Intermittency on catalysts

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Abstract

The present paper provides an overview of results obtained in four recent papers by the authors. These papers address the problem of intermittency for the Parabolic Anderson Model in a *time-dependent random medium*, describing the evolution of a "reactant" in the presence of a "catalyst". Three examples of catalysts are considered: (1) independent simple random walks; (2) symmetric exclusion process; (3) symmetric voter model. The focus is on the annealed Lyapunov exponents, i.e., the exponential growth rates of the successive moments of the reactant. It turns out that these exponents exhibit an interesting dependence on the dimension and on the diffusion constant.

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1 The Parabolic Anderson Model

1.1 Motivation

The *Parabolic Anderson Model* is the partial differential equation

$$\frac{\partial}{\partial t}u(x,t) = \kappa\Delta u(x,t) + \gamma\xi(x,t)u(x,t), \qquad x \in \mathbb{Z}^d,\ t \geq 0, \tag{1.1}$$

for the $\mathbb{R}$-valued random field

$$u = \{u(x,t)\colon\ x \in \mathbb{Z}^d,\ t \geq 0\}, \tag{1.2}$$

where $\kappa \in [0,\infty)$ is the diffusion constant, $\gamma \in [0,\infty)$ is the coupling constant, Δ is the discrete Laplacian, acting on u as

$$\Delta u(x,t) = \sum_{\substack{y \in \mathbb{Z}^d \\ \|y-x\|=1}} [u(y,t) - u(x,t)] \tag{1.3}$$

($\|\cdot\|$ is the Euclidian norm), while

$$\xi = \{\xi(x,t)\colon\ x \in \mathbb{Z}^d,\ t \geq 0\} \tag{1.4}$$

is an $\mathbb{R}$-valued random field that evolves with time and that drives the equation. As initial condition for (1.1) we take

$$u(\cdot\,,0) \equiv 1. \tag{1.5}$$

One interpretation of (1.1) and (1.5) comes from *population dynamics*. Consider a spatially homogeneous system of two types of particles, A (catalyst) and B (reactant), subject to:

(i) A-particles evolve autonomously, according to a prescribed stationary dynamics given by the ξ-field, with $\xi(x,t)$ denoting the number of A-particles at site x at time t;

(ii) B-particles perform independent simple random walks with jump rate $2d\kappa$ and split into two at a rate that is equal to γ times the number of A-particles present at the same location;

(iii) the initial density of B-particles is 1.

Then

$$u(x,t) \;=\; \begin{array}{l}\text{the average number of } B\text{-particles at site } x \text{ at time } t \\ \text{conditioned on the evolution of the } A\text{-particles.}\end{array} \tag{1.6}$$

It is possible to add that B-particles die at rate $\delta \in (0,\infty)$. This amounts to the trivial transformation

$$u(x,t) \to u(x,t)e^{-\delta t}. \tag{1.7}$$

What makes (1.1) particularly interesting is that the two terms in the right-hand side *compete with each other*: the diffusion (of B-particles) described by $\kappa\Delta$ tends to make u flat, while the branching (of B-particles caused by A-particles) described by ξ tends to make u irregular.

1.2 Intermittency

We will be interested in the presence or absence of *intermittency*. Intermittency means that for large t the branching dominates, i.e., the u-field develops sparse high peaks in such a way that u and its moments are each dominated by their own collection of peaks (see Gärtner and König [10], Section 1.3). In the *quenched* situation, i.e., conditional on ξ, this geometric picture of intermittency is well understood for several classes of *time-independent* random potentials ξ (see e.g. Sznitman [16] for Poisson clouds and Gärtner, König and Molchanov [11] for i.i.d. potentials with double-exponential and heavier upper tails; Gärtner and König [10] provides an overview). For *time-dependent* random potentials ξ, however, such a geometric picture is not yet available. Instead one restricts attention to understanding the phenomenon of intermittency indirectly by comparing the successive *annealed* Lyapunov exponents

$$\lambda_p = \lim_{t\to\infty} \Lambda_p(t), \qquad p \in \mathbb{N}, \tag{1.8}$$

with

$$\Lambda_p(t) = \frac{1}{t} \log \mathbb{E}\left([u(0,t)]^p\right)^{1/p}, \qquad p \in \mathbb{N},\ t > 0, \tag{1.9}$$

where $\mathbb{E}$ denotes expectation w.r.t. ξ. One says that the solution u is *p-intermittent* if

$$\lambda_p > \lambda_{p-1}, \tag{1.10}$$

and *intermittent* if (1.10) holds for all $p \in \mathbb{N} \setminus \{1\}$.

Carmona and Molchanov [2] succeeded to investigate the annealed Lyapunov exponents, and to obtain the qualitative picture of intermittency (in terms of these exponents), for potentials of the form

$$\xi(x,t) = \dot{W}_x(t), \tag{1.11}$$

where $\{W_x(t)\colon\, x \in \mathbb{Z}^d,\, t \geq 0\}$ denotes a collection of independent Brownian motions. (In this case, (1.1) corresponds to an infinite system of coupled Itô-diffusions.) They showed that for $d = 1, 2$ intermittency holds for all κ, whereas for $d \geq 3$ p-intermittency holds if and only if the diffusion constant κ is smaller than a critical threshold $\kappa_p = \kappa_p(d,\gamma)$ tending to infinity as $p \to \infty$. They also studied the asymptotics of the quenched Lyapunov exponent in the limit as $\kappa \downarrow 0$, which turns out to be singular. Subsequently, the latter was more thoroughly investigated in papers by Carmona, Molchanov and Viens [3], Carmona, Koralov and Molchanov [1], and Cranston, Mountford and Shiga [4].

In Sections 2–4 we consider three different choices for ξ, namely:

(1) Independent Simple Random Walks.

(2) Symmetric Exclusion Process.

(3) Symmetric Voter Model.

For each of these examples we study the annealed Lyapunov exponents as a function of d, κ and γ. Because of their *non-Gaussian* and *non-independent* spatial structure, these examples require techniques different from those developed for (1.11). Example (1) was studied earlier in Kesten and Sidoravicius [12]. We describe their work in Section 2.2.

By the Feynman-Kac formula, the solution of (1.1) and (1.5) reads

$$u(x,t) = E_x\left(\exp\left[\gamma \int_0^t ds\, \xi\left(X^\kappa(s), t-s\right)\right]\right), \tag{1.12}$$

where X^κ is simple random walk on $\mathbb{Z}^d$ with step rate $2d\kappa$ and E_x denotes expectation with respect to X^κ given $X^\kappa(0) = x$. This formula shows that understanding intermittency amounts to studying the *large deviation behavior* of a random walk *sampling* a time-dependent random field.

2 Independent Simple Random Walks

In this section we consider the case where ξ is a Poisson field of *Independent Simple Random Walks* (ISRW). We first describe the results obtained in Kesten and Sidoravicius [12]. After that we describe the refinements of these results obtained in Gärtner and den Hollander [6].

2.1 Model

ISRW is the Markov process with state space

$$\Omega = (\mathbb{N} \cup \{0\})^{\mathbb{Z}^d} \tag{2.1}$$

whose generator acts on cylindrical functions f as

$$(Lf)(\eta) = \frac{1}{2d} \sum_{(x,y)} \eta(x)[f(\eta^{x\frown y}) - f(\eta)], \tag{2.2}$$

where the sum runs over oriented bonds between neighboring sites, and

$$\eta^{x\frown y}(z) = \begin{cases} \eta(z) & \text{if } z \neq x, y, \\ \eta(x) - 1 & \text{if } z = x, \\ \eta(y) + 1 & \text{if } z = y, \end{cases} \tag{2.3}$$

i.e., $\eta^{x\frown y}$ is the configuration obtained from η by moving a particle from x to y. We choose $\xi(\cdot\,,0)$ according to the Poisson product measure with density $\rho \in (0,\infty)$, i.e., initially each site carries a number of particles that is Poisson distributed with mean ρ. For this choice, the ξ-field is stationary and reversible in time (see Kipnis and Landim [13]).

Under ISRW, particles move around independently as simple random walks, stepping at rate 1 and choosing from neighboring sites with probability $1/2d$ each.

2.2 Main theorems

Kesten and Sidoravicius [12] proved the following. They considered the language of A-particles and B-particles from population dynamics, as mentioned in Section 1.1, and included a death rate $\delta \in [0,\infty)$ for the B-particles (recall (1.7)).

(1) If $d = 1, 2$, then – for any choice of the parameters – the average number of B-particles per site tends to infinity at a rate that is faster than exponential.

(2) If $d \geq 3$, then – for γ sufficiently small and δ sufficiently large – the average number of B-particles per site tends to zero exponentially fast.

(3) If $d \geq 1$, then – conditional on the evolution of the A-particles – there is a phase transition: for small δ the B-particles locally survive, while for large δ they become locally extinct.

Properties (1) and (2) – which are annealed results – are implied by Theorems 2.2 and 2.3 below, while property (3) – which is a quenched result – is not. The main focus of [12] is on survival versus extinction. The approach in [12], being based on path estimates rather than on the Feynman-Kac representation, produces cruder results, but it is more robust against variations of the dynamics.

In Gärtner and den Hollander [6] the focus is on the annealed Lyapunov exponents. Theorems 2.1–2.3 below are taken from that paper.

Theorem 2.1. *Let $d \geq 1$, $\rho, \gamma \in (0,\infty)$ and $p \in \mathbb{N}$.*
(i) For all $\kappa \in [0,\infty)$, the limit in (1.8) *exist.*
(ii) If $\lambda_p(0) < \infty$, then $\kappa \to \lambda_p(\kappa)$ is finite, continuous, non-increasing and convex on $[0,\infty)$.

Let $p_t(x,y)$ denote the probability that simple random walk stepping at rate 1 moves from x to y in time t. Let

$$G_d = \int_0^\infty p_t(0,0)\, dt \tag{2.4}$$

be the Green function at the origin of simple random walk.

Theorem 2.2. *Let $d \geq 1$, $\rho, \gamma \in (0,\infty)$ and $p \in \mathbb{N}$. Then, for all $\kappa \in [0,\infty)$, $\lambda_p(\kappa) < \infty$ if and only if $p < 1/G_d\gamma$.*

It can be shown that if $p > 1/G_d\gamma$, then $\Lambda_p(t)$ in (1.9) grows exponentially fast with t, i.e., the p-th moment of $u(0,t)$ grows double exponentially fast with t. The constant in the exponent can be computed.

In the regime $p < 1/G_d\gamma$, $\kappa \mapsto \lambda_p(\kappa)$ has the following behavior (see Fig. 1):

Theorem 2.3. *Let $d \geq 1$, $\rho, \gamma \in (0,\infty)$ and $p \in \mathbb{N}$ such that $p < 1/G_d\gamma$.*
(i) $\kappa \mapsto \lambda_p(\kappa)$ is continuous, strictly decreasing and convex on $[0,\infty)$.
(ii) For $\kappa = 0$,

$$\lambda_p(0) = \rho\gamma\, \frac{(1/G_d)}{(1/G_d) - p\gamma}. \tag{2.5}$$

(iii) For $\kappa \to \infty$,

$$\lim_{\kappa\to\infty} 2d\kappa[\lambda_p(\kappa) - \rho\gamma] = \rho\gamma^2 G_d + 1_{d=3}\,(2d)^3(\rho\gamma^2 p)^2\,\mathcal{P}_3 \tag{2.6}$$

with

$$\mathcal{P}_3 = \sup_{\substack{f\in H^1(\mathbb{R}^3)\\ \|f\|_2=1}} \Big[\int_{\mathbb{R}^3} dx\,|f(x)|^2 \int_{\mathbb{R}^3} dy\,|f(y)|^2\, \frac{1}{4\pi\|x-y\|} - \int_{\mathbb{R}^3} dx\,|\nabla f(x)|^2 \Big]. \tag{2.7}$$

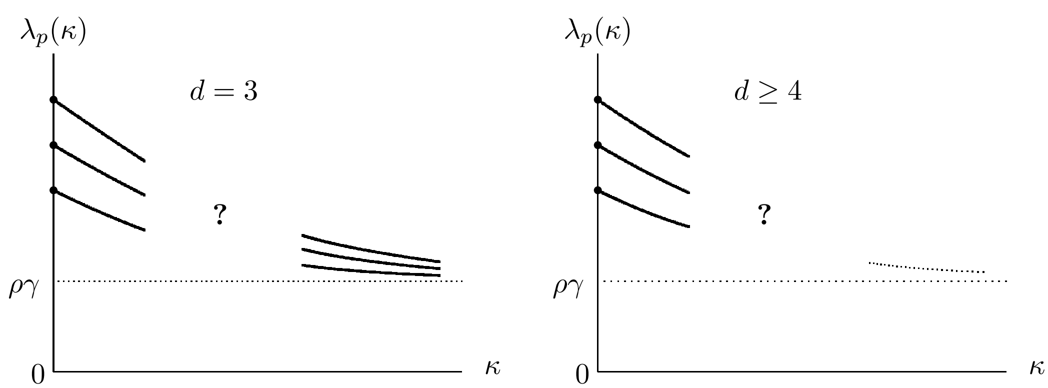

Figure 1: $\kappa \mapsto \lambda_p(\kappa)$ for $p = 1, 2, 3$ when $p < 1/G_d\gamma$ for simple random walk in $d = 3$ and $d \geq 4$.

2.3 Discussion

Theorem 2.2 says that if the catalyst is driven by a recurrent random walk ($G_d = \infty$), then it can pile up near the origin and make the reactant grow at an unbounded rate, while if the catalyst is driven by a transient random walk ($G_d < \infty$), then small enough moments of the reactant grow at a finite rate. We refer to this dichotomy as the *strongly catalytic*, respectively, the *weakly catalytic* regime.

Theorem 2.3(i) shows that, even in the weakly catalytic regime, some degree of *clumping* of the catalyst occurs, in that the growth rate of the reactant is $> \rho\gamma$, the average medium growth rate. As the diffusion constant κ of the reactant increases, the effect of the clumping of the catalyst on the reactant gradually diminishes, and the growth rate of the reactant gradually decreases to $\rho\gamma$.

Theorem 2.3(ii) shows that, again in the weakly catalytic regime, if the reactant stands still, then the system is intermittent. Apparently, the successive moments of the reactant are sensitive to *successive degrees of clumping*. By continuity, intermittency persists for small κ.

Theorem 2.3(iii) shows that all Lyapunov exponents decay to $\rho\gamma$ as $\kappa \to \infty$ in the same manner when $d \geq 4$ but not when $d = 3$. In fact, in $d = 3$ intermittency persists for large κ. It remains open whether the same is true for $d \geq 4$. To decide the latter, we need a finer asymptotics for $d \geq 4$. A large diffusion constant of the reactant hampers localization of the reactant around regions where the catalyst clumps, but it is not a priori clear whether this is able to destroy intermittency for $d \geq 4$. We conjecture:

Conjecture 2.4. *In $d = 3$, the system is intermittent for all $\kappa \in [0, \infty)$.*

Conjecture 2.5. *In $d \geq 4$, there exists a strictly increasing sequence $0 < \kappa_2 < \kappa_3 < \ldots$ such that for $p = 2, 3, \ldots$ the system is p-intermittent if and only if $\kappa \in [0, \kappa_p)$.*

In words, we conjecture that in $d = 3$ the curves in Fig. 1 never merge, whereas for $d \geq 4$ the curves merge successively.

What is remarkable about the scaling of $\lambda_p(\kappa)$ as $\kappa \to \infty$ in (2.6) is that $\mathcal{P}_3$ is the variational problem for the so-called *polaron model*. Here, one considers the quantity

$$\theta(t;\alpha) = \frac{1}{\alpha^2 t} \log E_0 \left(\exp\left[\alpha \int_0^t ds \int_s^t du \, \frac{e^{-(u-s)}}{|\beta(u) - \beta(s)|} \right] \right), \tag{2.8}$$

where $\alpha > 0$ and $(\beta(t))_{t \geq 0}$ is standard Brownian motion on $\mathbb{R}^3$ starting at $\beta(0) = 0$. Donsker and Varadhan [5] proved that

$$\lim_{\alpha\to\infty} \lim_{t\to\infty} \theta(t;\alpha) = 4\sqrt{\pi}\, \mathcal{P}_3. \tag{2.9}$$

Lieb [14] proved that (2.7) has a unique maximizer modulo translations and that the centered maximizer is radially symmetric, radially non-increasing, strictly positive and smooth. A deeper analysis shows that the link between the scaling of $\lambda_p(\kappa)$ for $\kappa \to \infty$ and the scaling of the polaron for $\alpha \to \infty$ comes from *moderate* deviation behavior of ξ and *large* deviation behavior of the occupation time measure of X^κ in (1.12). For details we refer to Gärtner and den Hollander [6].

3 Symmetric Exclusion Process

In this section we consider the case where ξ is the *Symmetric Exclusion Process* (SEP) in equilibrium. We summarize the results obtained in Gärtner, den Hollander and Maillard [7], [8].

3.1 Model

Let $p\colon \mathbb{Z}^d \times \mathbb{Z}^d \to [0,1]$ be the transition kernel of an irreducible symmetric random walk. SEP is the Markov process with state space

$$\Omega = \{0,1\}^{\mathbb{Z}^d} \tag{3.1}$$

whose generator L acts on cylindrical functions f as

$$(Lf)(\eta) = \sum_{\{x,y\}\subset\mathbb{Z}^d} p(x,y)\,\left[f\left(\eta^{x,y}\right) - f(\eta)\right], \tag{3.2}$$

where the sum runs over unoriented bonds between any pair of sites, and

$$\eta^{x,y}(z) = \begin{cases} \eta(z) & \text{if } z \neq x,y, \\ \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y. \end{cases} \tag{3.3}$$

In words, the states of x and y are interchanged along the bond $\{x,y\}$ at rate $p(x,y)$. We choose $\xi(\cdot\,,0)$ according to the Bernoulli product measure with density $\rho \in (0,1)$. For this choice, the ξ-field is stationary and reversible in time (see Liggett [15]).

Under SEP, particles move around independently according to the symmetric random walk transition kernel $p(\cdot,\cdot)$, but subject to the restriction that no two particles can occupy the same site. A special case is simple random walk

$$p(x,y) = \begin{cases} \frac{1}{2d} & \text{if } \|x-y\| = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

3.2 Main theorems

Theorem 3.1. *Let $d \geq 1$, $\rho \in (0,1)$, $\gamma \in (0,\infty)$ and $p \in \mathbb{N}$.*
(i) For all $\kappa \in [0,\infty)$, the limit in (1.8) *exists and is finite.*
(ii) On $[0,\infty)$, $\kappa \to \lambda_p(\kappa)$ is continuous, non-increasing and convex.

The following dichotomy holds (see Fig. 2):

Theorem 3.2. *Let $d \geq 1$, $\rho \in (0,1)$, $\gamma \in (0,\infty)$ and $p \in \mathbb{N}$.*
(i) If $p(\cdot,\cdot)$ is recurrent, then $\lambda_p(\kappa) = \gamma$ for all $\kappa \in [0,\infty)$.
(ii) If $p(\cdot,\cdot)$ is transient, then $\rho\gamma < \lambda_p(\kappa) < \gamma$ for all $\kappa \in [0,\infty)$. Moreover, $\kappa \mapsto \lambda_p(\kappa)$ is strictly decreasing with $\lim_{\kappa\to\infty} \lambda_p(\kappa) = \rho\gamma$. Furthermore, $p \mapsto \lambda_p(0)$ is strictly increasing.

For transient simple random walk, $\kappa \mapsto \lambda_p(\kappa)$ has the following behavior (similar as in Fig. 1):

Theorem 3.3. *Let $d \geq 3$, $\rho \in (0,1)$, $\gamma \in (0,\infty)$ and $p \in \mathbb{N}$. Assume* (3.4). *Then*

$$\lim_{\kappa\to\infty} 2d\kappa[\lambda_p(\kappa) - \rho\gamma] = \rho(1-\rho)\gamma^2 G_d + 1_{\{d=3\}}\,(2d)^3[\rho(1-\rho)\gamma^2 p]^2 \mathcal{P}_3 \tag{3.5}$$

with G_d and $\mathcal{P}_3$ as defined in (2.4) *and* (2.7).

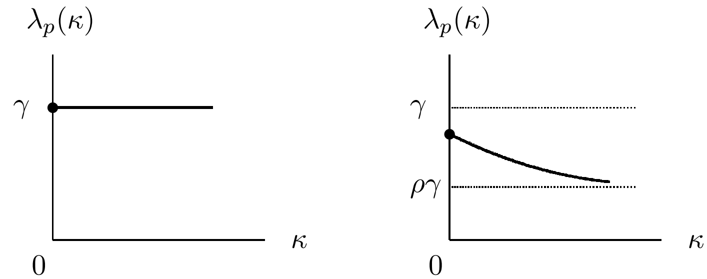

Figure 2: Qualitative picture of $\kappa \mapsto \lambda_p(\kappa)$ for recurrent, respectively, transient random walk.

3.3 Discussion

The intuition behind Theorem 3.2 is the following. If the catalyst is driven by a recurrent random walk, then it suffers from "traffic jams", i.e., with not too small a probability there is a large region around the origin that the catalyst fully occupies for a long time. Since with not too small a probability the simple random walk (driving the reactant) can stay inside this large region for the same amount of time, the average growth rate of the reactant at the origin is maximal. This phenomenon may be expressed by saying that *for recurrent random walk clumping of the catalyst dominates the growth of the moments.* For transient random walk, on the other hand, clumping of the catalyst is present (the growth rate of the reactant is $> \rho\gamma$), but it is *not* dominant (the growth rate of the reactant is $< \gamma$). Again, when the reactant stands still or moves slowly, the successive moments of the reactant are sensitive to successive degrees of clumping of the catalyst. As the diffusion constant κ of the reactant increases, the effect of the clumping of the catalyst on the reactant gradually diminishes and the growth rate of the reactant gradually decreases to $\rho\gamma$.

Theorem 3.3 has the same interpretation as its analogue Theorem 2.3(iii) for ISRW. We conjecture that the same behavior occurs for SEP as in Conjectures 2.4–2.5 for ISRW.

4 Symmetric Voter Model

In this section we consider the case where ξ is the *Symmetric Voter Model* (SVM) in equilibrium, or converging to equilibrium from a product measure. We summarize the results obtained in Gärtner, den Hollander and Maillard [9].

4.1 Model

As in Section 3, we abbreviate $\Omega = \{0,1\}^{\mathbb{Z}^d}$ and we let $p\colon \mathbb{Z}^d \times \mathbb{Z}^d \to [0,1]$ be the transition kernel of an irreducible symmetric random walk. The SVM is the Markov process on Ω whose generator L acts on cylindrical functions f as

$$(Lf)(\eta) = \sum_{x,y\in\mathbb{Z}^d} 1_{\{\eta(x)\neq\eta(y)\}}\, p(x,y)\ [f(\eta^y) - f(\eta)]\,, \tag{4.1}$$

where

$$\eta^y(z) = \begin{cases} \eta(z) & \text{if } z \neq y, \\ 1-\eta(y) & \text{if } z = y. \end{cases} \tag{4.2}$$

In words, site x imposes its state on site y at rate $p(x,y)$. The states 0 and 1 are referred to as opinions or, alternatively, as vacancy and particle. Contrary to ISRW and SEP, SVM is a non-conservative and non-reversible dynamics: opinions are not preserved.

We will consider two choices for the starting measure of ξ:

$$\begin{cases} \nu_\rho, \text{ the Bernoulli product measure with density } \rho \in (0,1), \\ \mu_\rho, \text{ the equilibrium measure with density } \rho \in (0,1). \end{cases} \tag{4.3}$$

The ergodic properties of the SVM are qualitatively different for recurrent and for transient transition kernels. In particular, when $p(\cdot,\cdot)$ is recurrent all equilibria are trivial, i.e., $\mu_\rho = (1-\rho)\delta_0 + \rho\delta_1$, while when $p(\cdot,\cdot)$ is transient there are also non-trivial equilibria, i.e., ergodic μ_ρ parameterized by the density ρ. When starting from ν_ρ, $\xi(\cdot\,,t)$ converges in law to μ_ρ as $t \to \infty$.

4.2 Main theorems

Theorem 4.1. *Let $d \geq 1$, $\kappa \in [0,\infty)$, $\rho \in (0,1)$, $\gamma \in (0,\infty)$ and $p \in \mathbb{N}$.*
(i) For all $\kappa \in [0,\infty)$, the limit in (1.8) *exists and is finite, and is the same for the two choices of starting measure in* (4.3)*.*
(ii) On $\kappa \in [0,\infty)$, $\kappa \to \lambda_p(\kappa)$ is continuous.

The following dichotomy holds (see Fig. 3):

Theorem 4.2. *Suppose that $p(\cdot,\cdot)$ has finite variance. Fix $\rho \in (0,1)$, $\gamma \in (0,\infty)$ and $p \in \mathbb{N}$.*
(i) If $1 \leq d \leq 4$, then $\lambda_p(\kappa) = \gamma$ for all $\kappa \in [0,\infty)$.
(ii) If $d \geq 5$, then $\rho\gamma < \lambda_p(\kappa) < \gamma$ for all $\kappa \in [0,\infty)$.

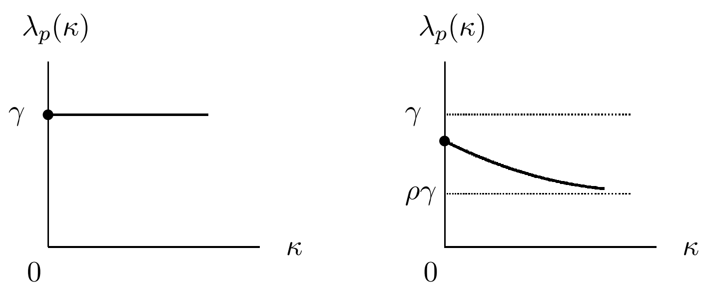

Figure 3: Qualitative picture of $\kappa \mapsto \lambda_p(\kappa)$ for symmetric random walk with finite variance in $d = 1,2,3,4$, respectively, $d \geq 5$.

Theorem 4.3. *Suppose that $p(\cdot,\cdot)$ has finite variance. Fix $\rho \in (0,1)$ and $\gamma \in (0,\infty)$. If $d \geq 5$, then $p \mapsto \lambda_p(0)$ is strictly increasing.*

4.3 Discussion

Theorem 4.2 shows that the Lyapunov exponents exhibit a dichotomy similar to those found for ISRW and SEP (see Fig. 3). The crossover in dimensions is at $d = 5$ rather than at $d = 3$. Theorem 4.3 shows that the system is intermittent at $\kappa = 0$ when the Lyapunov exponents are nontrivial, which is similar as well.

We conjecture that the following properties hold, whose analogues for ISRW and SEP are known to be true:

Conjecture 4.4. *On* $[0,\infty)$, $\kappa \mapsto \lambda_p(\kappa)$ *is strictly decreasing and convex with* $\lim_{\kappa\to\infty} \lambda_p(\kappa) = \rho\gamma$.

We close with a conjecture about the scaling behavior for $\kappa \to \infty$.

Conjecture 4.5. *Let* $d \geq 5$, $\rho \in (0,\infty)$ *and* $p \in \mathbb{N}$. *Assume* (3.4). *Then*

$$\lim_{\kappa\to\infty} 2d\kappa[\lambda_p(\kappa) - \rho\gamma] = \rho(1-\rho)\gamma^2 \frac{G_d^*}{G_d} + 1_{\{d=5\}}(2d)^3 \left[\rho(1-\rho)\gamma^2 \frac{1}{G_d} p\right]^2 \mathcal{P}_5 \tag{4.4}$$

with

$$\begin{aligned} G_d &= \int_0^\infty p_t(0,0)\,dt, \\ G_d^* &= \int_0^\infty t\,p_t(0,0)\,dt, \end{aligned} \tag{4.5}$$

and

$$\mathcal{P}_5 = \sup_{\substack{f\in H^1(\mathbb{R}^5)\\ \|f\|_2=1}} \left[\int_{\mathbb{R}^5} dx\,|f(x)|^2 \int_{\mathbb{R}^5} dy\,|f(y)|^2 \frac{1}{16\pi^2\|x-y\|} - \int_{\mathbb{R}^5} dx\,|\nabla f(x)|^2 \right]. \tag{4.6}$$

5 Concluding remarks

The theorems listed in Sections 2–4 show that the intermittent behavior of the reactant for the three types of catalyst exhibits interesting similarities and differences. ISRW, SEP and SVM each show a dichotomy of strongly catalytic versus weakly catalytic behavior, for ISRW between divergence and convergence of the Lyapunov exponents, for SEP and SVM between maximality and non-maximality. Each also shows an interesting dichotomy in the dimension for the scaling behavior at large diffusion constants, with $d = 3$ being critical for ISRW and SEP, and $d = 5$ for SVM. For ISRW and SEP the same polaron term appears in the scaling limit, while for SVM an analogous but different polaron-like term appears. Although the techniques we use for the three models differ substantially, there is a universal principle behind their scaling behavior. See the heuristic explanation offered in [6] and [7].

Both ISRW and SEP are conservative and reversible dynamics. The reversibility allows for the use of spectral techniques, which play a key role in the analysis. The SVM, on the other hand, is a non-conservative and irreversible dynamics. The non-reversibility precludes the use of spectral techniques, and this dynamics is therefore considerably harder to handle.

Both for SEP and SVM, the graphical representation is a powerful tool. For SEP this graphical representation builds on random walks, for SVM on coalescing random walks (see Liggett [15]).

The reader is invited to look at the original papers for details.

References

[1] Carmona, R.A., Koralov, L. and Molchanov, S.A. (2001). Asymptotics for the almost-sure Lyapunov exponent for the solution of the parabolic Anderson problem. *Random Oper. Stochastic Equations*, **9**, 77–86.

[2] Carmona, R.A. and Molchanov, S.A. (1994). *Parabolic Anderson Problem and Intermittency.* AMS Memoir 518. American Mathematical Society, Providence RI.

[3] Carmona, R.A., Molchanov S.A. and Viens, F. (1996). Sharp upper bound on the almost-sure exponential behavior of a stochastic partial differential equation. *Random Oper. Stochastic Equations*, **4**, 43–49.

[4] Cranston, M., Mountford, T.S. and Shiga, T. (2002). Lyapunov exponents for the parabolic Anderson model. *Acta Math. Univ. Comeniane*, **71**, 163–188.

[5] Donsker M.D. and Varadhan S.R.S. (1983). Asymptotics for the polaron. *Comm. Pure Appl. Math.*, **36**, 505–528.

[6] Gärtner, J. and den Hollander, F. (2006). Intermittency in a catalytic random medium. *Ann. Probab.*, **34**, 2219–2287.

[7] Gärtner, J., den Hollander, F. and Maillard, G. (2007). Intermittency on catalysts: symmetric exclusion. *Electr. J. Probab.*, **12**, 516–573.

[8] Gärtner, J., den Hollander, F. and Maillard, G. *Intermittency on catalysts: scaling for three-dimensional simple symmetric exclusion.* Preprint.

[9] Gärtner, J., den Hollander, F. and Maillard, G. *Intermittency on catalysts: voter model.* Work in progress.

[10] Gärtner, J. and König, W. (2005). The parabolic Anderson model. In: *Interacting Stochastic Systems* (J.-D. Deuschel and A. Greven, eds.). Springer, Berlin, pp. 153–179.

[11] Gärtner, J., König, W. and Molchanov, S.A. (2007). Geometric characterization of intermittency in the parabolic Anderson model. *Ann. Probab.*, **35**, 439–499.

[12] Kesten, H. and Sidoravicius, V. (2003). Branching random walk with catalysts. *Electr. J. Probab.*, **8**, 1–51.

[13] Kipnis, C. and Landim, C. (1999). *Scaling Limits of Interacting Particle Systems.* Grundlehren der Mathematischen Wissenschaften **320**. Springer, Berlin.

[14] Lieb, E.H. (1977). Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation. *Stud. Appl. Math.*, **57**, 93–105.

[15] Liggett, T.M. (1985). *Interacting Particle Systems.* Grundlehren der Mathematischen Wissenschaften **276**. Springer, New York.

[16] Sznitman, A.-S. (1998). *Brownian Motion, Obstacles and Random Media.* Springer, Berlin.