Local Well-posedness of Kinetic Chemotaxis Models

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Abstract

This paper presents a general functional analytic setting in which the Cauchy problem for mild solutions of kinetic chemotaxis models is well-posed, locally in time, in general physical dimensions. The models consist of a hyperbolic transport equation that is non-linearly and non-locally coupled to a reaction-diffusion system through kernel operators. Three examples are elaborated throughout the paper: the reaction-diffusion system is (1) a single linear equation, (2) a FitzHugh-Nagumo system with cubic nonlinearity and (3) a FitzHugh-Nagumo system with a piecewise linear approximation of a cubic nonlinearity. We use a limit argument to obtain solutions in $L^1 \cap L^\infty$. The presented results are a first step in further analysis of these coupled systems: global existence of solutions, positivity, attractors and limit approximations (e.g. diffusion and hydrodynamic limits).

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Contents

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1 Introduction

Chemotaxis is a process in biology in which moving organisms react to a chemical agent in their environment. It plays an important role in the aggregation of bacteria and of amoebae, like *Dicyostelium discoideum* (*Dd*). The latter normally lead a solitary life. When starving however, cells start to secrete, detect and move chemotactically to cyclic adenosine monophosphate (cAMP). After secretion it diffuses in the environment, where it is detected by other *Dd*-cells. These move in the direction of the concentration gradient and secrete additional cAMP in the environment. The amoebae also produce and secrete an enzyme (phosphodiesterase) in their environment that degrades intercellular cAMP. The combined processes lead to self-organised aggregation, see e.g. [3], [4],[17],[26].

Various mathematical models exist for this phenomenon, at different levels of detail. The family of macroscopic Keller-Segel type of reaction-diffusion models is well-known and has been studied intensively ([26], see [21],[22] for an extensive review). Lately, interest in (macroscopic) hyperbolic chemotaxis models, like the Cataneo models, have risen [14],[19]. They consist of a hyperbolic equation for the cell density and flow field that is coupled to a reaction-diffusion equation for the chemotactic signal. These models have the benefit of finite propation speed over the diffusion models, which allow for cell propagation with infinite speed.

Both types of models may be derived by means of singular perturbation methods from an underlying, mesoscopic, *kinetic chemotaxis model* (see e.g. [6], [14], [24], [29]). It is itself an approximation of a microscopic stochastic velocity-jump process [1],[28],[31].

The kinetic chemotaxis model describes the evolution of the position-velocity distribution $f = f(x, v, t)$ of cells at position $x \in \mathbb{R}^n$ with velocity $v \in V$ at time t. It is valid when cell densities are not too large and there is hardly interaction between cells. Accordingly these models and the Keller-Segel class of chemotaxis models can only cover the so-called 'early aggregation phase'. The objective of various researchers, including us, is to extend the kinetic models such that detailed signalling dynamics in the cell's interior and cell-cell interactions are included [13], [7]. Then also the later stage of aggregation and the streaming phase may be treated. It is one of the purposes of this paper to make appropriate functional analytic preparations for the early aggregation models in order to be able to deal with the latter cases thereafter.

The kinetic chemotaxis models are integro-differential equations of mixed type. If we denote by $S = S(x,t)$ the concentration of cAMP, then in the kinetic model f satisfies a *hyperbolic* transport equation of the form

$$\partial_t f(t) = -v \cdot \nabla_x f(t) + \mathcal{T}(t, S(t), f(t)), \tag{1}$$

where the *turning operator* $\mathcal{T}(t, S(t), f(t))$ is a kernel operator:

$$\mathcal{T}(S,f)(x,v) = -\int_V T[S,f](x,v',v)f(x,v) + T[S,f](x,v,v')f(x,v')d\mu(v') \tag{2}$$

$$= -\lambda[S,f](x,v) \cdot f(x,v) + \int_V T[S,f](x,v,v')f(x,v')d\mu(v'), \tag{3}$$

Here we suppressed the time dependence. The function $\lambda[S,f]$, given by

$$\lambda[S,f](x,v) := \int_V T[S,f](x,v',v)d\mu(v'), \tag{4}$$

is the *turning rate*. $T[S,f]$ is called the *turning kernel*. It models the interplay between the external chemotactic signal and the organisms' movement behaviour. The velocity space V is a compact subset of $\mathbb{R}^n$ (cells cannot move arbitrarily fast) that carries a Radon measure μ.

The evolutionary equation for f is coupled to a *parabolic* reaction-diffusion system for the signal S. In the Keller-Segel type of model originally there was a three-variable reaction-diffusion system, which was already simplified in [26] to a single reaction-diffusion equation. A linear reaction-diffusion equation for S,

$$\tau \partial_t S = d\Delta S + \alpha\rho - \beta S, \qquad (\alpha, \beta, \tau, d > 0), \tag{5}$$

in which ρ is the spatial cell distribution,

$$\rho(x,t) := \int_V f(x,v,t)dv, \tag{6}$$

has subsequently been studied most, also in combination with (1), see [27], [6], [20], [21],[22]. Note that the coupling between the f and S-equation is non-local. We shall refer to the coupled system (1)–(5) as the '*linear reaction-diffusion*' (kinetic chemotaxis) model. In various studies concerning the diffusion and hydrodynamic (hyperbolic) limit of the kinetic chemotaxis model, (5) has been replaced by its quasi-steady state approximation ('*instantaneous diffusion*') with $\beta = 0$ (e.g. [6],[24]):

$$-\Delta S = a\,\rho, \quad (a > 0). \tag{7}$$

The relay system for cAMP behaves like an excitable medium however. A detailed macroscopic model has been proposed in [32]. This forms a starting point for research on kinetic models for chemotaxis that include excitable effects. To that end, we discuss here a functional analytic set-up for the simplest of excitable models for the signalling, i.e. treat a FitzHugh-Nagumo type of charicature of the latter, instead of the more complicated one formulated in [32]. In this model, the signal S evolves according to

$$\tau_s\, \partial_t S = D\Delta S - \rho^\alpha(k_s\, \varphi(S) \,+\, k_r\, r), \tag{8}$$

$$\tau_r\, \partial_t r = \gamma S - r, \tag{9}$$

$\alpha, \tau_s, \tau_r, k_s, k_r, d, \gamma > 0)$, where φ is the cubic polynomial

$$\varphi(S) = S(S - S_1)(S - S_2) \tag{10}$$

or a piecewise linear approximation thereof. We shall call this model class the '*excitable medium model*'. It has been used in [34] and [35] for numerical simulations concerning aggregation and mound formation of *Dd* amoebae. Notice that the nonlinearity in (8) is locally Lipschitz continuous only when $\alpha \geq 1$.

In many examples in the literature V is the closure of a bounded domain equipped with the restriction of Lebesgue measure, or a sphere with its rotation invariant measure. In other settings, the symmetry condition $V = -V$ may be imposed, together with the assumption of invariance of μ with respect to this symmetry. μ may be normalised: $\mu(V) = 1$. Apart from compactness of V and that μ is a Radon measure on V, we do not impose any of these symmetry conditions here.

Turning operators that are linear in f, i.e. $T[S, f]$ does not depend on f, have been studied the most so far. We shall limit our treatment also to *time-independent linear turning operators* (which may depend non-linearly on S though). Turning operators that depend nonlinearly on f have been used for example to model blood vessel formation (see [14] and the reference given there).

This paper will present a precise functional analytic setting for the linear reaction-diffusion and excitable medium type of kinetic chemotaxis models for which we prove local well-posedness for mild solutions. The mixed, non-local setting makes this problem interesting and not straightforward. Hillen and Stevens [20] presented a proof for existence and uniqueness of solutions for $n = 1$, $V = \{-1, +1\}$ (μ is the counting measure), in the sense that

$$(f, S) \in L^\infty([0, T], L^\infty(\mathbb{R} \times V) \times W^{1,\infty}(\mathbb{R})), \tag{11}$$

under a suitable condition on the map $S \mapsto T[S]$. They used signal equation (5).

We extend these results to arbitrary higher physical dimensions n and general compact velocity spaces $V \subset \mathbb{R}^n$ using a different approach. We obtain local well-posedness results for mild solutions

$$(f, S) \in C([0, T], L^{p_0} \cap L^{p_0'}(\mathbb{R}^n \times V) \times W^{k,p_1} \cap C_0^k(\mathbb{R})).$$

for $k = 0$ or 1 in Section 4.4 and 4.5. Here k denotes the highest order spatial derivative of S on which $T[S]$ is allowed to depend.

Our conditions on the functional form $S \mapsto T[S]$, formulated in Assumption (AT) in Section 4.4, seem to be more general. We consider both the linear reaction-diffusion equation (5) and the excitable medium model (8)–(9). In Section 5 we present a practical class of functional forms for the map $S \mapsto T[S]$ and sufficient conditions under which the main assumption (AT) holds. Section 5.3 shows how various examples from the literature fit into this framework.

Both in the functional analytic set-up and the verification of Assumption (AT) we need results on local Lipschitz continuity of superposition mappings between L^∞-spaces. We present and prove these results in Section 2. Section 6 is devoted to showing that the semiflow (so far defined for finite time) leaves invariant the dense subspace $L^1 \cap L^\infty$ in the f-state space. That is, an initial condition f_0 in $L^1 \cap L^\infty$ yields a mild solution that has its f-component in $L^1 \cap L^\infty$ on its maximal time-interval of existence. In Section 6 we also show that our results imply existence (local-in-time) of solutions in the sense of (11).

The presented results serve as an initial step towards further research on the long-term behaviour of these kinetic chemotaxis models in relation to that of their limit approximations (e.g. diffusion or hydrodynamic limit) from a semigroup point of view. We shall consider global existence, positivity and attractors for these kinetic models. A clear functional analytic framework is then essential.

Conventions. $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of positive integers; $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We write $\mathbb{R}_+$ for the set of nonnegative real numbers. If (Ω, Σ, μ) is a measure space, we will occasionally write $L^p \cap L^q(\Omega, \mu)$ instead of the longer $L^p(\Omega, \mu) \cap L^q(\Omega, \mu)$. Similarly, if Ω is also a topological space or a manifold, we write $C^k \cap L^p(\Omega, \mu)$ instead of $C^k(\Omega) \cap L^p(\Omega, \mu)$. We denote by $W^{k,p}(\mathbb{R}^n)$ the usual Sobolev space of L^p-functions whose distributional derivatives up to order k are also L^p-functions. Here $\mathbb{R}^n$ is equipped with Lebesgue measure m. We use multi-index notation for partial differentiation on $\mathbb{R}^n$, i.e. $\partial_j := \frac{\partial}{\partial x_j}$ and $\partial^\alpha := \prod_{j=1}^n \partial_j^{\alpha_j}$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, $|\alpha| := \sum_j |\alpha_j|$. $C_0(\mathbb{R}^n)$ is the Banach space of continuous funtions on $\mathbb{R}^n$ that vanish at infinity. We denote by $C_0^k = C_0^k(\mathbb{R}^n)$ the space of functions on $\mathbb{R}^n$ for which $\partial^\alpha f$ is in $C_0(\mathbb{R}^n)$ for all α with $|\alpha| \leq k$. If X and Y are Banach spaces and X embeds continuously into Y, we call X a Banach subspace of Y.

For a separable Banach space X, and σ-finite measure space (Ω_1, μ_1) we denote by $L^\infty_{w^*}(\Omega_1, \mu_1, X^*)$ the Banach space consisting of equivalence classes of weak*-measurable functions $f : \Omega_1 \to X^*$ with norm given by the essential supremum of the measurable function $\omega \mapsto \sup_{x \in X : \|x\| \leq 1} |\langle f(\omega), x \rangle|$. The latter function is measurable because of the separability of X. If $\Omega_2, \mu_2)$ is also a σ-finite measure space then $L^\infty_{w^*}(\Omega_1, \mu_1; L^\infty(\Omega_2, \mu_2))$ can be identified with $L^\infty(\Omega_1 \times \Omega_2, \mu_1 \times \mu_2)$.

2 Lipschitz properties of superposition mappings

Let Ω_1 and Ω_2 be sets and let X and Y be vector spaces. Functions $\Phi : \Omega_2 \to \Omega_1$ and $F : \Omega_1 \times X \to Y$ define an *inner* and *outer superposition mapping* (or

Nemytskii-mapping)

$$\Phi^*(\varphi) := \varphi \circ \Phi, \quad N_F(\varphi)(\omega) := F(\omega, \varphi(\omega))$$

respectively, on X-valued functions φ on Ω_1. The Nemytskii mapping has been intensively studied in non-linear analysis, in the setting where the spaces Ω_i are measure spaces and X and Y are Banach spaces (see e.g. [2],[9],[33]). Let us denote by $\mathcal{M}(\Omega, X)$ the vector space of X-valued μ-measureable funtions on Ω in the sense of Bochner (i.e. functions that are the pointwise limit of a sequence of simple functions [10]). Let $M(\Omega, X)$ the vector space of equivalence classes in $\mathcal{M}(\Omega, X)$, where functions that are equal μ-almost everywhere are identified. The main questions that have been studied, are first the characterisation of functions Φ and F that result in superposition mappings that are well-defined on the space $M(\Omega_1, X)$ and map this space into a similar space $M(\Omega_2, Y)$, secondly a characterisation of these functions that result in maps between particular spaces of integrable functions (e.g. L^p-spaces, Sobolev spaces, etc.) and thirdly continuity and differentiability properties of the latter maps.

We shall encounter both types of superposition mappings in the sequel. There we need results on Lipschitz properties of these maps between vector-valued L^∞-spaces. This question seems to have received less attention, both in the case of finite and infinite-dimensional Banach spaces. In this section we present some new results on this topic. We start with outer superposition mappings.

2.1 Outer superposition mappings between L^∞-spaces

A *generalized Carathéodory function* is a function $F : \Omega \times X \to Y$ that satisfies

(GC1) For each $x \in X$, $F(\cdot, x) : \Omega \to Y$ is μ-measurable,

(GC2) For almost all $\omega \in \Omega$, $F(\omega, \cdot) : X \to Y$ is continuous.

The following lemma is fundamental:

Lemma 2.1 *If F is a generalized Carathéodory function and $\varphi : \Omega \to X$ is μ-measurable, then $\omega \mapsto F(\omega, \varphi(\omega)) : \Omega \to Y$ is μ-measurable. Moreover, N_F induces a well-defined map from $M(\Omega, X)$ into $M(\Omega, Y)$.*

Proof: Let (φ_n) be a sequence of simple functions from Ω to X such that φ_n converges pointwise to φ almost everywhere. $\omega \mapsto F(\omega, \varphi_n(\omega))$ is a μ-measurable function $\Omega \to Y$ for each n because of Assumption (GC1). For almost all ω, $F(\omega, \varphi_n(\omega))$ converges to $F(\omega, \varphi(\omega))$ because of Assumption (GC2). Thus $\omega \to F(\omega, \varphi(\omega))$ is μ-measurable.

Let $\varphi, \psi \in \mathcal{M}(\Omega, X)$ such that $\varphi = \psi$ a.e.. The set on which $N_F(\varphi)$ and $N_F(\psi)$ differ is measurable by the first part of the proof and it is contained in the nullset on which φ and ψ differ. Thus $N_F(\varphi)$ and $N_F(\psi)$ are in the same equivalence class in $M(\Omega, Y)$. □

We call a function $F : \Omega \times X \to Y$ *locally Lipschitz continuous in X, essentially uniformly on* Ω, if for each bounded set $B \subset X$ there exists a null set N_B and a constant L_B such that

$$\|F(\omega, x_1) - F(\omega, x_2)\|_Y \leq L_B \|x_1 - x_2\|_X$$

for all $x_1, x_2 \in B$ and all $\omega \in \Omega \setminus N_B$.

Proposition 2.2 *Let $F : \Omega \times X \to Y$ be a generalized Carathédory function and N_F the associated Nemytskii mapping. The following statements are equivalent:*

(i) N_F maps $L^\infty(\Omega, X)$ into $L^\infty(\Omega, Y)$ and is locally Lipschitz continuous,

(ii) (a) For all $x \in X$, $F(\cdot, x) \in L^\infty(\Omega, Y)$ and (b) F is locally Lipschitz continuous in X, essentially uniformly on Ω.

(iii) For all bounded sets $B \subset X$, there exists a null set N_B and constants C_B and L_B such that

$$\|F(\omega, x)\|_Y \leq C_B, \tag{12}$$

$$\|F(\omega, x) - F(\omega, x')\|_Y \leq L_B \|x - x'\|_X, \tag{13}$$

for all $x, x' \in B$ and all $\omega \in \Omega \setminus N_B$.

Proof: *(i) ⇒ (ii).* Let $B \subset X$ be non-empty and bounded. Define

$$\mathbb{B} := \{\varphi \in L^\infty(\Omega, X) | \ \varphi(\omega) \in B \text{ for almost all } \omega \in \Omega\}.$$

(Note that if $f(\omega) \in B$ for almost all $\omega \in \Omega$ for one representative f of φ, then it holds for all representatives of φ. Thus, $\mathbb{B}$ is well-defined). Then $\mathbb{B}$ is a non-empty bounded subset of $L^\infty(\Omega, X)$ and consequently there exists $L_{\mathbb{B}}$ such that $\|N_F(\varphi) - N_F(\psi)\|_\infty \leq L_{\mathbb{B}} \|\varphi - \psi\|_\infty$ for all $\varphi, \psi \in \mathbb{B}$. If $x, x' \in B$ then $\mathbb{B}$ contains the constant functions $\varphi_x \equiv x$, $\varphi_{x'} \equiv x'$. Thus for almost all $\omega \in \Omega$,

$$\begin{aligned} \|F(\omega, x) - F(\omega, x')\|_Y &\leq \|N_F(\varphi_x) - N_F(\varphi_{x'})\|_\infty \\ &\leq L_{\mathbb{B}} \|\varphi_x - \varphi_{x'}\|_\infty = L_{\mathbb{B}} \|x - x'\|_X. \end{aligned}$$

Thus F is locally Lipschitz continuous in X, essentially uniformly on Ω. If $x \in X$ and φ_x is the function constant x on Ω, then $F(\cdot, x) = N_F(\varphi_x) \in L^\infty(\Omega, Y)$.
(ii) ⇒ (iii). Let $\emptyset \neq B \subset X$ be bounded. We only have to prove (12). Put $M := \sup\{\|x\| \mid x \in B\}$ and pick $x_0 \in B$. There exists a nullset N_B and a constant L_B such that we have for all $\omega \in \Omega \setminus N_B$:

$$\begin{aligned} \|F(\omega, x)\| &\leq \|F(\omega, x) - F(\omega, x_0)\| + \|F(\omega, x_0)\| \\ &\leq L_B \|x\| + L_B \|x_0\| + \|F(\omega, x_0)\|. \end{aligned}$$

Moreover, there is a null set $N'_B \supset N_B$ such that $\|F(\omega, x_0)\| \leq \|F(\cdot, x_0)\|_\infty$ for all $\omega \notin N'_B$. Put

$$C_B := \max\bigl(L_B M, L_B \|x_0\| + \|F(\cdot, x_0)\|_\infty\bigr).$$

Then (12) holds for all $\omega \in \Omega \setminus N'_B$.
(iii) ⇒ (i). Let $\varphi \in L^\infty(\Omega, X)$. First we show that $N_F(\varphi) \in L^\infty(\Omega, Y)$. In fact, let $\tilde{\varphi}$ be a representative of φ. Then there exists a null set N (depending on $\tilde{\varphi}$), such that $\|\tilde{\varphi}(\omega)\| \leq \|\varphi\|_\infty$ for all $\omega \in \Omega \setminus N$. Therefore $B := \{\tilde{\varphi}(\omega) | \ \omega \in \Omega \setminus N\}$ is bounded in X and there exist C_B such that $\|F(\omega, \tilde{\varphi}(\omega))\| \leq C_B$ for all $\omega \notin N$, i.e. $N_F(\varphi) \in L^\infty(\Omega, Y)$, as desired.

We consider now the local Lipschitz continuity of N_F. Let $\mathbb{B} \subset L^\infty(\Omega, X)$ be bounded. Let M be such that $\|\varphi\|_\infty \leq M$ for all $\varphi \in \mathbb{B}$ and let $B := \{x \in X | \; \|x\| \leq M\}$. Let N_B, L_B as in part *(iii)* of the proposition. Pick $\varphi, \psi \in \mathbb{B}$ and representatives $\tilde{\varphi}$ and $\tilde{\psi}$ of φ and ψ respectively. Then $\tilde{\varphi}(\omega)$ and $\tilde{\psi}(\omega)$ are in B for almost all ω. Thus there exists a null set N (depending on B, $\tilde{\varphi}$ and $\tilde{\psi}$) such that for all $\omega \in \Omega \setminus N$,

$$\|F(\omega, \tilde{\varphi}(\omega) - f(\omega, \tilde{\psi}(\omega))\| \leq L_B \|\varphi(\omega) - \psi(\omega)\|_X \leq L_B \|\varphi - \psi\|_\infty.$$

We conclude that $\|N_F(\varphi) - N_F(\psi)\|_\infty \leq L_B \|\varphi - \psi\|_\infty$. □

The following proposition provides a useful criterion to check whether an outer superposition mapping is locally Lipschitz continuous between L^∞-spaces in applications.

Proposition 2.3 *Let $\Psi : \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function such that $s \mapsto \Psi(x, s)$ is continuously differentiable on $\mathbb{R}^m$ for almost all $x \in \Omega$ and such that for each bounded $B \subset \mathbb{R}^m$ there exists a null set N_B and nonnegative constants C_B and C'_B for which*

$$|\Psi(x, s)| \leq C_B, \qquad \|\nabla_s \Psi(x, s')\| \leq C'_B \tag{14}$$

for all $x \in \Omega \setminus N_B$ and for all $s \in B$, s' in the convex hull of B. Then N_Ψ maps $L^\infty(\Omega)^m$ into $L^\infty(\Omega)$ and is locally Lipschitz continuous.

Proof: Using the identification $L^\infty(\Omega, \mathbb{R}^m) \simeq L^\infty(\Omega)^m$ by means of a choice of basis in $\mathbb{R}^m$, we show that the conditions of Proposition 2.2 (*iii*) hold.

Let $B \subset \mathbb{R}^m$ be bounded and pick $s, s' \in B$. Define $f_x : [0, 1] \to \mathbb{R}$ by $f_x(\lambda) := \Psi(x, \lambda s + (1 - \lambda)s')$. f_x is continuously differentiable. For $x \in \Omega \setminus N_B$, we have

$$|\Psi(x, s) - \Psi(x, s')| = |f_x(1) - f_x(0)| \leq |f'_x(\theta_x)|$$

for some $0 < \theta_x < 1$. There exists $C > 0$ such that $\|x\|_2 \leq C\|x\|$ for all $x \in \mathbb{R}^m$. The estimates

$$|f'_x(\theta_x)| \leq C^2 \, \|\nabla_s \Psi(x, \theta_x s + (1 - \theta_x)s')\| \; \|s - s'\| \leq C^2 C'_B \|s - s'\|$$

now yield the desired result. □

A particular class of Carathéodory functions whose members satisfy the conditions of Proposition 2.3 is provided by functions $\Psi : \Omega \times \mathbb{R}^m \to \mathbb{R}$ given by a power series

$$\Psi(x, s) := \sum_{\alpha \in \mathbb{N}_0^m} a_\alpha(x) \, s^\alpha \tag{15}$$

with $a_\alpha \in L^\infty(\Omega)$ for all $\alpha \in \mathbb{N}_0^m$, and that is such that for all $s \in \mathbb{R}^m$, the series (15), viewed as a series in $L^\infty(\Omega)$, is absolutely convergent. In fact, let $B \subset \mathbb{R}^m$ be bounded, say $\|s\| \leq M$ for all $s \in B$. Let N^α be the the null set in Ω on which $|a_\alpha(x)| > \|a_\alpha\|_\infty$. Put $N_B := \bigcup_\alpha N^\alpha$. For $x \notin N_B$ and $s \in B$,

$$|\Psi(x, s)| \leq \sum_\alpha \|a_\alpha\|_\infty \, M^{|\alpha|} =: C_B < \infty$$

by assumption. Here $|\alpha| := \alpha_1 + \cdots + \alpha_m$. Moreover,

$$\|\nabla_s \Psi(x,s)\| \leq C \sum_\alpha |\alpha| M^{|\alpha|-1} \|a_\alpha\|_\infty =: C'_B < \infty$$

for all $x \in \Omega \setminus N_B$ and all s with $\|s\| \leq M$. This set of s contains the convex hull of B. Thus Ψ satisfies the conditions (14) of Proposition 2.3.

2.2 Inner superposition mappings between L^∞-spaces

Let $(\Omega_i, \Sigma_i, \mu_i)$ be measure spaces ($i = 1, 2$). If $\Phi : \Omega_2 \to \Omega_1$ is measurable, then the inner superposition mapping Φ^* defines a *linear* map from $\mathcal{M}(\Omega_1, X)$ into $\mathcal{M}(\Omega_2, X)$. The main problem is, whether Φ^* induces a *well-defined* map $M(\Omega_1, X) \to M(\Omega_2, X)$. If it does, then Φ^* is a non-expansive map between the corresponding L^∞-spaces. In particular it is *globally* Lipschitz continuous. The following observation is crucial in this respect:

Proposition 2.4 *Φ^* induces a well-defined linear map $M(\Omega_1, X) \to M(\Omega_2, X)$ if and only if the preimage under Φ of every null set in Ω_1 is a null set in Ω_2. In this case, if μ_2 is non-trivial, i.e. $\mu_2(\Omega_2) > 0$, then the induced linear map $\Phi^* : L^\infty(\Omega_1, X) \to L^\infty(\Omega_2, X)$ is non-trivial.*

The proof is straightforward. The verification of the necessary and sufficient condition in Proposition 2.4, also known as the inverse Lusin (N)-condition, may be cumbersome in examples however. The following particular result will be sufficient for our purposes:

Lemma 2.5 *Let $L : \mathbb{R}^d \to \mathbb{R}^n$ be a non-trivial, linear, surjective map. (Hence $1 \leq n \leq d$). If $S \subset \mathbb{R}^n$ is a Lebesgue null set, then $L^{-1}(S) \subset \mathbb{R}^d$ is a Lebesgue null set.*

Proof: We denote by $m_{(d')}$ Lebesgue measure on $\mathbb{R}^{d'}$, $d' \geq 0$ ($m_{(0)} = 0$). Let N be the kernel of L and let M be a complement of N in $\mathbb{R}^d$: $\mathbb{R}^d = M \oplus N$. Let P_M and $P_N = I - P_M$ be the corresponding projections. Because L is surjective, $\mathbb{R}^n$ is linearly isomorphic to M. The isomorphism is given by the map $\varphi = P_M \circ L^{-1}$. Let m_M be the push-forward of Lebesgue measure on $\mathbb{R}^n$ under φ. A measure m_N on N is obtained as push-forward of Lebesgue measure on $\mathbb{R}^{d'}$, $d' = \dim N$, after a choice of basis.

Under the decomposition $\mathbb{R}^d = M \oplus N$, $L^{-1}(S) \simeq P_M L^{-1}(S) \times N$. Moreover,

$$m_M(P_M L^{-1}(S)) = m_M(\varphi(S)) = m_{(n)}(\varphi^{-1} \circ \varphi(S)) = m_{(n)}(S) = 0.$$

Because of translation invariance, there exists a constant $c > 0$ such that $m_{(d)} = c m_M \times m_N$. We conclude that $m_{(d)}(L^{-1}(S)) = c \cdot 0 \cdot \infty = 0$ (in the measure theoretical convention). □

Lemma 2.6 *Let L be as in Lemma 2.5. If $(\Omega, \Sigma_\Omega, \mu_\Omega)$ is a measure space such that:*

(i) There exists a Lebesgue measurable embedding $j : \Omega \to \mathbb{R}^d$ such that $j(\Omega)$ is Lebesgue measurable,

(ii) The push-forward of μ_Ω to $\mathbb{R}^d$ under j is absolutely continuous with respect to Lebesgue measure m on $\mathbb{R}^d$.

Then the preimage under $\Phi = L \circ j : \Omega \to \mathbb{R}^n$ of every Lebesgue null set in $\mathbb{R}^n$ is a μ_Ω-null set in Ω.

Proof: Let $N \subset \mathbb{R}^n$ be a Lebesgue null set. According to Lemma 2.5, $L^{-1}(N)$ is a Lebesgue null set in $\mathbb{R}^d$. Then

$$\mu_\Omega(\Phi^{-1}(N)) = \mu_\Omega(j^{-1} \circ L^{-1}(N)) = 0$$

according to assumptions (i) and (ii). □

3 A general well-posedness result

We use a particular case of a quite general well-posedness result for mild solutions of non-autonomous semilinear abstract Cauchy problems that can be found in [25], Theorem 11.2, p. 450. We reformulate that result below in a particular case for convenience in view of the definition of the functional analytic framework. The setting is as follows.

Let Y be a Banach space in which acts a strongly continuous linear semigroup $(T(t))_{t\geq 0}$ with infinitesimal generator $(A, \mathcal{D}(A))$. We consider the semilinear Cauchy problem in Y:

$$\partial_t u(t) = Au(t) + F(t, u(t)), \quad t \geq 0, \tag{16}$$

$$u(0) = \phi, \tag{17}$$

where $F(t, \cdot)$ is a map from a Banach space X to Y which is continuous for almost all t. We assume that X embeds continuously into Y and that its image is dense. We identify the set X with its image in Y. In the applications that we study, X will be a proper subset of Y and $F(t, \cdot)$ will not map into X. Unfortunately, much of the available theory deals with appropriately continuous maps F defined *on all of* Y (cf. e.g. [30]) and therefore it cannot be applied directly in our setting. Following [25], we only consider intitial data ϕ in X. See e.g. [36] for interesting results for $\phi \in Y$ under the restrictive condition that $(T(t))_{t\geq 0}$ is analytic.

A *mild solution* of (16)-(17) on $[0, \tau]$ in X is a continuous map $u : [0, \tau] \to X$ such that $u(0) = \phi$ and that satisfies the Variation of Constants Formula

$$u(t) = T(t)\phi + \int_0^t T(t - s)F(s, u(s))ds, \quad 0 \leq t \leq \tau. \tag{18}$$

Let us formulate the assumptions and the particular case of the results of [25], Chapter 11, that we will use. Define for $1 \leq p \leq \infty$ and $t > 0$ the continuous linear map

$$\Psi_{p,t} : L^p([0, t], Y) \to Y : f \mapsto \int_0^t T(t - s)f(s)ds. \tag{19}$$

Note that the strong continuity of $(T(t))_{t\geq 0}$ in Y makes the Bochner integral in (19) well-defined. If the range of the map $\Psi_{p,t}$ is properly contained in Y, then we speak of a *regularising effect of integration* against this semigroup, or simply '*regularisation through integration*'. Moreover, if $\Psi_{p,t}$ maps into a Banach subspace X of Y, then application of the Closed Graph Theorem for linear maps between Banach spaces reveals that $\Psi_{p,t} : L^p([0,t], Y) \to X$ is continuous. Let us denote its norm by $\|\Psi_{p,t}\|_{Y,X}$. Notice that the range of $\Psi_{p,s}$ is contained in that of $\Psi_{p,t}$ for all $0 < s \leq t$.

Sufficient conditions under which one can prove local well-posedness for mild solutions in X of the Cauchy problem (16)-(17) are:

(A1) *The domain X of $F(t,\cdot)$ is invariant under $(T(t))_{t\geq 0}$ and the semigroup $(T(t)|_X)_{t\geq 0}$ on X, given by restriction to X, is strongly continuous.*

(A2) *There exists $1 \leq p < \infty$ and $T > 0$ such that:*
- *(a) $\Psi_{p,T}$ maps into X,*
- *(b) There is $M > 0$ such that $\|\Psi_{p,t}\|_{Y,X} \leq M$ for all $t \in (0,T]$.*

(A3)
- *(a) $F : [0,T] \times X \to Y$ is a generalized Carathéodory function,*
- *(b) the associated Nemytskii-mapping N_F maps $L^\infty([0,T], X)$ into $L^\infty([0,T], Y)$, and*
- *(c) $N_F : L^\infty([0,T], X) \to L^\infty([0,T], Y)$ is locally Lipschitz continuous.*

The result on well-posedness is ([25], Theorem 11.2, particular case):

Theorem 3.1 (Local well-posedness, mild solutions) *In the setting as described above, assume that (A1)–(A3) hold. Then the following statements hold:*

(i) For any $\gamma_0 > 0$ there exist constants $\tau = \tau(\gamma_0)$, $0 < \tau \leq T$, and $\gamma = \gamma(\gamma_0) \geq 0$ such that for all $\phi \in X$ with $\|\phi\|_X \leq \gamma_0$ there exists a mild solution $u(\cdot;\phi) \in C([0,\tau], X)$ of (16)-(17) on $[0,\tau]$ such that $\|u(t;\phi)\|_X \leq \gamma$ for $0 \leq t \leq \tau$.

(ii) For each $\phi \in X$ and $\tau > 0$ there exists at most one mild solution in $C([0,\tau], X)$.

(iii) For any $\gamma > 0$ and $\tau > 0$ there exists a constant $C = C(\gamma,\tau)$ such that

$$\|u(\cdot;\phi_1) - u(\cdot;\phi_2)\|_{C([0,\tau],X)} \leq C\ \|\phi_1 - \phi_2\|_X$$

for mild solutions $u(\cdot;\phi_j) \in C([0,\tau], X)$ with $\|u(t;\phi_j)\|_X \leq \gamma$, $0 \leq t \leq \tau$ $(j = 1,2)$.

Any mild solution has a maximal domain of existence, $[0, t_{max})$, and $t_{max} < \infty$ if and only if $\|u(t,\phi)\|_X \to \infty$ as $t \uparrow t_{\max}$. If Y is reflexive and the initial condition ϕ is in $\mathcal{D}(A) \cap X$ such that $A\phi + F(0,\phi) \in X$, then the unique mild solution $u(\cdot;\phi)$ in X is in $C^1((0,\tau); X) \cap C([0,\tau]; \mathcal{D}(A))$ for any $\tau \in [0, t_{max})$ and

$$\frac{d}{dt}u(t) = Au(t) + F(t, u(t)), \quad 0 \leq t \leq \tau.$$

That is, the mild solution in X is a classical solution (cf. [25], Theorem 11.4, p.456).

Remarks.
1.) In [18] we studied the question when the restriction of a strongly continuous linear semigroup to an invariant Banach subspace is strongly continuous.

2.) We can prove the results of Theorem 3.1 when (A2) is replaced by the weaker assumption

(A2') There exists $1 \le p < \infty$ *and* $T > 0$ *such that:*

(a) $\Psi_{p,T}$ *maps into* X*,*
(b) $\limsup_{t \downarrow 0} t^{1/p} \|\Psi_{p,t}\|_{Y,X} = 0$.

3.) The semigroups that we shall encounter all have a stronger property than (A2b), namely:

$$\lim_{t \to 0} \|\Psi_{p,t}\|_{Y,X} = 0 \tag{20}$$

when $p > 1$ is sufficiently large, for appropriate spaces X (see Lemma 4.7 for the heat semigroup; for the transport semigroup, $X = Y$ and a suitable estimate for $\|\Psi_{p,t}\|_{Y,Y}$ can be computed easily). It would be interesting to have a general result in this direction, or examples of semigroups for which $\|\Psi_{p,t}\|_{Y,X}$ is unbounded as $t \downarrow 0$, while Assumption (A2') still holds.

Notice, that if (20) holds, then the strong continuity of $(T(t)|_X)_{t\ge0}$ in Assumption (A1) is necessary (not only for the method of proof). If one assumes the existence of mild solutions in X for all $\phi \in X$ and the invariance of X under $(T(t))_{t\ge0}$, then strong continuity in X *follows* from (20) and (18), because then

$$\int_0^t T(t-s)F(s,u(s))ds = \Psi_{p,t}\big(N_F(u)\big) \to 0 \text{ in } X, \qquad \text{as } t \downarrow 0.$$

4.) According to Proposition 2.2, Assumption (A3) is equivalent to Assumption (A3) in [25], Chapter 11 (in slightly reformulated form): (1) $t \mapsto F(t,x)$ is strongly measurable on $[0,\tau]$ for all $x \in X$. (2) For any $\beta > 0$, there exists $k_\beta, k'_\beta > 0$ such that for all $x, y \in X$ with $\|x\| \le \beta$, $\|y\| \le \beta$,

$$\|F(t,x)\|_Y \le k_\beta(1 + \|x\|_X), \tag{21}$$
$$\|F(t,x) - F(t,y)\|_Y \le k'_\beta \|x-y\|_X, \tag{22}$$

for almost all $t \in [0,\tau]$.

5.) Assumptions (A2) imply that the map Ψ_p defined by $\Psi_p f(t) := \Psi_{p,t} f$ maps $L^p([0,T],Y)$ into $C([0,T],X)$ and is continuous, provided $p < \infty$. $\Psi_\infty f$ need not be continuous for all $f \in L^\infty([0,t],Y)$. It is for $f \in C([0,t],Y)$.

4 Set-up and proof of well-posedness

The next part of this paper will be primarily concerned with carefully describing a functional analytic set-up such that the 'linear reaction-diffusion' and 'excitable medium' models can be considered as a system of abstract equations in Banach space. That is, for both models we shall choose Banach spaces Y_i

and X_i and define maps $F_i : \mathbb{R}_+ \times X \to Y_i$, where $X = X_0 \oplus \cdots \oplus X_m$, and linear semigroup generators $(A_i, \mathcal{D}(A_i))$ in Y_i such that the system

$$\partial_t u_i(t) = A_i u_i(t) + F_i(t, u(t)), \qquad t \geq 0, \tag{23}$$

$$u_i(0) = \phi_i, \qquad (i = 0, \ldots, m) \tag{24}$$

corresponds to that particular model as an abstract system. Since we want to apply Theorem 3.1, the chosen set-up should satisfy Assumptions (A1)–(A3). In all cases, u_0 will correspond to f and u_1 to S.

4.1 The semigroups

First we discuss properties of the semigroups associated to $-v \cdot \nabla_x$ and Δ.

Let $\Phi : \mathbb{R}^n \times V \times \mathbb{R}_+ \to \mathbb{R}^n \times V$ be the flow $\Phi(x, v, t) = \Phi_t(x, v) := (x - vt, v)$. Φ defines a linear semigroup $(T_\Phi(t))_{t\geq 0}$ on the space of functions on $\mathbb{R}^n \times V$ by means of

$$T_\Phi(t) f(x, v) := f(\Phi(x, v, t)) = \Phi_t^* f(x, v).$$

It induces a semigroup of linear maps on the vector space $M(\mathbb{R}^n \times V)$ of equivalence classes of measurable functions on $\mathbb{R}^n \times V$. (One can verify that the conditions of Proposition 2.4 hold.) Note that $(T_\Phi(t))_{t\geq 0}$ is actually a *group*.

The following proposition summarises the main results that we will use:

Proposition 4.1 *$T_\Phi(t)$ is non-expansive in $L^p(\mathbb{R}^n \times V)$ for $1 \leq p \leq \infty$ and $t \in \mathbb{R}$. The (semi-)group $(T_\Phi(t))_{t\geq 0}$ is strongly continuous in $L^p(\mathbb{R}^n \times V)$ for $1 \leq p < \infty$, but not on $L^\infty(\mathbb{R}^n \times V)$. However, $(T_\Phi(t))_{t\geq 0}$ is strongly continuous on the closed invariant subspace $C_0(\mathbb{R}^n \times V)$. The space of Schwartz functions is a core for the infinitesimal generator A of $(T_\Phi(t))_{t\geq 0}$ in these spaces. $A = -v \cdot \nabla_x$ in the sense of distributions.*

The diffusion semigroup $(T_d(t))_{t\geq 0}$ on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ is defined by

$$T_d(t) f = h_d(\cdot, t) * f, \quad \text{for } t > 0, \tag{25}$$

where the heat or diffusion kernel h_d is given by

$$h_d(x, t) = (4\pi d t)^{-n/2}\, e^{-|x|^2/4dt} \tag{26}$$

(cf. eg. [16] or [12], Section II.2.13, p. 69 and Example II.4.10, p. 107). It is strongly continuous and analytic in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$ and on the closed subspace $C_0(\mathbb{R}^n)$ of $L^\infty(\mathbb{R}^n)$, but not on $L^\infty(\mathbb{R}^n)$ itself. Let $(A_p, \mathcal{D}(A_p))$ be its infinitesimal generator in $L^p(\mathbb{R}^n)$ or $C_0(\mathbb{R}^n)$ $(p = \infty)$. In each case, $\mathcal{S}(\mathbb{R}^n)$ is a core for A_p and $A_p f = d\Delta f$ on $\mathcal{S}(\mathbb{R}^n)$ (cf. [8],[12]).

Observe that $(T_d(t))_{t\geq 0}$ leaves $W^{k,p}(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$ and $C_0^k(\mathbb{R}^n)$ invariant. In particular, for a function f in these spaces, $\partial^\alpha(T_d(t)f) = T_d(t)(\partial^\alpha f)$ for all multi-indices α for which $|\alpha| \leq k$. We conclude that the restriction of $(T_d(t))_{t\geq 0}$ to $W^{k,r}(\mathbb{R}^n)$ or $C_0^k(\mathbb{R}^n)$ is strongly continuous.

In the following proposition we collect some results concerning the operators $T_d(t)$ that we will need later on.

Proposition 4.2 *Let $t > 0$, $k \in \mathbb{N}_0$ and $1 \le p < \infty$.*

(i) $T_d(t)$ maps $L^p(\mathbb{R}^n)$ into $C_0^\infty(\mathbb{R}^n)$.

(ii) $T_d(t)$ maps $L^p(\mathbb{R}^n)$ into $W^{k,r}(\mathbb{R}^n)$ for all r satisfying $p \le r \le \infty$. There exists a constant C, only depending on p, r, k and d, such that

$$\|T_d(t)\|_{\mathcal{L}(L^p, W^{k,r})} \;\le\; C\, t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{p})} \cdot \max(1, t^{-k/2}).$$

(iii) For any $f \in L^p(\mathbb{R}^n)$, the map $t \mapsto T_d(t)f$ is continuous as a map from $(0,\infty)$ into $W^{k,r}(\mathbb{R}^n)$ for all r such that $p \le r \le \infty$. In particular, it is continuous as map into $C_0^k(\mathbb{R}^n)$.

Proof: A result of this type for general analytic semigroups may be found in [30], Theorem 37.5, p.97. In our case one may exploit the explicit form (25) for the heat semigroup and the regularity of $h_d(\cdot, t)$ to derive explicit estimates of the form

$$\|\partial_x^\alpha h_d(\cdot, t)\|_{L^p(\mathbb{R}^n)} \le C t^{-\frac{n+|\alpha|}{2} + \frac{n}{2p}}$$

for all $t > 0$, where the constant C depends on $1 \le p \le \infty$, the multi-index $\alpha \in \mathbb{N}_0^n$ the dimension n and the diffusion constant d. One may then use the identity

$$\partial^\alpha\big(T_d(t)f\big) = \big(\partial^\alpha h_d(\cdot, t)\big) * f,$$

taken in the distributional sense. Then

$$\begin{aligned}\|T_d(t)f\|_{W^{k,r}} &\le \sum_{\alpha:|\alpha|\le k} \|\partial^\alpha h_D(\cdot,t)\|_q\;\|f\|_p \le \|f\|_p \sum_{\alpha:|\alpha|\le k} C_\alpha\, t^{-\frac{n+|\alpha|}{2}+\frac{n}{2q}} \\ &= \|f\|_p \;\cdot t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{p})} \sum_{0\le j\le k} C_j\, t^{-j/2},\end{aligned}$$

where $C_j := \sum_{\alpha:|\alpha|=j} C_\alpha$. Finally, define $C := (k+1)\max_{0\le j\le k} C_j$ in order to obtain the second result. The last result follows from the strong continuity of T_d in $L^r(\mathbb{R}^n)$ when $1 \le r < \infty$ and in $C_0(\mathbb{R}^n)$ when $r\infty$, the first part of the proposition and the identity $\partial^\alpha(T_d(t)f) = T_d(t)(\partial^\alpha f)$ when $f \in W^{k,r}(\mathbb{R}^n)$. □

4.2 Regularisation through integration

Recall that 'regularisation through integration' refers to the property of a strongly continuous linear semigroup, that the associated map $\Psi_{p,t}$ defined by (19) has range properly contained in Y. In that case, this range is contained in a Banach subspace X of Y. We shall consider this property for the transport (semi)group and the diffusion semigroup in this section. The presented results will assist in defining a functional analytic setting for the kinetic chemotaxis models in which Assumptions (A1) and (A2) hold.

The start with the following observation:

Lemma 4.3 *Let $t > 0$. The following statements hold:*

(i) $\{T(t)y \mid y \in Y\}$ is contained in the range of $\Psi_{p,t}$.

(ii) If $T(t)$ is invertible, then $\Psi_{p,t}$ is surjective.

Proof: Because $(T(t))_{t\geq 0}$ is strongly continuous in Y, the function $\mathcal{O}_y : s \mapsto T(s)y$ is in $C([0,t],Y) \subset L^p([0,t],Y)$ for all $y \in Y$ and $1 \leq p \leq \infty$. Moreover,

$$\Psi_{p,t}(\tfrac{1}{t}\mathcal{O}_y) = \frac{1}{t}\int_0^t T(t-s)T(s)y ds = T(t)y. \tag{27}$$

The statements in the lemma now directly follow. □

Note that if $T(t)$ is invertible for some $t > 0$, then it is invertible for all $t \geq 0$ and $(T(t))_{t\geq 0}$ can be extended to a strongly continuous group. Thus, if $(T(t))_{t\geq 0}$ extends to a group, then $\Psi_{p,t}$ is surjective for all $t > 0$ and the domain X of $F(t,\cdot)$ in the functional analytic set-up must be equal to Y. Because $(T_\Phi(t))_{t\geq 0}$ is a group, no 'regularisation through integration' can occur: the map $\Psi_{p,t}$ is surjective, according to Lemma 4.3.

Let us now consider the diffusion semigroup $(T_d(t))_{t\geq 0}$. It is an analytic semigroup and $A = -d\Delta$ with domain $\mathcal{D}(A) := W^{2,p}(\mathbb{R}^n)$ is a positive sectorial operator in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. The fractional power spaces $\mathcal{D}(A^\alpha)$, $\alpha \geq 0$, are invariant Banach subspaces of Y when equipped with the graph norm $\|u\|_\alpha := \|A^\alpha u\|_Y$. Moreover, the restriction of $(T(t))_{t\geq 0}$ to $\mathcal{D}(A^\alpha)$ is strongly continuous (cf. [30], Lemma 37.4, p.96). According to the Fundamental Theorem on Sectorial Operators (cf. [30], Theorem 37.5, p. 97), if $(A,\mathcal{D}(A))$ is a positive sectorial operator, then for any $\alpha \geq 0$, there is a constant M_α such that

$$\|e^{-At}\|_{\mathcal{L}(Y,\mathcal{D}(A^\alpha))} = \|A^\alpha e^{-At}\|_{\mathcal{L}(Y)} \leq M_\alpha t^{-\alpha} e^{-at} \tag{28}$$

for all $t > 0$. The constant a is positive and independent of α.

Lemma 4.4 *Let $(A,\mathcal{D}(A))$ be a positive sectorial operator in Y and let $T(t) = e^{-At}$ be the C_0-semigroup generated by A. For $1 < p \leq \infty$, $\Psi_{p,t}$ maps Y into $\mathcal{D}(A^\alpha)$ for $0 \leq \alpha < 1 - \frac{1}{p} =: \frac{1}{q}$ and*

$$\|\Psi_{p,t}\|_{Y,\mathcal{D}(A^\alpha)} \leq M_\alpha\, t^{\frac{1}{q}-\alpha}\big(\tfrac{1}{1-\alpha q}\big)^{1/q}.$$

Proof: If $f \in L^p([0,t],Y)$ and q is such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned}
\|\Psi_{p,t}(f)\|_{\mathcal{D}(A^\alpha)} &\leq \int_0^t \|T(s)\|_{\mathcal{L}(Y,\mathcal{D}(A^\alpha))}\ \|f(t-s)\|_Y\, ds \\
&\leq M_\alpha \int_0^t s^{-\alpha} e^{-as} \|f(t-s)\|_Y\, ds \\
&\leq \|f\|_p\ M_\alpha \Big(\int_0^t s^{-\alpha q} e^{-aqs} ds\Big)^{1/q} \\
&\leq \|f\|_p\ M_\alpha\ t^{\frac{1}{q}-\alpha}\big(\tfrac{1}{1-\alpha q}\big)^{1/q}
\end{aligned}$$

by using (28) and Hölder's Inequality. □

Thus if $(A,\mathcal{D}(A))$ is a positive sectorial operator in Y and $T(t) = e^{-At}$, then (20) holds for $X = \mathcal{D}(A^\alpha)$ for all $1 < p \leq \infty$ and $0 \leq \alpha < \frac{1}{q}$. We conclude that $(T(t))_{t\geq 0}$ satisfies assumptions (A1)–(A2) with $X = \mathcal{D}(A^\alpha)$ for any $0 \leq \alpha < 1$.

Instead of following the cumbersome approach of identifying the domains $\mathcal{D}(A^\alpha)$ (in particular in the case $p = 1$), we will make use of a less general result on regularisation through integration against the diffusion semigroup in $L^p(\mathbb{R}^n)$, $1 \le p < \infty$, that can be obtained through explicit computations and that will suit our needs. It is:

Proposition 4.5 *Let $1 \le p < \infty$. If $k \in \{0,1\}$ and $1 \le r \le \infty$ are such that*

$$\tfrac{k-2}{n} + \tfrac{1}{p} < \tfrac{1}{r} \le \tfrac{1}{p}, \tag{29}$$

then $q_0 := \big[1 + \frac{n}{2}\big(\frac{1}{r} - \frac{1}{p}\big) - \frac{k}{2}\big]^{-1}$ satisfies $1 \le q_0 < \infty$ and, as Bochner integral in $L^p(\mathbb{R}^n)$,

$$\int_0^\tau T_d(\tau - t) f(t) dt \ \in \ W^{k,r}(\mathbb{R}^n) \tag{30}$$

for all $\tau > 0$ and $f \in L^q([0,\tau], L^p(\mathbb{R}^n))$, whenever $1 \le q_0 < q \le \infty$. If $r = \infty$, then the integral (30) is actually in $C_0^k(\mathbb{R}^n) \subset W^{k,\infty}(\mathbb{R}^n)$.

Proof: The integrand in (30) is Bochner integrable in $L^p(\mathbb{R}^n)$. We show that under the given conditions the integrand is also Bochner integrable in $W^{k,r}(\mathbb{R}^n)$, hence in $L^p \cap W^{k,r}$, which is continuously embedded into L^p. It then follows that the image under this embedding of the value of the Bochner integral in $L^p \cap W^{k,r}$ coincides with the Bochner integral in L^p.

Let $\tau > 0$, $1 \le p \le r \le \infty$ and let $f : [0,\tau] \to L^p(\mathbb{R}^n)$ be a strongly measurable function. Proposition 4.2 *(ii)* implies that the map $[0,\tau) \to W^{k,r}(\mathbb{R}^n) : t \mapsto T_d(\tau - t) f(t)$ is strongly measurable and

$$\|T_d(\tau - t) f(t)\|_{W^{k,r}} \le \|T_d(\tau - t)\|_{\mathcal{L}(L^p, W^{k,r})} \, \|f(t)\|_p \ \le \ M(t) \, \|f(t)\|_p, \tag{31}$$

where

$$M(t) := C(\tau - t)^{\frac{n}{2}(\frac{1}{r} - \frac{1}{p})} \cdot \max\big(1, (\tau - t)^{-k/2}\big).$$

Applying Hölder's Inequality, the function $t \mapsto M(t)\, \|f(t)\|_p$ is in $L^1([0,\tau])$ if $f \in L^q([0,\tau], L^p(\mathbb{R}^n))$ and M is in $L^{q'}([0,\tau])$, where q' satisfies $\frac{1}{q} + \frac{1}{q'} = 1$. This yields the condition

$$q'\big(\tfrac{n}{2}(\tfrac{1}{r} - \tfrac{1}{p}) - \tfrac{k}{2}\big) > -1. \tag{32}$$

If there exists such q, q', $1 \le q, q' \le \infty$, satisfying (32), then necessarily

$$\tfrac{1}{q_0} := \tfrac{n}{2}\big(\tfrac{1}{r} - \tfrac{1}{p}\big) - \tfrac{k}{2} + 1 > 1 - \tfrac{1}{q'} = \tfrac{1}{q} \ge 0$$

and we must have

$$\tfrac{n}{2}\big(\tfrac{1}{r} - \tfrac{1}{p}\big) - \tfrac{k}{2} > -1.$$

The latter is equivalent to the lower estimate for $\frac{1}{r}$ in (29). Observe that $\frac{n}{2}(\frac{1}{r} - \frac{1}{p}) - \frac{k}{2} \le 0$ when $\frac{1}{r} \le \frac{1}{p}$. Therefore $1 \le q_0 < \infty$ and if q satisfies $1 \le q_0 < q \le \infty$, then q' satisfies (32). For such a q, the integrand in (30) is Bochner integrable in $L^p \cap W^{k,r}(\mathbb{R}^n)$ and we find that the Bochner integral in L^p is actually in $L^p \cap W^{k,r}$.

Consider the special case $r = \infty$. Replacing $W^{k,\infty}$ by C_0^k where necessary in the argument presented above shows that $t \mapsto T_d(\tau - t) f(t)$ is strongly measurable on $[0,\tau)$ as map into $C_0^k(\mathbb{R}^n)$. The latter space continuously embeds into $W^{k,\infty}(\mathbb{R}^n)$ and the estimates for $\|T_d(\tau-t)f(t)\|_{C_0^k}$ are similar to (31). □

Corollary 4.6 *Let $\tau > 0$. Let $1 \le p < \infty$ and $k \in \mathbb{N}_0$ satisfy $k < 2 - \frac{n}{p}$ (that is, k can be at most 1). Put $q_0 := \frac{2p}{p(2-k)-n}$. Then $q_0 \ge 1$. If q satisfies $q_0 < q \le \infty$, then*

$$\int_0^\tau T_d(\tau-t)f(t)dt \ \in \ C_0^k(\mathbb{R}^n)$$

as a Bochner integral in $L^p(\mathbb{R}^n)$ for all $f \in L^q([0,\tau], L^p(\mathbb{R}^n))$.

With regard to Assumption (A2b) for the diffusion semigroup, we have the following result:

Lemma 4.7 *If p, k, r and q satisfy the conditions of Proposition 4.5, then there exists a constant $C > 0$, such that*

$$\|\Psi_{q,t}\|_{L^p,W^{k,r}} \ \le \ C\, t^{\frac{n}{2}(\frac{1}{r}-\frac{1}{p})-\frac{k}{2}+1-\frac{1}{q}} \ = C\, t^{\frac{1}{q_0}-\frac{1}{q}}.$$

for $0 < t \le 1$ and for all q such that $q_0 < q \le \infty$. In particular, $t \mapsto \|\Psi_{q,t}\|_{L^p,W^{k,r}}$ is bounded on $(0,T]$ for any $T > 0$.

Proof: The results follow from Proposition 4.2 *(ii)* and Proposition 4.5 after a simple computation. □

4.3 Condition for a well-defined turning operator

Recall that we only consider the case where the turning operator $\mathcal{T}$ is linear in f, i.e. the kernel $T[S,f] : \mathbb{R}^n \times V \times V \to \mathbb{R}$ in (2) (suppressing time dependence) does not depend on f. Recall that the turning rate $\lambda[S] : \mathbb{R}^n \times V \to \mathbb{R}$ is given by (4). In this section we establish that $T[S] \in L^\infty(\mathbb{R}^n \times V \times V)$ is a sufficient condition on $T[S]$ such that $\mathcal{T}[S]$ is a well-defined bounded linear operator on $L^p(\mathbb{R}^n \times V)$ *for all* $1 \le p < \infty$.

Observe, that if (Ω, Σ, μ) is a *finite* measure space, then the map $L^\infty(\Omega \times \Omega) \to \mathcal{L}(L^p(\Omega)) : K \to L_K$ where L_K is given by

$$L_K\varphi(x) := \int_\Omega K(x,x')\varphi(x')d\mu(x') \tag{33}$$

is linear and continuous for all $1 \le p \le \infty$ and $\|L_K\| \le \mu(\Omega)\ \|K\|_\infty$. This result cannot be applied directly to our turning operator: we work in $\Omega = \mathbb{R}^n \times V$, which has infinite measure, and the turning operator only involves integration with respect to the v-variable. The following series of technical results establishes the equivalent of this observation in our case. The central idea is to view $L^p(\mathbb{R}^n \times V)$ as $L^p(\mathbb{R}^n, L^p(V))$, which is possible or $1 \le p < \infty$ and to apply the observation above pointwise (almost everywhere) to the range $L^p(V)$.

Proposition 4.8 *Let $1 \le p \le \infty$ and fix S_0. If the turning kernel $T[S_0]$ is in $L^\infty(\mathbb{R}^n \times V \times V)$, then the following statements hold:*

(i) For almost all $x \in \mathbb{R}^n$, $T[S_0](x,\cdot,\cdot) \in L^\infty(V \times V)$ defines a bounded linear operator $\tilde{\mathcal{T}}[S_0,x]$ on $L^p(V)$, given by

$$\tilde{\mathcal{T}}[S_0,x]f(v) := \int_V T[S_0](x,v,v')f(v')d\mu(v'), \tag{34}$$

with norm

$$\|\tilde{\mathcal{T}}[S_0,x]\|_{\mathcal{L}(L^p(V))} \le \mu(V)\, \|\, T[S_0]\, \|_{L^\infty(\mathbb{R}^n\times V\times V)}.$$

(ii) The map $x \mapsto \tilde{\mathcal{T}}[S_0,x]$ is in $L^\infty(\mathbb{R}^n, \mathcal{L}(L^p(V)))$.

(iii) The map $x \mapsto \tilde{\mathcal{T}}[S_0,x]f(x)$ is in $L^p(\mathbb{R}^n, L^p(V))$ for any function $f \in L^p(\mathbb{R}^n, L^p(V))$.

(iv) Let $p < \infty$. Identifying $L^p(\mathbb{R}^n, L^p(V))$ with $L^p(\mathbb{R}^n \times V)$, the linear operator $\hat{\mathcal{T}}[S_0]$ on $L^p(\mathbb{R}^n \times V)$ given by

$$\hat{\mathcal{T}}[S_0]f(x,v) := \tilde{\mathcal{T}}[S_0,x][f(x)](v) = \int_V T[S_0](x,v,v')f(x,v')d\mu(v')$$

is continuous with operator norm

$$\|\, \hat{\mathcal{T}}[S_0]\, \| \le \mu(V)\, \|T[S_0]\|_{L^\infty(\mathbb{R}^n\times V\times V)}.$$

(v) If $T[S_0] \in BC(\mathbb{R}^n, L^\infty(V \times V))$, then the map in (ii) is continuous.

Proof: *(i)* This follows directly from the observation made in relation to (33).
(ii) We need to show that the map $x \mapsto \tilde{\mathcal{T}}[S_0](x) : \mathbb{R}^n \to \mathcal{L}(L^p(V))$ is measurable in the uniform operator topology on $\mathcal{L}(L^p(V))$. This is the case, because $x \mapsto T[S_0](x,\cdot,\cdot) : \mathbb{R}^n \to L^\infty(V \times V)$ is strongly measurable and the map $K \to L_K : L^\infty(V \times V) \to \mathcal{L}(L^p(V))$ is continuous.
(iii) Let $f \in L^p(\mathbb{R}^n, L^p(V))$. In particular, f is strongly measurable. Because $x \mapsto \tilde{\mathcal{T}}[S_0](x)$ is strongly measurable in the uniform operator topology (part *(ii)*), the map $x \mapsto \tilde{\mathcal{T}}[S_0](x)[f(x)]$ is strongly measurable. Then, using part *(i)*, we obtain for almost all x,

$$\begin{aligned}\|\tilde{\mathcal{T}}[S_0,x]\big(f(x)\big)\|_p &\le \|\tilde{\mathcal{T}}[S_0,x]\|_{\mathcal{L}(L^p(V))}\, \|f(x)\|_p \\ &\le \mu(V)\, \|T[S_0]\|_\infty\, \|f(x)\|_p.\end{aligned} \tag{35}$$

The latter function of x is in $L^p(\mathbb{R}^n)$.
(iv) The required estimate of the operator norm is obtained from (35).
(v) Obvious when reviewing the proof of part *(ii)*. □

Proposition 4.9 *Let the assumptions of Proposition 4.8 hold. If $\theta \in L^\infty(V)$ and*

$$\Theta[S_0](x,v) := \int_V T[S_0](x,v',v)\,\theta(v')dv',$$

then:

(i) $\Theta[S_0] \in L^\infty(\mathbb{R}^n \times V)$ *and* $\| \Theta[S_0] \|_\infty \leq \mu(V) \, \|T[S_0]\|_\infty \, \|\theta\|_\infty$.

(ii) *Multiplication by* $\Theta[S_0]$ *defines a continuous linear operator* $M_\theta[S_0]$ *on* $L^p(\mathbb{R}^n \times V)$, *(*$1 \leq p \leq \infty$*) with norm*

$$\| M_\theta[S_0] \| \leq \mu(V) \, \|T[S_0]\|_{L^\infty(\mathbb{R}^n \times V \times V')} \, \|\theta\|_\infty. \tag{36}$$

(iii) *If* $T[S_0] \in BC(\mathbb{R}^n, L^\infty(V \times V))$*, then the map* $x \mapsto \Theta[S_0](x, \cdot) : \mathbb{R}^n \to L^\infty(V)$ *is continuous. In this case, multiplication by* $\Theta[S_0]$ *defines a continuous linear operator* $M_\theta[S_0]$ *on* $C_0(\mathbb{R}^n, L^p(V))$, $1 \leq p \leq \infty$*, with norm satisfying (36).*

Proof: *(i)* Note that $\Theta[S_0](x, \cdot) = \tilde{\mathcal{T}}[S_0, x]^* \theta$ for almost all x, where the asterix denotes operator adjoint in $\mathcal{L}(L^1(V))$. Because $x \mapsto \tilde{\mathcal{T}}[S_0, x] : \mathbb{R}^n \to \mathcal{L}(L^1(V))$ is strongly measurable in the uniform operator topology (Proposition 4.8 *(ii)*) and taking adjoints is continuous in this topology (since $\|A^*\| = \|A\|$), the map $x \mapsto \tilde{\mathcal{T}}[S_0, x]^*$ from $\mathbb{R}^n$ into $\mathcal{L}(L^1(V)^*) \simeq \mathcal{L}(L^\infty(V))$ is strongly measurable in the uniform topology. Thus $x \mapsto \Theta[S_0](x, \cdot)$ is a strongly measurable map from $\mathbb{R}^n$ into $L^\infty(V)$. Moreover,

$$\|\Theta[S_0](x, \cdot)\|_\infty \leq \|\tilde{\mathcal{T}}[S_0, x]\|_{\mathcal{L}(L^1(V))} \, \|\theta\|_\infty \leq \mu(V) \, \|T[S_0]\|_\infty \, \|\theta\|_\infty$$

for almost all x by Proposition 4.8. We conclude that $\Theta[S_0] \in L^\infty_{w^*}(\mathbb{R}^n, L^\infty(V))$ and that the stated norm estimate holds.
(*ii*). For any measure space Ω, a function $\phi \in L^\infty(\Omega)$ defines a continuous multiplication operator M_ϕ on $L^r(\Omega)$ for any $1 \leq r \leq \infty$, and $\|M_\phi\| \leq \|\phi\|_\infty$.
(iii) According to Proposition 4.8 *(iv)* the map $x \mapsto \tilde{\mathcal{T}}[S_0, x] : \mathbb{R}^n \to \mathcal{L}(L^1(V))$ is continuous in the uniform operator topology. A review of part *(i)* yields that $\Theta[S_0] \in BC(\mathbb{R}^n, L^\infty(V))$. □

The *turning operator* $\mathcal{T}[S_0]$ in (2)–(3) decomposes into

$$\mathcal{T}[S_0] \; := \; -M_1[S_0] \; + \; \hat{\mathcal{T}}[S_0] \tag{37}$$

in terms of the operators defined in Proposition 4.8 and 4.9. Thus,

Corollary 4.10 *If* $T[S_0] \in L^\infty(\mathbb{R}^n \times V \times V)$*, then the turning operator* $\mathcal{T}[S_0]$ *is a bounded linear operator on* $L^p(\mathbb{R}^n \times V)$ *for* $1 \leq p < \infty$*. If, a fortiori,* $T[S_0] \in BC(\mathbb{R}^n, L^\infty(V \times V))$*, then the turning operator is a bounded linear operator on* $C_0(\mathbb{R}^n, L^p(V))$ *for* $1 \leq p < \infty$ *as well.*

Corollary 4.11 *The map* $T[S_0] \to \mathcal{T}[S_0]$ *is linear and continuous from* $L^\infty(\mathbb{R} \times V \times V)$ *into* $\mathcal{L}(L^p(\mathbb{R}^n \times V))$ *with norm majorised by* $2\mu(V)$ *for all* $1 \leq p \leq \infty$*.*

4.4 The linear reaction-diffusion model

For the linear reaction-diffusion model the setting is the simplest of the two that we consider. Throughout this section the standing hypothesis on the dependence of the turning kernel on the signal S will be

(AT) The map $S \mapsto T[S]$ is defined on $W^{k,\infty}(\mathbb{R}^n)$, maps this space into $L^\infty(\mathbb{R}^n \times V \times V)$ and is locally Lipschitz continuous as a map from $W^{k,\infty}(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n \times V \times V)$.

In Section 5 we will provide a large class of examples of functional forms for the map $S \mapsto T[S]$ that satisfy Assumption (AT).

In defining the functional analytic set-up we will make use of the following result for a general measure space (Ω, Σ, μ):

Lemma 4.12 *Let $1 \leq p \leq q \leq \infty$. If $f \in L^p \cap L^q(\Omega, \mu)$, then $f \in L^s(\Omega, \mu)$ for all $p \leq s \leq q$ and $\|f\|_s \leq 2^{1/s}(\|f\|_p + \|f\|_q)$.*

We refer to the general framework for the set-up (23)–(24). Let $1 \leq p_0 \leq p_0' < \infty$ and

$$Y_0 \ := \ L^{p_0}(\mathbb{R}^n \times V) \cap L^{p_0'}(\mathbb{R}^n \times V). \tag{38}$$

The C_0-semigroup $(T_0(t))_{t\geq 0}$ acting in Y_0 is $(T_\Phi(t))_{t\geq 0}$ (see Section 4.1). It has infinitesimal generator $A_0 = -v \cdot \nabla_x$ (on a suitable domain). We observed in Section 4.2 that no 'regularisation through integration' can occur for this (semi-)group. So we have to take

$$X_0 \ := \ Y_0 \ = \ L^{p_0}(\mathbb{R}^n \times V) \cap L^{p_0'}(\mathbb{R}^n \times V). \tag{39}$$

The non-linearity F_0 is given by

$$F_0(t, f \oplus S) \ = \ F_0(f \oplus S) \ = \ \mathcal{T}[S]f.$$

The choice for Y_1 is restricted by the coupling given in (5): $f \mapsto \rho$ maps Y_0 onto $L^{p_0} \cap L^{p_0'}(\mathbb{R}^n)$. The latter space must be continuously embedded in Y_1, in order that (5) makes sense and yields a continuous map. A candidate for Y_1 is any $L^p(\mathbb{R}^n)$ with $p_0 \leq p \leq p_0'$, (or a finite intersection of these spaces) according to Lemma 4.12. In each of the latter spaces acts the strongly continuous semigroup $(T_{d_1}(t))_{t\geq 0}$ with $d_1 = D/\tau$ (see (5)) with infinitesimal generator $A_1 = d_1\Delta$ (on a suitable domain).

Now Assumption (A2) comes into play. According to Proposition 4.5, the range of the map $\Psi_{q,t}$ associated to $(T_{d_1}(t))_{t\geq 0}$ in $L^p(\mathbb{R}^n)$ is contained in $W^{k,r}(\mathbb{R}^n)$ for all $t > 0$, when $k \in \{0, 1\}$, r satisfies (29) and $q > 1$ is sufficiently large. The turning operator is defined for $S \in W^{k,\infty}(\mathbb{R}^n)$ however. Therefore we need that the range of $\Psi_{q,t}$ is contained in $W^{k,\infty}(\mathbb{R}^n)$ for q sufficiently large. A sufficient condition is $p > \frac{n}{2-k}$: the range is then contained in $W^{k,p}(\mathbb{R}^n) \cap C_0^k(\mathbb{R}^n)$ (see Proposition 4.5 and Corollary 4.6). Note that this will generally exclude the case $p = 1$.

In order to circumvent the condition $p > \frac{n}{2-k}$ and to allow the case $S \in L^1(\mathbb{R}^n)$, which is relevant for applications, we have to take an intersection of L^p-spaces for Y_1, similar to Y_0. For $1 \leq p_1 \leq p_1' < \infty$, define

$$Y_1 := L^{p_1}(\mathbb{R}^n) \cap L^{p_1'}(\mathbb{R}^n), \qquad X_1 := W^{k,p_1}(\mathbb{R}^n) \cap C_0^k(\mathbb{R}^n), \tag{40}$$

$$F_1(t, f \oplus S) \ = \ F_1(f \oplus S) \ = \ \tilde{\alpha}\rho - \tilde{\beta}S, \tag{41}$$

where $\tilde{\alpha} := \alpha/\tau$ and $\tilde{\beta} := \beta/\tau$.

The semigroup $(T_{d_1}(t))_{t\geq 0}$ acts strongly continuously in Y_1, X_1 is a dense invariant Banach subspace of Y_1 and the restriction of $(T_{d_1}(t))_{t\geq 0}$ to X_1 is strongly continuous. Note that we cannot take $X_1 = W^{k,p_1} \cap W^{k,\infty}(\mathbb{R}^n)$, because $(T_{d_1}(t))_{t\geq 0}$ is not strongly continuous there. If $k \in \{0,1\}$ and $p_1' > \frac{n}{2-k}$, then the range of the map $\Psi_{q,t}$ associated to $T_{d_1}(t)$ in Y_1 is contained in $W^{k,p_1} \cap C_0^k(\mathbb{R}^n)$ and satisfies (A2b) for sufficiently large q, see Proposition 4.5 and Lemma 4.7.

Define

$$Y := Y_0 \oplus Y_1, \quad X := X_0 \oplus X_1, \quad T(t) := T_\Phi(t) \oplus T_{d_1}(t), \tag{42}$$

and define $F : \mathbb{R}_+ \times X \to Y$ by

$$F(t, f \oplus S) := F_0(f \oplus S) \ \oplus \ F_1(f \oplus S). \tag{43}$$

Then we have

Theorem 4.13 (Well-posedness, linear reaction-diffusion model) *Let $k \in \{0,1\}$ and let (p_i, p_i'), $(i = 0,1)$, be such that*

$$1 \leq p_0 \leq p_1 \leq p_1' \leq p_0' < \infty, \qquad p_i' > \tfrac{n}{2-k}, \ (i = 0,1). \tag{44}$$

Let X, Y, $(T(t))_{t\geq 0}$ and F be defined by (38)–(43) and suppose that Assumption (AT) holds. Then Assumptions (A1)–(A3) hold for X, Y, $(T(t))_{t\geq 0}$ and F. In particular, mild solutions to the 'linear reaction-diffusion' model exist in X local in time, are unique and depend (locally Lipschitz) continuously on initial data.

4.5 The excitable medium model

Throughout this section we assume that Assumption (AT) holds. The main problem in defining the functional-analytic set-up for this class of models is to select the state spaces X_i and Y_i in the abstract setting (23) in such a way that the non-linear perturbation in (8) is well-defined on $X_0 \oplus X_1 \oplus X_2$, maps into Y_1, while regularisation through integration against the diffusion semigroup is able to bring the values in Y_1 back into X_1. The main result is Theorem 4.16, which holds for a rather general function φ. In Corollary 4.17 we consider the specific case of a cubic non-linearity φ and a piece-wise linear approximation, yielding the local-wellposedness for the excitable medium models.

Thus, first, let $\varphi : \mathbb{R} \to \mathbb{R}$ be any continuous function and N_φ the associated Nemytskii operator. The following two lemmas are essential to our set-up. The first can be found in [15].

Lemma 4.14 (Generalized Hölder Inequality) *Let (Ω, Σ, μ) be a measure space. Suppose that $1 \leq p_j \leq \infty$, $j = 1, \ldots, n$, are such that $\sum_j \frac{1}{p_j} = \frac{1}{r} \leq 1$. If $f_j \in L^{p_j}(\Omega, \mu)$ for $j = 1, \ldots, n$, then $f = \prod_j f_j \in L^r(\Omega, \mu)$ and $\|f\|_r \leq \prod_j \|f_j\|_{p_j}$.*

We use it to obtain:

Lemma 4.15 *Let* $1 \le p_0, p_1 \le \infty$ *and* $\alpha > 0$ *be such that* $1 \le p_1 \le \frac{p_0}{\alpha}$*. Define* $s := \frac{p_0 p_1}{p_0 - \alpha p_1}$*. Assume that* N_φ *maps a Banach space* W *of (equivalence classes of) measurable functions on* $\mathbb{R}^n$ *into* $L^s(\mathbb{R}^n)$*. Then the following statements hold:*

(i) $p_1 \le s \le \infty$.

(ii) If $N_\varphi : W \to L^s(\mathbb{R}^n)$ *is continuous, then* F_1*, given by*

$$F_1(t, f \oplus S \oplus r) = F_1(f \oplus S \oplus r) := -|\rho|^\alpha \big(\tilde{k}_s N_\varphi(S) \ + \ \tilde{k}_r \, r\big), \qquad (45)$$

is a continuous map from $L^{p_0}(\mathbb{R}^n \times V) \oplus W \oplus L^s(\mathbb{R}^n)$ *into* $L^{p_1}(\mathbb{R}^n)$.

(iii) If $\alpha \ge 1$ *and* $N_\varphi : W \to L^s(\mathbb{R}^n)$ *is locally Lipschitz continuous, then*

$$F_1 : L^{p_0}(\mathbb{R}^n \times V) \oplus W \oplus L^s(\mathbb{R}^n) \to L^{p_1}(\mathbb{R}^n)$$

is locally Lipschitz continuous.

Proof: *(i)* Elementary.
(ii) The map $f \mapsto -|\rho|^\alpha$ is continuous from $L^{p_0}(\mathbb{R}^n \times V)$ into $L^{p_0/\alpha}(\mathbb{R}^n)$. We have $F_1(f \oplus S \oplus r) = M_\psi(-|\rho|^\alpha)$ where M_ψ is the multiplication operator $g \mapsto \psi \cdot g$ defined by the function

$$\psi = \psi(S, r) \ := \ \tilde{k}_s \, N_\varphi(S) \ + \ \tilde{k}_r \, r.$$

We claim, that for the given choice of parameters, the map $(S, r) \mapsto M_{\psi(S,r)}$ is continuous from $W \oplus L^s(\mathbb{R}^n)$ into $\mathcal{L}(L^{p_0/\alpha}, L^{p_1})$, when the latter is equipped with the uniform operator topology. The estimate

$$\begin{aligned} &\|F_1(f_1 \oplus S_1 \oplus r_1) - F_1(f_2 \oplus S_2 \oplus r_2)\|_{p_1} \\ &\qquad \le \|M_{\psi_1} - M_{\psi_2}\| \cdot \| \, |\rho_1|^\alpha \|_{p_0/\alpha} \ + \ \|M_{\psi_2}\| \cdot \| \, |\rho_1|^\alpha - |\rho_2|^\alpha \|_{p_0/\alpha}, \end{aligned}$$

where $\psi_i := \psi(S_i, r_i)$, then yields the desired continuity result.

In order to prove this claim we use the Generalized Hölder Inequality, Lemma 4.14. It ensures that M_ψ maps $L^{p_0/\alpha}(\mathbb{R}^n)$ continuously into $L^{p_1}(\mathbb{R}^n)$ provided $\psi \in L^s(\mathbb{R}^n)$ and $0 \le \frac{1}{s} = \frac{1}{p_1} - \frac{\alpha}{p_0} \le 1$. In that case the linear map $\psi \mapsto M_\psi$ is bounded: $\|M_\psi\| \le \|\psi\|_s$. Rewriting the condition on s yields

$$s = \frac{p_0 \, p_1}{p_0 - \alpha p_1} \quad \text{and} \quad 1 \le s \le \infty. \qquad (46)$$

The inequalities in (46) are satisfied if and only if

$$\frac{p_0}{p_0 + \alpha} \ \le \ p_1 \ \le \ \frac{p_0}{\alpha}. \qquad (47)$$

Together with the initial assumptions on p_0 and α, this yields $1 \le p_1 \le \frac{p_0}{\alpha}$. Because the Nemytskii mapping $N_\varphi : W \to L^s(\mathbb{R}^n)$ is continuous we conclude that $(S, r) \mapsto \psi(S, r)$ is continuous from $W \oplus L^s(\mathbb{R}^n)$ to $L^s(\mathbb{R}^n)$. Because $\psi \mapsto M_\psi$ is linear and $\|M_\psi\| \le \|\psi\|_s$ we have proven the claim.

(iii) The map $f \mapsto |\rho|^\alpha$ is locally Lipschitz continuous if and only if $\alpha \geq 1$. If $N_\varphi : W \to L^s(\mathbb{R}^n)$ is locally Lipschitz continuous, then F_1 is a composition of locally Lipschitz continuous maps. □

Let us now describe the functional analytic set-up for the excitable medium model. Let $1 \leq p_0 \leq p_0' < \infty$ and define

$$Y_0 \;:=\; L^{p_0}(\mathbb{R}^n \times V) \cap L^{p_0'}(\mathbb{R}^n \times V). \tag{48}$$

In Y_0 acts the strongly continuous transport (semi-)group $(T_\Phi(t))_{t\geq 0}$. As in the linear reaction-diffusion case we have to take

$$X_0 \;:=\; Y_0 \;=\; L^{p_0}(\mathbb{R}^n \times V) \cap L^{p_0'}(\mathbb{R}^n \times V), \tag{49}$$

because no regularisation through integration can occur ($(T_\Phi(t))_{t\geq 0}$ is a group). Y_0 is continuously embedded into any $L^p(\mathbb{R} \times V)$ with $p_0 \leq p \leq p_0'$.

Define

$$\mathcal{P} := \{(p,p') \in [1,\infty) \times [1,\infty) \mid p_0 \leq p \leq p_0',\ 1 \leq p' \leq \tfrac{p}{\alpha}\}.$$

The set $\mathcal{P}$ is nonempty if and only if $\alpha \leq p_0'$. For each $(p,p') \in \mathcal{P}$,

$$s = s(p,p') := \frac{pp'}{p - \alpha p'}$$

satisfies $p' \leq s \leq \infty$ and if N_φ maps W continuously into $L^s(\mathbb{R}^n)$, then F_1 given by (45) maps $L^p(\mathbb{R}^n \times V) \oplus W \oplus L^s(\mathbb{R}^n)$ continuously into $L^{p'}(\mathbb{R}^n)$. If $N_\varphi : W \to L^s(\mathbb{R}^n)$ is locally Lipschitz continuous, then F_1 is also locally Lipschitz continuous on the domain just described (cf. Lemma 4.15).

Now, assume that $k \in \{0,1\}$ and

$$p_0' \geq \alpha, \qquad p_0' > \alpha \frac{n}{2-k}. \tag{50}$$

Take p_1 and p_1' such that

$$1 \leq p_1 \leq p_1' \leq \frac{p_0'}{\alpha}, \qquad p_1' > \frac{n}{2-k}, \tag{51}$$

(such exist because of (50)). Then for any $p^* \geq p_0$ with $\alpha p_1' \leq p^* \leq p_0'$, (p^*, p_1) and (p^*, p_1') are in $\mathcal{P}$. Put

$$\mathcal{S} := \{s = s(p,p_1) \mid (p,p_1) \in \mathcal{P}\}, \qquad \mathcal{S}' := \{s = s(p,p_1') \mid (p,p_1') \in \mathcal{P}\}.$$

Note that $\mathcal{S}$ and $\mathcal{S}'$ are nonempty subsets of $[p_1,\infty]$ and $[p_1',\infty]$ respectively. Define

$$p_* := \max(p_0, \alpha p_1), \qquad p_*' := \max(p_0, \alpha p_1').$$

$p_2 \in \mathcal{S}$ and $p_2' \in \mathcal{S}'$ if and only if

$$\frac{p_0' p_1}{p_0' - \alpha p_1} \leq p_2 \leq \frac{p_* p_1}{p_* - \alpha p_1}, \qquad \frac{p_0' p_1'}{p_0' - \alpha p_1'} \leq p_2' \leq \frac{p_*' p_1'}{p_*' - \alpha p_1'}. \tag{52}$$

In fact, the set $\mathcal{P}$ is the bounded simplex with edges given by $p = p_0$, $p = p_0'$, $p' = \alpha^{-1}p$ and $p' = 1$. It is easily seen that

$$P := \{p \mid (p, p_1) \in \mathcal{P}\} = [p_*, p_0'], \quad P' := \{p \mid (p, p_1') \in \mathcal{P}\} = [p_*', p_0'].$$

The function $x \mapsto s(x,p) = \frac{px}{x-\alpha p}$ is decreasing on $(\alpha p, \infty)$. This yields (52).

Because $p_1 \leq p_1'$ and the map $y \mapsto \frac{py}{p-\alpha y}$ is increasing on $(-\infty, \frac{p}{\alpha})$, we find that the minimal element of $\mathcal{S}$ is smaller than or equal to the minimal element of $\mathcal{S}'$. Thus we can choose $p_2 \in \mathcal{S}$, $p_2' \in \mathcal{S}'$ such that $p_2 \leq p_2'$. Define

$$Y_2 := L^{p_2}(\mathbb{R}^n) \cap L^{p_2'}(\mathbb{R}^n), \qquad X_2 := Y_2. \tag{53}$$

In Y_2 acts the trivial (semi)group $T(t) \equiv I$, $t \geq 0$. Moreover, define

$$Y_1 := L^{p_1}(\mathbb{R}^n) \cap L^{p_1'}(\mathbb{R}^n),$$

where p_1 and p_1' satisfy (51). Then F_1 maps $Y_0 \oplus W \oplus Y_2$ continuously into Y_1 when $N_\varphi : W \to L^{p_2} \cap L^{p_2'}(\mathbb{R}^n)$ is continuous, because there exist p and p' (possibly $p \neq p'$) such that both (p, p_1) and (p', p_1') are in $\mathcal{P}$ and

$$p_2 = s(p, p_1), \qquad p_2' = s(p', p_1').$$

If $\alpha \geq 1$ and $N_\varphi : W \to L^{p_2} \cap L^{p_2'}(\mathbb{R}^n)$ is locally Lipschitz continuous, then $F_1 : Y_0 \oplus W \oplus Y_2 \to Y_1$ is locally Lipschitz continuous (Lemma 4.15).

We further define

$$X_1 := W^{k,p_1}(\mathbb{R}^n) \cap C_0^k(\mathbb{R}^n). \tag{54}$$

Note that X_1 embeds continuously into Y_2, because $p_2' \geq p_2 \geq p_1$. Therefore F_2, given by

$$F_2(t, f \oplus S \oplus r) = F_2(S \oplus r) = \tau_r^{-1}(\gamma S - r), \tag{55}$$

is a well-defined, continuous linear map from $X_0 \oplus X_1 \oplus X_2$ into Y_2.

The diffusion semigroup $(T_{d_1}(t))_{t\geq 0}$, $d_1 = D/\tau_s$, is strongly continuous in Y_1. X_1 is an invariant Banach subspace of Y_1 such that the restriction of $(T_{d_1}(t))_{t\geq 0}$ is strongly continuous. Moreover, by construction, the map $\Psi_{q,t}$ associated to $(T_{d_1}(t))_{t\geq 0}$ in Y_1 maps into X_1 for all $t > 0$ and sufficiently large q (Proposition 4.5). Here we use the condition $p_1' > \frac{n}{2-k}$. It satisfies condition (A2b) (cf. Proposition 4.5 and Lemma 4.7).

We finally put

$$Y := Y_0 \oplus Y_1 \oplus Y_2, \quad X := X_0 \oplus X_1 \oplus X_2, \quad T(t) := T_\Phi(t) \oplus T_{d_1}(t) \oplus I \tag{56}$$

and define $F : \mathbb{R}_+ \times X \to Y$ by

$$F(t, f \oplus S \oplus r) := F_0(f \oplus S) \ \oplus \ F_1(f \oplus S \oplus r) \ \oplus \ F_2(S \oplus r). \tag{57}$$

The set-up has been made carefully such that we have

Theorem 4.16 (Well-posedness, general φ) *Let $\alpha \geq 1$, $k \in \{0,1\}$, $1 \leq p_i \leq p_i' < \infty$ $(i = 0,1)$, and $1 \leq p_2 \leq p_2' \leq \infty$ satisfy (50), (51) and (52). Let X, Y, $(T(t))_{t\geq 0}$ and F be defined as above. Suppose that* *(i)* *Assumption (AT) holds,*

(ii) X_1 *is a Banach subspace of the domain* W *of the Nemytskii map* $N_\varphi : W \to L^{p_2}(\mathbb{R}^n) \cap L^{p_2'}(\mathbb{R}^n)$*, and*

(iii) $N_\varphi : W \to L^{p_2}(\mathbb{R}^n) \cap L^{p_2'}(\mathbb{R}^n)$ *is locally Lipschitz continuous.*

Then Assumptions (A1)–(A3) hold for X*,* Y*,* $(T(t))_{t\geq 0}$ *and* F*. Therefore mild solutions to the excitable medium model exist locally in time, are unique and depend (locally Lipschitz) continuously on initial data.*

Corollary 4.17 (Well-posedness, FitzHugh-Nagumo) *Let* $\alpha \geq 1$*,* $k \in \{0,1\}$ *and* $1 \leq p_i \leq p_i' < \infty$ $(i = 0,1)$*,* $1 \leq p_2 \leq p_2' \leq \infty$ *satisfy (50), (51) and (52). Suppose that Assumption (AT) holds. Then the excitable medium model has mild solutions, local in time, which are unique and depend (locally Lipschitz) continuously on the initial data, when* $\varphi(x) = x(x - x_1)(x - x_2)$ *or a piecewise-linear approximation thereof with* $\varphi(0) = 0$*.*

Proof: Consider first $\varphi(x) = x(x-x_1)(x-x_2)$. There exists a constant $C > 0$ such that

$$|\varphi(x)| \leq C(|x| \ + \ |x|^3), \qquad \text{for all } x \in \mathbb{R}.$$

The Nemytskii operator N_φ is then a bounded and continuous (non-linear) map from $L^s \cap L^{3s}(\mathbb{R}^n)$ into $L^s(\mathbb{R}^n)$ for any $1 \leq s \leq \infty$ (cf. [2],[9] or [33]). Moreover, because

$$|\varphi'(x)| \ \leq \ C'(1 + |x|^2) \qquad \text{for all } x \in \mathbb{R},$$

one can conclude that N_φ is also locally Lipschitz continuous (e.g. [9] Theorem 2.6, p.13) in case $s < \infty$. One may use Proposition 2.3 to deduce this property in the case $s = \infty$. We conclude that $N_\varphi : W := L^{3p_2} \cap L^{3p_2'} \to L^{p_2} \cap L^{p_2'}$ is locally Lipschitz continuous. $X_1 = W^{k,p_1} \cap C_0^k$ embeds continuously into any $L^{s'}$ with $p_1 \leq s' \leq \infty$. The inequalities (52) imply that $p_2 \geq p_1$ and $p_2' \geq p_1' \geq p_1$. Thus X_1 embeds continuously into W and Theorem 4.16 can be applied.

A continuous piecewise linear function $\varphi : \mathbb{R} \to \mathbb{R}$ (with finite number of segments) is globally Lipschitz continuous. It is then easily seen, under the condition that $\varphi(0) = 0$, that N_φ maps $L^p(\Omega)$ into $L^p(\Omega)$ (for $1 \leq p \leq \infty$) and is *globally* Lipschitz continuous, with the same Lipschitz constant as φ. The space $W := L^{p_2} \cap L^{p_2'}(\mathbb{R}^n)$ will then satisfy the conditions (*ii*) and (*iii*) of Theorem 4.16, because of the inequality $p_2 \geq p_1$ and Lemma 4.12. □

Remarks. 1.) In some cases (not always!) one may take $p_2 = p_2'$. It is possible when $\mathcal{S} \cap \mathcal{S}' \neq \emptyset$. A necessary and sufficient condition for this situation is

$$\frac{p_0' p_1'}{p_0' - \alpha p_1'} \leq \frac{p_* p_1}{p_* - \alpha p_1}. \tag{58}$$

If $p_* = \alpha p_1$, then the right hand side in (58) is infinite and the condition holds. $\mathcal{S} \cap \mathcal{S}'$ then equals the infinite interval $[\frac{p_0' p_1'}{p_0' - \alpha p_1'}, \infty]$. If $p_* = p_0$, then (58) becomes the condition

$$\frac{p_0' p_1'}{p_0' - \alpha p_1'} \leq \frac{p_0 p_1}{p_0 - \alpha p_1}. \tag{59}$$

Thus either if $p_1 \geq \frac{p_0}{\alpha}$ or $p_1 < \frac{p_0}{\alpha}$ and (59) holds, then $\mathcal{S} \cap \mathcal{S}' \neq \emptyset$.
2.) Apparently one cannot have a set-up as described above in which both f and S are in the *full* L^1-space on $\mathbb{R}^n \times V$ and $\mathbb{R}^n$ respectively. The conditions are such that we do not have a result on local well-posedness of these models when $p_0' = p_0 = 1$, unless $k = 0$ and $n = 1$. In fact, p_0' must satisfy $p_0' > \frac{n}{2-k}$, which is generally larger than 1.

5 Turning kernels satisfying Assumption (AT)

Assumption (AT) in Section 4.4 is crucial in our well-posedness results. In this section we show that a large class of examples, including functional forms encountered in the literature, satisfy this assumption. These examples have inspired our general set-up for the functional form of the turning kernel, that we shall now introduce.

5.1 A useful class of functional forms

In the applications, the turning kernel will depend on the *function* S on $\mathbb{R}^n$, e.g. S and ∇S. We will present a particular class of functional forms that allows us to prove the measurability and integrability conditions on $T[S]$ and continuity properties for the map $S \mapsto T[S]$ as required in Assumption (AT).

Put $\Omega := \mathbb{R}^n \times V \times V$. Starting point is a finite set $\mathcal{D}$ of constant coefficient linear partial differential operators on $\mathbb{R}^n$. The turning kernel is allowed to depend on DS with $D \in \mathcal{D}$ only. Let k be the maximal order of elements in $\mathcal{D}$. Let N be the number of elements in $\mathcal{D}$ and let $D_1, \ldots, D_N$ be an enumeration of $\mathcal{D}$. The map $\boldsymbol{\partial}_{\mathcal{D}}$ is defined by $\boldsymbol{\partial}_{\mathcal{D}}(S) := (D_1 S, \ldots, D_N S)$. In order to obtain a kernel on Ω we need to map each function DS on $\mathbb{R}^n$ ($D \in \mathcal{D}$) to a function on Ω. This may be achieved by inner superposition operators $J^*_{D,i}$, $(i = 1, \ldots m_D)$, where $J_{D,i} : \Omega \to \mathbb{R}^n$ is a measurable function that satisfies the conditions of Proposition 2.4. We put

$$J^*_D : \; M(\mathbb{R}^n) \to M(\Omega)^{m_D} : S \mapsto (J^*_{D,1}(S), \ldots J^*_{D,m_D}(S))$$

and

$$\boldsymbol{J}^*(S) : \; \prod_{D \in \mathcal{D}} M(\mathbb{R}^n) \to \prod_{D \in \mathcal{D}} M(\Omega)^{m_D} : S \mapsto (J^*_{D_1}(S), \ldots, J^*_{D_N}(S)).$$

The construction is now completed by the prescription of a map

$$\boldsymbol{\Psi} : \; \prod_{D \in \mathcal{D}} M(\Omega)^{m_D} \to M(\Omega).$$

The class of functional forms for the kernel $T[S]$ that we consider are those given by the composition

$$T[S] := \boldsymbol{\Psi} \circ \boldsymbol{J}^* \circ \boldsymbol{\partial}_{\mathcal{D}}(S). \tag{60}$$

We call the triple $(\boldsymbol{\partial}_{\mathcal{D}}, \boldsymbol{J^*}, \boldsymbol{\Psi})$ '*datum of a functional form*' for the turning kernel.

Clearly, (60) is convenient, as it allows to break up the consideration of integrability properties (i.e. $S \mapsto T[S]$ maps a (subspace of an) L^p-space into an L^q-space) and related continuity properties of the map $S \mapsto T[S]$ to such questions for the maps $\boldsymbol{\Psi}$, $\boldsymbol{J}^*$ and $\boldsymbol{\partial}_{\mathcal{D}}$. On the other hand, (60) is sufficiently general such that it covers the functional forms for the turning kernel that one finds in the literature.

We will limit our attention by considering only mappings $\boldsymbol{\Psi}$ that are given as Nemytskii-mappings N_Ψ associated to a Carathéodory function $\Psi : \Omega \times \mathbb{R}^{N'} \to \mathbb{R}$, with $N' := \sum_{D\in\mathcal{D}} m_D$.

5.2 Sufficient conditions

In order that Assumption (AT) holds, N_Ψ must map into $L^\infty(\Omega)$. Generally however, a Nemtyskii mapping $N_\psi : L^p \to L^\infty$ will not be continuous if $1 \leq p < \infty$: it is only the case if N_ψ is constant (see [2], Theorem 3.17, p.110). Therefore N_Ψ should map $L^\infty(\Omega)^{N'} \to L^\infty(\Omega)$. An inner superposition operator cannot map an L^p space ($p < \infty$) into an L^∞-space. Thus it is necessary to assume that $\boldsymbol{J}^*$ maps $L^\infty(\mathbb{R})^N$ into $L^\infty(\Omega)^{N'}$. Then we have

Lemma 5.1 *Let $(\boldsymbol{\partial}_{\mathcal{D}}, \boldsymbol{J^*}, N_\Psi)$ be datum of a functional form for a turning kernel according to (60). Assume that $\boldsymbol{J}^*$ is a well-defined map from $L^\infty(\mathbb{R}^n)^N$ into $L^\infty(\Omega)^{N'}$ and that $\Psi : \Omega \times \mathbb{R}^{N'} \to \mathbb{R}$ is a Carathéodory function such that N_Ψ maps $L^\infty(\Omega)^{N'}$ into $L^\infty(\Omega)$ and is locally Lipschitz continuous. Then Assumption (AT) holds for the associated turning kernel.*

The main issue with regard to $\boldsymbol{J^*}$ is that each inner superposition operator $J^*_{D,i}$ must be a well-defined map $M(\mathbb{R}^n) \to M(\Omega)$. This property requires proof. See Section 2.2 for some useful (necessary and) sufficient conditions. Proposition 2.2 and 2.3 provide useful criteria for local Lipschitz continuity of the Nemytskii mapping N_Ψ.

5.3 Examples

Let us provide some examples of functional forms for the turning kernel as can be found in e.g. [6], [23], [24], [29], translated into the set-up just described.

Uniform redistribution.
Let $V := sS^{n-1}$, $s > 0$, the sphere in $\mathbb{R}^n$ of radius s, equipped with a rotation invariant measure μ. In this example,

$$T[S](x, v, v') := T_0(v, v') := \tfrac{1}{\mu(V)}. \tag{61}$$

Datum for the functional form may be taken as follows: $\boldsymbol{\partial}_{\mathcal{D}} := (I)$, the identity map, $m_I = 1$, $J_I(x, v, v') := x$ and $\boldsymbol{J^*} = (J^*_I)$. Note that J_I satisfies the conditions of Proposition 2.4. We take $\boldsymbol{\Psi} = N_\Psi$ for the constant map $\Psi \equiv \tfrac{1}{\mu(V)}$. N_Ψ is globally Lipschitz continuous.

A direction of anisotropy ([29]).
Let V and μ be as in the previous example. Let $b \in S^{n-1}$ be the direction of

anisotropy. The kernel is defined as $T[S] = T_0 + T_1$ with

$$T_1(x, v, v') = \kappa\,(v \cdot b)(v' \cdot b) \tag{62}$$

for some scalar κ. This example models the increased probability of choosing a new direction v in the direction of b or $-b$ if the current direction v' is also b or $-b$. we may take $\boldsymbol{\partial}_{\mathcal{D}}$ and $\boldsymbol{J^*}$ as in the previous example. Now,

$$\Psi : \Omega \times \mathbb{R} \to \mathbb{R} : (x, v, v', y) \mapsto \tfrac{1}{\mu(V)} + \kappa\,(v \cdot b)(v' \cdot b).$$

It satisfies the conditions of Proposition 2.3.

Bias towards the gradient of S ([29]).
Let V and μ are again as in the previous two examples. Now $T[S] = T_0 + T_1[S]$, where

$$T[S](x, v, v') = k(v', S(x))\, v \cdot \nabla S(x) \tag{63}$$

for a scalar function k. In this case, $\boldsymbol{\partial}_{\mathcal{D}} = (I, \partial_1, \ldots, \partial_n)$, $m_D = 1$ and $J_D(x, v, v') = x$ for all $D \in \mathcal{D}$. If $k : V \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, then

$$\Psi : \Omega \times \mathbb{R}^{n+1} \to \mathbb{R} : (x, v, v', (y_0, y)) \mapsto \tfrac{1}{\mu(V)} + k(v', y_0)\, v \cdot y$$

is a Carathéodory function and $\boldsymbol{\Psi} = N_\Psi$. If k satisfies the conditions of Proposition 2.3, then Ψ satisfies these conditions as well and consequently N_Ψ is a locally Lipschitz continuous map $L^\infty(\Omega)^{n+1} \to L^\infty(\Omega)$.

Spatial probing model.
Let $V \subset \mathbb{R}^n$ be a compact set with positive Lebesgue measure and let μ be the restriction of Lebesgue measure to V. In e.g. [23],[24] one encounters the functional form

$$T[S](x, v, v') = \phi(S(x \pm \varepsilon v), S(x \pm \varepsilon v'), \nabla S(x \pm \varepsilon v), \nabla S(x \pm \varepsilon v')) \tag{64}$$

for a suitable function $\phi : \mathbb{R}^{4(n+1)} \to \mathbb{R}$. It is obtained in our framework by taking $\boldsymbol{\partial}_{\mathcal{D}}$ as in the previous example, while $m_D = 4$ for all $D \in \mathcal{D}$ in this case. The maps

$$J_{\varepsilon,\varepsilon'} : \Omega \to \mathbb{R}^n : (x, v, v') := x + \varepsilon v + \varepsilon' v', \qquad (\varepsilon, \varepsilon' \in \mathbb{R}),$$

are measureable and we can apply Lemma 2.6 with $d = 3n$. Thus $J_{\varepsilon,\varepsilon'}$ induces a well-defined inner superposition mapping $J^*_{\varepsilon,\varepsilon'}$ from $M(\mathbb{R}^n)$ into $M(\Omega)$. Now take

$$J^*_I = J^*_{\partial_i} = (J^*_{\varepsilon,0}, J^*_{-\varepsilon,0}, J^*_{0,\varepsilon}, J^*_{0,-\varepsilon})$$

and $\boldsymbol{\Psi} = N_\phi$. If ϕ is continuously differentiable, then N_ϕ is a locally Lipschitz continous map from $L^\infty(\Omega)^{4(n+1)}$ into $L^\infty(\Omega)$ (Proposition 2.3).

6 Solutions in $L^1 \cap L^\infty$

The biological applications impose natural conditions on f and S in order for them to make physical sense. Because of their physical interpretations as density and concentration, $f(t) \geq 0$ and $S(t) \geq 0$ for all time t. A natural condition on $f(t)$ and $S(t)$ is that they are (essentially) bounded. They should also be integrable, because their integrals represent the total number of cells and the total amount of chemotactic agent present in the system respectively. These quantities are both finite. Summarising, natural conditions on f and S are:

$$f(t) \in L^1 \cap L^\infty(\mathbb{R}^n \times V), \quad S(t) \in L^1 \cap L^\infty(\mathbb{R}^n), \quad f(t), S(t) \geq 0. \tag{65}$$

The biological applications moreover require that the turning operator is allowed to depend on the function $S(t)$, $\nabla S(t)$.

Looking at (65), a natural candidate for the state spaces for f and S would therefore seem to be the Banach spaces $L^1 \cap L^\infty(\mathbb{R}^n \times V)$ and $L^1 \cap L^\infty(\mathbb{R}^n)$ respectively. However, the operator $-v \cdot \nabla_x$ (in a suitable domain) is not the infinitesimal generator of a C_0-semigroup in $L^1 \cap L^\infty(\mathbb{R}^n \times V)$, nor is $D\Delta$. Thus the theory cannot be applied directly and a different approach is needed.

Our idea is essentially to exploit that $L^1 \cap L^\infty(\mathbb{R}^n \times V)$ is naturally and continuously embedded in any $L^1 \cap L^{p_0'}(\mathbb{R}^n \times V)$ (Lemma 4.12) and that an initial condition in $L^1 \cap L^\infty$ yields a mild solution in any $L^1 \cap L^p$ (p sufficiently large). We show that the maximal existence time of these solutions cannot tend to zero when $p \to \infty$, in fact in our situation it turns out that they are equal, and that these solutions coincide on their common maximal interval of existence. We then use

Lemma 6.1 *Let $p_n \geq 1$, $n \in \mathbb{N}$, be such that $p_n \to \infty$. Let $f \in L^{p_n}(\Omega, \mu)$ for all $n \in \mathbb{N}$. If $\limsup_{n\to\infty} \|f\|_{p_n} < \infty$, then $f \in L^\infty(\Omega, \mu)$ and*

$$\|f\|_\infty = \limsup_{n\to\infty} \|f\|_{p_n} = \lim_{n\to\infty} \|f\|_{p_n}.$$

An estimate of the L^p-norms of the mild solutions that is independent of p is then used to conclude that these solutions take values in $L^1 \cap L^\infty$.

Let us consider the linear reaction-diffusion and excitable medium models under the conditions of Theorem 4.13 or Theorem 4.16 respectively, with $p_0 = p_1 = 1$. Let $X = X^{p_0'}$ be the state space given by (42) for the normal diffusion model, respectively (56) for the excitable medium model. Note that the value of p_1' does not influence X, but that it does change the ambient space Y_1, and that it should satisfy (44), respectively (51), in order to have well-posedness. Finally, note also that *the conditions on p_0' do not impose an upper bound.*

Let $\pi_0 : X \to X_0$ be the projection onto the f-state space and $\pi_1 : X \to X_1$ that onto the S-state space. For any $u \in X$, write $u_i := \pi_i(u)$, $(i = 0, 1)$. We also put $X_c := (I - \pi_0)X$ and write $u_c = (I - \pi_0)u$. Thus $X = X_0 \oplus X_c$ and $\|u\|_X = \|u_0\|_{X_0} + \|u_c\|_{X_c}$. Let us simply write p instead of p_0'.

For an initial condition $\phi \in X^p$ we have unique existence of a mild solution $u^p = u^p(\cdot;\phi)$ in X^p on $[0, \tau]$ by Theorem 3.1, with

$$u_0^p(t) \in L^1(\mathbb{R}^n \times V) \cap L^p(\mathbb{R}^n \times V), \quad u_1^p(t) \in W^{k,1}(\mathbb{R}^n) \cap C_0^k(\mathbb{R}^n)$$

for $t \in [0,\tau]$. Let $[0,\tau_\phi^p)$ be the maximal interval of existence for $u^p(\cdot;\phi)$. In general it may vary with p. In our situation however, it does not:

Lemma 6.2 *Assume that the conditions of Theorem 4.13 or 4.16 hold. Let p and p' both satisfy the conditions for p_0' as given by (44) or (50) and let $\phi \in X^p \cap X^{p'}$. Then $\tau_\phi^p = \tau_\phi^{p'} =: \tau_\phi$ and $u^p = u^{p'} =: u$ on $[0,\tau_\phi)$. Moreover, for r equal to p or p',*

$$\|u_0(t))\|_{L^1\cap L^r} \leq \|\phi_0\|_{L^1\cap L^r}\big[1+\beta(t)e^{\beta(t)}\big], \tag{66}$$

for all $0 \leq t < \tau_\phi$, with

$$\beta(t) := 2\mu(V)\int_0^t \|T[u_1(s)]\|_{L^\infty(\mathbb{R}^n\times V\times V)}\,ds. \tag{67}$$

Proof: Without loss of generality we may assume that $p' > p$. Let j be the continuous embedding of $X^{p'}$ into X^p given by Lemma 4.12. It is easily verified that the map $u := j\circ u^{p'}$ is a mild solution in X^p on $[0,\tau]$ with $u(0)=\phi$ for all $0<\tau<\tau_\phi^{p'}$. Thus $\tau_\phi^{p'} \leq \tau_\phi^p$. Uniqueness of mild solutions yields $u^p = u^{p'}$ on their common interval of definition, $[0,\tau_\phi^{p'})$.

Suppose that $\tau_\phi^{p'} < \tau_\phi^p$. Then $\tau_\phi^{p'}$ must be finite and $\|u^{p'}(t)\|_{X^{p'}} \to \infty$ as $t \uparrow \tau_\phi^{p'}$. Thus, $\|u_0^{p'}(t)\|_{L^1\cap L^{p'}} \to \infty$ or $\|u_c^{p'}(t)\|_{X_c^{p'}} \to \infty$ when $t\uparrow\tau_\phi^{p'}$. In the latter case, we have $\|u_c^p(t)\|_{X_c^p} = \|u_c^{p'}(t)\|_{X_c^{p'}} \to \infty$, because the space $X_c^{p'}$ does not depend on p' and $u^p = u^{p'}$ on $[0,\tau_\phi^{p'})$. It follows that $\|u^p(t)\|_{X^p}\to\infty$ as $t\uparrow\tau_\phi^{p'}$, which contradicts our assumption $\tau_\phi^{p'} < \tau_\phi^p$. So $\|u_c^{p'}(t)\|_{X_c^{p'}}$ must remain bounded on $[0,\tau_\phi^{p'})$ and $\|u_0^{p'}(t)\|_{L^1\cap L^{p'}}\to\infty$. Now,

$$\begin{aligned}\|u_0^{p'}(t))\|_{L^1\cap L^{p'}} &\leq \|T_\Phi(t)\phi_0\|_{L^1\cap L^{p'}} + \Big\|\pi_0\Big(\int_0^t T(t-s)F(s,u^{p'}(s))\,ds\Big)\Big\|_{L^1\cap L^{p'}}\\ &\leq \|\phi_0\|_{L^1\cap L^{p'}} + \Big\|\int_0^t T_\Phi(t-s)\pi_0\big(F(s,u^{p'}(s))\big)\,ds\Big\|_{L^1\cap L^{p'}}\\ &\leq \|\phi_0\|_{L^1\cap L^{p'}} + \int_0^t \big\|\mathcal{T}[u_1^{p'}(s)]u_0^{p'}(s)\big\|_{L^1\cap L^{p'}}\,ds \end{aligned} \tag{68}$$

by using twice that $(T_\Phi(t))_{t\geq 0}$ is non-expansive on all L^r, $1\leq r\leq\infty$. According to Proposition 4.8 and Proposition 4.9 (with θ the constant function 1), the integrand in (68) is majorized by

$$\begin{aligned}&\big(\|M_\theta[u_1^{p'}(s)]\|_{\mathcal{L}(L^1\cap L^{p'})} + \|\hat{\mathcal{T}}[u_1^{p'}(s)]\|_{\mathcal{L}(L^1\cap L^{p'})}\big)\,\|u_0^{p'}(s)\|_{L^1\cap L^{p'}}\\ &\qquad\leq\ 2\mu(V)\|T[u_1^{p'}(s)]\|_{L^\infty(\mathbb{R}^n\times V\times V)}\,\|u_0^{p'}(s)\|_{L^1\cap L^{p'}}.\end{aligned} \tag{69}$$

Application of Gronwall's Lemma (e.g. [30], Lemma D.2) then yields

$$\|u_0^{p'}(t))\|_{L^1\cap L^{p'}} \leq \|\phi_0\|_{L^1\cap L^{p'}}\big[1+\beta(t)e^{\beta(t)}\big], \tag{70}$$

for $0 \le t < \tau_\phi^{p'}$, with

$$\beta(t) := 2\mu(V) \int_0^t \|T[u_1^{p'}(s)]\|_{L^\infty(\mathbb{R}^n \times V \times V)}\, ds.$$

Above, we argued that the map $t \mapsto u_c^{p'}(t)$ remains bounded (in $X_c^{p'}$) on the interval $[0, \tau_\phi^{p'})$. In particular, $t \mapsto u_1^{p'}(t)$ remains bounded. Using Assumption (AT), we conclude that $\beta(t)$ is bounded on $[0, \tau_\phi^{p'})$. Estimate (70) now shows that $\|u_0^{p'}(t)\|_{L^1 \cap L^{p'}}$ cannot blow up, a contradiction.

We conclude that $\tau_\phi^{p'} = \tau_\phi^p$. Note that (68), (69) and therefore (70) remain valid when p' is replaced by p. □

The main result of this section is

Theorem 6.3 *Let the conditions of Theorem 4.13 or Theorem 4.16 be satisfied with $p_0 = p_1 = 1$. Fix p_0' and let $X = X^{p_0'}$ be given by (42), respectively (56). Let $\phi \in X$ be such that $\phi_0 \in L^1 \cap L^\infty(\mathbb{R}^n \times V)$ and let $u(\cdot;\phi) : [0, \tau_\phi) \to X$ be the unique mild solution in X with initial data ϕ and maximal interval of existence. Then the following statements hold:*

(i) $u_0(t, \phi) \in L^1 \cap L^\infty(\mathbb{R}^n \times V)$ *for all* $t \in [0, \tau_\phi)$ *and*

$$\|u_0(t;\phi)\|_\infty \ \le\ 3\|\phi_0\|_{L^1 \cap L^\infty}\left[1 + \beta(t) e^{\beta(t)}\right]$$

for $0 \le t < \tau_\phi$, *where* $\beta(t)$ *is given by (67).*

(ii) *If* $\tau_\phi < \infty$, *then* $\|u_1(t)\|_{X_1} \to \infty$ *as* $t \uparrow \tau_\phi$.

(iii) *For all* $0 < T < \tau_\phi$, $u_0(\cdot;\phi) \in L^\infty_{w^*}([0,T], L^\infty(\mathbb{R}^n \times V))$.

(iv) $u(\cdot;\phi) : [0, \tau_\phi) \to L^\infty(\mathbb{R}^n \times V)$ *is weak***-continuous.*

Proof: *(i).* Note first that for any $1 \le p \le \infty$ and $f \in L^1 \cap L^\infty$ one has, according to Lemma 4.12,

$$\|f\|_{L^1 \cap L^p} \ \le\ \|f\|_1 \ +\ 2^{1/p}\big(\|f\|_1 + \|f\|_\infty\big) \ \le 3\|f\|_{L^1 \cap L^\infty}. \qquad (71)$$

For any $p > p_0'$, let $u^p(t) := u^p(t;\phi)$ be the unique mild solution in X^p with $u^p(0) = \phi$ and maximal interval of existence. According to Lemma 6.2 this interval of existence equals $[0, \tau_\phi)$ and $u^p = u$. Norm estimate (66) in combination with (71), Lemma 6.1 and the observation that $\beta(t)$ is independent of p yield $u_0(t) \in L^\infty(\mathbb{R}^n \times V)$ for $0 \le t < \tau_\phi$ and the given estimate for the L^∞-norm.
(ii). Suppose $\tau_\phi < \infty$. If $\|u_c(t)\|_{X_c}$ remains bounded on $[0, \tau_\phi)$, then also $\|u_0(t)\|_{X_0}$, because of (66), (67) and Assumption (AT), and we cannot have blow-up. So, $\|u_c(t)\|_{X_c} \to \infty$ as $t \uparrow \tau_\phi$. In case of the linear reaction-diffusion model, the proof is complete. In case of the excitable medium model (with general nonlinearity φ, satisfying the conditions of Theorem 4.16), we obtain from the Variation of Constants Formula, that

$$\|u_2(t)\|_{X_2} \le\ \|u_2(0)\|_{X_2} + \tfrac{\gamma}{\tau_r} \int_0^t \|u_1(s)\|_{X_2} ds + \tfrac{1}{\tau_r} \int_0^t \|u_2(s)\|_{X_2} ds.$$

Gronwall's Lemma yields

$$\|u_2(t)\|_{X_2} \ \le\ a(t)\big[1 \ +\ \tfrac{1}{\tau_r}\, te^{t/\tau_r}\,\big], \tag{72}$$

where

$$a(t) := \|u_2(0)\|_{X_2} + \tfrac{C\gamma}{\tau_r}\int_0^t \|u_1(s)\|_{X_1} ds \tag{73}$$

and $C > 0$ is the operator norm of the continuous embedding $X_1 \mapsto X_2$. Thus, if $\|u_1(t)\|_{X_1}$ is bounded on $[0, \tau_\phi)$, then so is $\|u_2(t)\|_{X_2}$ and we cannot have blow-up. We conclude that $\|u_1(t)\|_{X_1} \to \infty$ as $t \uparrow \tau_\phi$.

(iii). The main point of the statement is the measurability. Now, $u_0 \in C([0, \tau_\phi), L^p(\mathbb{R}^n \times V)$ for some p. In particular u_0 is in $L^p([0, \tau_\phi), L^p(\mathbb{R}^n \times V))$, which can be identified with $L^p([0, \tau_\phi) \times \mathbb{R}^n \times V)$. Thus u_0 corresponds to a measurable function $\hat{u}_0$ on $[0, \tau_\phi) \times \mathbb{R}^n \times V$. According to (i), (ii) and Assumption (AT) it is essentially bounded on $[0, T] \times \mathbb{R} \times V$ for any $0 < T < \tau_\phi$.

(iv). Write $u(t) = u(t; \phi_0)$ and let u_0 and u_1 be the projections of u onto the X_0 and X_1-components of X. Let $t_0 \in [0, \tau_\phi)$ and let $h > 0$ such that $t_0 + h < \tau_\phi$. We denote the natural linear pairing between L^p and L^q, $\frac{1}{p} + \frac{1}{q} = 1$, by $\langle \cdot, \cdot \rangle_{p,q}$. For any $\psi \in L^1 \cap L^\infty(\mathbb{R}^n \times V)$ we have, using the Variation of Constants Formula in L^1 (where $(T_\Phi(t))_{t \ge 0}$ is strongly continuous):

$$\begin{aligned} |\langle u_0(t_0+h) - u_0(t_0), \psi\rangle_{1,\infty}| \ &\le\ \big|\langle T_\Phi(t_0+h)\phi_0 - T_\Phi(t_0)\phi_0, \psi\rangle_{1,\infty}\big| \\ &\quad + \ \Big|\int_0^{t_0+h} \langle T_\Phi(t_0+h-s)\mathcal{T}[u_1(s)]u_0(s), \psi\rangle_{1,\infty}\, ds \\ &\quad - \int_0^{t_0} \langle T_\Phi(t_0-s)\mathcal{T}[u_1(s)]u_0(s), \psi\rangle_{1,\infty}\, ds\Big| \end{aligned} \tag{74}$$

A simple computation shows that $\langle T_\Phi(t)f, g\rangle_{p,q} = \langle f, T_\Phi(-t)g\rangle_{p,q}$. Using this we obtain (because $\phi_0, \psi \in L^1 \cap L^\infty(\mathbb{R}^n \times V)$)

$$\begin{aligned} \big|\langle T_\Phi(t_0+h)\phi_0 - T_\Phi(t_0)\phi_0, \psi\rangle_{1,\infty}\big| &= \big|\langle \phi_0, T_\Phi(-t_0-h)\psi - T_\Phi(-t_0)\psi\rangle_{1,\infty}\big| \\ &= \big|\langle \phi_0, T_\Phi(-t_0-h)\psi - T_\Phi(-t_0)\psi\rangle_{\infty,1}\big| \\ &\le \|\phi_0\|_\infty\, \|T_\Phi(-h)T_\Phi(-t_0)\psi - T_\Phi(-t_0)\psi\|_1. \end{aligned} \tag{75}$$

Similarly, the whole second term between absolute value signs in the right hand side of (74) is majorized by

$$\begin{aligned} &\int_0^{t_0} \big|\langle \mathcal{T}[u_1(s)]u_0(s), T_\Phi(s-t_0)(T_\Phi(-h) - I)\psi\rangle_{\infty,1}\big|\, ds \\ &\qquad + \int_{t_0}^{t_0+h} \big|\langle \mathcal{T}[u_1(s)]u_0(s), T_\Phi(s-t_0-h)\psi\rangle_{\infty,1}\big|\, ds \end{aligned} \tag{76}$$

Proposition 4.8 and 4.9 yield

$$\begin{aligned} \|\mathcal{T}[u_1(s)]u_0(s)\|_\infty &\le \|\lambda[u_1(s)]\cdot u_0(s)\|_\infty \ +\ \|\hat{\mathcal{T}}[u_1(s)]u_0(s)\|_\infty \\ &\le 2\mu(V)\, \|T[u_1(s)]\|_{L^\infty(\Omega)}\, \|u_0(s)\|_\infty. \end{aligned}$$

Let τ be such that $t_0 + h < \tau < \tau_\phi$. According to (*ii*), $\|u_1(s)\|_{X_1}$ is uniformly bounded for $s \in [0, \tau]$. Assumption (AT) yields a constant C, depending on ϕ_0 and τ, but not h or ψ, such that

$$\|T[u_1(s)]\|_{L^\infty(\Omega)} \leq C \quad \text{for all } s \in [0, \tau].$$

Thus $\beta(t) \leq 2\mu(V)Ct$ and consequently

$$\|\mathcal{T}[u_1(s)]u_0(s)\|_\infty \leq 2\mu(V)C\|\phi_0\|_{L^1 \cap L^\infty}\left[1 + 2\mu(V)Cse^{2\mu(V)Cs}\right] \leq M \qquad (77)$$

for $0 \leq s \leq \tau$, for a constant $M > 0$. Define

$$M' := M \sup_{0 \leq s \leq \tau} \|T_\Phi(-s)\|_{\mathcal{L}(L^1)}.$$

Combining (74), (75), (76) and (77) yields

$$\begin{aligned} |\langle u_0(t_0+h) - u_0(t_0), \psi\rangle_{1,\infty}| \;\leq\; & \|\phi_0\|_\infty \, \|T_\Phi(-h)T_\Phi(-t_0)\psi - T_\Phi(-t_0)\psi\|_1 \\ & + \; M't_0\|T_\Phi(-h)\psi - \psi\|_1 \; + \; M'h\|\psi\|_1 \qquad (78) \end{aligned}$$

for any $h > 0$ such that $t_0 + h < \tau$ and $\psi \in L^1 \cap L^\infty(\mathbb{R}^n \times V)$. Notice however, that M' does not depend on h or ψ. By density of $L^1 \cap L^\infty$ in L^1 and the strong continuity of the *group* $(T_\Phi(t))_{t\in\mathbb{R}}$ in $L^1(\mathbb{R}^n \times V)$ we conclude that (78) also holds for $|\langle u_0(t_0+h) - u_0(t_0), \psi\rangle_{\infty,1}|$ *for all* $\psi \in L^1(\mathbb{R}^n \times V)$. We conclude that $t \mapsto u_0(t)$ is weak*-continuous from the right. In a similar way, weak* continuity from the left can be obtained. □

Remarks. 1.) Part *(iv)* of the theorem is an improvement over part *(iii)* as it shows the weak*-continuity in $L^\infty(\mathbb{R}^n \times V)$ of the mild solutions on bounded time intervals, instead of 'only' weak*-measurability. Note also that (strong) measurability is a stronger property than weak- or weak*-measurability. Accordingly, the part $u_0(\cdot, \phi)$ of the solution will not be in $L^\infty([0,T], L^\infty(\mathbb{R}^n \times V))$ in general.

2.) The linear reaction-diffusion and excitable medium models are also well-posed when the X_0 component of X equals $L^1 \cap C_0(\mathbb{R}^n \times V)$ and the turning kernel is a bounded continuous function from $\mathbb{R}^n$ into $L^\infty(V \times V)$. Thus, if ϕ is initial data such that $\phi_0 \in L^1 \cap C_0(\mathbb{R}^n \times V)$, then $u_0(t;\phi) \in L^1 \cap C_0(\mathbb{R}^n \times V)$ for $t \in [0, t_\phi)$. In this case, $u(\cdot;\phi)$ is a continuous map into $C_0(\mathbb{R}^n \times V)$, equipped with the uniform norm.

3.) The arguments presented above work for general $1 \leq p_0 < \infty$ and $1 \leq p_1 < \infty$, satisfying the conditions of Theorem 4.13 or Theorem 4.16, not only $p_0 = p_1 = 1$.

4.) Theorem 6.3 extends results of Hillen and Stevens, in particular Theorem 3.1 in [20].

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