CROSSED PRODUCTS OF BANACH ALGEBRAS. III.

MARCEL DE JEU AND MIEK MESSERSCHMIDT

Abstract. In earlier work a crossed product of a Banach algebra was constructed from a Banach algebra dynamical system (A, G, α) and a class $\mathcal{R}$ of continuous covariant representations, and its representations were determined. In this paper we adapt the theory to the ordered context. We construct a pre-ordered crossed product of a Banach algebra from a pre-ordered Banach algebra dynamical system (A, G, α) and a given uniformly bounded class $\mathcal{R}$ of continuous covariant representations of (A, G, α). If A has a positive bounded approximate left identity and $\mathcal{R}$ consists of non-degenerate continuous covariant representations, we establish a bijection between the positive non-degenerate bounded representations of the pre-ordered crossed product on pre-ordered Banach spaces with closed cones and the positive non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) on such spaces. Under mild conditions, we show that this pre-ordered crossed product is the essentially unique pre-ordered Banach algebra for which such a bijection exists. Finally, we study pre-ordered generalized Beurling algebras. We show that they are bi-positively topologically isomorphic to pre-ordered crossed products of Banach algebras associated with pre-ordered Banach algebra dynamical systems, and hence the general theory allows us to describe their positive representations on pre-ordered Banach spaces with closed cones.

1. Introduction

This paper is a continuation of [9] and [7], where, inspired by the theory of crossed products of C^*-algebras, the theory of crossed products of Banach algebras is developed. The lack of the convenient rigidity that C^*-algebras provide, where, e.g., morphisms are automatically continuous and even contractive, makes the task of developing the basics more laborious than it is for crossed products of C^*-algebras.

The paper [9] is for a large part concerned with one result: the General Correspondence Theorem [9, Theorem 8.1], most of which is formulated as Theorem 2.22 below. With (A, G, α) a Banach algebra dynamical system and $\mathcal{R}$ a uniformly bounded class of non-degenerate continuous covariant representations of (A, G, α) on Banach spaces – all notions will be reviewed in Section 2 – the General Correspondence Theorem, in the presence of a bounded approximate left identity of A, yields a bijection between the non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α), and the non-degenerate bounded representations of the crossed product Banach algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$ associated with (A, G, α) and $\mathcal{R}$.

2000 *Mathematics Subject Classification.* Primary 47L65; Secondary 22D12, 22D15, 43A20, 46B40, 46L55, 47B65.

Key words and phrases. Crossed product, (pre-)ordered Banach algebra dynamical system, positive representation in a (pre-)ordered Banach space, covariant representation, (pre-)ordered generalized Beurling algebra.

In [7] the theory established in [9] is developed further. Amongst others, there it is shown that (under mild conditions) the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ is the unique Banach algebra, up to topological isomorphism, which "generates" all non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) [7, Theorem 4.4]. Furthermore, given a weight ω on G and assuming α is uniformly bounded, for a particular choice of $\mathcal{R}$ it is shown that the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ is topologically isomorphic to a generalized Beurling algebra $L^1(G, A, \omega; \alpha)$ [7, Section 5]. These algebras, as introduced in [7], are weighted Banach spaces of (equivalence classes) of A-valued functions that are also associative algebras with a multiplication that is continuous in both variables, but they are not Banach algebras in general, since the norm need not be submultiplicative. The General Correspondence Theorem then provides a bijection between the non-degenerate continuous covariant representations of (A, G, α), of which the representation of G is bounded by a multiple of ω, and the non-degenerate bounded representations of $L^1(G, A, \omega; \alpha)$ [7, Theorem 5.20]. When A is taken to be the scalars, generalized Beurling algebras reduce to classical Beurling algebras, which are true Banach algebras, and then [7, Corollary 5.22] describes their non-degenerate bounded representations. In the case where $\omega = 1$ as well, this specializes to the classical bijection between uniformly bounded representations of G on Banach spaces and non-degenerate bounded representations of $L^1(G)$ (cf. [10, Assertion VI.1.32]).

In the current paper we adapt the theory developed in [9] and [7] to the ordered context: that of pre-ordered Banach spaces and algebras. Apart from its intrinsic interest, this is also motivated by the proven relevance of crossed products of C^*-algebras for unitary group representations. As is well known, a decomposition of a general unitary group representation into a direct integral of irreducible unitary representations is obtained via the group C^*-algebra (a particularly simple crossed product), and Mackey's Imprimitivity Theorem can, by Rieffel's work, now be conceptually interpreted in terms of (strong) Morita equivalence of a crossed product of a C^*-algebra and a group C^*-algebra. We hope that the results in the present paper will contribute to similar developments in the theory of positive representations of groups on pre-ordered Banach spaces (and Banach lattices in particular), which exist in abundance.

We are mainly concerned with four topics: Firstly, an adaptation of the construction of crossed products of Banach algebras from [9] to the ordered context (cf. Section 3). Secondly, proving a version of the General Correspondence Theorem in this context (cf. Theorem 3.13). Thirdly, for a pre-ordered Banach algebra dynamical system (A, G, α) and uniformly bounded class of positive continuous covariant representations $\mathcal{R}$, we establish (under mild conditions) the uniqueness, up to bipositive topological isomorphism, of the associated pre-ordered crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ as the unique pre-ordered Banach algebra which "generates" all positive non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) (cf. Theorem 4.7). And fourthly, we describe the positive non-degenerate bounded representations of a pre-ordered Beurling algebra $L^1(G, A, \omega; \alpha)$ in terms of positive non-degenerate continuous covariant representations of the pre-ordered Banach algebra dynamical system (A, G, α) to which $L^1(G, A, \omega; \alpha)$ is associated (cf. Section 5).

We now briefly describe the structure of the paper.

Section 2 contains all preliminary definitions and results concerning pre-ordered vector spaces and crossed products. Specifically, Sections 2.1–2.3 provide preliminary definitions and results concerning pre-ordered vector spaces and algebras and pre-ordered normed spaces and algebras. Some of the material is completely standard and/or elementary, but since the fields of representation theory and positivity seem to be somewhat disjoint, we have included it in an attempt to enhance the accessibility of this paper, which draws on both disciplines. Sections 2.4 and 2.5 provide a brief recapitulation of all relevant notions from [9] relating to Banach algebra dynamical systems and crossed products.

In Section 3 we define pre-ordered Banach algebra dynamical systems and provide the construction of pre-ordered crossed products associated with such systems. The construction is largely the same as in the general unordered case, but differs in keeping track of how order structures of (A, G, α) and $\mathcal{R}$ induce a natural cone, denoted $(A \rtimes_\alpha G)_+^{\mathcal{R}}$, which defines a pre-order on the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$. Theorem 3.8 collects properties of the cone $(A \rtimes_\alpha G)_+^{\mathcal{R}}$ (and thereby the order structure) of a pre-ordered crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ in terms of the order properties of the pre-ordered Banach algebra dynamical system (A, G, α) and uniformly bounded class $\mathcal{R}$ of continuous covariant representations. Finally, we adapt the General Correspondence Theorem to the ordered context. In the presence of a positive bounded approximate left identity of A, it gives a canonical bijection between the positive non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) on pre-ordered Banach spaces with closed cones, and positive non-degenerate bounded representations of the pre-ordered crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ on such spaces (cf. Theorem 3.13).

Paralleling work of Raeburn's [14], in Section 4 we show that (under mild additional hypotheses) the pre-ordered crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ associated with (A, G, α) and $\mathcal{R}$ is the unique pre-ordered Banach algebra, up to bipositive topological isomorphism, which "generates" all positive non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) (cf. Theorem 4.7).

Finally, in Section 5, we will study pre-ordered generalized Beurling algebras $L^1(G, A, \omega; \alpha)$. These algebras can be defined for any pre-ordered Banach algebra dynamical system (A, G, α) and weight ω on G, provided that α is uniformly bounded. If A has a bounded approximate right identity, for a specific choice of $\mathcal{R}$ the pre-ordered crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ is shown to be bipositively topologically isomorphic to a pre-ordered generalized Beurling algebra $L^1(G, A, \omega; \alpha)$ (cf. Theorem 5.7). In the presence of a positive bounded approximate left identity of A, our ordered version of the General Correspondence Theorem, Theorem 3.13, then provides a bijection between the positive non-degenerate bounded representations of $L^1(G, A, \omega; \alpha)$ and the positive non-degenerate continuous covariant representations of the pre-ordered Banach algebra dynamical system (A, G, α), where the representation of the group G is bounded by a multiple of ω (cf. Theorem 5.9). In the case where A is a Banach lattice algebra, it is shown that $L^1(G, A, \omega; \alpha)$ also becomes a Banach lattice (although it is not generally a Banach algebra), and, under further conditions, becomes a Banach lattice algebra (cf. Theorem 5.8). In the simplest case, where A is taken to be the real numbers and $\omega = 1$, our results reduce to a bijection between the positive strongly continuous uniformly bounded representations of G on pre-ordered Banach spaces with closed cones on the one hand, and the positive non-degenerate bounded representations of $L^1(G)$ on such spaces on the other hand; this also follows from [10, Assertion VI.1.32].

2. Preliminaries and recapitulation

In this section we will introduce the terminology and notation used in the rest of the paper and give a brief recapitulation of Banach algebra dynamical systems and their crossed products. Sections 2.1–2.3 will introduce general notions concerning pre-ordered (normed) vector spaces and algebras. Sections 2.4 and 2.5 will give a brief overview of results from [9] on Banach algebra dynamical systems and their crossed products.

Throughout this paper all vector spaces are assumed to be over the reals, and all locally compact topologies are assumed to be Hausdorff.

Let X and Y be normed spaces. The normed space of bounded linear operators from X to Y will be denoted by $B(X,Y)$, and by $B(X)$ if $X=Y$. The group of invertible elements in $B(X)$ will be denoted by $\mathrm{Inv}(X)$. If A is a normed algebra, by $\mathrm{Aut}(A)$ we will denote its group of bounded automorphisms. We do not assume algebras to be unital.

For a locally compact topological space Ω and topological vector space V, we will denote the space of all continuous compactly supported functions on Ω taking values in V by $C_c(\Omega, V)$. If $V=\mathbb{R}$, we write $C_c(\Omega)$ for $C_c(\Omega,\mathbb{R})$.

If G is a locally compact group, we will denote its identity element by $e\in G$. For $f\in C_c(G)$, we will write $\int_G f(s)\,ds$ for the integral of f with respect to a fixed left Haar measure μ on G.

2.1. **Pre-ordered vector spaces and algebras.**

We introduce the following terminology.

Let V be a vector space. A subset $C\subseteq V$ will be called a *cone* if $C+C\subseteq C$ and $\lambda C\subseteq C$ for all $\lambda\geq 0$. The pair (V,C) will be called a *pre-ordered vector space* and, for $x,y\in V$, by $y\leq x$ we mean $x-y\in C$. Elements of C will be called *positive.* We will often suppress mention of the cone C, and merely say that V is a pre-ordered vector space. In this case, we will denote the implicit cone by V_+ and refer to it as *the cone of* V. A cone $C\subseteq V$ will be said to be a *proper cone* if $C\cap(-C)=\{0\}$, in which case $\leq$ is a partial order, and then (V,C) will be called an *ordered vector space.* A cone $C\subseteq V$ will be said to be *generating* (*in* V) if $V=C-C$. If (V,C) is a pre-ordered vector space and V is also an associative algebra such that $C\cdot C\subseteq C$, we will say (V,C) is a *pre-ordered algebra.*

If (V_1,C_1) and (V_2,C_2) are pre-ordered vector spaces, we will say a linear map $T:V_1\to V_2$ is *positive* if $TC_1\subseteq C_2$. If T is injective and both $TC_1\subseteq C_2$ and $T^{-1}C_2\subseteq C_1$ hold, we will say T is *bipositive.*

With $W\subseteq V$ a subspace and $q:V\to V/W$ the quotient map, $q(C)\subseteq V/W$ will be called the *quotient cone.* Then $(V/W,q(C))$ is a pre-ordered vector space. Clearly $q:V\to V/W$ is positive and $q(C)$ is generating in V/W if C is generating in V.

2.2. **Pre-ordered normed spaces and algebras.**

We give a brief description of pre-ordered normed vector spaces and algebras. In Section 3 we will apply the results from this section to describe the order structure of crossed products associated with pre-ordered Banach algebra dynamical systems.

If A is a pre-ordered algebra that is also a normed algebra, then we will call A a *pre-ordered normed algebra,* and a *pre-ordered Banach algebra* if A is complete. The *positive automorphism group* (*of* A) is defined by $\mathrm{Aut}_+(A) := \{\alpha \in \mathrm{Aut}(A) : \alpha^{\pm 1}(A_+) \subseteq A_+\} \subseteq B(A)$.

Let X and Y be pre-ordered normed spaces. We will always assume that $B(X,Y)$ is endowed with the *natural operator cone* $B(X,Y)_+ := \{T \in B(X,Y) : TX_+ \subseteq Y_+\}$, so that $B(X,Y)$ is a pre-ordered normed space, and $B(X)$ is a pre-ordered normed algebra. We define the *group of bipositive invertible operators* on X by $\mathrm{Inv}_+(X) := \{T \in \mathrm{Inv}(X) : T^{\pm 1}X_+ \subseteq X_+\}$. We will say X_+ is *topologically generating* (*in* X) if $X = \overline{X_+ - X_+}$. If the ordering defined by X_+ is a lattice-ordering (i.e., if every pair of elements from X has a supremum, denoted by $\vee$) we will call X a *normed vector lattice* if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for all $x, y \in X$, where $|x| := x \vee (-x)$. A complete normed vector lattice will be called a *Banach lattice.* A pre-ordered Banach algebra that is also a Banach lattice will be called a *Banach lattice algebra.* A subspace $Y \subseteq X$ in a vector lattice X is called an *order ideal* if, for $g \in Y$ and $f \in X$, $|f| \leq |g|$ implies $f \in Y$.

We will need completions of pre-ordered normed spaces in Section 3, to be able to describe pre-ordered crossed products associated with pre-ordered Banach algebra dynamical systems.

Definition 2.1. Let V be a pre-ordered normed space. We define *the completion of* V by $(\overline{V}, \overline{V_+})$, where $\overline{V}$ denotes the usual metric completion of the normed space V, and $\overline{V_+}$ the closure of V_+ in $\overline{V}$.

The following two elementary observations are included for later reference.

Lemma 2.2. *Let V be a pre-ordered normed space, X a pre-ordered Banach space with a closed cone, and $T : V \to X$ a positive bounded linear operator. Then the bounded extension of T to the completion of V is a positive operator.*

Lemma 2.3. *If V is a pre-ordered normed algebra, then its completion is a pre-ordered Banach algebra with a closed cone.*

Together with Corollary 2.8, the following two elementary results will be used in Theorem 3.8 to give sufficient conditions for the cone of a crossed product of a pre-ordered Banach algebra dynamical system to be (topologically) generating.

Lemma 2.4. *Let V be a pre-ordered normed space. If V_+ is topologically generating in V, then V_+, and hence the cone $\overline{V_+}$, is topologically generating in the completion $\overline{V}$.*

Proof. Let $w \in \overline{V}$ be arbitrary, and let $(v_n) \subseteq V$ be such that $v_n \to w$. For every $n \in \mathbb{N}$, let $a_n, b_n \in V_+$ be such that $\|v_n - (a_n - b_n)\| < 2^{-n}$. Since

$$\|w - (a_n - b_n)\| \leq \|w - v_n\| + \|v_n - (a_n - b_n)\| < \|w - v_n\| + 2^{-n},$$

$(a_n - b_n) \subseteq V_+ - V_+$ converges to w. □

In certain cases the conclusion of the previous lemma for $\overline{V_+}$ may be strengthened.

Lemma 2.5. *Let V be a pre-ordered normed space and $(\cdot)^+ : V \to V_+$ a function such that $v \leq v^+$ for all $v \in V$. Then V_+ is generating in V, and if $(\cdot)^+ : V \to V_+$ maps Cauchy sequences to Cauchy sequences, then the cone $\overline{V_+}$ is generating in the completion $\overline{V}$.*

Proof. It is obvious that the fact that V_+ is generating in V is equivalent with the existence of a function $(\cdot)^+ : V \to V_+$ such that $v \leq v^+$ for all $v \in V$.

Assuming that $(\cdot)^+ : V \to V_+$ maps Cauchy sequences to Cauchy sequences, let $w \in \overline{V}$ be arbitrary and let $(v_n) \subseteq V$ be such that $v_n \to w$. The sequence $(v_n) \subseteq V$ is Cauchy, hence, by hypothesis, so is $(v_n^+) \subseteq V_+ \subseteq \overline{V_+}$. Since $\overline{V_+}$ is closed in $\overline{V}$, (v_n^+) converges to some $w' \in \overline{V_+}$. Since $v_n^+ - v_n \in V_+ \subseteq \overline{V_+}$, we have $w' - w = \lim_{n\to\infty}(v_n^+ - v_n) \in \overline{V_+}$. Writing $w = w' - (w' - w)$ yields the result. □

Remark 2.6. If V is a normed vector lattice, then the map $v \mapsto v \vee 0$ is uniformly continuous and hence maps Cauchy sequences to Cauchy sequences [16, Proposition II.5.2]. Hence $\overline{V_+}$ is generating in the completion $\overline{V}$. Since, in this case, $\overline{V}$ is actually a Banach lattice [16, Corollary 2, p. 84], this is not unexpected.

The following refinement of Andô's Theorem [2, Lemma 1] is a special case of [6, Theorem 4.1], of which the essence is that the decomposition of elements as the difference of positive elements can be chosen in a bounded, continuous and positively homogeneous manner. Its proof proceeds through applications of a generalization of the usual Open Mapping Theorem [6, Theorem 3.2] and the Michael Selection Theorem [1, Theorem 17.66]. It will be applied in Theorem 3.8 to prove that the cones of certain crossed products associated with pre-ordered Banach algebras are topologically generating.

Theorem 2.7. *Let X be a pre-ordered Banach space with closed generating cone. Then there exist a constant $\alpha > 0$ and continuous positively homogeneous maps $(\cdot)^\pm : X \to X_+$ such that $x = x^+ - x^-$ and $\|x^+\| + \|x^-\| \leq \alpha\|x\|$ for all $x \in X$.*

Simply through composition with the functions $(\cdot)^\pm : X \to X_+$, cones of continuous X_+-valued functions are then immediately seen to be generating in spaces of continuous X-valued functions. For example:

Corollary 2.8. *Let Ω be a locally compact Hausdorff space and X be a pre-ordered Banach space with closed generating cone. Then the cone $C_c(\Omega, X_+)$ is generating in $C_c(\Omega, X)$. In fact, there exists a constant $\alpha > 0$ with the property that, for every $f \in C_c(G, X)$, there exist $f^\pm \in C_c(G, X)$ such that $\|f^+(\omega)\| + \|f^-(\omega)\| \leq \alpha\|f(\omega)\|$ for all $\omega \in \Omega$. In particular, $\|f^\pm\|_\infty \leq \alpha\|f\|_\infty$ and $\operatorname{supp}(f^\pm) \subseteq \operatorname{supp}(f)$.*

Remark 2.9. The earliest results of this type known to the authors are [3, Theorem 2.3] and [18, Theorem 4.4]. Both results proceed through an application of Lazar's affine selection theorem to show that canonical cones of certain spaces of continuous affine functions are generating. The result [3, Theorem 2.3] shows, with K a Choquet simplex and X a pre-ordered Banach space with a closed cone, that the space $A(K, X)$ of continuous affine functions from K to X has $A(K, X_+)$ as a generating cone. By taking K to be the regular Borel probability measures on a compact Hausdorff space Ω, this result includes the case that $C(\Omega, X_+)$ is generating in $C(\Omega, X)$, which is part of the statement of Corollary 2.8.

We will now define normality and conormality properties for a pre-ordered Banach space X with a closed cone, and subsequently show in Theorem 2.12 how these properties imply normality properties of the pre-ordered normed space $B(X, Y)$. In Theorem 3.8 this will be used to conclude (conditional) normality properties of a pre-ordered crossed product.

Definition 2.10. Let X be a pre-ordered Banach space with closed cone and $\alpha > 0$. We define the following ***normality properties****:*

(1) We will say that X is *α-normal* if, for any $x, y \in X$, $0 \leq x \leq y$ implies $\|x\| \leq \alpha \|y\|$.
(2) We will that X is *α-absolutely normal* if, for any $x, y \in X$, $\pm x \leq y$ implies $\|x\| \leq \alpha \|y\|$.

We define the following ***conormality properties****:*

(1) We will say that X is *approximately α-absolutely conormal* if, for any $x \in X$ and $\varepsilon > 0$, there exists some $a \in X_+$ such that $\pm x \leq a$ and $\|a\| < \alpha \|x\| + \varepsilon$.
(2) We will say that X is *approximately α-sum-conormal* if, for any $x \in X$ and $\varepsilon > 0$, there exist some $a, b \in X_+$ such that $x = a - b$ and $\|a\| + \|b\| < \alpha \|x\| + \varepsilon$.

Remark 2.11. Normality (terminology due to Krein [12]) and (approximate) conormality (terminology due to Walsh [17]) are dual properties for pre-ordered Banach spaces with closed cones. Roughly speaking, a pre-ordered Banach space with a closed cone has some normality property precisely if its dual has a corresponding conormality property, and vice versa. The most complete reference for such duality relationships seems to be [4].

For a pre-ordered Banach space X with a closed cone, elementary arguments will show that α-absolutely normality of X implies that X is α-normal, which in turn implies that X_+ is a proper cone. Also, approximate α-sum-conormality of X implies that X is approximately α-absolute conormal, which in turn implies that X_+ is generating in X. An application of Andô's Theorem [2, Lemma 1] shows conversely that, if X_+ is generating in X, then there exists some $\beta > 0$ such that, for every $x \in X$, there exists $a, b \in X_+$ such that $x = a - b$ and $\max\{\|a\|, \|b\|\} \leq \beta \|x\|$ (another form of conormality, clearly implying approximate 2β-sum-conormality). We note that Banach lattices are always 1-absolutely normal and approximately 1-absolutely conormal.

The following results relate normality properties of spaces of operators to the normality and conormality properties of the underlying spaces. Part (2) is due to Wickstead [18, Theorem 3.1]. Part (3) is a slight refinement of a result due to Yamamuro [19, Theorem 1.3], where it is proven for the case $X = Y$ and $\alpha = \beta = 1$. No reference for part (4) is known to the authors. We include proofs for convenience of the reader.

Theorem 2.12. *Let X and Y be pre-ordered Banach spaces with closed cones and $\alpha, \beta > 0$.*

(1) *If X_+ is generating and Y_+ is a proper cone, then $B(X,Y)_+$ is a proper cone.*
(2) *If X_+ is generating and Y is α-normal, then there exists some $\gamma > 0$ for which $B(X,Y)$ is γ-normal.*
(3) *If X is approximately α-absolutely conormal and Y is β-absolutely normal, then $B(X,Y)$ is $\alpha\beta$-absolutely normal.*
(4) *If X is approximately α-sum-conormal and Y is β-normal, then $B(X,Y)$ is $\alpha\beta$-normal.*

Proof. We prove (1). Let $T \in B(X,Y)_+ \cap (-B(X,Y)_+)$. If $x \in X_+$, then $Tx \geq 0$ and $(-T)x \geq 0$. Hence $Tx = 0$, since Y_+ is proper. Since X_+ is generating, we have $T = 0$ as required.

We prove (2). By Andô's Theorem [2, Lemma 1], the fact that X_+ is generating in X implies that there exists some $\beta > 0$ such that, for every $x \in X$, there exist $a, b \in X_+$ such that $x = a - b$ and $\max\{\|a\|, \|b\|\} \leq \beta\|x\|$. Let $T, S \in B(X,Y)$ be such that $0 \leq T \leq S$. Then, for any $x \in X$, let $a, b \in X_+$ be such that $x = a - b$ and $\max\{\|a\|, \|b\|\} \leq \beta\|x\|$, so that $0 \leq Ta \leq Sa$ and $0 \leq Tb \leq Sb$. By α-normality of Y,

$$\|Tx\| \leq \|Ta\| + \|Tb\| \leq \alpha(\|Sa\| + \|Sb\|) \leq \alpha\|S\|(\|a\| + \|b\|) \leq 2\alpha\beta\|S\|\|x\|,$$

hence $\|T\| \leq 2\alpha\beta\|S\|$.

We prove (3). Let $T, S \in B(X,Y)$ satisfy $\pm T \leq S$. Let $x \in X$ be arbitrary. Then, for every $\varepsilon > 0$, there exists some $a \in X_+$ satisfying $\pm x \leq a$ and $\|a\| < \alpha\|x\| + \varepsilon$. Then

$$Tx = T\left(\frac{a+x}{2}\right) - T\left(\frac{a-x}{2}\right),$$

and hence

$$\pm Tx = \pm T\left(\frac{a+x}{2}\right) \mp T\left(\frac{a-x}{2}\right).$$

Since $(a+x)/2 \geq 0$, $(a-x)/2 \geq 0$ and $\pm T \leq S$, we find

$$\pm Tx \leq S\left(\frac{a+x}{2}\right) + S\left(\frac{a-x}{2}\right) = Sa.$$

Now, because Y is β-absolutely normal, we obtain

$$\|Tx\| \leq \beta\|Sa\| \leq \beta\|S\|\|a\| \leq \alpha\beta\|S\|\|x\| + \varepsilon\beta\|S\|.$$

Since $\varepsilon > 0$ was chosen arbitrarily, we conclude that $B(X,Y)$ is $\alpha\beta$-absolutely normal.

We prove (4). Let $T, S \in B(X,Y)$ satisfy $0 \leq T \leq S$. Let $x \in X$ be arbitrary. Then, for every $\varepsilon > 0$, there exist $a, b \in X_+$ such that $x = a - b$ and $\|a\| + \|b\| \leq \alpha\|x\| + \varepsilon$. Because $0 \leq T \leq S$, we have $0 \leq Ta \leq Sa$ and $0 \leq Tb \leq Sb$. Since Y is β-normal, we obtain

$$\|Tx\| \leq \|Ta\| + \|Tb\| \leq \beta\|Sa\| + \beta\|Sb\| \leq \beta\|S\|\left(\|a\| + \|b\|\right) \leq \alpha\beta\|S\|\|x\| + \varepsilon\beta\|S\|.$$

Since $\varepsilon > 0$ was chosen arbitrarily, we conclude that $B(X,Y)$ is $\alpha\beta$-normal. □

Remark 2.13. We note some specific cases of the above theorem. Any Banach lattice X is both approximately 1-absolutely conormal and 1-absolutely normal, therefore (3) in the previous result implies that $B(X)$ is always 1-absolutely normal in this case. Also, if X is a Banach lattice and Y and Z are pre-ordered Banach spaces with closed cones that are respectively α-absolutely normal and approximately α-absolutely conormal for some $\alpha > 0$, then $B(X,Y)$ and $B(Z,X)$ are α-absolutely normal, again by (3) above. If a Banach lattice X is an AL-space (i.e., $\|x+y\| = \|x\| + \|y\|$ for all $x, y \in X_+$), then it is approximately 1-sum-conormal, and hence, for any α-normal pre-ordered Banach space Y with a closed cone, $B(X,Y)$ is α-normal by (4) in the previous result.

Let $\mathcal{H}$ be real Hilbert space endowed with a Lorentz cone $\mathcal{L}_v := \{x \in \mathcal{H} : \langle v|x\rangle \geq \|Px\|\}$, where $v \in \mathcal{H}$ is any norm 1 element, and P the projection onto $\{v\}^\perp$. Although not a Banach lattice if $\dim \mathcal{H} \geq 3$, the ordered Banach space $(\mathcal{H}, \mathcal{L}_v)$ is

1-absolutely normal and approximately 1-absolutely conormal [13]. Hence, again by (3) above, $B(\mathcal{H})$ is 1-absolutely normal, and if Y and Z are pre-ordered Banach spaces with closed cones that are respectively α-absolutely normal and approximately α-absolutely conormal for some $\alpha > 0$, then $B(\mathcal{H}, Y)$ and $B(Z, \mathcal{H})$ are α-absolutely normal.

2.3. **Representations on pre-ordered normed spaces.** We will now introduce positive representations of groups and pre-ordered normed algebras on pre-ordered normed spaces. In Section 3 we will use the notions in this section to describe a bijection between the positive non-degenerate bounded representations of a crossed product associated with a pre-ordered Banach algebra dynamical system on the one hand, and positive non-degenerate covariant representations of the certain dynamical system on the other hand (cf. Theorem 3.13).

Definition 2.14. Let A be a normed algebra and X a normed space. An algebra homomorphism $\pi : A \to B(X)$ will be called a *representation of A on X*. We will write X_π for X, if the connection between X_π and π requires emphasis. We will say that π is *non-degenerate* if $\operatorname{span}\{\pi(a)x : a \in A, x \in X\}$ is dense in X.

If A is a pre-ordered normed algebra and X is a pre-ordered normed space, we will say that a representation π of A on X is *positive* if $\pi(A_+) \subseteq B(X)_+$.

Definition 2.15. Let G be a locally compact group and X a normed space. A group homomorphism $U : G \to \operatorname{Inv}(X)$ will be called a *representation* (*of G on X*).

If X is a pre-ordered normed space, a group homomorphism $U : G \to \operatorname{Inv}_+(X) \subseteq B(X)$ (cf. Section 2.2) will be called a *positive representation* of G on X.

For typographical reasons we will write U_s instead of $U(s)$ for $s \in G$.

Note that continuity is not included in the definition of representations of normed algebras and locally compact groups, and that representations of a unital algebra are not required to be unital.

The left centralizer algebra of a normed algebra, to be introduced next, plays a crucial role in the construction of the bijection mentioned in the first paragraph of this section.

Definition 2.16. Let A be a normed algebra. A bounded linear operator $L : A \to A$ will be called a *left centralizer of* A if $L(ab) = (La)b$ for all $a, b \in A$. The unital normed algebra of all left centralizers, with the operator norm inherited from $B(A)$, will be denoted $\mathcal{M}_l(A)$ and called the *left centralizer algebra of* A. The *left regular representation of* A, $\lambda : A \to \mathcal{M}_l(A)$, is defined by $\lambda(a)b := ab$ for $a, b \in A$.

If A is a pre-ordered normed algebra, we will always assume that $\mathcal{M}_l(A)$ is endowed with the cone $\mathcal{M}_l(A) \cap B(A)_+$. Then $\lambda : A \to B(A)$ is a positive contractive representation of A on itself.

Definition 2.17. If A is a pre-ordered normed algebra, we will say an approximate left (right) identity (u_i) of A is *positive* if $(u_i) \subseteq A_+$.

The result [8, Theorem 4.1] plays a key role in the proof of the General Correspondence Theorem (Theorem 2.22). We collect the relevant parts in Theorem 2.18, including how it can be applied to representations of pre-ordered normed algebras on pre-ordered Banach spaces with closed cones. This will be used to adapt the General Correspondence Theorem to the ordered context (cf. Theorem 3.13).

Theorem 2.18. *Let B be a normed algebra with an M-bounded approximate left identity (u_i) and X a Banach space. If $T : B \to B(X)$ is a non-degenerate bounded representation, then the map $\overline{T} : \mathcal{M}_l(B) \to B(X)$ defined by $\overline{T}(L) :=$ SOT-$\lim_i T(Lu_i)$ is the unique representation of $\mathcal{M}_l(B)$ on X such that the diagram*

$$\begin{array}{ccc} B & \xrightarrow{T} & B(X) \\ & \searrow^{\lambda} & \uparrow{\scriptstyle \overline{T}} \\ & & \mathcal{M}_l(B) \end{array}$$

commutes. Moreover, $\overline{T}$ is non-degenerate and bounded, with $\|\overline{T}\| \leq M\|T\|$, and satisfies $\overline{T}(L)T(b) = T(Lb)$ for all $b \in B$ and $L \in \mathcal{M}_l(B)$.

If, in addition, B is a pre-ordered normed algebra, (u_i) is positive, and X is a pre-ordered Banach space with a closed cone, then $\overline{T} : \mathcal{M}_l(B) \to B(X)$ is a positive non-degenerate bounded representation of $\mathcal{M}_l(B)$ on X.

2.4. **Banach algebra dynamical systems and crossed products.** We recall some basic definitions and results from [9].

Definition 2.19. Let A be a Banach algebra, G a locally compact group, and $\alpha : G \to \mathrm{Aut}(A)$ a strongly continuous representation of G on A. Then we will call the triple (A, G, α) a *Banach algebra dynamical system.*

If (A, G, α) is a Banach algebra dynamical system, $C_c(G, A)$ can be made into an associative algebra by defining the twisted convolution

$$(f * g)(s) := \int_G f(r)\alpha_r(g(r^{-1}s))\, dr \quad (f, g \in C_c(G, A),\ s \in G).$$

Here, as in [9], integrals of compactly supported continuous Banach space valued functions are defined by duality, following [15, Section 3].

Let (A, G, α) be a Banach algebra dynamical system and X a normed space. If $\pi : A \to B(X)$ and $U : G \to \mathrm{Inv}(X)$ are representations satisfying

$$\pi(\alpha_s(a)) = U_s \pi(a) U_s^{-1} \quad (a \in A,\ s \in G),$$

we will say that (π, U) is a *covariant representation* of (A, G, α) on X. We will say (π, U) is *continuous* if π is bounded and U is strongly continuous. We will say that (π, U) is *non-degenerate* if π is non-degenerate.

If (π, U) is a continuous covariant representation of (A, G, α) on a Banach space X, then, as in [9, Section 3], for every $f \in C_c(G, A)$, $\pi \rtimes U(f) \in B(X)$ is defined by

$$\pi \rtimes U(f)x := \int_G \pi(f(r))U_r x\, dr \quad (x \in X).$$

The map $\pi \rtimes U : C_c(G, A) \to B(X)$ is then a representation of the algebra $C_c(G, A)$ on X, and is called the *integrated form* of (π, U).

Let (A, G, α) be a Banach algebra dynamical system and $\mathcal{R}$ a class of continuous covariant representations of (A, G, α) on Banach spaces. We will always tacitly assume that $\mathcal{R}$ is non-empty. We will say $\mathcal{R}$ is a *uniformly bounded class* of continuous covariant representations if there exists a constant $C \geq 0$ and a function $\nu : G \to \mathbb{R}_{\geq 0}$, which is bounded on compact subsets of G, such that, for all $(\pi, U) \in \mathcal{R}$, $\|\pi\| \leq C$ and $\|U_r\| \leq \nu(r)$ for all $r \in G$.

Let (A, G, α) be a Banach algebra dynamical system and $\mathcal{R}$ a uniformly bounded class of continuous covariant representations of (A, G, α) on Banach spaces. It follows that $\|\pi \rtimes U(f)\| \leq C \left(\sup_{r \in \mathrm{supp}(f)} \nu(r)\right) \|f\|_1$ for all $(\pi, U) \in \mathcal{R}$ and $f \in C_c(G, A)$ [9, Remark 3.3]. We introduce the (hence finite) algebra seminorm $\sigma^{\mathcal{R}}$ on $C_c(G, A)$, defined by

$$\sigma^{\mathcal{R}}(f) := \sup_{(\pi, U) \in \mathcal{R}} \|\pi \rtimes U(f)\| \quad (f \in C_c(G, A)).$$

The kernel of $\sigma^{\mathcal{R}}$ is a two-sided ideal of $C_c(G, A)$, hence $C_c(G, A)/\ker \sigma^{\mathcal{R}}$ is a normed algebra with norm $\|\cdot\|^{\mathcal{R}}$ induced by $\sigma^{\mathcal{R}}$. Its completion is called the *crossed product* (*associated with* (A, G, α) *and* $\mathcal{R}$), and denoted by $(A \rtimes_\alpha G)^{\mathcal{R}}$. Multiplication in $(A \rtimes_\alpha G)^{\mathcal{R}}$ will be denoted by $*$. We denote the quotient map from $C_c(G, A)$ to $(A \rtimes_\alpha G)^{\mathcal{R}}$ by $q^{\mathcal{R}} : C_c(G, A) \to (A \rtimes_\alpha G)^{\mathcal{R}}$. For any Banach space X and linear map $T : C_c(G, A) \to X$, if T is bounded with respect to the $\sigma^{\mathcal{R}}$ seminorm, we will denote the canonically induced bounded linear map on $(A \rtimes_\alpha G)^{\mathcal{R}}$ by $T^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to X$, as detailed in [9, Section 3].

2.5. **Correspondence between representations of (A, G, α) and $(A \rtimes_\alpha G)^{\mathcal{R}}$.** We briefly describe the General Correspondence Theorem [9, Theorem 8.1], most of which is formulated as Theorem 2.22 below. In the presence of a bounded approximate left identity of A, the General Correspondence Theorem describes a bijection between the non-degenerate $\mathcal{R}$-continuous (to be defined below) covariant representations of a Banach algebra dynamical system (A, G, α) and the non-degenerate bounded representations of the associated crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$. In Section 3 we will adapt the results from this section to pre-ordered Banach algebra dynamical systems and the associated pre-ordered crossed products.

Definition 2.20. Let (A, G, α) be a Banach algebra dynamical system and $\mathcal{R}$ a uniformly bounded class of continuous covariant representations of (A, G, α) on Banach spaces. If (π, U) is a continuous covariant representation of (A, G, α) on a Banach space X, and $\pi \rtimes U : C_c(G, A) \to B(X)$ is bounded with respect to $\sigma^{\mathcal{R}}$, we will say (π, U) is $\mathcal{R}$-*continuous*.

The proof of the General Correspondence Theorem proceeds through an application of Theorem 2.18, which requires the existence of a bounded approximate left identity of $(A \rtimes_\alpha G)^{\mathcal{R}}$. The following theorem makes precise how this (and its right-sided counterpart) is inherited from A.

Theorem 2.21. [9, Theorem 4.4] *Let (A, G, α) be a Banach algebra dynamical system, and let $\mathcal{R}$ be a uniformly bounded class of continuous covariant representations of (A, G, α) on Banach spaces. Let A have a bounded approximate left (right) identity (u_i). Let $\mathcal{Z}$ be a neighbourhood base of $e \in G$ of which all elements are contained in a fixed compact subset of G. For each $V \in \mathcal{Z}$, let $z_V \in C_c(G)$ be positive, supported in V, and have integral equal to one. Then the net*

$$\left(q^{\mathcal{R}}(z_V \otimes u_i)\right),$$

where $(V, i) \leq (W, j)$ is defined to mean $i \leq j$ and $W \subseteq V$, is a bounded approximate left (right) identity of $(A \rtimes_\alpha G)^{\mathcal{R}}$.

Let (A, G, α) be a Banach algebra dynamical system. We define the maps $i_A : A \to \mathrm{End}(C_c(G, A))$ and $i_G : G \to \mathrm{End}(C_c(G, A))$ by

$$\begin{aligned} (i_A(a)f)(s) &:= af(s), \\ (i_G(r)f)(s) &:= \alpha_r(f(r^{-1}s)), \end{aligned}$$

for $f \in C_c(G, A)$, $a \in A$ and $r, s \in G$. The maps $i_A(a)$ and $i_G(r)$ are bounded on $C_c(G, A)$ with respect to $\sigma^{\mathcal{R}}$ [9, Lemma 6.3], hence we can define the maps $i_A^{\mathcal{R}} : A \to B((A \rtimes_\alpha G)^{\mathcal{R}})$ and $i_G^{\mathcal{R}} : G \to B((A \rtimes_\alpha G)^{\mathcal{R}})$, by $i_A^{\mathcal{R}}(a) := i_A(a)^{\mathcal{R}}$ and $i_G^{\mathcal{R}}(r) := i_G(r)^{\mathcal{R}}$ in the notation of Section 2.4, for $a \in A$ and $r \in G$. Moreover, the maps $a \mapsto i_A^{\mathcal{R}}(a)$ and $r \mapsto i_G^{\mathcal{R}}(r)$ map A and G into $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ [9, Proposition 6.4]. If A has a bounded approximate left identity and $\mathcal{R}$ is a uniformly bounded class of non-degenerate continuous covariant representations of (A, G, α) on Banach spaces, then $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ is a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on $(A \rtimes_\alpha G)^{\mathcal{R}}$, and the integrated form $(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})^{\mathcal{R}}$ equals the left regular representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ [9, Theorem 7.2].

This pair $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ can be used to "generate" non-degenerate continuous covariant representations of (A, G, α) from non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$. We will investigate this further in Section 4, but its key role becomes already apparent in the following result, giving an explicit bijection between the non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) and the non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$.

Theorem 2.22. (*General Correspondence Theorem, cf.* [9, Theorem 8.1]) *Let the triple (A, G, α) be a Banach algebra dynamical system, where A has a bounded approximate left identity. Let $\mathcal{R}$ be a uniformly bounded class of non-degenerate continuous covariant representations of (A, G, α) on Banach spaces. Then the map $(\pi, U) \mapsto (\pi \rtimes U)^{\mathcal{R}}$ is a bijection between the non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) on Banach spaces and the non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on such spaces.*

More precisely:

(1) *If the pair (π, U) is a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on a Banach space X_π, then $(\pi \rtimes U)^{\mathcal{R}}$ is a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on X_π, and*

$$(\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_A^{\mathcal{R}}, \overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_G^{\mathcal{R}}) = (\pi, U),$$

where $\overline{(\pi \rtimes U)^{\mathcal{R}}}$ is the non-degenerate bounded representation of $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ as in Theorem 2.18.

(2) *If T is a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on a Banach space X_T, then $(\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ is a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on X_T, and*

$$(\overline{T} \circ i_A^{\mathcal{R}} \rtimes \overline{T} \circ i_G^{\mathcal{R}})^{\mathcal{R}} = T$$

where $\overline{T}$ is the non-degenerate bounded representation of $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ as in Theorem 2.18.

3. Pre-ordered Banach algebra dynamical systems and crossed products

In this section we study pre-ordered Banach algebra dynamical systems and their associated crossed products. After the preliminary Section 3.1, in Section

3.2 we will define pre-ordered crossed products associated with pre-ordered Banach algebra dynamical systems, and describe properties of the cone of such pre-ordered crossed products (cf. Theorem 3.8). Finally, Theorem 3.13 in Section 3.3 will give an adaptation of the General Correspondence Theorem (Theorem 2.22) to the ordered context.

3.1. **Pre-ordered Banach algebra dynamical systems.** We introduce pre-ordered Banach algebra dynamical systems (A, G, α), and verify that the twisted convolution as defined in Section 2.4 gives $C_c(G, A)$ a pre-ordered algebra structure. Furthermore, Lemma 3.4 shows that positive continuous covariant representations of a pre-ordered Banach algebra dynamical systems (A, G, α) have positive integrated forms.

Definition 3.1. Let A be a pre-ordered Banach algebra with closed cone, G a locally compact group, and $\alpha : G \to \mathrm{Aut}_+(A)$ a strongly continuous positive representation of G on A. Then we will call the triple (A, G, α) a *pre-ordered Banach algebra dynamical system.*

Lemma 3.2. *If (A, G, α) is a pre-ordered Banach algebra dynamical system, with A having a closed cone, then $(C_c(G, A), C_c(G, A_+))$, with twisted convolution as defined in Section 2.4, is a pre-ordered algebra.*

Proof. Let $f, g \in C_c(G, A_+)$ with $f \neq 0$. By [15, Theorem 3.27], for every $s \in G$,

$$\frac{(f * g)(s)}{\mu(\mathrm{supp}(f))} = \int_{\mathrm{supp}(f)} f(r)\alpha_r(g(r^{-1}s)) \,\frac{dr}{\mu(\mathrm{supp}(f))},$$

where μ denotes the chosen left Haar measure on G, lies in the closed convex hull of $\{f(r)\alpha_r(g(r^{-1}s)) : r \in \mathrm{supp}(f)\} \subseteq A_+$. Since A_+ is itself closed and convex, the result follows. □

Definition 3.3. Let (A, G, α) be a pre-ordered Banach algebra dynamical system. Let (π, U) be a covariant representation of (A, G, α) on a pre-ordered normed space X. If both π and U are positive representations, we will say that the covariant representation (π, U) is *positive.*

Lemma 3.4. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system with A having a closed cone, and (π, U) a positive continuous covariant representation of (A, G, α) on a pre-ordered Banach space X with a closed cone. Then the integrated form $\pi \rtimes U : C_c(G, A) \to B(X)$ is a positive algebra representation.*

Proof. Let $f \in C_c(G, A_+)$. Since (π, U) is positive, we have $\pi(f(r))U_r x \in X_+$ for all $x \in X_+$ and $r \in G$. Since X_+ is closed and convex, we obtain $\int_G \pi(f(r))U_r x \,dr \in X_+$ as in the proof of Lemma 3.2. □

3.2. **Crossed products associated with pre-ordered Banach algebra dynamical systems.** In this section we will describe the construction of pre-ordered crossed products associated with pre-ordered Banach algebra dynamical systems. The construction as a Banach algebra is as described in Section 2.4, so we will focus mainly on the properties of the order structure.

Lemma 3.5. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone, and $\mathcal{R}$ a uniformly bounded class of continuous covariant representations of (A, G, α) on Banach spaces. Then the space $C_c(G, A)/\ker\sigma^{\mathcal{R}}$,* *with norm* $\|\cdot\|^{\mathcal{R}}$ *induced by* $\sigma^{\mathcal{R}}$ *and pre-ordered by the quotient cone* $q^{\mathcal{R}}(C_c(G, A_+))$, *is a pre-ordered normed algebra.*

Proof. As explained in Section 2.4, $C_c(G, A)/\ker \sigma^{\mathcal{R}}$ is a normed algebra with norm induced by $\sigma^{\mathcal{R}}$. That it is a pre-ordered algebra follows from the definition of the quotient cone and the fact that the twisted convolution of positive elements of $C_c(G, A)$ is again positive by Lemma 3.2. □

We can now describe $(A \rtimes_\alpha G)^{\mathcal{R}}$ as a pre-ordered Banach algebra:

Definition 3.6. Let (A, G, α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone, and $\mathcal{R}$ a uniformly bounded class of continuous covariant representations of (A, G, α) on Banach spaces. The completion of the pre-ordered normed algebra $(C_c(G, A)/\ker \sigma^{\mathcal{R}}, q^{\mathcal{R}}(C_c(G, A_+)))$ (in the sense of Definition 2.1), with norm $\|\cdot\|^{\mathcal{R}}$ induced by $\sigma^{\mathcal{R}}$, will be denoted by $(A \rtimes_\alpha G)^{\mathcal{R}}$, the pre-ordering being tacitly understood, and will be called the *pre-ordered crossed product* (*associated with* (A, G, α) *and* $\mathcal{R}$).

We recall the following result from [9], which will be used twice in the proof of Theorem 3.8:

Proposition 3.7. [9, Proposition 3.4] *Let* (A, G, α) *be a Banach algebra dynamical system, and let* $\mathcal{R}$ *be a uniformly bounded class of continuous covariant representations of* (A, G, α) *on Banach spaces. Then, for every* $d \in (A \rtimes_\alpha G)^{\mathcal{R}}$,

$$\|d\|^{\mathcal{R}} = \sup_{(\pi,U)\in\mathcal{R}} \|(\pi \rtimes U)^{\mathcal{R}}(d)\|.$$

The following theorem describes properties of the closed cone $(A \rtimes_\alpha G)^{\mathcal{R}}_+$ in a pre-ordered crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$.

Theorem 3.8. *Let* (A, G, α) *be a pre-ordered Banach algebra dynamical system with* A *having a closed cone. Let* $\mathcal{R}$ *be a uniformly bounded class of continuous covariant representations of* (A, G, α) *on Banach spaces. Then:*

(1) *The pre-ordered crossed product* $(A \rtimes_\alpha G)^{\mathcal{R}}$ *is a pre-ordered Banach algebra with a closed cone.*
(2) *If* A_+ *is generating in* A, *then* $(A \rtimes_\alpha G)^{\mathcal{R}}_+$ *is topologically generating in* $(A \rtimes_\alpha G)^{\mathcal{R}}$.
(3) *If* A_+ *is generating in* A *and* $(\cdot)^+ : q^{\mathcal{R}}(C_c(G, A)) \to q^{\mathcal{R}}(C_c(G, A_+))$ *is a function such that* $q^{\mathcal{R}}(f) \leq q^{\mathcal{R}}(f)^+$ *for all* $f \in C_c(G, A)$, *and which maps* $\|\cdot\|^{\mathcal{R}}$*-Cauchy sequences to* $\|\cdot\|^{\mathcal{R}}$*-Cauchy sequences, then the cone* $(A \rtimes_\alpha G)^{\mathcal{R}}_+$ *is generating in* $(A \rtimes_\alpha G)^{\mathcal{R}}$.
(4) *If* $\mathcal{R}$ *is a uniformly bounded class of positive continuous covariant representations of* (A, G, α) *on pre-ordered Banach spaces with closed cones such that, for every* $(\pi, U) \in \mathcal{R}$, *the cone* $B(X_\pi)_+$ *is proper, then the cone* $(A \rtimes_\alpha G)^{\mathcal{R}}_+$ *is a proper cone.*
(5) *If* $\beta > 0$ *and* $\mathcal{R}$ *is a uniformly bounded class of positive continuous covariant representations of* (A, G, α) *on pre-ordered Banach spaces with closed cones such that, for every* $(\pi, U) \in \mathcal{R}$, $B(X_\pi)$ *is* β-(*absolutely*) *normal, then* $(A \rtimes_\alpha G)^{\mathcal{R}}$ *is* β-(*absolutely*) *normal.*

Proof. As to (1): That $(A \rtimes_\alpha G)^{\mathcal{R}}$ is a pre-ordered Banach algebra with closed cone is immediate from Lemmas 3.5 and 2.3.

We prove (2). Let A_+ be generating in A. By Corollary 2.8, the cone $C_c(G, A_+)$ is generating in $C_c(G, A)$. Hence the quotient cone is generating in $C_c(G, A)/\ker \sigma^{\mathcal{R}}$, and by Lemma 2.4, $(A \rtimes_\alpha G)_+^{\mathcal{R}}$ is topologically generating in $(A \rtimes_\alpha G)^{\mathcal{R}}$.

The statement in (3) follows from Lemma 2.5.

We prove (4). Let $d \in (A \rtimes_\alpha G)^{\mathcal{R}}$ be such that $0 \leq d \leq 0$ in $(A \rtimes_\alpha G)^{\mathcal{R}}$. Let $(\pi, U) \in \mathcal{R}$ be arbitrary. By Lemma 3.4, $\pi \rtimes U : C_c(G, A) \to B(X_\pi)$ is a positive algebra representation. Therefore the induced map $(\pi \rtimes U)^{\mathcal{R}} : C_c(G, A)/\ker \sigma^{\mathcal{R}} \to B(X_\pi)$ is a positive bounded algebra representation. Since the cone of X_π is closed, so is the cone of $B(X_\pi)$, and therefore, by Lemma 2.2, $(\pi \rtimes U)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(X_\pi)$ is a positive algebra representation. Hence $0 \leq d \leq 0$ implies

$$0 \leq (\pi \rtimes U)^{\mathcal{R}}(d) \leq 0,$$

and since $B(X_\pi)_+$ is a proper cone, we obtain $(\pi \rtimes U)^{\mathcal{R}}(d) = 0$. Therefore, by Proposition 3.7, $\|d\|^{\mathcal{R}} = \sup_{(\pi,U)\in\mathcal{R}} \|(\pi \rtimes U)^{\mathcal{R}}(d)\| = 0$, and hence $d = 0$. We conclude that $(A \rtimes_\alpha G)_+^{\mathcal{R}}$ is a proper cone.

We prove (5). Let $\beta > 0$ be such that, for every $(\pi, U) \in \mathcal{R}$, $B(X_\pi)$ is β-absolutely normal. For any $(\pi, U) \in \mathcal{R}$, as previously, $(\pi \rtimes U)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(X_\pi)$ is a positive algebra representation. Hence, if $\pm d_1 \leq d_2$ for $d_1, d_2 \in (A \rtimes_\alpha G)^{\mathcal{R}}$, we have

$$\pm(\pi \rtimes U)^{\mathcal{R}}(d_1) \leq (\pi \rtimes U)^{\mathcal{R}}(d_2).$$

Since $B(X_\pi)$ is β-absolutely normal, we obtain $\|(\pi \rtimes U)^{\mathcal{R}}(d_1)\| \leq \beta\|(\pi \rtimes U)^{\mathcal{R}}(d_2)\|$. By Proposition 3.7, taking the supremum over $\mathcal{R}$ on both sides yields $\|d_1\|^{\mathcal{R}} \leq \beta\|d_2\|^{\mathcal{R}}$.

That $(A \rtimes_\alpha G)^{\mathcal{R}}$ is β-normal for some $\beta > 0$ under the assumption that, for every $(\pi, U) \in \mathcal{R}$, $B(X_\pi)$ is β-normal follows similarly. □

Under the hypotheses of Theorem 3.8, we see that it is relatively easy to have a topologically generating cone of $(A \rtimes_\alpha G)^{\mathcal{R}}$: It is sufficient that A_+ is generating in A. The condition in (3) under which the cone of $(A \rtimes_\alpha G)^{\mathcal{R}}$ is generating is less easily verified if $\sigma^{\mathcal{R}}$ is not a norm, but we will nevertheless see an example (where $\sigma^{\mathcal{R}}$ is a norm) in Section 5 where we conclude that the cone of $(A \rtimes_\alpha G)^{\mathcal{R}}$ is generating through a different method than provided by (3) in the theorem above. Furthermore, according to part (4) and Theorem 2.12, if every continuous covariant representation from $\mathcal{R}$ is positive and acts on a pre-ordered Banach space with a closed proper generating cone, then the cone of $(A \rtimes_\alpha G)^{\mathcal{R}}$ is proper. As to (5), an appeal to Remark 2.13 shows that $(A \rtimes_\alpha G)^{\mathcal{R}}$ is 1-absolutely normal if every continuous covariant representation from $\mathcal{R}$ is positive and acts on a Banach lattice. We collect the features of the latter case in the following result:

Corollary 3.9. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system with A having a closed cone. Let $\mathcal{R}$ be a uniformly bounded class of positive continuous covariant representations on Banach lattices. Then $(A \rtimes_\alpha G)_+^{\mathcal{R}}$ is a closed proper cone and $(A \rtimes_\alpha G)^{\mathcal{R}}$ is a 1-absolutely normal ordered Banach algebra, i.e., for $d_1, d_2 \in (A \rtimes_\alpha G)^{\mathcal{R}}$, if $\pm d_1 \leq d_2$, then $\|d_1\|^{\mathcal{R}} \leq \|d_2\|^{\mathcal{R}}$. If A_+ is generating in A, then $(A \rtimes_\alpha G)_+^{\mathcal{R}}$ is topologically generating in $(A \rtimes_\alpha G)^{\mathcal{R}}$.*

The following example shows that even with A a Banach lattice algebra and the positive representations from $\mathcal{R}$ acting on Banach lattices, $\ker \sigma^{\mathcal{R}}$ need not be an order ideal in the vector lattice $C_c(G, A)$ in general.

Example 3.10. Let $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$. We consider the pre-ordered Banach algebra dynamical system $(\mathbb{R}, \mathbb{Z}_2, \mathrm{triv})$ and $\mathcal{R} = \{(\mathrm{id}, \mathrm{triv})\}$ with $(\mathrm{id}, \mathrm{triv})$ the trivial positive non-degenerate continuous covariant representation of $(\mathbb{R}, \mathbb{Z}_2, \mathrm{triv})$ on $\mathbb{R}$. Then, for $f \in C_c(\mathbb{Z}_2)$, $\mathrm{id} \rtimes \mathrm{triv}(f) = f(0) + f(1)$, hence $f \in \ker \sigma^{\mathcal{R}}$ if and only if $f(0) = -f(1)$. In particular, since $f \in \ker \sigma^{\mathcal{R}}$ does not imply $|f| \in \ker \sigma^{\mathcal{R}}$, $\ker \sigma^{\mathcal{R}}$ is not an order ideal in the vector lattice $C_c(\mathbb{Z}_2)$.

3.3. **Correspondence between positive representations of** (A, G, α) **and** $(A \rtimes_\alpha G)^{\mathcal{R}}$**.** In this section we give an adaptation of the General Correspondence Theorem (Theorem 2.22) to the ordered context. As in the unordered context, Theorem 2.18 will be a crucial ingredient, which here will rely on the existence of a positive bounded approximate left identity of the pre-ordered crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$. The following result shows that this is inherited from A.

Proposition 3.11. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system with A having a closed cone, and let $\mathcal{R}$ be a uniformly bounded class of continuous covariant representations of (A, G, α) on Banach spaces. Let A have a positive bounded approximate left (right) identity (u_i). Then the net*

$$\left(q^{\mathcal{R}}(z_V \otimes u_i)\right),$$

as described in Theorem 2.21, is a positive bounded approximate left (right) identity of $(A \rtimes_\alpha G)^{\mathcal{R}}$.

Proof. Since the quotient map $q^{\mathcal{R}} : C_c(G, A) \to (A \rtimes_\alpha G)^{\mathcal{R}}$ is positive and $z_V \otimes u_i \in C_c(G, A_+)$ for all i and $V \in \mathcal{Z}$, we have $q^{\mathcal{R}}(z_V \otimes u_i) \in (A \rtimes_\alpha G)^{\mathcal{R}}_+$. That $\left(q^{\mathcal{R}}(z_V \otimes u_i)\right)$ is a bounded left (right) identity is the statement of Theorem 2.21. ☐

The following will be used in the proof of Theorem 3.13 and in Section 4.

Lemma 3.12. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone. With $C_c(G, A)$ pre-ordered by the cone $C_c(G, A_+)$, the representations $i_A : A \to \mathrm{End}(C_c(G, A))$ and $i_G : G \to \mathrm{End}(C_c(G, A))$ as in defined in Section 2.5 are positive.*

If $\mathcal{R}$ is a uniformly bounded class of non-degenerate continuous covariant representations of (A, G, α) on Banach spaces, and A has a bounded approximate left identity, then the pair $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ as defined in Section 2.5 is a positive non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on $(A \rtimes_\alpha G)^{\mathcal{R}}$ such that $i_A^{\mathcal{R}}(A), i_G^{\mathcal{R}}(G) \subseteq \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$.

Proof. That the maps $i_A : A \to \mathrm{End}(C_c(G, A))$ and $i_G : G \to \mathrm{End}(C_c(G, A))$ are positive is clear. By [9, Lemma 6.3] and Lemma 2.2 the operators $i_A^{\mathcal{R}}(a)$ and $i_G^{\mathcal{R}}(r)$ ($a \in A_+$, $r \in G$) on $(A \rtimes_\alpha G)^{\mathcal{R}}$ are positive. The remaining statement is contained in [9, Theorem 7.2]. ☐

We finally establish the following adaptation of the General Correspondence Theorem to the ordered context. Note that the class $\mathcal{R}$ is not required to consist of positive continuous covariant representations. Conditions in that vein affect the properties of the cone in $(A \rtimes_\alpha G)^{\mathcal{R}}$ (cf. Theorem 3.8, Corollary 3.9), but are not necessary for the correspondence.

Theorem 3.13. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone and a positive bounded approximate left identity. Let*

$\mathcal{R}$ be a uniformly bounded class of non-degenerate continuous covariant representations of (A, G, α) on Banach spaces. Then the map $(\pi, U) \mapsto (\pi \rtimes U)^{\mathcal{R}}$ is a bijection between the positive non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) on pre-ordered Banach spaces with closed cones and the positive non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on such spaces.

More precisely:

(1) *If (π, U) is a positive non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on a pre-ordered Banach space X_π with a closed cone, then $(\pi \rtimes U)^{\mathcal{R}}$ is a positive non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on X_π, and*

$$(\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_A^{\mathcal{R}}, \overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_G^{\mathcal{R}}) = (\pi, U),$$

where $\overline{(\pi \rtimes U)^{\mathcal{R}}}$ is the positive non-degenerate bounded representation of $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ as in Theorem 2.18.

(2) *If T is a positive non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on a pre-ordered Banach space X_T with a closed cone, then $(\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ is a positive non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on X_T, and*

$$(\overline{T} \circ i_A^{\mathcal{R}} \rtimes \overline{T} \circ i_G^{\mathcal{R}})^{\mathcal{R}} = T,$$

where $\overline{T}$ is the positive non-degenerate bounded representation of $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ as in Theorem 2.18.

Proof. We prove part (1). If (π, U) is a positive non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α), by Lemma 3.4 and Lemma 2.2 we obtain that $(\pi \rtimes U)^{\mathcal{R}}$ is a positive bounded representation bounded of $(A \rtimes_\alpha G)^{\mathcal{R}}$. That it is non-degenerate and that $(\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_A^{\mathcal{R}}, \overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_G^{\mathcal{R}}) = (\pi, U)$ follows by applying the General Correspondence Theorem (Theorem 2.22).

We prove part (2). Since it is assumed that A has a positive bounded approximate left identity, by Proposition 3.11, $(A \rtimes_\alpha G)^{\mathcal{R}}$ has a positive bounded approximate left identity. By Theorem 2.18, $\overline{T} : \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}}) \to B(X_T)$ is a positive representation. By Lemma 3.12, the maps $i_A^{\mathcal{R}} : A \to \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ and $i_G^{\mathcal{R}} : G \to \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ are both positive, and therefore $(\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ is a pair of positive representations of respectively A and G on X. The General Correspondence Theorem asserts that $(\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ is also a non-degenerate $\mathcal{R}$-continuous covariant representation, and that $(\overline{T} \circ i_A^{\mathcal{R}} \rtimes \overline{T} \circ i_G^{\mathcal{R}})^{\mathcal{R}} = T$. □

4. Uniqueness of the pre-ordered crossed product

Let (A, G, α) be a Banach algebra dynamical system and $\mathcal{R}$ a uniformly bounded class of non-degenerate continuous covariant representations of (A, G, α) on Banach spaces. In [7, Theorem 4.4] it was shown (under mild further hypotheses) that the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ is the unique Banach algebra (up to topological isomorphism) such that the triple $((A \rtimes_\alpha G)^{\mathcal{R}}, i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ generates all non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α), in the sense that, for every non-degenerate bounded representation T of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on a Banach space X, $(\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ is a non-degenerate $\mathcal{R}$-continuous representation of (A, G, α) on X, and that, moreover, all non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) are obtained in this way.

In this section we will adapt this to pre-ordered Banach algebra dynamical systems. If (A, G, α) is a pre-ordered Banach algebra dynamical system, with A having a closed cone, and $\mathcal{R}$ a uniformly bounded class of positive non-degenerate continuous covariant representations of (A, G, α) on pre-ordered Banach spaces with closed cones, then we will show that (under similar mild hypotheses as in the unordered case) the pre-ordered crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ is the unique pre-ordered Banach algebra (up to bipositive topological isomorphism) such that the triple $((A \rtimes_\alpha G)^{\mathcal{R}}, i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ generates all positive non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) as described above.

We begin with the general framework for generating positive non-degenerate $\mathcal{R}$-continuous representations from a suitable basic one as in [7, Lemma 4.1].

Lemma 4.1. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system with A having a closed cone, and let $\mathcal{R}$ be a uniformly bounded class of continuous covariant representations of (A, G, α) on Banach spaces. Let E be a pre-ordered Banach algebra (with a not necessarily closed cone) and positive bounded approximate left identity, and let (k_A, k_G) be a positive non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on the pre-ordered Banach space E such that $k_A(A), k_G(G) \subseteq \mathcal{M}_l(E)$. Suppose $T : E \to B(X)$ is a positive non-degenerate bounded representation of E on a pre-ordered Banach space X with a closed cone. Let $\overline{T} : \mathcal{M}_l(E) \to B(X)$ be the positive non-degenerate bounded representation of $\mathcal{M}_l(E)$ such that the following diagram commutes:*

$$\begin{array}{ccc} E & \xrightarrow{\;T\;} & B(X) \\ & \searrow^{\lambda} & \uparrow{\scriptstyle \overline{T}} \\ & & \mathcal{M}_l(E) \end{array}$$

Then the pair $(\overline{T} \circ k_A, \overline{T} \circ k_G)$ is a positive non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α), and $(\overline{T} \circ k_A) \rtimes (\overline{T} \circ k_G) = \overline{T} \circ (k_A \rtimes k_G)$.

Proof. That $(\overline{T} \circ k_A, \overline{T} \circ k_G)$ is a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) and that $(\overline{T} \circ k_A) \rtimes (\overline{T} \circ k_G) = \overline{T} \circ (k_A \rtimes k_G)$ follows from [7, Lemma 4.1]. That $(\overline{T} \circ k_A, \overline{T} \circ k_G)$ is positive follows from (k_A, k_G) being positive and $\overline{T} : \mathcal{M}_l(E) \to B(X)$ being positive by Theorem 2.18. □

Therefore, given a pre-ordered Banach algebra E with such a positive non-degenerate $\mathcal{R}$-continuous covariant representation (k_A, k_G) of (A, G, α) on E, positive non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) can be generated from positive non-degenerate bounded representations of E.

Clearly, any pre-ordered Banach algebra E' that is bipositively topologically isomorphic to E must also have a similar positive non-degenerate $\mathcal{R}$-continuous covariant generating pair (k'_A, k'_G). This is outlined in the following straightforward adaptation of [7, Lemma 4.2] to the ordered context.

Lemma 4.2. *Let (A, G, α), $\mathcal{R}$, E and (k_A, k_G) be as in Lemma 4.1. Suppose E' is a pre-ordered Banach algebra and $\psi : E \to E'$ is a bipositive topological isomorphism. Then:*

(1) *$\psi_l : \mathcal{M}_l(E) \to \mathcal{M}_l(E')$, defined by $\psi_l(L) := \psi L \psi^{-1}$ for $L \in \mathcal{M}_l(E)$, is a bipositive topological isomorphism.*

(2) *The pair* $(k'_A, k'_G) := (\psi_l \circ k_A, \psi_l \circ k_G)$ *is a positive non-degenerate* $\mathcal{R}$*-continuous covariant representation of* (A, G, α) *on* E' *such that* $k'_A(A) \subseteq \mathcal{M}_l(E')$ *and* $k'_G(G) \subseteq \mathcal{M}_l(E')$.

(3) *If* $T : E \to B(X)$ *is a positive non-degenerate bounded representation, then so is* $T' : E' \to B(X)$, *where* $T' := T \circ \psi^{-1}$.

(4) *If* $T : E \to B(X)$ *is a positive non-degenerate bounded representation on a pre-ordered Banach space with a closed cone, and* $\overline{T'} : \mathcal{M}_l(E') \to B(X)$ *is the positive non-degenerate bounded representation of* $\mathcal{M}_l(E')$ *such that the diagram*

$$\begin{array}{ccc} E' & \xrightarrow{T'} & B(X) \\ & \searrow^{\lambda} & \uparrow{\scriptstyle \overline{T'}} \\ & & \mathcal{M}_l(E') \end{array}$$

commutes, then $\overline{T} \circ k_A = \overline{T'} \circ k'_A$ *and* $\overline{T} \circ k_G = \overline{T'} \circ k'_G$.

If A has a positive bounded approximate left identity, then, according to Proposition 3.11 and Lemma 3.12, the triple $((A \rtimes_\alpha G)^{\mathcal{R}}, i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ satisfies the hypotheses of Lemma 4.1, and by Theorem 3.13 *all* positive non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) can be "generated" from positive non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$ as in Lemma 4.1. By Lemma 4.2, a bipositive topological isomorphism between $(A \rtimes_\alpha G)^{\mathcal{R}}$ and another pre-ordered Banach algebra yields a triple with the same properties. Our aim in the rest of this section is to establish the converse: If (E, k_A, k_G) (where now E has a closed cone) is a "generating triple" for *all* positive non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) as in Lemma 4.1, then, under mild additional hypotheses, this triple can be obtained from $((A \rtimes_\alpha G)^{\mathcal{R}}, i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ via a bipositive topological isomorphism as in Lemma 4.2 (cf. Corollary 4.8).

In order to do this, we will need the existence of a positive isometric representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on some pre-ordered Banach space with a closed cone. As in [7, Proposition 3.4], this is achieved through combining sufficiently many members of $\mathcal{R}$ into one suitable positive continuous covariant representation.

Definition 4.3. Let (A, G, α) be a Banach algebra dynamical system. Let $\mathcal{R}$ be a uniformly bounded class of possibly degenerate continuous covariant representations of (A, G, α) on Banach spaces. We define $[\mathcal{R}]$ to be the collection of all uniformly bounded classes S that are actually sets and satisfy $\sigma^{\mathcal{R}} = \sigma^S$ on $C_c(G, A)$. Elements of some $[\mathcal{R}]$ will be called *uniformly bounded sets of continuous covariant representations.*

We note that $[\mathcal{R}]$ is always non-empty: For every $f \in C_c(G, A)$, considering the set $\{\|\pi \rtimes U(f)\| : (\pi, U) \in \mathcal{R}\} \subseteq \mathbb{R}$ (subclasses of sets are sets), we may choose a sequence from $\{\|\pi \rtimes U(f)\| : (\pi, U) \in \mathcal{R}\}$ converging to $\sigma^{\mathcal{R}}(f)$, and consider only those corresponding covariant representations from $\mathcal{R}$. In this way we may choose a set S from $\mathcal{R}$ of cardinality at most $|C_c(G, A) \times \mathbb{N}|$ such that $\sigma^S(f) = \sigma^{\mathcal{R}}(f)$ for all $f \in C_c(G, A)$. Therefore the previous definition is non-void.

Definition 4.4. Let I be an index set and $\{X_i : i \in I\}$ a family of pre-ordered Banach spaces with closed cones. For $1 \leq p \leq \infty$, we will denote the ℓ^p-direct sum of $\{X_i : i \in I\}$ by $\ell^p\{X_i : i \in I\}$ and endow it with the cone $\ell^p\{(X_i)_+ : i \in I\}$, so that it is a pre-ordered Banach space with a closed cone.

Definition 4.5. Let (A, G, α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone, and $\mathcal{R}$ a uniformly bounded class of positive continuous covariant representations of (A, G, α) on pre-ordered Banach spaces with closed cones. For $S \in [\mathcal{R}]$ and $1 \leq p < \infty$, suppressing the p-dependence in the notation, we define the positive representations $(\oplus_S \pi) : A \to B(\ell^p\{X_\pi : (\pi, U) \in S\})$ and $(\oplus_S U) : G \to B(\ell^p\{X_\pi : (\pi, U) \in S\})$ by $(\oplus_S \pi)(a) := \bigoplus_{(\pi,U)\in S} \pi(a)$ and $(\oplus_S U)_r := \bigoplus_{(\pi,U)\in S} U_r$, for all $a \in A$ and $r \in G$ respectively.

It is easily seen that $((\oplus_S \pi), (\oplus_S U))$ is a positive continuous covariant representation, that

$$((\oplus_S \pi) \rtimes (\oplus_S U))(f) = \bigoplus_{(\pi,U)\in S} \pi \rtimes U(f),$$

and that $\|((\oplus_S \pi) \rtimes (\oplus_S U))(f)\| = \sigma^S(f) = \sigma^{\mathcal{R}}(f)$, for all $f \in C_c(G, A)$.

We hence obtain the following (where the statement concerning non-degeneracy is an elementary verification).

Proposition 4.6. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system, where A has a closed cone, and $\mathcal{R}$ a uniformly bounded class of positive continuous covariant representations of (A, G, α) on pre-ordered Banach spaces with closed cones. For any $S \in [\mathcal{R}]$ and $1 \leq p < \infty$, there exists a positive $\mathcal{R}$-continuous covariant representation of (A, G, α) on the pre-ordered Banach space $\ell^p\{X_\pi : (\pi, U) \in S\}$ with a closed cone, denoted $((\oplus_S \pi), (\oplus_S U))$, such that its positive integrated form satisfies $\|((\oplus_S \pi) \rtimes (\oplus_S U))(f)\| = \sigma^{\mathcal{R}}(f)$ for all $f \in C_c(G, A)$ and hence induces a positive isometric representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on $\ell^p\{X_\pi : (\pi, U) \in S\}$.*

If every element of S is non-degenerate, then $((\oplus_S \pi), (\oplus_S U))$ is non-degenerate.

In the following theorem we will give sufficient conditions under which a triple (E, k_A, k_G), generating all positive non-degenerate $\mathcal{R}$-continuous covariant representations of a pre-ordered Banach algebra dynamical system (A, G, α) as in Lemma 4.1, can be obtained from $((A \rtimes_\alpha G)^{\mathcal{R}}, i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ through a bipositive topological isomorphism as in Lemma 4.2.

Theorem 4.7. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system with A having a closed cone and a positive bounded approximate left identity. Let $\mathcal{R}$ be a uniformly bounded class of positive non-degenerate continuous covariant representations of (A, G, α) on pre-ordered Banach spaces with closed cones. Let E be a pre-ordered Banach algebra, with closed cone and positive bounded approximate left identity and such that $\lambda : E \to \lambda(E) \subseteq \mathcal{M}_l(E)$ is a bipositive topological embedding. Let (k_A, k_G) be a positive non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on the pre-ordered Banach space E such that:*

1. *$k_A(A), k_G(G) \subseteq \mathcal{M}_l(E)$,*
2. *$(k_A \rtimes k_G)(C_c(G, A)) \subseteq \lambda(E)$,*
3. *$(k_A \rtimes k_G)(C_c(G, A))$ is dense in $\lambda(E)$,*
4. *$(k_A \rtimes k_G)(C_c(G, A_+))$ is dense in $\lambda(E) \cap \mathcal{M}_l(E)_+$.*

Suppose that, for every positive non-degenerate $\mathcal{R}$-continuous covariant representation (π, U) of (A, G, α) on a pre-ordered Banach space X with a closed cone, there exists a positive non-degenerate bounded representation $T : E \to B(X)$ such that the positive non-degenerate bounded representation $\overline{T} : \mathcal{M}_l(E) \to B(X)$ in the *commuting diagram*

$$\begin{array}{ccc} E & \xrightarrow{T} & B(X) \\ & \searrow^{\lambda} & \uparrow \overline{T} \\ & & \mathcal{M}_l(E) \end{array}$$

generates (π, U) *as in Lemma 4.1, i.e., is such that* $\overline{T} \circ k_A = \pi$ *and* $\overline{T} \circ k_G = U$.

Then there exists a unique topological isomorphism $\psi : (A \rtimes_\alpha G)^{\mathcal{R}} \to E$ *such that the induced topological isomorphism* $\psi_l : \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}}) \to \mathcal{M}_l(E)$*, defined by* $\psi_l(L) := \psi L \psi^{-1}$ *for* $L \in \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$*, induces* (k_A, k_G) *from* $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ *as in Lemma 4.2, i.e., is such that* $k_A = \psi_l \circ i_A^{\mathcal{R}}$ *and* $k_G = \psi_l \circ i_G^{\mathcal{R}}$.

Moreover, ψ *is bipositive.*

The proof follows largely as in [7, Proposition 4.3], but with some modifications in the first part of the proof, which we now give.

Proof. By hypothesis $\mathcal{R}$ consists of positive non-degenerate continuous covariant representations of (A, G, α) on pre-ordered Banach spaces with closed cones. Hence Proposition 4.6 provides a positive non-degenerate $\mathcal{R}$-continuous covariant representation (π, U) of (A, G, α) on a pre-ordered Banach space X with a closed cone such that $(\pi \rtimes U)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(X)$ is a positive non-degenerate isometric representation. By hypothesis, there exists a positive non-degenerate representation $T : E \to B(X)$ such that $\overline{T} \circ k_A = \pi$ and $\overline{T} \circ k_G = U$. By Lemma 4.1, we obtain $\pi \rtimes U = (\overline{T} \circ k_A) \rtimes (\overline{T} \circ k_G) = \overline{T} \circ (k_A \rtimes k_G)$. Then, for any $f \in C_c(G, A)$,

$$\begin{aligned} \|q^{\mathcal{R}}(f)\| &= \|(\pi \rtimes U)^{\mathcal{R}}(q^{\mathcal{R}}(f))\| \\ &= \|\pi \rtimes U(f)\| \\ &= \|\overline{T} \circ (k_A \rtimes k_G)(f)\| \\ &\leq \|\overline{T}\| \|k_A \rtimes k_G(f)\| \\ &= \|\overline{T}\| \|(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f))\|. \end{aligned}$$

Since (k_A, k_G) was assumed to be $\mathcal{R}$-continuous, we obtain $\|(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f))\| \leq \|(k_A \rtimes k_G)^{\mathcal{R}}\| \|q^{\mathcal{R}}(f)\|$. Using (2), (3) and the fact that $\lambda(E)$ is closed, it now follows that $(k_A \rtimes k_G)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to \lambda(E)$ is a topological isomorphism.

Since (k_A, k_G) is positive and $\mathcal{R}$-continuous, and the cone of E is closed, by Lemmas 3.4 and 2.2, $(k_A \rtimes k_G)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to \lambda(E)$ is positive. We claim that $(k_A \rtimes k_G)^{\mathcal{R}}$ is bipositive. Let $b \in E$ be such that $\lambda(b) \in \lambda(E) \cap \mathcal{M}_l(E)_+$, hence by (4) there exists a sequence $(f_n) \subseteq C_c(G, A_+)$ such that $(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f_n)) = (k_A \rtimes k_G)(f_n) \to \lambda(b)$. Since $(k_A \rtimes k_G)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to \lambda(E)$ is a topological isomorphism, the sequence $(q^{\mathcal{R}}(f_n)) \subseteq (A \rtimes_\alpha G)_+^{\mathcal{R}}$ converges to some $d \in (A \rtimes_\alpha G)^{\mathcal{R}}$ and $(k_A \rtimes k_G)^{\mathcal{R}}(d) = \lambda(b)$. Moreover, since $(A \rtimes_\alpha G)_+^{\mathcal{R}}$ is closed and $(q^{\mathcal{R}}(f_n)) \subseteq (A \rtimes_\alpha G)_+^{\mathcal{R}}$, we have $d \in (A \rtimes_\alpha G)_+^{\mathcal{R}}$. We conclude that $(k_A \rtimes k_G)^{\mathcal{R}}$ is bipositive.

Since $\lambda : E \to \mathcal{M}_l(E)$ is assumed to be a bipositive topological embedding,

$$\psi := \lambda^{-1} \circ (k_A \rtimes k_G)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to E$$

is a bipositive topological isomorphism.

The remainder of the argument proceeds as in the proof of [7, Theorem 4.4]. □

Under the conditions on (A, G, α) and $\mathcal{R}$ as stated in the previous theorem, one would of course hope that the triple $((A \rtimes_\alpha G)^{\mathcal{R}}, i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ automatically satisfies the hypotheses on (E, k_A, k_G), as happens in the unordered context [7, Theorem 4.4]. Here, as there, the left regular representation $\lambda : (A \rtimes_\alpha G)^{\mathcal{R}} \to \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ is a topological embedding [7, Proposition 4.3], and, since $q^{\mathcal{R}}(C_c(G, A))$ is dense in $(A \rtimes_\alpha G)^{\mathcal{R}}$ and $(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})^{\mathcal{R}} = \lambda$ [9, Theorem 7.2], we have that $(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})(C_c(G, A))$ is dense in $\lambda((A \rtimes_\alpha G)^{\mathcal{R}})$. Hence (1), (2) and (3) in Theorem 4.7 are satisfied by $((A \rtimes_\alpha G)^{\mathcal{R}}, i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$. We claim that the additional assumption that A has a positive bounded approximate right identity gives the remaining conditions that $\lambda : (A \rtimes_\alpha G)^{\mathcal{R}} \to \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ is a bipositive topological embedding, and that (4) holds. Indeed, if this is the case, let $(u_i) \subseteq (A \rtimes_\alpha G)_+^{\mathcal{R}}$ be a positive approximate right identity of $(A \rtimes_\alpha G)^{\mathcal{R}}$ (which exists by Theorem 3.11), and let $d \in (A \rtimes_\alpha G)^{\mathcal{R}}$ be such that $\lambda(d) \geq 0$. Then $0 \leq \lambda(d)u_i = d * u_i \to d$, so that $d \in (A \rtimes_\alpha G)_+^{\mathcal{R}}$. We conclude that $\lambda^{-1} : \lambda((A \rtimes_\alpha G)^{\mathcal{R}}) \to (A \rtimes_\alpha G)^{\mathcal{R}}$ is also positive. Since $(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})^{\mathcal{R}} = \lambda$, this also gives that $(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})(C_c(G, A_+))$ is dense in $\lambda((A \rtimes_\alpha G)^{\mathcal{R}}) \cap \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})_+$. Hence we have the following uniqueness result:

Corollary 4.8. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone and both a positive bounded approximate left identity and a positive bounded approximate right identity. Let $\mathcal{R}$ be a uniformly bounded class of positive non-degenerate continuous covariant representations of (A, G, α) on pre-ordered Banach spaces with closed cones. Then $((A \rtimes_\alpha G)^{\mathcal{R}}, i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ satisfies all hypotheses on the triple (E, k_A, k_G) in Theorem 4.7. Hence triples (E, k_A, k_G) as in Theorem 4.7 exist, and every such "generating triple" for all positive non-degenerate $\mathcal{R}$-continuous representations of (A, G, α) originates from $((A \rtimes_\alpha G)^{\mathcal{R}}, i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ through a bipositive topological isomorphism $\psi : (A \rtimes_\alpha G)^{\mathcal{R}} \to E$ as in Theorem 4.7 (so that E necessarily has a positive bounded approximate right identity as well).*

5. Pre-ordered generalized Beurling algebras

In [7, Section 5] it was shown that a generalized Beurling algebra (to be defined below) is topologically isomorphic to a crossed product associated with a Banach algebra dynamical system, and the non-degenerate bounded representations of these algebras were described in terms of non-degenerate continuous covariant representations of the underlying Banach algebra dynamical system. We refer the reader to [7, Section 5] for a more complete treatment of generalized Beurling algebras and how they are constructed from Banach algebra dynamical systems.

In this section we will adapt the main results from [7, Section 5] to the case of pre-ordered Banach algebra dynamical systems and pre-ordered generalized Beurling algebras. Theorem 5.7 is the analogue of [7, Theorem 5.17] in the ordered context, and shows that a pre-ordered generalized Beurling algebra is bipositively topologically isomorphic to a crossed product associated with a pre-ordered Banach algebra dynamical system. In Theorem 5.9 we modify [7, Theorem 5.20] to explicitly describe a bijection between the positive non-degenerate continuous covariant representations of a pre-ordered Banach algebra dynamical system, where the group representation is bounded by a multiple of a fixed weight on the underlying group, and the positive non-degenerate bounded representations of the associated pre-ordered generalized Beurling algebra.

We begin with a brief description of pre-ordered generalized Beurling algebras and related spaces.

Definition 5.1. For a locally compact group G, let $\omega : G \to [0, \infty)$ be a non-zero submultiplicative Borel measurable function. Then ω is called a *weight* on G.

Note that we do not assume that $\omega \geq 1$, as is done in some parts of the literature. The fact that ω is non-zero readily implies that $\omega(e) \geq 1$. More generally, if $K \subseteq G$ is a compact set, there exist $a, b > 0$ such that $a \leq \omega(s) \leq b$ for all $s \in K$ [11, Lemma 1.3.3].

Definition 5.2. Let X be a pre-ordered Banach space with a closed cone, and $\omega : G \to [0, \infty)$ a weight on G. We define the weighted 1-norm on $C_c(G, X)$ by

$$\|h\|_{1,\omega} := \int_G \|h(s)\|\omega(s)\, ds \quad (h \in C_c(G, X)),$$

and define the pre-ordered Banach space $L^1(G, X, \omega)$ as the completion (in the sense of Definition 2.1) of the pre-ordered vector space $(C_c(G, X), C_c(G, X_+))$ with the $\| \cdot \|_{1,\omega}$-norm.

Given the prominent role of continuous compactly supported functions in the theory, the definition of $L^1(G, X, \omega)$ as the completion of the space $C_c(G, X)$ is clearly convenient. A drawback, however, is that it is then not clear that $L^1(G, X, \omega)_+$, which is, by definition, the closure of $C_c(G, X_+)$, is generating in $L^1(G, X, \omega)$ if X_+ is generating in X. From Corollary 2.8 we know that $C_c(G, X_+)$ is generating in $C_c(G, X)$, and then Lemma 2.4 yields that $L^1(G, X, \omega)_+$ is topologically generating in $L^1(G, X, \omega)$, but generalities do not seem to help us beyond this point. Similarly, it is not clear that $L^1(G, X, \omega)_+$ is a proper cone if X_+ is proper. To establish these results, we use the fact that, as already observed in [7, Remark 5.3], $L^1(G, X, \omega)$ is isometrically isomorphic to a Bochner space (also if the left Haar measure μ is not σ-finite, or X is not separable). We recall the relevant facts. A function $f : G \to X$ is Bochner integrable (with respect to $\omega d\mu$) if $f^{-1}(B)$ is a Borel subset of G for every Borel subset B of X, $f(G)$ is separable, and $\int_G \|f(s)\|\omega(s)\, d\mu(s) < \infty$ (the measurability of $s \mapsto \|f(s)\|$ is an automatic consequence of the Borel measurability of f). On identifying Bochner integrable functions that are equal $\omega d\mu$-almost everywhere, one obtains a Banach space $L^1(G, \mathcal{B}, \omega d\mu, X)$, where $\mathcal{B}$ is the Borel σ-algebra of G, and the norm is given by $\|[f]\| = \int_G \|f(s)\|\omega(s)\, d\mu(s)$, with f any representative of $[f] \in L^1(G, \mathcal{B}, \omega d\mu, X)$. Clearly the inclusion map of $(C_c(G, X), \| \cdot \|_{1,\omega})$ into $L^1(G, \mathcal{B}, \omega d\mu, X)$ is isometric, and the existence of the aforementioned isometric isomorphism between $L^1(G, X, \omega)$ and $L^1(G, \mathcal{B}, \omega d\mu, X)$ is then established by showing that $C_c(G, X)$ is dense in $L^1(G, \mathcal{B}, \omega d\mu, X)$. In the present context, if X is a pre-ordered Banach space, then $L^1(G, \mathcal{B}, \omega d\mu, X)$ has a natural cone

$$L^1(G, \mathcal{B}, \omega d\mu, X_+) := \{f \in L^1(G, \mathcal{B}, \omega d\mu, X) : f(s) \in X_+ \text{ for } \omega d\mu\text{-a.a. } s \in G\},$$

where, as usual we have ignored the distinction between equivalence classes and functions. As in the scalar case, a convergent sequence in $L^1(G, \mathcal{B}, \omega d\mu, X)$ has a subsequence that converges $\omega d\mu$-almost everywhere to the limit function. Hence if X_+ is closed, then so is $L^1(G, \mathcal{B}, \omega d\mu, X_+)$. We then have the following natural result.

Proposition 5.3. *Let X be a pre-ordered Banach space with a closed cone. Let G be a locally compact group and ω a weight on G. Then:*

(1) *The cone $C_c(G, X_+)$ is dense in the closed cone $L^1(G, \mathcal{B}, \omega d\mu, X_+)$.*

(2) *$(L^1(G,X,\omega), L^1(G,X,\omega)_+)$ and $(L^1(G,\mathcal{B},\omega d\mu,X), L^1(G,\mathcal{B},\omega d\mu,X_+))$ are pre-ordered Banach spaces with closed cones that are bipositively isometrically isomorphic through an isomorphism that is the identity on $C_c(G,X)$.*

Proof. For the first part we need, in view of the remarks preceding the proposition, only show that $C_c(G,X_+)$ is dense in $L^1(G,\mathcal{B},\omega d\mu,X_+)$. If $f \in L^1(G,\mathcal{B},\omega d\mu,X)$, then the proof of [5, Proposition E.2] shows that there exists a subset S of $\mathbb{Q}f(G)$ and a sequence of of simple functions (f_n), with values in S, such that $f_n(s) \to f(s)$ and $\|f_n(s)\| \leq \|f(s)\|$ for $\omega d\mu$-almost every $s \in G$. Hence by the dominated convergence theorem (see the argument on [5, p. 352]) $f_n \to f$. An inspection of the proof of [5, Proposition E.2] shows that, in fact, S can be chosen to be a subset of $\mathbb{Q}_{\geq 0}f(G)$. It is then clear that the (equivalence classes of) X_+-valued simple functions are dense in $L^1(G,\mathcal{B},\omega d\mu,X_+)$. Therefore, it is sufficient to show that the functions of the form $\chi_B \otimes x \in L^1(G,\mathcal{B},\omega d\mu,X_+)$, where $B \in \mathcal{B}$ and $x \in X_+$, can be approximated arbitrarily closely by elements of $C_c(G,X_+)$. As to this, since $C_c(G)$ is dense in the Beurling algebra $L^1(G,\omega)$ [11, Lemma 1.3.5], there exists a sequence $(g_n) \subseteq C_c(G)$ such that $g_n \to \chi_B$ in $L^1(G,\omega)$. Since $\chi_B \geq 0$ we clearly have $\|g_n^+ \otimes x - \chi_B \otimes x\|_{1,\omega} \leq \|g_n - \chi_B\|_{1,\omega}\|x\| \to 0$. Hence $g_n^+ \otimes x \to \chi_B \otimes x$, and the proof is complete.

The second part is immediate from the first. ☐

We can now settle the matters mentioned above.

Theorem 5.4. *Let X be a pre-ordered Banach space with a closed cone. Let G be a locally compact group and ω a weight on G.*

(1) *If X_+ is generating in X, then the closed cone $L^1(G,X,\omega)_+$ is generating in $L^1(G,X,\omega)$.*
(2) *If X_+ is a proper cone, then the closed cone $L^1(G,X,\omega)_+$ is proper.*

Proof. In view of Proposition 5.3, it is equivalent to prove the statements for the closed cone $L^1(G,\mathcal{B},\omega d\mu,X_+)$ of $L^1(G,\mathcal{B},\omega d\mu,X)$. Part (2) is then immediate. As to part (1), by Theorem 2.7, if X_+ is generating in X there exist continuous positively homogeneous functions $(\cdot)^{\pm} : X \to X_+$ and a constant $\alpha > 0$ such that $x = x^+ - x^-$ and $\|x^{\pm}\| \leq \alpha\|x\|$ for all $x \in X$. If $f \in L^1(G,\mathcal{B},\omega d\mu,X)$, we define $f^{\pm}(s) := (f(s))^{\pm}$ for all $s \in G$. Since the functions $(\cdot)^{\pm} : X \to X_+$ are continuous, the measurability of f implies the measurability of $f^{\pm}$, and the separability of $f(G)$ implies the separability of $f^{\pm}(G)$. The inequalities $\|x^{\pm}\| \leq \alpha\|x\|$ $(x \in X)$ imply $\|f^{\pm}\|_{1,\omega} \leq \alpha\|f\|_{1,\omega} < \infty$. We conclude that $f^{\pm} \in L^1(G,\mathcal{B},\omega d\mu,X_+)$. Since $f = f^+ - f^-$, the cone $L^1(G,\mathcal{B},\omega d\mu,X_+)$ is generating in $L^1(G,\mathcal{B},\omega d\mu,X)$. ☐

Thus, in particular, if (A,G,α) is a pre-ordered Banach algebra dynamical system with A having a closed cone, then $L^1(G,A,\omega)_+$ is generating (proper) in $L^1(G,A,\omega)$ if A_+ is generating (proper) in A.

We now turn to the definition of the multiplicative structure on $L^1(G,A,\omega)$ if α is uniformly bounded. Let (A,G,α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone, and ω a weight on G. If α is uniformly bounded, say $\|\alpha_s\| \leq C_\alpha$ for some $C_\alpha \geq 0$ and all $s \in G$, then, using the submultiplicativity of ω, it is routine to verify that

$$\|f * g\|_{1,\omega} \leq C_\alpha \|f\|_{1,\omega}\|g\|_{1,\omega} \quad (f,g \in C_c(G,A)).$$

Since $C_c(G,A)$ is a pre-ordered algebra by Lemma 3.2, it is now clear that the pre-ordered Banach space $L^1(G,A,\omega)$ can be supplied with the structure of a pre-ordered algebra with continuous multiplication. If $C_\alpha = 1$ (i.e., if α lets G act as bipositive isometries on A), then $L^1(G,A,\omega)$ is a pre-ordered Banach algebra. When $C_\alpha \neq 1$, as is well known, there is an equivalent norm on $L^1(G,A,\omega)$ such that it becomes a Banach algebra, which is a pre-ordered Banach algebra when endowed with the same cone $L^1(G,A,\omega)_+$. In [7, Theorem 5.17] it was shown that such a Banach algebra norm can be obtained from a topological isomorphism between $L^1(G,A,\omega)$ and the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ for a suitable choice of $\mathcal{R}$. In Theorem 5.7 below, we show that in the ordered context this topological isomorphism is bipositive.

Definition 5.5. Let (A,G,α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone and α uniformly bounded. Let ω be a weight on G. The pre-ordered Banach space $L^1(G,A,\omega)$ endowed with the continuous multiplication induced by the twisted convolution on $C_c(G,A)$, given by

$$[f*g](s) := \int_G f(r)\alpha_r(g(r^{-1}s))\,dr \quad (f,g\in C_c(G,A),\ s\in G),$$

will be denoted by $L^1(G,A,\omega;\alpha)$ and called a *pre-ordered generalized Beurling algebra.*

We note that if $A=\mathbb{R}$, the pre-ordered generalized Beurling algebra $L^1(G,A,\omega;\alpha)$ reduces to a classical Beurling algebra, which is a true Banach algebra.

Let (A,G,α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone. The following definition shows how to induce a continuous covariant representation of (A,G,α) from a positive bounded representation of A. Applying this construction to the left regular representation of A, and choosing (for instance) $\mathcal{R}$ to be the singleton containing this continuous covariant representation, yields the desired topological isomorphism (cf. [7, Theorem 5.13]). We keep track of possible order structures in order to show later that this topological isomorphism is bipositive.

Definition 5.6. Let (A,G,α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone, and let $\pi: A\to B(X)$ be a positive bounded representation of A on a pre-ordered Banach space X with a closed cone. We define the induced algebra representation $\tilde{\pi}$ and left regular group representation Λ on the space X^G of all functions from G to X by the formulae:

$$\begin{aligned}[][\tilde{\pi}(a)h](s) &:= \pi(\alpha_s^{-1}(a))h(s),\\ (\Lambda_r h)(s) &:= h(r^{-1}s),\end{aligned}$$

where $h: G\to X$, $r,s\in G$ and $a\in A$.

It is easy to see that $(\tilde{\pi},\Lambda)$ is covariant, and positive if X^G is endowed with the cone X^G_+. If α is uniformly bounded, then $(\tilde{\pi},\Lambda)$ yields a continuous covariant representation of A on $L^1(G,X,\omega)$ such that $\|\Lambda_r\|\leq\omega(r)$ for all $r\in G$, and if π is non-degenerate, so is $(\tilde{\pi},\Lambda)$ [7, Corollary 5.9]. Hence, if A has a bounded approximate left or right identity, then, with $\lambda: A\to B(A)$ denoting the left regular representation of A, $(\tilde{\lambda},\Lambda)$ is a positive non-degenerate continuous covariant representation of (A,G,α) on $L^1(G,A,\omega)$.

Let (A, G, α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone and a (not necessarily positive) bounded approximate right identity. Let ω be a weight on G and $\mathcal{R}$ a uniformly bounded class of non-degenerate continuous covariant representations of (A, G, α) on Banach spaces, such that $\sup_{(\pi,U)\in\mathcal{R}} \|U_r\| \leq \omega(r)$ for all $r \in G$. If $(\tilde{\lambda}, \Lambda)$ is $\mathcal{R}$-continuous, for instance if $\mathcal{R} = \{(\tilde{\lambda}, \Lambda)\}$, then the integrated form $\tilde{\lambda} \rtimes \Lambda : C_c(G, A) \to B(L^1(G, A, \omega))$ is faithful, and hence the seminorm $\sigma^{\mathcal{R}}$ is actually a norm on $C_c(G, A)$ and is equivalent to $\| \cdot \|_{1,\omega}$ [7, Theorem 5.13]. Furthermore, $\tilde{\lambda} \rtimes \Lambda$ extends to a topological embedding $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(L^1(G, A, \omega))$ [7, Theorem 5.13]. Since the norms $\sigma^{\mathcal{R}}$ and $\| \cdot \|_{1,\omega}$ are equivalent, the topological isomorphism between $(A \rtimes_\alpha G)^{\mathcal{R}}$ and $L^1(G, A, \omega; \alpha)$ which is the identity on the mutual dense subspace $C_c(G, A)$ is bipositive by construction, as the cones of both spaces are the closure of $C_c(G, A)$. Since the non-degenerate $\mathcal{R}$-continuous covariant representation $(\tilde{\lambda}, \Lambda)$ is positive, so is the topological embedding $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(L^1(G, A, \omega))$ by Lemmas 3.4 and 2.2.

Under the assumption that, in fact, A has a positive bounded approximate right identity, we claim that the positive topological embedding $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(L^1(G, A, \omega))$ is bipositive. Identifying $(A \rtimes_\alpha G)^{\mathcal{R}}$ with $L^1(G, A, \omega; \alpha)$ through the above bipositive topological isomorphism, the topological embedding $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}}$ is conjugate to the left regular representation $\lambda : L^1(G, A, \omega; \alpha) \to B(L^1(G, A, \omega; \alpha))$ through the bipositive map $\hat{\cdot} : L^1(G, A, \omega; \alpha) \to L^1(G, A, \omega; \alpha)$, determined by $\hat{h}(s) := \alpha_s(h(s))$ for $h \in C_c(G, A)$ and $s \in G$ [7, Remark 5.16]. We denote the inverse of $\hat{\cdot}$ by $\check{\cdot}$. With $(u_i) \subseteq L^1(G, A, \omega; \alpha)_+$ a positive approximate right identity (which exists by Proposition 3.11 and the fact that $(A \rtimes_\alpha G)^{\mathcal{R}}$ is bipositively topologically isomorphic to $L^1(G, A, \omega; \alpha)$, as described above), if $f \in L^1(G, A, \omega; \alpha)$ is such that $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}}(f) \geq 0$, then

$$0 \leq \left((\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}}(f) \check{u_i} \right)^{\wedge} = \lambda(f) u_i = f * u_i \to f.$$

Since $L^1(G, A, \omega; \alpha)_+$ is closed by construction, we obtain $f \in L^1(G, A, \omega; \alpha)_+$, and therefore the claim that $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(L^1(G, A, \omega))$ is a bipositive topological embedding follows.

We hence obtain the following ordered version of [7, Theorem 5.17]:

Theorem 5.7. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone and a (not necessarily positive) bounded approximate right identity. Let α be uniformly bounded and ω be a weight on G. Let the positive non-degenerate continuous covariant representation $(\tilde{\lambda}, \Lambda)$ of (A, G, α) on $L^1(G, A, \omega)$ be as yielded by Definition 5.6. Then the pre-ordered generalized Beurling algebra $L^1(G, A, \omega; \alpha)$ and the pre-ordered crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ with $\mathcal{R} := \{(\tilde{\lambda}, \Lambda)\}$ are bipositively topologically isomorphic via an isomorphism that is the identity on $C_c(G, A)$.*

Furthermore, the map $\tilde{\lambda} \rtimes \Lambda : C_c(G, A) \to B(L^1(G, A, \omega))$ extends to a positive topological embedding of $(A \rtimes_\alpha G)^{\mathcal{R}}$ into $B(L^1(G, A, \omega))$, and this extension is bipositive if A has a positive bounded approximate right identity.

If A has a 1-bounded right approximate identity, α lets G act as isometries on A and $\inf_{W\in\mathcal{Z}} \sup_{r\in W} \omega(r) = 1$, with $\mathcal{Z}$ denoting a neighbourhood base of $e \in G$

of which all elements are contained in a fixed compact set, then the bipositive topological isomorphism between $(A \rtimes_\alpha G)^{\mathcal{R}}$ *and* $L^1(G, A, \omega; \alpha)$ *and the above positive embedding of* $(A \rtimes_\alpha G)^{\mathcal{R}}$ *into* $B(L^1(G, A, \omega))$ *are both isometric.*

The following result gives some properties of the cones of pre-ordered generalized Beurling algebras. Here application of Theorem 5.4 yields stronger conclusions on the structure of the cone $L^1(G, A, \omega; \alpha)_+$ than can be concluded from the more generally applicable Theorem 3.8.

Theorem 5.8. *Let* (A, G, α) *be a pre-ordered Banach algebra dynamical system, with* A *having a closed cone and a (not necessarily positive) bounded approximate right identity. Let* α *be uniformly bounded and* ω *be a weight on* G.

If the cone A_+ *is generating (proper) in* A*, then the cone* $L^1(G, A, \omega; \alpha)_+$ *is generating (proper) in* $L^1(G, A, \omega; \alpha)$.

Furthermore, if A *is a Banach lattice algebra, then the pre-ordered generalized Beurling algebra* $L^1(G, A, \omega; \alpha)$*, viewed as pre-ordered Banach space, is a Banach lattice. If, in addition,* α *lets* G *act as bipositive isometries on* A*, the pre-ordered generalized Beurling algebra* $L^1(G, A, \omega; \alpha)$ *is a Banach lattice algebra.*

Proof. The conclusions on $L^1(G, A, \omega; \alpha)_+$ being generating or proper follow immediately from Theorem 5.4.

If A is a Banach lattice algebra, then $(C_c(G, A), C_c(G, A_+))$ with the norm $\|\cdot\|_{1,\omega}$ is a normed vector lattice. Therefore, by [16, Corollary 2, p. 84], $L^1(G, A, \omega; \alpha)$ is a Banach lattice. If α lets G act as bipositive isometries on A, then $L^1(G, A, \omega; \alpha)$ is also a pre-ordered Banach algebra as a consequence of Lemma 2.3 and the discussion preceding Definition 5.5. Therefore $L^1(G, A, \omega; \alpha)$ is a Banach lattice algebra. □

Through an application of Theorem 3.13, we can now adapt [7, Theorem 5.20] to the ordered context, and give an explicit description of the positive non-degenerate bounded representations of pre-ordered generalized Beurling algebras $L^1(G, A, \omega; \alpha)$ on pre-ordered Banach spaces with closed cones in terms of the positive non-degenerate continuous covariant representations of (A, G, α) on such spaces, where the group representation is bounded by a multiple of ω. The result is as follows:

Theorem 5.9. *Let* (A, G, α) *be a pre-ordered Banach algebra dynamical system, with* A *having a closed cone, a (not necessarily positive) bounded approximate right identity and a positive bounded approximate left identity. Let* α *be uniformly bounded and* ω *a weight on* G*. Then the following maps are mutual inverses between the positive non-degenerate continuous covariant representations* (π, U) *of* (A, G, α) *on a pre-ordered Banach space* X *with closed cone, satisfying* $\|U_r\| \leq C_U \omega(r)$ *for some* $C_U \geq 0$ *and all* $r \in G$*, and the positive non-degenerate bounded representations* $T : L^1(G, A, \omega; \alpha) \to B(X)$ *of the pre-ordered generalized Beurling algebra* $L^1(G, A, \omega; \alpha)$ *on* X*:*

$$(\pi, U) \mapsto \left(f \mapsto \int_G \pi(f(r)) U_r \, dr \right) =: T^{(\pi,U)} \quad (f \in C_c(G, A)),$$

determining a positive non-degenerate bounded representation $T^{(\pi,U)}$ *of the pre-ordered generalized Beurling algebra* $L^1(G, A, \omega; \alpha)$*, and,*

$$T \mapsto \left(\begin{array}{l} a \mapsto \text{SOT-lim}_{(V,i)} T(z_V \otimes a u_i), \\ s \mapsto \text{SOT-lim}_{(V,i)} T(z_V(s^{-1} \cdot) \otimes u_i) \end{array} \right) =: (\pi^T, U^T),$$

where $\mathcal{Z}$ is a neighbourhood base of $e \in G$, of which all elements are contained in a fixed compact subset of G, $z_V \in C_c(G, A)$ is chosen such that $z_V \geq 0$ is supported in $V \in \mathcal{Z}$, $\int_G z_V(r)dr = 1$, and (u_i) is any positive bounded approximate left identity of A.

Furthermore, if A has an M-bounded (not necessarily positive) approximate left identity, then the following bounds for $T^{(\pi,U)}$ and (π^T, U^T) hold:

(1) $\|T^{(\pi,U)}\| \leq C_U \|\pi\|$,
(2) $\|\pi^T\| \leq (\inf_{V\in\mathcal{Z}} \sup_{r\in V} \omega(r)) \|T\|$,
(3) $\|U_s^T\| \leq M (\inf_{V\in\mathcal{Z}} \sup_{r\in V} \omega(r)) \|T\|\omega(s) \quad (s \in G)$.

In the case where $(A, G, \alpha) = (\mathbb{R}, G, \mathrm{triv})$ with a weight ω on G, by Theorem 5.8 we obtain the (here rather obvious fact) fact that the classical Beurling algebra $L^1(G, \omega)$ is a Banach lattice algebra. Furthermore, Theorem 5.9 gives a bijection between the positive strongly continuous group representations of G on pre-ordered Banach spaces with closed cones that are bounded by a multiple of ω, and the positive non-degenerate bounded representations of $L^1(G, \omega)$ on such spaces. We hence obtain the following adaptation of [7, Corollary 5.22] to the ordered context:

Corollary 5.10. *Let ω be a weight on G. Let (z_V) be as in Theorem 5.9. The maps*

$$U \mapsto \left(f \mapsto \int_G f(r)U_r \, dr\right) =: T^U \quad (f \in C_c(G)),$$

determining a positive non-degenerate bounded representation T^U of the ordered Beurling algebra $L^1(G, \omega)$, and

$$T \mapsto \big(s \mapsto \text{SOT-lim}_V T(z_V(s^{-1}\cdot))\big) =: U^T$$

are mutual inverses between the positive strongly continuous group representations U of G on a pre-ordered Banach space X with closed cone, satisfying $\|U_r\| \leq C_U\omega(r)$, for some $C_U \geq 0$ and all $r \in G$, and the positive non-degenerate bounded representations $T : L^1(G, \omega) \to B(X)$ of the ordered Beurling algebra $L^1(G, \omega)$ on X.

If the weight satisfies $\inf_{W\in\mathcal{Z}} \sup_{r\in W} \omega(r) = 1$, where $\mathcal{Z}$ is a neighbourhood base of $e \in G$, of which all elements are contained in a fixed compact subset of G, then $\|T^U\| = \sup_{r\in G} \|U_r\|/\omega(r)$ and $\|U_r^T\| \leq \|T\|\omega(r)$ for all $r \in G$.

As a particular case, the uniformly bounded positive strongly continuous representations of G on a pre-ordered Banach space X with a closed cone are in natural bijection with the positive non-degenerate bounded representations of $L^1(G)$ on X; this also follows from [10, Assertion VI.1.32].

Finally, we note that [7, Theorem 8.3] gives a bijection between the non-degenerate bounded anti-representations of $L^1(G, A, \omega; \alpha)$ on Banach spaces, for a Banach algebra dynamical system (A, G, α) where A has a bounded two-sided approximate identity and α is uniformly bounded, and suitable (not covariant!) pairs (π, U) of anti-representations of A and G. As done above for [7, Theorem 5.20], an ordered version can be derived from this, but this is left to the reader for reasons of space.

Acknowledgements

The authors would like to thank Anthony Wickstead for helpful comments leading to Remark 2.9. Messerschmidt's research was supported by a Vrije Competitie grant of the Netherlands Organisation for Scientific Research (NWO).

References

1. C.D. Aliprantis and K.C. Border, *Infinite dimensional analysis*, third ed., Springer, Berlin, 2006.
2. T. Andô, *On fundamental properties of a Banach space with a cone*, Pacific J. Math. **12** (1962), 1163–1169.
3. L. Asimow and H. Atkinson, *Dominated extensions of continuous affine functions with range an ordered Banach space*, Quart. J. Math. Oxford Ser. (2) **23** (1972), 383–389.
4. C.J.K. Batty and D.W. Robinson, *Positive one-parameter semigroups on ordered Banach spaces*, Acta Appl. Math. **2** (1984), 221–296.
5. D.L. Cohn, *Measure theory*, Birkhäuser, Boston, Mass., 1980.
6. M. de Jeu and M. Messerschmidt, *Right inverses of surjections from cones onto Banach spaces*, Arxiv preprint arXiv:1302.2822 (2013).
7. M. de Jeu, M. Messerschmidt, and M. Wortel, *Crossed products of Banach algebras. II.*, Arxiv preprint arXiv:1305.2304 (2013).
8. S. Dirksen, M. de Jeu, and M. Wortel, *Extending representations of normed algebras in Banach spaces*, Operator structures and dynamical systems (Leiden, 2008), Contemp. Math. **503**, American Mathematical Society, Providence, RI, 2009, pp. 53–72.
9. S. Dirksen, M. de Jeu, and M. Wortel, *Crossed products of Banach algebras. I.*, to appear in Dissertationes Math., Arxiv preprint arXiv:1104.5151 (2011).
10. A. Ya. Helemski, *Banach and locally convex algebras*, Oxford University Press, Oxford, 1993.
11. E. Kaniuth, *A course in commutative Banach algebras*, Springer, New York, 2009.
12. M. Krein, *Propriétés fondamentales des ensembles coniques normaux dans l'espace de Banach*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **28** (1940), 13–17.
13. M. Messerschmidt, *Normality and conormality of pre-ordered banach spaces, normality of spaces of operators, and quasi-lattices*, to appear.
14. I. Raeburn, *On crossed products and Takai duality*, Proc. Edinburgh Math. Soc. (2) **31** (1988), no. 2, 321–330.
15. W. Rudin, *Functional analysis*, second ed., McGraw-Hill Inc., New York, 1991.
16. H.H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, New York, 1974.
17. B. Walsh, *Ordered vector sequence spaces and related classes of linear operators*, Math. Ann. **206** (1973), 89–138.
18. A.W. Wickstead, *Spaces of linear operators between partially ordered Banach spaces*, Proc. London Math. Soc. (3) **28** (1974), 141–158.
19. S. Yamamuro, *On linear operators on ordered Banach spaces*, Bull. Austral. Math. Soc. **27** (1983), no. 2, 285–305.

Marcel de Jeu, Mathematical Institute, Leiden University, P.O. Box 9512, 2300 RA Leiden, The Netherlands
E-mail address: mdejeu@math.leidenuniv.nl

Miek Messerschmidt, Mathematical Institute, Leiden University, P.O. Box 9512, 2300 RA Leiden, The Netherlands
E-mail address: mmesserschmidt@gmail.com