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The Lovász number of the Keller graphs

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Preface

This thesis is written to conclude the Master study Mathematics at the University of Leiden. First of all I want to thank Dr. D.C. Gijswijt for offering me this interesting subject and for his guidance during the whole project.

As the title suggests the subject of this thesis is about the *Lovász number* and *Keller graphs*. The Lovász number is a value that is defined for every graph and is known to be an upper bound for the clique number of the complement of the graph.

We are hoping that the Lovász number gives us a good enough upper bound for the clique number of the Keller graphs. The motivation for this will be explained in Chapter 1.

In Chapter 2 we give some definitions and preliminary information for the subjects graph theory, linear algebra, linear programming and semidefinite programming.

After this we will describe in Chapter 3 what these Keller's graphs actually are and we will also study them in more detail.

In Chapter 4 we will deal with the other important subject of this thesis, namely the Lovász number. We will see that this value is defined as the optimal value of a semidefinite program.

Semidefinite programming is a generalization of linear programming. We will show that the semidefinite program to calculate the Lovász number of the (complement of) the Keller graph can easily be reduced to a linear program. How this works will be described in Chapter 5 and in Chapter 6 we will give the results.

In Chapter 7 we will study a generalization of the Keller graphs and we will also show how to calculate the Lovász number of these graphs.

In the next chapter we will try to improve the upper bound for the clique number of the Keller graphs. (The inclusion of this chapter is a subtle indication that we were not as successful as were hoping.)

This thesis will be ended with some conclusions in Chapter 9.

Contents

1 Introduction: Keller's conjecture

In this chapter we will describe where the *Keller graphs*, which we will encounter in Chapter 3, originally came from.

Definition 1.1 *A* tiling *of* $\mathbb{R}^n$ *by n-dimensional unit (hyper)cubes is a set of unit cubes such that every point in* $\mathbb{R}^n$ *is covered by one of the cubes, while no overlap of the interior of any two cubes is allowed.*

Intuitively it is clear that we can assume that all the cubes are aligned parallel to the coordinate axes. If we say that a point $c = (c_1, c_2, ..., c_n)$ is in the tiling, we mean that there is a cube with *center* c in the tiling. The corresponding cube itself is thus the set

$$\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n | c_i - \tfrac{1}{2} \leq x_i \leq c_i + \tfrac{1}{2}, i \in \{1, 2, ..., n\}\}.$$

Without loss of generality we only consider tilings that contain the point $(0, 0, ..., 0)$.

Definition 1.2 *Two cubes* meet in a an $n-1$ dimensional face *if the two corresponding centers differ by exactly* 1 *in one coordinate, while the other* $n-1$ *coordinates are the same.*

In 1930 Keller [4] conjectured the following statement.

Conjecture 1.3 (Keller's conjecture)
Any tiling of $\mathbb{R}^n$ *by unit cubes contains two cubes that meet in an* $n-1$ *dimensional face.*

Keller's conjecture was proven by Perron [11] for $n \leq 6$ in 1940. But in 1992 by Lagarias and Shor [5] it was disproven for $n = 10$ and later in 2002 even for $n = 8$ by Mackey [9]. This actually implies that Keller's conjecture is also false for all $n \geq 8$. The case $n = 7$ is still an open question.

Corrádi and Szabó [2] have reduced Keller's conjecture to a combinatorial problem. In this chapter we will shortly describe how this is done, for this I have used [15].

Definition 1.4 *A cube tiling has* period $k \in \mathbb{N}$ *if it satisfies the condition that if* x *is in the tiling, then every point of the form* $x + (ka_1, ka_2, ..., ka_n)$ *with* $a_1, ..., a_n \in \mathbb{Z}$ *is also in the tiling.*

Proposition 1.5
There exists an n-dimensional cube tiling such that no two cubes meet in an $n-1$ *dimensional face if and only if there is such an n-dimensional cube tiling with period* 2.

Proposition 1.5 can be proven by showing that for any tiling we can consider only the points in the interval $[0, 2)^n$. These cubes will generate a tiling of period 2 and form a counterexample to Keller's conjecture if and only if the original tiling is a counterexample.

An n-dimensional cube tiling of period 2 can be represented by listing all the centers having only values in the interval $[0, 2)$. The interval $[0, 2)^n$ always contains (the centers of) 2^n cubes, this should be intuitively clear.

It is easy to see that in any tiling if a center x in $[0, 2)^n$ has a value $x_i \in [0, 1)$ then there must be another center y with $y_i = x_i + 1$ (and vice versa). But it appears that in any tiling it must be that for any two cubes there is a coordinate in which they differ by 1.

Proposition 1.6
A set of 2^n n-dimensional cubes in the interval $[0, 2)^n$ is such that for any two cubes there is at least one coordinate which differs by exactly 1 $\Leftrightarrow$ these 2^n cubes generates a tiling of period 2.

If the 2^n cubes of a tiling of period 2 is given then for a counterexample of Keller's conjecture we have to check if there is a pair of centers for which two coordinates are different and we must also check if for any pair of centers there is a coordinate in which they differ by 1.

Proposition 1.7
For every $n \in \mathbb{N}$ we have that all tilings of $\mathbb{R}^n$ by unit cubes contain two cubes that meet in an $n - 1$ dimensional face.
$\Leftrightarrow$
For every $n \in \mathbb{N}$ we have that all tilings of $\mathbb{R}^n$ by unit cubes with only half integral centers contain two cubes that meet in an $n - 1$ dimensional face.

Apparently Keller's conjecture can be reduced to the situation where all coordinates only have half integral values. So Keller's conjecture equivalent with the following conjecture.

Conjecture 1.8 (Szabó)
Any tiling of $\mathbb{R}^n$ by unit cubes with only half integral centers contains two cubes that meet in an $n - 1$ dimensional face.

Unfortunately we don't have the following statement for an $n \in \mathbb{N}$:
All tilings of $\mathbb{R}^n$ by unit cubes contain two cubes that meet in an $n-1$ dimensional face $\Leftrightarrow$ all tilings of $\mathbb{R}^n$ by unit cubes with only half integral centers contain two cubes that meet in an $n - 1$ dimensional face.

The problem is that during the reduction the dimension of the tiling will be increased, which we will explain.

Assume a tiling T_1 of period 2 is given where all cubes only have the values 0, 1, a and $a + 1$ for an $a \in (0, 1)$. If we replace this value a by any other real number $b \in (0, 1)$, we call the resulting set T_2, then it is easy to see that T_2 is also a tiling. But we also have T_1 is a counterexample of Keller's conjecture if and only if T_2 is one. (So the value $\frac{1}{2}$ in Conjecture 1.8 could have been replaced by any other real number in $(0,1)$.) The same holds if a tiling contains different values $a_1, ..., a_n \in (0,1)$. The values a_i can be replaced by any real numbers in $(0,1)$ as long you make sure that no two a_i's are the same. From now on we will assume that all values are fractional numbers.

If one of the coordinates in a tiling contains 2^k different values in $(0,1)$, then by increasing the dimension by $k-1$ we can create a tiling such that the new coordinates only have values 0, $\frac{1}{2}$, 1 and $\frac{3}{2}$.

In [15] the reduction is described as:
We will replace the set $\{a_i, 1+a_i\}$ with the 2^k sequences of k coordinates which all have the same set of fractional values. There are 2^k possible sequences of length k of the fractional values $0, \frac{1}{2}$, so we can assign one of these sequences to each a_i. We next need to split this set of 2^k sequences up into two sets such that no two sequences in the same set differ by 1 in exactly 1 place. There is only one way to do this, which is to split them up by the parity of the number of entries $\{0, \frac{1}{2}\}$ and the number of entries $\{1, \frac{3}{2}\}$.

We will give an example. Assume that a coordinate, let's say the first one, contains eight different values $a_0 = 0, a_1, ..., a_7$ in $[0,1)$ then if the set $\{a_i, 1+a_i\}$ is for example assigned to the sequence $(0, \frac{1}{2}, 0)$ we have the following replacing table:

value	replace by
a_i	$0\frac{1}{2}0$; $0\frac{3}{2}1$; $1\frac{1}{2}1$; $1\frac{3}{2}0$
$1+a_i$	$1\frac{3}{2}1$; $1\frac{1}{2}0$; $0\frac{3}{2}0$; $0\frac{1}{2}1$

For instance the cube $(a_1, 0, 1)$ will then be replaced by the cubes $(0, \frac{1}{2}, 0, 0, 1)$, $(0, \frac{3}{2}, 1, 0, 1)$, $(1, \frac{1}{2}, 1, 0, 1)$ and $(1, \frac{3}{2}, 0, 0, 1)$.

It is easy to check that after the described reduction we still get a tiling (thus it contains the right number of cubes and between every two cubes there is a coordinate which differs exactly by 1) and that it is a counterexample to Keller's conjecture if and only if the original tiling is one.

In Chapter 3 we will see that Conjecture 1.8 can be reduced to a maximum clique problem of Keller graphs (see Proposition 3.1).

The goal of this research project will be to investigate if we can solve Conjecture 1.8 (for the cases $n \leq 7$) by finding an upper bound for the clique numbers of the Keller graphs. This upper bound we will try to find via semidefinite programming or to be more specific via de Lovász number of the complement of the Keller graph.

2 Preliminaries

2.1 Graph theory

Definition 2.1 *A* graph $G = (V, E)$ *consists of a finite nonempty set* V*, and a set* E *of unordered pairs of elements of* V*. The set* V *is called the* vertex set *of* G *and* E *is called the* edge set *of* G*. The vertex set of* G *can also be denoted by* $V(G)$*, and the edge set of* G *by* $E(G)$*.*

Throughout this chapter $G = (V, E)$ will be a graph.

Definition 2.2 *An element* $v \in V$ *is called a* vertex *or a* node *of* G*. An element* $e = \{x, y\} \in E$ *with* $x, y \in V$ *is called an* edge *of* G*, we say that the nodes* x *and* y *are* connected *by the edge* e*. The nodes* x *and* y *are the* endpoints *of* e*.*

Definition 2.3 *The complement of* G*, denoted by* $\overline{G} = (\overline{V}, \overline{E})$*, is defined as* $\overline{V} = V$ *and* $\{i, j\} \in \overline{E}$ *if and only if* $\{i, j\} \notin E$ *for all* $i, j \in V$ *with* $i \neq j$*.*

Definition 2.4 *The* degree $\deg(v)$ *of a vertex* $v \in V$ *is the number of edges that have* v *as one of its endpoints. If every vertex of* G *has the same degree we say that* G *is* regular*. If this degree is* k *we say that the degree* $\deg(G)$ *of* G *is* k *and that* G *is* k-regular*.*

Definition 2.5 *Let* $W \subseteq V$ *be a set of vertices. We say that a graph* G_W *is* induced *by* W *if* $V(G_W) = W$ *and two vertices of* G_W *are connected if and only if they are in* G*.*

Definition 2.6 *We say that* G *is* complete *if for all* $v, w \in V$ *with* $v \neq w$ *we have* $\{v, w\} \in E$*.*

Definition 2.7 *A set of vertices* $W \subseteq V$ *is called a* clique *of* G *if for all* $w_1, w_2 \in W$ *with* $w_1 \neq w_2$ *we have* $\{w_1, w_2\} \in E$*. The maximum cardinality of any clique of* G *is denoted by* $\omega(G)$ *and is called the* clique number *of* G*.*

Definition 2.8 *A set of vertices* $W \subseteq V$ *is called a* stable set *of* G *if for all* $w_1, w_2 \in W$ *with* $w_1 \neq w_2$ *we have* $\{w_1, w_2\} \notin E$*. The maximum cardinality of any stable set of* G *is denoted by* $\alpha(G)$ *and is called the* stability number *of* G*.*

Definition 2.9 *A* (vertex)coloring *of* G *is an assignment of a color to every vertex of* G *such that no two connected vertices have the same color. The minimum numbers of colors needed to give such an assignment is called the* chromatic number *of* G *and is denoted by* $\chi(G)$*.*

Alternatively you can say that a coloring of G is a partitioning of V where each partition class is a stable set.

Proposition 2.10
The following (in)equalities hold:

1. $\omega(G) = \alpha(\overline{G})$.

2. $\omega(G) \leq \chi(G)$.

Definition 2.11 *Let $H = (W, F)$ be a graph. Then the* strong product $G \boxtimes H$ *of G and H is a graph with vertex set $V \times W$ (Cartesian product of V and W). And two vertices (v_1, w_1) and (v_2, w_2), with $v_1, v_2 \in V$ and $w_1, w_2 \in W$, are connected by an edge if and only if $\{v_1, v_2\} \in E$ and $\{w_1, w_2\} \in F$ or $v_1 = v_2$ and $\{w_1, w_2\} \in F$ or $\{v_1, v_2\} \in E$ and $w_1 = w_2$.*

The strong product has the associative property. To be more precise, $(G_1 \boxtimes G_2) \boxtimes G_3$ is isomorphic with $G_1 \boxtimes (G_2 \boxtimes G_3)$. Thus we can speak of the strong product of n copies of G, which will be denoted by G^n.

Definition 2.12 *An* automorphism *of G is a bijective mapping $\varphi : V \to V$ from the vertex set to itself such that $\{v, w\} \in E \Leftrightarrow \{\varphi(v), \varphi(w)\} \in E$ for all $v,w \in V$.*

The set of all automorphisms of G form a group and will be denoted by $\mathrm{Aut}(G)$.

Definition 2.13 *We say that G is* vertex transitive *if for every $v, w \in V$ there exists an automorphism φ of G with $\varphi(v) = w$.*

If G is vertex transitive then G is regular.

Definition 2.14 *The adjacency matrix* $\mathrm{Adj}(G)$ *of G is a $V \times V$ matrix defined as*

$$(\mathrm{Adj}(G))_{vw} = \begin{cases} 1 & \{v, w\} \in E; \\ 0 & \textit{otherwise}. \end{cases}$$

2.2 Linear algebra

The element of a matrix M located in the i-th row and in the j-th column will be denoted by M_{ij} and the i-th element of a (row or column) vector v will be denoted by v_i. An n-dimensional vector will be a column vector. We only consider matrices and vectors that contain elements from $\mathbb{R}$. Further the *transpose* of a matrix M will be denoted by M^{T} and the *inverse* of M will be denoted by M^{-1}.

The $n \times n$ unit matrix (which has an 1 on the diagonal elements and further only 0's) is denoted by I_n and the all-ones $n \times n$ matrix is denoted by J_n. (Sometimes the subscript will be omitted, in this case it will be clear from the context what the sizes of the matrices are.) An all-ones vector will be denoted by $\mathbf{1}$ and a zero vector by $\mathbf{0}$.

Definition 2.15 *Let M be a real $n \times n$ matrix. A scalar $\lambda \in \mathbb{C}$ is called an* eigenvalue *of M if there exists a nonzero n-dimensional vector x such that $Mx = \lambda x$. Such a vector x is called an* eigenvector *of M corresponding to λ.*

Definition 2.16 *A matrix D is called* diagonal *if all its non-diagonal elements are 0.*

Definition 2.17 *A square matrix A is* diagonalizable *if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal.*

A square matrix B is orthogonally diagonalizable *if there exists an orthogonal matrix Q such that $Q^\top BQ$ is diagonal.*

Note that for a orthogonal matrix P we have $P^\top = P^{-1}$.

Theorem 2.18
Let A be a square matrix, then we have the following statement:
Matrix A is diagonalizable, and P is an invertible matrix P such that $D := P^{-1}AP$ is diagonal.
$\Longleftrightarrow$
The the columns of P form a linearly independent set of eigenvectors of A and the diagonal elements of D are the eigenvalues of A. Further the eigenvalue D_{ii} correspond to the eigenvector P_i where P_i is the i-th column of P.

Definition 2.19 *Let A be an $n \times n$ matrix. The* trace *of A, denoted by* $\mathrm{tr}(A)$, *is defined as*

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

Proposition 2.20
Let A be an $n \times n$ matrix and let $\lambda_1, ..., \lambda_n$ be the eigenvalues of A. Then we have

$$\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i.$$

Definition 2.21 *Let A and B be $n \times n$ matrices. The* inner product $A \cdot B$ *of A and B is defined as*

$$A \cdot B := \mathrm{tr}(A^\top B) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}B_{ij}.$$

Definition 2.22 *A matrix M is called* symmetric *if $M = M^\top$.*

Proposition 2.23
A real matrix M is symmetric if and only if M is orthogonally diagonalizable.

Proposition 2.24
If M is a real symmetric matrix, then M only has real eigenvalues.

Definition 2.25 *A symmetric real matrix M is called* positive semidefinite *if all its eigenvalues are nonnegative and this will be denoted by $M \succeq 0$. If all eigenvalues of M are strict positive then M is* positive definite *and will be denoted by $M \succ 0$.*

Theorem 2.26
Let M be an $n \times n$ matrix. Then the following statements are equivalent:

1. *M is positive semidefinite;*

2. *For every $x \in \mathbb{R}^n$ we have $x^\top Mx \geq 0$;*

3. *M can be written as the Gram matrix of n vectors $v_1, ..., v_n \in \mathbb{R}^m$ for some m, e.g. $M_{ij} = v_i^\top v_j$. Equivalently we have $M = V^\top V$ for some matrix V;*

4. *M can be written as a nonnegative linear combination of matrices of the form $xx^\top$.*

Proposition 2.27
Let A be a symmetric matrix that is of the form

$$A = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & A_n \end{pmatrix},$$

where the A_i's are square matrices (not necessary all of the same size). Then we have:
A is positive semidefinite $\Leftrightarrow$ $A_1, ..., A_n$ are positive semidefinite.

Theorem 2.28
For all $n \in \mathbb{N}$ the set $\mathbb{S}_+^n$ of all positive semidefinite $n \times n$ matrices form a convex cone. This means if $A, B \in \mathbb{S}_+^n$ and $c \in \mathbb{R}^+$ then also $A + B \in \mathbb{S}_+^n$ and $cA \in \mathbb{S}_+^n$.

Definition 2.29 *Let A be an $m \times n$ matrix and B a $p \times q$ matrix. Then the* Kronecker product *(also called* Tensor product*) $A \otimes B$ of A and B is defined as*

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{pmatrix}.$$

Note that $A \otimes B$ is an $mp \times nq$ matrix.

Proposition 2.30
Let A, B, C and D be matrices and let $k \in \mathbb{R}$. Then the following properties hold for Kronecker products:

1. $A \otimes (B + C) = A \otimes B + A \otimes C$;

2. $(A + B) \otimes C = A \otimes C + B \otimes C$;

3. $(kA) \otimes B = A \otimes (kB) = k(A \otimes B)$;

4. $(A \otimes B) \otimes C = A \otimes (B \otimes C)$;

5. *If A, B, C and D have such dimensions that that the matrix products AC and BD are well defined then we have: $(A \otimes B)(C \otimes D) = AC \otimes BD$;*

6. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$;

7. $(A \otimes B)^{\mathsf{T}} = A^{\mathsf{T}} \otimes B^{\mathsf{T}}$.

Proposition 2.31
Let A and B be diagonalizable matrices and let P_A and P_B be matrices such that $D_A := P_A^{-1} A P_A$ and $D_B := P_B^{-1} B P_B$ are diagonal. Let $P_C := P_A \otimes P_B$ and $C := A \otimes B$, then $D_C := P_C^{-1} C P_C$ is diagonal.

Proof
We will prove the following identity

$$(P_A \otimes P_B)^{-1}(A \otimes B)(P_A \otimes P_B) = D_A \otimes D_B.$$

Note that $D_A \otimes D_B$ is diagonal. We have

$$\begin{aligned} D_A \otimes D_B &= (P_A^{-1} A P_A) \otimes (P_B^{-1} B P_B) \\ &= (P_A^{-1} \otimes P_B^{-1})(A P_A \otimes B P_B) \\ &= (P_A^{-1} \otimes P_B^{-1})(A \otimes B)(P_A \otimes P_B) \\ &= (P_A \otimes P_B)^{-1}(A \otimes B)(P_A \otimes P_B). \end{aligned}$$

□

Corollary 2.32
Let A be a diagonalizable $m \times m$ matrix and B a diagonalizable $n \times n$ matrix. Further denote the eigenvalues of A by $\lambda_1, ..., \lambda_m$ and the eigenvalues of B by $\mu_1, ..., \mu_n$. Then the eigenvalues of $A \otimes B$ are $\{\lambda_i \mu_j | i \in \{1, ..., m\}, j \in \{1, ..., n\}\}$.

2.3 Linear programming

A *linear program*, abbreviated by *LP*, is an optimization problem which can be written in the following form (called *primal form*):

$$\max \left\{ c^{\mathsf{T}} x \;\middle|\; \begin{array}{l} a_i^{\mathsf{T}} x \geq b_i, i \in M_1 \\ a_i^{\mathsf{T}} x \leq b_i, i \in M_2 \\ a_i^{\mathsf{T}} x = b_i, i \in M_3 \\ x_j \geq 0, j \in N_1 \\ x_j \leq 0, j \in N_2 \\ x_j \text{ free}, j \in N_3 \end{array} \right\}. \tag{1}$$

Here A is a given $m \times n$ matrix, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ are given vectors, $x \in \mathbb{R}^n$, and a_i is the i-th row of A. Further the M_i's form a given partitioning of the set $\{1, 2, ..., m\}$ and the N_i's is a given partitioning of $\{1, 2, ..., n\}$.

The term $c^{\mathsf{T}} x$ is called the *objective function* of LP (1) and the terms $a_i^{\mathsf{T}} x \geq b_i$, $a_i^{\mathsf{T}} x \leq b_i$, $a_i^{\mathsf{T}} x = b_i$, $x_j \geq 0$ and $x_j \leq 0$ are its *constraints*. The expression "x_j free" means that the variable x_j is allowed to be any number of $\mathbb{R}$. (Usually when we are writing down an LP we will not mention it explicitly when a certain variable is free.) A vector x that satisfies all the constraints is called a *feasible* solution. If no such x exists then we say that the LP is *infeasible*. If the value of the objective function can be arbitrary large (convention: optimal objective value is ∞) the LP is called *unbounded*.

The *dual form* of the LP (1) is defined as

$$\min \left\{ b^\top y \;\middle|\; \begin{array}{l} y_i \leq 0, i \in M_1 \\ y_i \geq 0, i \in M_2 \\ y_i \text{ free}, i \in M_3 \\ y^\top A_j \geq c_i, j \in N_1 \\ y^\top A_j \leq c_i, j \in N_2 \\ y^\top A_j = c_i, j \in N_3 \end{array} \right\}, \tag{2}$$

where $y \in \mathbb{R}^m$ and A_j is the j-th column of A. Note that (2) is also a LP, because we can just maximize the objective function $-b^\top y$.

Remark: An LP in the form of (1) can be "reduced" to the following form

$$\max \left\{ c^\top x \,|Ax \leq b; x \geq 0 \right\},$$

where A is an $m \times n$ matrix, $c \in \mathbb{R}^n$, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. In this case the dual form is

$$\min \left\{ b^\top y \,\middle|\, A^\top y \geq c; y \geq 0 \right\},$$

where $y \in \mathbb{R}^m$. This is the form (or a variant of it) you will likely come more often across in the literature.

Theorem 2.33 *The following statements hold:*

1. ***Weak duality theorem:*** *If $\overline{x}$ is a feasible solution of LP (1) and $\overline{y}$ a feasible solution of (2), then we have $c^\top \overline{x} \leq b^\top \overline{y}$;*

2. ***Strong duality theorem:*** *Assume that both LP's (1) and (2) have a finite solution. Let x^* be an optimal solution of (1) and let y^* be an optimal solution of (2). Then we have $c^\top x^* = b^\top y^*$;*

3. *If an LP is unbounded then its dual LP is infeasible;*

4. *If an LP is infeasible then its dual LP is either infeasible or unbounded.*

Theoretically every LP can be solved in polynomial time by *interior points methods*. However in practice more often the *simplex method* is used, which in general does not run in polynomial time, but usually works very well in practice. For more information about the theory of linear programming see for example [1].

2.4 Semidefinite programming

A *semidefinite program*, abbreviated by *SDP*, is an optimization problem which can be written in the following form:

$$\inf\left\{c^{\mathsf{T}}x \,|x_1A_1 + ... + x_nA_n - B \succeq 0\right\}, \tag{3}$$

where $A_1, ..., A_n, B$ are given symmetric $m \times m$ matrices, and $c \in \mathbb{R}^n$ is a given vector. Let

$$X := x_1A_1 + ... + x_nA_n - B.$$

The term $c^{\mathsf{T}}x$ is called the *objective function* and $X \succeq 0$ is a *constraint* of SDP (3).

In contrast to linear programming if the infimum value of an SDP is finite, it does not have to be attained by a solution x. The vector x is *feasible* if $X \succeq 0$, and x is *strictly feasible* if $X \succ 0$.

The dual form of SDP (3) is defined as

$$\sup\left\{ B \cdot Y \left| \begin{array}{rcl} Y & \succeq & 0 \\ A_1 \cdot Y & = & c_1 \\ & \vdots & \\ A_n \cdot Y & = & c_n \end{array} \right.\right\}, \tag{4}$$

where Y is a symmetric $m \times m$ matrix.

Note that if the A_i's en B are diagonal matrices then (3) is an LP. Because then the matrix X is diagonal where each diagonal element is a linear expression on the variables x_i, which are all required to be nonnegative by the constraint $X \succeq 0$. So we can consider semidefinite programming as a generalization of linear programming.

If we add linear constraints on the variables x_i to SDP (3) it is still an SDP. How this works will hopefully become more clear with the following example. Assume that we have the SDP

$$\inf\left\{3x_1 + x_2 \left| x_1 \begin{pmatrix} 1 & 3 \\ 3 & -4 \end{pmatrix} + x_2 \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 6 & 1 \\ 1 & -9 \end{pmatrix} \succeq 0 \right.\right\},$$

and we want to add the following two linear constraints to the problem:

$$4x_1 - 3x_2 \geq 5,$$

$$-7x_1 + 5x_2 \geq 0.$$

Then we can take the following SDP:

$$\inf\left\{3x_1 + x_2 \,|x_1A_1 + x_2A_2 - B \succeq 0\right\},$$

with

$$A_1 = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 3 & -4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -7 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}, B = \begin{pmatrix} 6 & 1 & 0 & 0 \\ 1 & -9 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can write SDP (4) also in the form of SDP (3). This will again be illustrated by an example. Take

$$\sup \left\{ \begin{pmatrix} 6 & 1 \\ 1 & -9 \end{pmatrix} \cdot Y \;\middle|\; \begin{array}{rcl} \begin{pmatrix} 1 & 3 \\ 3 & -4 \end{pmatrix} \cdot Y & = & 3 \\ \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \cdot Y & = & 1 \\ Y & \succeq & 0 \end{array} \right\},$$

Let

$$Y := \begin{pmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{pmatrix},$$

then the SDP can be rewritten as

$$\inf \left\{ -6y_{11} - 2y12 + 9y_{22} \;\middle|\; \begin{array}{rcl} y_{11} + 6y_{12} - 4y_{22} & = & 3 \\ -y_{11} + 3y_{22} & = & 1 \\ y_{11}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + y_{12}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + y_{22}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \succeq & 0 \end{array} \right\}.$$

We already know that this can be rewritten again so that it will have the same form as SDP (3), so it is justified to call the optimization problem (4) an SDP.

Also in semidefinite program there is a *Weak duality theorem* and in certain circumstances also a *Strong duality theorem.*

Theorem 2.34
Denote the infimum value of SDP (3) by v_p and the supremum value of SDP (4) by v_d (if they exist). Then we have the following statements:

1. ***Weak duality theorem:*** *If both the primal and dual SDP's have feasible solutions then we have $v_p \geq v_d$.*

2. ***Strong duality theorem:*** *If the primal SDP has a strictly feasible solution (this is called the* Slater condition*), then the dual optimum is attained and $v_p = v_d$.*

The supremum/infimum value of an SDP can be approximated in polynomial time by the *ellipsoid method* and *interior point/barrier methods*, see for example [12, 13].

3 Keller graphs

In Chapter 1 we have mentioned that the half integral version of Keller's conjecture can be reduced to a maximum clique problem of graphs. In this chapter we will study those graphs more in detail.

Let

$$S := \left\{0, \tfrac{1}{2}, 1, \tfrac{3}{2}\right\} \subset \mathbb{Q}/2\mathbb{Z},$$

then we will define the Keller graph $G_n = \{V_n, E_n\}$ of dimension $n \in \mathbb{N}$ as

$$V_n = \{(x_1, ..., x_n) | x_i \in S, i \in \{1, ..., n\}\},$$

$$E_n = \{\{v, w\} | \exists i \in \{1, ...n\} : |v_i - w_i| = 1, \exists j \in \{1, ...n\} : i \neq j \wedge v_j \neq w_j\}.$$

Note that we have $|V_n| = 4^n$. For convenience an n-tuple $(x_1, x_2, ..., x_n)$ will sometimes be written as $x_1 x_2 ... x_n$. The i-th element of an n-tuple x will be denoted by x_i. The vertex (0,0,...,0) will be abbreviated by $\mathbf{0}$. Also from now on we will use the symbol a for the value $\frac{1}{2}$ and the symbol b for $\frac{3}{2}$. So we rather have

$$S = \{0, 1, a, b\}.$$

Thus for $x, y \in S$, with $x \neq y$, we have the relation $|x - y| = 1$ if and only if both x and y are numbers or if both are characters.

For a given $n \in \mathbb{N}$ Conjecture 1.8 can be reduced to a maximum clique problem of the graph G_n.

Proposition 3.1
Conjecture 1.8 is false for dimension $n \Leftrightarrow \omega(G_n) = 2^n$.

Proof
By the definition of V_n all the half integral points in $[0, 2)^n$ is a vertex of G_n and vice versa. Further by the definition of E_n two vertices are connected if and only if they have a coordinate which differ by 1 (condition of a tiling) and two coordinates which are different (without this condition these two cubes would have met in an $n - 1$ dimensional face). Because a tiling in $[0, 2)^n$ must contain exactly 2^n cubes we see that the statement must be true.

□

Definition 3.2 *Let $x, y \in S$. If $x \neq y$ then we say that x and y are* different. *If $|x - y| = 1$ we say that x and y are* opposite, *and if we have $x \neq y$ and $|x - y| \neq 1$ then we say that x and y are* type different.

So two vertices $v, w \in V_n$ are connected if there is at least one coordinate $i \in \{1, ...n\}$ such that v_i and w_i are *opposite* and there is another coordinate $j \neq i$ such that v_j and w_j are *different.* Note that for every element $s \in S$ there is one corresponding *opposite* element in S, while the remaining two other elements of S are *type different* from s.

Proposition 3.3
We have $\chi(G_n) \leq 2^n$.

Proof
Consider the nodes which only contain the numbers 0 and 1, there are 2^n of these. Give them all a different color. For any other vertex $v = (v_1, v_2, ..., v_n)$ give it the same color as $\lfloor v \rfloor := (\lfloor v_1 \rfloor, \lfloor v_2 \rfloor, ..., \lfloor v_n \rfloor)$. This is a legal coloring of G_n and therefore we must have $\chi(G_n) \leq 2^n$.

□

For example a coloring of V_3 with 8 colors is (vertices in the same column will have the same color):

000	001	010	011	100	101	110	111
$00a$	$00b$	$01a$	$01b$	$10a$	$10b$	$11a$	$11b$
$0a0$	$0a1$	$0b0$	$0b1$	$1a0$	$1a1$	$1a0$	$1b1$
$0aa$	$0ab$	$0ba$	$0bb$	$1aa$	$1ab$	$1ba$	$1bb$
$a00$	$a01$	$a10$	$a11$	$b00$	$b01$	$b10$	$b11$
$a0a$	$a0b$	$a1a$	$a1b$	$b0a$	$b0b$	$b1a$	$b1b$
$aa0$	$aa1$	$ab0$	$ab1$	$ba0$	$ba1$	$bb0$	$bb1$
aaa	aab	aba	abb	baa	bab	bba	bbb

Proposition 3.4
If G_n *has a clique of order* k*, then* G_{n+1} *has a clique of order* $2k$.

Proof
The following proof is given in [3] (Theorem 4.2). Let K be a clique of order k. Let

$$X := \{(0, v_1, ..., v_n) | v \in K\}$$

and

$$Y := \{(1, w_1 + a, ..., w_n + a) | w \in K\}.$$

It is easy to see that if $p, q \in X$ or $p, q \in Y$, then we have $\{p, q\} \in E_{n+1}$. Assume that there exist an $x \in X$ and a $y \in Y$ such that $\{v, w\} \notin E_{n+1}$. Then we must have $x = (0, s_1, ..., s_n)$ and $y = (1, s_1, ..., s_n)$ for an $s \in K$, while we also have $t := (s_1 - a, ..., s_n - a) \in K$. As we have $s, t \in K$ we must have $\{s, t\} \in E_n$, but it is clear that this cannot be true. So we can conclude that for all $x \in X$ and $y \in Y$ we have $\{v, w\} \in E_{n+1}$, and therefore $X \cup Y$ is a clique of order $2r$ in G_{n+1}.

□

Definition 3.5 *Two vertices* $v, w \in V_n$ *have* relation $r(v, w) = (x, y, z)$ *if* $|\{i | v_i = w_i\}| = x$, $|\{j | v_j$ *and* w_j *are opposite*$\}| = y$ *and* $|\{k | v_k$ *and* w_k *are type different*$\}| = z$.

For example $r(0aabb, 1b0ab) = (1, 3, 1)$ and $r(001, 00b) = (2, 0, 1)$. Note that if $v, w \in G_n$ and $r(v, w) = (i, j, k)$ then $i + j + k = n$ holds. Further we always have $r(v, w) = r(w, v)$.

Proposition 3.6
Let φ be a bijection from V_n to V_n. If $r(v,w) = r(\varphi(v), \varphi(w))$ for every $v, w \in V_n$ then we have $\varphi \in \mathrm{Aut}(G_n)$.

Proof
If we have $r(v,w) = r(\varphi(v), \varphi(w))$ for every $v, w \in V_n$, then v and w have an opposite coordinate and another coordinate that is different if and only if $\varphi(v)$ and $\varphi(w)$ have an opposite coordinate and another coordinate that is different.

And thus also $\{v, w\} \in E_n$ if and only if $\{\varphi(v), \varphi(w)\} \in E_n$. Because φ is also a bijection then by definition φ is now an automorphism of G_n.

□

We can easily create automorphisms of G_n that satisfy the condition of Proposition 3.6 by the following operations (the correctness can easily be checked):

- In a coordinate $i \in \{1, ..., n\}$ adds a number $s \in S$.
- In a coordinate $i \in \{1, ..., n\}$ swap an $s \in S$ with $s + 1$ (thus 0 with 1 or a with b).
- Permutate the coordinates of every $v \in V_n$ in the same way.
- Compositions of any of the above mentioned operations.

The automorphisms of G_n that changes something in a coordinate $i \in \{1, ..., n\}$ together with the identity map form a group which is isomorphic to D_4 (the eight symmetries of a square with corners named clockwise as 0, a, 1 and b).

The automorphisms of G_n that will only permutate the coordinates is isomorphic to the group S_n (permutation group of n elements), which has $n!$ elements.

Note that if $\varphi_i \in \mathrm{Aut}(G_n)$ is an automorphism that only changes something in coordinate i, then we have $\varphi_i \circ \varphi_j = \varphi_j \circ \varphi_i$ for all $i, j \in \{1, ..., n\}$ with $i \neq j$. But if $\pi \in \mathrm{Aut}(G_n)$ is a permutation automorphism then in general we don't have $\varphi_i \circ \pi = \pi \circ \varphi_i$.

Let R be the set of all automorphisms of G_n that satisfy the condition $r(v,w) = r(\varphi(v), \varphi(w))$. We can now conclude that

$$S_n \ltimes \underbrace{D_4 \times ... \times D_4}_{n \text{ times}} \subseteq R.$$

But we will show that these two groups are actually isomorphic.

Define the following sets of vertices of G_n:

$$N_{xyz} = \{v \in V_n | r(\mathbf{0}, v) = (x, y, z)\}.$$

Note that an n-tuple from N_{xyz} will contain x zeros, y ones and z characters.

Lemma 3.7
If $\psi \in \mathrm{Aut}(G_n)$ is an automorphism with $r(v,w) = r(\psi(v),\psi(w))$ and has the property if $v \in N_{xyz}$ then also $\psi(v) \in N_{xyz}$ for all $v,w \in V_n$, then for $1 \le i < n$ and $0 \le j \le n-i-1$ we have:

$$\psi(v) = v \text{ for all } v \in N_{n-i-j,i,j} \Longrightarrow \psi(w) = w \text{ for all } w \in N_{n-i-j-1,i+1,j}.$$

Proof
Assume that $\psi(v) = v$ for all $v \in N_{n-i-j,i,j}$. Take a $w \in N_{n-i-j-1,i+1,j}$, then w will be of the form (or where the elements of the n-tuple are permutated differently in which case the argument will be very similar):

$$w = (\underbrace{0,...,0}_{n-i-j-1}, \underbrace{1,...,1}_{i+1}, l_1,...,l_j)$$

where the l_i's are characters. Let

$$s := (\underbrace{0,...,0}_{n-i-j}, \underbrace{1,...,1}_{i}, l_1,...,l_j) \text{ and } t := (\underbrace{0,...,0}_{n-i-j-1}, 1, 0, \underbrace{1,...,1}_{i-1}, l_1,...,l_j).$$

Then by assumption we have $\psi(s) = s$ and $\psi(t) = t$. Further we have $r(w,s) = r(w,t) = (n-1,1,0)$ and because ψ respects the relation between two n-tuples we now get $r(s,\psi(w)) = r(t,\psi(w)) = (n-1,1,0)$.

From $r(s,\psi(w)) = (n-1,1,0)$ and $\psi(w) \in N_{n-i-j-1,i+1,j}$ we can derive that $\psi(w)$ is of the following form:

$$\psi(w) = (w_1,...,w_{n-i-j}, \underbrace{1,...,1}_{i}, l_1,...,l_j).$$

And from $r(t,\psi(w)) = (n-1,1,0)$ and $\psi(w) \in N_{n-i-j-1,i+1,j}$ we can see that $\psi(w)$ looks like:

$$\psi(w) = (w_1,...,w_{n-i-j-1}, 1, w_{n-i+1}, \underbrace{1,...,1}_{i-1}, l_1,...,l_j).$$

This combined means that $\psi(w)$ is of the form

$$\psi(w) = (w_1,...,w_{n-i-j-1}, \underbrace{1,...,1}_{i+1}, l_1,...,l_j),$$

but by using $r(s,\psi(w)) = (n-1,1,0)$ we now can conclude that we then must have $\psi(w) = w$.

□

Lemma 3.8
If $\psi \in \mathrm{Aut}(G_n)$ is an automorphism with $r(v,w) = r(\psi(v),\psi(w))$ and has the property if $v \in N_{xyz}$ then also $\psi(v) \in N_{xyz}$ for all $v,w \in V_n$, then for $1 \le i < n$ we have:

$$\psi(x) = x \text{ for all } x \in N_{n-i-j,j,i}, \text{ with } 0 \le j \le n-1$$
$$\Longrightarrow \psi(y) = y \text{ for all } y \in N_{n-i-k-1,k,i+1}, \text{ with } 0 \le k \le n-i-1.$$

Proof
Assume that $\psi(x) = x$ for all $x \in N_{n-i-j,j,i}$. Take a $y \in N_{n-i-k-1,k,i+1}$, then y will be of the form (or a permutation of it):

$$y = (\underbrace{0, ..., 0}_{n-i-k-1}, \underbrace{1, ..., 1}_{k}, l_1, ..., l_{i+1}),$$

where the l_i's are characters. Let

$$p := (\underbrace{0, ..., 0}_{n-i-k-1}, \underbrace{1, ..., 1}_{k}, 0, l_2, ..., l_j) \text{ and } q := (\underbrace{0, ..., 0}_{n-i-k-1}, \underbrace{1, ..., 1}_{k}, l_1, 0, l_3, ..., l_{i+1}).$$

Then by assumption we have $\psi(p) = p$ and $\psi(q) = q$. Further we have $r(y, p) = r(y, q) = (n-1, 0, 1)$ and thus also $r(p, \psi(y)) = r(q, \psi(y)) = (n-1, 0, 1)$. With similar technique as in the proof of Lemma 3.7 we can derive that $\psi(y) = y$.

□

Lemma 3.9
Let ψ be an automorphism of G_n such that $r(v, w) = r(\psi(v), \psi(w))$ and that has the property if $v \in N_{xyz}$ then also $\psi(v) \in N_{xyz}$ for all $v, w \in V_n$. If we have $\psi(x) = x$ for all $x \in N_{n-1,1,0} \cup N_{n-1,0,1}$ then ψ is the identity map.

Proof
Combining $\psi(\mathbf{0}) = \mathbf{0}$ with Lemma 3.7 with $j = 0$ we can conclude that $\psi(p) = p$ if p is a vertex that only has values 0 and 1.

Using $\psi(x) = x$ for all $x \in N_{n-1,0,1}$ and Lemma 3.7 with $j = 1$ we can conclude that $\psi(q) = q$ if q is a vertex that only has exactly one character. And combining the last fact with Lemma 3.8 we can conclude that ψ is the identity map.

□

Theorem 3.10
Let R be the group of automorphisms of G_n that satisfy the condition $r(v, w) = r(\varphi(v), \varphi(w))$ for all $v, w \in V_n$ and $\varphi \in R$. Then we have $R \cong H$, where

$$H := S_n \ltimes \underbrace{D_4 \times ... \times D_4}_{n \text{ times}}.$$

Proof
We already know that $R \supseteq H$. Assume that there exists an automorphism $\psi \notin H$ of G_n with $r(v, w) = r(\psi(v), \psi(w))$ for all $v, w \in V_n$. If $\psi(\mathbf{0}) = z$ we can take $\chi \in H$ defined as $\chi(v) = v - z$ so that $\chi(\psi(\mathbf{0})) = \mathbf{0}$. So we can assume that $\psi(\mathbf{0}) = \mathbf{0}$ and because of this we also have: if $\psi(v) = w$ and $v \in N_{xyz}$ then we have $w \in N_{xyz}$.

We will now concentrate on the vertices of $N_{n-1,1,0}$. Let $e_i \in N_{n-1,1,0}$ be the n-tuple with an 1 at its i-th coordinate (the other elements are 0). We can assume that $\psi(e_j) = e_j$ for all $1 \leq j \leq n$. If not there exists a $\tau \in H$ (to be more specific $\tau \in S_n$) that will permutate the coordinates in such a way that $\tau(\psi(e_j)) = e_j$. Note that we then also have $\tau(\psi(\mathbf{0})) = \mathbf{0}$.

Now we will examine the elements of $N_{n-1,0,1}$. Let $a_j \in N_{n-1,0,1}$ be the n-tuple with an a at its j-th coordinate and b_j the n-tuple with a b at its j-th coordinate. Because $\psi(e_j) = e_j$ holds for each $j \in \{1, ..., n\}$ we have: $\psi(a_j) = a_j$ and $\psi(b_j) = b_j$ or $\psi(a_j) = b_j$ and $\psi(b_j) = a_j$.

We can assume that it is the first option for all $j \in \{1, ..., n\}$. Because if $\psi(a_k) = b_k$ and $\psi(b_k) = a_k$ for a k we can take $\sigma \in H$ where σ is the automorphism that will swap the elements a and b in the k-th coordinate so that we have $\sigma(\psi(a_k)) = a_k$ and $\sigma(\psi(b_k)) = b_k$.

But by Lemma 3.9 we now actually have $\psi(v) = v$ for all $v \in V_n$. From this follows that the set H don't have the identity map ψ as an element. This is of course not true, so the assumption that there exists a $\psi \notin H$ with $r(v, w) = r(\psi(v), \psi(w))$ for all $v, w \in V_n$ is false.

□

Proposition 3.11
The graph G_n is vertex transitive and regular of degree $4^n - 3^n - n$.

Proof
Let $s, t \in V_n$. The automorphism φ of G_n given by $x \mapsto x + (t - s)$ will map s to t. Thus G_n is vertex transitive, so we can indeed speak of the degree of G_n. We will calculate the degree of vertex $v := \mathbf{0}$.

We know that G_n has 4^n nodes. If a $w \in V_n$ is not connected with v then we have two situations:

- There is an $i \in \{1, ...n\}$ such that $|v_i - w_i| = 1$, but there is no $j \in \{1, ...n\}$ with $i \neq j$ and $v_j \neq w_j$. Thus the n-tuple of w must contain exactly one 1 and all the other numbers are 0. There are n of such nodes.
- There does not exist an $i \in \{1, ...n\}$ such that $|v_i - w_i| = 1$. In this case the n-tuple of w only has the values 0, a or b. There are 3^n of such nodes.

In conclusion the degree of v is $4^n - 3^n - n$ and thus $\deg(G_n) = 4^n - 3^n - n$.

□

Theorem 3.12
Let $v, w, x, y \in V_n$. If $r(v, w) = r(x, y)$ holds, then there exists an automorphism $\varphi \in \mathrm{Aut}(G_n)$ with $\varphi(v) = x$ and $\varphi(w) = y$.

Proof
We can assume that for all $i \in \{1, ..., n\}$ the relation (same, opposite or type different) between v_i and w_i is the same as the relation between x_i and y_i. Otherwise

we can take an automorphism $\chi \in \mathrm{Aut}(G_n)$ that will permutate the coordinates in such a way that this will be the case.

Further we can assume that $v = x$, if not we can take the automorphism $\psi \in \mathrm{Aut}(G_n)$ given by $\psi(z) = z + (x - v)$, so that $\psi(v) = x$. Thus we now have $r(x, w) = r(x, y)$.

So we only need to show that there exists an automorphism $\varphi \in \mathrm{Aut}(G_n)$ with $\varphi(x) = x$ and $\varphi(w) = y$. Let φ be the automorphism that will do the following: for every i with $w_i \neq y_i$ (note that w_i and y_i must be opposite then) swap w_i and y_i in coordinate i. Then we indeed have $\varphi(v) = x$ and $\varphi(w) = y$.

□

Proposition 3.6 gives a sufficient condition for a bijection $\varphi : V_n \to V_n$ to be an automorphism of G_n, but it is not a necessary condition. A clear example is in G_1, which only contains four isolated vertices, thus every bijection $\psi : V_1 \to V_1$ is an automorphism. However only the identity map respects the condition in Proposition 3.6.

But I have also found a counterexample for $n = 2$. Let $\phi : V_2 \to V_2$ given by:

$$\begin{array}{llll} 00 \mapsto 00, & 10 \mapsto 10, & a0 \mapsto ab, & b0 \mapsto bb, \\ 01 \mapsto 0a, & 11 \mapsto 1a, & a1 \mapsto a1, & b0 \mapsto b1, \\ 0a \mapsto 01, & 1a \mapsto 11, & aa \mapsto aa, & b0 \mapsto ba, \\ 0b \mapsto 0b, & 1b \mapsto 1b, & ab \mapsto a0, & b0 \mapsto b0. \end{array}$$

Note that $r(00, 01) \neq r(\phi(00), \phi(01)) = r(00, 0a)$, thus the condition in Proposition 3.6 is not fulfilled. This finding implies that for every $\{v, w\} \in E_2$ and $\{x, y\} \in E_2$ there exists an automorphism ψ of G_2 with $\psi(v) = x$ and $\psi(w) = y$. It's not known to me if there are also counterexamples for $n > 2$.

4 The Lovász number of a graph

It is well known that finding the clique number of a graph belongs to the *NP-hard* problems. So it is very unlikely that there will be an algorithm to solve this in polynomial time in terms of the number of nodes of a graph. But in this chapter we will see how semidefinite programming can be used to get an upper bound for the clique number of a graph.

Consider the following SDP for a graph $G = (V, E)$:

$$\min \left\{ t \;\middle|\; \begin{array}{l} Y \text{ is a } V \times V \text{ matrix} \\ Y \succeq 0 \\ Y_{ij} = -1 \text{ for all } \{i,j\} \in \overline{E} \\ Y_{ii} = t - 1 \text{ for all } i \in V \end{array} \right\}. \tag{5}$$

The dual problem of SDP (5) is:

$$\max \left\{ \sum_{i \in V} \sum_{j \in V} Z_{ij} \;\middle|\; \begin{array}{l} Z \text{ is a } V \times V \text{ matrix} \\ Z \succeq 0 \\ Z_{ij} = 0 \text{ for all } \{i,j\} \in E \\ \operatorname{tr}(Z) = 1 \end{array} \right\}. \tag{6}$$

The optimal value of both SDP's is denoted by $\vartheta(G)$ and is called the *Lovász number* of the graph G, see [7]. The value $\vartheta(G)$ is sometimes also called the *Lovász theta function* in the literature. The two SDP's do have the same optimal value because $Y := |V|I - J$ is strictly feasible in SDP (5) and $Z = \frac{1}{|V|}I$ is strictly feasible in SDP (6). By the strong duality theorem the two optimal values are equal and in both problems the optimal value is attained, so it is justified to speak about a minimum and a maximum.

To see that SDP (6) is the dual of SDP (5), let A_{vw} be a $|V| \times |V|$ matrix defined as

$$(A_{vw})_{ij} = \begin{cases} 1 & (i,j) = (v,w); \\ 1 & (i,j) = (w,v); \\ 0 & \text{otherwise.} \end{cases}$$

Then SDP (5) can be rewritten as

$$\min \left\{ t \;\middle|\; tI + \sum_{\{v,w\} \in E} t_{vw} A_{vw} - (I + \operatorname{Adj}(\overline{G})) \succeq 0 \right\}.$$

The dual SDP is then

$$\max \left\{ (I + \operatorname{Adj}(\overline{G})) \cdot Z \;\middle|\; \begin{array}{l} Z \succeq 0 \\ I \cdot Z = 1 \\ A_{vw} \cdot Z = 0 \text{ for all } \{v,w\} \in E \end{array} \right\}.$$

The constraint $I \cdot Z = 1$ is equivalent with $\operatorname{tr}(Z) = 1$. The constraint $A_{vw} \cdot Z = 0$ for all $\{v,w\} \in E$ can be reduced to $Z_{ij} = 0$ for all $\{i,j\} \in E$. Because of the last constraint the objective function $(I + \text{Adj}(\overline{G})) \cdot Z$ can be replaced by

$$\sum_{i \in V} \sum_{j \in V} Z_{ij}.$$

So we can conclude that SDP (6) is indeed the dual of SDP (5).

Remark: The number $\vartheta(G)$ was introduced by Lovász [6] as an upper bound for the *Shannon capacity* $\Theta(G)$ of a graph G, which is defined as

$$\Theta(G) := \lim_{k \to \infty} \sqrt[k]{\alpha(G^k)}.$$

The limit always exists, although there is no efficient algorithm known to compute this value, but the problem is not known to be NP-hard either. We always have the following inequalities:

$$\alpha(G) \leq \Theta(G) \leq \vartheta(G).$$

Theorem 4.1 (Sandwich Theorem)
Let $G = (V, E)$ be a graph. Then we have $\omega(G) \leq \vartheta(\overline{G}) \leq \chi(G)$.

Proof

- Let S be a maximal stable set of G. Then

  $$Z_{ij} := \begin{cases} 1/|S| & \text{if } i, j \in S; \\ 0 & \text{otherwise,} \end{cases}$$

  is feasible in SDP (6). To see that $Z \succeq 0$ we can write

  $$Z = \frac{1}{|S|} x_s x_s^\mathsf{T},$$

  where x_s is a $|V|$ dimensional vector defined as

  $$(x_s)_i = \begin{cases} 1 & i \in S; \\ 0 & \text{otherwise,} \end{cases}$$

  and then use Theorem 2.26.4. Further Z contains $|S|^2$ elements which have value $\frac{1}{|S|}$ and the other elements are 0. Thus the corresponding objective value is $|S| = \alpha(G)$. As SDP (6) is a maximum problem we can conclude $\alpha(G) \leq \vartheta(G)$ for any graph G which is equivalent with $\omega(G) \leq \vartheta(\overline{G})$.

- Consider a minimal coloring $f : V \to \{1, 2, ..., k\}$ of $\overline{G}$ with $k = \chi(\overline{G})$. Now take

  $$Y_{vw} = \begin{cases} k - 1 & \text{if } v \text{ and } w \text{ have the same color;} \\ -1 & \text{otherwise.} \end{cases}$$

The constraint $Y_{ij} = -1$ for all $\{i, j\} \in \overline{E}$ is clearly satisfied. Further we have

$$Y = \sum_{1 \le i < j \le k} A_{ij},$$

where A_{ij} is defined as

$$(A_{ij})_{uv} := \begin{cases} 1 & f(u) = f(v) = i \text{ or } f(u) = f(v) = j; \\ -1 & f(u) = i, f(v) = j \text{ or } f(u) = j, f(v) = i; \\ 0 & \text{otherwise.} \end{cases}$$

Note that we have $A_{ij} = a_{ij} a_{ij}^{\mathsf{T}}$, where a_{ij} is a $|V|$ dimensional vector defined as

$$(a_{ij})_v := \begin{cases} 1 & f(v) = i; \\ -1 & f(v) = j; \\ 0 & \text{otherwise.} \end{cases}$$

Thus we can write

$$Y = \sum_{1 \le i < j \le k} a_{ij} a_{ij}^{\mathsf{T}},$$

and by Theorem 2.26.4 we have $Y \succeq 0$. Thus Y is feasible in SDP (5) with corresponding objective value k. As SDP (5) is a minimum problem we can conclude $\vartheta(\overline{G}) \le \chi(G)$.

□

Corollary 4.2
Let G_n be the Keller graph of dimension n, then we have $\vartheta(\overline{G}_n) \le 2^n$.

Proof
We have $\chi(G_n) \le 2^n$ (Proposition 3.3) and also $\vartheta(\overline{G}) \le \chi(G)$ (Theorem 4.1).

□

Theorem 4.3
There exists an optimal solution Z^* for SDP (6) which is invariant under the automorphism group of G. That means $Z^*_{ij} = Z^*_{\varphi(i)\varphi(j)}$ for all $\varphi \in \mathrm{Aut}(G)$.

Proof
Let F be the (nonempty) set of all feasible solutions of SDP (6) with optimal objective value. It is easy to see that F is a convex set, that means if $X_1, X_2 \in F$ and $c \in [0, 1]$ then also $cX_1 + (1 - c)X_2 \in F$. Further the objective function of (6) is invariant under $\mathrm{Aut}(G)$. Let $M \in F$ and let M_τ, with $\tau \in \mathrm{Aut}(G)$, be the matrix we get after applying τ to (the rows and columns) of M. Note that M_τ is feasible in (6). Then

$$Z^* := \frac{1}{|\mathrm{Aut}(G)|} \sum_{\varphi \in \mathrm{Aut}(G)} M_\varphi$$

is an feasible solution which is invariant under $\mathrm{Aut}(G)$ with optimal objective value.

□

Proposition 4.4
Let G_1 and G_2 be graphs. Then we have $\vartheta(G_1 \boxtimes G_2) = \vartheta(G_1)\vartheta(G_2)$.

Proof
Let denote the corresponding SDP (6) for a graph G as P_G. Let X_{G_i} be an optimal solution of P_{G_i} for $i \in \{1, 2\}$. Then $X_{G_1} \otimes X_{G_2}$ is a feasible solution of $P_{G_1 \boxtimes G_2}$ with objective value $\vartheta(G_1)\vartheta(G_2)$. Thus we have $\vartheta(G_1 \boxtimes G_2) \geq \vartheta(G_1)\vartheta(G_2)$.

For $\vartheta(G_1 \boxtimes G_2) \leq \vartheta(G_1)\vartheta(G_2)$ see the proof of Lemma 2 in [6]. (It is much easier to proof using other characterizations of $\vartheta(G)$.)

□

Lemma 4.5
Let $G = (V, E)$ be a vertex transitive graph. If X is a $V \times V$ matrix that is invariant under the automorphisms of G, then X and $J_{|V|}$ have a same set of n orthogonal eigenvectors.

Proof
Because X is invariant under the automorphisms of G, then for all $i, j, p \in V$ we have $X_{ij} = X_{pq}$ for a $q \in V$. Because of this the sum of every row of X is the same. Therefore $\mathbf{1}$ is an eigenvector of X (as well of $J_{|V|}$). As X is symmetric it has n eigenvectors that form an orthonormal set. All vectors orthonormal to $\mathbf{1}$ are also eigenvectors of $J_{|V|}$.

□

Proposition 4.6
Let $G = (V, E)$ be a graph and let $n := |V|$. Then we have $\vartheta(G)\vartheta(\overline{G}) \geq n$.
If G is vertex transitive then $\vartheta(G)\vartheta(\overline{G}) = n$.

Proof
Let H be the graph defined as $H := G \boxtimes \overline{G}$. The set $\{(x, x) | x \in G\} \subset V(H)$ contain n elements and form a stable set in H. Thus by Theorem 4.1 we have $\vartheta(G)\vartheta(\overline{G}) = \vartheta(G \boxtimes \overline{G}) \geq n$.

Assume that G is vertex transitive. We have

$$\vartheta(\overline{G}) = \min \left\{ t \;\middle|\; \begin{array}{l} X \text{ is a } V \times V \text{ matrix} \\ X - J \succeq 0 \\ X_{ij} = 0 \text{ for all } \{i, j\} \in E(G) \\ X_{ii} = t \text{ for all } i \in V \end{array} \right\}, \tag{7}$$

and

$$\vartheta(G) = \max \left\{ \sum_{i \in V} \sum_{j \in V} Z_{ij} \;\middle|\; \begin{array}{l} Z \text{ is a } V \times V \text{ matrix} \\ Z \succeq 0 \\ Z_{ij} = 0 \text{ for all } \{i, j\} \in E(G) \\ \operatorname{tr}(Z) = 1 \end{array} \right\}. \tag{8}$$

By Theorem 4.3 there exists an optimal solution Z of (8) which is invariant under the automorphisms of G. Thus we have $Z_{ii} = \frac{1}{n}$ for all $i \in \{1, ...n\}$. By Lemma 4.5 Z and J have a same set of n orthogonal eigenvectors. Let

$$X := \frac{n^2}{\vartheta(G)} Z.$$

To prove that X is feasible in (7), we need to show that $\frac{n^2}{\vartheta(G)} Z - J \succeq 0$.

The matrix J has one eigenvalue n corresponding to eigenvector $\mathbf{1}$, and the other eigenvalues are 0. So it is enough to show that the eigenvalue corresponding to eigenvector $\mathbf{1}$ of $\frac{n^2}{\vartheta(G)} Z - J$ is at least 0, which is equivalent with

$$\mathbf{1}^\mathsf{T} \left(\frac{n^2}{\vartheta(G)} Z - J \right) \mathbf{1} \geq 0.$$

This is indeed the case because $\mathbf{1}^\mathsf{T} Z \mathbf{1} = \vartheta(G)$ and $\mathbf{1}^\mathsf{T} J \mathbf{1} = n^2$. The corresponding objective value of X in (7) is $\frac{n}{\vartheta(G)}$, thus we have $\vartheta(\overline{G}) \leq \frac{n}{\vartheta(G)}$. We have already proven $\vartheta(G)\vartheta(\overline{G}) \geq n$, so we can conclude that we have $\vartheta(G)\vartheta(\overline{G}) = n$.

□

Proposition 4.7
Let $G = (V, E)$ be a graph and $H = (W, F)$ be an induced subgraph of G. Then we have $\vartheta(H) \leq \vartheta(G)$.

Proof
Assume that Z_H is an optimal solution for the corresponding SDP (6) for H. Let the vertices of G be ordered in such a way that the first $|W|$ vertices are all in W. Then it is easy to see that

$$Z_G := \begin{pmatrix} Z_H & 0 \\ 0 & 0 \end{pmatrix}$$

is a feasible solution for the corresponding SDP (6) for G. Thus we have $\vartheta(H) \leq \vartheta(G)$.

□

5 Reducing the SDP of the Keller graph to an LP

From Chapter 4 we know that $\vartheta(\overline{G}_n)$ will give an upper bound for the size of a maximal clique in the Keller graph $G_n = (V_n, E_n)$ of dimension n and that $\vartheta(\overline{G}_n)$ can be calculated via the SDP

$$\max \left\{ Z \cdot J \;\middle|\; \begin{array}{l} Z \text{ is a } V_n \times V_n \text{ matrix} \\ Z \succeq 0 \\ Z_{ij} = 0 \text{ for all } \{i,j\} \in E(\overline{G}_n) \\ \mathrm{tr}(Z) = 1 \end{array} \right\}. \tag{9}$$

The number of constraints of this SDP is not polynomial in n, because the number of unordered pairs $\{i,j\} \in V_n \times V_n$ which are not in $E(\overline{G}_n)$ is also not. In this chapter we will show that SDP (9) can be reduced to an LP which is practically much easier to handle.

Let R_{ijk} with $i+j+k=n$ be a $V_n \times V_n$ matrix defined as

$$(R_{ijk})_{vw} = \begin{cases} 1 & \text{if } r(v,w) = (i,j,k); \\ 0 & \text{if } r(v,w) \neq (i,j,k). \end{cases}$$

Note that R_{ijk} is a symmetric matrix.

Theorem 5.1
There exists an optimal solution for SDP (9) such that the matrix Z has the form

$$Z = \sum_{i+j+k=n} x_{ijk} R_{ijk}$$

for some real numbers x_{ijk}.

Proof
By Proposition 3.6 and Theorem 4.3 we see that there exists an optimal solution for (9) such that $Z_{vw} = Z_{pq}$ if $r(v,w) = r(p,q)$. By the definition of R_{ijk} we see that Z is of the form of

$$Z = \sum_{i+j+k=n} x_{ijk} R_{ijk}.$$

□

Let

$$Q_n := \{(i,j,k) \in \mathbb{N}_0^3 | i+j+k=n, j \geq 1, j+k \geq 2\}.$$

Note that for all $v,w \in V_n$ we have $r(v,w) \in Q_n$ if and only if $\{v,w\} \in E_n$.

Proposition 5.2
There exists an optimal solution for SDP (9) such that the matrix Z has the form

$$Z = x_{n00} R_{n00} + \sum_{(i,j,k) \in Q_n} x_{ijk} R_{ijk}, \tag{10}$$

for some real numbers x_{ijk}.

Proof
This follows from Theorem 5.1 and the constraint $Z_{ij} = 0$ for all $\{i, j\} \in E(\overline{G}_n)$ of SDP (9).

□

For example for $n = 4$ we only need to consider matrices Z that have the following form:

$$Z = x_{400}R_{400} + x_{220}R_{220} + x_{211}R_{211} + x_{130}R_{130} + x_{121}R_{121}$$
$$+x_{112}R_{112} + x_{040}R_{040} + x_{031}R_{031} + x_{022}R_{022} + x_{013}R_{013}$$

Proposition 5.3
If Z is of the form of (10), then the constraint $\mathrm{tr}(Z) = 1$ *in SDP (9) reduces to* $4^n x_{n00} = 1$.

Proof
Because we have $r(v, v) = (n, 0, 0)$ for every $v \in V_n$ by Proposition 5.2 all the diagonal elements of Z should have the same value, which we call x_{n00}. As Z is a $4^n \times 4^n$ matrix we indeed have $4^n x_{n00} = 1$.

□

Convention: *In a $V_n \times V_n$ matrices the vertices are ordered lexicographic with the following alphabet: $\{0,1,a,b\}$.*

For example in dimension 2 the lexicographical order of vertices is:
$00, 01, 0a, 0b, 10, 11, 1a, 1b, a0, a1, aa, ab, b0, b1, ba, bb$.

Proposition 5.4
The matrices R_{ijk} can be calculated by the following recursion formula:

$$\begin{cases} R_{100} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, R_{010} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, R_{001} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}; \\ R_{ijk} = \begin{pmatrix} R_{i-1,j,k} & R_{i,j-1,k} & R_{i,j,k-1} & R_{i,j,k-1} \\ R_{i,j-1,k} & R_{i-1,j,k} & R_{i,j,k-1} & R_{i,j,k-1} \\ R_{i,j,k-1} & R_{i,j,k-1} & R_{i-1,j,k} & R_{i,j-1,k} \\ R_{i,j,k-1} & R_{i,j,k-1} & R_{i,j-1,k} & R_{i-1,j,k} \end{pmatrix}. \end{cases}$$

If $i < 0$ or $j < 0$ or $k < 0$, then R_{ijk} is the $4^{i+j+k} \times 4^{i+j+k}$ zero matrix.

Proof
The correctness of the given matrices for R_{100}, R_{010} and R_{001} can easily be verified by the definition of R_{ijk}.

Let $i + j + k = n$ and write R_{ijk} as

$$R_{ijk} = \begin{pmatrix} S_{00} & S_{01} & S_{0a} & S_{0b} \\ S_{10} & S_{11} & S_{1a} & S_{1b} \\ S_{a0} & S_{a1} & S_{aa} & S_{ab} \\ S_{b0} & S_{b1} & S_{ba} & S_{bb} \end{pmatrix},$$

where each S_{pq} is a $4^{n-1} \times 4^{n-1}$ submatrix of R_{ijk}. Note that the rows of S_{pq} belongs to vertices for which the first element of the n-tuple is a p and the columns of S_{pq} belongs to vertices which have a q as the first element. If we delete the first element of the n-tuples then S_{pq} can be considered as a $V_{n-1} \times V_{n-1}$ matrix (because the vertices are still lexicographic ordered correctly).

We can have the following situations for S_{pq}:

- If $p = q$, then we have

  $$r(pv_2...v_n, qw_2...w_n) = (i, j, k) \Leftrightarrow r(v_2...v_n, w_2...w_n) = (i - 1, j, k).$$

  From this we can conclude $S_{pq} = R_{i-1,j,k}$.

- If p and q are opposite then

  $$r(pv_2...v_n, qw_2...w_n) = (i, j, k) \Leftrightarrow r(v_2...v_n, w_2...w_n) = (i, j - 1, k).$$

  Thus we have $S_{pq} = R_{i,j-1,k}$.

- If i and j are type different, then

  $$r(pv_2...v_n, qw_2...w_n) = (i, j, k) \Leftrightarrow r(v_2...v_n, w_2...w_n) = (i, j, k - 1).$$

  Therefore we have $S_{pq} = R_{i,j,k-1}$.

Further it is clear that if $i < 0$ or $j < 0$ or $k < 0$, then R_{ijk} must be the zero matrix.

□

We will now reduce the objective function of SDP (9) to a linear function on the variables x_{ijk}.

Lemma 5.5
Let c_{ijk} be the number of ones in the matrix R_{ijk}. Then we have

$$c_{ijk} = 4^n \cdot 2^k \cdot \frac{n!}{i!j!k!}, \text{ with } n = i + j + k.$$

Proof
It is clear that every row of R_{ijk} has the same number of ones, this for example just follows from the recursion formula in Proposition 5.4. The number of ones in the first row of R_{ijk} is the same as the number of n-tuples containing exactly i zeros, j ones and k character's (a or b). If we don't distinguish the character a from b, then it is known in the theory of Combinatorics that the number of such n-tuples is equal to the multinomial coefficient

$$\left(\begin{array}{c} n \\ i, j, k \end{array} \right) := \frac{n!}{i!j!k!}.$$

But because we have two distinct characters we should multiply this number by 2^k. Further the matrix R_{ijk} has 4^n rows so the number of ones in R_{ijk} is

$$4^n \cdot 2^k \cdot \frac{n!}{i!j!k!}.$$

□

Proposition 5.6
If Z is of the form of (10), then the objective function

$$Z \cdot J$$

of SDP (9) can be reduced to

$$4^n x_{n00} + \sum_{(i,j,k)\in Q_n} 4^n \cdot 2^k \cdot \frac{n!}{i!j!k!} x_{ijk}.$$

Proof
Because R_{ijk} is a 0-1 matrix we only need to know how many elements of R_{ijk} are an 1. This number is the corresponding coefficient of the variable x_{ijk}. Thus

$$Z \cdot J = \sum_{v\in V_n} \sum_{w\in V_n} Z_{vw}$$

can be replaced by

$$c_{n00} x_{n00} + \sum_{(i,j,k)\in Q_n} c_{ijk} x_{ijk},$$

where c_{ijk} is the number of ones in the matrix R_{ijk}, which is already determined in Lemma 5.5.

□

Now we arrived at the more difficult part. How to convert the constraint $Z \succeq 0$ in the SDP (9) to linear constraints?

Theorem 5.7
R_{ijk} can be calculated by the following recursion formula:

$$\begin{cases} R_{100} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, R_{010} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, R_{001} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}; \\ \\ R_{ijk} = R_{100} \otimes R_{i-1,j,k} + R_{010} \otimes R_{i,j-1,k} + R_{001} \otimes R_{i,j,k-1}. \end{cases}$$

If $i < 0$ or $j < 0$ or $k < 0$, then R_{ijk} is the $4^{i+j+k} \times 4^{i+j+k}$ zero matrix.

Proof
This is just a rewriting of Proposition 5.4 in terms of Kronecker products.

□

Proposition 5.8
For every $n \in \mathbb{N}$ there exists an orthogonal matrix P_n such that the matrix $D_{ijk} := P_n^{\mathsf{T}} R_{ijk} P_n$ is diagonal for all $i, j, k \in \mathbb{N}_0$ with $i + j + k = n$.

Proof
We will proof this with induction to n:

- Let
$$P_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}, \text{ which gives } P_1^{-1} = P_1^{\mathsf{T}},$$
then we have
$$D_{100} := P_1^{\mathsf{T}} R_{100} P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, D_{010} := P_1^{\mathsf{T}} R_{010} P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
and
$$D_{001} := P_1^{\mathsf{T}} R_{001} P_1 = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$
- Assume that all matrices R_{xyz} with $x + y + z = N - 1$ can be orthogonally diagonalized by P_{N-1}, thus the matrix $D_{xyz} := P_{N-1}^{\mathsf{T}} R_{x,y,z} P_{N-1}$ is diagonal. Let $i + j + k = N$, then by Proposition 2.31 we can take $P_N := P_1 \otimes P_{N-1}$ to get (note that we have $P_N^{\mathsf{T}} = P_N^{-1}$):
$$\begin{aligned} & P_N^{\mathsf{T}} R_{ijk} P_N \\ &= P_N^{\mathsf{T}} (R_{100} \otimes R_{i-1,j,k} + R_{010} \otimes R_{i,j-1,k} + R_{001} \otimes R_{i,j,k-1}) P_N \\ &= D_{100} \otimes D_{i-1,j,k} + D_{010} \otimes D_{i,j-1,k} + D_{001} \otimes D_{i,j,k-1} := D_{ijk} \end{aligned}$$
Thus the defined matrix P_N is indeed the required matrix.

□

Corollary 5.9
The eigenvalues of R_{ijk} can be calculated via the recursion formula

$$\begin{cases} L_{100} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, L_{010} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, L_{001} = \begin{pmatrix} -2 \\ 0 \\ 0 \\ 2 \end{pmatrix}; \\ L_{ijk} = L_{100} \otimes L_{i-1,j,k} + L_{010} \otimes L_{i,j-1,k} + L_{001} \otimes L_{i,j,k-1}. \end{cases}$$

If $i < 0$ or $j < 0$ or $k < 0$, then L_{ijk} is the zero vector.
The vector L_{ijk} then contains all the eigenvalues of R_{ijk}.

Proof
The matrix D_{ijk}, defined in the the proof of Proposition 5.8, contains the eigenvalues of R_{ijk}. In the same proof also a recursion formula for D_{ijk} is given. If we take

$$(L_{ijk})_v = (D_{ijk})_{vv},$$

we see that the given recursion formula for L_{ijk} is correct.

□

Proposition 5.10
If Z is of the form of (10), then all eigenvalues of Z are listed in the vector $L(Z)$, which is defined as

$$L(Z) = x_{n00} L_{n00} + \sum_{(i,j,k) \in Q_n} x_{ijk} L_{ijk},$$

where the vectors L_{ijk} are defined in Corollary 5.9.

Proof
By Proposition 5.8 we know that all matrices R_{ijk} with $i + j + k = n$ have a same orthonormal set of 4^n eigenvectors. Further in Corollary 5.9 all the eigenvalues of R_{ijk} are listed in the vector L_{ijk}. Also the eigenvectors in a vector L_{ijk} is ordered in such a way that for every $v \in \mathbb{N}$ the v-th eigenvalue listed in all the L_{ijk} vectors belong to the same eigenvector. Therefore we can easily see that the statement is correct.

□

Let the elements of Q_n be renamed as $\{q_1, q_2, q_3, ...\}$ and let $(n, 0, 0) := q_0$. Also define the vector x and the matrix B as

$$x =: (x_{q_0}, x_{q_1}, x_{q_2}, ...)^\top \text{ and } B := (L_{q_0}, L_{q_1}, L_{q_2}, ...).$$

Then we can replace the constraint

$$Z \succeq 0$$

in the SDP (9) with the linear constraints $Bx \geq 0$. The matrix B has 4^n rows which is not polynomial in n. But if B has duplicate rows then these rows just give the same constraint. Let A be the matrix which we get by deleting the duplicate rows of B. Then $Bx \geq 0$ is just equivalent with

$$Ax \geq 0.$$

Proposition 5.11
The matrix A defined above has at most $\frac{1}{2}n^2 + \frac{3}{2}n + 1$ rows.

Proof
In Corollary 5.9 we can see that the second and third element of L_{100}, L_{010} and L_{001} is the same for each vector. So we can use the following recursion formula instead to already get rid of many duplicate rows of the matrix B:

$$\begin{cases} L_{100} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, L_{010} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, L_{001} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}; \\ L_{ijk} = L_{100} \otimes L_{k-1,j,k} + L_{010} \otimes L_{i,j-1,k} + L_{001} \otimes L_{i,j,k-1}. \end{cases}$$

From this recursion formula we can derive a direct formula for each of the L_{ijk}. For convenience we will use the following renaming: $L_1 := L_{100}$, $L_2 := L_{010}$ and $L_3 := L_{001}$. It is easy to see that if $i + j + k = n$ we have

$$L_{ijk} = \sum_{\substack{i\ t_l\text{'s are } 1 \\ j\ t_l\text{'s are } 2 \\ k\ t_l\text{'s are } 3}} L_{t_1} \otimes L_{t_2} \otimes ... \otimes L_{t_n}.$$

For example we have

$$L_{201} = L_1 \otimes L_1 \otimes L_3 + L_1 \otimes L_3 \otimes L_1 + L_3 \otimes L_1 \otimes L_1.$$

Thus every element of L_{ijk} is of the form of

$$T(m_1, m_2, ..., m_n) := \sum_{\substack{i\ t_i\text{'s are } 1 \\ j\ t_i\text{'s are } 2 \\ k\ t_i\text{'s are } 3}} (L_{t_1})_{m_1} \cdot (L_{t_2})_{m_2} \cdot ... \cdot (L_{t_n})_{m_n},$$

for some $m_1, m_2, ..., m_n \in \{1, 2, 3\}$. The value $T(m_1, m_2, ..., m_n)$ is invariant under a permutation of the m_i's. So the number of unique values of L_{ijk} is the number of unordered n-tuples having values 1,2 or 3. There are $\frac{1}{2}n^2 + \frac{3}{2}n + 1$ of such n-tuples, so we can conclude that the matrix A has (at most) $\frac{1}{2}n^2 + \frac{3}{2}n + 1$ rows.

□

We can conclude that the SDP

$$\max \left\{ Z \cdot J \;\middle|\; \begin{array}{l} Z \text{ is a } V_n \times V_n \text{ matrix} \\ Z \succeq 0 \\ Z_{ij} = 0 \text{ for all } \{i,j\} \in E(\overline{G}_n) \\ \operatorname{tr}(Z) = 1 \end{array} \right\}.$$

to calculate $\vartheta(\overline{G}_n)$ can be reduced to the LP

$$\max \left\{ c_{n00}x_{n00} + \sum_{(i,j,k)\in Q_n} c_{ijk}x_{ijk} \;\middle|\; \begin{array}{l} 4^n x_{n00} = 1 \\ Ax \geq 0 \end{array} \right\}, \text{where } c_{ijk} = 4^n \cdot 2^k \cdot \frac{n!}{i!j!k!}.$$

The number of variables of this LP is of order $\mathrm{O}(n^2)$ (it is easy to determine that Q_n has $\frac{1}{2}n^2 + \frac{1}{2}n - 1$ elements) and the same holds for the number of constraints of this LP.

Remark: It is also possible to calculate $\vartheta(G_n)$ to determine the value of $\vartheta(\overline{G}_n)$ (by using Proposition 4.6). The corresponding LP for $\vartheta(G_n)$ will looks very similar. Let

$$\overline{Q}_n := \{(i,j,k) \in \mathbb{N}_0^3 | i+j+k = n, (i,j,k) \neq (n,0,0), j = 0 \text{ or } j+k < 2\}.$$

Note that for $v, w \in V_n$ with $v \neq w$ we have $r(v,w) \in \overline{Q}_n$ if and only if $\{v,w\} \notin E_n$.

Let the elements of $\overline{Q}_n$ be renamed as $\{\overline{q}_1, \overline{q}_2, \overline{q}_3 \ldots\}$ and let $(n,0,0) := \overline{q}_0$. Also define the vector $\overline{x}$ and the matrix $\overline{B}$ as

$$\overline{x} =: (x_{\overline{q}_0}, x_{\overline{q}_1}, x_{\overline{q}_2}, \ldots)^{\mathsf{T}} \text{ and } \overline{B} := (L_{\overline{q}_0}, L_{\overline{q}_1}, L_{\overline{q}_2}, \ldots).$$

Further let $\overline{A}$ be the matrix which we get by deleting the duplicate rows of $\overline{B}$. Then we can use the following LP to calculate $\vartheta(G_n)$:

$$\max \left\{ c_{n00}x_{n00} + \sum_{(i,j,k)\in \overline{Q}_n} c_{ijk}x_{ijk} \;\middle|\; \begin{array}{l} 4^n x_{n00} = 1 \\ \overline{A}\overline{x} \geq 0 \end{array} \right\}, \text{where } c_{ijk} = 4^n \cdot 2^k \cdot \frac{n!}{i!j!k!}.$$

As a matter of fact it is more efficient to calculate $\vartheta(G_n)$ than to calculate $\vartheta(\overline{G}_n)$ as $\overline{Q}_n$ only has $n+1$ elements. Thus if $n \geq 3$ the corresponding LP to calculate $\vartheta(G_n)$ has less variables than the LP of $\vartheta(\overline{G}_n)$, while both LP's have $\frac{1}{2}n^2 + \frac{3}{2}n + 1$ eigenvalue constraints.

Conclusion: *In this chapter we have shown that the SDP to calculate* $\vartheta(\overline{G}_n)$ *can easily be reduced to an LP. The number of variables and constraints of this LP is polynomial in* n*. So* $\vartheta(G_n)$ *can be computed in polynomial time.*

6 The Lovász number of the Keller graphs

In this chapter we will determine the Lovász number of the complement of the Keller graphs, which by Theorem 4.1 give us an upper bound for the clique numbers of the Keller graphs.

Theorem 6.1
We have $\vartheta(\overline{G}_1) = 1$.

Proof
The graph $\overline{G}_1$ is complete and the Lovász number of such graphs is always 1.

□

Theorem 6.2
We have $\vartheta(\overline{G}_2) = \frac{8}{3}$.

Proof
The corresponding LP to calculate $\vartheta(\overline{G}_2)$ is

$$\max\left\{ 16x_{200} + 16x_{020} + 64x_{011} \;\middle|\; \begin{array}{r} 16x_{200} + 0x_{020} + 0x_{011} = 1 \\ x_{200} + x_{020} - 4x_{011} \geq 0 \\ x_{200} - x_{020} + 2x_{011} \geq 0 \\ x_{200} + x_{020} + 0x_{011} \geq 0 \\ x_{200} - x_{020} - 2x_{011} \geq 0 \\ x_{200} + x_{020} + 4x_{011} \geq 0 \end{array} \right\}$$

and its dual problem is

$$\min\left\{ y_0 \;\middle|\; \begin{array}{r} 16y_0 + 1y_1 + 1y_2 + 1y_3 + 1y_4 + 1y_5 = 16 \\ 0y_0 + 1y_1 - 1y_2 + 1y_3 - 1y_4 + 1y_5 = 16 \\ 0y_0 - 4y_1 + 2y_2 + 0y_3 - 2y_4 + 4y_5 = 64 \\ y_1, y_2, y_3, y_4, y_5 \leq 0 \end{array} \right\}.$$

We have the following feasible solutions

$$x_{200} = \tfrac{3}{48}, x_{020} = \tfrac{1}{48}, x_{011} = \tfrac{1}{48},$$

$$y_0 = \tfrac{8}{3}, y_1 = -\tfrac{16}{3}, y_2 = 0, y_3 = 0, y_4 = -\tfrac{64}{3}, y_5 = 0.$$

In both cases the objective value is $\frac{8}{3}$, so by the strong duality theorem the optimal value for both problems is $\frac{8}{3}$. Thus we have $\vartheta(\overline{G}_2) = \frac{8}{3}$.

□

Theorem 6.3
We have $\vartheta(\overline{G}_3) = \frac{48}{7}$.

Proof
The the corresponding LP to calculate $\vartheta(\overline{G}_3)$ is

$$\max\left\{\begin{pmatrix}64\\192\\768\\64\\384\\768\end{pmatrix}^{\mathsf{T}}\begin{pmatrix}x_{300}\\x_{120}\\x_{111}\\x_{030}\\x_{021}\\x_{012}\end{pmatrix}\;\middle|\;\begin{matrix}64x_{300}=1\\ \begin{pmatrix}1&3&-12&1&-6&12\\1&-1&0&-1&4&-4\\1&3&-4&1&-2&-4\\1&-1&4&1&-2&0\\1&-1&0&-1&0&4\\1&3&4&1&2&-4\\1&3&0&-1&0&0\\1&-1&-4&1&2&0\\1&-1&0&-1&-4&-4\\1&3&12&1&6&12\end{pmatrix}\begin{pmatrix}x_{300}\\x_{120}\\x_{111}\\x_{030}\\x_{021}\\x_{012}\end{pmatrix}\geq 0\end{matrix}\right\}.$$

and its dual problem is

$$\min\left\{y_0\;\middle|\;\begin{matrix}\begin{pmatrix}64&0&0&0&0&0\\1&3&-12&1&-6&12\\1&-1&0&-1&4&-4\\1&3&-4&1&-2&-4\\1&-1&4&1&-2&0\\1&-1&0&-1&0&4\\1&3&4&1&2&-4\\1&3&0&-1&0&0\\1&-1&-4&1&2&0\\1&-1&0&-1&-4&-4\\1&3&12&1&6&12\end{pmatrix}^{\mathsf{T}}\begin{pmatrix}y_0\\y_1\\y_2\\y_3\\y_4\\y_5\\y_6\\y_7\\y_8\\y_9\\y_{10}\end{pmatrix}=\begin{pmatrix}64\\192\\768\\64\\384\\768\end{pmatrix}\\ y_i\leq 0, i\in\{1,...,10\}\end{matrix}\right\}.$$

We have the following feasible solutions for the primal and dual problem which in both cases give objective value $\frac{48}{7}$:

$$x_{300}=\tfrac{7}{448}, x_{120}=\tfrac{1}{448}, x_{111}=\tfrac{2}{448}, x_{030}=\tfrac{2}{448}, x_{021}=0, x_{012}=\tfrac{1}{448},$$
$$y_0=\tfrac{48}{7}, y_1=-\tfrac{128}{7}, y_2=-\tfrac{384}{7}, y_3=-\tfrac{192}{7}, y_4=0, y_5=0,$$
$$y_6=0, y_7=0, y_8=-\tfrac{768}{7}, y_9=-\tfrac{1152}{7}, y_{10}=0.$$

Thus we have $\vartheta(\overline{G}_3)=\frac{48}{7}$.

□

Theorem 6.4
We have $\vartheta(\overline{G}_n)=2^n$ *for* $n\in\{4,5,6,7\}$.

Proof
The corresponding LP's (from practical reasons I will not show them here) gives optimal objective value 2^n. Thus we can conclude $\vartheta(\overline{G}_n)=2^n$ for $n\in\{4,5,6,7\}$.

□

Theorem 6.5
We have $\vartheta(\overline{G}_n) = 2^n$ *for* $n \geq 8$.

Proof
A (largest) clique of order 256 in G_8 has been found by Mackey [9], so by Proposition 3.4 we also have $\omega(G_n) = 2^n$ for $n > 8$. By the sandwich theorem (Theorem 4.1) and Corollary 4.2 we can conclude $\vartheta(\overline{G}_n) = 2^n$ for $n \geq 8$.

□

Apparently we always have $\vartheta(\overline{G}_{n+1}) \geq 2\vartheta(\overline{G}_n)$ for all $n \geq 1$, but I could not find a general argument for this.

Corollary 6.6
We have $\vartheta(G_1) = 4$, $\vartheta(G_2) = 6$, $\vartheta(G_3) = \frac{28}{3}$ *and* $\vartheta(G_n) = 2^n$ *for* $n \geq 4$.

Proof
This follows from Proposition 4.6.

□

Corollary 6.7
We have the following upper bounds for the clique number of the Keller graphs:
$\omega(G_1) \leq 1$, $\omega(G_2) \leq 2$, $\omega(G_3) \leq 6$ *and* $\omega(G_n) \leq 2^n$ *for* $n \in \{4, 5, 6, 7\}$.

And for the chromatic number we have:
$\chi(G_1) \geq 1$, $\chi(G_2) \geq 3$, $\chi(G_3) \geq 7$ *and* $\chi(G_n) = 2^n$ *for* $n \geq 4$.

Proof
This follows from the sandwich theorem (Theorem 4.1) and Proposition 3.3.

□

At the end of Chapter 3 we saw that if $n = 2$ (and $n = 1$) the automorphism group of G_n is bigger than

$$S_n \ltimes \underbrace{D_4 \times ... \times D_4}_{n \text{ times}}.$$

Because of this for $n = 2$ there exists an optimal solution of SDP (9) which is of the form

$$Z = x_1 I + x_2 \text{Adj}(G_n),$$

for some $x_1, x_2 \in \mathbb{R}$. If Z has this form then SDP (9) reduces to

$$\max\left\{4^n x_1 + 4^n(4^n - 3^n - n)x_2 \;\middle|\; \begin{array}{l} 4^n x_1 = 1 \\ x_1 + \lambda_i x_2 \geq 0, i \in \{1, ..., 4^n\} \end{array}\right\},$$

where the λ_i are the eigenvalues of $\mathrm{Adj}(G_n)$. The coefficient of x_2 in the objective function follows from Proposition 3.11. It is easy to see that this LP can be reduced to

$$\max\left\{4^n x_1 + 4^n \kappa_n x_2 \;\middle|\; \begin{array}{l} 4^n x_1 = 1 \\ x_1 + \mu_n x_2 \geq 0 \end{array}\right\}, \tag{11}$$

where μ_n is the smallest eigenvalue and κ_n the largest eigenvalue of $\mathrm{Adj}(G_n)$. (Note that μ_n is always a negative number by Proposition 2.20 and that $\kappa_n = \deg(G_n)$.) Let τ_n be the optimal objective value of LP (11). Then we have

$$\tau_n = 1 - \frac{\kappa_n}{\mu_n},$$

with

$$x_1 = \frac{1}{4^n} \text{ and } x_2 = \frac{1}{-4^n \mu_n}.$$

Of course we should have $\tau_2 = \vartheta(\overline{G}_2) = \frac{8}{3}$, but it is interesting to find out if we have $\vartheta(\overline{G}_n) = \tau_n$ or not for $n > 2$. So we first need to calculate the eigenvalues of G_n.

Proposition 6.8
The eigenvalues of $\mathrm{Adj}(G_n)$ *are listed in the vector*

$$L(G_n) := \sum_{(i,j,k)\in Q_n} L_{ijk},$$

and the eigenvalues of of $\mathrm{Adj}(\overline{G}_n)$ *are listed in the vector*

$$L(\overline{G}_n) := \sum_{(i,j,k)\in \overline{Q}_n} L_{ijk}.$$

Proof
We have the identities

$$\mathrm{Adj}(G_n) = \sum_{(i,j,k)\in Q_n} R_{ijk} \text{ and } \mathrm{Adj}(\overline{G}_n) = \sum_{(i,j,k)\in \overline{Q}_n} R_{ijk}.$$

From this the claims follow easily.

□

We have the following values for the τ_n's:

$$\tau_2 = 2\tfrac{2}{3}, \tau_3 = 4\tfrac{2}{5}, \tau_4 = 6\tfrac{26}{29}, \tau_5 = 10\tfrac{5}{21}, \tau_6 = 14\tfrac{163}{256}, \tau_7 = 20\tfrac{122}{367}.$$

We see that we don't have $\vartheta(\overline{G}_n) = \tau_n$ for $n \geq 3$. This might be an indication that for those n the automorphism group $\mathrm{Aut}(G_n)$ is not bigger than

$$S_n \ltimes \underbrace{D_4 \times ... \times D_4}_{n \text{ times}}.$$

At least it is not always the case that if $\{v, w\} \in E_n$ and $\{x, y\} \in E_n$ there exists an automorphism $\psi \in \text{Aut}(G_n)$ with $\psi(v) = x$ and $\psi(w) = y$.

Conclusion: *With the Lovász number of the complement of the Keller graphs we are able to proof the correctness of the half integral version of Keller's conjecture (Conjecture 1.8) in the case $1 \leq n \leq 3$. Unfortunately it will not give a decisive answer for $4 \leq n \leq 7$.*

In reality the clique numbers of all Keller graphs are already determined by other people. They are listed in the table below.

n	$\vartheta(\overline{G}_n)$	$\omega(G_n)$
1	1	1
2	$2\frac{2}{3}$	2
3	$6\frac{6}{7}$	5
4	16	12
5	32	28
6	64	60
7	128	124
$n \geq 8$	2^n	2^n

The cases $1 \leq n \leq 5$ were solved by Corrádi and Szabó via a computer search and the case $n = 6$ is solved by David Applegate and Peter Shor. The clique number for G_7 is published in 2010 [3]. And as earlier mentioned a clique of order 256 has been found in G_8, see [9].

So the half integral version of Keller's conjecture (Conjecture 1.8) is correct for $1 \leq n \leq 7$, but false for $n \geq 8$.

The only remaining open question is if Keller's conjecture is true or not for $n = 7$.

7 The Lovász number of the extended Keller graphs

In Chapter 1 we saw that Keller's conjecture can be reduced to the situation where all coordinates have half integral values. But the problem is that the dimension of Keller's conjecture doesn't match anymore with the dimension of the reduced problem as in the reduction the dimension will be increased.

This can be prevented by allowing the coordinates to have more fractional values (if a fractional value p is allowed then also $p+1$ is allowed). It is not known to me how many fractional values are exactly needed for a given dimension n, but it can not be more than 2^n, which is the number of cubes in $[0, 2)^n$ in any tiling.

We can convert this new problem to a graph, which we will call *extended Keller graph*, and then try to find the size of the largest clique and see if it is smaller than 2^n or not. But this is of course not so easy, it was already very hard for the original Keller graph.

However in this chapter we will show that it is not harder at all to calculate the Lovász number of the extended Keller graph.

7.1 Extended Keller graphs

Let

$$S^{(m)} := \left\{ \frac{0}{m}, \frac{1}{m}, ..., \frac{2m-1}{m} \right\} \subset \mathbb{Q}/2\mathbb{Z}.$$

Then we will define the extended Keller graph $G_n^{(m)} = \{V_n^{(m)}, E_n^{(m)}\}$ with m fractional numbers of dimension n as

$$V_n^{(m)} = \{(x_1, ..., x_n) | x_i \in S^{(m)}, i \in \{1, ..., n\}\},$$

$$E_n^{(m)} = \{\{v, w\} | \exists i \in \{1, ...n\} : |v_i - w_i| = 1, \exists j \in \{1, ...n\} : i \neq j \wedge v_j \neq w_j\}.$$

Note that when $m = 2$ we have our original Keller graph. In this chapter we will again use the terms *different*, *opposite* and *type different*, which are defined in Chapter 3. For every $s \in S^{(m)}$ there is exactly one corresponding opposite vertex, while the other $2m - 2$ vertices are type different from s.

Proposition 7.1
We have $\chi(G_n^{(m)}) \leq 2^n$.

Proof
The proof of Proposition 3.3 can also be used here.

□

Corollary 7.2
We have $\omega(G_n^{(m)}) \leq 2^n$.

Corollary 7.3
We have $\vartheta(\overline{G}_n^{(m)}) = 2^n$ *for all* $n \geq 4$ *and* $m \geq 2$.

Proof
Let H be the subgraph of $\overline{G}_n^{(m)}$ induced by the following set of vertices:

$$\left\{ \frac{0}{m}, \frac{1}{m}, \frac{m}{m}, \frac{m+1}{m} \right\}.$$

It is easy to see that H is isomorph with $G_n^{(2)} = G_n$. In Chapter 6 we saw that $\vartheta(\overline{G}_n) = 2^n$ for all $n \geq 4$. So by Proposition 4.7 and $\omega(G_n^{(m)}) \leq 2^n$ it follows that $\vartheta(\overline{G}_n^{(m)}) = 2^n$ for all $n \geq 4$ and $m \geq 2$.

□

Proposition 7.4
Let φ be a bijection from $V_n^{(m)}$ to $V_n^{(m)}$. If $r(v,w) = r(\varphi(v), \varphi(w))$ for every $v, w \in V_n^{(m)}$ then φ is an automorphism of $G_n^{(m)}$.

Proof
See the proof of Proposition 3.6.

□

Some examples of automorphism of $G_n^{(m)}$ that satisfy the condition $r(v,w) = r(\varphi(v), \varphi(w))$ for every $v, w \in V_n^{(m)}$ are:

- In one of the coordinates add a number $s \in S^{(m)}$;
- In one of the coordinates swap an element $s \in S^{(m)}$ with $s+1$;
- In one of the coordinates swap an $s \in S^{(m)}$ and a $t \in S^{(m)}$, with t type different from s, and also swap $s+1$ and $t+1$;
- Permutate the coordinates of every $v \in V_n^{(m)}$ in the same way.

Proposition 7.5
The graph $G_n^{(m)}$ is vertex transitive and regular of degree $(2m)^n - (2m-1)^n - n$.

Proof
See the proof of Proposition 3.11. The only difference is that $V_n^{(m)}$ now has $(2m)^n$ nodes and that there are $(2m-1)^n$ n-tuples that only has values different than 1.

□

Theorem 7.6
Let $v, w, x, y \in V_n^{(m)}$. If $r(v,w) = r(x,y)$ holds, then there exists an automorphism $\varphi \in \mathrm{Aut}(G_n^{(m)})$ with $\varphi(v) = x$ and $\varphi(w) = y$.

Proof

As in the proof of Theorem 3.12 we can assume that for all $i \in \{1, ..., n\}$ the relation between v_i and w_i is the same as the relation between x_i and y_i and that $v = x$.

So we only need to show that there exists an automorphism $\varphi \in \mathrm{Aut}(G_n^{(m)})$ with $\varphi(x) = x$ and $\varphi(w) = y$. Let φ be the automorphism that will do the following. For every i with $w_i \neq y_i$ (this can only be if x_i and w_i are type different):

- if w_i and y_i are opposite, then swap w_i and y_i in coordinate i;
- if w_i and y_i are type different, then swap w_i and y_i and also swap $w_i + 1$ and $y_i + 1$ in coordinate i.

Then we indeed have $\varphi(v) = x$ and $\varphi(w) = y$.

□

7.2 Reducing the SDP to an LP

Similar as in Chapter 5 we will let R_{ijk} with $i+j+k = n$ be a $V_n^{(m)} \times V_n^{(m)}$ matrix defined as

$$(R_{ijk})_{vw} = \begin{cases} 1 & \text{if } r(v,w) = (i,j,k); \\ 0 & \text{if } r(v,w) \neq (i,j,k). \end{cases}$$

Convention: *In a $V_n^{(m)} \times V_n^{(m)}$ matrix the vertices are ordered lexicographic with the following alphabet:*

$$\left\{ \frac{0}{m}, \frac{0+m}{m}, \frac{1}{m}, \frac{1+m}{m}, ..., \frac{m-1}{m}, \frac{(m-1)+m}{m} \right\}.$$

So the corresponding pairs of opposite vertices are always right next to each other.

Theorem 7.7

R_{ijk} can be calculated by the recursion formula

$$\begin{cases} R_{100} = I_{2m}, R_{010} = I_m \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, R_{001} = J_{2m} - R_{100} - R_{010}; \\ R_{ijk} = R_{100} \otimes R_{i-1,j,k} + R_{010} \otimes R_{i,j-1,k} + R_{001} \otimes R_{i,j,k-1}. \end{cases}$$

If $i < 0$ or $j < 0$ or $k < 0$, then R_{ijk} is the zero matrix.

Proof

The identities for R_{100}, R_{010} and R_{001} can easily be verified. Further the proof is very similar to the proof of Proposition 5.4.

□

To calculate $\vartheta(\overline{G}_n^{(m)})$ we can use the SDP

$$\max \left\{ Z \cdot J \;\middle|\; \begin{array}{l} Z \text{ is a } V_n^{(m)} \times V_n^{(m)} \text{ matrix} \\ Z \succeq 0 \\ Z_{ij} = 0 \text{ for all } \{i,j\} \in E(\overline{G}_n^{(m)}) \\ \operatorname{tr}(Z) = 1 \end{array} \right\}. \tag{12}$$

Let the sets Q_n and $\overline{Q}_n$ be defined in the same way as in Chapter 5. By Theorem 7.6 we now know that to solve SDP(12) we only have to consider matrices Z that are of the form

$$Z = x_{n00} R_{n00} + \sum_{(i,j,k) \in Q_n} x_{ijk} R_{ijk}. \tag{13}$$

It is easy to see that the constraint $\operatorname{tr}(Z) = 1$ then reduces to $(2m)^n x_{n00} = 1$.

Proposition 7.8
Assume that Z is of the form of (13), then the objective function $Z \cdot J$ of SDP (12) can be reduced to

$$(2m)^n x_{n00} + \sum_{(i,j,k) \in Q_n} c_{ijk} x_{ijk},$$

where

$$c_{ijk} = (2m)^n \cdot (2m-2)^k \cdot \frac{n!}{i!j!k!}.$$

Proof
See the proof of Lemma 5.5 and Proposition 5.6. The proof is very similar, we only have to apply two small differences in the proof of Lemma 5.5. First is the fact that R_{ijk} now contains $(2m)^n$ rows instead of 4^n. And second that there are $2m-2$ elements in $S^{(m)}$ that are type different from 0, therefore the term $(2m-2)^k$ in the identity for c_{ijk}.

□

Now we only have to deal with the $Z \succeq 0$ constraint of SDP (12).

Proposition 7.9
There exist a set of $2m$ orthonormal vectors that are at the same time eigenvectors of R_{100}, R_{010} and R_{001}.

Proof
We have

$$R_{100} = I_{2m} = I_m \otimes I_2,$$

$$R_{010} = I_m \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$R_{001} = J_{2m} - R_{100} - R_{010} = J_m \otimes J_2 - R_{100} - R_{010}.$$

Let P_{J_m} be an orthogonal matrix such that $P_{J_m}^{\mathsf{T}} J_m P_{J_m}$ is diagonal, then clearly $P^{-1} I_m P$ is diagonal. Further

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

form an orthonormal set of vectors that are also eigenvectors of I_2, J_2 and

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

So if we take

$$P_1 = P_{J_m} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

then the matrices $P_1^{\mathsf{T}} R_{100} P_1$, $P_1^{\mathsf{T}} R_{010} P_1$ and $P_1^{\mathsf{T}} R_{001} P_1$ are diagonal.

□

Corollary 7.10
Let L_{100}, L_{010} and L_{001} be vectors containing all the eigenvalues of R_{100}, R_{010} and R_{001} (where the eigenvalues are ordered in such a way that the eigenvalues $(L_{100})_i$, $(L_{010})_i$ and $(L_{001})_i$ correspond to the same eigenvector), then the eigenvalues of R_{ijk} can be calculated with the following recursion formula:

$$L_{ijk} = L_{100} \otimes L_{i-1,j,k} + L_{010} \otimes L_{i,j-1,k} + L_{001} \otimes L_{i,j,k-1}.$$

If $i < 0$ or $j < 0$ or $k < 0$, then L_{ijk} is the zero vector.
The vector L_{ijk} then contains all the eigenvalues of R_{ijk}.

What remains is to determine the values of L_{100}, L_{010} and L_{001}. It is clear that L_{100} only contains 1's. Further from

$$R_{010} = I_m \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we can be derive that L_{010} only contains -1 and 1, both with multiplicity m. The values of L_{001} can be calculated via $R_{001} = J_{2m} - R_{100} - R_{010}$. The eigenvalues of J_{2m} are $2m$ and 0 (with multiplicity $2m - 1$). Using the eigenvector $\mathbf{1}$ we see that L_{001} contains the value $2m - 1 - 1 = 2m - 2$. The other values must then be $0 - 1 - 1 = -2$ (with multiplicity $m - 1$) and $0 - 1 - (-1) = 0$ (with multiplicity m).

So we can take (the duplicate eigenvalues are deleted):

$$L_{100} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, L_{010} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, L_{001} = \begin{pmatrix} 2m-2 \\ -2 \\ 0 \end{pmatrix}.$$

Let the elements of Q_n be renamed as $\{q_1, q_2, q_3, ...\}$ and let $(n, 0, 0) := q_0$ and define the vector x and the matrix B as

$$x =: (x_{q_0}, x_{q_1}, x_{q_2}, ...)^{\mathsf{T}} \text{ and } B := (L_{q_0}, L_{q_1}, L_{q_2}, ...).$$

Let A be the matrix which we get by deleting the duplicate rows of B. And let $\overline{x}$ and $\overline{A}$ be defined similar for $\overline{Q}_n$ We can now conclude that to calculate $\vartheta(\overline{G}_n^{(m)})$ we can solve the LP

$$\max\left\{ c_{n00}x_{n00} + \sum_{(i,j,k)\in Q_n} c_{ijk}x_{ijk} \;\middle|\; \begin{array}{l} (2m)^n x_{n00} = 1 \\ Ax \geq 0 \end{array} \right\},$$

and for $\vartheta(G_n^{(m)})$ we can use the LP

$$\max\left\{ c_{n00}x_{n00} + \sum_{(i,j,k)\in \overline{Q}_n} c_{ijk}x_{ijk} \;\middle|\; \begin{array}{l} (2m)^n x_{n00} = 1 \\ \overline{A}\overline{x} \geq 0 \end{array} \right\},$$

where

$$c_{ijk} = (2m)^n \cdot (2m-2)^k \cdot \frac{n!}{i!j!k!}.$$

Further because $G_n^{(m)}$ is vertex transitive we have

$$\vartheta(G_n^{(m)}) \cdot \vartheta(\overline{G}_n^{(m)}) = (2m)^n.$$

We can see that the LP's are almost the same as in the case for the original Keller graphs. The only changes are that the values of the c_{ijk}'s are different, the constraint $4^n x_{n00} = 1$ is replaced by $(2m)^n x_{n00} = 1$ and that the values in the vectors L_{ijk} are different. But the number of unique triples $((L_{100})_i, (L_{010})_i, (L_{001})_i)$ of eigenvalues is the same for every m. So the LP's still have the same number of constraints and because the set Q_n and $\overline{Q}_n$ are not changed it will also have the same number of variables.

Some of the Lovász numbers are listed in the table below:

n	$\vartheta(\overline{G}_n^{(2)})$	$\vartheta(\overline{G}_n^{(3)})$	$\vartheta(\overline{G}_n^{(4)})$
1	1	1	1
2	$2\frac{2}{3}$	3	$3\frac{1}{5}$
3	$6\frac{6}{7}$	8	8
$n \geq 4$	2^n	2^n	2^n

Conclusion: *With the Lovász number of the complement of the extended Keller graphs we are able to proof that Keller's conjecture is true for $n = 2$, but it will not give a decisive answer for $n \geq 3$.*

8 Trying to improve the upper bound for the clique number

We saw in section 6 that $\vartheta(G_n) = 2^n$ for $n \geq 4$, thus $\omega(G_n) \leq 2^n$. Unfortunately this fact is not so useful for finding out if the half integral version of Keller's conjecture is true or false for those dimensions n. But we can try other ways to improve the upper bound for the clique number.

Proposition 8.1
Let H_n be the subgraph of G_n induced by the vertex set $N(\mathbf{0})$ where $N(\mathbf{0})$ is the set of nodes that are connected with $\mathbf{0}$ in G_n. Then we have $\omega(G) = \omega(H) + 1$.

Proof
Because G_n is vertex transitive we know that there is a largest clique K which contains the the node $\mathbf{0}$. Then it is obvious that $K \backslash \{\mathbf{0}\}$ is a largest clique in H_n.

□

Proposition 8.2
We have $\chi(H_n) \leq 2^n - 1$.

Proof
We can use the same coloring as in the proof of Proposition 3.3, because none of the vertices belonging the corresponding color class of $\mathbf{0}$ in G_n are in $V(H_n)$.

□

So it is an idea to calculate $\vartheta(\overline{H}_n)$, because $1 + \vartheta(\overline{H}_n)$ is then an upper bound for $\omega(G_n)$. By $\chi(H_n) \leq 2^n - 1$ we know that $1 + \vartheta(\overline{H}_n) \leq 2^n$. Using this method we do get a better upper bound for $n = 2$, namely $\omega(G_n) \leq 2$, but in this case we are just lucky that H_2 is a graph without edges. For $n \geq 3$ I think it is not so easy to determine $\vartheta(\overline{H}_n)$ like the way we did to calculate $\vartheta(\overline{G}_n)$. The problem is that H_n has much less symmetries than G_n, for example H_n is not even regular. Further it is also not clear if the constraint $Z \succeq 0$ can easily be reduced to linear constraints. So I have not tried this method to get a possible better upper bound for $\omega(G_n)$.

What we also can try is to add more constraints to SDP

$$\max \left\{ \sum_{i \in V} \sum_{j \in V} Z_{ij} \;\middle|\; \begin{array}{l} Z \text{ is a } V \times V \text{ matrix} \\ Z \succeq 0 \\ Z_{ij} = 0 \text{ for all } \{i,j\} \in E(\overline{G}) \\ \operatorname{tr}(Z) = 1 \end{array} \right\},$$

which we used to calculate $\vartheta(\overline{G})$. To start with we can add nonnegative constraints for the elements of Z. Let

$$\vartheta'(G) := \max \left\{ \sum_{i \in V} \sum_{j \in V} Z_{ij} \;\middle|\; \begin{array}{l} Z \text{ is a } V \times V \text{ matrix} \\ Z \succeq 0 \\ Z_{ij} = 0 \text{ for all } \{i,j\} \in E(G) \\ Z_{ij} \geq 0 \text{ for all } i,j \in V(G) \\ \operatorname{tr}(Z) = 1 \end{array} \right\}. \tag{14}$$

Proposition 8.3
Let G be a graph, then we have $\omega(G) \leq \vartheta'(\overline{G}) \leq \vartheta(\overline{G})$.

Proof
The inequality $\vartheta'(\overline{G}) \leq \vartheta(\overline{G})$ is obvious. For $\omega(G) \leq \vartheta'(\overline{G})$ we can reuse the proof of Theorem 4.1 as the there defined matrix Z is still feasible for SDP (14).

□

In the literature the value $\vartheta'(G)$ is mentioned in [10, 14].

Unfortunately these nonnegative constraints did not sharpen the the upper bound for the clique number of the Keller graphs, as to my findings we have $\vartheta'(\overline{G}_n) = \vartheta(\overline{G}_n)$ for $1 \leq n \leq 7$.

But we can also consider the following set of constraints:

$$\sum_{j \in S} x_{ji} \leq x_{ii}, \text{ for all } i \in V \text{ and for all stable sets } S \text{ of } G. \tag{15}$$

Using the fact that no two vertices of a clique are in a same stable set we can easily see that the matrix Z defined in the proof of Theorem 4.1 also satisfies these constraints. But the other good news is that by adding constraints (15) to SDP (14) it is easy to see that Theorem 4.3 will still be applicable.

So in the case of the Keller graph we can still assume that Z is of the form

$$Z = x_{n00} R_{n00} + \sum_{(i,j,k) \in Q_n} x_{ijk} R_{ijk}.$$

The only thing is that we don't have the list of all possible stable sets of G_n. But for every stable set of G_n we do know we can already try to translate the corresponding constraints in terms of the x_{ijk} variables with $(i, j, k) \in Q_n$ and just add these to the LP.

In the proof of Proposition 3.3 a coloring of the graph G_n with 2^n colors is given. Every color class is a stable set, so we will now determine what the corresponding constraints are.

We will first examine the case $i = \mathbf{0}$ in (15). Define the following sets of vertices of G_n:

$$N_{xyz} := \{v \in V_n | r(\mathbf{0}, v) = (x, y, z)\}.$$

We will denote the corresponding color class of a vertex u by C_u. A color class contains exactly one 0-1 vertex v. Note that we have $v \in N_{pq0}$ for a p and q with $0 \leq p \leq n$, $0 \leq p \leq n$ and $p + q = n$. All the other elements of C_v can be constructed by replacing some of the 0's of v by a's and by replacing some of the 1's by b's. Thus for every element of w of C_v we have $w \in N_{xyz}$ for an x, y and z with $0 \leq x \leq p$, $0 \leq y \leq q$ and $x + y + z = n$.

We can calculate how many elements of C_v are in an N_{xyz}. If $w \in N_{xyz}$ then it means that x 0's (of the total of p) of v are not changed to an a, there are $\binom{p}{i}$ ways to do this. Further there are y 1's (of the total of q) which are not changed to a b, so this gives $\binom{q}{y}$ possibilities. So for a color class C_v with $v \in N_{pq0}$ we have the constraint

$$\sum_{\substack{i\in\{0,\ldots,p\}\\ j\in\{0,\ldots,q\}\\ (i,j,k)\in Q_n}} \binom{p}{i}\binom{q}{j} x_{ijk} \le x_{n00}.$$

Note that if two 0-1 vertices are from a same set N_{pq0}, then the corresponding color classes give the same constraint. For a given dimension n the case $i = \mathbf{0}$ will give n constraints (if we don't count the color class $C_{\mathbf{0}}$ which gives the constraint $x_{n00} \le x_{n00}$). For example for $n = 4$ we have the following constraints:

$$\begin{aligned}
&C_{310} : 3x_{211} + 3x_{112} + x_{013} \le x_{400},\\
&C_{220} : x_{220} + 2x_{211} + 2x_{121} + 4x_{112} + x_{022} + 2x_{013} \le x_{400},\\
&C_{130} : x_{130} + 3x_{121} + 3x_{112} + x_{031} + 3x_{022} + 3x_{013} \le x_{400},\\
&C_{040} : x_{040} + 4x_{031} + 6x_{022} + 4x_{013} \le x_{400}.
\end{aligned}$$

We will now show that if we take an other vertex for i in (15) we will get the same constraints as the case $i = \mathbf{0}$. Assume that we have two vertices $x, y \in V_n$ which are in the same color class and let $p \in V_n$ be a vertex. Let $q \in C_p$ be such that that $p_i \ne q_i$ if and only if $x_i \ne y_i$ (q is unique), then we have $r(x, p) = r(y, q)$. Because of this the cases $i = x, S = C_p$ and $i = y, S = C_p$ give the same constraints, so the cases $i = x$ and $i = y$ are just the same. So we can conclude that for any color class C we only need to consider the case $i = u_C$, where u_C is the unique 0-1 vertex of C.

Let $w, z \in V_n$ be 0-1 vertices and let $s, s' \in V_n$ be vertices which are in a same color class. Let $t \in V_n$ be such that $t_i = s_i + 1$ if $w_i \ne z_i$ and $t_i = s_i$ if $w_i = z_i$ and let $t' \in V_n$ be similar defined. Then t and t' belong to the same color class and we also have $r(w, s) = (z, t)$ and $r(w, s') = (z, t')$. Therefor the cases $i = w, S = C_s$ and $i = z, S = C_t$ give the same constraints and eventually the cases $i = w$ and $i = z$ are just the same.

So we can conclude that it is enough to consider the case $i = \mathbf{0}$. But unfortunately by adding the constraints in (15), using the stable sets we have found in the proof of Proposition 3.3, the optimal objective value of the LP's remains the same for all dimensions n.

9 Conclusions

As I have mentioned earlier at the end of Chapter 6 all the clique numbers of the Keller graphs are known. But most of them are found via a computer search which is from complexity point of view not a very fast method. Finding the maximum clique in a graph is unfortunately one of those problems which is known to be NP-hard, so it is very unlikely that a polynomial algorithm will ever be found that works for all graphs.

For the cases $1 \leq n \leq 6$ it appeared to be very doable to find the clique number of G_n via the computer, where G_n is the Keller graph of dimension n. But for the case $n = 7$ they already needed 15 hours to conclude that the clique number of G_7 is less than 128, see [3]. This doesn't seem that long, but a computer with 64 CPU's was used. And it took even 109 days to find out that $\omega(G_7) = 124$. Can you imagine how long it would take for the cases $n \geq 8$? It is that these cases are already solved by other smart methods.

In this thesis we have tried a heuristic, namely calculating the Lovász number of the complement of the Keller graph, in the hope that we could find a faster way to solve the half integral version of Keller's conjecture (Conjecture 1.8). Actually when I started this master project it was still a open question if we have $\omega(G_7) < 128$ or not, so we also had the hope that we could solve Conjecture 1.8 for $n = 7$. We thought that it is potentially a good heuristic because we knew beforehand that $\vartheta(\overline{G}_n) \leq 2^n$.

We have shown that it is not so difficult to compute $\vartheta(\overline{G}_n)$, because the corresponding SDP can easily be reduced to an LP. Despite the fact that G_n has an exponential number of vertices, the LP only has a polynomial number of variables and constraints in terms of n. Also for the extended Keller graphs it is very easy to compute the the Lovász number, which could potentially help solving Keller's conjecture. At this moment only the case $n = 7$ is still an open question.

Unfortunately it appeared that our heuristic failed for solving Conjecture 1.8 for the cases $n \geq 4$. And in the case of the extended Keller graph with one extra pair of fractal values it even failed for $n = 3$.

In Chapter 8 we have tried some methods to improve our upper bound for $\omega(G_n)$, but these tries also weren't successful.

But I have some suggestions for further research. As I already had mentioned in Chapter 8 it can be an idea to calculate the Lovász number of the complement of the subgraph H_n of G_n induced by the neighbors of vertex $\mathbf{0}$. Further in general for a graph G there exist SDP hierarchies converging to $\alpha(G)$ (see [8]), so the values you will find via this method will be between $\alpha(G)$ and $\theta(G)$. So this method can also be tried.

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