Algebraic cycles, Chow motives, and *L*-functions

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Contents

Signalons aussi que les définitions et conjectures ci-dessus peuvent être données dans la cadre des “motifs” de Grothendieck, c’est-à-dire, grosso modo, des facteurs directs de H^m fournis par des projecteurs algébriques. Ce genre de généralisation est utile si l’on veut, par example, discuter des propriétés des produits tensoriels de groupes de cohomologie, ou, ce qui revient au même, des variétés produits. (Serre [Se70])

Introduction

A theme showing up in several places of the mathematical landscape is that of *L-functions*. These complex functions go back to the days of DIRICHLET and his study of primes in arithmetic progressions. Since then, L-functions have been associated to number fields k (the Dedekind zeta function $\zeta_k(s)$, with as a famous case the Riemann zeta function, when $k = \mathbb{Q}$), Galois representations, modular forms, varieties, and more. Several of these cases have later been unified, with as a well-known example the modularity theorem (the L-function of an elliptic curve coincides with that of the associated modular form).

Central to all these L-functions is that they are defined by some series, converging on some right half of the complex plane. They should (sometimes this is known, sometimes it is a conjecture) satisfy a meromorphic continuation and a *functional equation*. Interesting theory comes from the study of their zeroes and poles, and their *values at the integers* (most notably those closest to the centre of reflection of the functional equation).

The first four sections of this thesis are in some sense a "toy model" for the later sections. Here we associate an L-function to every abelian variety over a number field, and show that it contains information about the reduction behaviour of the abelian variety. In the case of elliptic curves (dimension 1) we also state part of the Birch and Swinnerton-Dyer conjecture. It relates the order of vanishing of the L-function at its special value (analysis) and the rank of the Mordell–Weil group (geometry).

GROTHENDIECK came up with the vision of a universal cohomology theory for smooth projective varieties over a field, which he christened *motives*. All 'nice' cohomology theories (Weil cohomology theories) should factor through this category of motives, which should have lots of properties in common with the categories in which cohomology theories take values. I. e., it should be abelian, have a tensor structure, and possibly even be equivalent to the category of representations of some affine group scheme.

Later, *mixed motives* were proposed, for the singular, non-projective case.

So far, there are several candidates for the category of motives, each with its own advantages and disadvantages. The idea has shown itself to be a guiding principle in arithmetic geometry over the last few decades. In the rest of this thesis we use GROTHENDIECK's proposal, named *Chow motives*. We associate L-functions to them, and formulate a conjecture by BEILINSON and BLOCH about the special values of these L-functions, and so called *Chow groups* (akin to the Mordell–Weil group). In sections 7 and 8 we do some explicit computations with Chow motives and algebraic cycles. Finally, in the last section we apply the computations to the Fermat quartic, and use the conjecture to predict the existence of cycles.

1 Conductors

The Galois group G of a finite Galois extension l/k of local fields contains rich information about ramification behaviour in the form of a finite filtration of *higher ramification groups*

We refer to [Se79, §IV.1] for more information about this filtration.

$$G = G_{-1} \supset G_0 \supset G_1 \supset \ldots \supset G_n = 1.$$

To define this filtration, denote with $(\mathcal{O}_l, \mathfrak{p})$ the valuation ring of l, and its maximal ideal. Then G_i is the set of $\sigma \in G$ such that σ acts trivially on $\mathcal{O}_l/\mathfrak{p}^{i+1}$. It follows that G_i is a normal subgroup of G. One can show that $G_i = 1$ for sufficiently large i. For $\sigma \neq \mathrm{id}$, we write $i_G(\sigma)$ for the smallest integer i such that $\sigma \notin G_i$.

Observe that the higher ramification groups indeed give information about the ramification behaviour. After all, G_0 is the inertia group, and therefore l/k is unramified if and only if G_0 is trivial. The extension is tamely ramified precisely if G_1 is trivial.

Given a finite-dimensional Galois representation V (of G) over some field K with $\mathrm{char}(K) \neq p$, it is a natural question to ask how the G_i act on V. Let us first introduce some terminology. The Galois representation V is said to be *unramified* if G_0 acts trivially on V, and *tamely ramified* if G_1 acts trivially. Observe that this generalises the classical terminology of unramified (resp. tamely ramified) extensions. This leads naturally to the definition of the *conductor* of V; a measure of the ramification behaviour of V.

If we write g_i for the cardinality $\#G_i$, we may define the *measure of wild ramification* by putting

$$\delta(V) = \frac{1}{g_0} \sum_{i=1}^{\infty} g_i \dim(V/V^{G_i}).$$

Observe that the sum is finite, because the filtration of higher ramification groups is finite. In a similar nature, we define the *measure of tame ramification* as

$$\varepsilon(V) = \dim(V/V^{G_0}).$$

The *conductor* of V is then defined to be $f(V) = \varepsilon(V) + \delta(V)$.

If V is a complex representation, i. e., $K = \mathbb{C}$, there is a scalar product on the space of complex valued class functions of G, given by

For a detailed and more general exposition, see [Se79, Og67, Se60].

$$(\phi, \psi) = \frac{1}{\#G} \sum_{\sigma \in G} \phi(\sigma)\overline{\psi(\sigma)}.$$

We can use this to give an alternative computation of the conductor via the character χ of V. Crucial in this approach is the *Artin character*:

$$a_G \colon G \to \mathbb{Z}$$
$$\sigma \mapsto \begin{cases} -[G:G_0] \cdot i_G(\sigma) & \text{if } \sigma \neq \mathrm{id} \\ [G:G_0] \cdot \sum_{\tau \neq \mathrm{id}} i_G(\tau) & \text{if } \sigma = \mathrm{id}\,. \end{cases}$$

The other ingredient is the following lemma.

1.1 Lemma. *For every non-negative integer i we have*

$$g_i \dim(V/V^{G_i}) = \sum_{\sigma \in G_i} (\chi(\mathrm{id}) - \chi(\sigma))\,.$$

Proof. We have a split exact sequence of G_i-representations

$$0 \to U_i \to \mathbb{C}[G_i] \to \mathbb{C} \to 0,$$

The *augmentation representation* U_i is defined as the kernel of the map $\mathbb{C}[G_i] \to \mathbb{C}, \sigma \mapsto 1$.

We apply the contravariant functor $\mathrm{Hom}_{\mathbb{C}}(\_, V)$ to get

$$0 \to V \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[G_i], V) \to \mathrm{Hom}_{\mathbb{C}}(U_i, V) \to 0.$$

Taking G_i-invariants (which is left exact, therefore preserving split exact sequences), we get the exact sequence

$$0 \to V^{G_i} \to V \to \mathrm{Hom}_{G_i}(U_i, V) \to 0.$$

Write u_i for the character of U_i. If we decompose $U_i = \oplus_j U_{i,j}$ and $V = \oplus_k V_k$ into irreducible representations, with characters $u_{i,j}$ and χ_k, then we find

$$\dim \mathrm{Hom}_{G_i}(U_i, V) = \sum_{j,k} \dim \mathrm{Hom}_{G_i}(U_{i,j}, V_k) = \sum_{j,k} (u_{i,j}, \chi_k) = (u_i, \chi).$$

The lemma now follows from explicitly computing

$$(u_i, \chi) = \frac{1}{g_i} \left(g_i \chi(\mathrm{id}) - \sum_{\sigma \in G_i} \chi(\sigma) \right).$$ □

A direct computation now shows that

$$\begin{aligned} (\chi, a_G) &= \frac{1}{\#G} \sum_{\sigma \in G} \chi(\sigma) \overline{a_G(\sigma)} \\ &= \frac{1}{\#G} \sum_{i=0}^{\infty} \sum_{\sigma \in G_i} \frac{\#G}{g_0} (\chi(\mathrm{id}) - \chi(\sigma)) \\ &= \frac{1}{g_0} \sum_{i=0}^{\infty} \sum_{\sigma \in G_i} \chi(\mathrm{id}) - \chi(\sigma). \end{aligned}$$

We conclude that the conductor $f(V)$ equals (χ, a_G).

2 Abelian varieties

Let S be a scheme. An *abelian scheme* over S of dimension d is a proper smooth finitely presented commutative group scheme over S whose fibres are geometrically connected and of dimension d. An *abelian variety* over a field k is an abelian scheme over k. It can be shown that an abelian variety A/k is projective as k-variety.

2.1 Example. An elliptic curve over a field k is defined as a proper variety E/k that is smooth of relative dimension 1, of which the geometric fibre $E_{\bar{k}}$ is connected and has genus 1, together with a given point $0 \in E(k)$. One can show that elliptic curves over k are precisely the 1-dimensional abelian varieties over k. Elliptic curves form an important class of objects in the study of abelian varieties. Abelian varieties are a generalization of elliptic curves to higher dimensions. «

Let A/S be an abelian scheme of dimension d, and n an integer. The endomorphism $[n]\colon A \to A$, defined by $A(T) \xrightarrow{\cdot n} A(T)$, for S-schemes T, is called the *multiplication by n map*. We denote the kernel of $[n]$ by $A[n]$. It is important to observe that $A[n]$ is in general not an abelian scheme.

Assume S is the spectrum of a field k, and fix a separable closure k^{s} of k. Let ℓ be a prime number coprime to $\operatorname{char} k$. Multiplication by ℓ defines canonical maps $A[\ell^{n+1}](k^{\mathrm{s}}) \to A[\ell^n](k^{\mathrm{s}})$, for every non-negative integer n. We define the ℓ-adic *Tate module* $T_\ell A$ of A/k to be $\lim A[\ell^n](k^{\mathrm{s}})$. This is a free $\mathbb{Z}_\ell$-module of rank $2d$ that comes equipped with a natural continuous action of $\operatorname{Gal}(k^{\mathrm{s}}/k)$. There is also a canonical isomorphism of Galois representations between $\mathrm{H}^1_{\text{ét}}(A_{k^{\mathrm{s}}}, \mathbb{Q}_\ell)$ and the dual of $T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Moreover, there are canonical isomorphisms of Galois representations

This isomorphism $\mathrm{H}^1_{\text{ét}}(A_{k^{\mathrm{s}}}, \mathbb{Q}_\ell) \cong (T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)^\vee$ can be deduced from e. g., results and computations in [SZ11, §2].

$$\mathrm{H}^i_{\text{ét}}(A_{k^{\mathrm{s}}}, \mathbb{Q}_\ell) \cong \bigwedge^i \mathrm{H}^1_{\text{ét}}(A_{k^{\mathrm{s}}}, \mathbb{Q}_\ell),$$

and in particular $\mathrm{H}^\bullet_{\text{ét}}(A, \mathbb{Q}_\ell) \cong \bigwedge^\bullet \mathrm{H}^1_{\text{ét}}(A, \mathbb{Q}_\ell)$. Thus the ℓ-adic Tate module contains all the information of the ℓ-adic cohomology.

Néron models

Let R be a Dedekind domain, k its field of fractions, A an abelian variety over k. A *Néron model* of A over R is a scheme $\mathcal{A}$ representing the functor

$$\begin{aligned} \{\text{smooth } R\text{-schemes}\} &\to \mathbf{Set} \\ T &\mapsto \operatorname{Hom}_k(T_k, A), \end{aligned}$$

I. e., if T is a smooth R-scheme with a morphism $T_k \to A$, it can be extended uniquely to a morphism $T \to \mathcal{A}$. One should note that, although A is proper over k, we do not require $\mathcal{A}$ to be proper over R, and in general it is not.

2.2 **Theorem.** *Let R be a Dedekind domain, k its field of fractions, A an abelian variety over k. Then there exists a Néron model of A over R and this Néron model is unique up to unique isomorphism.*

Proof. For the existence I will not give a proof. See [BLR90] for a proof. The uniqueness is the usual exercise in abstract nonsense. □

The theorem shows that it makes sense to speak of *the* Néron model of A.

2.3 **Proposition.** *The Néron model of an abelian variety is a quasi-projective commutative group scheme over R.*

Proof. Omitted. See [BLR90]. □

3 Reduction of abelian varieties

In section 1 we defined for a finite Galois extension of local fields when a Galois representation is unramified. In this section we will prove an important theorem that links the ramification behaviour of Tate modules of an abelian variety with its reduction at primes (to be defined). This is the so called *criterion of Néron–Ogg–Shafarevich*.

An excellent treatment of this criterion is given by Serre and Tate [ST68]. The proof in this paper is essentially the same, and is longer only because it spells out some facts in more detail.

Notation

Let k be a field, v a discrete valuation of k, and $\mathcal{O}_v$ the valuation ring of v. Fix an extension $\overline{v}$ of v to k^s. Let G be the absolute Galois group $\mathrm{Gal}(k^s/k)$, and let D, and I denote the decomposition group and the inertia group of $\overline{v}$. (Observe that this can be done since v corresponds to a prime of $\mathcal{O}_v$.) Denote the residue field $\mathcal{O}_v/\mathfrak{m}_v$ by $\kappa(v)$. We assume that $\kappa(v)$ is perfect. Let $\overline{\kappa(v)} = \kappa(\overline{v})$ be a separable closure of it. Sometimes we will just write $\mathcal{O}$, κ or $\overline{\kappa}$, to reduce notation. Let A/k be an abelian variety, A_v the Néron model of A with respect to v, and furthermore, let $\tilde{A}$ denote $A_v \times_{\mathcal{O}_v} \kappa(v)$, the reduction modulo v. The connected component (for the Zariski topology) of the identity of $\tilde{A}$ is denoted $\tilde{A}^0$. Let n be a non-zero integer. We write A_n for $A(k^s)[n] = A[n](k^s)$.

Criterion of Néron–Ogg–Shafarevich

3.1 **Definition.** A has *good reduction at v* if there exists an abelian scheme over $\mathcal{O}_v$ whose generic fibre is isomorphic to A. «

It is a theorem that A has good reduction at v if and only if the Néron model A_v is proper over $\mathcal{O}_v$.

3.2 **Lemma.** *Let k^{unr} be the maximal unramified extension of k in k^s. The ring $\mathcal{O}_{k^{\mathrm{unr}}}$ of $\overline{v}$-integers in k^{unr} is strict henselian.*

See appendix A for a definition and some facts about henselian rings.

Proof. Let v' denote the restriction of $\overline{v}$ to k^{unr}. Let $f \in \mathcal{O}_{k^{\mathrm{unr}}}[X]$ be a monic irreducible polynomial with a simple root a_0 modulo v'. Then the derivative of f is non-zero modulo v', so it is non-zero itself. Hence f is separable, and therefore its roots are in k^{s}. Since f is monic all roots lie in the integral closure of $\mathcal{O}_{k^{\mathrm{unr}}}$ in k^{s}, i.e., in $\mathcal{O}_{\overline{v}}$. In particular, f factors in linear factors over $\mathcal{O}_{\overline{v}}$, and this factorisation reduces to a factorisation over $\overline{\kappa}$. Since f is monic, $\deg f = \deg \overline{f}$, and consequently there exists a unique root a of f that reduces to a_0.

Since $D/I \cong \mathrm{Gal}(\overline{\kappa}/\kappa)$ it follows that for each $\sigma \in I$ we have $\sigma(a) \equiv a \pmod{v'}$. Because $\sigma(a)$ is a root of f that reduces to a_0 it follows that σ fixes a, and therefore $a \in k^{\mathrm{unr}}$, hence $a \in \mathcal{O}_{k^{\mathrm{unr}}}$. Since the residue field of $\mathcal{O}_{k^{\mathrm{unr}}}$ is the separably closed field $\overline{\kappa}$, we conclude that $\mathcal{O}_{k^{\mathrm{unr}}}$ is strict henselian. □

3.3 **Lemma.** *With notation as in the beginning of this section, let l/k be a unramified field extension, v' an extension of v to l. Write $\mathcal{O}_{v'}$ for the valuation ring of v'. Let n be any integer.*

The reduction map $\mathcal{O}_{v'} \to \kappa(v')$ induces a map $A(l)[n] \to \tilde{A}(\kappa(v'))[n]$, which we will also call a reduction map. Moreover, if l is the maximal unramified extension over k, and n is invertible in $\mathcal{O}_v$, this map is bijective.

Proof. Since l/k is unramified, $\operatorname{Spec} \mathcal{O}_{v'}$ is smooth over $\operatorname{Spec} \mathcal{O}_v$. Further $\mathcal{O}_{v'} \otimes_{\mathcal{O}_v} k \cong l$, since taking field of fractions is localizing at the zero ideal.

By the universal property of Néron models $A_v(\mathcal{O}_{v'}) \cong A(l)$. Observe that $\tilde{A}(\kappa(v')) \cong A_v(\kappa(v'))$ by the universal property of fibred products. Therefore we have $A(l)[n] \cong A_v(\mathcal{O}_{v'})[n] \cong A_v[n](\mathcal{O}_{v'})$ and $\tilde{A}(\kappa(v'))[n] \cong A_v(\kappa(v'))[n] \cong A_v[n](\kappa(v'))$.

The reduction map $\mathcal{O}_{v'} \to \kappa(v')$ induces a reduction map $A_v[n](\mathcal{O}_{v'}) \to A_v[n](\kappa(v'))$. The composition of this reduction map with the above isomorphisms gives the reduction map $A(l)[n] \to \tilde{A}(\kappa(v'))[n]$.

The last statement of the lemma is an immediate consequence of Hensel's lemma. By lemma 3.2 the ring $\mathcal{O}_{v'}$ is henselian. Since $n \in \mathcal{O}_v^*$ it follows that $A_v[n]$ is étale over $\mathcal{O}_{v'}$ (see [KM85, thm 2.3.1] for a proof in the case of elliptic curves, or [Ntor, thm 10] for the general case). Now corollary A.4 asserts that the reduction map is bijective. □

Recall that we write $A_n = A(k^{\mathrm{s}})[n]$, and so we will write $\tilde{A}_n = \tilde{A}(\overline{\kappa})[n]$, and $\tilde{A}_n^0 = \tilde{A}^0(\overline{\kappa})[n]$. Let A_n^I denote the set of elements in A_n that are invariant under the action of the inertia group. Throughout the discussion, n is an integer that is coprime to the residue characteristic.

3.4 **Lemma.** *The reduction map induces an isomorphism of A_n^I onto $\tilde{A}_n$.*

Proof. Let k^{unr} be the field fixed by the inertia group I and the residue field of k^{unr} is $\overline{\kappa}$. By lemma 3.3 there is a bijective reduction map $A(k^{\mathrm{unr}})[n] \to \tilde{A}(\overline{\kappa})[n] = \tilde{A}_n$. Observe that $A_n^I = A(k^{\mathrm{unr}})[n]$. It follows that $\tilde{A}_n$ is isomorphic to A_n^I. □

By proposition 2.3, $\tilde{A}^0$ is a connected smooth commutative group scheme. Therefore it is an extension of an abelian variety B by an affine closed subgroup scheme H due to lemma B.1. Observe that H is again smooth and commutative, and therefore can be decomposed as $S \times U$ for some torus S and unipotent group scheme U, by lemma B.2.

3.5 Lemma. *The index of $\tilde{A}^0$ in $\tilde{A}$ is finite, and $\tilde{A}_n$ is an extension of a group of order dividing $[\tilde{A} : \tilde{A}^0]$ by a free $\mathbb{Z}/n\mathbb{Z}$-module of rank* $\dim S + 2\dim B$.

Proof. By proposition 2.3 we see that A_v is of finite type over $\operatorname{Spec} R$. Therefore the index of $\tilde{A}^0$ in $\tilde{A}$ is finite. The inclusion $\tilde{A}_n \to \tilde{A}(\overline{\kappa})$ induces a map

$$\tilde{A}_n/\tilde{A}_n^0 \to \tilde{A}(\overline{\kappa})/\tilde{A}^0(\overline{\kappa}),$$

which shows that the index c of $\tilde{A}_n^0$ in $\tilde{A}_n$ divides $[\tilde{A}(\overline{\kappa}) : \tilde{A}^0(\overline{\kappa})] = [\tilde{A} : \tilde{A}^0]$.

Observe that taking n-torsion is left exact, and therefore we have the exact sequence

$$0 \to H_n \to \tilde{A}_n^0 \to B_n.$$

To show that the last map is surjective, let $x \in B_n$ be given. Since $\tilde{A}^0(\overline{\kappa}) \to B(\overline{\kappa})$ is surjective, we have a preimage y in $\tilde{A}^0$. As n is coprime to $\operatorname{char} \kappa$ the group $H(\overline{\kappa})$ is n-divisible. Therefore there exists an $h \in H_n$ such that $nh = -ny$. We conclude that $y + h$ is an element of $\tilde{A}_n^0$ mapping to x, which proves surjectivity.

$H_{\overline{\kappa}}$ is an iterated extension of copies of $\mathbb{G}_a$. Extensions of n-divisible groups are n-divisible groups.

Observe that H_n is a free $\mathbb{Z}/n\mathbb{Z}$-module of rank $\dim S$, and B_n is free of rank $2\dim B$. It turns out that $\tilde{A}_n^0$ is a free $\mathbb{Z}/n\mathbb{Z}$-module, so it follows that it is free of rank $\dim S + 2\dim B$. □

3.6 Lemma. *Let R be a discrete valuation ring, with residue field κ and field of fractions k. Let X be a smooth separated R-scheme, and suppose that the generic fibre X_k is geometrically connected, and the special fibre X_κ is proper and non-empty. Then X is proper over R and X_κ is geometrically connected.*

Proof. We first show that we may assume R to be complete. Write $\hat{R}$ for the completion of R, and $\hat{\kappa}$ for the residue field of $\hat{R}$. Observe that $\operatorname{Spec} \hat{R} \to \operatorname{Spec} R$ is faithfully flat and quasi-compact (since affine). Observe that $X_{\hat{k}}$ is geometrically connected over $\hat{k}$, and $X_{\hat{\kappa}}$ is proper over $\hat{\kappa}$. After proving the lemma for $\hat{R}$, [EGA4, prop 2.7.1] shows that X is proper over R. Since the residue field $\hat{\kappa}$ of $\hat{R}$ equals κ it also follows that X_κ would then be geometrically connected over κ. By [EGA3, cor 5.5.2], there exist open disjoint subschemes Y and Z of X, with $X = Y \cup Z$, Y proper, and $X_\kappa \subset Y$. Since X is smooth over R, observe that $Y \cap X_k$ is non-empty. (Because smooth morphisms are flat and locally of finite presentation, hence open, cf. [EGA4, thm 2.4.6], and R is a discrete valuation ring.) Since X_k is connected, we conclude that $Z \cap X_k$ is empty, hence $X = Y$. In particular X is proper.

The fact that X_κ is geometrically connected follows from Zariski's connectedness theorem (cf. [EGA3, thm 4.3.1]), since R is noetherian. □

3.7 Proposition. *Suppose A_n is unramified at v for infinitely many n coprime to* $\operatorname{char}\kappa(v)$. *Then A has good reduction at v.*

Proof. By assumption there exists an integer n such that

- $n > [\tilde{A} : \tilde{A}^0]$;
- n is coprime to $\operatorname{char}\kappa$;
- $A_n = A_n^I$.

By lemma 3.4 we know that $A_n = A_n^I$ is isomorphic to $\tilde{A}_n$. Combined with lemma 3.5 this gives

$$n^{2\dim A} = cn^{\dim S + 2\dim B},$$

where c is equal to $[\tilde{A}_n : \tilde{A}_n^0]$ and divides $[\tilde{A} : \tilde{A}^0]$. By assumption $n > [\tilde{A} : \tilde{A}^0] \geq c$. Hence $c = 1$ and $\dim S + 2\dim B = 2\dim A$. Since

$$\dim A = \dim \tilde{A}^0 = \dim B + \dim S + \dim U,$$

we have $S = U = 0$. Therefore $\tilde{A}^0$ is isomorphic to B, and hence proper over $\kappa(v)$. Since the index of $\tilde{A}^0$ in $\tilde{A}$ is finite, we conclude that $\tilde{A}$ is proper over $\kappa(v)$. It remains to prove that A_v is proper over $\mathcal{O}_v$ and $\tilde{A}$ is geometrically connected over $\kappa(v)$.

Since A is geometrically connected over k and $\tilde{A}$ is proper over $\kappa(v)$ we are in a situation to apply lemma 3.6, and the result follows. □

3.8 Theorem (Néron–Ogg–Shafarevich). *Let ℓ be a prime number different from the residue characteristic* $\operatorname{char}\kappa(v)$. *The following are equivalent.*

1. *A has good reduction at v;*
2. *A_n is unramified at v for all n coprime to* $\operatorname{char}\kappa(v)$*;*
3. *A_n is unramified at v for infinitely many n coprime to* $\operatorname{char}\kappa(v)$*;*
4. *$T_\ell A$ is unramified at v.*

Proof. First observe that item 4 is equivalent to A_{ℓ^i} being unramified for all $i \in \mathbb{Z}_{\geq 0}$; this follows immediately from the definition of Galois action on $T_\ell A$. From this we deduce that item 2 implies item 4, which in turn implies item 3. Also, item 3 implies item 1 by proposition 3.7.

We now continue with (1) $\implies$ (2). Note that because A has good reduction at v, A_v is an abelian scheme over $\mathcal{O}_v$. Thus $\tilde{A}$ is an abelian variety over $\kappa(v)$. For any n coprime to $\operatorname{char}\kappa(v)$, we then know that $\tilde{A}_n$ is a free $\mathbb{Z}/n\mathbb{Z}$-module of rank $2\dim\tilde{A} = 2\dim A$. By lemma 3.4 this also holds for A_n^I. Hence A_n^I must be all of A_n. We conclude that item 1 implies item 2. □

4 L-functions of abelian varieties

Let k be a global field. Let Σ_k (respectively Σ_k^∞) be the set of ultrametric (respectively archimedean) places of k. For a valuation $v \in \Sigma_k \cup \Sigma_k^\infty$, we write k_v for the completion at v.

Let v be an ultrametric place of k. We write $\mathcal{O}_v$, $\kappa(v)$, and p_v for the corresponding valuation ring of k_v, its residue field, and its residue characteristic. Let q_v denote the cardinality of $\kappa(v)$, and let ℓ be a prime number different from p_v. Finally, let A be an abelian variety over k of dimension d.

The ℓ-adic Tate module $T_\ell A$ (being dual to the ℓ-adic cohomology) comes with an action of the absolute Galois group G_k. Write $I = I_v$ for the inertia group of v in G_k. Write $\rho\colon I \to \operatorname{Aut}(T_\ell A_v)$ for the restriction to the inertia group. By a theorem of Deligne there is a subgroup $I' \subset I$ of finite index so that for all $\sigma \in I'$ the action of σ is unipotent, i.e., $\rho(\sigma)$ is unipotent. Consequently the character of ρ factors via I/I'. Analogously to section 1 we may then associate a conductor $f(v)$ to the pair (A, v). We define the *conductor* of A to be the cycle $\mathfrak{f} = \sum_v f(v) \cdot v$. The sum is finite, as the criterion of Néron–Ogg–Shafarevich (theorem 3.8) implies that $f(v) = 0$ if and only if A has good reduction at v. (And A has good reduction at almost all v.)

For a more general treatment of conductors of varieties, see [Se70].

Local factors of the L-function

First assume that v is a finite place. We can choose a lift, π_ℓ, of the Frobenius endomorphism of $\kappa(v)$ to $\operatorname{Gal}(k^s/k)$, which acts on $V_\ell A = T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. We now define the polynomial

$$P_{v,\ell}(T) = \det(1 - \pi_\ell T),$$

which turns out to have coefficients in $\mathbb{Z}$ and to be independent of the choice of ℓ and the lift π_ℓ. We will therefore denote it with $P_v(T)$.

In case A has good reduction at v, we define the local factor at v to be

$$L_v(s) = \frac{1}{P_v(q_v^{-s})}.$$

If A does not have good reduction at v, we proceed as follows. Write $A_v = A \times_k k_v^s$, and recall that $T_\ell A_v$ is a Galois representation. Therefore $V = V_\ell A_v = T_\ell A_v \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is also a Galois representation. Let $I = I(k_v^s/k_v)$ denote the inertia group. Let π denote the restriction of the geometric Frobenius to the coinvariants V_I. Then we define $P_{v,\ell}(T) = \det(1 - \pi T)$ as before. Again this polynomial has coefficients in $\mathbb{Z}$, and is independent of the choice of ℓ, so that it makes sense to call it P_v. Just as we did above, we define

$$L_v(s) = \frac{1}{P_v(q_v^{-s})}.$$

Note that this approach generalizes the case of good reduction.

Now assume that v is an infinite place. Define $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$, where $\Gamma(s)$ denotes the usual gamma function. For $v \in \Sigma_k^\infty$ we know that k_v is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$. In case $k_v \cong \mathbb{R}$ we define $\Gamma_v(s) = \Gamma_{\mathbb{C}}(s)^d$. If, on the other hand, $k_v \cong \mathbb{C}$, we define $\Gamma_v(s) = \Gamma_{\mathbb{C}}(s)^{2d}$. It appears that these definitions coincide with the more general definitions in [Se70, §3] (as can be read in one of the examples there).

The function $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,\mathrm{d}t$ is a generalisation of the factorial: $\Gamma(n) = (n-1)!$.

L-functions and their functional equation

To come to a definition of global L-functions, we define a few more invariants.

$$D = \begin{cases} |d_{k/\mathbb{Q}}|, & \text{abs. value of the discriminant, if } k \text{ is a number field,} \\ q^{2g-2}, & \text{if } k \text{ is a function field of genus } g \text{ over } \mathbb{F}_q. \end{cases}$$
$$C = N(\mathfrak{f}) \cdot D^{2d}.$$

Finally, after putting

$$L'(s) = \prod_{v \in \Sigma_k} L_v(s)$$

we define the global L-function to be

$$L(A,s) = C^{s/2}\mathbb{L}'(s) \prod_{v \in \Sigma_k^\infty} \Gamma_v(s).$$

It is proven that the L-function attached to A converges on some right half plane of the complex numbers. Conjecturally every L-function satisfies a functional equation, which allows us to extend it meromorphically to all of $\mathbb{C}$.

The functional equation is given by

$$L(A,s) = \varepsilon \cdot L(A, 2-s), \qquad \text{with } \varepsilon = \pm 1.$$

Note that ε does not depend on s. It is called the *sign* of the functional equation.

Example: L-functions of elliptic curves

An abelian variety of dimension 1 is also called an elliptic curve. It is possible to give the L-function of an elliptic curve $E/\mathbb{Q}$ explicitly.

First consider a finite place v. Then $\kappa(v)$ is finite of cardinality q_v. Let a_v denote $q_v + 1 - \#\tilde{E}(\kappa(v))$. It has been shown that

$$P_v(T) = \begin{cases} 1 - a_vT + q_vT^2 & \text{good} \\ 1 - T & \text{split multiplicative} \\ 1 + T & \text{non-split multiplicative} \\ 1 & \text{additive.} \end{cases} \qquad \text{(Reduction of } E \text{ at } v\text{:)}$$

Observe that $L_v(1) = P_v(1/q_v)^{-1} = q_v/\#\tilde{E}(\kappa(v))$, no matter what the reduction type of E at v is.

A quick inspection of the previous section shows that the contribution from the only archimedean place is $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$. So we come to the following expression

$$L(E,s) = N(\mathfrak{f})(2\pi)^{-s}\Gamma(s)\prod_p \frac{1}{P_p(p^{-s})}.$$

By the modularity theorem we know that this L-function coincides with the L-function of a modular form associated to E. In particular it follows that $L(E,s)$ has a meromorphic continuation to the entire complex plane, and that there is a functional equation

$$L(E,s) = \varepsilon L(E, 2-s), \qquad \varepsilon = \pm 1.$$

Birch and Swinnerton-Dyer conjecture

We continue with the notation of section 4.

4.1 Conjecture (Birch and Swinnerton-Dyer). *For an elliptic curve $E/\mathbb{Q}$ with L-function $L(E,s)$ the following two quantities are equal:*

- $\operatorname{ord}_{s=1} L(E,s)$, *the order of vanishing at $s = 1$;*
- $\operatorname{rk} E(\mathbb{Q})$, *the rank of the Mordell–Weil group.* «

For more details on the statement and an overview of the results concerning this conjecture, see [Wi06].

As a consequence of the conjecture, if ε (the sign of the functional equation) is equal to -1 we deduce from the functional equation that $L(E,1) = 0$. Then we can conclude that there are elements of infinite order in $E(\mathbb{Q})$.

4.2 Remark. We have only stated half of the Birch and Swinnerton-Dyer conjecture. In its full glory, this conjecture involves an invariant differential ω_E, the order of the Tate–Shafarevich group of E, a certain regulator and other constants. We will not go into that here. For more information we refer to [Si10, conj 16.5]. «

5 Gamma factors of Hodge structures

The local factors of the L-function of an abelian variety at finite primes are complemented by gamma factors for the infinite primes. These gamma factors are naturally attached to the Hodge structures on the de Rham cohomology at the infinite primes, just as the local factors at finite primes are attached to Galois representations on the ℓ-adic cohomology.

A $\mathbb{C}$-Hodge structure is a finite-dimensional $\mathbb{C}$-vector space V, with a decomposition $V = \oplus_{p,q\in\mathbb{Z}} V^{p,q}$. Writing $h^{p,q}$ for $\dim V^{p,q}$, we define the *gamma factor* attached to V by

This terminology might be slightly unconventional, but we follow Serre [Se70].

Recall: $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$.

$$\Gamma_V(s) = \prod_{p,q\in\mathbb{Z}} \Gamma_{\mathbb{C}}(s - \min\{p,q\}))^{h(p,q)}.$$

An $\mathbb{R}$-Hodge structure is a finite-dimensional $\mathbb{C}$-vector space V, with a decomposition $V = \oplus_{p,q\in\mathbb{Z}} V^{p,q}$, and an automorphism σ of V, such that $\sigma^2 = 1$ and $\sigma(V^{p,q}) = V^{q,p}$, for all $p, q \in \mathbb{Z}$. Again, we write $h(p,q)$ for $\dim V^{p,q}$.

Just like Galois representations, $\mathbb{R}$-Hodge structures are representations of a group scheme: the *Deligne torus* $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$.

If V is an $\mathbb{R}$-Hodge structure, defining its gamma factor requires a bit more work. As before, we have a factor $\prod_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h(p,q)}$. The space $V^{p,p}$ is fixed by σ, and as such gives an $(-1)^p$-eigenspace $V^{p,+}$, and a $(-1)^{p+1}$-eigenspace $V^{p,-}$. Put $h(p,+) = \dim V^{p,+}$, and $h(p,-) = \dim V^{p,-}$, so that $h(p,p) = h(p,+) + h(p,-)$. The contribution from these subspaces is defined to be $\Gamma_{\mathbb{R}}(s-p)^{h(p,+)}\Gamma_{\mathbb{R}}(s+1-p)^{h(p,-)}$. Together, this gives the *gamma factor*

Define: $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$.

$$\Gamma_V(s) = \prod_p \Gamma_{\mathbb{R}}(s-p)^{h(p,+)}\Gamma_{\mathbb{R}}(s+1-p)^{h(p,-)} \prod_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h(p,q)}.$$

6 Chow motives

Algebraic cycles and intersection theory

Let k be a field, and let X be a k-scheme, projective, geometrically integral, and smooth of dimension n. A *prime cycle* is a closed integral subscheme of X over k. The *codimension* of a prime cycle $Z \subset X$ is defined as $\dim X - \dim Z$, and denoted $\operatorname{codim} Z$. Write $\mathcal{Z}^r(X)$ for the free abelian group generated by the prime cycles of codimension r. An element of $\mathcal{Z}^\bullet(X) = \oplus_{r\in\mathbb{Z}} \mathcal{Z}^r(X)$ is called an *algebraic cycle*.

A prime cycle may be identified with its generic point; and as such, algebraic cycles are formal sums of (not necessarily closed) points of X.

For two prime cycles Z_1 and Z_2, and an irreducible component W of $Z_1 \cap Z_2$, we always have

$$\operatorname{codim} W \leq \operatorname{codim} Z_1 + \operatorname{codim} Z_2.$$

The cycles Z_1 and Z_2 are said to *intersect properly* if equality holds for all irreducible components W. Two algebraic cycles $\gamma_1, \gamma_2 \in Z^\bullet(X)$ *intersect properly* if all prime cycles occurring in γ_1 intersect properly with all prime cycles occurring in γ_2.

We now want to define the intersection product of two properly intersecting cycles. The naive set-theoretic intersection does not take multiplicity into account. A correct (but admittedly, pretty dense) definition is the following, due to Serre. Let Z_1 and Z_2 be two prime cycles on X, that intersect properly. Let W be an irreducible component of $Z_1 \cap Z_2$. Let w be the generic point of W, and let R denote the stalk $\mathcal{O}_{X,w}$. Let $\mathfrak{a}_1$ be the ideal of R corresponding to Z_1, and $\mathfrak{a}_2$ the ideal of Z_2. The *intersection multiplicity* of Z_1 and Z_2 along W is defined as

For more information we refer to [Se65], where Serre (among other things) proves that this is a non-negative integer.

$$\sum_{i=0}^{\infty} (-1)^i \operatorname{length} \operatorname{Tor}_i^R(R/\mathfrak{a}_1, R/\mathfrak{a}_2),$$

and is denoted by $i(Z_1, Z_2; W)$.

To deal with the fact that cycles do not intersect properly, we need to be able to move them around.

6.1 Definition. Let γ_1 and γ_2 be two algebraic cycles on X. We say that γ_1 is *rationally equivalent* to γ_2 if there exists an algebraic cycle V on $X_{\mathbb{P}^1}$, all whose components are flat over $\mathbb{P}^1$, such that $\gamma_1 - \gamma_2 = (V \cap X_0) - (V \cap X_\infty)$, where the intersection $\cap$ is in the cycle-theoretic sense, as defined above. «

We identify the fibres X_0 and X_∞ with X, so that the equality makes sense.

The subset of rationally trivial cycles forms a subgroup, and we write $\mathrm{CH}^r(X)$ for the quotient with $\mathbb{Q}$-coefficients, i.e., $(\mathcal{Z}^r(X)/\sim) \otimes_{\mathbb{Z}} \mathbb{Q}$. We call $\mathrm{CH}^r(X)$ the *r-th Chow group of X*.

6.2 Lemma. *(Chow's moving lemma) Let $[\gamma_1] \in \mathrm{CH}^r(X)$ and $[\gamma_2] \in \mathrm{CH}^s(X)$ be given. Then there exists representatives in $\mathcal{Z}^r(X)$ and $\mathcal{Z}^s(X)$ that intersect properly.*

Proof. Murre, Nagel, and Peters [MNP13, §1.2] give a sketch of a proof by Roberts [Ro72]. □

Together with lemma 6.2 this enables us to define a product structure on $\mathrm{CH}^\bullet(X) = \oplus_{r=0}^n \mathrm{CH}^r(X)$, which on graded components is given by

$$\begin{aligned} \mathrm{CH}^r(X) \times \mathrm{CH}^s(X) &\to \mathrm{CH}^{r+s}(X) \\ ([\gamma_1], [\gamma_2]) &\mapsto [\gamma_1 \cap \gamma_2]. \end{aligned}$$

By lemma 6.2 we may assume that γ_1 and γ_2 intersect properly, and we obtain a well-defined product. This is clearly commutative, and we argue below that it is also associative. We call $\mathrm{CH}^\bullet(X)$ the *Chow ring of X*, and will denote the intersection product with $\cdot$ instead of $\cap$.

Let $f \colon X \to Y$ be a morphism of smooth projective geometrically integral schemes over k. We associate to f two maps f_* and f^*. Let γ be a prime cycle on X. If $\dim f(\gamma) < \dim \gamma$, we put $f_*(\gamma) = 0$; if $\dim f(\gamma) = \dim \gamma$, then the function field $K(\gamma)$ is a finite extension of $K(f(\gamma))$ and we put

$$f_*(\gamma) = [K(\gamma) : K(f(\gamma))] \cdot f(\gamma).$$

By linearity, this extends to a linear map $f_* \colon \mathcal{Z}(X) \to \mathcal{Z}(Y)$. It turns out that f_* respects rational equivalence, so that we also obtain a linear map $f_* \colon \mathrm{CH}(X) \to \mathrm{CH}(Y)$. We stress that this is in general not a ring map.

For the definition of f^*, first observe that $X \times Y$ is smooth and projective. Let pr_1 and pr_2 be the projections from $X \times Y$ to X and Y respectively. Let Γ_f be the graph of f, viewed as cycle on $X \times Y$. We then define the ring map

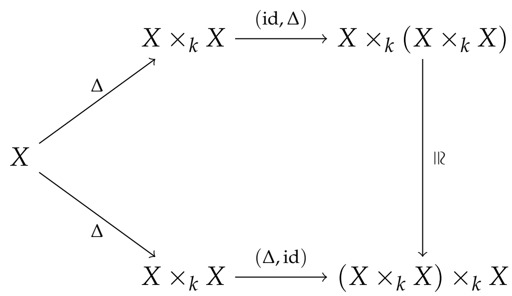
$$\begin{aligned} f^* \colon \mathrm{CH}(Y) &\to \mathrm{CH}(X) \\ \gamma &\mapsto \mathrm{pr}_{1*}(\Gamma_f \cdot \mathrm{pr}_2^{-1}(\gamma)). \end{aligned}$$

We now observe that the intersection product is associative, since one can show that it is equal to the composition [Fu98, §8.1]

$$\mathrm{CH}^r(X) \times \mathrm{CH}^s(X) \xrightarrow{\times} \mathrm{CH}^{r+s}(X \times X) \xrightarrow{\Delta^*} \mathrm{CH}^{r+s}(X)$$

where Δ is the diagonal map $X \to X \times X$.

Indeed, the commutative diagram



implies that for $\alpha, \beta, \gamma \in \mathrm{CH}^\bullet(X)$

$$\alpha \cdot (\beta \cdot \gamma) = \Delta^*(\alpha \times \Delta^*(\beta \times \gamma) = \Delta^*(\Delta^*(\alpha \times \beta) \times \gamma) = (\alpha \cdot \beta) \cdot \gamma.$$

Chow motives

Denote with $\mathcal{V}(k)$ the category of smooth projective geometrically integral schemes over k. Let X and Y be two such schemes. A *correspondence* from Y to X of degree r is a cycle in $\mathrm{CH}^{\dim Y + r}(Y \times_k X)$. Correspondences are composed via the rule

Excellent introductions to Chow motives are given in [An04, MNP13, Sc94].

$$\mathrm{CH}^{\dim Z + r}(Z \times_k Y) \times \mathrm{CH}^{\dim Y + s}(Y \times_k X) \to \mathrm{CH}^{\dim Z + r + s}(Z \times_k X)$$
$$(g, f) \mapsto \mathrm{pr}_{ZX,*}(\mathrm{pr}_{ZY}^*(g) \cdot \mathrm{pr}_{YX}^*(f)).$$

The preadditive category of correspondences $\mathcal{C}(k)$ has as objects smooth projective schemes over k, and as morphisms correspondences of degree 0. There is a natural functor $c \colon \mathcal{V}(k)^{\mathrm{opp}} \to \mathcal{C}(k)$, sending a morphism of schemes to the transpose of its graph.

We proceed to the category of Chow motives in two steps. First we formally add the kernels of all idempotent endomorphisms in $\mathcal{C}(k)$ to obtain a category $\mathcal{M}_{\mathrm{eff}}(k)$ of effective motives. Since p is idempotent if and only if $\mathrm{id} - p$ is idempotent, all idempotent endomorphisms in $\mathcal{M}_{\mathrm{eff}}(k)$ also have images (in the categorical sense). We denote objects of this category as pairs (X, p), where $p \colon c(X) \to c(X)$ satisfies $p \circ p = p$. Note that (X, p) is not the kernel, but the image, of the endomorphism p. A morphism $(X, p) \to (Y, q)$ is a correspondence $f \colon c(X) \to (Y)$ of degree 0, such that $f \circ p = f = q \circ f$. If

(X, p) and (Y, q) are effective motives, then $(X \sqcup Y, p \sqcup q)$ is their biproduct. Moreover

$$(X, p) \otimes (Y, q) = (X \times Y, p \times q)$$

defines a monoidal structure, with unit $\mathbb{1} = (\operatorname{Spec} k, \Gamma_{\mathrm{id}}^{\mathrm{t}})$. (Strictly speaking, one should also give diagrams for the commutativity and associativity of this tensor structure. However, we will not do that here.)

For the next step, we decompose the effective motive $(\mathbb{P}_k^1, \Gamma_{\mathrm{id}}^{\mathrm{t}})$. Let $x \in \mathbb{P}_k^1(k)$ be a rational point, and denote with $f\colon \mathbb{P}_k^1 \to \operatorname{Spec} k$ the structure morphism. Then $f \circ x$ is the identity on k, and $x \circ f$ is an idempotent morphism, whose transposed graph we will denote with p. It follows that $\mathbb{1}$ and $(\mathbb{P}_k^1, p)$ are isomorphic as effective motives (since $p \circ x = x$). Now $p + p^t$ is rationally equivalent to the diagonal $\Delta = \Gamma_{\mathrm{id}}^t$. Therefore $(\mathbb{P}_k^1, \mathrm{id})$ decomposes as $(\mathbb{P}_k^1, p) \oplus (\mathbb{P}_k^1, p^t)$. The summand $(\mathbb{P}_k^1, p^t)$ is called the *Lefschetz motive*, $\mathbb{L}$. We remark that $\mathbb{L}$ does not depend on the choice of x, since all rational points of $\mathbb{P}^1$ are rationally equivalent.

Without loss of generality, assume $x = \infty$. Now take the algebraic cycle on $(\mathbb{P}_k^1 \times \mathbb{P}_k^1) \times \mathbb{P}_k^1$, given by $y = x + c$, for $c \neq \infty$, and $\mathbb{P}_k^1 \times \infty + \infty \times \mathbb{P}_k^1$ for the fibre at $c = \infty$.

We form the category $\mathcal{M}(k)$ of *Chow motives* by formally adjoining an inverse $\mathbb{L}^{-1}$ for the tensor product. The objects are triples (X, p, m), where m is an integer. A morphism $(X, p, m) \to (Y, q, n)$ is a correspondence $f\colon X \to Y$ of degree $n - m$, such that $f \circ p = f = q \circ f$. We write $h\colon \mathcal{V}(k)^{\mathrm{opp}} \to \mathcal{M}(k)$ for the functor sending X to $(X, \mathrm{id}, 0)$, and morphisms to the transpose of their graph.

The Lefschetz motive $\mathbb{L}$ is isomorphic to $(\operatorname{Spec} k, \mathrm{id}, -1)$, which can be seen by looking at the graphs of x and the structure morphism (as opposed to the transposed graphs). If M is a Chow motive, it is common to write $M(n)$ for $M \otimes \mathbb{L}^{\otimes -n}$. By definition of the hom-sets in $\mathcal{M}(k)$ we have $\mathrm{CH}^i(X) = \mathrm{Hom}(\mathbb{1}, h(X)(i))$, and in general the i-th Chow group of a motive M is defined as $\mathrm{CH}^i(M) = \mathrm{Hom}(\mathbb{1}, M(i))$. For any smooth projective geometrically integral scheme X/k of dimension d, the choice of a rational point $e \in X(k)$ (or even a cycle in $\mathrm{CH}^d(X)$ of degree 1) induces projectors p and p^t as in the case of $\mathbb{P}^1$, and one shows that $(X, p, 0) \cong \mathbb{1}$, and $(X, p^t, 0) \cong \mathbb{L}^{\otimes d}$. Assume X is a curve, i. e., $d = 1$. In general $p + p^t$ is not rationally equivalent to the diagonal, which gives rise to another idempotent $\Delta - p - p^t$, cutting out a motive $h^1(X)$. In this way (for curves), we arrive at a decomposition $h(X) = h^0(X) \oplus h^1(X) \oplus h^2(X)$, with $h^0(X) \cong \mathbb{1}$ and $h^2(X) \cong \mathbb{L}$. Moreover, for $i \neq 1$, the Chow group $\mathrm{CH}^i(h^1(X))$ is trivial. In light of the decomposition of $h(\mathbb{P}_k^1)$ computed above, we have $h^1(\mathbb{P}^1) = 0$.

See [Sc94, Mu90] for more details about this *Künneth decomposition*. Such a decomposition is also known for e. g., the motive of a surface and the motive of a Jacobian variety, but (though evident on the side of cohomology) remains a conjecture in general.

Tate twists

As noted above, if M is a motive, we write $M(n)$ for $M \otimes \mathbb{L}^{-n}$. This notation is not only used for motives, but also for ℓ-adic representations and Hodge structures. Indeed, in the ℓ-adic setting one defines $\mathbb{Q}_\ell(-1)$ to be the cyclotomic representation, which is isomorphic to $\mathrm{H}^2_{\text{ét}}(\mathbb{P}^1_{\overline{\mathbb{Q}_\ell}}, \mathbb{Q}_\ell)$. We write $\mathbb{Q}_\ell(n)$ for $\mathbb{Q}_\ell(-1)^{\otimes -n}$. If V is a ℓ-adic represention, then $V(n)$ denotes $V \otimes \mathbb{Q}(n)$.

In the setting of Hodge structures, $\mathbb{Q}(-1)$ denotes the 1-dimensional Hodge structure $\mathbb{Q}$ of type $(1,1)$. Here, we have $\mathbb{Q}(-1) \cong \mathrm{H}^2(\mathbb{P}^1_{\mathbb{C}}, \mathbb{Q})$, and we write $\mathbb{Q}(n)$ for $\mathbb{Q}(-1)^{\otimes -n}$. Finally, if V is a Hodge structure, then $V(n)$ denotes $V \otimes \mathbb{Q}(n)$.

L-functions of Chow motives over number fields

Assume k is a number field. Let $M = (X, p, m)$ be a motive over k. For each complex embedding $\sigma\colon k \to \mathbb{C}$ we can associate Betti cohomology (with a Hodge structure) to $X \times_{k,\sigma} \operatorname{Spec} \mathbb{C}$. Further, there is the ℓ-adic cohomology of X. We want to generalise these cohomology groups to M, and we call them the *Hodge realisations* and *ℓ-adic realisations* of M.

Let H be one of these realisations, i. e., $\mathrm{H}(\_)$ denotes one of $\mathrm{H}(\_ \times_{k,\sigma} \mathbb{C}, \mathbb{Q})$ or $\mathrm{H}_{\text{ét}}(\_, \mathbb{Q}_\ell)$. For each integer i, and for all $X \in \mathcal{V}(k)$ there is a cycle map $c_X\colon \mathrm{CH}^i(X) \to \mathrm{H}^{2i}(X)(i)$. For a motive $M = (X, p, m)$ there are induced idempotent endomorphisms

$$\begin{aligned} p_{*,i}\colon \mathrm{H}^i(X) &\to \mathrm{H}^i(X) \\ \alpha &\mapsto \mathrm{pr}_{2,*}(c_{X\times X}(p) \cup \mathrm{pr}_1^*(\alpha)), \end{aligned}$$

with pr_j the projection of $X \times X$ to the j-th factor. We extend the cohomology to $\mathcal{M}(k)$ by

$$\mathrm{H}^i(M) = \underbrace{\mathrm{im}(p_{*,i+2m})}_{\subset \mathrm{H}^{i+2m}(X)} \otimes \mathbb{Q}(m).$$

This is indeed an extension, since $\mathrm{H}^i(h(X)) = \mathrm{H}^i(X)$. As a reality check, note that $\mathrm{H}^2(\operatorname{Spec} k, \mathrm{id}, -1) = \mathrm{H}^0(\mathbb{1}) \otimes \mathbb{Q}(1)$. In general, for a curve X, one has $\mathrm{H}^i(h^i(X)) = \mathrm{H}^i(X)$, using the decomposition $h(X) = h^0(X) \oplus h^1(X) \oplus h^2(X)$.

Fix a motive M, and an integer n. We proceed with the definition of the L-function $L(M, n, s)$. To do so, assume the following conjecture, which is a generalisation of [Se70, hypothesis H_ρ].

6.3 Conjecture. *Let v be a non-archimedean place of k, and let ℓ be a prime not lying under v. There exists a finite index subgroup $I' \subset I_v$ of the inertia group $I_v \subset G_k$, so that I' acts on $\mathrm{H}^n_\ell(M_{k^s}, \mathbb{Q}_\ell)$ by unipotent automorphisms.* «

As in the case of abelian varieties, one associates to M a conductor $\mathfrak{f} = \sum_v f(v, \mathrm{H}^n_{\text{ét}}(M_{k^s}, \mathbb{Q}_\ell)) \cdot v$, where the sum ranges over the finite places of k. We now want to associate local zeta factors to M for every finite place v of k. We have an induced action of Frobenius on the invariants of the inertia group

In contrast to the case of abelian varieties, it is not known (but conjectured) that this sum is finite.

$I \subset \mathrm{Gal}(l/k_v)$, and we put

$$P_v(T) = \det(1 - \pi T | \, \mathrm{H}^n_{\text{ét}}(M_{k^s}, \mathbb{Q}_\ell)^I),$$
$$\zeta_v(s) = \frac{1}{1 - P_v(q_v^{-s})},$$

where q_v is the number of elements of the residue field of v. It is conjectured that $P_v(T)$ (and a fortiori $\zeta_v(s)$) does not depend on the choice of prime $\ell \nmid q_v$.

For a complex embedding $\sigma\colon k \to \mathbb{C}$, such that $\sigma(k)$ is not contained in $\mathbb{R}$, we obtain a $\mathbb{C}$-Hodge structure $\mathrm{H}^n_\sigma(M_{\sigma,\mathbb{C}}, \mathbb{C})$, and a corresponding gamma factor, in the sense of section 5. If $\overline{\sigma}$ denotes the conjugate embedding, then $\mathrm{H}^n_{\overline{\sigma}}(M_{\overline{\sigma},\mathbb{C}}, \mathbb{C})$ gives the same gamma factor. Thus we may attach a gamma factor $\Gamma_v(s)$ to M, n and a complex place v.

If $\sigma\colon k \to \mathbb{C}$ is a real embedding, then we obtain an $\mathbb{R}$-Hodge structure on $\mathrm{H}^n_\sigma(M_{\sigma,\mathbb{C}}, \mathbb{C})$, via the $\mathbb{R}$-automorphism $M_{\sigma,\mathbb{C}} \to M_{\sigma,\mathbb{C}}$ induced by complex conjugation. So to a real place v, M, and n, we also associate a gamma factor $\Gamma_v(s)$, using the definition of section 5.

Let B_n denote the dimension of the n-th Betti number, $\dim \mathrm{H}^n_\sigma(M)$, of the motive M (which does not depend on the complex embedding σ). Put $C = N(\mathfrak{f}) \cdot |d_{k/\mathbb{Q}}|^{B_n}$, and define the L-function of M and n to be

$$L(M, n, s) = C^{s/2} \prod_{v<\infty} \zeta_v(s) \prod_{v|\infty} \Gamma_v(s).$$

This product should converge on $\{s \in \mathbb{C} | \Re(s) > n/2 + 1\}$. Conjecturally it has a meromorphic continuation to $\mathbb{C}$, and satisfies a functional equation

$$L(M, n, s) = \varepsilon \cdot L(M, n, n + 1 - s), \qquad \varepsilon \in \mathbb{C}^*.$$

Conjectures on algebraic cycles and L-functions

A central landmark in the theory on algebraic cycles and L-functions is due to Bloch [Bl84], which he calls a *recurring fantasy*. It states that the dimension of the kernel of the cycle map $\mathrm{CH}^i(X) \to \mathrm{H}^{2i}(X)(i)$ is equal to the order of vanishing at $s = i$ of the L-function, $L(\mathrm{H}^{2i-1}(X), s)$, attached to $\mathrm{H}^{2i-1}(X)$. When we try to cast this in the language of Chow motives, we first extend the cycle map to Chow motives. The cycle maps extend to motives, in the sense that for a motive M and an integer i, there are cycle maps $c\colon\ \mathrm{CH}^i(M) \to \mathrm{H}^{2i}(M)(i)$ for all suitable cohomology theories H. Conjecturally, the kernel of these cycle maps is independent of the choice of H, and we denote it with $\mathrm{CH}^i(M)^0$.

A *suitable* cohomology theory is a *Weil cohomology theory*, see e. g., [Ano4]. As before, we are interested in étale, de Rham, and Betti cohomology, which are examples of Weil cohomology theories.

6.4 Conjecture. *Let X be a projective smooth geometrically integral scheme over a number field k, and let i be an integer. Then* $\dim \mathrm{CH}^i(h(X))^0$ *and* $\mathrm{ord}_{s=i} L(h^{2i-1}(X), 2i - 1, s)$ *are finite and equal.* «

The problem with this conjecture is the motive $h^{2i-1}(X)$, of which we do not know the existence in general (though we do when $i = 1$, or $i = \dim X$).

It is conjectured by Murre that such Künneth projectors always exist in $\mathrm{Corr}^0(X, X) = \mathrm{End}_{\mathcal{M}(k)}(X)$. An even stronger conjecture then conjecture 6.4 is the following.

6.5 Conjecture (Motivic friend of recurring fantasy). *Let k be a number field, and let M be a Chow motive over k, and let i be an integer. Write $n = 2i - 1$. Then $\dim \mathrm{CH}^i(M)^0$ and $\mathrm{ord}_{s=i} L(M, n, s)$ are finite and equal.* «

To understand the strength of this conjecture, let X be a surface over k, take $M = h^2(X)$ and $i = 2$. (Note that $h^2(X)$ exists, by work of Murre [Mu90].) We have $n = 3$ in the above conjecture, and by construction $L(h^2(X), 3, s) = 1$. The conjecture then says that $\mathrm{CH}^2(h^2(X))$ is trivial. However, Murre [Mu90] shows that $\mathrm{CH}^2(h^2(X))$ consists of those homologically trivial cycles that are in the kernel of the Abel–Jacobi map. Thus, the conjecture implies that over a number field the Abel–Jacobi map is injective (at least up to torsion, but the general statement follows).

Observe that conjecture 4.1 is a special case of conjectures 6.4 and 6.5. We have $X = E$, $i = 1$, and indeed $\mathrm{CH}^1(E)^0$ is precisely the Mordell–Weil group.

The Mordell–Weil group $E(k)$ is $\mathrm{Pic}^0_{E/k}(k)$, which are the degree-0 cycles in $\mathrm{CH}^1(E)$. The homology class only sees the degree, so this is the homologically trivial part.

I am not exactly confident whether conjecture 6.5 is equivalent to the so called Beilinson–Bloch conjecture. The formulations that I have seen are either remarkably close to conjecture 6.4 or they involve motivic cohomology and extensions of motives, of which I do not know anything. I have the impression that these conjectures are not formulated for Chow motives, but other categories of motives. The interested reader is referred to [Ne94] for a good introduction to this theory and conjectures.

Further I want to stress that all conjectures in this section are nonsense when one removes the assumption that k is a number field. For an example, let us give a small preview of section 8. If X be a curve, then we will exhibit a cycle in $\mathrm{CH}^2(h^2(X^2))$. There exist curves for which this cycle is non-trivial, by work of Green and Griffiths [GG03]. Moreover, Mumford proved that on a surface the kernel of the Abel–Jacobi map may even have infinite dimension. This indicates that in conjecture 6.5 even the finiteness of $\dim \mathrm{CH}^i(M)^0$ is a very strong conjecture.

7 The Chow motive of the triple product of a curve

Let k be a field, and X/k a smooth, projective, geometrically integral curve. After fixing a degree 1 cycle $e \in \mathrm{CH}^1(X)$, we have a decomposition,

The results of this section do not depend on the conjectures in section 6; indeed the ground field is arbitrary.

$$h(X) = h^0(X) \oplus h^1(X) \oplus h^2(X),$$

of the Chow motive of X, depending on e. This provides us with a Künneth decomposition of the Chow motive $h(X^3)$, of the triple self-product of X, namely

$$
\begin{aligned}
h(X^3) &= h(X)^{\otimes 3} = \bigoplus_{i=0}^{6} h^i(X^3), \quad \text{where} \\
h^i(X^3) &= \bigoplus_{a+b+c=i} h^a(X) \otimes h^b(X) \otimes h^c(X).
\end{aligned} \tag{7.1}
$$

It is a natural question to ask how the Chow groups of X^3 relate with this decomposition. Since the Chow groups are actually hom-sets in the category $\mathcal{M}_k$, we get a similar decomposition of Chow groups. We would like to know which of them are 0. I. e., we would like to fill out the table

	$h^0(X^3)$	$h^1(X^3)$	$h^2(X^3)$	$h^3(X^3)$	$h^4(X^3)$	$h^5(X^3)$	$h^6(X^3)$
CH^0							
CH^1							
CH^2							
CH^3							

To complete this job, we first compute a similar table for the curve X. It looks like

	$h^0(X)$	$h^1(X)$	$h^2(X)$
CH^0	$\mathbb{Q}$	0	0
CH^1	0	$\mathrm{CH}^1(X)^0$	$\mathbb{Q}$

In particular

$$\mathrm{CH}^1(X) = \mathrm{CH}^1(h^1(X) \oplus h^2(X)) \to \mathrm{CH}^1(h^2(X)) \cong \mathbb{Q}$$

is the degree map. Using the fact that $\mathrm{Hom}(\mathbb{L}^{\otimes i}, \mathbb{L}^{\otimes j}) = 0$ if $i \neq j$, we can now fill out quite a part of the table for X^3.

	$h^0(X^3)$	$h^1(X^3)$	$h^2(X^3)$	$h^3(X^3)$	$h^4(X^3)$	$h^5(X^3)$	$h^6(X^3)$
CH^0	$\mathbb{Q}$	0	0	0	0	0	0
CH^1	0	$*$	$*$	?	0	0	0
CH^2	0	0	$+/?$	$+$	$*$	0	0
CH^3	0	0	?	?	?	$\triangle$	$\mathbb{Q}$

(Here $*$ means that there are certainly curves X for which this is non-zero. The $\triangle$ indicates that it is not yet obvious from the previous what this group is. A $+$ means that we will exhibit a cycle in this group, and give situations in which it is non-trivial. A ? means that conjecturally (6.5) these groups are trivial when the ground field k is a number field.)

It turns out that $\triangle$ consists of homologically trivial cycles of codimension 3, modulo the kernel of the Abel–Jacobi map, [Mu90].

It may be very enlightening for the reader to study the table for a surface, which may for example be found in [Sc94, §4.6]. This table is due to

Murre [Mu90], and he showed that $\mathrm{CH}^3(h^5(X^3))$ is isomorphic to the quotient $\mathrm{CH}^3(X^3)^0/\ker(\mathrm{Alb}(X^3))$, where Alb denotes the map to the Albanese variety.

We will not go into the theory of the Albanese variety here, but note that it is dual to the Picard variety.

In the remaining sections of this thesis we will focus on the two $+$'s in the table. We will exhibit algebraic cycles in $\mathrm{CH}^2(h^2(X^3))$ and $\mathrm{CH}^2(h^3(X^3))$ that are known to be non-trivial for certain curves X. Before we do so, we exploit the action of S_n on X^n. For each $\sigma \in S_n$, there is a morphism

$$\begin{aligned} X^n &\to X^n \\ (x_i)_i &\mapsto (x_{\sigma(i)})_i. \end{aligned}$$

Denote the graph of this map with Γ_σ, and its transpose with Γ^t_σ. The element $\frac{1}{n!}\sum_{\sigma\in S_n}\sigma$ in the group ring $\mathbb{Q}[S_n]$ is idempotent, and therefore $\pi_n = \frac{1}{n!}\sum_{\sigma\in S_n}\Gamma^t_\sigma$ is an idempotent correspondence $X^n \to X^n$. We conclude that $\mathrm{Sym}^n h(X) = (X^n, \pi_n, 0)$ is a Chow motive. We now specialise to $n = 3$.

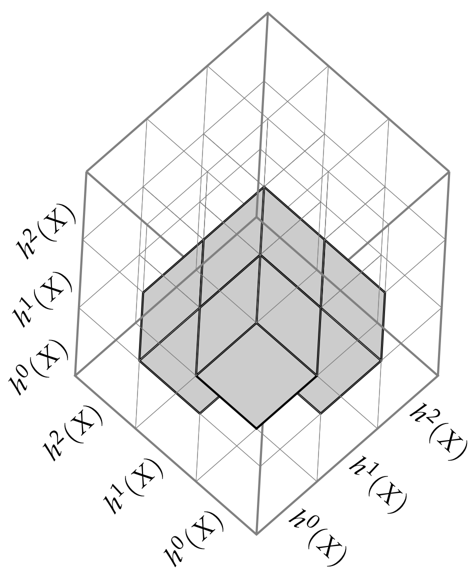


Figure 1: A graphical presentation of the Chow motive $h(X^3)$. Each cube represents one component in the Künneth decomposition (7.1). The vertical level represents the weight, e. g., the cube at the bottom is $h^0(X^3)$, and the topmost cube is $h^6(X^3)$. Identifying orbits under the action of S_3 on the axes, one finds the decomposition (7.2), corresponding to $\mathrm{Sym}^3 h(X)$. In this decomposition the shaded cubes represent $\mathrm{Sym}^2 h^1(X) \oplus \mathrm{Sym}^3 h^1(X)$.

This motive naturally decomposes as

$$\begin{aligned} \mathrm{Sym}^3 h(X) = \mathbb{1} \oplus h^1(X) \oplus \mathbb{L} \oplus \mathrm{Sym}^2 h^1(X) \oplus h^1(X)(-1) \oplus \mathrm{Sym}^3 h^1(X) \\ \oplus \mathrm{Sym}^2 h^1(X)(-1) \oplus \mathbb{L}^{\otimes 2} \oplus h^1(X)(-2) \oplus \mathbb{L}^{\otimes 3}. \quad (7.2) \end{aligned}$$

We can decompose $\mathrm{Sym}^3 h^1(X)$ even further into a primitive part and a non-primitive part. We do this, by considering a correspondence $X \to X^3$. Let f be the image of

$$\begin{aligned} X^2 &\to X \times_k X^3 \\ (x,y) &\mapsto (x) \times (x,y,y). \end{aligned}$$

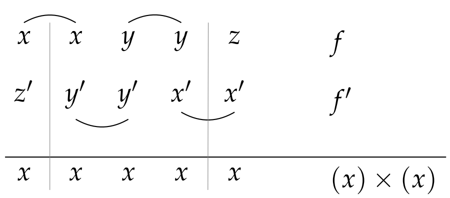
Essentially, this is because $h^1(X) \cong h^1(\mathrm{Jac}\, X)$, and $\mathrm{Sym}^3 h^1(\mathrm{Jac}\, X) \cong h^3(\mathrm{Jac}\, X)$. Then one uses the Lefschetz decomposition, which is known for Jacobian varieties, by Künnemann [Kü93]. Also see [Sc94, §5] or [Kü94].

Observe that f is a degree 1 correspondence from X to X^3, and thus a morphism $h(X)(-1) \to h(X^3)$. Similarly, define f' to be the degree 1 correspondence $X^3 \to X$ that is the image of

$$\begin{aligned} X^2 &\to X^3 \times_k X \\ (x,y) &\mapsto (y,y,x) \times (x). \end{aligned}$$

This is a morphism $h(X^3) \to h(X)(-1)$. We claim that $f' \circ f = \mathrm{id}$, so that $f \circ f'$ is an idempotent correspondence from $h(X^3)$ to itself, cutting out a motive isomorphic to $h(X)(-1)$. Indeed the following computation shows that $f' \circ f = \mathrm{id}$.

See appendix C for an explanation of these diagrams.

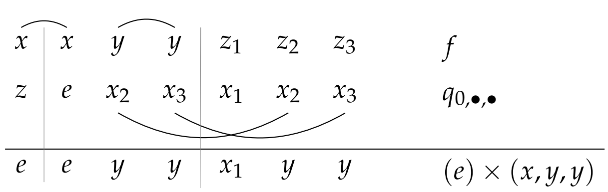


Recall that the projector of $\mathrm{Sym}^3 h(X)$ is denoted with π_3. Denote the projector of $h^i(X)$ with p_i, and the projector of $h^a(X) \otimes h^b(X) \otimes h^c(X)$ with $q_{a,b,c}$. (Here a, b, c, and i are taken from the set $\{0, 1, 2, \bullet\}$, with the understanding that $h^\bullet(X) = h(X)$.) Then the projector of $\mathrm{Sym}^3 h^1(X)$ is given by $q_{1,1,1} \circ \pi_3$. Observe that $g = q_{1,1,1} \circ \pi_3 \circ f \circ p_1$ is a morphism $h^1(X)(-1) \to \mathrm{Sym}^3 h^1(X)$. Write g' for $p_1 \circ f' \circ \pi_3 \circ q_{1,1,1}$. We claim that $g' \circ g$ is a non-zero multiple of $\mathrm{id}_{h^1(X)(-1)}$. Note that

We leave it to the reader to verify that $q_{1,1,1} \circ \pi_3 = \pi_3 \circ q_{1,1,1}$.

$$g' \circ g = p_1 \circ f' \circ \pi_3 \circ q_{1,1,1} \circ f \circ p_1.$$

We continue with the computation of $q_{1,1,1} \circ f$. Essential are the following four computations.



$$\begin{array}{c|ccc|ccc l}
x & x & y & y & z_1 & z_2 & z_3 & f \\
z & x_1 & x_2 & x_3 & e & x_2 & x_3 & q_{2,\bullet,\bullet} \\
\hline
x & x & y & y & e & y & y & (x) \times (e, y, y)
\end{array}$$

$$\begin{array}{c|ccc|ccc l}
x & x & y & y & z_1 & z_2 & z_3 & f \\
z & x_1 & e & x_3 & x_1 & x_2 & x_3 & q_{\bullet,0,\bullet} \\
\hline
x & x & e & e & x & x_2 & e & (x) \times (x, y, e)
\end{array}$$

$$\begin{array}{c|ccc|ccc l}
x & x & y & y & z_1 & z_2 & z_3 & f \\
z & x_1 & x_2 & x_3 & x_1 & e & x_3 & q_{\bullet,2,\bullet} \\
\hline
x & x & y & y & x & e & y & (x) \times (x, e, y)
\end{array}$$

By symmetry one finds $q_{\bullet,\bullet,0} \circ f = q_{\bullet,2,\bullet} \circ f$ and $q_{\bullet,\bullet,2} \circ f = q_{\bullet,0,\bullet} \circ f$. It is then immediate that

$$q_{\bullet,0,0} \circ f = q_{\bullet,0,\bullet} \circ q_{\bullet,\bullet,0} \circ f = 0$$
$$q_{\bullet,2,2} \circ f = q_{\bullet,2,\bullet} \circ q_{\bullet,\bullet,2} \circ f = 0.$$

Further, observe that

$$q_{0,\bullet,0} \circ f = q_{0,2,0} \circ f = q_{0,2,\bullet} \circ f, \qquad q_{0,\bullet,2} \circ f = q_{0,0,2} \circ f = q_{0,0,\bullet} \circ f,$$
$$q_{2,\bullet,0} \circ f = q_{2,2,0} \circ f = q_{2,2,\bullet} \circ f, \qquad q_{2,\bullet,2} \circ f = q_{2,0,2} \circ f = q_{2,0,\bullet} \circ f.$$

Notice that all these correspondences as cycles on $X \times X^3$ have an e in one of the first two coordinates, and an e in one of the last two coordinates; and the remaining two coordinates are independent. E. g., the three on the first line are computed below.

$$\begin{array}{c|ccc|ccc l}
e & x & y & y & z_1 & z_2 & z_3 & q_{0,\bullet,\bullet} \circ f \\
z & x_1 & x_2 & x_3 & x_1 & e & x_3 & q_{\bullet,2,\bullet} \\
\hline
e & x & y & y & x & e & y & (e) \times (x, e, y)
\end{array}$$

The reader may now check that for any of these correspondences, composing with $f' \circ \pi_3$ gives one of p_0, p_2, or 0. (The latter for dimension reasons.) Finally composing with p_1 will always give the 0-morphism. Another such computation kills the contribution of $q_{0,1,1}$ and $q_{2,1,1}$.

Since $q_{\bullet,2,2} \circ f = 0$, we have $q_{\bullet,2,1} \circ f = (q_{\bullet,2,\bullet} - q_{\bullet,2,0}) \circ f$. However, as observed above, $q_{\bullet,2,\bullet} \circ f = q_{\bullet,2,0} \circ f$. This kills the contribution of $q_{\bullet,2,1}$, and for symmetrical reasons, that of $q_{\bullet,1,2}$, $q_{\bullet,0,1}$, and $q_{\bullet,1,0}$. We are left with $q_{1,2,0}$, $q_{1,1,1}$, and $q_{1,0,2}$.

Next, observe that

$$q_{1,2,0} \circ f = q_{\bullet,2,0} \circ f, \qquad q_{1,0,2} \circ f = q_{\bullet,0,2} \circ f.$$

It is immediate that $\pi_3 \circ q_{1,2,0} = \pi_3 \circ q_{1,0,2}$. Further, one computes that $f' \circ \pi_3 \circ q_{1,2,0} = \frac{1}{3}(p_\bullet + p_0 + p_2)$. Composing this with p_1 gives $\frac{1}{3}p_1$. We conclude that

$$g' \circ g = (p_1 \circ f' \circ \pi_3 \circ f \circ p_1) - 2 \cdot \frac{1}{3} p_1.$$

A quick glance at the computation of $f' \circ f$ on page 25 reveals that $f' \circ \pi_3 \circ f$ is a multiple of id. This proves our claim that $g' \circ g$ is a non-zero multiple of $p_1 = \mathrm{id}_{h^1(X)(-1)}$. Thus, up to a invertible scalar $g \circ g'$ is idempotent, and we have a decomposition $\mathrm{Sym}^3 h^1(X) = M \oplus M'$, where $g \colon h^1(X)(-1) \to M'$ is an isomorphism by construction (with as inverse some non-zero multiple of g').

I have not proven that the motive M gives rise to the primitive part on the side of cohomology, but it is suggested in [Zh10, §5.1].

8 Explicit calculations on certain symmetric cycles

Faber–Pandharipande cycle

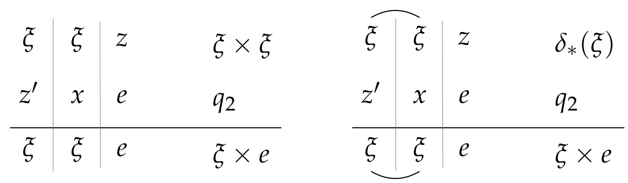
The results of this section do not depend on the conjectures in section 6; indeed, the ground field is arbitrary. Moreover, in light of said conjectures the first part of this section is trivial over a number field.

Let X/k be a curve of genus $g > 2$ over a field k. Let K be the canonical divisor, which has degree $2g - 2$, and write ξ for $K/(2g-2)$ in $\mathrm{CH}^1(X)$. Denote the diagonal embedding $X \to X \times X$ with δ. On $X \times X$ we have a cycle $Z = \xi \times \xi - \delta_*(\xi)$ in $\mathrm{CH}^2(X \times X)$, which was first considered by Faber and Pandharipande. This cycle is algebraically trivial, and conjecturally, if k is a number field, Z is rationally trivial. However, Green and Griffiths showed in [GG03] that Z is rationally non-trivial if X is generic and $g \geq 4$. Recently, Yin [Yi13] gave a short and elegant proof of this fact.

Using the decomposition $h(X) = h^0(X) \oplus h^1(X) \oplus h^2(X)$, induced by a cycle $e \in \mathrm{CH}^1(X)$ of degree 1, we obtain a decomposition

$$h(X \times X) = \left(h^0(X) \otimes h(X)\right) \oplus \left(h^1(X) \otimes h(X)\right) \oplus \left(h^2(X) \otimes h(X)\right).$$

If we write q_i for the projector of the term $h^i(X) \otimes h(X)$, it is easy to see that $q_0(Z) = 0$, because the intersection is empty (after moving e if necessary). On the other hand, a quick computation shows that $q_2(Z) = \xi \times e - \xi \times e = 0$. By symmetry we may thus conclude that Z can only have a non-zero component in $\mathrm{CH}^2(h^1(X) \otimes h^1(X))$.



Gross–Schoen cycles

Gross and Schoen define a modified diagonal on the triple product of a pointed curve, and show that it is homologically trivial. Let X/k be a curve of genus g. For every rational point $e \in X(k)$ put

$$\begin{aligned}
\Delta^e_{123} &= \{(x,x,x) : x \in X\}, \\
\Delta^e_{12} &= \{(x,x,e) : x \in X\}, \\
\Delta^e_{23} &= \{(e,x,x) : x \in X\}, \\
\Delta^e_{31} &= \{(x,e,x) : x \in X\}, \\
\Delta^e_{1} &= \{(x,e,e) : x \in X\}, \\
\Delta^e_{2} &= \{(e,x,e) : x \in X\}, \\
\Delta^e_{3} &= \{(e,e,x) : x \in X\}.
\end{aligned}$$

Observe that Δ^e_{123} is the diagonal, hence independent of e. We will also write Δ_{123} for the diagonal.

Define

$$\Delta_e = \Delta^e_{123} - (\Delta^e_{12} + \Delta^e_{23} + \Delta^e_{31}) + \Delta^e_1 + \Delta^e_2 + \Delta^e_3.$$

Gross and Schoen show that this cycle is homologically trivial (see [GS95]). They also show that there is a degree 1 correspondence from X to X^3 given by the cycle

$$\{\{e\} \times \Delta_e : e \in X\} \subset X \times X^3.$$

In particular, this gives a map of Chow motives $h(X)(-1) \to h(X^3)$, and therefore a map $\mathrm{CH}^1(X) \to \mathrm{CH}^2(X^3)$, which is the linear extension of $x \mapsto \Delta_x$.

As before, let K denote the canonical divisor of X, and write $\xi = K/(2g-2) \in \mathrm{CH}^1(X)$. The above map $\mathrm{CH}^1(X) \to \mathrm{CH}^2(X^3)$ associates a Gross–Schoen cycle $\Delta_{\mathrm{GS}} \in \mathrm{CH}^2(X^3)$ to ξ. We stress that if we write $K = \sum_{x \in X} n_x x$, with $n_x \in \mathbb{Z}$, then $\Delta_{\mathrm{GS}} = \frac{1}{2g-2} \sum_{x \in X} n_x \Delta_x$. Write $\mathrm{pr}_{123}, \mathrm{pr}_{12}, \ldots, \mathrm{pr}_3$ for the projections onto the corresponding components. Write $\delta \colon X \to X^2$ for the diagonal embedding. We see that

$$\Delta_{\mathrm{GS}} = \Delta_{123} - (\mathrm{pr}_{12}^*(\delta_*(X)) \cdot \mathrm{pr}_3^*(\xi) + \text{cyclic permutations}) + \mathrm{pr}_{12}^*(\delta_*(\xi)) + \mathrm{pr}_{13}^*(\delta_*(\xi)) + \mathrm{pr}_{23}^*(\delta_*(\xi)).$$

Zhang also defines a cycle on X^3, very similar to the one we considered above. Again, write $K = \sum_{x \in X} n_x x, n_x \in \mathbb{Z}$. Now he defines

$$\Delta_{\mathrm{Z}} = \Delta_{123} - (\mathrm{pr}_{12}^*(\delta_*(X)) \cdot \mathrm{pr}_3^*(\xi) + \text{cyclic permutations}) + (X \times \xi \times \xi) + (\xi \times X \times \xi) + (\xi \times \xi \times X).$$

We see that $\Delta_{\mathrm{Z}} - \Delta_{\mathrm{GS}}$ equals

$$((\xi \times \xi \times X) - \mathrm{pr}_{12}^*(\delta_*(\xi))) + \text{cyclic permutations},$$

or rather

$$\Delta_{\mathrm{Z}} - \Delta_{\mathrm{GS}} = \mathrm{pr}_{12}^*(Z) + \mathrm{pr}_{13}^*(Z) + \mathrm{pr}_{23}^*(Z)$$

where Z is the Faber–Pandharipande cycle introduced above. Let i_{12} be the inclusion $X^2 \to X^3, (x,y) \mapsto (x,y,e)$. Observe that $\pi_{12} \circ i_{12} = \mathrm{id}_{X^2}$. It follows that $i_{12}^* \pi_{12}^*(Z) = Z$. In particular the Δ_{Z} and Δ_{GS} coincide if and only if the Faber–Pandharipande cycle Z vanishes.

Recall the notation $q_{2,\bullet,\bullet}$ for the projector of $h^2(X) \otimes h(X) \otimes h(X)$ as submotive of $h(X)^{\otimes 3}$. The reader may compute that $q_{2,\bullet,\bullet}(\Delta_{\mathrm{GS}}) = 0$. (Indeed, intuitively, when projecting Δ_{GS} to a plane, all the partial diagonals cancel.) The table on page 23 shows that Δ_{GS} has no components in the CH^2 of $h^0(X) \oplus h^1(X)$ viewed as submotives of $\mathrm{Sym}^3 h(X)$.

We conclude that Δ_{GS} is contained in $\mathrm{CH}^2(\mathrm{Sym}^2 h^1(X) \oplus \mathrm{Sym}^3 h^1(X))$ (the shaded area of the cube in section 7). Since this is exactly where $\Delta_{\mathrm{Z}} - \Delta_{\mathrm{GS}}$ is located, we conclude that Δ_{Z} resides there as well.

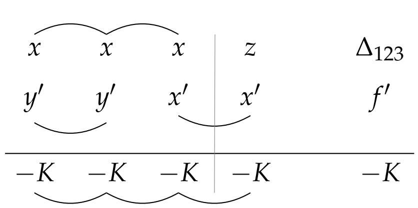
The reader may verify that both Δ_{Z} and Δ_{GS} have (in general) a non-zero component in $\mathrm{CH}^2(\mathrm{Sym}^2 h^1(X))$.

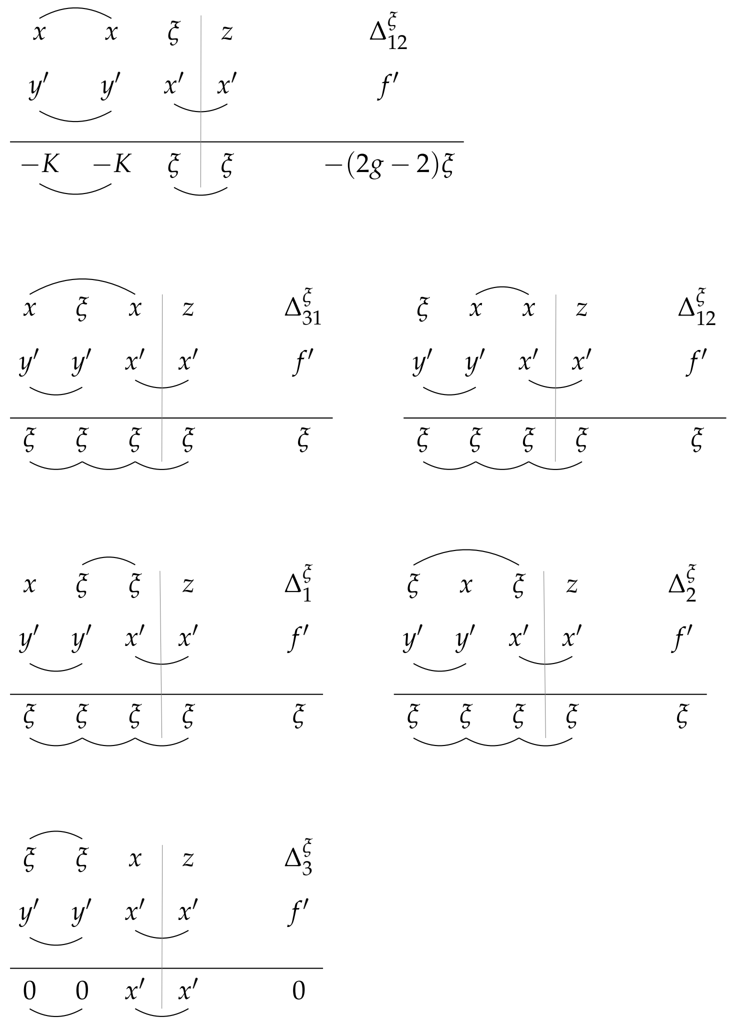
In the previous section, we gave a decomposition $\mathrm{Sym}^3 h^1(X) = M \oplus M'$, where $M' \cong h^1(X)(-1)$. Recall that the isomorphism $h^1(X)(-1) \to M'$ was induced by a correspondence f, defined as the image of

$$\begin{aligned} X^2 &\to X \times_k X^3 \\ (x,y) &\mapsto (x) \times (x,y,y). \end{aligned}$$

We claim that the projection of Δ_{GS} (and therefore Δ_{Z}) to $\mathrm{CH}^2(M')$ is trivial. We do this by computing the following intersections.

Recall that the self-intersection of the diagonal on $X \times X$ is $-K$.





We conclude that

$$f'(\Delta_{\mathrm{GS}}) = -K + (2g-2)\xi - \xi - \xi + \xi + \xi + 0 = 0.$$

Thus the projector of M' kills Δ_{GS}, and therefore Δ_{Z}. We summarise the above computations by saying that Δ_{GS} and Δ_{Z} are contained in $\mathrm{CH}^2(\mathrm{Sym}^2 h^1(X) \oplus M)$.

9 Application to the Fermat quartic

Denote with F_4 the smooth projective geometrically connected curve over $\mathbb{Q}$ defined by $x^4 + y^4 + z^4 = 0$. We call this curve the *Fermat quartic*. This curve, which has genus 3, is studied in some examples in [Li06], and the results there are of particular interest to us. Indeed observe that F_4 has good reduction at all odd primes. Further [Li06, p. 549] shows what kind of field extension is needed over $\mathbb{Q}_2$ to obtain a semistable model. Carrying out the computations we find a field extension $k/\mathbb{Q}$ with $[k:\mathbb{Q}] = 64$ such that F_4 has semistable reduction over k. However, this particular field is not of very much interest to us. The only important fact is that $\zeta_8 \in k$, and therefore

- k has no real places;
- $[k:\mathbb{Q}] = 4 \cdot [k:\mathbb{Q}(\zeta_8)]$, so k has an even number of complex places.

Write X for the curve $F_4 \times_{\mathbb{Q}} \operatorname{Spec} k$ over k. Let M be the primitive part of $\operatorname{Sym}^3 h^1(X)$, as in section 7. Throughout this section we assume conjecture 6.5. Hence $\mathrm{CH}^2(\operatorname{Sym}^2 h^1(X))$ is trivial, and in particular the Faber–Pandharipande cycle vanishes. Consequently we have $\Delta_Z \in \mathrm{CH}^2(M)$. We recall that Δ_Z is homologically trivial, so that it is actually contained in $\mathrm{CH}^2(M)^0$. With the following lemma we can compute the sign of the functional equation of $L(M,3,s)$.

9.1 Lemma. *Let v be a place of k. The local root factor ε_v in the functional equation of $L(M,3,s)$ is given by*

$$\varepsilon_v = \begin{cases} -1, & \text{if } v \text{ is complex,} \\ 1, & \text{if } v \text{ is real,} \\ (-1)^{e(e-1)(e-2)/6+3e}\tau^{(e-1)(e-2)/2+3}, & \text{if } v \text{ is finite.} \end{cases}$$

Here e is the rank of the first homology group of the reduction graph of X at v, and τ is the determinant of the action of Frobenius on the e-dimensional character group of the toric part of the reduction of the Jacobian $\operatorname{Jac}(X)$ *at v.*

Proof. Recall that the genus of F_4 is 3, and apply [Zh10, lem 5.2.1 and 5.2.2]. □

Since k has an even number of complex places, the total contribution of local root factors of infinite places is 1. If v is a finite place and does not lie above 2, then F_4 has good reduction at v, and therefore $\varepsilon_v = 1$. If v lies above 2, then [Li06, p. 549] shows us the semistable model of F_4 over k_v. It consists of three disjoint genus-1 curves that intersect one $\mathbb{P}^1$. Consequently the reduction graph is acyclic, and therefore the τ in lemma 9.1 is 1, and $e = 0$. Hence $\varepsilon_v = 1$. As a corollary, the sign of the functional equation of the L-function of M is 1.

9.2 Lemma. *The canonical Gross–Schoen cycle is rationally non-trivial on F_4.*

Proof. Bloch proves in [Bl84, thm 4.1] that a certain cycle (the Ceresa cycle) is rationally non-torsion on $\operatorname{Jac}(F_4)$. By [Zh10, thm 1.5.5] it follows that hence the canonical Gross–Schoen cycle is rationally non-trivial. □

As a consequence of conjecture 6.5, we get the following theorem.

9.3 Theorem. *Conjecture 6.5 implies that the dimension of* $\mathrm{CH}^2(M)^0$ *is strictly larger than* 1. *In particular there exists a homologically trivial algebraic cycle that is (in the Chow group) linearly independent of the canonical Gross–Schoen cycle.*

Proof. In this case conjecture 6.5 states that

$$\dim \mathrm{CH}^2(M)^0 = \mathrm{ord}_{s=2} L(M,3,s).$$

Since the sign of the functional equation of $L(M,3,s)$ is 1, the order of vanishing is even. As the canonical Gross–Schoen cycle is non-trivial, the result follows. □

We conclude with a final remark about the conjectures involved in theorem 9.3. We assume conjecture 6.5 to prove theorem 9.3. In fact, assuming Bloch's recurring fantasy is enough. The motive M is not a Künneth component of X^3, but it is obtained from the Künneth component $\mathrm{Sym}^3 h^1(X)$ (which, after all, is isomorphic to $h^3(\mathrm{Jac}\, X)$), by cutting away another Künneth component: $h^1(X)(-1)$. We can divide the L-function of $\mathrm{Sym}^3 h^1(X)$ by the L-function of $h^1(X)(-1)$, to obtain the L-function of M. Observe that $\mathrm{CH}^2(h^1(X)(-1))^0 = \mathrm{CH}^1(h^1(X))^0 = \mathrm{CH}^1(h^1(X))$, so that we get a decomposition

$$\mathrm{CH}^2(\mathrm{Sym}^3 h^1(X))^0 \cong \mathrm{CH}^2(M)^0 \oplus \mathrm{CH}^2(h^1(X)(-1))^0.$$

In this way, we get a recurring fantasy for M.

A Some facts about henselian rings

A *henselian ring* is a local ring satisfying Hensel's lemma. I. e., a local ring $(R, \mathfrak{m}, \kappa)$ is henselian if for every monic polynomial $f \in R[x]$ and every simple root a_0 of $\overline{f}$ in $\kappa[x]$ there exists a root $a \in R$ of f such that $a_0 = \overline{a}$. (That a_0 is a simple root means that $\overline{f}'(a_0) \neq 0$.) Typical examples of henselian local rings are fields, complete local rings, and quotients of henselian rings. A henselian ring is called a *strict henselian ring* if its residue field is separably closed.

A.1 Lemma. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. The following are equivalent.*

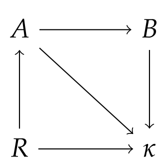
- *R is henselian;*
- *for any $f \in R[x]$ and every factorisation $\overline{f} = g_0 h_0$ in $\kappa[x]$ in coprime factors (i. e., $\gcd(g_0, h_0) = 1$) there exist $g, h \in R[x]$, with $f = gh$, such that $g_0 = \overline{g}$ and $h_0 = \overline{h}$;*
- *for any étale ring map $R \to S$ and prime $\mathfrak{q}$ of S lying over $\mathfrak{m}$ with $\kappa = \kappa(\mathfrak{q})$ there exists a section $S \to R$ of $R \to S$;*
- *any finite R-algebra is a finite product of local rings.*

Proof. See [Stacks, 04GG]. □

A geometric formulation of Hensel's lemma is found in the following propositions.

A.2 Proposition. *Let $(R, \mathfrak{m}, \kappa)$ be a henselian ring. If X is a smooth R-scheme then the reduction map $X(R) \to X(\kappa)$ is surjective.*

Proof. Let a point $x \in X(\kappa)$ be given. Let $\operatorname{Spec} A \subset X$ be a smooth affine open neighbourhood of x. By [Stacks, 07M7] there exists an étale R-algebra B such that the following diagram commutes.



Since $R \to B$ is étale, there is a section $B \to R$, cf. lemma A.1. The composition $A \to B \to R$ induces a map $\operatorname{Spec} R \to \operatorname{Spec} A \subset X$ that reduces to x. □

A.3 Proposition. *Let $(R, \mathfrak{m}, \kappa)$ be a local ring. If X is a formally unramified R-scheme then the reduction map $X(R) \to X(\kappa)$ is injective.*

Proof. Let $x, y \in X(R)$ be given. Let x_n and y_n denote the image of x respectively y in $X(R/\mathfrak{m}^n)$. Suppose $x_1 = y_1$, and observe that therefore $x_n = y_n$ for all n, since X is formally unramified over R. Indeed, both x_n and y_n reduce to x_1, and there can be only one such lift, hence $x_n = y_n$. We conclude that $(x_n)_n = (y_n)_n$ in $X(\hat{R})$, and therefore $x = y$. □

A.4 Corollary. *Let $(R, \mathfrak{m}, \kappa)$ be a henselian ring. If X is an étale R-scheme then the reduction map $X(R) \to X(\kappa)$ is bijective.*

Proof. Since X is étale over R, it is both smooth and unramified (hence formally unramified). Now use the above two propositions. □

B Some facts about group schemes

B.1 Lemma. *Let G be a connected smooth group scheme over a perfect field k. Then there exists a unique, connected, smooth, affine, normal, closed subgroup scheme L of G and an abelian variety A fitting into an exact sequence of group schemes*

$$0 \to L \to G \to A \to 0.$$

Proof. See [Co02, thm 1.1]. □

Let k be a field, and let k^{s} be a separable closure of k. A group scheme T/k is called a *torus* if $\overline{T} = T \times_k k^{\mathrm{s}}$ is isomorphic to $\mathbb{G}_{\mathrm{m}}{}_{k^{\mathrm{s}}}^{r}$ for some integer r.

Let $\overline{k}$ be an algebraic closure of k. A group scheme $G/\overline{k}$ is called *unipotent* if it is of finite type, and there exist $\overline{k}$-group schemes $G_0, \ldots, G_n$, such that

- $G_0 = 1$,
- $G_n = U$, and
- for $i \in \{1, \ldots n\}$ the group scheme G_i is an extension of G_{i-1} by a subgroup scheme of $\mathbb{G}_{\mathrm{a}\overline{k}}$.

A k-group scheme U is called *unipotent* if the base change to $\overline{k}$ is unipotent. If U is connected, smooth, and affine, then $U_{\overline{k}}$ is an iterated extension of copies of $\mathbb{G}_{\mathrm{a}\overline{k}}$, [SGA3-2, XVII 4.1.1].

B.2 Lemma. *Let L be a geometrically reduced commutative affine group scheme over a perfect field k. Then there exists a torus T/k and a unipotent group scheme U/k such that $L \cong T \times_k U$.*

Proof. See [Wa79, p. 70]. □

C Graphical presentation of the computation of intersections

When writing about the composition of correspondences (or more general the intersection of algebraic cycles) the author faces the problem of communicating the computations to his readers in a way that is intuitive and clear. In this thesis I have tried to do this with tabular diagrams that I will explain below.

This method is particularly apt for computations involving tautological cycles (e. g., partial diagonals on self-products of a variety).

The input for the computation consists of three schemes X, Y, and Z, and two correspondences $f \in \mathrm{CH}^\bullet(X \times Y)$ and $g \in \mathrm{CH}^\bullet(Y \times Z)$. These are place in a diagram as follows.

f_X	f_Y	Z	f
X	g_Y	g_Z	g
$f_X \cdot X$	$f_Y \cdot g_Y$	$Z \cdot g_Z$	$g \circ f$

The first column corresponds to X, the middle to Y and the last column corresponds to Z. To the right one places the names of the correspondences. Usually the components f_X, f_Y, g_Y, and g_Z are given with coordinates, and the vertical lines help to indicate the three factors X, Y, and Z. If any of X, Y, or Z is 0-dimensional, then the corresponding column, together with a vertical line will be removed from the table. To indicate partial diagonals the coordinates in question are linked together by a bended line, as can be seen in the example below.

For each coordinate the intersection is computed, and written below the horizontal line. Any partial diagonals are then accounted for, to obtain the intersection as cycle on $X \times Y \times Z$. The composition of the correspondences f and g is then written in the lower right corner, and this amounts to omitting the middle column (corresponding to Y).

One drawback of these diagrams is that they do not show which coordinates are free, and which are points. We hope that this is clear from the context of the diagram.

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