Gibbs-non-Gibbs dynamical transitions for mean-field interacting Brownian motions

F. den Hollander [1]
F. Redig [2]
W. van Zuijlen [1]

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Abstract

We consider a system of real-valued spins interacting with each other through a mean-field Hamiltonian that depends on the empirical magnetization of the spins via a general potential. The system is subjected to a stochastic dynamics where the spins perform independent Brownian motions. As in [9], which considers the Curie-Weiss model with Ising spins interacting via a quadratic potential and subjected to independent spins flips, we follow the program outlined in [7]. We show that in the thermodynamic limit the system is non-Gibbs at time $t \in (0, \infty)$ if and only if there exists an $\alpha \in \mathbb{R}$ such that the large deviation rate function for the trajectory of the magnetization conditional on hitting the value α at time t has multiple global minimizers. We further show that different minimizing trajectories are different at time $t = 0$. We give conditions on the potential under which the system is Gibbs at time $t = 0$, classify the possible scenarios of being Gibbs at time $t \in (0, \infty)$ in terms of the second difference quotient of the potential, and show that the system cannot become Gibbs once it has become non-Gibbs, i.e., there is a unique and explicitly computable crossover time $t_c \in [0, \infty]$ from Gibbs to non-Gibbs. We give examples of immediate loss of Gibbsianness ($t_c = 0$), short-time conservation of Gibbsianness, large-time loss of Gibbsianness ($t_c \in (0, \infty)$), and preservation of Gibbsianness ($t_c = \infty$). Depending on the potential, the system can be Gibbs or non-Gibbs at the cross-over time time $t = t_c$.

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1 Introduction and main results

1.1 Background

Gibbs states are mathematical tools to describe physical interacting particle systems. When such systems evolve over time according to a stochastic dynamics, it may happen that the time-evolved state no longer is Gibbs. This phenomenon was originally discovered and described

[1] Mathematical Institute, Leiden University, P.O. Box 9512, 2300 RA, Leiden, The Netherlands.
[2] Delft Institute of Applied Mathematics, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands.

for heating dynamics by van Enter, Fernández, den Hollander and Redig [6]. In this paper, a low-temperature Ising model is subjected to a high-temperature Glauber spin-flip dynamics. The state remains Gibbs for short times, but becomes non-Gibbs after a finite time. If the magnetic field is zero, then Gibbsianness once lost is never recovered. But if the magnetic field is non-zero and small enough, then Gibbsianness is recovered at later times.

By now results of this type are available for a variety of interacting particle systems, both in the lattice setting and in the mean-field setting. Both for heating dynamics and for cooling dynamics estimates are available on transition times, as well as characterizations of the so-called *bad configurations* leading to non-Gibbsianness (i.e., the "points of essential discontinuity of the conditional probabilities"). It has become clear that Gibbs-non-Gibbs transitions are the rule rather than the exception. We refer the reader to the recent overview by van Enter [5].

The ubiquity of the Gibbs-non-Gibbs phenomenon calls for a better understanding of its causes and consequences. In many papers non-Gibbsianness is proved by looking at the evolving system at two times, the initial time and the final time, and applying techniques from equilibrium statistical mechanics. This is a *static* approach that does not illuminate the relation between the Gibbs-non-Gibbs phenomenon and the dynamical effects responsible for its occurrence, in particular, the *nature-versus-nurture* scenario suggested in [6]. This unsatisfactory situation was addressed in Enter, Fernández, den Hollander and Redig [7], where possible dynamical mechanisms were proposed and a *program* was put forward to develop a theory of Gibbs-non-Gibbs transitions in terms of *large deviations for trajectories of relevant physical quantities.*

Fernández, den Hollander and Martínez [9], [10], building on earlier work by Külske and Le Ny [12] and Ermolaev and Külske [8], showed that this program can be fully carried out for the Curie-Weiss model of Ising spins subjected to an infinite-temperature spin-flip dynamics, and also for a Kac-type version of the Curie-Weiss model. The present paper extends the work on this program to systems of continuous spins that interact with each other through a *general* mean-field interaction potential and perform independent *Brownian motions*. The fact that we consider Brownian motions allows us to obtain a *complete characterization* of passages from Gibbs to non-Gibbs. The key notions of interest are *good magnetizations* and *bad magnetizations* in the thermodynamic limit. Gibbsianness corresponds to having only good magnetizations, while non-Gibbsianness corresponds to having at least one bad magnetization.

1.2 Outline

The definition of Gibbs for mean-field models differs from that for lattice models because the interaction depends on the size of the system. In Section 1.3 we introduce the notions of mean-field Gibbs with a potential, good magnetizations, bad magnetizations and *sequentially Gibbs*, and show that mean-field Gibbs with a differentiable potential whose derivative is locally bounded implies sequentially Gibbs. In Section 1.4 we define the Brownian motion dynamics. We show that a magnetization $\alpha \in \mathbb{R}$ is bad at time t if and only if the large deviation rate function for the magnetization at time 0 conditional on the magnetization at time t being α has multiple global minimizers. We further show that the system is sequentially Gibbs at time t if and only if all magnetizations are good at time t. In Section 1.5 we show that a magnetization α is bad at time t if and only if the large deviation rate function for the *trajectory* of the magnetization conditional on hitting the value α at time t has multiple global minimizers. We further show that different minimizing trajectories are different at time 0. In Section 1.6 we show that Gibbsianness can be classified in terms of the *second difference quotient of the potential.* With the help of this classification we show that there exists a unique time $t_c \in [0, \infty]$ at which the system changes from Gibbs to non-Gibbs, and give a characterization of t_c in terms of the potential associated with the starting measures. In Section 1.7 we give examples for which $t_c = 0$, $t_c \in (0, \infty)$ and $t_c = \infty$. In Section 1.8 we discuss our results and indicate possible future research. Proofs are given in Sections 2–5. Appendix A collects a few key formulas that are needed along the way.

1.3 Mean-field Gibbs, Potential, Sequentially Gibbs

In this section we give the definition of mean-field Gibbs sequences (Definition 1.1), and of good/bad magnetizations and sequentially Gibbs sequences (Definition 1.2). We show that a sequentially Gibbs sequence has a weakly continuous specification kernel (Lemma 1.3). We show that for all differentiable potentials with locally bounded derivative mean-field Gibbs sequences are sequentially Gibbs (Theorem 1.4).

In what follows, we write $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_{\geq 2} = \mathbb{N} \setminus \{1\}$. For $n \in \mathbb{N}$, $\mathcal{B}(\mathbb{R}^n)$ denotes the Lebesgue measurable subsets of $\mathbb{R}^n$, and $\mu_{\mathcal{N}(v,A)}$ denotes the normal distribution on $\mathcal{B}(\mathbb{R}^n)$ with mean vector $v \in \mathbb{R}^n$ and covariance matrix $A \in \mathbb{R}^{n\times n}$. We write I_n for the identity matrix in $\mathbb{R}^{n\times n}$. For $\alpha \in \mathbb{R}$ and $\epsilon > 0$, $B(\alpha, \epsilon)$ denotes the open ball of radius ϵ centered at α.

Definition 1.1. For $n \in \mathbb{N}$, the *empirical magnetization* $m_n \colon \mathbb{R}^n \to \mathbb{R}$ is given by

$$m_n(x_1, \dots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i \qquad \big((x_1, \dots, x_n) \in \mathbb{R}^n\big). \tag{1.1}$$

For $n \in \mathbb{N}$, let μ_n be a probability measure on $\mathcal{B}(\mathbb{R}^n)$. Let $V \colon \mathbb{R} \to [0, \infty)$ be a Borel measurable function. The sequence $(\mu_n)_{n\in\mathbb{N}}$ is called *mean-field Gibbs* with *mean-field interaction potential* V and *reference measures* $\mu_{\mathcal{N}(0,I_n)}$ if

$$\mu_n(A) = \frac{1}{Z_n} \int_{\mathbb{R}^n} \mathbf{1}_A(x)\, e^{-n(V\circ m_n)(x)}\, \mathrm{d}\mu_{\mathcal{N}(0,I_n)}(x) \qquad (A \in \mathcal{B}(\mathbb{R}^n), n \in \mathbb{N}), \tag{1.2}$$

where $Z_n \in (0, \infty)$ is the *normalizing constant.*

Note that μ_n in (1.2) does not change when V is replaced by $V + c$ for some $c \in \mathbb{R}$. Therefore our assumption that $V \geq 0$ is equivalent to the assumption that V is bounded from below.

Definition 1.2. For $n \in \mathbb{N}$, let ρ_n be a probability measure on $\mathcal{B}(\mathbb{R}^n)$, and let $\pi_{(2:n)} \colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ be defined by

$$\pi_{(2:n)}(y_1, \dots, y_n) = (y_2, \dots, y_n) \qquad \big((y_1, \dots, y_n) \in \mathbb{R}^n\big). \tag{1.3}$$

Suppose that for every $n \in \mathbb{N}_{\geq 2}$ there exists a weakly continuous proper regular conditional probability $\gamma_n \colon \mathbb{R}^{n-1} \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ under ρ_n of the first spin given the other spins, i.e., γ_n is the unique weakly continuous probability kernel for which (see Appendix B)

$$\rho_n(A \times B) = \int_{\mathbb{R}^n} \mathbf{1}_B(y_2, \dots, y_n)\, \gamma_n\big((y_2, \dots, y_n), A\big)\, \mathrm{d}\left[\rho_n \circ \pi_{(2:n)}^{-1}\right](y_2, \dots, y_n) \qquad (A \in \mathcal{B}(\mathbb{R}),\ B \in \mathcal{B}(\mathbb{R}^{n-1})). \tag{1.4}$$

- $\alpha \in \mathbb{R}$ is called a *good magnetization* for the sequence $(\rho_n)_{n\in\mathbb{N}}$ when there exists a probability measure $\gamma_\alpha\colon \mathcal{B}(\mathbb{R}) \to [0,1]$ for which the sequence of measures $(\gamma_n(v_{n-1},\cdot))_{n\in\mathbb{N}_{\geq 2}}$ weakly converges to γ_α for all sequences $(v_{n-1})_{n\in\mathbb{N}_{\geq 2}}$ with $v_{n-1} \in \mathbb{R}^{n-1}$ for which the empirical magnetization of v_n converges to α, i.e., $m_{n-1}(v_{n-1}) \to \alpha$.
- $\alpha \in \mathbb{R}$ is called a *bad magnetization* when it is not a good magnetization.
- The sequence $(\rho_n)_{n\in\mathbb{N}}$ is called *sequentially Gibbs* when all $\alpha \in \mathbb{R}$ are good magnetizations.

The notion of Gibbs for a mean-field model was introduced by Kulske and Le Ny [12, Definition 2.1] and is the same as our definition of sequentially Gibbs (even though our definition of good magnetization is slightly different).

The following lemma shows that, in the thermodynamic limit $n \to \infty$, the probability measure of the first spin given the magnetization of the other spins is a transition kernel that depends weakly continuously on the magnetization of the other spins. This lemma will be proved in Section 2.

Lemma 1.3. *Let $(\rho_n)_{n\in\mathbb{N}}$ be sequentially Gibbs. With the same notation as in Definition* 1.2 *define $\gamma\colon \mathbb{R}\times\mathcal{B}(\mathbb{R}) \to [0,1]$ by letting $\gamma(\alpha,\cdot) = \gamma_\alpha$. Then $\alpha \mapsto \gamma(\alpha,\cdot)$ is weakly continuous and, consequently, γ is a transition kernel (called the* specification kernel*).*

Our first main result, whose proof will be given in Section 3, shows that for a differentiable potential with a locally bounded derivative, mean-field Gibbs implies sequentially Gibbs.

Theorem 1.4. *Let $(\mu_n)_{n\in\mathbb{N}}$ be mean-field Gibbs with potential $V\colon \mathbb{R} \to [0,\infty)$.*

(a) *Define $\overline{\gamma}_n\colon \mathbb{R}\times\mathcal{B}(\mathbb{R}) \to [0,1]$ by*

$$\overline{\gamma}_n(\alpha, A) = \frac{\int_{\mathbb{R}} \mathbf{1}_A(x) e^{-nV(\frac{n-1}{n}\alpha + \frac{x}{n})} e^{-x^2/2}\,\mathrm{d}x}{\int_{\mathbb{R}} e^{-nV(\frac{n-1}{n}\alpha + \frac{y}{n})} e^{-y^2/2}\,\mathrm{d}y} \qquad (\alpha \in \mathbb{R}, A \in \mathcal{B}(\mathbb{R})). \tag{1.5}$$

Then $\gamma_n\colon \mathbb{R}^{n-1}\times\mathcal{B}(\mathbb{R}) \to [0,1]$ defined by $\gamma_n(v,A) = \overline{\gamma}_n(m_{n-1}(v),A)$ for $v \in \mathbb{R}^{n-1}$ and $A \in \mathcal{B}(\mathbb{R})$ is the weakly continuous proper conditional probability under ρ_n of the first spin given the other spins.

(b) *If V is differentiable and V' is bounded on a neighborhood of $\alpha \in \mathbb{R}$, then $\overline{\gamma}_n(\alpha_n,\cdot)$ converges weakly (even strongly) to $\mu_{\mathcal{N}(-V'(\alpha),1)}$ for all sequences $(\alpha_n)_{n\in\mathbb{N}}$ that converge to α (in particular, α is a good magnetization for $(\mu_n)_{n\in\mathbb{N}}$).*

(c) *If V is differentiable and V' is locally bounded, then $(\mu_n)_{n\in\mathbb{N}}$ is sequentially Gibbs.*

In Section 1.7 we give an example of a non-differentiable potential for which $(\mu_n)_{n\in\mathbb{N}}$ is mean-field Gibbs but not sequentially Gibbs (Example 1.17).

1.4 Brownian motion dynamics

In this section we introduce the Brownian motion dynamics, give the essential tools for identifying good magnetizations (Lemma 1.5) and global minimizers of a certain tilted form of the potential (Lemma 1.6), and show that a magnetization is good if and only if the tilted potential has a unique global minimizer (Theorem 1.7).

For $n \in \mathbb{N}$, $\mu_{n,0}$ represents the law of the n spins at time $t = 0$. We assume that $(\mu_{n,0})_{n\in\mathbb{N}}$ is a mean-field Gibbs sequence with potential V. Let $\mu_{n,t}$ be the evolved law at time $t \in (0, \infty)$ when the n spins perform independent Brownian motions, i.e.,

$$\mu_{n,t}(A) = \frac{1}{Z_n} \int_{\mathbb{R}^n} p_n(t, z, A)\, e^{-n(V\circ m_n)(z)}\ \mathrm{d}\mu_{\mathcal{N}(0,I_n)}(z) \qquad (A \in \mathcal{B}(\mathbb{R}^n)) \tag{1.6}$$

(recall (1.2)), where

$$p_n(t, z, A) = \mu_{\mathcal{N}(z,tI_n)}(A) = (2\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathbf{1}_A(y)\, e^{-\frac{\|y-z\|^2}{2t}}\ \mathrm{d}y \qquad (z \in \mathbb{R}^n, A \in \mathcal{B}(\mathbb{R}^n)). \tag{1.7}$$

There exists a weakly continuous proper regular conditional probability $\gamma_{n,t}$ under $\mu_{n,t}$ of the first spin given the other spins, for which $\gamma_{n,t}(u, \cdot) = \gamma_{n,t}(v, \cdot)$ for all $u, v \in \mathbb{R}^{n-1}$ with $m_{n-1}(u) = m_{n-1}(v)$ (a proof and an expression for $\gamma_{n,t}$ are given in Appendix A). Therefore we can determine whether or not $(\mu_{n,t})_{n\in\mathbb{N}}$ is sequentially Gibbs by looking at the sequence $(\overline{\gamma}_{n,t})_{n\in\mathbb{N}}$ of probability kernels $\mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0, 1]$, where $\overline{\gamma}_{n,t}(\alpha, \cdot) = \gamma_{n,t}(v, \cdot)$ for all $v \in \mathbb{R}^{n-1}$ and $\alpha \in \mathbb{R}$ with $m_{n-1}(v) = \alpha$ (an expression for $\overline{\gamma}_{n,t}$ is given in Appendix A). This is formalized in the following lemma.

Lemma 1.5. *Let $t \in (0, \infty)$. Then $\alpha \in \mathbb{R}$ is a good magnetization for $(\mu_{n,t})_{n\in\mathbb{N}}$ if and only if there exists a measure $\gamma_\alpha\colon \mathcal{B}(\mathbb{R}) \to [0, 1]$ such that the sequence $(\overline{\gamma}_{n,t}(\alpha_n, \cdot))_{n\in\mathbb{N}}$ converges weakly to γ_α for all sequences $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ that converge to α.*

$\eta_{n,t}\colon\ \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ defined for $n \in \mathbb{N}$ and $t \in (0, \infty)$ by

$$\eta_{n,t}(\alpha, A) = \frac{\int_{\mathbb{R}} \mathbf{1}_A(s)\, e^{-n\left[V(s)+\frac{s^2}{2}+\frac{(s-\alpha)^2}{2t}\right]}\ \mathrm{d}s}{\int_{\mathbb{R}} e^{-n\left[V(s)+\frac{s^2}{2}+\frac{(s-\alpha)^2}{2t}\right]}\ \mathrm{d}s} \qquad (\alpha \in \mathbb{R}, A \in \mathcal{B}(\mathbb{R})). \tag{1.8}$$

is the weakly continuous proper regular conditional probability of the magnetization at time 0 given the magnetization at time t (see Appendix A). By Ellis [4, Theorem II.7.2] or den Hollander [11, Theorem III.17], the sequence $(\eta_{n,t}(\alpha, \cdot))_{n\in\mathbb{N}}$ satisfies the large deviation principle with rate n and rate function

$$r \mapsto V(r) + \frac{r^2}{2} + \frac{(r-\alpha)^2}{2t} - \inf_{s\in\mathbb{R}} \Big[V(s) + \frac{s^2}{2} + \frac{(s-\alpha)^2}{2t}\Big]. \tag{1.9}$$

With this notation, $\overline{\gamma}_{n,t}$ can be written as (see Appendix A)

$$\overline{\gamma}_{n,t}(\alpha, B) = \frac{\int_{\mathbb{R}} \mu_{\mathcal{N}(s,t)}(B)\ g_{n,t}(\alpha, s)\ \mathrm{d}\mu_{\mathcal{N}(0,1)}(s)}{\int_{\mathbb{R}} g_{n,t}(\alpha, s)\ \mathrm{d}\mu_{\mathcal{N}(0,1)}(s)} \qquad (\alpha \in \mathbb{R}, B \in \mathcal{B}(\mathbb{R})), \tag{1.10}$$

where $g_{n,t}\colon\ \mathbb{R}^2 \to \mathbb{R}$ is given by

$$g_{n,t}(\alpha, s) = \frac{\int_{\mathbb{R}} e^{-n\left[V(r+\frac{1}{n}(s-r))-V(r)\right]}\ e^{-V(r)}\ \mathrm{d}[\eta_{n-1,t}(\alpha, \cdot)]\,(r)}{\int_{\mathbb{R}} e^{-V(r)}\ \mathrm{d}[\eta_{n-1,t}(\alpha, \cdot)]\,(r)} \qquad (\alpha, s \in \mathbb{R}). \tag{1.11}$$

The following lemma will be proved in Section 4.

Lemma 1.6. *Let $V \in C^1(\mathbb{R}, [0, \infty))$, $t \in (0, \infty)$ and $\alpha \in \mathbb{R}$.*

(a) *If* (1.9) *has a unique global minimizer* $q \in \mathbb{R}$, *then there exists a* $\mu_{\mathcal{N}(0,1)}$*-integrable function* $h\colon \mathbb{R} \to [0,\infty)$ *such that*

$$(1.12) \qquad \begin{aligned} g_{n,t}(\alpha_n, s) &\to e^{-sV'(q)} && (s \in \mathbb{R}), \\ g_{n,t}(\alpha_n, s) &\le h(s) && (n \in \mathbb{N}, s \in \mathbb{R}), \end{aligned}$$

for all sequences $(\alpha_n)_{n\in\mathbb{N}}$ *that converge to* α.

(b) *Let* q_1, q_2 *be the smallest, respectively, the largest global minimizer of* (1.9). *Then there exists a* $\mu_{\mathcal{N}(0,1)}$*-integrable function* $h\colon \mathbb{R} \to [0,\infty)$, *and sequences* $(\alpha_n)_{n\in\mathbb{N}}$ *and* $(\beta_n)_{n\in\mathbb{N}}$ *both converging to* α, *for which* (1.12) *holds with* $q = q_1$ *and with* $q = q_2$ *(where* α_n *is replaced by* β_n*).*

Our second main result shows that sequentially Gibbs is equivalent to uniqueness of the global minimizer of (1.9).

Theorem 1.7. *Let* $V \in C^1(\mathbb{R}, [0,\infty))$. *Then for every* $t \in (0,\infty)$

(a) $\alpha \in \mathbb{R}$ *is a good magnetization for* $(\mu_{n,t})_{n\in\mathbb{N}}$ *if and only if* (1.9) *has a unique global minimizer.*

(b) $(\mu_{n,t})_{n\in\mathbb{N}}$ *is sequentially Gibbs if and only if* (1.9) *has a unique global minimizer for all* $\alpha \in \mathbb{R}$.

The claim in Theorem 1.7 follows from Lemma 1.6, (1.10) and Lebesgue's Dominated Convergence Theorem after we note that if $q_1, q_2 \in \mathbb{R}$ with $q_1 \neq q_2$ are global minimizers of (1.9), then $V'(q_1) - V'(q_2) = (q_2 - q_1)(1 + t^{-1}) \neq 0$.

Note that the statements in Theorem 1.7 are independent of whether (V is such that) $(\mu_{n,0})_{n\in\mathbb{N}}$ is sequentially Gibbs or not.

1.5 Trajectories of the magnetization

In this section we consider the probability measure on the set of trajectories of the magnetization between time 0 and time t. We show the equivalence of uniqueness of the minimizing magnetization at time 0 and uniqueness of the minimizing trajectory of the magnetization (Theorem 1.8). This characterizes good and bad magnetizations in terms of the trajectory of the magnetization (Corollary 1.9).

Let μ_n be the law on $C([0,\infty), \mathbb{R}^n)$ of the paths of the independent Brownian motions performed by the n spins with initial distribution $\mu_{n,0}$. Thus, with $P(x,\cdot)$ denoting the law of the Brownian motion on $C([0,\infty),\mathbb{R})$ and $\mathfrak{S}_{C([0,\infty),\mathbb{R}^n)}$ denoting the Skorohod σ-algebra on $C([0,\infty),\mathbb{R})$, we have

$$(1.13) \qquad \mu_n(A) = \int_{\mathbb{R}} \left(\bigotimes_{i=1}^n P(x_i, \cdot) \right)(A) \; \mathrm{d}\mu_{n,0}(x_1, \dots, x_n) \qquad (A \in \mathfrak{S}_{C([0,\infty),\mathbb{R}^n)}).$$

Let $t \in (0,\infty)$. Let $Q_{n,t}\colon \mathbb{R} \times \mathfrak{S}_{C([0,t],\mathbb{R})} \to [0,1]$ be the transition kernel where $Q_{n,t}(s,\cdot)$ is the probability measure of a Brownian motion with variance $\frac{1}{n}$ starting at s. We write m_n also for the function $C([0,t],\mathbb{R}^n) \to C([0,t],\mathbb{R})$ given by

$$(1.14) \qquad m_n(\phi_1, \dots, \phi_n) = \frac{1}{n} \sum_{i=1}^n \phi_i, \qquad (\phi_1, \dots, \phi_n) \in C([0,t],\mathbb{R}^n).$$

Then $Q_{n,t}(s,A) = \left[\otimes_{i=1}^{n} P(x_i,\cdot)\right]\left(\pi_{[0,t]}^{-1}(m_n^{-1}(A))\right)$ for all $A \in \mathfrak{S}_{C([0,t],\mathbb{R})}$ and all $s \in \mathbb{R}$ and $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ with $m_n(x) = s$, where $\pi_{[0,t]}\colon C([0,\infty),\mathbb{R}^n) \to C([0,t],\mathbb{R}^n)$ is given by $\pi_{[0,t]}(\phi) = \phi|_{[0,t]}$. We have

$$\mu_n \circ \pi_{[0,t]}^{-1}\left(m_n^{-1}(A)\right) = \int_{\mathbb{R}} Q_{n,t}(s,A)\, \mathrm{d}\big[\mu_{n,0} \circ m_n^{-1}\big](s) \qquad (A \in \mathfrak{S}_{C([0,t],\mathbb{R})}). \tag{1.15}$$

Let $\pi_t\colon C([0,t],\mathbb{R}) \to \mathbb{R}$ be the projection on the endpoint of the path, i.e., $\pi_t(\phi) = \phi(t)$.

Theorem 1.8. *For every $n \in \mathbb{N}$ there exists a weakly continuous proper regular conditional probability $\rho_n\colon \mathbb{R} \times \mathfrak{S}_{C([0,t),\mathbb{R})} \to [0,1]$ under $\mu_n \circ \pi_{[0,t]}^{-1} \circ m_n^{-1}$ given π_t (given the endpoint of the trajectory). For all $\alpha \in \mathbb{R}$, $(\rho_n(\alpha,\cdot))_{n\in\mathbb{N}}$ satisfies the large deviation principle with rate n and rate function $C([0,t),\mathbb{R}) \to [0,\infty]$ given by*

(1.16)
$$\phi \mapsto \begin{cases} V(\phi(0)) + \frac{\phi(0)^2}{2} + \frac{1}{2}\int_0^t \dot{\phi}^2(s)\,\mathrm{d}s - C_{t,\alpha}, & \text{if } \phi \in \mathcal{AC}([0,t),\mathbb{R}) \text{ and } \lim_{s\uparrow t}\phi(s) = \alpha, \\ \infty, & \text{otherwise,} \end{cases}$$

where $\mathcal{AC}([0,t),\mathbb{R})$ is the set of absolutely continuous functions from $[0,t]$ to $\mathbb{R}$ (restricted to $[0,t)$), and

$$C_{t,\alpha} = \inf_{\phi\in\mathcal{AC}([0,t],\mathbb{R}),\phi(t)=\alpha} V(\phi(0)) + \tfrac{1}{2}\,\phi(0)^2 + \tfrac{1}{2}\int_0^t \dot{\phi}^2(s)\,\mathrm{d}s. \qquad (\alpha \in \mathbb{R}). \tag{1.17}$$

Futhermore, the function

$$\mathcal{AC}([0,t],\mathbb{R}) \to \mathbb{R}, \qquad \phi \mapsto \int_0^t \dot{\phi}^2(s)\,\mathrm{d}s \tag{1.18}$$

is strictly convex. Since the contraction principle implies that the infimum of (1.16) *over all paths $\phi \in C([0,t),\mathbb{R})$ with $\phi(0) = x$ is equal to* (1.9) *(see* (1.15)*), it follows that, for every $\alpha \in \mathbb{R}$,* (1.16) *has a unique global minimizer if and only if* (1.9) *has a unique global minimizer.*

Proof. The existence of ρ_n is proved in Appendix A. The proof that $(\rho_n(\alpha,\cdot))_{n\in\mathbb{N}}$ satisfies the large deviation principle with rate n and rate function (1.16) is similar to the proof of Schilder's Theorem. A sketch of the proof is given in Appendix A. It is easily checked that (1.18) is strictly convex by the inequality $2ab < a^2 + b^2$ for $a,b \in \mathbb{R}$ with $a \neq b$. □

As a consequence of Theorem 1.8, we can refine the result of Theorem 1.7.

Corollary 1.9. *Let $V \in C^1(\mathbb{R},[0,\infty))$. Then for every $t \in (0,\infty)$*

(a) *For $\alpha \in \mathbb{R}$ the following are equivalent:*
   (a1) *$\alpha \in \mathbb{R}$ is a good magnetization for $(\mu_{n,t})_{n\in\mathbb{N}}$,*
   (a2) (1.9) *has a unique global minimizer,*
   (a3) (1.16) *has a unique global minimizer.*

(b) *The following are equivalent:*
   (b1) *$(\mu_{n,t})_{n\in\mathbb{N}}$ is sequentially Gibbs,*
   (b2) (1.9) *has a unique global minimizer for all $\alpha \in \mathbb{R}$,*
   (b3) (1.16) *has a unique global minimizer for all $\alpha \in \mathbb{R}$.*

1.6 Uniqueness of the minimizers of the rate function

In this section we give a necessary and sufficient condition in terms of the second difference quotient of V (Definition 1.10) to have uniqueness of the global minimizers of (1.9) (Theorem 1.11 and Corollary 1.12). From this condition it follows that Gibbsianness can never be recovered once it is lost.

Definition 1.10. Let $f\colon \mathbb{R} \to \mathbb{R}$. The *second difference quotient* of f is the function

$$\begin{aligned} &\Phi_2 f\colon \{(x,y,z) \in \mathbb{R}^3\colon x < y < z\} \to \mathbb{R}, \\ &(x,y,z) \mapsto \frac{1}{z-x}\left(\frac{f(z)-f(y)}{z-y} - \frac{f(y)-f(x)}{y-x}\right). \end{aligned} \tag{1.19}$$

Our third main result, whose proof will given in Section 5, is the following classification of Gibbsianness.

Theorem 1.11. *Fix $t \in (0,\infty)$. There exists an $\alpha \in \mathbb{R}$ for which* (1.9) *has multiple global minimizers if and only if $\Phi_2 V \not> -\frac{1+t}{2t}$, i.e., if and only if there exist $a,b,c \in \mathbb{R}$ with $a < b < c$ for which $\Phi_2 V(a,b,c) \leq -\frac{1+t}{2t}$. Consequently, there exists a crossover time $t_c \in [0,\infty]$ such that $(\mu_{n,t})_{n\in\mathbb{N}}$ is sequentially Gibbs for $t \in (0,t_c)$ and not sequentially Gibbs for $t \in (t_c,\infty)$.*

At $t = t_c$, $(\mu_{n,t})_{n\in\mathbb{N}}$ may be sequentially Gibbs or not sequentially Gibbs. Both scenarios are possible (see Example 1.14). Theorem 1.11 yields the following.

Corollary 1.12. *Fix $t \in (0,\infty)$. For all $\alpha \in \mathbb{R}$,* (1.9) *has a unique global minimizer if and only if $\Phi_2 V > -\frac{1+t}{2t}$. Consequently, the following scenarios occur (where $M \in (\frac{1}{2},\infty)$):*

- (a) $(\mu_{n,t})_{n\in\mathbb{N}}$ *is sequentially Gibbs*
  - (a1) *for* $t \in (0,\infty)$ *when* $\Phi_2 V \geq -\frac{1}{2}$,
  - (a2) *for* $t \in (0,(M-\frac{1}{2})^{-1})$ *when* $\Phi_2 V \geq -M$,
  - (a3) *for* $t \in (0,(M-\frac{1}{2})^{-1}]$ *when* $\Phi_2 V > -M$.
- (b) $(\mu_{n,t})_{n\in\mathbb{N}}$ *is not sequentially Gibbs*
  - (b1) *for* $t \in ((M-\frac{1}{2})^{-1},\infty)$ *when* $\Phi_2 V \not\geq -M$,
  - (b2) *for* $t \in [(M-\frac{1}{2})^{-1},\infty)$ *when* $\Phi_2 V \not> -M$,
  - (b3) *for* $t \in (0,\infty)$ *when* $\Phi_2 V$ *is not bounded from below.*

Note that if V is convex, then $(\mu_{n,t})_{n\in\mathbb{N}}$ is sequentially Gibbs for all $t \in (0,\infty)$. We will see at the end of Section 5 that if $V \in C^2(\mathbb{R},[0,\infty))$, then (a1),(a2) and (b1),(b2) hold with $\Phi_2 V$ replaced by V''.

1.7 Examples

In this section we give examples of continuously differentiable potentials for each of the scenarios described in Corollary 1.12 (Examples 1.13–1.16).

Example 1.13. [Polynomial potentials: $t_c \in (0,\infty]$, sequentially Gibbs at $t = t_c$] Let $m \in \mathbb{N}$, $a_{2m} \in (0,\infty)$, $a_{2m-1},\ldots,a_2,a_1 \in \mathbb{R}$. Let $a_0 \in \mathbb{R}$ be such that

$$V\colon \mathbb{R} \to \mathbb{R},\ r \mapsto a_m r^{2m} + a_{m-1} r^{2m-2} + \cdots a_1 r^2 + a_0 \tag{1.20}$$

satisfies $V \geq 0$. Since V'' is a polynomial of even degree, it is bounded from below, say $V'' \geq -M$ for some $M \in (0, \infty)$. Hence, if V is such a polynomial, then the crossover time t_c is strictly positive, i.e., $t_c \in (0, \infty]$. For example, for the potentials $V(r) = 0$, $V(r) = r^2$ and $V(r) = r^4 - \frac{1}{2}r^2 + 1$, $(\mu_{n,t})_{n\in\mathbb{N}}$ is sequentially Gibbs for all $t \in [0, \infty)$, while for the potentials $V(r) = r^4 - 4r^2 + 3$ and $V(r) = (r^2 - 9)^2$ there exists a $t_c \in (0, \infty)$ for which $(\mu_{n,t})_{n\in\mathbb{N}}$ is sequentially Gibbs for $t \in [0, t_c)$ and not sequentially Gibbs for $t \in (t_c, \infty)$.

If $m = 1$, then $\Phi_2 V = a_1 > 0$. Hence $t_c = \infty$ by Corollary 1.12.

If $m \geq 2$, then V'' is a polynomial of even degree at least 2. Hence, if $\beta = -\frac{1}{2}\inf_{r\in\mathbb{R}} V''(r)$, then the set $\{r \in \mathbb{R}\colon V''(r) = -2\beta\}$ is finite. By Lemmas 5.9–5.10, we therefore have that $\Phi_2 V > -\beta$ and $\Phi_2 \not\geq -M$ for all $M < \beta$. So if $\beta \in (-\infty, \frac{1}{2}]$, then $t_c = \infty$, while if $\beta \in (\frac{1}{2}, \infty)$, then $t_c = (\beta - \frac{1}{2})^{-1}$ and $(\mu_{n,t})_{n\in\mathbb{N}}$ is sequentially Gibbs for $t = t_c$ by Corollary 1.12.

Example 1.14. [Other potentials: $t_c \in (0, \infty]$, sequentially Gibbs at $t = t_c$]
Consider the potential $V(r) = 2\beta(1 + \cos r)$ for some $\beta \in (0, \infty)$. Then $V'' \geq -2\beta$ and $V'' \not\geq -M$, and hence $\Phi_2 V \geq -\beta$ and $\Phi_2 V \not\geq -M$ for $M < \beta$ (see Lemma 5.9). So, for $\beta \in (0, \frac{1}{2}]$ we have $t_c = \infty$, while for $\beta \in (\frac{1}{2}, \infty)$ we have $t_c = (\beta - \frac{1}{2})^{-1}$ by Corollary 1.12. Morover, if $\beta \in (\frac{1}{2}, \infty)$, then by Lemma 5.10 it follows that $\Phi_2 V > -\beta$, and hence $(\mu_{n,t})_{n\in\mathbb{N}}$ is sequentially Gibbs for $t = t_c$.

In the previous two examples the sequence $(\mu_{n,t})_{n\in\mathbb{N}}$ is sequentially Gibbs at $t = t_c$. This is not always the case, as we show in Example 1.15 below.

Example 1.15. [Other potentials: $t_c \in (0, \infty]$, not sequentially Gibbs at $t = t_c$]
Let $g\colon \mathbb{R} \to \mathbb{R}$ be given by

$$g(r) = \begin{cases} e^{-\frac{1}{|r|-1} + |r| - 1} & r \in (-\infty, -1) \cup (1, \infty), \\ 0 & r \in [-1, 1]. \end{cases} \tag{1.21}$$

Because

$$\frac{d}{dr} e^{-\frac{1}{r} + r} = (1 + r^{-2}) e^{-\frac{1}{r} + r} \qquad (r \in (0, \infty)), \tag{1.22}$$

by L'Hôpital's rule $\lim_{r\downarrow 0} \frac{d}{dr} e^{-\frac{1}{r}+r} = 0 = \lim_{r\uparrow 0} \frac{d}{dr} 0$. Hence $g \in C^1(\mathbb{R}, [0, \infty))$. Furthermore

$$\frac{d^2}{dr^2} e^{-\frac{1}{r} + r} = r^{-4}(1 - 2r + 2r^2 + r^4) e^{-\frac{1}{r} + r} \geq 0 \qquad (r \in (0, \infty)). \tag{1.23}$$

So g is a convex function with $\Phi_2 g \geq 0$ (see Lemma 5.4) and $\Phi_2 g|_{[-1,1]} = 0$. Hence $\Phi_2 g \not> 0$. Note also that $\lim_{r\to\infty} r^{-2} e^{-\frac{1}{r}+r} = \infty$ by L'Hôpital's rule. Therefore, for all $\beta \in (0, \infty)$,

$$\lim_{|r|\to\infty} g(r) - 2\beta r^2 = \infty. \tag{1.24}$$

Let $\beta \in (0, \infty)$ and consider $V \in C^1(\mathbb{R}, [0, \infty))$ given by

$$V(r) = g(r) - \beta r^2 - C_\beta \qquad (r \in \mathbb{R}), \tag{1.25}$$

where $C_\beta = \inf_{s\in\mathbb{R}} g(s) - \beta s^2$ (which exists because of (1.24)). By Lemma 5.9, $\Phi_2 V \geq -\beta$ and $\Phi_2 V|_{[-1,1]} = -\beta$ and thus also $\Phi_2 V \not> -\beta$. So, for $\beta \in (0, \frac{1}{2}]$ we have $t_c = \infty$, while for $\beta \in (\frac{1}{2}, \infty)$ we have $t_c = (\beta - \frac{1}{2})^{-1}$ and $(\mu_{n,t})_{n\in\mathbb{N}}$ is not sequentially Gibbs for $t = t_c$ by Corollary 1.12.

Example 1.16. [Other potentials: $t_c = 0$]
Consider the potential $V(r) = 1 - \cos(r^2)$. Then

$$
\begin{aligned}
V'(r) &= 2r\sin(r^2) && (r \in \mathbb{R}),\\
V''(r) &= 2\left[2r^2\cos(r^2) - \sin(r^2)\right] && (r \in \mathbb{R}),\\
V''(\pm\sqrt{\pi k}) &= (-1)^k 4\pi k && (k \in \mathbb{N}).
\end{aligned}
\tag{1.26}
$$

Hence $V'' \ngeq -M$ for all $M \in (0,\infty)$, and hence $(\mu_{n,t})_{n\in\mathbb{N}}$ is sequentially Gibbs for $t = 0$ but not for $t \in (0,\infty)$ (see Corollary 1.12).

We end with an example of a sequence that is mean-field Gibbs but not sequentially Gibbs.

Example 1.17. [mean-field Gibbs, not sequentially Gibbs]
Let $V \in C(\mathbb{R},[0,\infty))$ be given by $V(r) = \frac{1}{8}|r|$ for $r \in \mathbb{R}$. Then $(\mu_{n,0})_{n\in\mathbb{N}}$ is mean-field Gibbs but not sequentially Gibbs as we will show here. Indeed, for all sequences $(\alpha_n)_{n\in\mathbb{N}}$,

$$
\begin{aligned}
&\int_{\mathbb{R}} \mathbf{1}_A(r) e^{-nV(\frac{n-1}{n}\alpha_n + \frac{r}{n})} e^{-r^2/2}\, \mathrm{d}r && (A \in \mathcal{B}(\mathbb{R})).\\
&= e^{\frac{(n-1)\alpha_n}{8}} \int_{-\infty}^{-(n-1)\alpha_n} \mathbf{1}_A(r) e^{-\frac{r^2}{2}+\frac{r}{8}}\, \mathrm{d}r + e^{-\frac{(n-1)\alpha_n}{8}} \int_{-(n-1)\alpha_n}^{\infty} \mathbf{1}_A(r) e^{-\frac{r^2}{2}-\frac{r}{8}}\, \mathrm{d}r.
\end{aligned}
\tag{1.27}
$$

Note that

$$
\begin{aligned}
\int_{-\infty}^{-(n-1)\alpha_n} e^{-r^2/2+r}\, \mathrm{d}r &= \int_{-\infty}^{0} e^{-\frac{(r-(n-1)\alpha_n)^2}{2} + \frac{r-(n-1)\alpha_n}{8}}\, \mathrm{d}r\\
&\le e^{-\frac{((n-1)\alpha_n)^2}{2} - \frac{(n-1)\alpha_n}{8}} \int_{-\infty}^{0} e^{-r^2/2} e^{(n-1)\alpha_n r}\, \mathrm{d}r.
\end{aligned}
\tag{1.28}
$$

Hence, if $\alpha_n = (n-1)^{-\frac{1}{2}}$ for $n \ge 2$, then $\alpha_n \downarrow 0$, $(n-1)\alpha_n \to \infty$ and, for $A \in \mathcal{B}(\mathbb{R})$,

$$
0 \le \lim_{n\to\infty} e^{2\frac{(n-1)\alpha_n}{8}} \int_{-\infty}^{-(n-1)\alpha_n} \mathbf{1}_A(r) e^{-\frac{r^2}{2}+\frac{r}{8}}\, \mathrm{d}r \le \lim_{n\to\infty} \sqrt{2\pi} e^{-\frac{((n-1)\alpha_n)^2}{4} - \frac{(n-1)\alpha_n}{8}} = 0,
\tag{1.29}
$$

and hence ($\overline{\gamma}_n$ as in (1.5))

$$
\begin{aligned}
\lim_{n\to\infty} \overline{\gamma}_n(\alpha_n, A) &= \lim_{n\to\infty} \frac{e^{\frac{2(n-1)\alpha_n}{8}} \int_{-\infty}^{-(n-1)\alpha_n} \mathbf{1}_A(r) e^{-\frac{r^2}{2}+\frac{r}{8}}\, \mathrm{d}r + \int_{-(n-1)\alpha_n}^{\infty} \mathbf{1}_A(r) e^{-\frac{r^2}{2}-\frac{r}{8}}\, \mathrm{d}r}{e^{\frac{2(n-1)\alpha_n}{8}} \int_{-\infty}^{-(n-1)\alpha_n} e^{-\frac{r^2}{2}+\frac{r}{8}}\, \mathrm{d}r + \int_{-(n-1)\alpha_n}^{\infty} e^{-\frac{r^2}{2}-\frac{r}{8}}\, \mathrm{d}r}\\
&= \mu_{\mathcal{N}(-1,1)}(A) \qquad (A \in \mathcal{B}(\mathbb{R})).
\end{aligned}
\tag{1.30}
$$

Similarly, if $\alpha_n = -(n-1)^{-\frac{1}{2}}$ for $n \ge 2$, then $\alpha_n \uparrow 0$, $(n-1)\alpha_n \to -\infty$ and $\overline{\gamma}_n(\alpha_n, A) \to \mu_{\mathcal{N}(1,1)}(A)$ for $A \in \mathcal{B}(\mathbb{R})$. From this we conclude that $(\mu_{n,0})_{n\in\mathbb{N}}$ is not sequentially Gibbs.

1.8 Discussion

1. If V has a power series expansion $V(x) = \sum_{k\in\mathbb{N}} J_k x^k$, $x \in \mathbb{R}$, then

$$
-n(V \circ m_n)(x_1,\dots,x_n) = -\sum_{k\in\mathbb{N}} \frac{J_k}{n^{k-1}} \sum_{i_1,\dots,i_k=1}^{n} \prod_{j=1}^{k} x_{i_j},
\tag{1.31}
$$

i.e., the system with n spins has a mean-field k-spin interaction of strength J_k/n^{k-1} for $k \in \mathbb{N}$. The special case with $J_k \geq 0$ for all $k \in \mathbb{N}$ is called the ferromagnetic model.

2. Redig and Wang [15] analyzed our model for a restricted class of potentials. Short-time Gibbsianness (i.e., the time-evolved state is Gibbs up to a strictly positive time) was proved under the condition that the second derivative of the potential exists and is bounded from below. Several scenarios of Gibbs-non-Gibbs transitions were discussed. Our paper considers a very general class of positive potentials and provides the precise connection between bifurcation of minimizing trajectories and loss of Gibbsianness.

3. Our paper contains the first example of an initial Gibbs state and a stochastic dynamics for which there is immediate loss of Gibbsianness. For all the models that were considered in the literature so far short-time Gibbsianness occurs. See e.g. [3], [6], [9], [13], [14].

4. In case the independent Brownian motions are replaced by independent Ornstein-Uhlenbeck processes, we get

$$r \mapsto V(r) + \frac{r^2}{2} + \frac{(e^t r - \alpha)^2}{e^{2t} - 1} - \inf_{s \in \mathbb{R}} V(s) + \frac{s^2}{2} + \frac{(e^t s - \alpha)^2}{e^{2t} - 1} \tag{1.32}$$

instead of (1.9) (cf. [15, Eq. (25)]), and so we obtain completely analogous results (in Corollary 1.12 the condition $\Phi_2 V > -\frac{1+t}{2t}$ is replaced by $\Phi_2 V > -(e^{2t} - 1)^{-1}$). In a forthcoming paper we will investigate what happens when the independent Brownian motions are replaced by independent diffusions.

5. For $n \in \mathbb{N}$ and $t > 0$, we can write $\mu_{n,t}$ as (compare with (1.2))

$$\mu_{n,t}(A) = \frac{1}{Z_n} \int_{\mathbb{R}^n} \mathbf{1}_A(x)\, e^{-n(V_{n,t} \circ m_n)(x)}\, \mathrm{d}\mu_{\mathcal{N}(0,(1+t)I_n)}(x) \qquad (A \in \mathcal{B}(\mathbb{R}^n)) \tag{1.33}$$

with

$$V_{n,t}(r) = -\frac{1}{n} \log \left[\int_{\mathbb{R}} e^{-nV(s)}\, \mathrm{d}\mu_{\mathcal{N}(\frac{r}{1+t}, \frac{t}{n(1+t)})}(s) \right] \qquad (r \in \mathbb{R}) \tag{1.34}$$

(see (A.11) in Appendix A). The sequence

$$\left(\mu_{\mathcal{N}(\frac{r}{1+t}, \frac{t}{n(1+t)})} \right)_{n \in \mathbb{N}} \tag{1.35}$$

satisfies the large deviation principle with rate n and rate function $s \mapsto \frac{1}{2}(s - \frac{r}{1+t})^2 \left(\frac{1+t}{t}\right)$. Therefore, by Varadhan's Lemma (see den Hollander [11, Theorem III.13]),

$$\lim_{n \to \infty} V_{n,t}(r) = \inf_{s \in \mathbb{R}} \left[V(s) + \tfrac{1}{2} \left(s - \frac{r}{1+t} \right)^2 \left(\frac{1+t}{t} \right) \right] \qquad (r \in \mathbb{R}). \tag{1.36}$$

Note that, in the context of Definition 1.1, we are interested in the behavior of μ_n for large n only. Therefore, looking back at Definition 1.1, we may generalize the notion of a mean-field Gibbs sequence by replacing V in (1.2) by V_n, provided V_n converges to V as $n \to \infty$ in an appropriate sense. Then $(\mu_{n,t})_{n \in \mathbb{N}}$ becomes a "generalized" mean-field Gibbs sequence with potential $V_t(r) = \lim_{n \to \infty} V_{n,t}(r)$ as given in (1.36).

2 Proof of Lemma 1.3

Lemma 2.1. *Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces. Let $f_n \colon \mathcal{X} \to \mathcal{Y}$ for $n \in \mathbb{N}$ and suppose that there exists an $f \colon \mathcal{X} \to \mathcal{Y}$ such that, for all $x \in \mathcal{X}$ and for all sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{X}$ with $x_n \to x$ we have $f_n(x_n) \to f(x)$. Then f is continuous.*

Proof. The proof is elementary. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{X}$ that converges to an element $x \in \mathcal{X}$. We first prove that $f_{k_n}(x_n) \to f(x)$ for all strictly increasing sequences $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$. To that end, define the sequence $(y_m)_{m\in\mathbb{N}}$ in $\mathcal{X}$ by putting $y_m = x$ for $m \in \mathbb{N} \setminus \{k_n \colon n \in \mathbb{N}\}$ and $y_{k_n} = x_n$ for $n \in \mathbb{N}$. Then $y_m \to x$, hence $f_m(y_m) \to f(x)$, in particular, $f_{k_n}(x_n) = f_{k_n}(y_{k_n}) \to f(x)$. Since $f_k(x_n) \xrightarrow{k\to\infty} f(x_n)$ for all $n \in \mathbb{N}$, we can find a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ for which $d_{\mathcal{Y}}(f_{k_n}(x_n), f(x_n)) < \frac{1}{n}$ for all $n \in \mathbb{N}$. Hence $d_{\mathcal{Y}}(f(x_n), f(x)) \le d_{\mathcal{Y}}(f_{k_n}(x_n), f(x_n)) + d_{\mathcal{Y}}(f_{k_n}(x_n), f(x)) \to 0$. □

Proof of Lemma 1.3. The proof of weak continuity of the map $\alpha \mapsto \gamma(\alpha, \cdot)$ is an adaptation of the proof of Lemma 2.1. Weak continuity of the map $\alpha \mapsto \gamma(\alpha, \cdot)$ implies continuity of the maps $\alpha \mapsto \int_{\mathbb{R}} f(x) \, \mathrm{d}[\gamma(\alpha, \cdot)](x)$ for $f \in C_b(\mathbb{R})$. For open $A \in \mathcal{B}(\mathbb{R})$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_b(\mathbb{R})$ with $f_n \uparrow \mathbf{1}_A$ (pointwise). It follows that $\int_{\mathbb{R}} f_n(x) \, \mathrm{d}[\gamma(\alpha, \cdot)](x) \uparrow \gamma(\alpha, A)$ for all $\alpha \in \mathbb{R}$ and open A, and so $\alpha \mapsto \gamma(\alpha, A)$ is measurable for all $A \in \mathcal{B}(\mathbb{R})$ (since the open sets generate the Borel sigma-algebra). □

3 Proof of Theorem 1.4

Proof of Theorem 1.4. It is not hard to check that γ_n is a regular conditional probability measure under ρ_n of the first coordinate given the magnetization of the other coordinates. To see that γ_n is proper and weakly continuous, we refer to Appendix B. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence that converges to α. Let $\delta > 0$ be such that V is differentiable and such that V' is bounded on $B(\alpha, 2\delta)$. Then

$$\lim_{n\to\infty} \mathbf{1}_{[-n\delta,n\delta]}(y)\mathbf{1}_A(y)e^{-n[V(\alpha_n+\frac{y}{n})-V(\alpha)]} = \mathbf{1}_A(y)e^{-yV'(\alpha)} \qquad (y \in \mathbb{R}, A \in \mathcal{B}(\mathbb{R})). \tag{3.1}$$

Let $N \in \mathbb{N}$ be such that $\alpha_n \in B(\alpha, \delta)$ for all $n \ge N$. Then, by the mean value theorem,

$$e^{-n[V(\alpha_n+\frac{y}{n})-V(\alpha)]} \le e^{\sup_{s\in B(\alpha,2\delta)} |V'(s)||y|} \qquad (y \in [-n\delta, n\delta], n \ge N). \tag{3.2}$$

Since $y \mapsto e^{\sup_{s\in B(\alpha,2\delta)} |V'(s)||y|}$ is $\mu_{\mathcal{N}(0,1)}$-integrable, Lebesgue's Dominated Convergence Theorem implies

$$\lim_{n\to\infty} \int_{[-n\delta,n\delta]} \mathbf{1}_A(y)\, e^{-n[V(\alpha_n+\frac{y}{n})-V(\alpha)]} e^{-y^2/2} \, \mathrm{d}y = \int_{\mathbb{R}} \mathbf{1}_A(y)\, e^{-yV'(\alpha)}\, e^{-y^2/2} \, \mathrm{d}y \qquad \big(A \in \mathcal{B}(\mathbb{R})\big). \tag{3.3}$$

Note that

$$\lim_{n\to\infty} \int_{[-n\delta,n\delta]^c} e^{-yV'(\alpha)} e^{-y^2/2} \, \mathrm{d}y = 0. \tag{3.4}$$

Furthermore (because $n \le e^n$, $n^2 = n(n-1)+n$ and $V \ge 0$)

$$\int_{[-n\delta,n\delta]^c} e^{-n[V(\alpha_n+\frac{y}{n})-V(\alpha)]} e^{-y^2/2}\,\mathrm{d}y = n\int_{[-\delta,\delta]^c} e^{-n[V(\alpha_n+z)-V(\alpha)]} e^{-n^2z^2/2}\,\mathrm{d}z \tag{3.5}$$
$$\le n\int_{[-\delta,\delta]^c} e^{nV(\alpha)} e^{-n^2z^2/2}\,\mathrm{d}z \le e^{-n\left[\frac{n-1}{2}\delta^2-(V(\alpha)+1)\right]}\int_{[-\delta,\delta]^c} e^{-nz^2/2}\,\mathrm{d}z,$$

where the last term converges to 0 as $n\to\infty$. So, by (3.3) - (3.5),

$$\int_{\mathbb{R}} \mathbf{1}_A(y)\, e^{-n[V(\alpha_n+\frac{y}{n})-V(\alpha)]}\, e^{-y^2/2}\,\mathrm{d}y \to \int_{\mathbb{R}} \mathbf{1}_A(y)\, e^{-yV'(\alpha)} e^{-y^2/2}\,\mathrm{d}y \qquad \big(A\in\mathcal{B}(\mathbb{R})\big), \tag{3.6}$$

and hence, by (1.5), $\lim_{n\to\infty}\overline{\gamma}_n(\alpha_n,A) = \mu_{\mathcal{N}(-V'(\alpha),1)}(A)$ for all $A\in\mathcal{B}(\mathbb{R})$, i.e., $(\overline{\gamma}_n(\alpha_n,\cdot))_{n\in\mathbb{N}}$ converges strongly (and hence weakly) to $\mu_{\mathcal{N}(-V'(\alpha),1)}$. □

4 Proof of Lemma 1.6

Section 4.1 contains two preparatory lemmas (Lemmas 4.1–4.2) that provide estimates on $g_{n,t}$ in (1.11). These lemmas will be needed in Section 4.2 to give the proof.

4.1 Two preparatory lemmas

Define $I_{t,\alpha}\colon \mathbb{R}\to[0,\infty)$ for $t\in(0,\infty)$ and $\alpha\in\mathbb{R}$ by

$$I_{t,\alpha}(r) = V(r) + \left(r-\frac{\alpha}{1+t}\right)^2\frac{1+t}{2t} \qquad (r\in\mathbb{R}). \tag{4.1}$$

Note that $r\mapsto I_{t,\alpha}(r) - \inf_{s\in\mathbb{R}} I_{t,\alpha}(s)$ is equal to (1.9). Hence (see (1.8))

$$\eta_{n,t}(\alpha,A) = \frac{\int_{\mathbb{R}} \mathbf{1}_A(s)\, e^{-nI_{t,\alpha}(s)}\,\mathrm{d}s}{\int_{\mathbb{R}} e^{-nI_{t,\alpha}(s)}\,\mathrm{d}s} \qquad \big(\alpha\in\mathbb{R}, A\in\mathcal{B}(\mathbb{R}), n\in\mathbb{N}, t\in(0,\infty)\big). \tag{4.2}$$

Lemma 4.1. *For every $t\in(0,\infty)$ there exists an $L>0$ such that, for all $n\in\mathbb{N}_{\ge2}$,*

$$g_{n,t}(\alpha,s) \le L e^{-\frac{\alpha}{t}s+\frac14 s^2} G_t(n,\alpha) \qquad (\alpha,s\in\mathbb{R}), \tag{4.3}$$

where $G_t\colon \mathbb{N}\times\mathbb{R}\to\mathbb{R}$ is given by

$$G_t(n,\alpha) = \frac{\int_{\mathbb{R}} e^{\left(\frac{1+t}{t}\right)^2 z^2} e^{-nV(z)} e^{-(n-1)\left(z-\frac{\alpha}{1+t}\right)^2\frac{1+t}{2t}}\,\mathrm{d}z}{\int_{\mathbb{R}} e^{-nV(r)} e^{-(n-1)\left(r-\frac{\alpha}{1+t}\right)^2\frac{1+t}{2t}}\,\mathrm{d}r} \qquad (n\in\mathbb{N},\alpha\in\mathbb{R}). \tag{4.4}$$

Consequently, if for a bounded sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ the sequence $(G_t(n,\alpha_n))_{n\in\mathbb{N}}$ is bounded as well, then there exists a $\mu_{\mathcal{N}(0,1)}$-integrable function $h\colon\mathbb{R}\to[0,\infty)$ for which, for all $n\in\mathbb{N}$,

$$g_{n,t}(\alpha_n,s) \le h(s) \qquad (s\in\mathbb{R}). \tag{4.5}$$

Proof. After some elementary computations (see (A4) in Appendix A), we may rewrite (1.11) as

$$(4.6)\quad g_{n,t}(\alpha,s) = \frac{n}{n-1} e^{-\frac{\alpha}{t}s} \frac{\int_{\mathbb{R}} e^{\left[-2z^2+2(s+\frac{\alpha}{1+t})z-\frac{1}{n-1}(z-s)^2\right]\frac{1+t}{2t}} e^{-nV(z)} e^{-(n-1)\left(z-\frac{\alpha}{1+t}\right)^2\frac{1+t}{2t}}\,\mathrm{d}z}{\int_{\mathbb{R}} e^{-nV(r)} e^{-(n-1)\left(r-\frac{\alpha}{1+t}\right)^2\frac{1+t}{2t}}\,\mathrm{d}r} \qquad (\alpha, s\in\mathbb{R}).$$

Since $-z^2 + 2\frac{\alpha}{1+t}z = -(z-\frac{\alpha}{1+t})^2 + (\frac{\alpha}{1+t})^2$ and $\frac{1+t}{t}sz \le \frac14 s^2 + (\frac{1+t}{t})^2 z^2$, we get

$$(4.7)\quad g_{n,t}(\alpha,s) \le 2e^{\left(\frac{\alpha}{1+t}\right)^2} e^{-\frac{\alpha}{t}s} e^{\frac14 s^2} \frac{\int_{\mathbb{R}} e^{\left(\frac{1+t}{t}\right)^2 z^2} e^{-nV(z)} e^{-(n-1)\left(z-\frac{\alpha}{1+t}\right)^2\frac{1+t}{2t}}\,\mathrm{d}z}{\int_{\mathbb{R}} e^{-nV(r)} e^{-(n-1)\left(r-\frac{\alpha}{1+t}\right)^2\frac{1+t}{2t}}\,\mathrm{d}r} \qquad (\alpha, s\in\mathbb{R}),$$

which yields (4.3). The claim in (4.5) follows from (4.3) because $s\mapsto Le^{l|s|+\frac14 s^2}$ is $\mu_{\mathcal{N}(0,1)}$-integrable for all $l\in\mathbb{R}$. ☐

Lemma 4.2. *Let $V\in C^1(\mathbb{R},[0,\infty))$ and $t\in(0,\infty)$. For all $q,s,\alpha\in\mathbb{R}$, all sequences $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n\to\alpha$ and all $\epsilon>0$, there exist $\delta>0$, $N\in\mathbb{N}$ and $M>0$ such that for all $n\ge N$,*

$$(4.8)\quad \begin{aligned} &\left|g_{n,t}(\alpha_n,s) - e^{-sV'(q)}\right| \vee \left|G_t(n,\alpha_n) - e^{\left(\frac{1+t}{t}\right)^2 q^2}\right| \\ &\le \epsilon + M\frac{\int_{B(q,\delta)^c} e^{2\left(\frac{1+t}{t}\right)^2(r-\alpha_n)^2} e^{-(n-1)V(r)\wedge V\left(r+\frac1n(s-r)\right)} e^{-(n-1)(r-\frac{\alpha_n}{1+t})^2\frac{(1+t)}{2t}}\,\mathrm{d}r}{\int_{B(q,\delta)} e^{-V(r)} e^{-(n-1)\left[V(r)+(r-\frac{\alpha_n}{1+t})^2\frac{(1+t)}{2t}\right]}\,\mathrm{d}r}, \end{aligned}$$

where $G_t(n,\alpha)$ is as in (4.4).

Proof. Let $q,s,\alpha\in\mathbb{R}$, let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ with $\alpha_n\to\alpha$, and let $\epsilon>0$. Let $\delta>0$ be such that

$$(4.9)\qquad \left|e^{-sV'(q)} - e^{-sV'(r)}\right| < \epsilon, \qquad \left|e^{\left(\frac{1+t}{t}\right)^2 r^2} - e^{\left(\frac{1+t}{t}\right)^2 q^2}\right| < \epsilon \qquad (r\in B(q,2\delta)).$$

Let $N\in\mathbb{N}$ be such that $\frac{|s|+|q|+\delta}{N}<\delta$. By the Mean Value Theorem, we have

$$(4.10)\qquad \sup_{r\in B(q,\delta)} \left|e^{-n\left(V(r+\frac1n(s-r))-V(r)\right)} - e^{-sV'(q)}\right| < \epsilon \qquad (n\ge N).$$

Hence

$$(4.11)\qquad \begin{aligned} &\int_{\mathbb{R}} \left|e^{-n\left[V(r+\frac1n(s-r))-V(r)\right]} - e^{-sV'(q)}\right| e^{-V(r)} e^{-(n-1)\left[V(r)+\left(r-\frac{\alpha_n}{1+t}\right)^2\frac{(1+t)}{2t}\right]}\,\mathrm{d}r \\ &\le \epsilon\int_{B(q,\delta)} e^{-(n-1)\left[V(r)+\left(r-\frac{\alpha_n}{1+t}\right)^2\frac{(1+t)}{2t}\right]}\,\mathrm{d}r \\ &\quad + e^{-sV'(q)}\int_{B(q,\delta)^c} e^{-(n-1)\left[V(r)+\left(r-\frac{\alpha_n}{1+t}\right)^2\frac{(1+t)}{2t}\right]}\,\mathrm{d}r \\ &\quad + \int_{B(q,\delta)^c} e^{-(n-1)\left[V(r+\frac1n(s-r))+\left(r-\frac{\alpha_n}{1+t}\right)^2\frac{(1+t)}{2t}\right]}\,\mathrm{d}r \end{aligned}$$

and

$$\int_{\mathbb{R}} \left| e^{\left(\frac{1+t}{t}\right)^2 r^2} - e^{\left(\frac{1+t}{t}\right)^2 q^2} \right| \; e^{-V(r)} e^{-(n-1)\left[V(r)+\left(r-\frac{\alpha_n}{1+t}\right)^2 \frac{(1+t)}{2t}\right]} \,\mathrm{d}r$$
$$\leq \epsilon \int_{B(q,\delta)} e^{-(n-1)\left[V(r)+\left(r-\frac{\alpha_n}{1+t}\right)^2 \frac{(1+t)}{2t}\right]} \,\mathrm{d}r$$
$$+ e^{\left(\frac{1+t}{t}\right)^2 q^2} \int_{B(q,\delta)^c} e^{-(n-1)\left[V(r)+\left(r-\frac{\alpha_n}{1+t}\right)^2 \frac{(1+t)}{2t}\right]} \,\mathrm{d}r$$
$$+ \int_{B(q,\delta)^c} e^{\left(\frac{1+t}{t}\right)^2 r^2} e^{-(n-1)\left[V(r+\frac{1}{n}(s-r))+\left(r-\frac{\alpha_n}{1+t}\right)^2 \frac{(1+t)}{2t}\right]} \,\mathrm{d}r. \tag{4.12}$$

Because $e^{-(n-1)V(r)} \vee e^{-(n-1)V(r+\frac{1}{n}(s-r)} \leq e^{-(n-1)[V(r)\wedge V(r+\frac{1}{n}(s-r))]}$, we obtain

$$\left| g_{n,t}(\alpha_n, s) - e^{-sV'(q)} \right| \vee \left| G_t(n,\alpha_n) - e^{\left(\frac{1+t}{t}\right)^2 q^2} \right|$$
$$\leq \epsilon + K \frac{\int_{B(q,\delta)^c} e^{\left(\frac{1+t}{t}\right)^2 r^2} e^{-(n-1)\left[V(r)\wedge V\left(r+\frac{1}{n}(s-r)\right)+(r-\frac{\alpha_n}{1+t})^2 \frac{(1+t)}{2t}\right]} \,\mathrm{d}r}{\int_{B(q,\delta)} e^{-V(r)} e^{-(n-1)\left[V(r)+(r-\frac{\alpha_n}{1+t})^2 \frac{(1+t)}{2t}\right]} \,\mathrm{d}r} \tag{4.13}$$

with $K = e^{-sV'(q)} + e^{\left(\frac{1+t}{t}\right)^2 q^2} + 1$ (see (A.15) in Appendix A). Because $r^2 \leq 2\left(r - \frac{\alpha_n}{1+t}\right)^2 + 2\left(\frac{\alpha_n}{1+t}\right)^2$ and $(\alpha_n)_{n\in\mathbb{N}}$ is bounded, we get (4.8). □

4.2 Proof of Lemma 1.6

In the proof we use the identity

$$\left(r - \frac{\alpha_n}{1+t}\right)^2 \frac{1+t}{2t} = \left(r - \frac{\alpha}{1+t}\right)^2 \frac{1+t}{2t} + \frac{1}{t}\left(r - \frac{\alpha}{1+t}\right)(\alpha - \alpha_n) + \frac{(\alpha-\alpha_n)^2}{2t(1+t)}, \tag{4.14}$$

which implies

$$I_{t,\alpha_n}(r) = I_{t,\alpha}(r) + \frac{1}{t}\left(r - \frac{\alpha}{1+t}\right)(\alpha-\alpha_n) + \frac{(\alpha-\alpha_n)^2}{2t(1+t)}. \tag{4.15}$$

Proof of Lemma 1.6. Let $s, q \in \mathbb{R}$ be the smallest global minimizer of (1.9), i.e.,

$$q = \inf\left\{ r \in \mathbb{R}\colon\; I_{t,\alpha}(r) = \inf_{s\in\mathbb{R}} I_{t,\alpha}(s) \right\}. \tag{4.16}$$

(A similar argument works for the largest global minimizer.) By Lemmas 4.1–4.2 it suffices to show that, for all $\delta > 0$,

$$\frac{\int_{B(q,\delta)^c} e^{2\left(\frac{1+t}{t}\right)^2\left(r-\frac{\alpha_n}{1+t}\right)^2} e^{-(n-1)\left[V(r)\wedge V\left(r+\frac{1}{n}(s-r)\right)+(r-\frac{\alpha_n}{1+t})^2 \frac{(1+t)}{2t}\right]} \,\mathrm{d}r}{\int_{B(q,\delta)} e^{-V(r)} e^{-(n-1)\left[V(r)+(r-\frac{\alpha_n}{1+t})^2 \frac{(1+t)}{2t}\right]} \,\mathrm{d}r} \to 0. \tag{4.17}$$

For Part (b) we need to consider a particular sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ converging to α, while for Part (a) we need to consider all sequences $(\alpha_n)_{n\in\mathbb{N}}$ converging to α. In both cases, for

$\delta > 0$ we provide a sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ for which we check the following three steps, which together yield (4.17):

Step 1: Find $R > 0$, $C_1 > 0$ and $N_1 \in \mathbb{N}$ for which

$$\int_{B(0,R)^c} e^{2\left(\frac{1+t}{t}\right)^2 (r-\alpha_n)^2} e^{-(n-1)\left[V(r)\wedge V\left(r+\frac{1}{n}(s-r)\right)+(r-\frac{\alpha_n}{1+t})^2\frac{(1+t)}{2t}-c_n\right]}\,\mathrm{d}r \le C_1 \quad (n \ge N_1). \tag{4.18}$$

Step 2: Find $C_2 > 0$ and $N_2 \in \mathbb{N}$ for which

$$\int_{B(q,\delta)^c\cap B(0,R)} e^{2\left(\frac{1+t}{t}\right)^2 (r-\alpha_n)^2} e^{-(n-1)\left[V(r)\wedge V\left(r+\frac{1}{n}(s-r)\right)+\left(r-\frac{\alpha_n}{1+t}\right)^2\frac{1+t}{2t}-c_n\right]}\,\mathrm{d}r \le C_2 \qquad (n \ge N_2). \tag{4.19}$$

Step 3: Find $N_3 \in \mathbb{N}$ and a sequence $(\Gamma_n)_{n\in\mathbb{N}}$ with $\Gamma_n \to \infty$ for which

$$\int_{B(q,\delta)} e^{-V(r)} e^{-(n-1)\left[V(r)+(r-\frac{\alpha_n}{1+t})^2\frac{(1+t)}{2t}-c_n\right]}\,\mathrm{d}r \ge \Gamma_n \qquad (n \ge N_3). \tag{4.20}$$

Abbreviate

$$c = I_{t,\alpha}(q) = \inf_{r\in\mathbb{R}} I_{t,\alpha}(r) \in [0,\infty). \tag{4.21}$$

• **Step 1 for (a) and (b).** For all bounded sequences $(\alpha_n)_{n\in\mathbb{N}}$ (in particular those that converge to α) there exists an $R > 0$ such that

$$\left(r - \frac{\alpha_n}{1+t}\right)^2 \frac{1}{2t} > c + 1 \qquad (r \in B(0,R)^c, n \in \mathbb{N}). \tag{4.22}$$

Therefore, for all sequences $(c_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with $c_n \le c + 1$ for all $n \in \mathbb{N}$,

$$V(r) \wedge V\left(r + \frac{1}{n}(s-r)\right) + \left(r - \frac{\alpha_n}{1+t}\right)^2 \frac{1+t}{2t} > c_n + \left(r - \frac{\alpha_n}{1+t}\right)^2 \frac{1}{2}, \qquad (r \in B(0,R)^c, n \in \mathbb{N}). \tag{4.23}$$

Let $N_1 \in \mathbb{N}$ be such that $N_1 - 1 > 4(\frac{1+t}{t})^2 + 1$. Then

$$\begin{aligned}&\int_{B(0,R)^c} e^{2\left(\frac{1+t}{t}\right)^2\left(r-\frac{\alpha_n}{1+t}\right)^2} e^{-(n-1)\left[V(r)\wedge V\left(r+\frac{1}{n}(s-r)\right)+\left(r-\frac{\alpha_n}{1+t}\right)^2\frac{1+t}{2t}-c_n\right]}\,\mathrm{d}r \\ &\quad\le \int_{\mathbb{R}} e^{\left(4\left(\frac{1+t}{t}\right)^2-(n-1)\right)\left(r-\frac{\alpha_n}{1+t}\right)^2\frac{1}{2}}\,\mathrm{d}r \le \int_{\mathbb{R}} e^{-\left(r-\frac{\alpha_n}{1+t}\right)^2\frac{1}{2}}\,\mathrm{d}r = \sqrt{2\pi} \qquad (n \ge N_1).\end{aligned} \tag{4.24}$$

• **Step 2 for (a).** Because $\lim_{r\to\pm\infty} I_{t,\alpha}(r) = \infty$, $I_{t,\alpha}$ is continuous and $I_{t,\alpha}$ attains its global minimum at q, there exists a $\rho \in (0, \frac{1}{5})$ for which

$$I_{t,\alpha}(r) > c + 5\rho \qquad (r \in B(q,\delta)^c). \tag{4.25}$$

Here, and in Step 3 for (a) below, we pick $c_n = c + 3\rho$ for $n \in \mathbb{N}$. Note that $c_n \le c + 1$ for all $n \in \mathbb{N}$. By (4.14) and the continuity of V there exists an $N_2 \in \mathbb{N}$ such that, for all $n \ge N_2$,

$$V(r) \wedge V\left(r + \frac{1}{n}(s-r)\right) + \left(r - \frac{\alpha_n}{1+t}\right)^2 \frac{1+t}{2t} > I_{t,\alpha}(r) - \rho > c + 4\rho \qquad \big(r \in B(q,\delta)^c \cap B(0,R)\big). \tag{4.26}$$

Moreover, there exists an $\Upsilon > 0$ such that $e^{2\left(\frac{1+t}{t}\right)^2\left(r-\frac{\alpha_n}{1+t}\right)^2} \le \Upsilon$ for all $n \in \mathbb{N}$ and all $r \in B(0,R)$. Hence we obtain (4.19) with $C_2 = 2R\Upsilon$ (and $c_n = c + 3\rho$ for $n \in \mathbb{N}$).

• **Step 2 for (b).** Here, and in Step 3 for (b) below, we consider $\alpha_n = \alpha - \frac{1}{\sqrt{n}}$ for $n \in \mathbb{N}$, and

(4.27)
$$c_n = I_{t,\alpha_n}(q) + \frac{\delta}{\sqrt{n}t} = I_{t,\alpha}(q) + \frac{1}{t}\left(q - \frac{\alpha}{1+t}\right)(\alpha-\alpha_n) + \frac{(\alpha-\alpha_n)^2}{2t(1+t)} + \frac{\delta}{\sqrt{n}t} \qquad (n \in \mathbb{N}).$$

Note that $c_n \to c$, and so there exists an $N_1 \in \mathbb{N}$ for which $N_1 - 1 > 4(\frac{1+t}{t})^2 + 1$ and $c_n \le c+1$ for $n \ge N_1$ (and thus (4.24) holds). For $r \in B(q,\delta)^c \cap B(0,R)$ we write

$$\begin{aligned}(4.28)\qquad & V(r) \wedge V\left(r + \frac{1}{n}(s-r)\right) + \left(r - \frac{\alpha_n}{1+t}\right)^2 \frac{1+t}{2t} - c_n \\ &= \left(V(r) \wedge V\left(r + \frac{1}{n}(s-r)\right) - V(r)\right) + \left(I_{t,\alpha_n}(r) - c_n\right).\end{aligned}$$

For the left part (of the righthandside of (4.28)) we have

$$(4.29)\qquad V(r) \wedge V\left(r + \frac{1}{n}(s-r)\right) - V(r) \ge -\frac{1}{n}\Theta$$

with $\Theta = (\sup_{u \in B(0,R+|s|)} |V'(u)|)(R+|s|)$. For the right part first note that, by the definition of q and the continuity of $I_{t,\alpha}$, there exists a $\rho > 0$ such that

$$(4.30)\qquad I_{t,\alpha}(r) > I_{t,\alpha}(q) + \rho \qquad \big(r \in (-\infty, q-\delta)\big).$$

Because $(\alpha - \alpha_n)\frac{1}{t} = \frac{1}{\sqrt{n}t}$, by (4.15) we have for the right part, for $r \in B(0,R)$,

$$\begin{aligned}(4.31)\qquad I_{t,\alpha_n}(r) - c_n &= I_{t,\alpha}(r) - I_{t,\alpha}(q) + (r-q)\frac{1}{\sqrt{n}t} - \frac{\delta}{\sqrt{n}t} \\ &\ge \begin{cases} \rho - (R+\delta)\frac{1}{\sqrt{n}t} & r < q-\delta, \\ 0 & r > q+\delta. \end{cases}\end{aligned}$$

Let $N_2 \in \mathbb{N}$ be such that $(R+\delta)\frac{1}{\sqrt{n}t} < \rho$ for $n \ge N_2$. Then, for $r \in B(q,\delta)^c \cap B(0,R)$,

$$(4.32)\qquad V(r) \wedge V\left(r + \frac{1}{n}(s-r)\right) + \left(r - \frac{\alpha_n}{1+t}\right)^2 \frac{1+t}{2t} - c_n \ge -\frac{1}{n}\Theta \qquad (n \ge N_2).$$

Moreover, there exists a $\Upsilon > 0$ such that $e^{2\left(\frac{1+t}{t}\right)^2\left(r-\frac{\alpha_n}{1+t}\right)^2} \le \Upsilon$ for all $n \in \mathbb{N}$ and all $r \in B(0,R)$. Therefore we obtain (4.19) with $C_2 = 2R\Upsilon e^{\Theta}$.

• **Step 3 for (a).** For $r \in A = B(q,\delta) \cap \{r \in \mathbb{R} \colon I_{t,\alpha}(r) < c + \rho\}$ there exists an $N_3 \in \mathbb{N}$ for which $I_{t,\alpha_n}(r) < c + 2\rho$ for all $n \ge N_3$. Hence

$$(4.33)\qquad \int_{B(q,\delta)} e^{-V(r)} e^{-(n-1)\left[V(r) + (r - \frac{\alpha_n}{1+t})^2 \frac{(1+t)}{2t} - (c+3\rho)\right]} \, dr \ge e^{(n-1)\rho} \int_A e^{-V(r)} \, dr \qquad (n \ge N_3).$$

• **Step 3 for (b).** There exists a $K > 0$ such that, for all $n \in \mathbb{N}$ and $r \in B(q, \frac{\delta}{n})$,

$$I_{t,\alpha_n}(r) - c_n = I_{t,\alpha}(r) - I_{t,\alpha}(q) + (r-q)\frac{1}{\sqrt{n}t} - \frac{\delta}{\sqrt{n}t} \tag{4.34}$$
$$< \frac{\delta}{n}\sup_{s\in B(q,\delta)}\left|\frac{d}{ds}I_{t,\alpha}(s)\right| + \frac{\delta}{tn\sqrt{n}} - \frac{\delta}{\sqrt{n}t}$$
$$< \frac{1}{n}K - \frac{\delta}{\sqrt{n}t} = \frac{1}{\sqrt{n}}\left(\frac{K}{\sqrt{n}} - \frac{\delta}{t}\right).$$

Let $N_3 \in \mathbb{N}$ be such that $\frac{K}{\sqrt{n}} < \frac{1}{2}\frac{\delta}{t}$ for $n \geq N_3$. Then, for $r \in B(q, \frac{\delta}{n})$,

$$V(r) + \left(r - \frac{\alpha_n}{1+t}\right)^2 \frac{1+t}{2t} - c_n < -\frac{1}{2}\frac{\delta}{t}\frac{1}{\sqrt{n}} \qquad (n \geq N_3). \tag{4.35}$$

Let $\kappa > 0$ be such that $e^{-V(r)} > \kappa$ for all $r \in B(q,\delta)$. Then

$$\int_{B(q,\frac{\delta}{n})} e^{-V(r)} e^{-(n-1)\left[V(r)+(r-\frac{\alpha_n}{1+t})^2\frac{(1+t)}{2t}-c_n\right]}\,\mathrm{d}r \geq \frac{2\delta}{n}\kappa e^{(\sqrt{n}-1)\frac{1}{2}\frac{\delta}{t}} \qquad (n \geq N_3). \tag{4.36}$$

□

5 Tools from convex analysis: proof of Theorem 1.11

In this section we state a definition (Definition 5.1) and several lemmas (Lemmas 5.2–5.8) that are based on convex analysis, and use these to give the proof of Theorem 1.11. After that we prove the claim made below Corollary 1.12 (Lemma 5.9) and make an additional observation (Lemma 5.10) that can be used to determine whether $(\mu_{n,t})_{n\in\mathbb{N}}$ is sequentially Gibbs at $t = t_c$.

Definition 5.1. Let $f\colon \mathbb{R} \to \mathbb{R}$. Then $a \in \mathbb{R}$ is called a *supporting point* for f if there exists a linear function $l\colon \mathbb{R} \to \mathbb{R}$ with $l(a) = f(a)$ and $l(x) \leq f(x)$, $x \in \mathbb{R}$.

Lemma 5.2. *Let* $f\colon \mathbb{R} \to \mathbb{R}$. *Then*

(a) *for* $x, y, z \in \mathbb{R}$ *with* $x < y < z$:

$$\Phi_2 f(x,y,z) = \frac{f(x)}{(x-y)(x-z)} + \frac{f(y)}{(y-x)(y-z)} + \frac{f(z)}{(z-x)(z-y)}.$$

(b) *for* $a, b, c, d \in \mathbb{R}$ *with* $a < b < c < d$:

$$(d-a)\Phi_2 f(a,b,d) = (b-a)\Phi_2 f(a,b,c) + (d-c)\Phi_2 f(b,c,d),$$
$$(d-a)\Phi_2 f(a,c,d) = (c-a)\Phi_2 f(a,b,c) + (d-b)\Phi_2 f(b,c,d).$$

(c) *for* $g\colon \mathbb{R} \to \mathbb{R}$, $\theta, \kappa \in \mathbb{R}$:

$$\Phi_2(\theta f + \kappa g) = \theta\Phi_2 f + \kappa\Phi_2 g.$$

(d) *for* $g(x) = x^2$, $\Phi_2 g = 1$ *and* $\Phi_2 h = 0$ *if* $h(x) = \alpha x + \beta$ *for* $\alpha, \beta \in \mathbb{R}$.

Proof. The proof can be done by hand. See also Schikhof [17, Lemma 29.2]. ☐

Lemma 5.3. *Let $f\colon \mathbb{R} \to \mathbb{R}$ and $y \in \mathbb{R}$. Then the following are equivalent:*

(a) *y is a supporting point for f,*

(b) $\dfrac{f(z) - f(y)}{z - y} \geq \dfrac{f(y) - f(x)}{y - x} \qquad (x, z \in \mathbb{R},\ x < y < z),$

(c) $\Phi_2 f(\cdot, y, \cdot) \geq 0$.

Proof. Straightforward. ☐

Lemma 5.4. *A function $f\colon \mathbb{R} \to \mathbb{R}$ is convex if and only if $\Phi_2 f \geq 0$. Moreover, f is strictly convex if and only if $\Phi_2 f > 0$.*

Proof. See Schikhof and van Rooij [16, Theorem 2.2]. ☐

Lemma 5.5. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be lower semicontinuous with $\lim_{|x| \to \infty} f(x) = \infty$. Suppose that f is bounded from below. Then there exists an $a \in \mathbb{R}$ for which $f(a) = \inf_{x \in \mathbb{R}} f(x)$. In particular, a is a supporting point for f.*

Proof. Let $c = \inf_{x \in \mathbb{R}} f(x)$. Define $A_n = \{x \in \mathbb{R}\colon\ f(x) \leq c + \frac{1}{n}\}$, $n \in \mathbb{N}$. Then A_n is compact and $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}$. Therefore there exists an $a \in \mathbb{R}$ for which $a \in \bigcap_{n \in \mathbb{N}} A_n$. ☐

Lemma 5.6. *Let $f\colon \mathbb{R} \to [0, \infty)$ be lower semicontinuous with $\lim_{|x| \to \infty} f(x) = \infty$. Then the following are equivalent:*

(a) *There exists an $\alpha \in \mathbb{R}$ for which $x \mapsto f(x) - \alpha x$ has multiple global minimizers.*

(b) *There exists a linear $l\colon \mathbb{R} \to \mathbb{R}$ for which $\#\{x \in \mathbb{R}\colon\ l(x) = f(x)\} \geq 2$ and $l \leq f$.*

(c) *There exist $a, b, c \in \mathbb{R}$ with $a < b < c$ and $\Phi_2 f(\cdot, a, \cdot) \geq 0$, $\Phi_2 f(\cdot, c, \cdot) \geq 0$, $\Phi_2 f(\cdot, b, \cdot) \not\geq 0$.*

(d) *There exist $a, x, b, y, c \in \mathbb{R}$ with $a \leq x < b < y \leq c$ and $\Phi_2 f(\cdot, a, \cdot) \geq 0$, $\Phi_2 f(\cdot, c, \cdot) \geq 0$, $\Phi_2 f(x, b, y) \leq 0$.*

Proof. The equivalence (a) $\iff$ (b) and the implication (d) $\Rightarrow$ (c) are trivial.
(c) $\Rightarrow$ (d). Assume (c). Then there exist $x, y \in \mathbb{R}$ with $x < b < y$ for which $\Phi_2 f(x, b, y) \leq 0$. If $x < a$ and/or $y > c$, then $\Phi_2 f(a, b, y) \leq 0$ and/or $\Phi_2(x, b, c) \leq 0$ by Lemma 5.2(b). Therefore we may assume that $x \geq a$ and $y \leq c$, i.e., we obtain (d).
(b) $\Rightarrow$ (c). Assume (b). Let $a, c \in \{x \in \mathbb{R}\colon\ l(x) = f(x)\}$ with $a < c$. Let $b \in (a, c)$. Then

$$\frac{f(c) - f(a)}{c - a} = \frac{l(c) - l(a)}{c - a} = \frac{l(b) - l(a)}{b - a} \leq \frac{f(b) - f(a)}{b - a}, \tag{5.1}$$

i.e., $\Phi_2 f(a, b, c) \leq 0$.
(d) $\Rightarrow$ (b). Define $w, z \in \mathbb{R}$ by

$$\begin{aligned} w &= \sup\{s \leq b\colon\ \Phi_2 f(\cdot, s, \cdot) \geq 0\}, \\ z &= \inf\{s \geq b\colon\ \Phi_2 f(\cdot, s, \cdot) \geq 0\}. \end{aligned} \tag{5.2}$$

Because f is lower semicontinuous, we have $\liminf_{s\uparrow w} f(s) \geq f(w)$. Hence, by Lemma 5.2(a), we have, for $q, r \in \mathbb{R}$ with $q < w < r$,

$$
\begin{aligned}
(5.3) \qquad 0 &\leq \limsup_{s\uparrow w} \Phi_2 f(q,s,r) \\
&= \frac{f(q)}{(q-w)(q-r)} + \frac{f(r)}{(r-w)(r-q)} - \frac{\liminf_{s\uparrow w} f(s)}{(r-w)(w-q)} \leq \Phi_2 f(q,w,r).
\end{aligned}
$$

So $\Phi_2 f(\cdot, w, \cdot) \geq 0$. Similarly $\Phi_2 f(\cdot, z, \cdot) \geq 0$. If $w = b$, then $z = b$, and vice versa.

- Assume that $w = b = z$. Then f is convex and $\Phi_2 f(x, b, y) = 0$. With $l\colon \mathbb{R} \to \mathbb{R}$, $s \mapsto f(x) + \frac{f(y)-f(x)}{y-x}(s-x)$ one then has $l \leq f$ and $l(s) = f(s)$ for all $s \in [x, y]$, since

$$
(5.4) \qquad \frac{f(b)-f(x)}{b-x} \leq \frac{f(s)-f(b)}{s-b} \leq \frac{f(y)-f(b)}{y-b} = \frac{f(b)-f(x)}{b-x}.
$$

- Assume that $w < b < z$. Define $l\colon \mathbb{R} \to \mathbb{R}$, $s \mapsto f(w) + \frac{f(z)-f(w)}{z-w}(s-w)$. Then $l \leq f$ on $(w,z)^c$. Note that $f - l|_{[w,z]}$ is lower semicontinuous and bounded from below. By Lemma 5.5, it attains its infimum at some $a \in [w, z]$. This a is a supporting point of f, and hence $a = w$ or $a = z$ by Lemma 5.3. Thus $l(s) \leq f(s)$ for all $s \in \mathbb{R}$. □

Lemma 5.7. *Let $f\colon \mathbb{R} \to [0,\infty)$ be lower semicontinuous. Let $r \in \mathbb{R}$ and $\beta > 0$. Then there exist $q, s \in \mathbb{R}$ with $q < r < s$ that are supporting points of $x \mapsto f(x) + \beta x^2$, i.e., $\Phi_2 f(\cdot, q, \cdot) \geq -\beta$, $\Phi_2 f(\cdot, s, \cdot) \geq -\beta$.*

Proof. Since $x \mapsto f(x) + \beta x^2$ is lower semicontinuous and $\lim_{|x|\to\infty}[f(x) + \beta x^2] = \infty$, by Lemma 5.5 there exists an $a \in \mathbb{R}$ for which a is a global minimum and thus a supporting point for $x \mapsto f(x) + \beta x^2$. There exists a (large enough) $\theta > 0$ such that

$$
(5.5) \qquad \{x \in \mathbb{R} \colon\ f(a) - 1 + \theta(x - r) = \beta x^2\}
$$

has two elements, say x_1, x_2 with $x_1 < x_2$. By the definition of a, we have $x_1 > r$. By Lemma 5.5, there exists an $s \in \mathbb{R}$ that is a global minimum and a supporting point of

$$
(5.6) \qquad x \mapsto f(x) + \beta x^2 - (f(a) - 1 + \theta(x - r)).
$$

Hence s is also a supporting point of $x \mapsto f(x) + \beta x^2$. Because (5.6) is strictly negative on (x_1, x_2) and non-negative on $[x_1, x_2]^c$, we have $s \in [x_1, x_2]$. Therefore $s > r$. There also exists a (small enough) $\theta < 0$ for which (5.5) has two elements. In the same way we can prove that there is an $q < r$ that is also a supporting point of $x \mapsto f(x) + \beta x^2$. The last part of the statement is a consequence of Lemma 5.2. □

Lemma 5.8. *Let $f\colon \mathbb{R} \to [0,\infty)$ be lower semicontinuous and let $\beta \in (0,\infty)$. Then there exists an $\alpha \in \mathbb{R}$ for which $x \mapsto f(x) + \beta x^2 - \alpha x$ has multiple global minimizers if and only if $\Phi_2 f \not\geq -\beta$.*

Proof. This is a consequence of Lemmas 5.6–5.7. □

Proof of Theorem 1.11. The claim in Theorem 1.11 follows by applying Lemma 5.8 with $\beta = \frac{1+t}{2t}$ to the lower semicontinuous function $r \mapsto V(r) + \frac{1}{2}r^2$. □

The following observation proves the claim made below Corollary 1.12.

Lemma 5.9. *Let $f\colon \mathbb{R}\to\mathbb{R}$ be twice differentiable. Then $f''\geq 2\beta$ if and only if $\Phi_2 f\geq\beta$ for all $\beta\in\mathbb{R}$.*

Proof. By Lemma 5.4, $\Phi_2 g\geq 0$ if and only if g is convex. Since a twice differentiable function g is convex if and only if $g''\geq 0$, this implies the equivalence $\Phi_2 f\geq 0 \iff f''\geq 0$. Let $\beta\in\mathbb{R}$ and let $g\colon \mathbb{R}\to\mathbb{R}$ be given by $g(r)=f(r)-\beta r^2$. Then, by Lemma 5.2, we have $f''\geq 2\beta \iff g''\geq 0 \iff \Phi_2 g\geq 0 \iff \Phi_2 f\geq\beta$. □

In contrast to Lemma 5.9, we can have $\Phi_2 f>\beta$ but not $f''>2\beta$ (take e.g. $\beta=0$ and $f(x)=x^4$, in which case $\Phi_2 f>0$ by Lemma 5.4 but $f''(0)=0$). However, according to the next observation the second derivative of f can be used to determine whether $\Phi_2 f>\beta$. This observation can be used to determine whether $(\mu_{n,t})_{n\in\mathbb{N}}$ is sequentially Gibbs at $t=t_c$.

Lemma 5.10. *Let $f\colon \mathbb{R}\to\mathbb{R}$. Let $a,b,c\in\mathbb{R}$ with $a<b<c$, and $\beta\in\mathbb{R}$.*

(a) *If $\Phi_2 f|_{(a,b)}>\beta$, $\Phi_2 f|_{(a,b]}\geq\beta$, $\Phi_2 f|_{(b,c)}>\beta$, $\Phi_2 f|_{[b,c)}\geq\beta$ and $\Phi_2 f|_{(a,c)}(\cdot,b,\cdot)\geq 0$, then $\Phi_2 f|_{(a,c)}>\beta$.*

(b) *If f is upper semicontinuous and $\Phi_2 f|_{(a,b)}\geq\beta$, then $\Phi_2 f|_{[a,b]}\geq\beta$.*

(c) *If f is twice differentiable on (a,b) and ${f|_{(a,b)}}''>\beta$, then $\Phi_2 f|_{(a,b)}>\beta$.*

Proof. Without loss of generality we may assume $b=0$.
(a) Let $x,y,z\in(a,c)$. If $x<y<0<z$ or $x<0<y<z$, then with Lemma 5.2(b) we easily get $\Phi_2 f(x,y,z)>\beta$. If $y=0$, then $x<\frac{x}{2}<0<z$, and hence $\Phi_2(x,\frac{x}{2},0)>0$. Again with Lemma 5.2(b), we get $\Phi_2(x,0,z)>0$.
(b) If f is upper semicontinuous, then $\limsup_{s\uparrow b} f(s)\leq f(b)$ and $\limsup_{s\downarrow a} f(s)\leq f(a)$. Together with Lemma 5.2(a) this proves the second statement.
(c) If ${f|_{(a,b)}}''>0$, then f is strictly convex, and with Lemma 5.4 this implies (c) in case $\beta=0$. Replacing f by $g(r)=f(r)-\frac{\beta}{2}r^2$, we obain (c) for $\beta\neq 0$ (see Lemma 5.2). □

A Key formulas

In this appendix we derive a few formulas that were used in the main body of the paper.

A1. We derive formulas for $\gamma_{n,t}$ and $\overline{\gamma}_{n,t}$ described in Section 1.4.
Inserting (1.7) into (1.6) we get, for $A\in\mathcal{B}(\mathbb{R}^n)$,

$$\begin{aligned}\mu_{n,t}(A)&=\frac{1}{Z_n}\int_{\mathbb{R}^n}\Big[(2\pi t)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\mathbf{1}_A(y)e^{-\frac{\|y-z\|^2}{2t}}\,\mathrm{d}y\Big]e^{-n(V\circ m_n)(z)}\,\mathrm{d}\mu_{\mathcal{N}(0,I_n)}(z)\\&=\frac{1}{Z_n}\int_{\mathbb{R}^n}\mathbf{1}_A(y)\Big[(2\pi)^{-n}t^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{-\frac{\|y-z\|^2}{2t}}e^{-\frac{\|z\|^2}{2}}e^{-n(V\circ m_n)(z)}\,\mathrm{d}z\Big]\,\mathrm{d}y.\end{aligned}\tag{A.1}$$

Since $\frac{\|y-z\|^2}{2t}+\frac{\|z\|^2}{2}=\frac{\|y\|^2}{2(1+t)}+\frac{\|\frac{y}{1+t}-z\|^2(1+t)}{2t}$ for $y,z\in\mathbb{R}^n$, we get, for $A\in\mathcal{B}(\mathbb{R}^n)$,

$$\mu_{n,t}(A)=\frac{1}{Z_n}\int_{\mathbb{R}^n}\mathbf{1}_A(y)\Big[(2\pi)^{-n}t^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{-\frac{\|y\|^2}{2(1+t)}}e^{-\frac{\|\frac{y}{1+t}-z\|^2(1+t)}{2t}}e^{-n(V\circ m_n)(z)}\,\mathrm{d}z\Big]\,\mathrm{d}y.\tag{A.2}$$

Then it is not hard to check that $\gamma_{n,t} : \mathbb{R}^{n-1} \times \mathcal{B}(\mathbb{R})$ defined for $y_2, \ldots, y_n \in \mathbb{R}$ and $B \in \mathcal{B}(\mathbb{R})$ by

$$\text{(A.3)}\qquad \gamma_{n,t}((y_2,\ldots,y_n),B) = \frac{(2\pi(1+t))^{-\frac{n}{2}} \int_{\mathbb{R}} \mathbf{1}_B(x)\, e^{-\frac{x^2}{2(1+t)}} \int_{\mathbb{R}^n} e^{-n(V\circ m_n)(z)}\, \mathrm{d}\mu_{\mathcal{N}(\frac{(x,y_2,\ldots,y_n)}{1+t},\frac{t}{1+t}I_n)}(z)\, \mathrm{d}x}{(2\pi(1+t))^{-\frac{n}{2}} \int_{\mathbb{R}} e^{-\frac{x^2}{2(1+t)}} \int_{\mathbb{R}^n} e^{-n(V\circ m_n)(z)}\, \mathrm{d}\mu_{\mathcal{N}(\frac{(x,y_2,\ldots,y_n)}{1+t},\frac{t}{1+t}I_n)}(z)\, \mathrm{d}x}.$$

is the weakly continuous proper conditional probability measure under $\mu_{n,t}$ of the first spin given the other spins. Using the identities

$$\text{(A.4)}\qquad \mu_{\mathcal{N}\left(\frac{(x,y_2,\ldots,y_n)}{1+t},\frac{t}{1+t}I_n\right)} = \mu_{\mathcal{N}(\frac{x}{1+t},\frac{t}{1+t})} \otimes \mu_{\mathcal{N}\left(\frac{(y_2,\ldots,y_n)}{1+t},\frac{t}{1+t}I_{n-1}\right)},$$

$$\text{(A.5)}\qquad \mu_{\mathcal{N}\left(\frac{(y_2,\ldots,y_n)}{1+t},\frac{t}{1+t}I_{n-1}\right)} \circ m_{n-1}^{-1} = \mu_{\mathcal{N}\left(\frac{m_{n-1}(y_2,\ldots,y_n)}{1+t},\frac{t}{(n-1)(1+t)}\right)},$$

$$\text{(A.6)}\qquad m_n(z_1,\ldots,z_n) = \frac{z_1}{n} + \frac{n-1}{n} m_{n-1}(z_2,\ldots,z_n),$$

we obtain the expression

(A.7)

$$\gamma_{n,t}((y_2,\ldots,y_n),B) = \frac{\int_{\mathbb{R}} \mathbf{1}_B(x) \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-nV(\frac{1}{n}s+\frac{n-1}{n}r)}\, \mathrm{d}\mu_{\mathcal{N}\left(\frac{m_{n-1}(y_2,\ldots,y_n)}{1+t},\frac{t}{(n-1)1+t}\right)}(r)\, \mathrm{d}\mu_{\mathcal{N}(\frac{x}{1+t},\frac{t}{1+t})}(s)\, \mathrm{d}\mu_{\mathcal{N}(0,1+t)}(x)}{\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-nV(\frac{1}{n}s+\frac{n-1}{n}r)}\, \mathrm{d}\mu_{\mathcal{N}\left(\frac{m_{n-1}(y_2,\ldots,y_n)}{1+t},\frac{t}{(n-1)1+t}\right)}(r)\, \mathrm{d}\mu_{\mathcal{N}(\frac{x}{1+t},\frac{t}{1+t})}(s)\, \mathrm{d}\mu_{\mathcal{N}(0,1+t)}(x)}.$$

We see that $\gamma_{n,t}(u,\cdot) = \gamma_{n,t}(v,\cdot)$ for all $u, v \in \mathbb{R}^{n-1}$ with $m_{n-1}(v) = m_{n-1}(u)$. Hence we can define $\overline{\gamma}_{n,t} \colon \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0,1]$ by letting $\overline{\gamma}_{n,t}(\alpha,B) = \gamma_{n,t}(v,B)$ for $\alpha \in \mathbb{R}$ and $B \in \mathcal{B}(\mathbb{R})$, where $v \in \mathbb{R}^{n-1}$ is such that $m_{n-1}(v) = \alpha$, i.e.,

$$\text{(A.8)}\qquad \overline{\gamma}_{n,t}(\alpha,B) = \frac{\int_{\mathbb{R}} \mathbf{1}_B(x) \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-nV(\frac{1}{n}s+\frac{n-1}{n}r)}\, \mathrm{d}\mu_{\mathcal{N}\left(\frac{\alpha}{1+t},\frac{t}{(n-1)1+t}\right)}(r)\, \mathrm{d}\mu_{\mathcal{N}(\frac{x}{1+t},\frac{t}{1+t})}(s)\, \mathrm{d}\mu_{\mathcal{N}(0,1+t)}(x)}{\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-nV(\frac{1}{n}s+\frac{n-1}{n}r)}\, \mathrm{d}\mu_{\mathcal{N}\left(\frac{\alpha}{1+t},\frac{t}{(n-1)1+t}\right)}(r)\, \mathrm{d}\mu_{\mathcal{N}(\frac{x}{1+t},\frac{t}{1+t})}(s)\, \mathrm{d}\mu_{\mathcal{N}(0,1+t)}(x)}.$$

A2. We show that $\eta_{n,t}$ is indeed the weakly continuous proper regular conditional probability of the magnetization of the n spins at time 0 given the magnetization at time t.
Let μ_n be the law on $C([0,\infty),\mathbb{R}^n)$ of the paths of the independent Brownian motions performed by the n spins with initial distribution $\mu_{n,0}$, i.e., μ_n is given by (1.13). The joint law of the process at time 0 and time t is given by

$$\text{(A.9)}\qquad \mu_{n,(0,t)}(A) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_A(x,y)\, \mathrm{d}\left[p_n(t,x,\cdot)\right](y)\, \mathrm{d}\mu_{n,0}(x) \qquad (A \in \mathcal{B}((\mathbb{R}^2)^n)).$$

We write m_n also for the function $(\mathbb{R}^2)^n \to \mathbb{R}^2$ given by

$$\text{(A.10)}\qquad m_n((x_1,y_1),\ldots,(x_n,y_n)) = \frac{1}{n}\sum_{i=1}^n (x_i,y_i) \qquad (x_1,y_1,\ldots,x_n,y_n \in \mathbb{R}).$$

Let $\overline{\mu}_{n,(0,t)} = \mu_{n,(0,t)} \circ m_n^{-1}$. Since $p_n(t,x,\cdot) \circ m_n^{-1} = \mu_{\mathcal{N}(x,tI_n)} \circ m_n^{-1} = \mu_{\mathcal{N}(m_n(x),\frac{t}{n})}$ and $\mu_{\mathcal{N}(0,I_n)} \circ m_n^{-1} = \mu_{\mathcal{N}(0,\frac{1}{n})}$, we have

$$\text{(A.11)}\quad \overline{\mu}_{n,(0,t)}(A) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_A(s,\alpha)\ \mathrm{d}\mu_{\mathcal{N}(s,\frac{t}{n})}(\alpha) e^{-nV(s)}\ \mathrm{d}\mu_{\mathcal{N}(0,\frac{1}{n})}(s)$$
$$= \frac{1}{\sqrt{2\pi\frac{t}{n}}} \frac{1}{\sqrt{2\pi\frac{1}{n}}} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_A(s,\alpha)\, e^{-n[V(s)+\frac{s^2}{2}+\frac{(s-\alpha)^2}{2t}]}\ \mathrm{d}s\ \mathrm{d}\alpha \qquad (A \in \mathcal{B}(\mathbb{R}^2)).$$

From this it follows that $\eta_{n,t}$ given in (1.8) is the weakly continuous proper regular conditional probability under $\overline{\mu}_{n,(0,t)}$ of the first coordinate given the second, i.e., the weakly continuous proper regular conditional probability of the magnetization of the n spins at time 0 given the magnetization at time t.

A3. We verify (1.11) and (1.10).
An elementary computation gives that, for $\alpha, s \in \mathbb{R}$, $t \in (0,\infty)$ and $n \in \mathbb{N}$,

(A.12)
$$\int_{\mathbb{R}} e^{-nV(\frac{1}{n}s+\frac{n-1}{n}r)}\ \mathrm{d}\mu_{\mathcal{N}\left(\frac{\alpha}{1+t},\frac{t}{(n-1)1+t}\right)}(r)$$
$$= \sqrt{\frac{(n-1)(1+t)}{2\pi t}} \int_{\mathbb{R}} e^{-nV\left(\frac{1}{n}s+\frac{n-1}{n}r\right)} e^{-\left(r-\frac{\alpha}{1+t}\right)^2\frac{(n-1)(1+t)}{2t}}\ \mathrm{d}r$$
$$= \sqrt{\frac{(n-1)(1+t)}{2\pi t}} e^{-(n-1)\frac{\alpha^2}{1+t}} \int_{\mathbb{R}} e^{-n\left[V(r+\frac{1}{n}(s-r))-V(r)\right]}\, e^{-V(r)}\, e^{-(n-1)[V(r)+\frac{r^2}{2}+\frac{(r-\alpha)^2}{2t}]}\ \mathrm{d}r.$$

Hence, for $n \in \mathbb{N}$ and $t \in (0,\infty)$, we can write

(A.13)
$$\overline{\gamma}_{n,t}(\alpha,B) = \frac{\int_{\mathbb{R}} \mathbf{1}_B(x) \int_{\mathbb{R}} g_{n,t}(\alpha,s)\ \mathrm{d}\mu_{\mathcal{N}\left(\frac{x}{1+t},\frac{t}{1+t}\right)}(s)\ \mathrm{d}\mu_{\mathcal{N}(0,1+t)}(x)}{\int_{\mathbb{R}} \int_{\mathbb{R}} g_{n,t}(\alpha,s)\ \mathrm{d}\mu_{\mathcal{N}\left(\frac{x}{1+t},\frac{t}{1+t}\right)}(s)\ \mathrm{d}\mu_{\mathcal{N}(0,1+t)}(x)} \qquad (\alpha \in \mathbb{R}, B \in \mathcal{B}(\mathbb{R})),$$

where $g_{n,t} \colon \mathbb{R}^2 \to \mathbb{R}$ is as in (1.11). With Fubini's Theorem we have

$$\text{(A.14)}\quad \int_{\mathbb{R}} \mathbf{1}_B(x) \int_{\mathbb{R}} g_{n,t}(\alpha,s)\ \mathrm{d}\mu_{\mathcal{N}\left(\frac{x}{1+t},\frac{t}{1+t}\right)}(s)\ \mathrm{d}\mu_{\mathcal{N}(0,1+t)}(x)$$
$$= \frac{1}{2\pi\sqrt{t}} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_B(x)\, g_{n,t}(\alpha,s) e^{-\left(s-\frac{x}{1+t}\right)^2\frac{1+t}{2t}}\, e^{-x^2\frac{1}{2(1+t)}}\ \mathrm{d}x\ \mathrm{d}s$$
$$= \frac{1}{2\pi\sqrt{t}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathbf{1}_B(x)\, e^{2xs\frac{1}{2t}}\, e^{-x^2\frac{1}{2t}}\ \mathrm{d}x \right) g_{n,t}(\alpha,s)\, e^{-s^2\frac{1+t}{2t}}\ \mathrm{d}s$$
$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mu_{\mathcal{N}(s,t)}(B)\, e^{s^2\frac{1}{2t}}\, g_{n,t}(\alpha,s)\, e^{-s^2\frac{1+t}{2t}}\ \mathrm{d}s$$
$$= \int_{\mathbb{R}} \mu_{\mathcal{N}(s,t)}(B)\, g_{n,t}(\alpha,s)\ \mathrm{d}\mu_{\mathcal{N}(0,1)}(s) \qquad (B \in \mathcal{B}(\mathbb{R})).$$

With this we obtain (1.10).

A4. Let $n \in \mathbb{N}_{\geq 2}$ and $t \in (0,\infty)$. Note that, with (4.1) and (4.2), $g_{n,t}$ is given by

$$\text{(A.15)}\qquad g_{n,t}(\alpha,s) = \frac{\int_{\mathbb{R}} e^{-n[V(r+\frac{1}{n}(s-r))]}\, e^{-(n-1)\left(r-\frac{\alpha}{1+t}\right)^2\frac{1+t}{2t}}\ \mathrm{d}r}{\int_{\mathbb{R}} e^{-nV(r)} e^{-(n-1)\left(r-\frac{\alpha}{1+t}\right)^2\frac{1+t}{2t}}\ \mathrm{d}r} \qquad (\alpha, s \in \mathbb{R}).$$

The numerator equals

(A.16)
$$\int_{\mathbb{R}} e^{-n\left[V(r+\frac{1}{n}(s-r))\right]}\, e^{-(n-1)\left(r-\frac{\alpha}{1+t}\right)^2\frac{1+t}{2t}}\ \mathrm{d}r = \frac{n}{n-1}\int_{\mathbb{R}} e^{-nV(z)}\, e^{-(n-1)\left(\frac{n}{n-1}z-\frac{1}{n-1}s-\frac{\alpha}{1+t}\right)^2\frac{1+t}{2t}}\ \mathrm{d}z.$$

Via the identities

(A.17)
$$\begin{aligned}&-(n-1)\left(\frac{n}{n-1}z-\frac{1}{n-1}s-\frac{\alpha}{1+t}\right)^2\\ &=-(n-1)\left(z+\frac{1}{n-1}(z-s)-\frac{\alpha}{1+t}\right)^2\\ &=-(n-1)\left(z-\frac{\alpha}{1+t}\right)^2-2\left(z-\frac{\alpha}{1+t}\right)(z-s)-\frac{1}{n-1}(z-s)^2\\ &=-(n-1)\left(z-\frac{\alpha}{1+t}\right)^2-2z^2+2\left(s+\frac{\alpha}{1+t}\right)z-2\frac{\alpha}{1+t}s-\frac{1}{n-1}(z-s)^2,\end{aligned}$$

we get

(A.18)
$$g_{n,t}(\alpha,s)=\frac{n}{n-1}\,e^{-\frac{\alpha}{t}s}\,\frac{\int_{\mathbb{R}} e^{\left[-2z^2+2(s+\frac{\alpha}{1+t})z-\frac{1}{n-1}(z-s)^2\right]\frac{1+t}{2t}}\ e^{-nV(z)}\ e^{-(n-1)(z-\frac{\alpha}{1+t})^2\frac{1+t}{2t}}\ \mathrm{d}z}{\int_{\mathbb{R}} e^{-nV(r)}\, e^{-(n-1)\left(r-\frac{\alpha}{1+t}\right)^2\frac{1+t}{2t}}\ \mathrm{d}r}\qquad(\alpha,s\in\mathbb{R}).$$

A5. We prove existence of ρ_n mentioned in Theorem 1.8 and prove that for $\alpha\in\mathbb{R}$ the large deviation principle holds for $(\rho_n(\alpha,\cdot))_{n\in\mathbb{N}}$ with rate n and rate function given in (1.16).
Let $\mathfrak{J}=\{(0,t_1,\cdots,t_k,t)\colon k\in\mathbb{N}, 0<t_1<\cdots<t_k<t\}$, and define $\pi_j\colon C([0,\infty],\mathbb{R})\to\mathbb{R}^j$ by

(A.19) $\pi_{(0,t_1,\ldots,t_k,t)}(f)=(f(0),f(t_1),\ldots,f(t_k))\qquad (f\in C([0,t],\mathbb{R}), 0<t_1<\cdots<t_k<t).$

Similarly as in item A2, with $\overline{\mu}_{n,j}=\mu_n\circ\pi_j^{-1}\circ m_n^{-1}=\mu_n\circ m_n^{-1}\circ\pi_j^{-1}$ for $j\in\mathfrak{J}$, where $j=(0,t_1,\cdots,t_k,t)$, we get

(A.20)
$$\begin{aligned}\overline{\mu}_{n,j}(A)=&\int_{\mathbb{R}^{\#j}}\mathbf{1}_A(s_0,s_1,\ldots,s_k,s_{k+1})\sqrt{\frac{n}{2\pi(t-t_k)}}\,e^{-n\frac{(s_{k+1}-s_k)^2}{2(t-t_k)}}\prod_{i=1}^k\left[\sqrt{\frac{n}{2\pi(t_i-t_{i-1})}}\,e^{-n\frac{(s_i-s_{i-1})^2}{2(t_i-t_{i-1})}}\right]\\ &\times\frac{1}{Z_n}e^{-n\left[V(s_0)+\frac{s_0^2}{2}\right]}\ \mathrm{d}s_{k+1}\ \mathrm{d}s_k\cdots\mathrm{d}s_1\ \mathrm{d}s_0\qquad(A\in\mathcal{B}(\mathbb{R}^{\#j})).\end{aligned}$$

Then $\rho_{n,t,j}\colon\mathbb{R}\times\mathcal{B}(\mathbb{R}^{\#j-1})\to[0,1]$ defined by

(A.21)
$$\rho_{n,t,j}(\alpha,A)=\frac{\int_{\mathbb{R}^{\#j-1}}\mathbf{1}_A(s_0,s_1,\ldots,s_k)e^{-n\frac{(\alpha-s_k)^2}{2(t-t_k)}}\prod_{i=1}^k\left[e^{-n\frac{(s_i-s_{i-1})^2}{2(t_i-t_{i-1})}}\right]e^{-n\left[V(s_0)+\frac{s_0^2}{2}\right]}\ \mathrm{d}s_k\cdots\mathrm{d}s_1\ \mathrm{d}s_0}{\int_{\mathbb{R}^{\#j-1}}e^{-n\frac{(\alpha-s_k)^2}{2(t-t_k)}}\prod_{i=1}^k\left[e^{-n\frac{(s_i-s_{i-1})^2}{2(t_i-t_{i-1})}}\right]e^{-n\left[V(s_0)+\frac{s_0^2}{2}\right]}\ \mathrm{d}s_k\cdots\mathrm{d}s_1\ \mathrm{d}s_0}$$
$$(A\in\mathcal{B}(\mathbb{R}^{\#j-1})),$$

is the weakly continuous proper regular conditional probability under $\overline{\mu}_{n,j}$ given the coordinate at time t. For all $j \in \mathfrak{J}$ and all $\alpha \in \mathbb{R}$, by Ellis [4, Theorem II.7.2] or den Hollander [11, Theorem III.17], the sequence $(\rho_{n,t,j}(\alpha,\cdot))_{n\in\mathbb{N}}$ satisfies the large deviation principle with rate n and rate function $I_j\colon \mathbb{R}^{\#j-1} \to [0,\infty]$ given by

$$(s_0, s_1, \dots, s_k) \mapsto V(s_0) + \frac{s_0^2}{2} + \sum_{i=1}^{k} \left[\frac{(s_i - s_{i-1})^2}{2(t_i - t_{i-1})}\right] + \frac{(\alpha - s_k)^2}{2(t - t_k)} - \mathfrak{C}_j, \tag{A.22}$$

where $\mathfrak{C}_j$ is such that (A.22) has infimum 0. By Kolmogorov's Theorem (e.g. Bogachev [1, Theorem 7.7.2]), there exists a measure $\rho_{n,t}(\alpha,\cdot)$ on $C([0,t),\mathbb{R})$ (see e.g. [1, Theorem 7.7.4], and use the Cauchy-Schwarz inequality to prove the condition) for which $\rho_{n,t}(\alpha,\cdot) \circ \pi_j^{-1} = \rho_{n,t,j}(\alpha,\cdot)$ for all $j \in \mathfrak{J}$. Because $\alpha \mapsto \rho_{n,t,j}(\alpha,\cdot)$ is strongly continuous for all $n \in \mathbb{N}$ and $j \in \mathfrak{J}$ (see Appendix B), the map $\alpha \mapsto \rho_{n,t}(\alpha,\cdot)$ is (strongly and hence) weakly continuous, i.e., $\rho_{n,t}$ is the weakly continuous proper regular conditional probability of μ_n under π_t. To derive the first statement of Theorem 1.8, we can follow the lines of Dembo and Zeitouni [2, Section 5.1]. First define an exponentially equivalent sequence of measures $(\tilde{\rho}_{n,t}(\alpha,\cdot))_{n\in\mathbb{N}}$ and use that the increments under $\rho_{n,t}(\alpha,\cdot)$ are independent, to prove the analogue of [2, Lemma 5.1.7], i.e., the sequence $(\tilde{\rho}_{n,t}(\alpha,\cdot))_{n\in\mathbb{N}}$ is exponentially tight. Therefore we know that also $(\tilde{\rho}_{n,t}(\alpha,\cdot))_{n\in\mathbb{N}}$ is exponentially tight. With the help of the Dawson-Gärtner projective limit theorem [2, Theorem 4.6.1] we know that the sequence $(\rho_{n,t}(\alpha,\cdot))_{n\in\mathbb{N}}$ satisfies the large deviation principle with rate n and rate function $C([0,t),\mathbb{R}) \to [0,\infty]$ given by $\phi \mapsto \sup_{j\in\mathfrak{J}} I_j(\phi)$. To prove that this rate function is the same as (1.16), we note that if $\phi \in \mathcal{AC}([0,t),\mathbb{R})$ and $\phi(s)$ does not converge to α as $s \uparrow t$, then $\sup_{j\in\mathfrak{J}} I_j(\phi) = \infty$. For the rest we can follow the lines of [2, Proof of Lemma 5.1.6].

B Proper weakly continuous regular conditional probabilities

Definition B.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces with Borel sigma-algebras $\mathcal{B}(\mathcal{X})$ and $\mathcal{B}(\mathcal{Y})$. Equip $\mathcal{X} \times \mathcal{Y}$ with the product topology. Then $\mathcal{B}(\mathcal{X} \times \mathcal{Y}) = \mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{Y})$ (i.e., the smallest sigma-algebra containing all sets $A \times B$ with $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$. Let μ be a probability measure on $\mathcal{B}(\mathcal{X} \times \mathcal{Y})$ and let $\pi\colon \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$ the canonical projection. Then $\gamma\colon \mathcal{Y} \times \mathcal{B}(\mathcal{X}) \to [0,1]$ is called a *regular conditional probability* under μ of the first coordinate given the second, when γ is a transition kernel and

$$\mu(A \times B) = \int \mathbf{1}_B(y)\gamma(y,A)\ \mathrm{d}[\mu \circ \pi^{-1}](y) \qquad \big(A \in \mathcal{B}(\mathcal{X}), B \in \mathcal{B}(\mathcal{Y})\big). \tag{B.1}$$

γ is called *proper* when $\gamma(y,\cdot) = 0$ for all $y \in \operatorname{supp}(\mu \circ \pi^{-1})^c$, where

$$\operatorname{supp}(\nu) = \mathcal{Y} \setminus \bigcup \big\{U \subset \mathcal{Y} \colon U \text{ is open and } \nu(U) = 0\big\} \tag{B.2}$$

for measures ν on $\mathcal{B}(\mathcal{Y})$. γ is called *weakly continuous* when the map $\alpha \to \gamma(\alpha,\cdot)$ is weakly continuous.

Lemma B.2. *With the notation as in Definition* B.1, *if* $\gamma_1, \gamma_2\colon \mathcal{Y} \times \mathcal{B}(\mathcal{X}) \to [0,1]$ *are two proper regular conditional probabilities under* μ *of the first coordinate given the second, then* $\gamma_1(y,\cdot) = \gamma_2(y,\cdot)$ *for* $\mu \circ \pi^{-1}$*-a.e.* $y \in Y$*. Consequently, if there exists a weakly continuous proper regular conditional probability of* μ *under* π*, then it is unique.*

Proof. The first statement can be found in Bogachev [1, Section 10.4]. The second statement follows from the fact that if γ_1 and γ_2 are proper regular conditional probabilities, then $\mu(B) = 1$ for $B = \{y \in \mathrm{supp}(\mu) \colon \gamma_1(y, \cdot) = \gamma_2(y, \cdot)\}$, and hence B is dense in $\mathrm{supp}(\mu)$. So if γ_1 and γ_2 are weakly continuous, then $B = \mathrm{supp}(\mu)$, i.e., $\gamma_1 = \gamma_2$. □

We will use the following lemma to conclude that regular conditional probabilities with a continuous bounded density are weakly continuous. This lemma is an easy consequence of Lebesgue's Dominated Convergence Theorem.

Lemma B.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces with Borel sigma-algebras $\mathcal{B}(\mathcal{X})$ and $\mathcal{B}(\mathcal{Y})$. Let μ be a probability measure on $\mathcal{B}(\mathcal{X})$. Let $f \in C_b(\mathcal{X} \times \mathcal{Y}, \mathbb{R})$. If $\gamma \colon \mathcal{Y} \times \mathcal{B}(\mathcal{X}) \to [0, 1]$ is given by*

$$\gamma(y, A) = \frac{\int \mathbf{1}_A(x) f(y, x)\, \mathrm{d}\mu(x)}{\int f(y, x)\, \mathrm{d}\mu(x)} \qquad \big(y \in \mathcal{Y}, A \in \mathcal{B}(\mathcal{X})\big), \tag{B.3}$$

then γ is weakly continuous (even strongly continuous, i.e., $y \mapsto \gamma(y, A)$ is continuous for all $A \in \mathcal{B}(\mathcal{X})$).

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