \in \mathbb{N}} n \pi_n(x)}, & \text{if } \alpha > 2. \end{cases} \tag{2.32}$$

This gives an explicit description of the constant K_α that was introduced in Theorem 1.5. For a sufficiently large spread-out constant L we can prove that the sums on the right-hand side are finite (see Section 4.2), so that $0 < K_\alpha < \infty$ as soon as L is sufficiently large.

When we apply this result to (2.27) we get

$$\lim_{n \to \infty} \hat{\varrho}_n(k_n) = \frac{\hat{\Psi}_1(0)}{\hat{A}(0)} \exp\big\{-K_\alpha |k|^{2 \wedge \alpha}\big\}. \tag{2.33}$$

Finally, if we set $k=0$ in (2.33) and we use $\hat{\varrho}_n(0)=1$, we get $\hat{\Psi}_1(0)=\hat{A}(0)$, and so we have identified the first limit in (1.20) subject to Propositions 2.3 and 2.4. Next we consider the limit behaviour of $\hat{\tau}_n$.

Analyzing the expansion for T_z. Taking the Fourier transformation of (2.10) we get

$$\hat{\mathrm{T}}_z(k)^{-1}=\frac{1}{\hat{\Pi}_z(k)}-zp_c\hat{D}(k). \tag{2.34}$$

Recall that $p_c\hat{\Pi}_1(0)=1$. Like we did for $\hat{\mathrm{P}}_z(k)$, we write $\hat{\mathrm{T}}(k)^{-1}=(1-z)\hat{A}'(k)+\hat{B}'(k)+\hat{E}'_z(k)$ where

$$\hat{A}'(k) \equiv -\partial_z\left(\frac{1}{\hat{\Pi}_z(k)}-zp_c\hat{D}(k)\right)\bigg|_{z=1}=\frac{\partial_z\hat{\Pi}_z(k)|_{z=1}}{\hat{\Pi}_1(k)^2}+p_c\hat{D}(k), \tag{2.35}$$

$$\hat{B}'(k) \equiv \frac{1}{\hat{\Pi}_1(k)}-\frac{1}{\hat{\Pi}_1(0)}+p_c[1-\hat{D}(k)], \tag{2.36}$$

$$\hat{E}'_z(k) \equiv \left(\frac{1}{\hat{\Pi}_z(k)}-\frac{1}{\hat{\Pi}_1(k)}\right)+(1-z)\partial_z\frac{1}{\hat{\Pi}_z(k)}\bigg|_{z=1}. \tag{2.37}$$

Now we repeat the same analysis as for $\hat{\mathrm{P}}_z(k)$. Indeed, if we define $\hat{\Theta}'_z(k)$ and $\hat{\theta}'_n(k)$ analogous to (2.24) and (2.26), and if we follow the same steps as in the proof of Proposition 2.3, we can easily show that also $|\hat{\theta}'_n(k)|\le O(n^{-\varepsilon}+|k|^{\varepsilon}\log n)$ for some $\varepsilon>0$. In particular, the bounds in Section 5 and the proof of Lemma 5.1 require only minor modifications to yield the correct bounds, if we replace $\hat{\Psi}$ by $\hat{\Pi}$, $\hat{A}$ by $\hat{A}'$, $\hat{B}$ by $\hat{B}'$, and $\hat{E}_z$ by $\hat{E}'_z$. It follows that if we use (2.18) and if we reason like (2.25)–(2.33), we find that $\lim_{n\to\infty}\hat{\tau}_n(k_n)=A\exp\{-K_\alpha|k|^{2\wedge\alpha}\}$ with

$$A=\lim_{|k|\to 0}\frac{1}{\hat{A}'(k)+\hat{B}'(k)}=\frac{1}{\hat{A}'(0)}=\frac{1}{p_c^2\sum_{x\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}}n\pi_n(x)+p_c} \tag{2.38}$$

and

$$K_\alpha=\lim_{|k|\to 0}\left(\frac{\hat{B}'(k)}{1-\hat{D}(k)}\cdot\frac{1}{\hat{A}'(k)+\hat{B}'(k)}\right)=\begin{cases}\dfrac{1+p_c}{1+p_c\sum_{x\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}}n\pi_n(x)}, & \text{if } \alpha\le 2;\\[2ex] \dfrac{1+p_c(2dv_\alpha)^{-1}\sum_{x\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}}|x|^2\pi_n(x)}{1+p_c\sum_{x\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}}n\pi_n(x)}, & \text{if } \alpha>2.\end{cases} \tag{2.39}$$

The limit in (2.39) is equal to the limit in (2.32).

3. Lace expansion for the backbone two-point function

In this section we derive the lace expansion for the backbone two-point function. Recall (1.19), and write $\tau_m(x,y)=\tau_m(y-x)$. We prove the existence of a family $\pi_l(x,y)$ ($l\in\mathbb{N}$, $x,y\in\mathbb{Z}^d$) such that

$$\tau_m(x,y)=\pi_m(x,y)+\sum_{l=0}^{m-1}\sum_{b\in\mathbb{B}}\pi_l(x,\underline{b})\,p\,D(b)\tau_{m-l-1}(\overline{b},y) \tag{3.1}$$

for the coefficients π_l that are given in (3.57) and (3.58) below. We achieve this goal by a lace expansion. The lace expansion below is *novel* for percolation.

We can take the Fourier transform of (3.1) to get

$$\hat{\tau}_m(k)=\hat{\pi}_m(k)+\sum_{l=0}^{m-1}\hat{\pi}_m(k)\,p\,\hat{D}(k)\,\hat{\tau}_{m-l-1}(k). \tag{3.2}$$

Equation (3.2) is the starting point of our analysis for the two-point function.

This section is divided into three parts. In the first part (Section 3.1) we rewrite the two-point function τ_m in such a way that we can expand it. The representation that we use is similar to that for self-avoiding walk, but is new in the context of percolation. In the second part (Section 3.2), we use this representation to derive the lace expansion for the two-point function using the algebraic expansion that was first derived by Brydges and Spencer in [6]. In the third part (Section 3.3) we adapt the argument to work for ϱ_n.

3.1. The fundamental rewrite of the two-point function

We write $x \Leftrightarrow y$ when there exist two bond-disjoint paths of occupied bonds that connect x to y, and we adopt the convention that $\{x \Leftrightarrow x\}$ is the full probability space. For $x, y \in \mathbb{Z}^d$, we write $\mathrm{Piv}(x, y)$ for the ordered list of *occupied and pivotal* (directed) bonds, i.e.,

$$\mathrm{Piv}(x,y) := \begin{cases} (b_1, b_2, \dots, b_m) & \text{if } x \Leftrightarrow \underline{b}_1, \overline{b}_1 \Leftrightarrow \underline{b}_2, \dots, \overline{b}_m \Leftrightarrow y, \text{ and } b_1, \dots, b_m \text{ are occupied;} \\ () & \text{if } x \Leftrightarrow y; \\ \varnothing & \text{if } x \not\leftrightarrow y. \end{cases} \tag{3.3}$$

Mind the difference between "$()$" (x and y are connected, but without any pivotal bonds) and "$\varnothing$" (x and y are not connected). With this definition, we partition τ_m according to the positions of the m pivotal bonds,

$$\tau_m(x,y) = \sum_{b_1,\dots,b_m} \mathbb{P}_p\big(\mathrm{Piv}(x,y) = (b_1,\dots,b_m)\big), \qquad m \ge 1, \tag{3.4}$$

and $\tau_0(x,y) = \mathbb{P}_p\big(\mathrm{Piv}(x,y) = ()\big)$.

The Factorization Lemma. Before we can state the lace expansion, we need to introduce some notation and recall a useful lemma.

Definition 3.1. *(i) Given a (deterministic or random) set of vertices A and a bond configuration ω, we define ω_A, the restriction of ω to A, to be*

$$\omega_A(\{x,y\}) = \begin{cases} \omega(\{x,y\}) & \text{if } x, y \in A, \\ 0 & \text{otherwise,} \end{cases} \tag{3.5}$$

for every nearest-neighbor pair x, y. In other words, we get ω_A from ω by making every bond that does not have both endpoints in A vacant.

(ii) Given a (deterministic or random) set of vertices A and an event E, we say that E occurs in *A, and write $\{E \text{ in } A\}$, if $\omega_A \in E$. In other words, $\{E \text{ in } A\}$ means that E occurs on the (possibly modified) configuration in which every bond that does not have both endpoints in A is made vacant. We adopt the convention that $\{x \leftrightarrow x \text{ in } A\}$ occurs if and only if $x \in A$. We further say that E* occurs off *A, and write $\{E \text{ off } A\}$, when E occurs in A^c.*

(iii) Given a bond configuration and $x \in \mathbb{Z}^d$, we define $\mathcal{C}(x)$ to be the set of vertices to which x is connected, i.e., $\mathcal{C}(x) = \{y \in \mathbb{Z}^d : x \leftrightarrow y\}$. Given a bond configuration and a bond b, we define $\tilde{\mathcal{C}}^b(x)$ to be the set of vertices $y \in \mathcal{C}(x)$ to which x is connected in the (possibly modified) configuration in which b is made vacant.

We often use the following, easily checked rules for "occurs in": for all events E, F and sets of vertices A, B,

$$\{E \text{ in } A\} \cap \{F \text{ in } A\} = \{E \cap F \text{ in } A\}, \tag{3.6}$$

$$\{E \text{ in } A\} \cup \{F \text{ in } A\} = \{E \cup F \text{ in } A\}, \tag{3.7}$$

$$\{E \text{ in } A\}^c = \{E^c \text{ in } A\}, \tag{3.8}$$

$$\big\{\{E \text{ in } A\} \text{ in } B\big\} = \{E \text{ in } A \cap B\}. \tag{3.9}$$

Equations (3.6)–(3.9) imply that "occurs in" is well behaved under set operations.

In terms of the above definition,

$$\{b_1 \text{ first occ. and piv. for } \{x \leftrightarrow y\}\} \\ = \{x \Leftrightarrow \underline{b}_1 \text{ in } \tilde{\mathcal{C}}^{b_1}(x)\} \cap \{b_1 \text{ occupied}\} \cap \{\overline{b}_1 \leftrightarrow y \text{ off } \tilde{\mathcal{C}}^{b_1}(x)\}. \tag{3.10}$$

Similarly, we get the following crucial identity:

$$\begin{aligned}&\{\mathrm{Piv}(x,y)=(b_1,\dots,b_m)\}\\&\qquad=\{x\Leftrightarrow \underline{b}_1 \text{ in } \tilde{\mathcal{C}}^{b_1}(x)\}\cap\{b_1 \text{ occupied}\}\cap\{\mathrm{Piv}(\overline{b}_1,y)=(b_2,\dots,b_m)\text{ off }\tilde{\mathcal{C}}^{b_1}(x)\}.\end{aligned}\tag{3.11}$$

Hence, we can rewrite

$$\begin{aligned}\tau_m(x,y)=\sum_{b_1,\dots,b_m}&\mathbb{P}_p\big(\{x\Leftrightarrow \underline{b}_1 \text{ in } \tilde{\mathcal{C}}^{b_1}(x)\}\\&\cap\{b_1 \text{ occupied}\}\cap\{\mathrm{Piv}(\overline{b}_1,y)=(b_2,\dots,b_m)\text{ off }\tilde{\mathcal{C}}^{b_1}(x)\}\big).\end{aligned}\tag{3.12}$$

We next investigate probabilities on the right-hand side of (3.12). A useful tool in this analysis is the *Factorization Lemma.* This lemma is the workhorse of our expansion method. In its statement, we write $\theta(p)=\mathbb{P}_p(|\mathcal{C}(x)|=\infty)$ for the probability that the cluster of $x\in\mathbb{Z}^d$ is infinite. For a proof, see [26, Lemma 2.2].

Lemma 3.2 [Factorization Lemma]. *For any p such that $\theta(p)=0$, any bond (u,v), vertex y and events E, F,*

$$\mathbb{P}_p\big(E\text{ in }\tilde{\mathcal{C}}^{(u,v)}(y), F\text{ off }\tilde{\mathcal{C}}^{(u,v)}(y)\big)=\mathbb{E}_0\Big(\mathbb{1}_{\{E\text{ in }\tilde{\mathcal{C}}_0^{(u,v)}(y)\}}\mathbb{P}_1\big(F\text{ off }\tilde{\mathcal{C}}_0^{(u,v)}(y)\big)\Big).\tag{3.13}$$

Moreover, when $E\subseteq\{u\in\tilde{\mathcal{C}}^{(u,v)}(y), v\notin\tilde{\mathcal{C}}^{(u,v)}(y)\}$, the event on the left-hand side of (3.13) is independent of the occupation status of (u,v).

In the nested expectation on the right-hand side of (3.13), the set $\tilde{\mathcal{C}}_0^{(u,v)}(y)$ is *random* with respect to the outer expectation, but *deterministic* with respect to the inner expectation. We have added a subscript “0” to $\tilde{\mathcal{C}}_0^{(u,v)}(y)$ and subscripts “0” and “1” to the expectations on the right-hand side of (3.13) to emphasize this difference. The inner expectation on the right-hand side is with respect to a second, independent percolation model on a second lattice. The second model interacts with the first model via the set $\tilde{\mathcal{C}}_0^{(u,v)}(y)$.

By the Factorization Lemma we may thus rewrite (3.12) as

$$\tau_m(x,y)=\sum_{b_1,\dots,b_m}pD(b_1)\mathbb{E}_0\Big[\mathbb{1}_{\{x\Leftrightarrow\underline{b}_1\text{ in }\tilde{\mathcal{C}}^{b_1}(x)\}}\mathbb{P}_1\big(\mathrm{Piv}(\overline{b}_1,y)=(b_2,\dots,b_m)\text{ off }\tilde{\mathcal{C}}_0^{b_1}(x)\big)\Big].\tag{3.14}$$

We can replace the event $\{x\Leftrightarrow\underline{b}_1\text{ in }\tilde{\mathcal{C}}_0^{b_1}(x)\}$ by the event $\{x\Leftrightarrow\underline{b}_1\}$, since if $\{x\Leftrightarrow\underline{b}_1\}$ occurs, but $\{x\Leftrightarrow\underline{b}_1\text{ in }\tilde{\mathcal{C}}_0^{b_1}(x)\}$ doesn't, then $\overline{b}_1\in\tilde{\mathcal{C}}_0^{b_1}(x)$. But if this is the case, then

$$\mathbb{P}_1\big(\mathrm{Piv}(\overline{b}_1,y)=(b_2,\dots,b_m)\text{ off }\tilde{\mathcal{C}}_0^{b_1}(x)\big)=0.\tag{3.15}$$

Therefore,

$$\tau_m(x,y)=\sum_{b_1,\dots,b_m}pD(b_1)\mathbb{E}_0\Big[\mathbb{1}_{\{x\Leftrightarrow\underline{b}_1\}}\mathbb{P}_1\big(\mathrm{Piv}(\overline{b}_1,y)=(b_2,\dots,b_m)\text{ off }\tilde{\mathcal{C}}_0^{b_1}(x)\big)\Big].\tag{3.16}$$

When $m=1$, the situation simplifies somewhat, because then $\mathrm{Piv}(\overline{b}_1,y)=()$, so that we can replace $\{\mathrm{Piv}(\overline{b}_1,y)=(b_2,\dots,b_m)\}$ by $\{\mathrm{Piv}(\overline{b}_1,y)=()\}=\{\overline{b}_1\Leftrightarrow y\}$.

The iteration. The probability $\mathbb{P}_1\big(\mathrm{Piv}(\overline{b}_1,y)=(b_2,\dots,b_m)\text{ off }\tilde{\mathcal{C}}_0^{b_1}(x)\big)$ is similar to the probability that we started with, $\mathbb{P}\big(\mathrm{Piv}(x,y)=(b_1,\dots,b_m)\big)$, but the space is smaller. We can continue by repeating the steps described above. We start by generalizing the setting so we can iterate these steps. For any fixed subset $A\subseteq\mathbb{Z}^d$ we have

$$\begin{aligned}\{\mathrm{Piv}(x,y)=(b_1,\dots,b_m)\text{ off }A\}=\{&\{x\Leftrightarrow\underline{b}_1\text{ off }A\}\text{ in }\tilde{\mathcal{C}}^{b_1}(x)\}\\&\cap\{b_1\text{ occupied}\}\cap\{\{\mathrm{Piv}(\overline{b}_1,y)=(b_2,\dots,b_m)\text{ off }A\}\text{ off }\tilde{\mathcal{C}}^{b_1}(x)\}.\end{aligned}\tag{3.17}$$

By the Factorization Lemma,

$$\mathbb{P}_1\big(\mathrm{Piv}(x,y)=(b_1,\dots,b_m)\text{ off }A\big) = pD(b_1)\mathbb{E}_0\Big[\mathbb{1}_{\{\{x\Leftrightarrow \underline{b}_1\text{ off }A\}\text{ in }\tilde{\mathcal{C}}^{b_1}(x)\}}\mathbb{P}_1\big(\{\mathrm{Piv}(\overline{b}_1,y)=(b_2,\dots,b_m)\text{ off }A\}\text{ off }\tilde{\mathcal{C}}_0^{b_1}(x)\big)\Big]. \tag{3.18}$$

Again, we can replace $\mathbb{1}_{\{\{x\Leftrightarrow \underline{b}_1\text{ off }A\}\text{ in }\tilde{\mathcal{C}}^{b_1}(x)\}}$ by $\mathbb{1}_{\{x\Leftrightarrow \underline{b}_1\text{ off }A\}}$ to arrive at

$$\mathbb{P}_1\big(\mathrm{Piv}(x,y)=(b_1,\dots,b_m)\text{ off }A\big) = pD(b_1)\mathbb{E}_0\Big[\mathbb{1}_{\{x\Leftrightarrow \underline{b}_1\text{ off }A\}}\mathbb{P}_1\big(\{\mathrm{Piv}(\overline{b}_1,y)=(b_2,\dots,b_m)\text{ off }A\}\text{ off }\tilde{\mathcal{C}}_0^{b_1}(x)\big)\Big]. \tag{3.19}$$

Furthermore, by (3.9),

$$\{\{\mathrm{Piv}(\overline{b}_1,y)=(b_2,\dots,b_m)\text{ off }A\}\text{ off }\tilde{\mathcal{C}}_0^{b_1}(x)\}=\{\mathrm{Piv}(\overline{b}_1,y)=(b_2,\dots,b_m)\text{ off }A\cup\tilde{\mathcal{C}}_0^{b_1}(x)\}. \tag{3.20}$$

As a result, we get

$$\tau_m(x,y)=\sum_{b_1,\dots,b_m} pD(b_1)pD(b_2)\mathbb{E}_0\Big[\mathbb{1}_{\{x\Leftrightarrow\underline{b}_1\}}\mathbb{E}_1\Big[\mathbb{1}_{\{\overline{b}_1\Leftrightarrow\underline{b}_2\text{ off }\tilde{\mathcal{C}}_0^{b_1}(x)\}} \times\mathbb{P}_2(\mathrm{Piv}(\overline{b}_2,y)=(b_3,\dots,b_m)\text{ off }\tilde{\mathcal{C}}_0^{b_1}(x)\cup\tilde{\mathcal{C}}_1^{b_2}(\overline{b}_1))\Big]\Big], \tag{3.21}$$

where, when $m=2$, we must replace $\{\mathrm{Piv}(\overline{b}_2,y)=(b_3,\dots,b_m)\text{ off }\tilde{\mathcal{C}}_0^{b_1}(x)\cup\tilde{\mathcal{C}}_1^{b_2}(\overline{b}_1)\}$ by $\{\overline{b}_2\Leftrightarrow y\text{ off }\tilde{\mathcal{C}}_0^{b_1}(x)\cup\tilde{\mathcal{C}}_1^{b_2}(\overline{b}_1)\}$.

The expansion for τ_m follows when we repeat the above steps m times. To make this precise we need some additional notation. Let $\overline{b}_0=x$ and $\underline{b}_{m+1}=y$, and for $j=0,\dots,m$ write

$$\tilde{\mathcal{C}}_j=\tilde{\mathcal{C}}_j^{b_{j+1}}(\overline{b}_j)\qquad\text{and}\qquad \tilde{\mathcal{C}}_{[s,j]}=\bigcup_{i=s}^{j}\tilde{\mathcal{C}}_i \tag{3.22}$$

where $[a,b]$ in the above definition means the set of *integers* $\{z\in\mathbb{Z}\mid a\le z\le b\}$). We get

$$\tau_m(x,y)=\sum_{b_1,\dots,b_m}\Big[\prod_{i=1}^m pD(b_i)\Big]\mathbb{E}_0\Big[\mathbb{1}_{\{x\Leftrightarrow\underline{b}_1\}}\mathbb{E}_1\Big[\mathbb{1}_{\{\overline{b}_1\Leftrightarrow\underline{b}_2\text{ off }\tilde{\mathcal{C}}_1\}_0} \times\cdots\times\mathbb{E}_{m-1}\Big[\mathbb{1}_{\{\overline{b}_{m-1}\Leftrightarrow\underline{b}_m\text{ off }\tilde{\mathcal{C}}_{[0,m-1]}\}_{m-2}}\mathbb{P}_m\big(\overline{b}_m\Leftrightarrow y\text{ off }\tilde{\mathcal{C}}_{[0,m-1]}\big)\Big]\cdots\Big]\Big]. \tag{3.23}$$

Let $\mathbb{P}^{\otimes[0,m]}$ be the product measure $\mathbb{P}_0\otimes\cdots\otimes\mathbb{P}_m$. We append a subscript i to an event to indicate that it is an event of the ith copy of the percolation model. Fubini's Theorem then allows us to rewrite (3.23) as

$$\tau_m(x,y)=\sum_{b_1,\dots,b_m}\Big[\prod_{i=1}^m pD(b_i)\Big]\mathbb{E}^{\otimes[0,m]}\Big[\prod_{i=0}^m\mathbb{1}_{\{\overline{b}_i\Leftrightarrow\underline{b}_{i+1}\text{ off }\tilde{\mathcal{C}}_{[0,i-1]}\}_i}\Big], \tag{3.24}$$

where $\overline{b}_0=x$, $\underline{b}_{m+1}=y$, and, by convention, $\tilde{\mathcal{C}}_{[0,-1]}=\varnothing$.

Equation (3.24) is the fundamental rewrite of the two-point function with a fixed number of pivotals. It is the basis of the lace expansion.

Self-repellence in percolation. In the following lemma we state a powerful consequence of (3.24):

Lemma 3.3 [Self-repellence]. *For every $x\in\mathbb{Z}^d$, $m\ge1$, and $s=0,\dots,m-1$,*

$$\tau_m(x)\le\sum_b\tau_s(\underline{b})\,pD(b)\,\tau_{m-s-1}(x-\overline{b})=\big(\tau_s*pD*\tau_{m-s-1}\big)(x). \tag{3.25}$$

Furthermore, writing $\mathcal{S}^x_{[i,j]}(y) = \mathcal{B}^x_j \setminus \mathcal{B}^x_{i-1}$ *for the subset of* $\mathcal{C}(y)$ *that consists of the vertices between the* i*th and* j*th pivotal for* $y \leftrightarrow x$*, we have for any increasing event* E *and all* $0 \le i \le j \le n$*,*

$$\begin{aligned}\mathbb{P}_p(0 \leftrightarrow x \text{ with } n \text{ pivotals}, E \text{ on } \mathcal{S}^x_{[i,j]}(0)) \\ \le \sum_{y,z} (pD * \tau_{i-1})(y)\mathbb{P}_p(y \leftrightarrow z \text{ with } i-j-1 \text{ pivotals}, E \text{ on } \mathcal{S}^z_{[0,i-j]}(y)) \\ \times (pD * \tau_{n-j-2})(x-y).\end{aligned} \tag{3.26}$$

We will use this lemma in Section 7.2 and the proof of Lemma 8.1 below.

The bound (3.25) gives a kind of self-repellence that is also present in (and highly useful for) self-avoiding walk.

Proof. Since $\tilde{\mathcal{C}}_{[s,i-1]} \subseteq \tilde{\mathcal{C}}_{[0,i-1]}$ for any $s = 0, 1, \ldots, i-1$, and since $\{x \Leftrightarrow y\}$ is an increasing event, we get

$$\mathbb{1}_{\{\overline{b}_i \Leftrightarrow \underline{b}_{i+1} \text{ off } \tilde{\mathcal{C}}_{[0,i-1]}\}_i} \le \mathbb{1}_{\{\overline{b}_i \Leftrightarrow \underline{b}_{i+1} \text{ off } \tilde{\mathcal{C}}_{[s+1,i-1]}\}_i}. \tag{3.27}$$

Thus, using (3.24),

$$\tau_m(x,y) \le \sum_{b_1,\ldots,b_m} \prod_{i=1}^m pD(b_i)\mathbb{E}^{\otimes[0,m]}\Big[\prod_{i=0}^s \mathbb{1}_{\{\overline{b}_i \Leftrightarrow \underline{b}_{i+1} \text{ off } \tilde{\mathcal{C}}_{[0,i-1]}\}_i} \prod_{j=s+1}^m \mathbb{1}_{\{\overline{b}_j \Leftrightarrow \underline{b}_{j+1} \text{ off } \tilde{\mathcal{C}}_{[s+1,j-1]}\}_j}\Big], \tag{3.28}$$

where, by convention, $\tilde{\mathcal{C}}_{[s+1,s]} = \varnothing$. Furthermore, the product

$$\prod_{i=0}^s \mathbb{1}_{\{\overline{b}_i \Leftrightarrow \underline{b}_{i+1} \text{ off } \tilde{\mathcal{C}}_{[0,i-1]}\}_i} \tag{3.29}$$

only depends on the occupation statuses of bonds described by $\mathbb{P}_0 \otimes \cdots \otimes \mathbb{P}_s$, while

$$\prod_{j=s+1}^m \mathbb{1}_{\{\overline{b}_j \Leftrightarrow \underline{b}_{j+1} \text{ off } \tilde{\mathcal{C}}_{[s+1,j-1]}\}_j} \tag{3.30}$$

only depends on the occupation statuses of bonds described by $\mathbb{P}_{s+1} \otimes \cdots \otimes \mathbb{P}_m$. Hence, the expectation factorizes and (3.25) follows.

The proof of (3.26) is similar to that of (3.25) if we rewrite the function on the left-hand side of (3.26) in the same way as we rewrote τ_m in the preceding paragraphs. But we need to keep in mind that we may only rewrite the first $i-1$ and the last $n-j-1$ sausages, since sausages i through j are affected by the event $\{E \text{ on } \mathcal{S}^x_{[i,j]}(0)\}$. □

Introducing pair interactions. We proceed with our expansion of τ_m. Consider a set of fixed bonds $e_1, \ldots, e_m$ and $i \in \{1, \ldots, m\}$. We write

$$\mathbb{1}_{\{\overline{e}_i \Leftrightarrow \underline{e}_{i+1} \text{ off } \tilde{\mathcal{C}}_{[0,i-1]}\}_i} = \mathbb{1}_{\{\overline{e}_i \Leftrightarrow \underline{e}_{i+1}\}_i}\big(1 - \mathbb{1}_{\{\overline{e}_i \Leftrightarrow \underline{e}_{i+1} \text{ off } \tilde{\mathcal{C}}_{[0,i-1]}\}_i^c}\big), \tag{3.31}$$

and define, for $s < i$,

$$U_{si} = \mathbb{1}_{\{\overline{e}_i \Leftrightarrow \underline{e}_{i+1} \text{ off } \tilde{\mathcal{C}}_{[s,i-1]}\}_i^c} \mathbb{1}_{\{\overline{e}_i \Leftrightarrow \underline{e}_{i+1} \text{ off } \tilde{\mathcal{C}}_{[s+1,i-1]}\}_i}. \tag{3.32}$$

The following claim brings us to the heart of the expansion.

Claim 3.4. *Upon the event* $\{\overline{e}_i \Leftrightarrow \underline{e}_{i+1}\}$*,*

$$\mathbb{1}_{\{\overline{e}_i \Leftrightarrow \underline{e}_{i+1} \text{ off } \tilde{\mathcal{C}}_{[0,i-1]}\}_i} = 1 - \mathbb{1}_{\{\overline{e}_i \Leftrightarrow \underline{e}_{i+1} \text{ off } \tilde{\mathcal{C}}_{[0,i-1]}\}_i^c} = \prod_{0 \le s \le i-1} (1 - U_{si}). \tag{3.33}$$

Proof. For fixed $i \in \{0, \ldots, m\}$, the event

$$F_s \equiv \{\overline{e}_i \Leftrightarrow \underline{e}_{i+1} \text{ off } \tilde{\mathcal{C}}_{[s,i-1]}\}_i \tag{3.34}$$

is increasing in s. Hence,

$$U_{si} = \mathbb{1}_{F_s^c \cap F_{s+1}} = \mathbb{1}_{F_{s+1}} - \mathbb{1}_{F_s}, \tag{3.35}$$

and so, on the event $\{\overline{e}_i \Leftrightarrow \underline{e}_{i+1}\}$,

$$\begin{aligned}\sum_{s=0}^{i-1} U_{si} &= \mathbb{1}_{F_i} - \mathbb{1}_{F_0} = \mathbb{1}_{\{\overline{e}_i \Leftrightarrow \underline{e}_{i+1} \text{ off } \varnothing\}_i} - \mathbb{1}_{\{\overline{e}_i \Leftrightarrow \underline{e}_{i+1} \text{ off } \tilde{\mathcal{C}}_{[0,i-1]}\}_i} \\ &= \mathbb{1}_{\{\overline{e}_i \Leftrightarrow \underline{e}_{i+1}\}_i} - \mathbb{1}_{\{\overline{e}_i \Leftrightarrow \underline{e}_{i+1} \text{ off } \tilde{\mathcal{C}}_{[0,i-1]}\}_i} = \mathbb{1}_{\{\overline{e}_i \Leftrightarrow \underline{e}_{i+1} \text{ off } \tilde{\mathcal{C}}_{[0,i-1]}\}_i^c}. \end{aligned} \tag{3.36}$$

This, together with the fact that $U_{si}U_{s'i} = 0$ whenever $s \neq s'$, implies the claim. □

Claim 3.4 implies that

$$\mathbb{1}_{\{\overline{b}_i \Leftrightarrow \underline{b}_{i+1} \text{ off } \tilde{\mathcal{C}}_{[0,i-1]}\}_i} = \mathbb{1}_{\{\overline{b}_i \Leftrightarrow \underline{b}_{i+1}\}_i} \prod_{0 \le s \le i-1} (1 - U_{si}), \tag{3.37}$$

so it follows that

$$\prod_{i=0}^{m} \mathbb{1}_{\{\overline{b}_i \Leftrightarrow \underline{b}_{i+1} \text{ off } \tilde{\mathcal{C}}_{[0,i-1]}\}_i} = \prod_{i=0}^{m} \mathbb{1}_{\{\overline{b}_i \Leftrightarrow \underline{b}_{i+1}\}_i} \prod_{0 \le s < t \le m} (1 - U_{st}). \tag{3.38}$$

If we substitute the above identity into (3.24) we get

$$\tau_m(x,y) = \sum_{b_1,\dots,b_m} \Big[\prod_{i=1}^{m} pD(b_i)\Big] \mathbb{E}^{\otimes[0,m]}\Big[\prod_{i=0}^{m} \mathbb{1}_{\{\overline{b}_i \Leftrightarrow \underline{b}_{i+1}\}_i} \prod_{0 \le s < t \le m} (1 - U_{st})\Big]. \tag{3.39}$$

This expression of the two-point function τ_m in (3.39) is very similar to the expression that you get when you do a lace expansion for self-avoiding walk. In particular, the presence of the *pair-interaction term*

$$K[a,b](\omega) = \prod_{a \le s < t \le b} (1 - U_{st}(\omega)), \tag{3.40}$$

together with the *independence* between the different probability distributions in (3.24), means that we can use the same, standard expansion for $K[a,b]$ that also works for self-avoiding walk in high dimensions. We recall this expansion – originally due to Brydges and Spencer [6] – in the next subsection. The expansion below is also similar to the expansion for lattice trees and lattice animals as first performed in [16].

3.2. The algebraic expansion

In this section we define the lace expansion coefficients $\pi_m(x)$ and prove (3.2).

We start by rewriting (3.39) in terms of $K[a,b]$, to get

$$\tau_n(0,x) = \sum_{b_1,\dots,b_n} \Big[\prod_{i=1}^{n} pD(b_i)\Big] \mathbb{E}^{\otimes[0,n]}\Big[\prod_{i=0}^{n} \mathbb{1}_{\{\overline{b}_i \Leftrightarrow \underline{b}_{i+1}\}_i} K[0,n](\omega)\Big], \tag{3.41}$$

where, under the measure $\mathbb{P}^{\otimes[0,n]}$, the configurations $(\omega_0,\dots,\omega_n)$ are *independent.* (Here and throughout the paper we will adopt the conventions that the empty sum equals 0, and that the empty product equals 1.)

The lace expansion is most easily described in terms of *graphs*, as we now explain.

Definition 3.5 [Graphs and laces]. *Given an interval $I = [a,b]$ of integers with $0 \le a \le b$, we refer to a pair $\{s,t\}$ of integers in I with $s < t$ as an* edge. *To simplify notation, we write st for $\{s,t\}$. A set of edges is called a* graph. *A graph Γ on $[a,b]$ is said to be* connected *if both a and b are endpoints of edges in Γ and if, in addition, for any $c \in [a,b]$ there is an edge $st \in \Gamma$ such that $s \le c \le t$. We write $\mathcal{M}[a,b]$ for the set of all graphs on $[a,b]$, and we write $\mathcal{G}[a,b]$ for the set of all connected graphs on $[a,b]$.*

A lace *is a minimally connected graph, i.e., a connected graph that would not be a connected graph if any of its edges would be removed. The set of laces on $[a,b]$ is denoted by $\mathcal{L}[a,b]$.*

This notion of connectivity is slightly weaker than the one used for self-avoiding walks in [6], but it agrees with the one used for lattice trees and lattice animals in [16].

The expansion is crucially based on the expansion of large products. In general, for any finite set of indices $\mathcal{I}$, and any $g_i, h_i \in \mathbb{R}$ $(i \in \mathcal{I})$, we have the following identity:

$$\prod_{i\in\mathcal{I}}(g_i+h_i)=\sum_{I\subseteq\mathcal{I}}\prod_{i\in I}g_i\prod_{j\in\mathcal{I}\setminus I}h_i. \tag{3.42}$$

Applying this formula with $\mathcal{I}=\mathcal{M}[a,b]$, $g_{st}=-U_{st}(\omega)$, $h_{st}=1$, we get

$$K[a,b](\omega)=\sum_{\Gamma\in\mathcal{M}[a,b]}\prod_{st\in\Gamma}(-U_{st}(\omega)). \tag{3.43}$$

For $0\le a<b$, we define an analogous quantity, where the sum over graphs is restricted to *connected* graphs, namely,

$$J[a,b](\omega)=\sum_{\Gamma\in\mathcal{G}[a,b]}\prod_{st\in\Gamma}(-U_{st}(\omega)). \tag{3.44}$$

From now on we will not write ω if it is not needed for the argument. We claim that

$$K[0,n+1]=K[1,n+1]+J[0,n+1]+\sum_{m=1}^{n}J[0,m]K[m+1,n+1]. \tag{3.45}$$

To prove (3.45), we note from (3.43) that the contribution to $K[0,n+1]$ from all graphs Γ for which 0 is not in an edge is exactly $K[1,n+1]$. The resummation of the contributions from the remaining graphs goes as follows.

When Γ contains an edge ending at 0, we write $m(\Gamma)$ for the largest value of m such that the set of edges in Γ with at least one end in the interval $[0,m]$ forms a connected graph on $[0,m]$. When $m=n+1$, resummation over graphs on $[0,n+1]$ gives $J[0,n+1]$. When $m\le n$, resummation over graphs on $[m+1,n+1]$ gives

$$K[0,n+1]=K[1,n+1]+J[0,n+1]+\sum_{m=1}^{n}\sum_{\Gamma\in\mathcal{G}[0,m]}\prod_{st\in\Gamma}(-U_{st})K[m+1,n+1]. \tag{3.46}$$

This, together with (3.44), proves (3.45).

Define

$$\pi_m(x)=\sum_{b_1,\dots,b_m}\Big[\prod_{i=1}^{m}pD(b_i)\Big]\mathbb{E}^{\otimes[0,m]}\Big[\prod_{i=0}^{m}\mathbb{1}_{\{\overline{b}_i\Leftrightarrow\underline{b}_{i+1}\}_i}J[0,m](\omega)\Big],\qquad m\ge1, \tag{3.47}$$

and

$$\pi_0(x)=\mathbb{P}(0\Leftrightarrow x). \tag{3.48}$$

Inserting (3.45) into (3.41) gives (3.1) with coefficients $\pi_m(x)$ as defined above if we factorize the expectation over ω. We can factorize the expectation because $(\omega_0,\dots,\omega_s)$ and $(\omega_{s+1},\dots,\omega_m)$ are independent for all $s\in[0,m-1]$ under the measure $\mathbb{P}^{\otimes[0,m]}$. The quantity $\pi_m(x)$ is sometimes called the *irreducible two-point function.*

We continue by rewriting $\pi_m(x)$ in a more convenient form using *laces.* Given a connected graph Γ, we can associate a unique lace L_Γ to Γ: The lace L_Γ consists of edges $s_1t_1, s_2t_2,\dots$, with $t_1,s_1,t_2,s_2,\dots$ determined, in that order, by

$$t_1=\max\{t: at\in\Gamma\},\quad s_1=a, \tag{3.49}$$

$$t_{i+1}=\max\{t:\exists s\le t_i \text{ such that } st\in\Gamma\},\quad s_{i+1}=\min\{s: st_{i+1}\in\Gamma\}. \tag{3.50}$$

Given a lace L, the set of all edges $st\notin L$ such that $\mathsf{L}_{L\cup\{st\}}=L$ is denoted $\mathrm{Comp}(L)$. We say of an edge in $\mathrm{Comp}(L)$ that it is *compatible* with L.

Note the following equivalence:

$$\mathsf{L}_\Gamma=L \qquad\Longleftrightarrow\qquad \Gamma\setminus L\subseteq\mathrm{Comp}(L). \tag{3.51}$$

This equivalence is due to the fact that we get the lace L_Γ by checking maxima and minima criteria. Moreover, $\mathsf{L}_\Gamma = L$ is equivalent to the statement that an edge that is *not* in L is never chosen in (3.49) and (3.50), so it suffices to check each of the edges individually.

Using (3.51), we resum the right-hand side of (3.44) partially, to get

$$\begin{aligned} J[a,b] &= \sum_{L\in\mathcal{L}[a,b]} \sum_{\Gamma:\mathsf{L}_\Gamma=L} \prod_{st\in L}(-U_{st}) \prod_{s't'\in\Gamma\setminus L}(-U_{s't'}) \\ &= \sum_{L\in\mathcal{L}[a,b]} \prod_{st\in L}(-U_{st}) \sum_{C\subseteq \mathrm{Comp}(L)} \prod_{s't'\in C}(-U_{s't'}). \end{aligned} \tag{3.52}$$

In the next step we factor the sum over compatible edges using (3.42),

$$\sum_{C\subseteq \mathrm{Comp}(L)} \prod_{s't'\in C}(-U_{s't'}) = \prod_{s't'\in \mathrm{Comp}(L)}(1-U_{s't'}), \tag{3.53}$$

so that finally

$$J[a,b] = \sum_{L\in\mathcal{L}[a,b]} \prod_{st\in L}(-U_{st}) \prod_{s't'\in \mathrm{Comp}(L)}(1-U_{s't'}). \tag{3.54}$$

We see that the interaction is restored along the compatible edges. We finally identify $\pi_m^{(N)}$. For $0\le a<b$, we define $J^{(N)}[a,b]$ as the contribution to (3.52) that comes from laces that consist of exactly N edges, i.e.,

$$J^{(N)}[a,b] = \sum_{L\in\mathcal{L}^{(N)}[a,b]} \prod_{st\in L} U_{st} \prod_{s't'\in \mathrm{Comp}(L)}(1-U_{s't'}), \qquad N\ge 1, \tag{3.55}$$

where $\mathcal{L}^{(N)}[a,b]$ is the set of laces that consist of precisely N edges. Then

$$J[a,b] = \sum_{N=1}^{\infty}(-1)^N J^{(N)}[a,b]. \tag{3.56}$$

Hence by (3.47), for $m\ge 0$,

$$\pi_m(x) = \sum_{N=0}^{\infty}(-1)^N \pi_m^{(N)}(x), \tag{3.57}$$

where we define (using the convention that $\overline{b}_0 = 0$ and $\underline{b}_{m+1} = x$)

$$\begin{aligned} \pi_m^{(N)}(x) &= \sum_{b_1,\dots,b_m}\Big[\prod_{i=1}^m pD(b_i)\Big]\mathbb{E}^{\otimes[0,m]}\Big[\prod_{i=0}^m \mathbb{1}_{\{\overline{b}_i\Leftrightarrow \underline{b}_{i+1}\}_i} J^{(N)}[0,m]\Big], \\ &= \sum_{b_1,\dots,b_m}\Big[\prod_{i=1}^m pD(b_i)\Big]\mathbb{E}^{\otimes[0,m]}\Big[\prod_{i=0}^m \mathbb{1}_{\{\overline{b}_i\Leftrightarrow \underline{b}_{i+1}\}_i} \sum_{L\in\mathcal{L}^{(N)}[0,m]}\prod_{st\in L}U_{st}\prod_{s't'\in\mathrm{Comp}(L)}(1-U_{s't'})\Big] \end{aligned} \tag{3.58}$$

for $m\ge 1$ and

$$\pi_0^{(N)}(x) = \delta_{0,N}\,\mathbb{P}_p(0\Leftrightarrow x). \tag{3.59}$$

This completes the algebraic derivation of the lace expansion for $\tau_m(x)$, proves (3.1) and identifies $\pi_m(x)$.

3.3. The expansion for $\varrho_n(x)$

We can extend the above expansion to an expansion for $\varrho_n(x)$ and thus prove (2.7). Recall the notion of *backbone-pivotal bonds* from Section 1.1. A problem arises because the construction in (1.9) applies to events that depend on *finitely* many bonds, but the event {bond b is backbone-pivotal} is not of that type. Nevertheless, the *backbone limit reversal lemma* in [20, Lemma 4.2 and Corollary 4.3(ii)] establishes for any $n\in\mathbb{N}$ and bonds $\{b_1,\dots,b_n\}\subset\mathbb{B}$ that

$$\begin{aligned} &\mathbb{Q}_{\mathrm{IIC}}(b_1,\dots,b_n \text{ first } n \text{ backbone-pivotals}) \\ &\qquad = \lim_{p\nearrow p_c}\frac{1}{\chi(p)}\sum_{y\in\mathbb{Z}^d}\mathbb{P}_p(b_1,\dots,b_n \text{ first } n \text{ pivotals for } 0\leftrightarrow y, 0\leftrightarrow y). \end{aligned} \tag{3.60}$$

By following the arguments leading to (3.39) we get

$$\mathbb{P}_p(b_1,\dots,b_n \text{ first } n \text{ pivotals for } 0\leftrightarrow y, 0\leftrightarrow y)$$
$$= \Big[\prod_{i=1}^n pD(b_i)\Big]\mathbb{E}_p^{\otimes[0,n+1]}\Big[\mathbb{1}_{\{\overline{b}_n\leftrightarrow y \text{ off } \tilde{\mathcal{C}}_{[0,n]}\}_{n+1}}\prod_{i=0}^n \mathbb{1}_{\{\overline{b}_i\Leftrightarrow \underline{b}_{i+1}\}_i}\prod_{0\le s<t\le n}(1-U_{st})\Big]. \tag{3.61}$$

The main difference between this equation and (3.39) is that here we have a factor $\mathbb{1}_{\{\overline{b}_n\leftrightarrow y \text{ off } \tilde{\mathcal{C}}_{[0,n]}\}_{n+1}}$ present. This factor is here because we need a connection to y. We bound $\mathbb{1}_{\{\overline{b}_n\leftrightarrow y \text{ off } \tilde{\mathcal{C}}_{[0,n]}\}_{n+1}}$ from above by $\mathbb{1}_{\{\overline{b}_n\leftrightarrow y\}_{n+1}}$ and apply the resulting independence to get that

$$\mathbb{P}_p(b_1,\dots,b_n \text{ first } n \text{ pivotals for } 0\leftrightarrow y, 0\leftrightarrow y)$$
$$\le \Big[\prod_{i=1}^n pD(b_i)\Big]\mathbb{E}_p^{\otimes[0,n+1]}\Big[\prod_{i=0}^n \mathbb{1}_{\{\overline{b}_i\Leftrightarrow \underline{b}_{i+1}\}_i}\prod_{0\le s<t\le n}(1-U_{st})\Big]\tau_p(\overline{b}_n,y). \tag{3.62}$$

Once we substitute the right-hand into (3.60) we can use the dominated convergence theorem to conclude

$$\varrho_n(x)\le p_c\tau_{n-1}(x). \tag{3.63}$$

This bound is useful because it lets us derive results for $\varrho_n(x)$ from results for $\tau_{n-1}(x)$ (see for instance the proof of Theorem 1.6).

Before we complete the expansion, we investigate the self-repellence of the IIC backbone.

Lemma 3.6 [Self-repellence for backbone-pivotal bonds]. *For every $x_i\in\mathbb{Z}^d$, $0\le n_1<n_2<\ldots<n_l$,*

$$\mathbb{Q}_{\mathrm{IIC}}(S_{n_1}=x_1,S_{n_2}=x_2,\dots,S_{n_\ell}=x_\ell)\le\tau_{n_1}(x_1)\prod_{i=2}^{\ell}(p_cD*\tau_{n_i-n_{i-1}-1})(x_i-x_{i-1}). \tag{3.64}$$

The proof is performed by first applying [20, Lemma 4.2] and then apply (3.26) iteratively. This is straightforward, so we omit the details of this proof. We will use this lemma crucially in Section 7.2, when we prove tightness of the sequence X_n.

We proceed with the expansion of $\varrho_n(x)$, by rewriting the factor $\mathbb{1}_{\{\overline{b}_n\leftrightarrow y \text{ off } \tilde{\mathcal{C}}_{[0,n]}\}_{n+1}}$ as

$$\mathbb{1}_{\{\overline{b}_n\leftrightarrow y \text{ off } \tilde{\mathcal{C}}_{[0,n]}\}_{n+1}}=\mathbb{1}_{\{\overline{b}_n\leftrightarrow y\}_{n+1}}(1-\mathbb{1}_{\{\overline{b}_n\leftrightarrow y \text{ off } \tilde{\mathcal{C}}_{[0,n]}\}^c_{[n+1]}}). \tag{3.65}$$

Define

$$V_{i,n+1}=\mathbb{1}_{\{\overline{b}_n\leftrightarrow y \text{ off } \tilde{\mathcal{C}}_{[i,n]}\}^c_{[n+1]}}\mathbb{1}_{\{\overline{b}_n\leftrightarrow y \text{ off } \tilde{\mathcal{C}}_{[i+1,n]}\}_{n+1}}. \tag{3.66}$$

Similar to (3.33),

$$1-\mathbb{1}_{\{\overline{b}_n\leftrightarrow y \text{ off } \tilde{\mathcal{C}}_{\le n}\}^c_{[n+1]}}=\prod_{0\le s\le n}(1-V_{i,n+1}). \tag{3.67}$$

Define

$$K_V[0,n+1]=K[0,n]\prod_{0\le s\le n}(1-V_{i,n+1})=\prod_{0\le s<t\le n+1}(1-V_{st}), \tag{3.68}$$

where for $t\le n$ we define $V_{st}=U_{st}$. Let

$$\varrho_{p,n}(x)=\frac{1}{\chi(p)}\sum_{y\in\mathbb{Z}^d}\sum_{b_1,\dots,b_n:\underline{b}_n=x}\mathbb{P}_p(b_1,\dots,b_n \text{ first } n \text{ pivotals for } 0\leftrightarrow y, 0\leftrightarrow y) \tag{3.69}$$

with $\chi(p)=\sum_{x\in\mathbb{Z}^d}\mathbb{P}_p(0\leftrightarrow x)$ denoting the expected cluster size (or *susceptibility*). Then, the above rewrites yield

$$\varrho_{p,n}(x)=\frac{1}{\chi(p)}\sum_{y\in\mathbb{Z}^d}\sum_{b_1,\dots,b_n:\,\underline{b}_n=x}\Big[\prod_{i=1}^n pD(b_i)\Big]\mathbb{E}_p^{\otimes[0,n+1]}\Big[\mathbb{1}_{\{\overline{b}_n\leftrightarrow y\}_{n+1}}\prod_{i=0}^m \mathbb{1}_{\{\overline{b}_i\Leftrightarrow \underline{b}_{i+1}\}_i}K_V[0,n+1]\Big]. \tag{3.70}$$

Now we are ready to complete the expansion for $\varrho_n(x)$. Similar to (3.45), we get

$$K_V[0,n+1] = K_V[1,n+1] + J_V[0,n+1] + \sum_{m=1}^{n} J_V[0,m]K_V[m+1,n+1]. \tag{3.71}$$

Now, since $V_{st} = U_{st}$ when $t \le n$, we get that $J_V[0,m] = J[0,m]$ for $m \le n$. This leads to (2.7), where $\pi_m(x)$ is indeed the same coefficient that was determined in the expansion of τ_n, while

$$\psi_{p,n}(x) = \frac{1}{\chi(p)} \sum_{y\in\mathbb{Z}^d} \sum_{b_1,\dots,b_n\colon \underline{b}_n = x} \Big[\prod_{i=1}^{n} pD(b_i)\Big]\mathbb{E}_p^{\otimes[0,n+1]}\Big[\mathbb{1}_{\{\overline{b}_n \leftrightarrow y\}_{n+1}} \prod_{i=0}^{n} \mathbb{1}_{\{\overline{b}_i \Leftrightarrow \underline{b}_{i+1}\}_i} J_V[0,n+1]\Big]. \tag{3.72}$$

Define the translation $\mathrm{Tr}_z\colon \mathbb{Z}^d \to \mathbb{Z}^d$ by the operation $y \mapsto y+z$ and define $\mathbb{P}^{\otimes[0,n+1]}_{\mathrm{IIC},\overline{b}_n} = \mathbb{P}^{\otimes[0,n]}_{p_c} \otimes (\mathbb{Q}_{\mathrm{IIC}} \circ \mathrm{Tr}_{-\overline{b}_n})$ with corresponding expectation $\mathbb{E}^{\otimes[0,n+1]}_{\mathrm{IIC},\overline{b}_n}$. Then, in the limit $p \nearrow p_c$,

$$\varrho_n(x) = \sum_{b_1,\dots,b_n\colon \underline{b}_n = x} \Big[\prod_{i=1}^{n} p_c D(b_i)\Big]\mathbb{E}^{\otimes[0,n+1]}_{\mathrm{IIC},\overline{b}_n}\Big[\prod_{i=0}^{n} \mathbb{1}_{\{\overline{b}_i \Leftrightarrow \underline{b}_{i+1}\}_i} K_V[0,n+1]\Big] \tag{3.73}$$

and

$$\psi_n(x) = \sum_{b_1,\dots,b_n\colon \underline{b}_n = x} \Big[\prod_{i=1}^{n} p_c D(b_i)\Big]\mathbb{E}^{\otimes[0,n+1]}_{\mathrm{IIC},\overline{b}_n}\Big[\prod_{i=0}^{n} \mathbb{1}_{\{\overline{b}_i \Leftrightarrow \underline{b}_{i+1}\}_i} J_V[0,n+1]\Big], \tag{3.74}$$

where, in the definition of $V_{i,n+1}$ in (3.66), we must replace y by ∞. Thus, the terms $K_V[0,n+1]$ and $J_V[0,n+1]$ enforce that the connection from $\overline{b}_n \leftrightarrow \infty$ that is present $\mathbb{Q}_{\mathrm{IIC}}$-a.s. occurs off $\tilde{\mathcal{C}}_{[0,n]}$. This completes the derivation of (2.7), and identifies $\psi_n(x)$.

4. Bounds on the lace-expansion coefficients

This section consists of two parts: in the first part we determine bounds on the lace-expansion coefficients. These bounds will be in terms of convolutions of simpler functions, such as $D(x)$ and $\tau(x)$. They have a simple, diagrammatic representation and in the literature they are hence known as *diagrammatic bounds*. In the second part we will use these bounds to derive technical results that are needed in the sections that follow.

4.1. Derivation of the diagrammatic bounds

Van den Berg and Kesten determined a very handy inequality for percolation: if events A and B always occur on disjoint sets of edges, then we write $A \circ B$ and we have $\mathbb{P}_p(A\circ B) \le \mathbb{P}_p(A)\mathbb{P}_p(B)$ [3]. This inequality is commonly known as the *BK-inequality*. A usable diagrammatic bound for $\pi_m^{(0)}$ follows immediately after applying the BK-inequality to the right-hand side of (3.59).

In this section we will from now on assume that $N \ge 1$. Recall that in (3.58) we described the lace expansion coefficient $\pi_m^{(N)}(x)$ in an algebraic manner, that is,

$$\pi_m^{(N)}(x) = \sum_{b_1,\dots,b_m} \Big[\prod_{i=1}^{m} pD(b_i)\Big]\mathbb{E}^{\otimes[0,m]}\Big[\prod_{i=0}^{m} \mathbb{1}_{\{\overline{b}_i \Leftrightarrow \underline{b}_{i+1}\}_i} \sum_{L\in\mathcal{L}^{(N)}[0,m]} \prod_{st\in L} U_{st}(\omega) \prod_{s't'\in\mathrm{Comp}(L)} (1-U_{s't'}(\omega))\Big]. \tag{4.1}$$

Our first step is now to reinterpret $\pi_m^{(N)}(x)$ in terms of percolation configurations, so that we can use the BK inequality to extract an upper bound from the identity (4.1). Before we start, we recall the following important facts: (1) the expectation $\mathbb{E}^{\otimes[0,m]}$ is taken with respect to the (product) measure of $m+1$ percolation configurations $\omega_0,\dots,\omega_m$, (2) the (restricted) cluster $\tilde{\mathcal{C}}_i$ 'lives' on ω_i $(i=0,\dots,m)$, and (3) the coupling between the clusters is encoded in the indicator functions U_{st} and $U_{s't'}$.

Suppose $L \in \mathcal{L}^{(N)}[0,m]$ is a lace with N edges on the vertices $0,1,\dots,m$. The starting vertex of the ith edge is denoted by k_{i-1}, and its ending vertex is denoted by m_i $(i=1,\dots,N)$. See Figure 1 for an example.

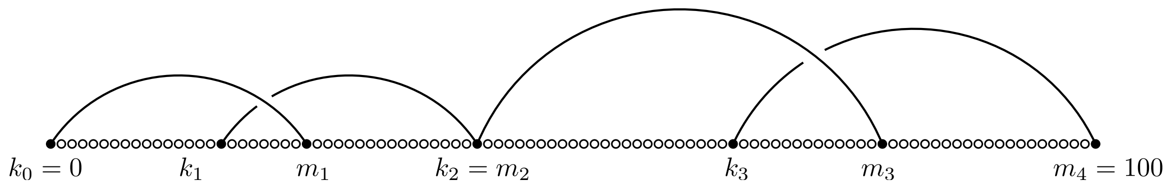

Figure 1. A lace $L \in \mathcal{L}^{(4)}[0,100]$ with 4 edges on the vertices $0,\dots,100$.

Since a lace graph does not contain removable edges, we must have that

$$0 = k_0 < k_1 \le m_1 < k_2 \le m_2 < \cdots < k_i \le m_i < \cdots < k_{N-1} \le m_{N-1} < m_N. \tag{4.2}$$

Note the alternating occurrence of $<$ and $\le$. Instead of summing over laces $L \in \mathcal{L}^{(N)}[0,m]$, we express (4.1) as a sum over all N-dimensional vectors of nonnegative integers $\mathsf{k} = (k_0,\dots,k_{N-1})$ and $\mathsf{m} = (m_1,\dots,m_N)$ that satisfy (4.2) and $m_N = m$. We get an upper bound on (4.1) by restricting the product over compatible edges $s't' \in \mathrm{Comp}(L)$ to edges $s't'$ with $0 \le s' < t' < k_1$ or $k_i \le s' < t' < m_i$ or $m_{i-1} \le s' < t' < k_i$ or $m_{N-1} \le s' < t' < m_N$ $(i = 1,\dots,N-1)$; these edges are all compatible with the lace L. For all other compatible edges we simply bound $1 - U_{s't'}(\omega)$ by 1.

This results in the upper bound

$$\pi_m^{(N)}(x) \le \sum_{\substack{(\mathsf{k},\mathsf{m})\\ m_N=m}} \sum_{b_1,\dots,b_m} \Big[\prod_{i=1}^m pD(b_i)\Big] \mathbb{E}^{\otimes[0,m]}\Big[\prod_{j=1}^N U_{k_{j-1},m_j}\, \mathbb{1}_{\{\overline{b}_{k_{j-1}} \Leftrightarrow \underline{b}_{k_{j-1}+1}\}_{k_{j-1}}}\, \mathbb{1}_{\{\overline{b}_{m_j} \Leftrightarrow \underline{b}_{m_j+1}\}_{m_j}}$$
$$\times \prod_{i=m_{j-1}+1}^{m_j-1} \mathbb{1}_{\{\overline{b}_i \Leftrightarrow \underline{b}_{i+1}\}_i} \prod_{s=m_{j-1}+1}^{i} (1 - U_{s,i})\Big], \tag{4.3}$$

where the first sum is taken over all vectors (k,m) that satisfy (4.2) and $m_N = m$. We can split the joint expectation $\mathbb{E}^{\otimes[0,m]}$ in the above expression into configurations that correspond to either the first or the second line of the upper bound in (4.3), respectively. To deal with the contributions that come from the second line of (4.3), we introduce the quantity $A_{st}(y,x;z)$ for $z \in [0,1]$, integers $s \le t$ and sites $y, x \in \mathbb{Z}^d$ as $A_{ss} \equiv 1$ and, for $s < t$,

$$A_{st}(y,x;z) \equiv z^{t-s} \sum_{\substack{b_{s+1}:\underline{b}_{s+1}=y\\ b_t:\overline{b}_t=x}} \sum_{b_{s+2},\dots,b_{t-1}} \prod_{i=s+1}^{t-1} pD(b_i)$$
$$\times\, \mathbb{E}^{\otimes[s+1,t-1]}\Big[\mathbb{1}_{\{\overline{b}_i \Leftrightarrow \underline{b}_{i+1}\}_i} \prod_{k=s}^{i-1}(1 - U_{ki})\Big]\, pD(b_t). \tag{4.4}$$

From (2.11), (3.24) and (3.33) we recall

$$\mathrm{T}_z(x) = \sum_{m=0}^{\infty} \tau_m(0,x) z^m = \sum_{m=0}^{\infty} z^m \sum_{b_1,\dots,b_m} \mathbb{E}^{\otimes[0,m]}\Big[\mathbb{1}_{\{0 \Leftrightarrow \underline{b}_1\}_0} \prod_{i=1}^m pD(b_i) \mathbb{1}_{\{\overline{b}_i \Leftrightarrow \underline{b}_{i+1} \text{ off } \tilde{\mathcal{C}}_{\le i}\}_i}\Big]$$
$$= \sum_{m=0}^{\infty} z^m \sum_{b_1,\dots,b_m} \mathbb{E}^{\otimes[0,m]}\Big[\mathbb{1}_{\{0 \Leftrightarrow \underline{b}_1\}_i} \prod_{i=1}^m pD(b_i) \mathbb{1}_{\{\overline{b}_i \Leftrightarrow \underline{b}_{i+1}\}_i} \prod_{s=0}^{i-1}(1 - U_{si})\Big] \tag{4.5}$$

with $x = \underline{b}_{m+1}$. It follows that

$$\sum_{t=s+1}^{\infty} A_{st}(y,x;z) = zpD(x-y) + z^2 (pD * \mathrm{T}_z * pD)(x-y), \tag{4.6}$$

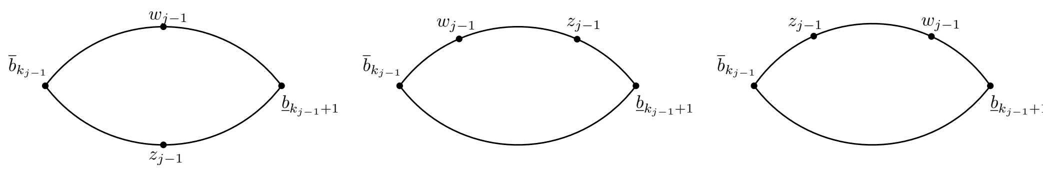

Figure 2. The three possible configurations in (4.12).

where zpD comes from the summand $t = s+1$ (i.e., when the connection is formed by a single edge). Define

$$\tilde{\mathrm{T}}_z(x) \equiv 2zp_c^2(D * \mathrm{T}_z)(x) \tag{4.7}$$

and observe that $zpD(x) + z^2(pD * \mathrm{T}_z * pD)(x) \leq \tilde{\mathrm{T}}_z(x)$ for all $z \in [0,1]$ and all $p \leq p_c$.

To simplify notation, we will rewrite (4.3) before we analyze its first line. For a given pair (k,m), we write v for the vector of length $2N$ that we get by interlacing k and m in the order described in (4.2) (with $v_1 = k_0$ and $v_{2N} = m$). When we substitute (4.4) and v into (4.3) we get

$$\begin{aligned}\sum_{m=0}^{\infty} \pi_m^{(N)}(x) z^m \leq \sum_{(\mathsf{k},\mathsf{m})} \sum_{\underline{b}_1} \sum_{\bar{b}_{k_1},\dots,\bar{b}_{k_N}} \sum_{\underline{b}_{k_1+1},\dots,\underline{b}_{k_{N-1}+1}} \sum_{\bar{b}_{m_1},\dots,\bar{b}_{m_{N-1}}} \sum_{\underline{b}_{m_1+1},\dots,\underline{b}_{m_{N-1}+1}} \sum_{\underline{b}_{m_N}} \\ \times \mathbb{E}^{\otimes\{k_0,k_1,m_1,\cdots,k_{N-1},m_{N-1},m_N\}}\Bigg[\prod_{j=1}^{N} U_{k_{j-1},m_j} \mathbb{1}_{\{\bar{b}_{k_{j-1}} \Leftrightarrow \underline{b}_{k_{j-1}+1}\}_{k_{j-1}}} \mathbb{1}_{\{\bar{b}_{m_j} \Leftrightarrow \underline{b}_{m_j+1}\}_{m_j}}\Bigg] \\ \times \Bigg[\prod_{i=1}^{2N} A_{v_i v_{i+1}}(\underline{b}_{v_i}, \bar{b}_{v_{i+1}}; z)\Bigg].\end{aligned} \tag{4.8}$$

We continue by investigating the second line in (4.8). Suppose that for some $j = 1, \dots, N$

$$U_{k_{j-1},m_j} = 1 \qquad \text{and} \qquad \{\bar{b}_{k_{j-1}} \Leftrightarrow \underline{b}_{k_{j-1}+1}\}_{k_{j-1}} \cap \{\bar{b}_{m_j} \Leftrightarrow \underline{b}_{m_j+1}\}_{m_j} \text{ occurs.} \tag{4.9}$$

This means that the double connection in ω_{m_j} is intersected by the cluster $\tilde{\mathscr{C}}_{k_{j-1}}$. Hence, there exists a vertex z_j such that

$$\Big\{\{\bar{b}_{m_j} \leftrightarrow z_j\} \circ \{z_j \leftrightarrow \underline{b}_{m_j+1}\} \circ \{\bar{b}_{m_j} \leftrightarrow \underline{b}_{m_j+1}\}\Big\}_{m_j} \text{ and } z_j \in \tilde{\mathscr{C}}_{k_{j-1}}. \tag{4.10}$$

What does the cluster $\tilde{\mathscr{C}}_{k_{j-1}}$ look like? If $k_{j-1} < m_{j-1}$ or if $j = 1$, then there exists a site w_{j-1} such that the following event occurs:

$$\Big\{\{\bar{b}_{k_{j-1}} \leftrightarrow w_{j-1}\} \circ \{w_{j-1} \leftrightarrow \underline{b}_{k_{j-1}+1}\} \circ \{\bar{b}_{k_{j-1}} \leftrightarrow \underline{b}_{k_{j-1}+1}\} \circ \{w_{j-1} \leftrightarrow z_j\}\Big\}_{k_{j-1}}. \tag{4.11}$$

If, on the other hand, $k_{j-1} = m_{j-1}$, then $\tilde{\mathscr{C}}_{k_{j-1}}$ contains a site w_{j-1} (from which an arm connects to z_j) and a site z_{j-1} (from which an ω_{k_j-2}-arm connects "back" to w_{j-2}). There are three different configurations possible for the position of these two sites in $\tilde{\mathscr{C}}_{k_{j-1}}$, cf. Figure 2,

$$\begin{aligned}&\Big\{\{\bar{b}_{k_{j-1}} \leftrightarrow w_{j-1}\} \circ \{w_{j-1} \leftrightarrow \underline{b}_{k_{j-1}+1}\} \circ \{\bar{b}_{k_{j-1}} \leftrightarrow z_{j-1}\} \circ \{z_{j-1} \leftrightarrow \underline{b}_{k_{j-1}+1}\}\Big\}_{k_{j-1}} \\ \cup &\Big\{\{\bar{b}_{k_{j-1}} \leftrightarrow w_{j-1}\} \circ \{w_{j-1} \leftrightarrow z_{j-1}\} \circ \{z_{j-1} \leftrightarrow \underline{b}_{k_{j-1}+1}\} \circ \{\bar{b}_{k_{j-1}} \leftrightarrow \underline{b}_{k_{j-1}+1}\}\Big\}_{k_{j-1}} \\ \cup &\Big\{\{\bar{b}_{k_{j-1}} \leftrightarrow z_{j-1}\} \circ \{z_{j-1} \leftrightarrow w_{j-1}\} \circ \{w_{j-1} \leftrightarrow \underline{b}_{k_{j-1}+1}\} \circ \{\bar{b}_{k_{j-1}} \leftrightarrow \underline{b}_{k_{j-1}+1}\}\Big\}_{k_{j-1}}.\end{aligned} \tag{4.12}$$

The three events in (4.12) are not a partition, because the clusters can intersect each other at more than one site. But we do get an upper bound by summing the probability over each of the three events in (4.12). The final step is to apply the BK-inequality. The result is complicated, so

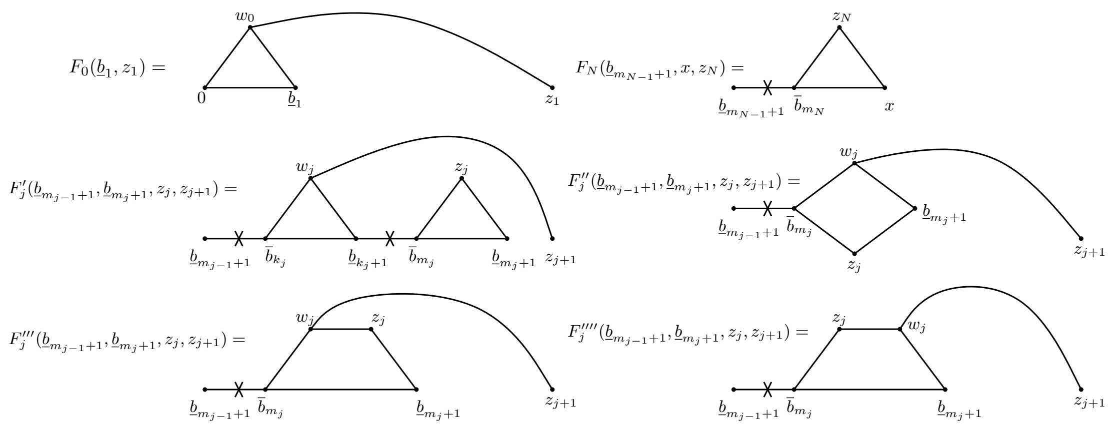

Figure 3. Diagrams of $F_0,\dots,F_N$. A line between points x and y describes to the two-point function $\mathbb{P}(x \leftrightarrow y)$. A line with an "X" describes $\tilde{\mathrm{T}}_z(y-x)$ (see (4.7) for the definition of $\tilde{\mathrm{T}}_z$).

we define the following functions that we will use to streamline the notation. These functions are drawn as diagrams in Figure 3,

$$\begin{aligned} F_j'(\underline{b}_{m_{j-1}+1}, \underline{b}_{m_j+1}, z_j, z_{j-1}; z) \equiv \sum_{w_j} \sum_{\overline{b}_{m_j}} \sum_{\overline{b}_{k_j}} \sum_{\underline{b}_{k_j+1}} &\tilde{\mathrm{T}}_z(\overline{b}_{k_j} - \underline{b}_{m_{j-1}+1}) \tilde{\mathrm{T}}_z(\overline{b}_{m_j} - \underline{b}_{k_j+1}) \\ &\times \tau(\underline{b}_{k_j+1} - \overline{b}_{k_j}) \tau(w_j - \overline{b}_{k_j}) \tau(\underline{b}_{k_j+1} - w_j) \\ &\times \tau(z_j - \overline{b}_{m_j}) \tau(\underline{b}_{m_j+1} - z_j) \tau(\underline{b}_{m_j+1} - \overline{b}_{m_j}) \tau(z_{j+1} - w_j); \end{aligned} \tag{4.13}$$

$$\begin{aligned} F_j''(\underline{b}_{m_{j-1}+1}, \underline{b}_{m_j+1}, z_j, z_{j-1}; z) \equiv \sum_{w_j} \sum_{\overline{b}_{m_j}} &\tilde{\mathrm{T}}_z(\overline{b}_{m_j} - \underline{b}_{m_{j-1}+1}) \tau(w_j - \overline{b}_{m_j}) \tau(\underline{b}_{m_j+1} - w_j) \\ &\times \tau(z_j - \overline{b}_{m_j}) \tau(\underline{b}_{m_j+1} - z_j) \tau(z_{j+1} - w_j); \end{aligned} \tag{4.14}$$

$$\begin{aligned} F_j'''(\underline{b}_{m_{j-1}+1}, \underline{b}_{m_j+1}, z_j, z_{j-1}; z) \equiv \sum_{w_j} \sum_{\overline{b}_{m_j}} &\tilde{\mathrm{T}}_z(\overline{b}_{m_j} - \underline{b}_{m_{j-1}+1}) \tau(w_j - \overline{b}_{m_j}) \tau(z_j - w_j) \\ &\times \tau(\underline{b}_{m_j+1} - z_j) \tau(\underline{b}_{m_j+1} - \overline{b}_{m_j}) \tau(z_{j+1} - w_j); \end{aligned} \tag{4.15}$$

$$\begin{aligned} F_j''''(\underline{b}_{m_{j-1}+1}, \underline{b}_{m_j+1}, z_j, z_{j-1}; z) \equiv \sum_{w_j} \sum_{\overline{b}_{m_j}} &\tilde{\mathrm{T}}_z(\overline{b}_{m_j} - \underline{b}_{m_{j-1}+1}) \tau(z_j - \overline{b}_{m_j}) \tau(w_j - z_j) \\ &\times \tau(\underline{b}_{m_j+1} - w_j) \tau(\underline{b}_{m_j+1} - \overline{b}_{m_j}) \tau(z_{j+1} - w_j); \end{aligned} \tag{4.16}$$

$$F_0(\underline{b}_1, z_1) \equiv \sum_{w_0} \tau(\underline{b}_1) \tau(w_0) \tau(\underline{b}_1 - w_0) \tau(z_1 - w_0); \tag{4.17}$$

$$F_j(\underline{b}_{m_{j-1}+1}, \overline{b}_{m_j}, z_j, z_{j-1}; z) \equiv F_j' + F_j'' + F_j''' + F_j''''; \tag{4.18}$$

$$F_N(\underline{b}_{m_{N-1}+1}, x, z_N; z) \equiv \sum_{\overline{b}_{m_N}} \tilde{\mathrm{T}}_z(\overline{b}_{m_N} - \underline{b}_{m_{N-1}+1}) \tau(z_N - \overline{b}_{m_N}) \tau(x - \overline{b}_{m_N}) \tau(x - z_N), \tag{4.19}$$

for $j=1,\dots,N-1$. Now we can bound (4.8) from above by

$$\sum_{m=0}^{\infty}\pi_m^{(N)}(x)z^m \le \sum_{0<m_1<m_2<\cdots<m_N\in\mathbb{N}} \; \sum_{\substack{\underline{b}_1,\overline{b}_{m_1},\underline{b}_{m_1+1},\overline{b}_{m_2},\underline{b}_{m_2+1},\dots,\\ \overline{b}_{m_{N-1}},\underline{b}_{m_{N-1}+1},\underline{b}_{Nm}}\in\mathbb{Z}^d} \; \sum_{z_1,z_2,\dots,z_N\in\mathbb{Z}^d} F_0(\underline{b}_1,z_1)\left(\prod_{j=1}^{N-1}F_j(\underline{b}_{m_{j-1}+1},\overline{b}_{m_j},z_j,z_{j-1};z)\right)F_N(\underline{b}_{m_{N-1}+1},x,z_N;z). \tag{4.20}$$

The bound in (4.20) is an important step, but we can still improve on this bound by reorganizing our notation. To do this we introduce the following functions that are similar to those in other lace expansions (see e.g. [33, pp. 109–110]):

$$A_3(s,u,v) = \tau(v-s)\,\tau(u-s)\,\tau(v-u), \tag{4.21}$$
$$B_1(s,t,u,v;z) = \tau(u-s)\,\tilde{\mathrm{T}}_z(v-t), \tag{4.22}$$
$$B_2^{(0)}(u,v,s,t;z) = \tau(t-u)\,\tau(s-v)\sum_{a,b\in\mathbb{Z}^d}\tilde{\mathrm{T}}_z(b-a)\,\tau(a-u)\,\tau(t-a)\,\tau(b-v)\,\tau(s-b), \tag{4.23}$$
$$B_2^{(1)}(u,v,s,t;z) = \tau(t-u)\,\tau(s-v)\,\tau(v-u)\,\tau(s-t), \tag{4.24}$$
$$B_2^{(2)}(u,v,s,t;z) = \tau(t-v)\,\tau(s-u)\,\tau(v-u)\,\tau(s-t), \tag{4.25}$$
$$B_2^{(3)}(u,v,s,t;z) = \tau(t-u)\,\tau(s-v)\,\tau(v-t)\,\tau(s-u), \tag{4.26}$$
$$B_2(u,v,s,t;z) = \sum_{i=0}^{3}B_2^{(i)}(u,v,s,t;z). \tag{4.27}$$

In Figure 4 we show drawings of these functions.

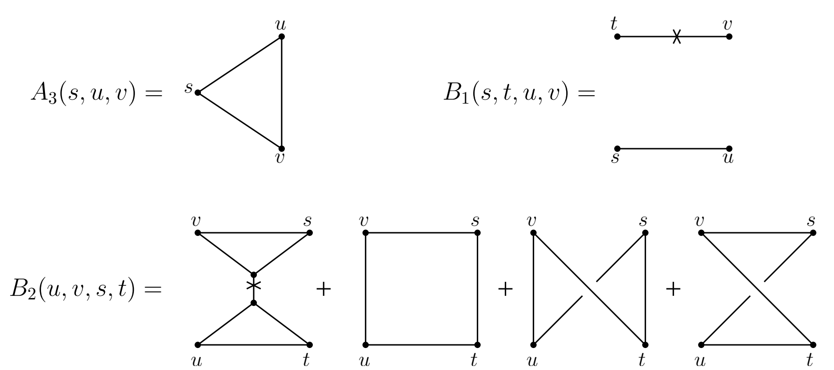

Figure 4. Diagrammatic representation of $A_3(s,u,v)$, $B_1(s,t,u,v)$, and $B_2(u,v,s,t)$.

Using (4.21)–(4.27), we can rewrite the right hand side in (4.20) as follows,

$$\sum_{m=0}^{\infty}\pi_m^{(N)}(x)z^m \le \sum_{s_1,\dots,s_N}\sum_{t_1,\dots,t_N}\sum_{u_1,\dots,u_N}\sum_{v_1,\dots,v_N}A_3(0,s_1,t_1) \times\prod_{j=1}^{N-1}\Big[B_1(s_j,t_j,u_j,v_j;z)\,B_2(u_j,v_j,s_{j+1},t_{j+1};z)\Big] \times B_1(s_N,t_N,u_N,v_N;z)\,A_3(u_N,v_N,x). \tag{4.28}$$

Again, note the similarity with the usual lace expansion for percolation, e.g. [33, (10.53)]. The resulting diagrams for $N=1$ and $N=2$ are shown in Figure 5.

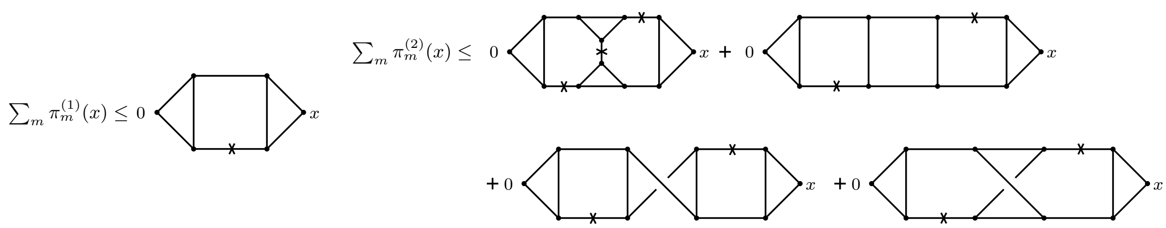

Figure 5. The diagrams bounding $\sum_{m=0}^{\infty} \pi_m^{(1)}(x)$ and $\sum_{m=0}^{\infty} \pi_m^{(2)}(x)$.

Diagrammatic bounds for ψ_m. The algebraic expansion for ϱ_m in Section 3.3 shows that the structure of $\psi_m^{(N)}$ is very similar to that of $\pi_m^{(N+1)}$. In particular, all interactions that do not involve the sausage that ends at b_m are the same. This can be seen by comparing, for instance, (3.58) and (3.74). Interactions that involve the final sausage are different for $\psi_m^{(N)}$ and $\pi_m^{(N+1)}$, as can be seen from (3.66) that $V_{i,m+1}$ requires that there is an intersection between the ith sausage and the path from x to ∞. But the diagrammatic bound on $\psi_m^{(N)}$ can be performed in very much the same way as was done for $\pi_m^{(N+1)}$, and the final intersection event can be bounded in almost the same way as any other interaction. Therefore, we do not give the entire diagrammatic expansion, but we rather just show its conclusion. Define

$$B_3(s,t,x,u,v;z) \equiv z p_c D(x-t)\tau(v-x)\tau(s-u), \tag{4.29}$$

then we have the following diagrammatic bound for $\psi_m^{(N)}$:

$$\begin{aligned}\sum_x \sum_{m=0}^{\infty} \psi_m^{(N)}(x) z^m \le \sum_x \sum_{s_1,\dots,s_{N+1}} \sum_{t_1,\dots,t_{N+1}} \sum_{u_1,\dots,u_{N+1}} \sum_{v_1,\dots,v_{N+1}} \sum_a A_3(0,s_1,t_1)\\ \times \prod_{j=1}^{N} \Big(B_1(s_j,t_j,u_j,v_j;z)\, B_2(u_j,v_j,s_{j+1},t_{j+1};z) \Big)\\ \times B_3(s_{N+1},t_{N+1},x,u_{N+1},v_{N+1};z)\, A_3(u_{N+1},v_{N+1},a).\end{aligned} \tag{4.30}$$

The resulting diagrams for $N=0$ and $N=1$ are shown in Figure 6. Note the similarities with Figure 5.

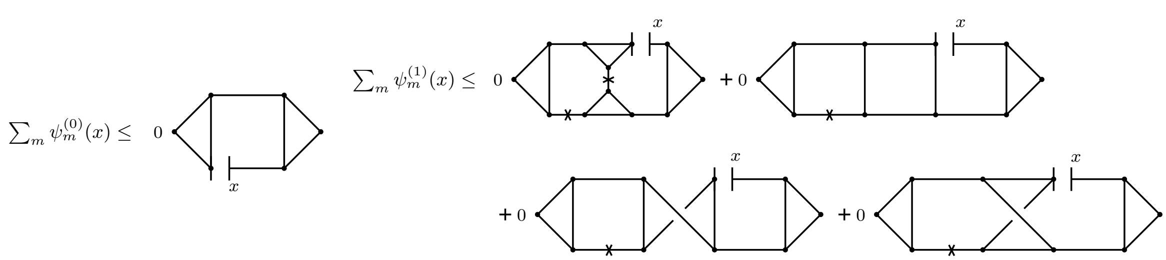

Figure 6. The diagrams bounding $\sum_{m=0}^{\infty} \psi_m^{(0)}(x)$ and $\sum_{m=0}^{\infty} \psi_m^{(1)}(x)$. Two parallel dashes with a gap in between denote a factor D.

4.2. Properties of the lace-expansion coefficients

We now use the diagrammatic bounds that we derived in the first part of this section to prove several technical results that we will use in the upcoming sections. These results are stated in the four propositions below. The order in which the results are proved is very important. In particular, the proofs of Propositions 4.2 and 4.3 use Proposition 4.1. In turn, the proof of Proposition

4.4 uses Proposition 4.2. The proofs in this section also make strong use of results from [20] and [23].

Proposition 4.1 [Basic properties of the lace-expansion coefficients]. *Under the assumptions of Theorem 1.1 there exists $\tilde{c} > 0$ such that uniformly in $k \in \mathbb{R}^d$ and $z \in [0,1]$,*

$$\sum_{x\in\mathbb{Z}^d}\sum_{m=0}^{\infty}|\pi_m(x)|z^m \quad\le\quad 1+\tilde{c}\beta^{1/2}; \tag{4.31}$$

$$\sum_{x\in\mathbb{Z}^d}\sum_{m=1}^{\infty}m|\pi_m(x)| \quad\le\quad \tilde{c}\beta^{1/2}. \tag{4.32}$$

The proof of this proposition can be found in the supplementary material to this paper, [22]. The bounds in Proposition 4.1 can be improved to $1 + O(\beta)$ and $O(\beta)$, respectively, but this requires significantly more effort, and we do not require such strong bounds.

Proposition 4.2. *Under the assumptions of Theorem 1.1: (i) There exists $\tilde{c} > 0$ such that uniformly in $k \in \mathbb{R}^d$,*

$$|\hat{\Pi}_1(k) - \hat{\Pi}_z(k)| \le \tilde{c}\beta^{1/2}(1-z), \qquad \textit{and} \qquad |\hat{\Pi}_z(k) - 1| \le \tilde{c}\beta^{1/2}. \tag{4.33}$$

(ii) There exists constants $C, C' > 0$ such that uniformly in $k \in \mathbb{R}^d$,

$$0 \le \hat{\mathrm{T}}_z(k) \le \frac{C}{1-z} \quad \textit{and} \quad \hat{\mathrm{T}}_z(k) \le \frac{C'}{1-\hat{D}(k)}. \tag{4.34}$$

Proof of Proposition 4.2 subject to Proposition 4.1. (i) We start with the proof of the first inequality of (4.33). We bound

$$|\hat{\Pi}_1(k) - \hat{\Pi}_z(k)| = \left|\sum_{x\in\mathbb{Z}^d}\sum_{m=1}^{\infty}(1-z^m)\pi_m(x)\mathrm{e}^{ik\cdot x}\right| \le \sum_{x\in\mathbb{Z}^d}\sum_{m=1}^{\infty}(1-z^m)|\pi_m(x)|. \tag{4.35}$$

Using that $(1-z^m) \le m(1-z)$ for $z \ge 0$ and Proposition 4.1, it follows that

$$\sum_{x\in\mathbb{Z}^d}\sum_{m=1}^{\infty}(1-z^m)|\pi_m(x)| \le (1-z)\sum_{x\in\mathbb{Z}^d}\sum_{m=1}^{\infty}m|\pi_m(x)| \le \tilde{c}\beta^{1/2}(1-z). \tag{4.36}$$

Now we turn to the second inequality in (4.33): since $\pi_0(0) = 1$ and $\pi_m(0) = 0$ for $m \ge 1$,

$$\left|\hat{\Pi}_z(k) - 1\right| = \left|\sum_{x\in\mathbb{Z}^d}\sum_{m=0}^{\infty}(1-\delta_{0,x}\delta_{0,m})\pi_m(x)z^m\mathrm{e}^{ik\cdot x}\right| \le \left|\sum_{x\in\mathbb{Z}^d}\sum_{m=1}^{\infty}\pi_m(x)z^m\right| \le \tilde{c}\beta^{1/2}, \tag{4.37}$$

where we have used (4.32) in the last step.

(ii) We start by proving that $\hat{\mathrm{T}}_z(k) \ge 0$. Note that $\hat{\mathrm{T}}_1(k) = \hat{\tau}_{p_c}(k)$. It is a well-known fact that $\hat{\tau}_{p_c}(k) \ge 0$ for all $k \in \mathbb{R}^d$ [1, Lemma 3.3]. To prove that this inequality also holds for $z \in [0,1)$ we start by showing that $\hat{\mathrm{T}}_z(k)$ is continuous for $z \in [0,1)$. By (2.15) it suffices to show that $\Pi_z(k)$ is continuous, and that $1 - zp_c\hat{D}(k)\hat{\Pi}_z(k)$ is continuous and non-vanishing. Continuity of both functions follows from Proposition 4.1 and the Uniform Convergence Theorem. Now we use (2.15), the right-hand bound of (4.33) and the left-hand bound of (4.34) to bound

$$|1 - zp_c\hat{D}(k)\hat{\Pi}_z(k)| = \frac{|\hat{\Pi}_z(k)|}{|\hat{\mathrm{T}}_z(k)|} \ge \frac{|\hat{\Pi}_z(k)|}{C}(1-z) > 0, \tag{4.38}$$

where positivity of $|\hat{\Pi}_z(k)|$ is a consequence of (4.33). We conclude that $\hat{\mathrm{T}}_z(k)$ is continuous in z on $[0,1)$.

Because $\hat{\mathrm{T}}_z(k)$ is continuous, $|\hat{\Pi}_z(k)| > 0$, and $|1 - zp_c\hat{D}(k)\Pi_z(k)| > 0$, we conclude that $\hat{\mathrm{T}}_z(k)$ is either always positive or always negative for z on the interval $[0,1)$ (for all $k \in \mathbb{R}^d$). We also know that

$$\hat{\mathrm{T}}_0(0) = \sum_{x\in\mathbb{Z}^d}\tau_0(x) = \sum_{x\in\mathbb{Z}^d}\mathbb{P}_{p_c}(0 \Leftrightarrow x) \ge 1, \tag{4.39}$$

and hence, $\hat{\mathrm{T}}_z(k) \geq 0$ for all $z \in [0,1)$ and $k \in \mathbb{R}^d$.

Now we prove that $\hat{\mathrm{T}}_z(k) \leq C(1-z)^{-1}$. We bound

$$\hat{\mathrm{T}}_z(k) = |\hat{\mathrm{T}}_z(k)| \leq |\hat{\mathrm{T}}_z(0)| = \left|\frac{\hat{\Pi}_z(0)}{1 - zp_c\hat{\Pi}_z(0)}\right| \leq \frac{1 + |\hat{\Pi}_z(0) - 1|}{|(1-z) + zp_c(\hat{\Pi}_1(0) - \hat{\Pi}_z(0))|}, \tag{4.40}$$

where we have used (2.18) for the last inequality. We use the bound $p_c \leq 1 + O(\beta)$ (cf. [15] [23]) and the bounds from part (i) of the proposition to conclude that there exists a constant $0 \leq c' < 1$ such that

$$\frac{1 + \tilde{c}\beta^{1/2}}{|(1-z) - zp_c|\hat{\Pi}_1(0) - \hat{\Pi}_z(0)||} \leq \frac{1 + \tilde{c}\beta^{1/2}}{(1-z)|1 - zp_c c'\beta^{1/2}|} \leq \frac{C}{1-z}. \tag{4.41}$$

Finally, we prove that $\hat{\mathrm{T}}_z(k) \leq C'[1 - \hat{D}(k)]^{-1}$. By a similar argument as the previous bound, and since $|\hat{D}(k)| \leq 1$,

$$\hat{\mathrm{T}}_z(k) \leq \left|\frac{\hat{\Pi}_z(0)}{1 - zp_c\hat{D}(k)\hat{\Pi}_z(0)}\right| \leq \frac{1 + |\hat{\Pi}_z(0) - 1|}{(1 - \hat{D}(k))(1 + (1-z)|1 - zp_c c'\beta^{1/2}|)} \leq \frac{C'}{1 - \hat{D}(k)}. \tag{4.42}$$

This completes the proof of (4.34). □

The following proposition deals with *spatial fractional derivatives.*

Proposition 4.3 [Bounds on spatial fractional derivatives]. *Under the assumptions of Theorem 1.1 there exist $\delta_i > 0$ for $i = 1,2,3$ such that*

(i)
$$\sum_{x \in \mathbb{Z}^d} |x|^{(2 \wedge \alpha) + \delta_1} |\Pi_1(x)| < \infty; \tag{4.43}$$

(ii)
$$\sum_{x \in \mathbb{Z}^d} \sum_{m=0}^{\infty} |x|^{\delta_2} |\pi_m(x)| < \infty; \tag{4.44}$$

(iii)
$$\sum_{x \in \mathbb{Z}^d} \sum_{m=0}^{\infty} |x|^{\delta_3} |\psi_m(x)| < \infty. \tag{4.45}$$

(iv) Furthermore, let $\delta_4 > 0$ and define $\pi_{m,n}(x,y)$ as in (3.47) *but leave out the summation over $\underline{b}_m$ and write y for the free variable that denotes the position of $\underline{b}_m$, then, uniformly in $n \geq 1$,*
$$\sum_{x,y \in \mathbb{Z}^d} \sum_{m=0}^{n} (|x|^{\delta_4} + |y|^{\delta_4}) |\pi_{m,n}(y,x)| < C. \tag{4.46}$$

The bound (4.43) has been proved in [20, Proposition 2.5]. In particular, see [20, Remark 2.6], and observe that $\Pi_1(x)$ in the current paper is equal to $\Pi^{\text{classical}}(x)$ in [20]. We give an outline of the proofs of the three other bounds in the supplementary material [22].

The next Proposition 4.4 is the most involved bound of this section. It gives bounds on *temporal fractional derivatives* (with m playing the role of "time"). Its proof crucially uses the new lace expansion.

Proposition 4.4 [Bounds on temporal fractional derivatives]. *Under the conditions of Theorem 1.1, there exists $\tilde{c} > 0$ and $\varepsilon \in (0, d - 3(2 \wedge \alpha) \wedge 1)$ such that*

$$\sum_{x \in \mathbb{Z}^d} \sum_{m=1}^{\infty} m^{1+\varepsilon} |\pi_m(x)| \leq \tilde{c}\beta^{1/2} \quad \textit{and} \quad \sum_{x \in \mathbb{Z}^d} \sum_{m=1}^{\infty} m^{1+\varepsilon} |\psi_m(x)| \leq \tilde{c}\beta^{1/2}. \tag{4.47}$$

The proof of this proposition uses the full power of our new lace expansion, since only this lace expansion makes the m-dependence explicit. The proof of (4.47) is based on the following lemma that combines the *temporal fractional derivatives* as described in [32, Section 6.3] and the diagrammatic bounds developed in the first half of this section.

Define for $n = 0,1,2$, the *square diagrams*

$$\square_z^{(n)} \equiv \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi^d)} |\hat{D}(k)|^n \hat{\tau}(k)^3 z \hat{\mathrm{T}}_z(k). \tag{4.48}$$

Lemma 4.5 [Diagrammatic bounds]. *Under the conditions of Theorem 1.1, there exists a constant $\beta > 0$ such that for $N \geq 0$, $C_1, C_2 > 0$ and $|z| \leq 1$,*

$$\sum_{x\in\mathbb{Z}^d} \sum_{m=1}^{\infty} m^2 \pi_m^{(N)}(x) z^m \leq C_1 N^2 \left(C_2 \beta^{1/4}\right)^{(N-2)\vee 0} \sqrt{\square_z^{(0)} \square_z^{(2)}}. \tag{4.49}$$

Here is a very brief outline of the proof: We distribute the factor m^2 over the diagram as follows: we "mark" to two lines in the π-diagram with an indicator function in such a way that we can retrieve the factor m^2 by summing over all m^2 possible combinations of such marks. Then we bound the resulting triangle and square diagrams (weighted with the factor z). The full proof is presented in the supplementary material to this article, [22].

Proof of Proposition 4.4 subject to Lemma 4.5. We start with the bound on the left-hand side of (4.47). For $\varepsilon \in (0,1)$ we have the identity (cf. [32, (6.3.5)])

$$m^{\varepsilon} = \frac{m}{(1-\varepsilon)\Gamma(1-\varepsilon)} \int_0^{\infty} \mathrm{d}\lambda\, \mathrm{e}^{-m\lambda^{1/(1-\varepsilon)}}, \tag{4.50}$$

which gets

$$\sum_{N=0}^{\infty} \sum_{x\in\mathbb{Z}^d} \sum_{m=1}^{\infty} m^{1+\varepsilon} \pi_m^{(N)}(x) = \frac{1}{(1-\varepsilon)\Gamma(1-\varepsilon)} \sum_{N=0}^{\infty} \sum_{x\in\mathbb{Z}^d} \sum_{m=1}^{\infty} \int_0^{\infty} \mathrm{d}\lambda\, m^2 \pi_m^{(N)}(x) \mathrm{e}^{-m\lambda^{1/(1-\varepsilon)}}. \tag{4.51}$$

Applying Lemma 4.5 with $z_\lambda = \mathrm{e}^{-\lambda^{1/(1-\varepsilon)}}$ to the right-hand side gives

$$C \sum_{N=0}^{\infty} \sum_{x\in\mathbb{Z}^d} \sum_{m=1}^{\infty} \int_0^{\infty} \mathrm{d}\lambda\, m^2 \pi_m^{(N)}(x) \mathrm{e}^{-m\lambda^{1/(1-\varepsilon)}} \leq \sum_{N=0}^{\infty} C_1 N^2 \left(C_2 \beta^{1/4}\right)^{(N-2)\vee 0} \int_0^{\infty} \mathrm{d}\lambda \sqrt{\square_{z_\lambda}^{(0)} \square_{z_\lambda}^{(2)}}. \tag{4.52}$$

By Cauchy-Schwarz, Fubini, and the bound $\hat{\tau}(k) \leq C/(1-\hat{D}(k))$ (cf. (4.34) or [15],[23]),

$$\begin{aligned} \int_0^{\infty} \mathrm{d}\lambda \sqrt{\square_{z_\lambda}^{(0)} \square_{z_\lambda}^{(2)}} \leq C & \left(\int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi^d)} \frac{1}{[1-\hat{D}(k)]^3} \int_0^{\infty} \mathrm{d}\lambda\, z_\lambda \hat{\mathrm{T}}_{z_\lambda}(k) \right)^{1/2} \\ & \times \left(\int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi^d)} \frac{\hat{D}(k)^2}{[1-\hat{D}(k)]^3} \int_0^{\infty} \mathrm{d}\lambda\, z_\lambda \hat{\mathrm{T}}_{z_\lambda}(k) \right)^{1/2}. \end{aligned} \tag{4.53}$$

The aim is now to show that the integral over λ is small compared to $[1-\hat{D}(k)]$, so that the integral over k is effectively the same as the integral over a triangle diagram. When $d > d_c$, we know that a triangle diagram is of order 1 when there are no factors $\hat{D}(k)$ and that it is of order β when there are two factors $\hat{D}(k)$. For any $\varepsilon \in (0,1)$ we can use both upper bounds in Proposition

4.2(ii) to bound the integral over λ:

$$\begin{aligned}\int_0^\infty \mathrm{d}\lambda\, z_\lambda \hat{\mathrm{T}}_{z_\lambda}(k) &= \int_0^\infty \mathrm{d}s\,(1-\varepsilon)s^{-\varepsilon}\mathrm{e}^{-s}\hat{\mathrm{T}}_{\mathrm{e}^{-s}}(k) \\ &\le c_2(1-\varepsilon)\int_0^{1-\hat{D}(k)} \mathrm{d}s\,\frac{s^{-\varepsilon}}{1-\hat{D}(k)} + c_1(1-\varepsilon)\int_{1-\hat{D}(k)}^\infty \mathrm{d}s\,\frac{s^{-\varepsilon}\mathrm{e}^{-s}}{1-\mathrm{e}^{-s}} \\ &\le c_2[1-\hat{D}(k)]^{-\varepsilon} + c_1(1-\varepsilon)\int_{1-\hat{D}(k)}^{1} \mathrm{d}s\left(s^{-1-\varepsilon}+\frac{s^{-\varepsilon}}{2}+o(1)\right) + c_2(1-\varepsilon)\int_1^\infty \mathrm{d}s\,\mathrm{e}^{-s} \\ &\le c_2[1-\hat{D}(k)]^{-\varepsilon} + \frac{c_1(1-\varepsilon)}{\varepsilon}[1-\hat{D}(k)]^{-\varepsilon} + O(1) \le C_\varepsilon[1-\hat{D}(k)]^{-\varepsilon},\end{aligned} \tag{4.54}$$

where C_ε is a constant that depends only on ε. When we apply this bound to (4.53) we get an upper bound on (4.52),

$$\begin{aligned}\sum_{N=0}^\infty \sum_{x\in\mathbb{Z}^d}\sum_{m=1}^\infty m^{1+\varepsilon}\pi_m^{(N)}(x) &\le \frac{C_1(5+C_2^2\beta^{1/2}-4C_2\beta^{1/4})}{(1-C_2\beta^{1/4})^3} \\ &\quad\times\left(\int_{[-\pi,\pi]^d}\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{[1-\hat{D}(k)]^{3+\varepsilon}}\right)^{1/2}\left(\int_{[-\pi,\pi]^d}\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\hat{D}(k)^2}{[1-\hat{D}(k)]^{3+\varepsilon}}\right)^{1/2} \le C\beta^{1/2},\end{aligned} \tag{4.55}$$

where the final inequality holds when β is small enough and $d > 3(2\wedge\alpha)$ by (1.7) (see [23, Proposition 2.2] for details). This completes the proof of Proposition 4.4(i).

Recall (4.30). For the proof of the right-hand bound of (4.47) we note that since $z \le 1$ and (4.7),

$$\sum_x zp_cD(x-t)\tau(v-x) = zp_cD(v-t) + z(p_cD * p_cD * \tau)(v-t) \le 2\tilde{\mathrm{T}}_z(v-t) \tag{4.56}$$

so that $\sum_x B_3(s,t,x,u,v;z) \le 2B_1(s,t,u,v;z)$ and thus

$$\sum_x\sum_{m=0}^\infty m^{1+\varepsilon}|\psi_m^{(N)}(x)| \le 2\sum_a\sum_{m=0}^\infty m^{1+\varepsilon}|\pi_m^{(N+1)}(a)| \tag{4.57}$$

and the claim follows. □

5. Error bound for percolation lace expansion: proof of Proposition 2.3

In this section we prove Proposition 2.3. We start by stating a lemma that contains the key bounds used in this proof.

Lemma 5.1. *Assume that Propositions 4.1 and 4.4 hold, then there exists $\tilde{c}, c', c'' > 0$ such that for all $k\in\mathbb{R}^d$ and $z\in[0,1)$,*

$$|\hat{E}_z(k)| \le \tilde{c}\beta^{1/2}(1-z)^{1+\varepsilon}; \tag{5.1}$$

$$|1-zp_c\hat{D}(k)\hat{\Pi}_z(k)| \ge (1+c'\beta^{1/2})(1-z); \tag{5.2}$$

$$|(1-z)\hat{A}(k)+\hat{B}(k)| \ge c''(1-z). \tag{5.3}$$

We prove this lemma at the end of the section.

Proof of Proposition 2.3 subject to Lemma 5.1. Let $f(z) = \sum a_n z^n$ have radius of convergence 1. It is proved in [10, Lemma 3.2] that if, for $z\in[0,1)$, we have the bound $|f(z)| \le C(1-z)^{-b}$, then it follows that $|a_n| \le C'n^{b-1}$ if $b<1$ and $|a_n| \le C'\log n$ if $b=1$.

The power series

$$\hat{\Theta}_z(k) = \sum_{n=0}^{\infty} \hat{\theta}_n(k) z^n \tag{5.4}$$

has radius of convergence 1. But before try to determine bounds on $|\hat{\theta}_n(k)|$, we first rewrite $\hat{\Theta}_z(k)$. Using the decomposition in (2.24), we write

$$\hat{\Theta}_z(k) = \hat{\Theta}_z^{(1)}(k) + \hat{\Theta}_z^{(2)}(k) \tag{5.5}$$

with

$$\hat{\Theta}_z^{(1)}(k) = \frac{\hat{\Psi}_1(0)\,\hat{E}_z(k)}{((1-z)\hat{A}(k)+\hat{B}(k))^2} \qquad \text{and} \qquad \hat{\Theta}_z^{(2)}(k) = \frac{\hat{\Psi}_1(0)-\hat{\Psi}_z(k)}{1-zp_c\hat{D}(k)\hat{\Pi}_z(k)}. \tag{5.6}$$

To prove the proposition it is sufficient to show that there exist $0<\varepsilon<1$ such that

$$|\hat{\Theta}_z^{(1)}(k)| \le C(1-z)^{-(1-\varepsilon)} \qquad \text{and} \qquad |\hat{\Theta}_z^{(2)}(k)| \le C(|k|^{\varepsilon}(1-z)^{-1}+(1-z)^{-(1-\varepsilon)}) \tag{5.7}$$

when z is close to 1.

We start by bounding $\hat{\Theta}_z^{(1)}(k)$. It is a simple consequence of Proposition 4.3(iii) that

$$|\hat{\Psi}_1(0)| \le C. \tag{5.8}$$

Hence, by these two bounds, and by (5.1) and (5.3),

$$|\hat{\Theta}_z^{(1)}(k)| \le |\hat{\Psi}_1(0)|\cdot|\hat{E}_z(k)|\cdot\frac{1}{|(1-z)\hat{A}(k)+\hat{B}(k)|^2} \le \frac{C}{(1-z)^{1-\varepsilon}}. \tag{5.9}$$

We now establish the upper bound on $|\hat{\Theta}_z^{(2)}(k)|$. For the numerator on the right-hand side of (5.6) we bound

$$|\hat{\Psi}_1(0)-\hat{\Psi}_z(k)| \le |\hat{\Psi}_1(0)-\hat{\Psi}_1(k)| + |\hat{\Psi}_1(k)-\hat{\Psi}_z(k)|. \tag{5.10}$$

When $\varepsilon \in (0, 2\wedge\delta_3]$, then Proposition 4.3(iii) implies

$$\begin{aligned}|\hat{\Psi}_1(0)-\hat{\Psi}_1(k)| &\le \sum_{x\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}}[1-\cos(k\cdot x)]\,|\psi_n(x)| \le \sum_{x\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}}(k\cdot x)^{\varepsilon}\,|\psi_n(x)| \\ &\le |k|^{\varepsilon}\sum_{x\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}}|x|^{\varepsilon}\,|\psi_n(x)| \le C\,|k|^{\varepsilon}.\end{aligned} \tag{5.11}$$

For any $z,\varepsilon\in(0,1)$ and integers $n\ge\ell\ge0$,

$$1-z^{\ell} \le 1-z^n = \left(1-z^n\right)^{1-\varepsilon}\left(\frac{1-z^n}{1-z}\right)^{\varepsilon}(1-z)^{\varepsilon} \le \left(\sum_{\ell=0}^{n-1} z^{\ell}\right)^{\varepsilon}(1-z)^{\varepsilon} \le n^{\varepsilon}\,(1-z)^{\varepsilon}. \tag{5.12}$$

This bound, together with Proposition 4.4 gives

$$|\hat{\Psi}_1(k)-\hat{\Psi}_z(k)| \le \sum_{x\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}}\left(1-z^n\right)|\psi_n(x)| \le (1-z)^{\varepsilon}\sum_{x\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}} n^{\varepsilon}\,|\psi_n(x)| \le C\,(1-z)^{\varepsilon}. \tag{5.13}$$

To bound the denominator in (5.6) we use (5.3). Combined with (5.13) and (5.11), this yields

$$|\hat{\Theta}_z^{(2)}(k)| \le C(|k|^{\varepsilon}(1-z)^{-1}+(1-z)^{-(1-\varepsilon)}). \tag{5.14}$$

This proves (5.7), and thus completes the proof. □

Proof of Lemma 5.1. Proof of (5.1). Recall the definition of $\hat{E}_z$, (2.22). We start by bounding

$$\begin{aligned}|\hat{E}_z(k)| &= \left|p_c\hat{D}(k)\left(\hat{\Pi}_1(k)-\hat{\Pi}_z(k)\right)-(1-z)p_c\hat{D}(k)\partial_z\hat{\Pi}_z(k)|_{z=1}-(1-z)p_c\hat{D}(k)\left(\hat{\Pi}_z(k)-\hat{\Pi}_z(k)\right)\right| \\ &\le (1-z)p_c|\hat{D}(k)|\left|\frac{\hat{\Pi}_1(k)-\hat{\Pi}_z(k)}{1-z}-\partial_z\hat{\Pi}_z(k)|_{z=1}\right| + (1-z)p_c|\hat{D}(k)|\left|\hat{\Pi}_1(k)-\hat{\Pi}_z(k)\right| \\ &\le (1-z)p_c|\hat{D}(k)|\left|\frac{\hat{\Pi}_1(k)-\hat{\Pi}_z(k)}{1-z}-\partial_z\hat{\Pi}_z(k)|_{z=1}\right| + \tilde{c}\beta^{1/2}(1-z)^2,\end{aligned} \tag{5.15}$$

where the final inequality follows from Proposition 4.2(i). To bound the remaining term on the right-hand side of (5.15), we choose ε such that Proposition 4.4 holds. Then, by (5.12) and Proposition 4.4,

$$\left|\partial_z \hat{\Pi}_z(k)|_{z=1} - \frac{\hat{\Pi}_1(k)-\hat{\Pi}_z(k)}{1-z}\right| = \left|\sum_{x\in\mathbb{Z}^d}\sum_{n=1}^{\infty} n\pi_n(x)\mathrm{e}^{ik\cdot x} - \sum_{x\in\mathbb{Z}^d}\sum_{n=1}^{\infty}\left(\frac{1-z^n}{1-z}\right)\pi_n(x)\mathrm{e}^{ik\cdot x}\right|$$

$$= \left|\sum_{x\in\mathbb{Z}^d}\sum_{n=1}^{\infty}\left(\sum_{\ell=0}^{n-1}(1-z^\ell)\right)\pi_n(x)\mathrm{e}^{ik\cdot x}\right| \le (1-z)^\varepsilon \sum_{x\in\mathbb{Z}^d}\sum_{n=1}^{\infty} n(n-1)^\varepsilon\,|\pi_n(x)| \le \tilde{c}\beta^{1/2}(1-z)^\varepsilon. \tag{5.16}$$

Combining (5.15) and (5.16) completes the proof.

Proof of (5.2). By (2.15) and Proposition 4.2(i) and (ii), there exists a $c'>0$ such that

$$|1-zp_c\hat{D}(k)\hat{\Pi}_z(k)| = \left|\frac{\hat{\Pi}_z(k)}{\hat{\mathrm{T}}_z(k)}\right| \ge (1+c'\beta^{1/2})(1-z), \tag{5.17}$$

where for the last inequality we used Proposition 4.2(i) and (ii) to bound the numerator and the denominator, respectively.

Proof of (5.3). By (2.15) and (2.19) we can bound

$$|(1-z)\hat{A}(k)+\hat{B}(k)| \ge \left|\frac{\hat{\Pi}_z(k)}{\hat{\mathrm{T}}_z(k)}\right| - |\hat{E}_z(k)| \ge (1+c'\beta^{1/2})(1-z) - \tilde{c}\beta^{1/2}(1-z)^{1+\varepsilon} \ge c''(1-z), \tag{5.18}$$

when β is small enough. For the second inequality we used the bounds from parts (i) and (ii) of the lemma. □

Proof of Proposition 2.4. The proof of Proposition 2.4 follows from (1.7), the spatial symmetries of the model and Proposition 4.3. It is identical to the proof of [19, Proposition 2.3]. □

6. THE MEAN-r DISPLACEMENT: PROOF OF THEOREM 1.6

Proof of Theorem 1.6. We follow the proof of [19, Theorem 1.4]. Write x_1 for the first coordinate of the vector $x\in\mathbb{Z}^d$. Since $|x_1|^r \le |x|^r \le d^{r/2}\sum_{i=1}^d |x_i|^r$, and the model is invariant under rotation by $\pi/2$, it is sufficient to prove the existence of constants $c, C>0$ such that

$$c\,f_\alpha(n)^{-r} \overset{(a)}{\le} \sum_{x\in\mathbb{Z}^d}|x_1|^r\varrho_{n+1}(x) \overset{(b)}{\le} p_c\sum_{x\in\mathbb{Z}^d}|x_1|^r\tau_n(x) \overset{(c)}{\le} Cf_\alpha(n)^{-r}. \tag{6.1}$$

Inequality (b) is a simple consequence of (3.63).

Now we prove inequality (a). Write u_n for the vector $(f_\alpha(n),0,\dots,0)\in\mathbb{R}^d$. Theorem 1.5 implies

$$\lim_{n\to\infty} 1-\hat{\varrho}_n(u_n) = 1-\mathrm{e}^{-K_\alpha} > 0. \tag{6.2}$$

Hence, for n sufficiently large,

$$0 < \frac{1}{2}(1-\mathrm{e}^{-K_\alpha}) \le 1-\hat{\varrho}_n(u_n) = \sum_{x\in\mathbb{Z}^d}\big[1-\cos(f_\alpha(n)\,x_1)\big]\varrho_n(x) \le \sum_{x\in\mathbb{Z}^d} f_\alpha(n)^r|x_1|^r\varrho_n(x), \tag{6.3}$$

where the last bound follows from $1-\cos(t)\le|t|^r$ for any $r\le(2\wedge\alpha)$. This implies the lower bound.

Inequality (c) in (6.1) requires a little more work.

We start by observing that for any $r\in(0,2)$ there exists $c_r\in(0,\infty)$ such that

$$t^r = c_r\int_0^\infty \frac{1-\cos(ut)}{u^{1+r}}\,\mathrm{d}u \tag{6.4}$$

for all $t>0$.

Write $\vec{u} = (u, 0, \dots, 0)$ for $u > 0$, and consider the generating function

$$H_{z,r} \equiv \sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty} |x_1|^r \tau_n(x) z^n = c_r \sum_{x\in\mathbb{Z}^d}\sum_{n=0}^{\infty}\int_0^{\infty} \frac{\mathrm{d}u}{u^{1+r}}\,[1-\cos(\vec{u}\cdot x)]\,\tau_n(x)\,z^n, \quad |z|<1, \tag{6.5}$$

where the last identity uses (6.4). It remains to prove that for any $r \in (0,(2\wedge\alpha))$ there exists a constant $C > 0$ such that

$$H_{z,r} \le C(1-z)^{-1-r/(2\wedge\alpha)} \tag{6.6}$$

for $\alpha \neq 2$ and $H_{z,r} \le C(1-z)^{-1-r/2}\log(1-z)^{-1/2}$ for $\alpha = 2$ because if this bound holds we can apply [10, Lemma 3.2] (cf. the opening paragraphs of Section 5) to get the bound $\sum_x |x_1|^r \tau_n(x) \le Cn^{r/(2\wedge\alpha)}$ when $r < (2\wedge\alpha)$ and to get $\sum_x |x_1|^r \tau_n(x) \le Cn\log n$ when $r = (2\wedge\alpha)$.

We will now prove (6.6). First consider the case $\alpha \neq 2$. Using $1-\cos t \le 2$ we get the upper bound

$$H_{z,r} \le c_r \int_0^{(1-z)^{1/(2\wedge\alpha)}} \Big(\hat{\mathrm{T}}_z(0)-\hat{\mathrm{T}}_z(\vec{u})\Big)\frac{\mathrm{d}u}{u^{1+r}} + c_r\int_{(1-z)^{1/(2\wedge\alpha)}}^{\infty} 2\,\hat{\mathrm{T}}_z(0)\frac{\mathrm{d}u}{u^{1+r}} \tag{6.7}$$

Applying Proposition 4.2(ii) to the second integral in (6.7) gives

$$\int_{(1-z)^{1/(2\wedge\alpha)}}^{\infty} 2\,\hat{\mathrm{T}}_z(0)\frac{\mathrm{d}u}{u^{1+r}} \;\le\; C(1-z)^{-1}\int_{(1-z)^{1/(2\wedge\alpha)}}^{\infty}\frac{\mathrm{d}u}{u^{1+r}} \;=\; \frac{C(2\wedge\alpha)}{r}\,(1-z)^{-1-r/(2\wedge\alpha)}, \tag{6.8}$$

as claimed. For the first integral in (6.7) we first rewrite the integrand using (2.34), and then we use Proposition 4.2(ii) to bound

$$\hat{\mathrm{T}}_z(0)-\hat{\mathrm{T}}_z(\vec{u}) = \hat{\mathrm{T}}_z(0)\,\hat{\mathrm{T}}_z(\vec{u})\left(\hat{\mathrm{T}}_z(\vec{u})^{-1}-\hat{\mathrm{T}}_z(0)^{-1}\right) \le \frac{C}{(1-z)^2}\left(\frac{\hat{\Pi}_z(0)-\hat{\Pi}_z(\vec{u})}{\hat{\Pi}_z(0)\,\hat{\Pi}_z(\vec{u})} + zp_c[1-\hat{D}(\vec{u})]\right). \tag{6.9}$$

Observe that it follows from (2.28) that

$$\hat{\Pi}_z(0)-\hat{\Pi}_z(\vec{u}) \le C\,[1-\hat{D}(\vec{u})]. \tag{6.10}$$

We apply this bound, (1.7), and Proposition 4.2(i) to the right-hand side of (6.9) to get $C(1-z)^{-2}|u|^{2\wedge\alpha}$ as an upper bound. Hence, using $r < (2\wedge\alpha)$, the first integral in (6.7) is bounded above by

$$\int_0^{(1-z)^{1/(2\wedge\alpha)}}\Big(\hat{\mathrm{T}}_z(0)-\hat{\mathrm{T}}_z(\vec{u})\Big)\frac{\mathrm{d}u}{u^{1+r}} \;\le\; C(1-z)^{-2}\int_0^{(1-z)^{1/(2\wedge\alpha)}}\frac{u^{2\wedge\alpha}}{u^{1+r}}\,\mathrm{d}u \;\le\; C\,\frac{(1-z)^{-1-r/(2\wedge\alpha)}}{(2\wedge\alpha)-r}. \tag{6.11}$$

We combine (6.7), (6.8) and (6.11) to get the desired bound (6.6). This finishes the argument for $\alpha \neq 2$.

To prove Theorem 1.6 for $\alpha = 2$, we have to take the logarithmic corrections into account. The way to do this has been demonstrated in [19, Theorem 1.4], so we omit the proof here. □

7. Convergence as a stochastic process: Proof of Theorem 1.1

This section is devoted to the proofs of Proposition 2.1 and Corollary 2.2, thus completing the proof of Theorem 1.1.

7.1. Finite-dimensional distributions

We start with the proof of Proposition 2.1. Instead of proving it directly, we consider the following generalized version:

Proposition 7.1 [Finite-dimensional distributions, generalized version]. *Let N be a positive integer, $k^{(1)}, \dots, k^{(N)} \in \mathbb{R}^d$, $0 = t^{(0)} < t^{(1)} < \cdots < t^{(N)} < 1$, and $g = (g_n)$ a sequence of real numbers satisfying $0 \le g_n \le \log n/n$. We write*

$$\mathsf{k}_n = \left(k_n^{(1)},\dots,k_n^{(N)}\right) = f_\alpha(n)\left(k^{(1)},\dots,k^{(N)}\right), \tag{7.1}$$

$$n\mathsf{T} = \left(\lfloor nt^{(1)}\rfloor,\dots,\lfloor nt^{(N-1)}\rfloor,\lfloor nT\rfloor\right) \tag{7.2}$$

with $T = t^{(N)}(1-g_n)$. Under the conditions of Theorem 1.1,

$$\lim_{n\to\infty} \underline{\hat{\varrho}}^{(N)}_{n\mathsf{T}}(\mathsf{k}_n) = \frac{1}{A} \lim_{n\to\infty} \underline{\hat{\tau}}^{(N)}_{n\mathsf{T}}(\mathsf{k}_n) = \exp\Big\{ - K_\alpha \sum_{j=1}^{N} |k^{(j)}|^{(2\wedge\alpha)} \, (t^{(j)} - t^{(j-1)}) \Big\} \tag{7.3}$$

hold uniformly in g, where A is the same constant as in Theorem 1.5.

The proof is carried out by induction on N. We use sequence g to ensure that we can advance the induction, because g gives us a little flexibility in our choice of the location of the end-point.

Proof of Proposition 7.1. The proof that we present here takes a well-known approach, see for example the proofs of [32, Theorem 6.6.2] or [19, Theorem 1.6]. We will first give the proof for $\underline{\hat{\tau}}^{(N)}_{n\mathsf{T}}$, and then discuss the necessary changes for $\underline{\hat{\varrho}}^{(N)}_{n\mathsf{T}}$. For convenience, we write $nt^{(j)}$ and nT instead of $\lfloor nt^{(j)} \rfloor$ and $\lfloor nT \rfloor$.

The proof is by induction on N. We start the induction by applying Theorem 1.5: since $\underline{\hat{\tau}}^{(1)}_{n\mathsf{T}}(\mathsf{k}_n) = \hat{\tau}_{nT}(k_n^{(1)})$, we can replace n by nT in (2.31) and the claim follows.

To advance the induction we use a *KJK-expansion,*

$$K[0,n] = \sum_{\substack{I=[I_1,I_2] \\ 0\le I_1 \le nt^{(N-1)} \le I_2 \le n}} K[0, I_1 - 1]\, J[I_1, I_2]\, K[I_2+1, n] \tag{7.4}$$

where $I = [I_1, I_2]$ is an interval of *integers*, and $K[a, a-1] = 1$ for all a. We get (7.4) if we take the sum in (3.43) and partition off the terms that correspond to connected components with the start vertex to the left of $nt^{(N-1)}$ and the end vertex to the right of $nt^{(N-1)}$. Compare (7.4) with [32, Lemma 5.2.5], where a similar bound in the context of self-avoiding walks is derived (but mind that the formula there is slightly different due to a different notion of connectivity of graphs).

When we combine (3.41), (3.47) and (7.4) we get

$$\tau_n(x) = \sum_{\substack{I=[I_1,I_2] \\ 0\le I_1 \le nt^{(N-1)} \le I_2 \le n}} \sum_{b_{I_1}, b_{I_2+1}} \big[p_c D(b_{I_1})\big] \big[p_c D(b_{I_2+1})\big] \times \tau_{I_1-1}(\underline{b}_{I_1})\, \pi_{|I|}(\underline{b}_{I_2+1} - \overline{b}_{I_1})\, \tau_{n-I_2-1}(x - \overline{b}_{I_2+1}). \tag{7.5}$$

We can similarly rewrite the characteristic function for increments, $\underline{\hat{\tau}}^{(N)}_{n\mathsf{T}}(\mathsf{k}_n)$ with the KJK-expansion, i.e.,

$$\underline{\hat{\tau}}^{(N)}_{n\mathsf{T}}(\mathsf{k}_n) = \sum_{\substack{I=[I_1,I_2] \\ 0\le I_1 \le nt^{(N-1)} \le I_2 \le n}} \big[p_c \hat{D}(k_n^{(N-1)})\big] \big[p_c \hat{D}(k_n^{(N)})\big] \times \underline{\hat{\tau}}^{(N-1)}_{(nt^{(1)},\dots,nt^{(N-2)},I_1-1)}\big(k_n^{(1)},\dots,k_n^{(N-1)}\big) \underline{\hat{\pi}}_{(nt^{(N-1)}-I_1,|I|)}(k_n^{(N-1)}, k_n^{(N)})\, \hat{\tau}_{nT-I_2-1}(k_n^{(N)}), \tag{7.6}$$

where, like (3.47),

$$\underline{\hat{\pi}}_{(m,n)}(k_1,k_2) = \sum_{\substack{\text{bonds} \\ b_1,\dots,b_n}} \sum_{\underline{b}_{n+1}\in\mathbb{Z}^d} \exp\{i(k_1\cdot \underline{b}_m + k_2 \cdot (\underline{b}_n - \underline{b}_m))\} \Big[\prod_{i=1}^{n} p_c D(b_i)\Big] \times \mathbb{E}^{\otimes[0,n]}_{p_c}\Big[\mathbb{1}_{\{0 \Leftrightarrow \underline{b}_1\}_0} \prod_{i=0}^{n} \mathbb{1}_{\{\overline{b}_i \Leftrightarrow \underline{b}_{i+1}\}_i} J[0,n+1]\Big], \qquad 0 \le m \le n;\ k_1, k_2 \in \mathbb{R}^d. \tag{7.7}$$

Observe that $\underline{\hat{\pi}}_{(m,n)}(0,0) = \hat{\pi}_n(0)$.

Using (7.6) we can split up $\underline{\hat{\tau}}^{(N)}_{n\mathsf{T}}(\mathsf{k}_n)$ into the contribution of short intervals I and long intervals. We write $\underline{\sigma}^{(N)}_{n\mathsf{T}}(\mathsf{k}_n)$ for the contribution to $\underline{\hat{\tau}}^{(N)}_{n\mathsf{T}}(\mathsf{k}_n)$ that comes from intervals with length $|I| = I_2 - I_1 \le \log n$ and we write $\underline{\lambda}^{(N)}_{n\mathsf{T}}(\mathsf{k}_n)$ for the contributions from the intervals with length $|I| > \log n$.

We start by showing that $\underline{\lambda}^{(N)}_{n\mathsf{T}}(\mathsf{k}_n)$ is negligible. It follows from Theorem 1.5 that $\hat{\tau}_m(k) \le \hat{\tau}_m(0) \le C$ for some $C > 0$. Using this bound and (7.5),

$$\underline{\lambda}^{(N)}_{n\mathsf{T}}(\mathsf{k}_n) \le \underline{\lambda}^{(N)}_{n\mathsf{T}}(\underline{0}) = \sum_{\substack{I \ni nt^{(N-1)} \\ |I| > \log n}} p_c^2 \hat{\tau}_{I_1-1}(0)\, |\hat{\pi}_{|I|}(0)|\, \hat{\tau}_{nT-I_2-1}(0) \le p_c^2 C^2 \sum_{m=\log n+1}^{\infty} (m+1)\, |\hat{\pi}_m(0)|. \quad (7.8)$$

We get the factor $m+1$ here because there are precisely $m+1$ ways to choose the interval $I \ni nt^{(N-1)}$ under the restriction $|I| = m$. Since the right-hand side of (7.8) is finite for all $n \ge 1$ by Proposition 4.4, it vanishes as $n \to \infty$.

We now establish the bound for $\underline{\sigma}^{(N)}_{n\mathsf{T}}(\mathsf{k}_n)$. Assume that n is large enough that both $(nt^{(N-1)} - nt^{(N-2)}) \ge \log n$ and $(nt^{(N)} - nt^{(N-1)}) \ge \log n$. The induction hypothesis is that

$$\underline{\hat{\tau}}^{(N-1)}_{(nt^{(1)},\dots,nt^{(N-2)},I_1-1)}\big(k_n^{(1)},\dots,k_n^{(N-1)}\big) = A \exp\Big\{ -K_\alpha \sum_{j=1}^{N-1} |k^{(j)}|^{2\wedge\alpha}\, (t^{(j)} - t^{(j-1)}) \Big\} + E_1(I), \quad (7.9)$$

where $E_1(I)$ is an error term that converges to 0 as $n \to \infty$ uniformly in $|I| \le \log n$.

A slight generalization of the case $N = 1$ shows that

$$\hat{\tau}_{nT-I_2-1}(k_n^{(N)}) = A \exp\Big\{ -K_\alpha |k^{(N)}|^{2\wedge\alpha}\, (t^{(N)} - t^{(N-1)}) \Big\} + E_2(I), \quad (7.10)$$

where $E_2(I)$ is an error term that is due to Theorem 1.5. Note that $E_2(I)$ converges to 0 as $n \to \infty$ uniformly in $|I| \le \log n$. Hence,

$$\begin{aligned}\underline{\sigma}^{(N)}_{n\mathsf{T}}(\mathsf{k}_n) = \Bigg(A^2 \exp\Big\{ -K_\alpha \sum_{j=1}^{N} |k^{(j)}|^{2\wedge\alpha}\, (t^{(j)} - t^{(j-1)}) \Big\} + E_3 \Bigg) \\ \times \sum_{\substack{I \ni nt^{(N-1)} \\ |I| \le \log n}} \big[p_c \hat{D}(k_n^{(N-1)})\big] \big[p_c \hat{D}(k_n^{(N)})\big]\, \underline{\hat{\pi}}_{(nt^{(N-1)}-I_1,|I|)}(k_n^{(N-1)}, k_n^{(N)}), \quad (7.11)\end{aligned}$$

where E_3 is the error term that comes from E_1 and E_2. Note that E_3 is uniform in the sequences g that satisfy $g_n \le \log n/n$.

The proof is complete when we show that the second line in (7.11) converges to $1/A$. We begin by writing

$$\begin{aligned}&\sum_{\substack{I \ni nt^{(N-1)} \\ |I| \le \log n}} \big[p_c \hat{D}(k_n^{(N-1)})\big] \big[p_c \hat{D}(k_n^{(N)})\big]\, \underline{\hat{\pi}}_{(nt^{(N-1)}-I_1,|I|)}(k_n^{(N-1)}, k_n^{(N)}) \\ &\quad = \sum_{\substack{I \ni nt^{(N-1)} \\ |I| \le \log n}} p_c^2 \hat{\pi}_{|I|}(0) - \sum_{\substack{I \ni nt^{(N-1)} \\ |I| \le \log n}} p_c^2 \Big(\hat{\pi}_{|I|}(0) - \underline{\hat{\pi}}_{(nt^{(N-1)}-I_1,|I|)}(k_n^{(N-1)}, k_n^{(N)}) \Big) \\ &\qquad - \sum_{\substack{I \ni nt^{(N-1)} \\ |I| \le \log n}} p_c^2 \big[1 - \hat{D}(k_n^{(N-1)})\big] \big[1 - \hat{D}(k_n^{(N)})\big]\, \underline{\hat{\pi}}_{(nt^{(N-1)}-I_1,|I|)}(k_n^{(N-1)}, k_n^{(N)}). \quad (7.12)\end{aligned}$$

The first term converges to $1/A$. This follows from (2.38) and the fact that by Proposition 4.4, $\sum_{m=n}^{\infty} (m+1)|\hat{\pi}_m(0)| \to 0$ as $n \to \infty$. Indeed,

$$\sum_{\substack{I \ni nt^{(N-1)} \\ |I| \le \log n}} p_c^2 \hat{\pi}_{|I|}(0) = \sum_{I \ni nt^{(N-1)}} p_c^2 \hat{\pi}_{|I|}(0) - \sum_{\substack{I \ni nt^{(N-1)} \\ |I| > \log n}} p_c^2 \hat{\pi}_{|I|}(0) \quad \stackrel{n\to\infty}{\longrightarrow} \quad p_c^2 \sum_{m \ge 0} (m+1) \hat{\pi}_m(0) = 1/A, \quad (7.13)$$

where the factor $m+1$ arises because there are $m+1$ intervals of length $|I| = m$ that contain the point $nt^{(N-1)}$.

Now we show that the second term on the right-hand side of (7.12) converges to 0. Recall the definition of $\pi_{m,n}(y,x)$ in Proposition 4.3(iv). We use the spatial symmetry of the model to replace the exponential factor in (7.7) by a cosine. We also use $|1-\cos(a)\cos(b)| \le 2|a|^\delta + 2|b|^\delta$ for all $\delta \in [0,2]$ to get for $m \le n$,

$$\Big|\hat\pi_n(0) - \underline{\hat\pi}_{m,n}(k_1,k_2)\Big| \le 2\sum_{y,x}(|k\cdot y|^\delta + |(k_2-k_1)\cdot x|^\delta)\pi_{m,n}(y,x)|. \tag{7.14}$$

Again, there are $m+1$ of intervals of length $|I| = m$, hence, using (7.14), uniformly in $k^{(N-1)}, k^{(N)} \in [-\pi,\pi]^d$,

$$\begin{aligned}\sum_{\substack{I \ni nt^{(N-1)}\\ |I|\le \log n}} p_c^2 \Big|\underline{\hat\pi}_{(nt^{(N-1)}-I_1,|I|)}(k_n^{(N-1)},k_n^{(N)}) - \hat\pi_{|I|}(0)\Big|& \\ \le C\sum_{m=0}^{\log n}(m+1)\sum_{x,y}(|f_\alpha(n)\,x|^\delta + |f_\alpha(n)\,y|^\delta)|\pi_{m,n}(y,x)|& \\ \le C f_\alpha(n)^\delta\,(\log n+1)\sum_{m=0}^{\log n}\sum_{x,y}(|x|^\delta+|y|^\delta)|\pi_{m,n}(y,x)|,&\end{aligned} \tag{7.15}$$

and this converges to 0 as $n\to\infty$ when δ is sufficiently small since the sum is uniformly bounded in n by Proposition 4.3(iv). It now follows from $\lim_{n\to\infty} n\,[1-\hat D(k_n)] = |k|^{2\wedge\alpha}$ (cf. (2.30)), and (7.15) that the second line on the right-hand side of (7.12) vanishes as $n\to\infty$. This completes the proof that the second line in (7.11) converges to $1/A$, and thus we have completed the advancement of the induction.

With the result for $\underline{\hat\tau}^{(N)}_{n\mathsf{T}}$ in hand, we can derive the statement for $\underline{\hat\varrho}^{(N)}_{n\mathsf{T}}$. Instead of (7.4) we now have the identity

$$K_V[0,n] = \sum_{\substack{I=[I_1,I_2]\\ 0\le I_1\le nt^{(N-1)}\le I_2\le n}} K_V[0,I_1-1]\,J_V[I_1,I_2]\,K_V[I_2+1,n]. \tag{7.16}$$

Recall that $K_V[0,I_1-1] = K[0,I_1-1]$ and $J_V[I_1,I_2] = J[I_1,I_2]$ unless $I_2 = n$. Thus, using (3.73),

$$\begin{aligned}\underline{\hat\varrho}^{(N)}_{n\mathsf{T}}(\mathsf{k}_n) = \sum_{\substack{I=[I_1,I_2]\\ 0\le I_1\le nt^{(N-1)}\le I_2\le n}} \big[p_c\hat D(k_n^{(N-1)})\big]\big[p_c\hat D(k_n^{(N)})\big]\,\underline{\hat\tau}^{(N-1)}_{(nt^{(1)},\dots,nt^{(N-2)},I_1-1)}\big(k_n^{(1)},\dots,k_n^{(N-1)}\big)& \\ \times\,\underline{\hat\pi}_{(nt^{(N-1)}-I_1,|I|)}(k_n^{(N-1)},k_n^{(N)})\,\hat\varrho_{nT-I_2-1}(k_n^{(N)}),&\end{aligned} \tag{7.17}$$

The term in (7.17) involving $\underline{\hat\tau}$ gives rise to a factor

$$A\exp\big\{-K_\alpha\sum_{j=1}^{N-1}|k^{(j)}|^{2\wedge\alpha}\,(t^{(j)}-t^{(j-1)})\big\} + \tilde E, \tag{7.18}$$

where $\tilde E$ is an error term, similar to (7.9). Likewise, the term involving $\underline{\hat\pi}$ gives rise to the factor $1/A$, by (7.13). Finally, by Theorem 1.5,

$$\hat\varrho_{nT-I_2-1}(k_n^{(N)}) \to \exp\Big\{-K_\alpha\,|k^{(N)}|^{2\wedge\alpha}\,(t^{(N)}-t^{(N-1)})\Big\} \qquad \text{as } n\to\infty, \tag{7.19}$$

so that the statement for $\underline{\hat\varrho}^{(N)}_{n\mathsf{T}}$ follows. □

7.2. Tightness

In this section we prove the tightness of X_n and Y_n. We claim that tightness follows from the bound

$$\mathbb{E}_{\mathrm{IIC}}\big[|X_n(t_2)-X_n(t_1)|^r\,|X_n(t_3)-X_n(t_2)|^r\big] \le C|t_3-t_1|^a, \tag{7.20}$$

for some $r>0$, $a>1$, and $C>0$. This claim is proved in [4, Theorem 13.5] (where (13.13) is replaced by the stronger moment condition (13.14)) and also in [19, Section 5]. By Lemma 3.6, we get that

$$\mathbb{E}_{\text{IIC}}\big[|X_n(t_2)-X_n(t_1)|^r|X_n(t_3)-X_n(t_2)|^r\big] \\ \leq p_c n^{-2r/(2\wedge\alpha)}\Big(\sum_x \tau_{nt_1}(x)\Big)\Big(\sum_x |x|^r\tau_{n(t_2-t_1)}(x)\Big)\Big(\sum_x |x|^r\tau_{n(t_3-t_2)}(x)\Big). \quad (7.21)$$

By Theorem 1.6, for any $0 \leq r < (2\wedge\alpha)$, there exists C_r such that

$$\sum_x |x|^r(pD*\tau_n)(x) \leq C_r n^{r/(2\wedge\alpha)}. \quad (7.22)$$

Therefore,

$$\mathbb{E}_{\text{IIC}}\big[|X_n(t_2)-X_n(t_1)|^r|X_n(t_3)-X_n(t_2)|^r\big] \\ \leq p_c C_r^2 C_0 n^{-2r/(2\wedge\alpha)}[n(t_2-t_1)]^{r/(2\wedge\alpha)}[n(t_3-t_2)]^{r/(2\wedge\alpha)} \\ \leq C|t_2-t_1|^{r/(2\wedge\alpha)}|t_3-t_2|^{r/(2\wedge\alpha)}. \quad (7.23)$$

Tightness of X_n follows when we choose $r > (2\wedge\alpha)/2$, so that $2r/(2\wedge\alpha) > 1$, as required.

The proof for Y_n is similar and follows when we replace $\mathbb{E}_{\text{IIC}}$ by $\mathbb{E}^*_{p_c,n}$ in the above proof and we apply Lemma 3.3 instead of Lemma 3.6. □

8. CONVERGENCE OF THE BACKBONE AS A SET

In this section we prove Theorem 1.3 and Proposition 1.4 under Hypothesis H. We only prove the convergence of the processes restricted to the time-interval $[0,1]$, for the reasons that we give at the beginning of Section 2.

Our main goal is to show that, under Hypothesis H, the "sausages" $(\mathcal{S}_i)_{0\leq i\leq n}$ are all small compared to the scale of the pivotal walk $S_0, S_1, \ldots, S_n$. This is formalized as follows.

Lemma 8.1. *Under Hypothesis H, it holds that*

$$f_\alpha(n)\max_{0\leq i\leq n}\operatorname{diam}(\mathcal{S}_i) \longrightarrow 0 \quad (8.1)$$

as $n\to\infty$, in probability under $\mathbb{Q}_{\text{IIC}}$, where for $A\subset\mathbb{R}^d$, $\operatorname{diam}(A)$ *denotes the diameter of A.*

Assuming this lemma, we now prove Proposition 1.4. Note that $\mathbb{R}^d$ is isometrically embedded in $(\mathcal{K}, d_H)$ by the mapping $x\mapsto\{x\}$. Because of this embedding, the convergence in distribution of the process $(X_n(t), 0\leq t\leq 1)$ in the space $\mathbb{D}([0,1],\mathbb{R}^d)$ implies the convergence of $(\{X_n(t)\}, 0\leq t\leq 1)$ to $(\{B_t^{(2\wedge\alpha)}\}, 0\leq t\leq 1)$ in the space $\mathbb{D}([0,1],\mathcal{K})$.

Next, Lemma 8.1 implies that

$$f_\alpha(n)\sup_{0\leq t\leq 1} d_H(\{S_{\lfloor nt\rfloor}\},\mathcal{S}_{\lfloor nt\rfloor}) \leq f_\alpha(n)\max_{0\leq i\leq n}\operatorname{diam}(\mathcal{S}_i) \longrightarrow 0 \quad (8.2)$$

in $\mathbb{P}_{\text{IIC}}$-probability as $n\to\infty$, because $S_i\in\mathcal{S}_i$ by definition. Since the latter uniform estimate dominates the Skorokhod distance, we get that $(f_\alpha(n)\mathcal{S}_{\lfloor nt\rfloor})_{0\leq t\leq 1}$ converges in distribution in $\mathbb{D}([0,1],\mathcal{K})$ to $(\{B_t^{(2\wedge\alpha)}\}, 0\leq t\leq 1)$. This implies the first statement of Proposition 1.4.

Now, if g is a càdlàg function from an interval I to $\mathcal{K}$, the historical path is the function $\tilde{g}\colon I\to\mathcal{K}$ defined by

$$\tilde{g}(t) = \Big(\bigcup_{s\in I, s\leq t} g(s)\Big)^{\text{cl}}. \quad (8.3)$$

Indeed, $\tilde{g}$ takes its values in $\mathcal{K}$, because the function $t\mapsto\operatorname{diam}(g(t))$ is right-continuous with left limits, and therefore it is bounded. And the same goes for the function $t\mapsto d(0,g(t))$, where by definition $d(x,A) = \inf_{y\in A}|x-y|$. This comes directly from the fact that $A\mapsto\operatorname{diam}(A)$ and $A\mapsto d(0,A)$ are continuous functions on $\mathcal{K}$. Noting that $(\mathcal{B}_{\lfloor nt\rfloor}, t\geq 0)$ is the historical path associated with $(\mathcal{S}_{\lfloor nt\rfloor}, t\geq 0)$, we see that the second statement of Proposition 1.4 is an immediate consequence of the first statement and the following lemma:

Lemma 8.2. *Let $(g_n)_{n\geq 1}$ be a sequence of functions converging in $\mathbb{D}([0,1],\mathcal{K})$ to a limit g. Then the historical paths $\tilde{g}_n$ converge to $\tilde{g}$ in $\mathbb{D}([0,1],\mathcal{K})$ as well.*

Proof. The Skorokhod convergence of g_n to g means that there exists a sequence of time-changes $\lambda_n, n\geq 1$, i.e., a sequence of increasing continuous functions from $[0,1]$ onto $[0,1]$, such that λ_n converges uniformly to the identity, and such that

$$\varepsilon_n = \sup_{0\leq t\leq 1} d_H(g_n\circ\lambda_n(t), g(t)) \longrightarrow 0. \tag{8.4}$$

Now we have, for every $t\in[0,1]$,

$$d_H(\tilde{g}_n\circ\lambda_n(t),\tilde{g}(t)) = d_H\Big(\bigcup_{0\leq s\leq\lambda_n(t)} g_n(s), \bigcup_{0\leq s\leq t} g(s)\Big) = d_H\Big(\bigcup_{0\leq s\leq t} g_n\circ\lambda_n(s), \bigcup_{0\leq s\leq t} g(s)\Big). \tag{8.5}$$

This equality holds because $d_H(A,B) = d_H(A^{\mathrm{cl}}, B^{\mathrm{cl}})$ for any two subsets $A,B\subseteq\mathbb{R}^d$. By definition, if $x\in\bigcup_{0\leq s\leq t} g_n\circ\lambda_n(s)$ then it is in $g_n\circ\lambda_n(s)$ for some $s\in[0,t]$, and thus we can find some y in $g(s)$ at distance at most ε_n from x. It follows that the converse also holds when we exchange the roles of $g_n\circ\lambda_n$ and g. This shows that $\sup_{0\leq t\leq 1} d_H(\tilde{g}_n\circ\lambda_n(t),\tilde{g}(t))\leq\varepsilon_n\longrightarrow 0$ as $n\to\infty$, as desired. □

We can now prove Theorem 1.3. Recall that the process $B^{(2\wedge\alpha)}$ is a.s. continuous at time 1. It immediately follows that the historical process $(\{B_s^{(2\wedge\alpha)}: 0\leq s\leq t\}, 0\leq t\leq 1)$ is almost surely continuous at time 1 as well. From this, we deduce that the projection $g\mapsto g(1)$ from $\mathbb{D}([0,1],\mathcal{K})$ to $\mathcal{K}$ is almost everywhere continuous with respect to the law of the limiting process of $(f_\alpha(n)\mathcal{B}_{\lfloor nt\rfloor}, t\geq 0)$. By standard properties of weak convergence of probability measures, we conclude that $f_\alpha(n)\mathcal{B}_n$ converges to $\{B_s^{(2\wedge\alpha)}: 0\leq s\leq 1\}$. The convergence of $f_\alpha(n)\mathcal{B}_{\lfloor nT\rfloor}$ for a general $T>0$ follows by a scaling argument.

This finishes the proof of Proposition 1.4 and Theorem 1.3, and it remains to prove Lemma 8.1.

Proof of Lemma 8.1. For $\varepsilon>0$ we may apply the union bound,

$$\mathbb{Q}_{\mathrm{IIC}}\Big(f_\alpha(n)\max_{0\leq i\leq n}\mathrm{diam}(\mathcal{S}_i)\geq\varepsilon\Big)\leq n\max_{0\leq i\leq n}\mathbb{Q}_{\mathrm{IIC}}(f_\alpha(n)\,\mathrm{diam}(\mathcal{S}_i)\geq\varepsilon). \tag{8.6}$$

By the Backbone Limit Reversal Lemma [20, Lemma 4.2], we can write

$$\begin{aligned}\mathbb{Q}_{\mathrm{IIC}}(f_\alpha(n)\,\mathrm{diam}(\mathcal{S}_i)\geq\varepsilon) &= \lim_{p\nearrow p_c}\frac{1}{\chi(p)}\sum_{x\in\mathbb{Z}^d}\mathbb{P}_p\left(\{f_\alpha(n)\,\mathrm{diam}(\mathcal{S}_i^x)\geq\varepsilon\}\cap\{0\leftrightarrow x\}\right)\\ &= \lim_{p\nearrow p_c}\frac{1}{\chi(p)}\sum_{x\in\mathbb{Z}^d}\sum_{\ell=0}^{\infty}\mathbb{P}_p\left(\{f_\alpha(n)\,\mathrm{diam}(\mathcal{S}_i^x)\geq\varepsilon\}\cap\{0\leftrightarrow x\}\cap\{|\mathrm{Piv}(0,x)=\ell|\}\right)\end{aligned} \tag{8.7}$$

where $\mathcal{S}_i^x$ is the ith sausage along the path $0\leftrightarrow x$ (but recall that $\mathcal{S}_i^x=\varnothing$ when there are fewer than i pivotals). Thus, applying (3.26) and performing the summation over ℓ, we get the upper bound

$$\lim_{p\nearrow p_c}\frac{1}{\chi(p)}\sum_{x\in\mathbb{Z}^d}\sum_{e,b}\tau_{i-1}(\underline{e})pD(e)\mathbb{P}_p(\exists y\colon |y|>\varepsilon f_\alpha(n)^{-1}\colon\{0\leftrightarrow y\}\circ\{0\leftrightarrow\underline{b}\}\circ\{y\leftrightarrow\underline{b}\})pD(b)\tau(x-\overline{b}). \tag{8.8}$$

Summing over x and letting $p\nearrow p_c$, we get

$$\begin{aligned}&\mathbb{Q}_{\mathrm{IIC}}\left(f_\alpha(n)\,\mathrm{diam}(\mathcal{S}_i)\geq\varepsilon\right)\\ &\qquad\leq p_c\sum_u(p_cD*\tau_{i-1})(u)\sum_v\mathbb{P}_{p_c}(\exists y\colon |y|>\varepsilon f_\alpha(n)^{-1}\colon\{0\leftrightarrow y\}\circ\{0\leftrightarrow v\}\circ\{y\leftrightarrow v\}).\end{aligned} \tag{8.9}$$

We use that by Theorem 1.5, $\sum_u\tau_{i-1}(u)\leq C$ for some C to arrive at

$$\begin{aligned}\mathbb{Q}_{\mathrm{IIC}}(f_\alpha(n)\,\mathrm{diam}(\mathcal{S}_i)\geq\varepsilon) &\leq C\sum_v\mathbb{P}_{p_c}(\exists y\colon |y|>\varepsilon f_\alpha(n)^{-1}\colon\{0\leftrightarrow y\}\circ\{0\leftrightarrow v\}\circ\{y\leftrightarrow v\})\\ &= C\sum_v\mathbb{P}_{p_c}\left(\{0\leftrightarrow v\}\circ\{\exists y\colon |y|>\varepsilon f_\alpha(n)^{-1}\colon\{0\leftrightarrow y\}\circ\{y\leftrightarrow v\}\}\right),\end{aligned} \tag{8.10}$$

so that

$$\mathbb{Q}_{\rm IIC}\Big(f_\alpha(n)\max_{0\le i\le n}\mathrm{diam}(\mathcal{S}_i)\ge\varepsilon\Big)\le Cn\sum_v\mathbb{P}_{p_c}\left(\{0\leftrightarrow v\}\circ\{\exists y\colon |y|>\varepsilon f_\alpha(n)^{-1}\colon \{0\leftrightarrow y\}\circ\{y\leftrightarrow v\}\}\right). \tag{8.11}$$

Thus, we are left to show that

$$\sum_z\mathbb{P}_{p_c}\left(\{0\leftrightarrow v\}\circ\{\exists y\colon |y|>\varepsilon f_\alpha(n)^{-1}\colon \{0\leftrightarrow y\}\circ\{y\leftrightarrow v\}\}\right)=o(1/n). \tag{8.12}$$

By the BK-inequality, we have the upper bound

$$C\sum_v\tau(v)\mathbb{P}_{p_c}\left(\exists y\colon |y|>\varepsilon f_\alpha(n)^{-1}\colon \{0\leftrightarrow y\}\circ\{y\leftrightarrow v\}\right). \tag{8.13}$$

Let $a_n=\frac{\varepsilon}{2}(n/\log n)^{1/(2\wedge\alpha)}$. We split the sum over v into $|v|\le a_n$ and $|v|>a_n$. For $|v|>a_n$, we bound the sum by

$$\sum_{|v|>a_n}\tau(v)^2\le a_n^{-(2\wedge\alpha)-\delta}\sum_{|v|>a_n}|v|^{(2\wedge\alpha)+\delta}\tau(v)^2=O(1)a_n^{-(2\wedge\alpha)-\delta}=o(1/n), \tag{8.14}$$

where, for the second inequality we used (4.43) and the fact that $\tau(v)^2$ is the upper bound on $\sum_n\pi_n^{(0)}(v)$ that is used in the proof of (4.43) (see [20, Section 7]).

Uniformly in $|v|\le a_n$, we use that under Hypothesis H

$$\mathbb{P}_{p_c}(\exists y\colon |y|>\varepsilon f_\alpha(n)^{-1}\colon \{0\leftrightarrow y\}\circ\{y\leftrightarrow v\})\le C/(\varepsilon f_\alpha(n)^{-1})^{2(2\wedge\alpha)}, \tag{8.15}$$

and [20, Theorem 1.5] to bound

$$\begin{aligned}\sum_{|v|\le a_n}\tau(v)\mathbb{P}_{p_c}(\exists y\colon |y|>\varepsilon f_\alpha(n)^{-1}\colon \{0\leftrightarrow y\}\circ\{y\leftrightarrow v\})&\le C/(\varepsilon f_\alpha(n)^{-1})^{2(2\wedge\alpha)}\sum_{|v|\le a_n}\tau(v)\\&\le C/(\varepsilon f_\alpha(n)^{-1})^{2(2\wedge\alpha)}a_n^{(2\wedge\alpha)}=o(1/n).\end{aligned} \tag{8.16}$$

This completes the proof of Lemma 8.1. □

We complete this section with a proof of Proposition 1.2.

Proof of Proposition 1.2. For finite-range percolation, under the strong triangle condition, we use the result from [31] that the extrinsic one-arm probability is bounded by C/r^2, i.e.,

$$\mathbb{P}_{p_c}(0\leftrightarrow Q_r^c)\le C/r^2, \tag{8.17}$$

where Q_r is the Euclidean ball of radius r and Q_r^c is its complement. For $|x|\le m$, we can apply the BK-inequality and (8.17) to bound

$$\begin{aligned}\mathbb{P}_{p_c}(\exists y\in\mathbb{Z}^d:|y|>2m,\{0\leftrightarrow y\}\circ\{x\leftrightarrow y\})&\le\mathbb{P}_{p_c}(\{0\leftrightarrow Q_m^c\}\circ\{x\leftrightarrow Q_m(x)^c\})\\&\le\mathbb{P}_{p_c}(0\leftrightarrow Q_m^c)^2\le C^2/m^4,\end{aligned} \tag{8.18}$$

where $Q_m(x)$ is the Euclidean ball of radius m around x. This proves the claim in the finite-range case.

In the long-range case, we bound

$$\begin{aligned}\mathbb{P}_{p_c}(\exists y\in\mathbb{Z}^d:|y|>2m,\{0\leftrightarrow y\}\circ\{x\leftrightarrow y\})&\le\sum_{|y|>2m}\tau_{p_c}(y)\tau_{p_c}(y-x)\\&\le m^{-2(2\wedge\alpha)}\sum_{|y|>2m}|y|^{(2\wedge\alpha)}\tau_{p_c}(y)|y-x|^{(2\wedge\alpha)}\tau_{p_c}(y-x)\\&\le m^{-2(2\wedge\alpha)}\sup_x\sum_y|y|^{(2\wedge\alpha)}\tau_{p_c}(y)|y-x|^{(2\wedge\alpha)}\tau_{p_c}(y-x).\end{aligned} \tag{8.19}$$

We claim that for $d>4(2\wedge\alpha)$,

$$\sup_x\sum_y|y|^{(2\wedge\alpha)}\tau_{p_c}(y)|y-x|^{(2\wedge\alpha)}\tau_{p_c}(y-x)<\infty. \tag{8.20}$$

This bound completes the proof. In [20, Proposition 2.5(ii)] it is proved for $d > 3(2\wedge\alpha)$ that

$$\sup_x \sum_y |y|^{(2\wedge\alpha)+\delta} \tau_{p_c}(y)\tau_{p_c}(y-x) < \infty. \tag{8.21}$$

The proof of (8.20) is similar to this bound and is omitted here. □

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