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Radical Galois groups and cohomology

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Introduction

Throughout this thesis, all modules considered will be left modules.

1. Summary

Let $\overline{\mathbf{Q}}$ be an algebraic closure of $\mathbf{Q}$. For $r \in \mathbf{Q}^*$, let

$$R_{\infty,r} = \{x \in \overline{\mathbf{Q}}^* : x^n \in \langle r \rangle \text{ for some } n \in \mathbf{Z}_{\geq 1}\}$$

be the subgroup of $\overline{\mathbf{Q}}^*$ consisting of all radicals of r. Moreover, let

$$F_{\infty,r} = \mathbf{Q}(R_{\infty,r})$$

be the Galois extension of $\mathbf{Q}$ generated by all radicals of r in $\overline{\mathbf{Q}}$. In the present thesis we are focused on describing the structure of the Galois group $\mathrm{Gal}(F_{\infty,r}/\mathbf{Q})$.

Theorem 1. *Let G be a profinite group. Then the following are equivalent.*

(a) *There exists $r \in \mathbf{Q}^* \setminus \{\pm 1\}$ such that*

$$G \cong \mathrm{Gal}\left(\mathbf{Q}(R_{\infty,r})/\mathbf{Q}\right)$$

as profinite groups.

(b) *There is a short exact sequence of profinite groups*

$$0 \longrightarrow \widehat{\mathbf{Z}} \xrightarrow{f} G \xrightarrow{g} \widehat{\mathbf{Z}}^* \longrightarrow 1,$$

where $\widehat{\mathbf{Z}}$ is the profinite completion of $\mathbf{Z}$ and $\widehat{\mathbf{Z}}^*$ is its group of units, such that

(i) *the induced action of $\widehat{\mathbf{Z}}^*$ on $\widehat{\mathbf{Z}}$ is the natural action, that is, for all $x \in \widehat{\mathbf{Z}}$ and $\sigma \in G$ we have $\sigma f(x)\sigma^{-1} = f(g(\sigma) \cdot x)$, where $\cdot$ is the multiplication of the ring $\widehat{\mathbf{Z}}$;*

(ii) *the sequence is not semisplit, that is, there is no continuous group homomorphism $h\colon \widehat{\mathbf{Z}}^* \to G$ satisfying $g \circ h = \mathrm{id}_{\widehat{\mathbf{Z}}^*}$.*

In §2.2 we briefly explain the proof of (a) implying (b) that was given in [LMS13, §2]. As announced in the same article, the other implication is proven in the present thesis; see §2.3. The main tool in our proof is the algebraic cohomology of topological groups acting continuously on topological modules, which one calls *continuous cochain cohomology*. Given a topological group G and a topological G-module A, the *continuous cochain cohomology of G with coefficients in A* is the cohomology obtained from the complex

$$0 \longrightarrow A \xrightarrow{d_0} \mathrm{C}^1(G,A) \xrightarrow{d_1} \mathrm{C}^2(G,A) \xrightarrow{d_2} \mathrm{C}^3(G,A) \xrightarrow{d_3} \mathrm{C}^4(G,A) \xrightarrow{d_4} \dots$$

where for $n \in \mathbf{Z}_{\geq 1}$ the group $\mathrm{C}^n(G, A)$ consists of all continuous maps of G^n to A, and d_n is the standard coboundary map one also has in non-continuous group cohomology. For $n \in \mathbf{Z}_{\geq 0}$, the cohomology groups of this complex are denoted by $\mathrm{H}^n(G, A)$. See section §1.1 for more details.

For a topological group G and a topological G-module A, we give in 1.31 and 1.33 the definition of *topological group extensions of G by A* and equivalence classes thereof. Moreover, as in non-continuous group cohomology, the set of those equivalence classes may be identified with $\mathrm{H}^2(G, A)$.

One easily sees that the other implication of Theorem 1 is implied by the following theorem.

Theorem 2. *There exists a group isomorphism $\varphi\colon \mathrm{H}^2(\widehat{\mathbf{Z}}^*, \widehat{\mathbf{Z}}) \to \mathbf{Q}^*/\pm\mathbf{Q}^{*2}$ that for every $r \in \mathbf{Q}^* \setminus \{\pm 1\}$ maps the equivalence class of any short exact sequence*

$$0 \longrightarrow \widehat{\mathbf{Z}} \longrightarrow \mathrm{Gal}(F_{\infty,r}/\mathbf{Q}) \longrightarrow \widehat{\mathbf{Z}}^* \longrightarrow 1$$

satisfying condition (i) *of* Theorem 1(b)*, to $\pm r_0\mathbf{Q}^{*2}$, where $r_0 \in \mathbf{Q}^*$ is such that*

$$R_{\infty,r} \cap \mathbf{Q}^* = \langle -1, r_0 \rangle.$$

For the proof, see Theorem 2.14.

One of the key steps in the proof of Theorem 2 is to show that $\mathrm{H}^2(\widehat{\mathbf{Z}}^*, \widehat{\mathbf{Z}})$ has exponent dividing 2. We generalize this result in the following way.

Let M be a profinite abelian group, and let Γ be a closed subgroup of $\widehat{\mathbf{Z}}^*$. There is a natural topological $\widehat{\mathbf{Z}}$-module structure on M, and this action induces both a topological Γ-module structure on M and a $\widehat{\mathbf{Z}}$-module structure on each $\mathrm{H}^n(\Gamma, M)$. We define

$$W_\Gamma = \overline{\sum_{\gamma \in \Gamma} \widehat{\mathbf{Z}}(\gamma - 1)},$$

which is the closure of the $\widehat{\mathbf{Z}}$-ideal generated by $\{\gamma - 1 : \gamma \in \Gamma\}$. For example, one has $W_{\widehat{\mathbf{Z}}^*} = 2\widehat{\mathbf{Z}}$, so the desired relation $2 \cdot \mathrm{H}^2(\widehat{\mathbf{Z}}^*, \widehat{\mathbf{Z}}) = 0$ is a special case of the following theorem.

Theorem 3. *Let Γ be a closed subgroup of $\widehat{\mathbf{Z}}^*$, and let M be a profinite abelian group with the Γ-module structure defined above. Then we have*

$$W_\Gamma \cdot \mathrm{H}^n(\Gamma, M) = 0$$

for all $n \in \mathbf{Z}_{\geq 0}$.

For the proof, see Theorem 2.16. Its main ingredient is an extension to the present context of a result on the conjugation map that is familiar from non-continuous group cohomology (see 1.29). We give two proofs of this extended result. The first proof makes use of a dimension shifting technique for continuous cochain cohomology

that appears to be new (see §1.4). One may hope that this technique admits further applications, although it is restricted to the case of locally compact G. The second proof, which works for general G, proceeds by a direct calculation with cocycles.

Thus far Theorem 3 is just group-theoretic. Relating Theorem 3 to fields in the following way, serves as a first step to generalizing Theorem 1 to any field of characteristic 0, that is, replacing $\mathbf{Q}$ with any field extension of it.

Let K be any field of characteristic 0, and let $\overline{K}$ be an algebraic closure of K. Let μ be the subgroup of $\overline{K}^*$ consisting of all roots of unity; it is a discrete topological $\widehat{\mathbf{Z}}$-module, which we write additively (see §2.1). There is an inclusion-reversing bijection between the set of subgroups of μ and the set of closed $\widehat{\mathbf{Z}}$-ideals that sends a subgroup ν of μ to the $\widehat{\mathbf{Z}}$-annihilator

$$\mathrm{Ann}_{\widehat{\mathbf{Z}}}(\nu) = \{x \in \widehat{\mathbf{Z}} : x \cdot \nu = 0\}$$

of ν, the inverse bijection sending a closed $\widehat{\mathbf{Z}}$-ideal W to the W-torsion subgroup

$$\mu[W] = \{\xi \in \mu : W \cdot \xi = 0\}$$

of μ (see Proposition 2.8).

The maximal cyclotomic extension $K(\mu)$ of K is Galois with a Galois group that may be viewed as a closed subgroup of $\widehat{\mathbf{Z}}^*$, which we shall denote by Γ or Γ_K (see §2.4). The interest of the closed $\widehat{\mathbf{Z}}$-ideal W_Γ, defined earlier, is that under the bijection above it corresponds to the subgroup $\mu(K) = \mu \cap K^*$ of μ, as expressed by the following theorem.

Theorem 4. *Let K be a field of characteristic 0, and let $\Gamma \subset \widehat{\mathbf{Z}}^*$ be its maximal cyclotomic Galois group. Then the closed ideal W_Γ is equal to the $\widehat{\mathbf{Z}}$-annihilator $\mathrm{Ann}_{\widehat{\mathbf{Z}}}(\mu(K))$ of the group $\mu(K)$ of roots of unity in K^*.*

For the proof, see Theorem 2.17.

Combining Theorem 3 and Theorem 4, we obtain $\mathrm{Ann}_{\widehat{\mathbf{Z}}}(\mu(K)) \cdot \mathrm{H}^n(\Gamma, M) = 0$ for all M as in Theorem 3, and all $n \in \mathbf{Z}_{\geq 0}$. If K contains only finitely many roots of unity, then one has $\mathrm{Ann}_{\widehat{\mathbf{Z}}}(\mu(K)) = \#\mu(K) \cdot \widehat{\mathbf{Z}}$, so that $\#\mu(K)$ annihilates each $\mathrm{H}^n(\Gamma, M)$. In particular, as there are exactly two roots of unity in $\mathbf{Q}$, we see again that $2 \cdot \mathrm{H}^n(\widehat{\mathbf{Z}}^*, \widehat{\mathbf{Z}}) = 0$ for all $n \in \mathbf{Z}_{\geq 0}$.

2. Overview

The present thesis is organized as follows.

In Chapter 1 our main goal is to prove that the conjugation map on cohomology is the identity, in section 5. To this end, we establish, in section 4, the technique of dimension shifting for locally compact groups acting on topological modules. We introduce the necessary tools for this, such as the definition of the continuous

cohomology groups and the long exact sequence in this cohomology, in sections 1 to 3. In the last section of Chapter 1 we briefly address the extension theory of topological groups.

In Chapter 2 we prove the theorems given above. Section 1 covers the preliminaries of the chapter. In section 2 the first half of the proof of Theorem 1 is shown, and in section 3 the rest of the proof is given. The last section is concerned with Theorem 3 and 4.

3. Acknowledgements

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CHAPTER 1

Continuous cochain cohomology

In this chapter we first cover some basic continuous group cohomology. This theory is treated more deeply in [Wilson, Chapter 9] for the particular case of profinite groups acting on abelian topological groups. However, the notions and proofs of our interest in this book apply without change to arbitrary topological groups acting on abelian topological groups, since no properties of profinite groups are used in the proofs. Therefore, we do refer to [Wilson, Chapter 9] for some omitted proofs in this chapter.

1. Continuous cohomology groups

Let G and H be topological groups. Let $\mathrm{C}(G,H)$ be the set of continuous functions from G to H. Define multiplication on $\mathrm{C}(G,H)$ by defining $\varphi\cdot\psi$ as the map sending $g\in G$ to $\varphi(g)\cdot\psi(g)$; this is well-defined since $\varphi\cdot\psi$ is the composition of the continuous map $G\longrightarrow H\times H$ sending $g\in G$ to $(\varphi(g),\psi(g))$ and the continuous map defining the operation of H. It is easy to see that $\mathrm{C}(G,H)$ is a group under this operation. The identity element is given by the map sending all elements to $1\in H$, and the inverse of an element $f\in\mathrm{C}(G,H)$ is given by sending $g\in G$ to $f(g)^{-1}$. Observe that $\mathrm{C}(G,H)$ is abelian if and only if H is abelian. When $\mathrm{C}(G,H)$ is an abelian group, we use additive notation and terminology.

Moreover, let $\mathrm{CHom}(G,H)$ denote the subset of $\mathrm{C}(G,H)$ consisting of continuous group homomorphisms from G to H, and note that it is a subgroup of $\mathrm{C}(G,H)$ when H is abelian.

Let G be a topological group. A G-module A is called *topological* if A is a topological group and the action of G on A, viewed as a map $G\times A\longrightarrow A$, is continuous. A *morphism of topological G-modules* is a G-module homomorphism that is continuous. This defines the category G-**TMod** of topological G-modules. We let $\mathrm{CHom}_G(A,B)$ be the group of morphisms of two objects A and B in G-**TMod**. It is easy to check that G-**TMod** is an additive category, and that this category is not abelian in general.

Let A be a topological G-module. For $n\in\mathbf{Z}_{\geq 0}$, endow G^n with the product topology and let $\mathrm{C}^n(G,A)$ denote the group $\mathrm{C}(G^n,A)$ of continuous functions from G^n to A. The elements of $\mathrm{C}^n(G,A)$ are called *continuous n-cochains*. It is clear that $\mathrm{C}^0(G,A)$ is canonically isomorphic to A as a group.

For $n\in\mathbf{Z}_{\geq 0}$ define the *boundary map* $d_n\colon\ \mathrm{C}^n(G,A)\longrightarrow\mathrm{C}^{n+1}(G,A)$ by

$$(d_n\varphi)(g_1,\dots,g_{n+1})=g_1\cdot\varphi(g_2,\dots,g_{n+1})+\textstyle\sum_{i=1}^n(-1)^i\varphi(g_1,\dots,g_ig_{i+1},\dots,g_{n+1})+$$

$$+(-1)^{n+1}\varphi(g_1,\dots,g_n),$$

whose kernel is the group of *continuous n-cocycles*, and is denoted by $\mathrm{Z}^n(G, A)$. One proves that $d_{n+1} \circ d_n = 0$ for every $n \in \mathbf{Z}_{\geq 0}$ in the same tedious way this is done in non-continuous group cohomology (see [EM47]). Hence, for $n \in \mathbf{Z}_{\geq 1}$ the image of d_{n-1}, denoted by $\mathrm{B}^n(G, A)$, is contained in $\mathrm{Z}^n(G, A)$; its elements are called the *continuous n-coboundaries*. Moreover, the group of continuous 0-coboundaries $\mathrm{B}^0(G, A)$ is defined as the trivial group. For $n \in \mathbf{Z}_{\geq 0}$, we define the *$n$-th continuous cochain cohomology group of G with coefficients in A* as the quotient $\mathrm{Z}^n(G, A)/\mathrm{B}^n(G, A)$, denoted by $\mathrm{H}^n(G, A)$. One notes that for $n \in \mathbf{Z}_{\geq 0}$ the groups $\mathrm{Z}^n(G, A)$, $\mathrm{B}^n(G, A)$ and $\mathrm{H}^n(G, A)$ are abelian groups.

We will often omit 'continuous' in the above defined objects. Note that if G is a discrete topological group, the notions above coincide with the usual group cohomology notions.

The cohomology group $\mathrm{H}^0(G, A)$ will often be identified with A^G via the group isomorphism $\varphi \mapsto \varphi(1)$. Moreover, one easily checks that if G acts trivially on A, then $\mathrm{H}^1(G, A) = \mathrm{CHom}(G, A)$.

2. Compatible pairs

Let $\varphi\colon H \longrightarrow G$ and $\psi\colon A \longrightarrow B$ be continuous group homomorphisms, where A and B are topological modules over G and H, respectively. The pair (φ, ψ) is called *compatible* if for all $h \in H$ and $a \in A$ we have $\psi(\varphi(h)a) = h\psi(a)$.

Examples. (1) Let G be a topological group, and let A be a topological G-module. Then the identity id_G on G and the identity id_A on A form a compatible pair $(\mathrm{id}_G, \mathrm{id}_A)$.

(2) Let B be another topological G-module and suppose that $\psi\colon A \longrightarrow B$ is a continuous group homomorphism. Then the pair (id_G, ψ) is compatible if and only if ψ is G-linear.

(3) Let $\varphi\colon H \longrightarrow G$ be a continuous group homomorphism, and let $(\varphi, \chi\colon A \longrightarrow B)$ and $(\varphi, \chi'\colon A \longrightarrow B)$ be two compatible pairs. Then the pair $(\varphi, \chi + \chi')$ is compatible.

(4) Now, let K be a normal subgroup of G, and let $\pi\colon G \longrightarrow G/K$ denote the quotient map. Then the fixed point subgroup A^K of A is a topological G/K-module. Let ι denote the inclusion of A^K in A. Then the pair (π, ι) is compatible.

Lemma 1.1. *Let $\varphi\colon H \longrightarrow G$ and $\psi\colon A \longrightarrow B$ be a compatible pair. Then the following statements hold.*

(a) *For each $n \in \mathbf{Z}_{\geq 0}$ there is an induced group homomorphism*

$$\mathrm{C}^n(\varphi, \psi)\colon\ \mathrm{C}^n(G, A) \longrightarrow \mathrm{C}^n(H, B)$$

given by

$$\mathrm{C}^n(\varphi,\psi)(f) = \psi \circ f \circ \varphi_n,$$

where $\varphi_n\colon H^n \longrightarrow G^n$ *sends* $(h_1,\ldots,h_n) \in H^n$ *to* $(\varphi(h_1),\ldots,\varphi(h_n))$.

(b) *For each* $n \in \mathbf{Z}_{\geq 0}$ *the diagram*

$$\begin{tikzcd} \mathrm{C}^n(G,A) \arrow[r, "d_n"] \arrow[d, "\mathrm{C}^n(\varphi,\psi)"'] & \mathrm{C}^{n+1}(G,A) \arrow[d, "\mathrm{C}^{n+1}(\varphi,\psi)"] \\ \mathrm{C}^n(H,B) \arrow[r, "d_n"'] & \mathrm{C}^{n+1}(H,B) \end{tikzcd}$$

is commutative.

(c) *For each* $n \in \mathbf{Z}_{\geq 0}$ *there is an induced homomorphism*

$$\mathrm{H}^n(\varphi,\psi)\colon\ \mathrm{H}^n(G,A) \longrightarrow \mathrm{H}^n(H,B)$$

defined by sending $[f] \in \mathrm{H}^n(G,A)$ *to* $[\mathrm{C}^n(\varphi,\psi)(f)]$.

Proof. See [Wil98, Lemma 9.2.1]. ■

Lemma 1.2. *Let* $(\varphi\colon H \longrightarrow G, \psi\colon A \longrightarrow B)$ *and* $(\varphi'\colon I \longrightarrow H, \psi'\colon B \longrightarrow C)$ *be compatible pairs. Then the following hold.*

(a) *The pair* $(\varphi\circ\varphi', \psi'\circ\psi)$ *is compatible.*

(b) *For* $n \in \mathbf{Z}_{\geq 0}$ *we have* $\mathrm{C}^n(\varphi\circ\varphi', \psi'\circ\psi) = \mathrm{C}^n(\varphi',\psi')\circ\mathrm{C}^n(\varphi,\psi)$.

(c) *For* $n \in \mathbf{Z}_{\geq 0}$ *we have* $\mathrm{H}^n(\varphi\circ\varphi', \psi'\circ\psi) = \mathrm{H}^n(\varphi',\psi')\circ\mathrm{H}^n(\varphi,\psi)$.

Proof. Let $i \in I$ and $a \in A$. Then

$$(\psi'\circ\psi)((\varphi\circ\varphi')(i)a) = \psi'(\psi(\varphi(\varphi'(i))a)) = \psi'(\varphi'(i)\psi(a)) = i\psi'(\psi(a)) = i(\psi'\circ\psi)(a),$$

which proves (a). Moreover, (b) and (c) are clear by definition of the maps $\mathrm{C}^n(\varphi\circ\varphi', \psi'\circ\psi)$ and $\mathrm{H}^n(\varphi\circ\varphi', \psi'\circ\psi)$. ■

Let $\mathcal{C}$ be the category defined as follows. Let the objects of $\mathcal{C}$ be all pairs (G,A) where G is a topological group and A is a topological G-module. The morphisms in this category are given by the following compatible pairs: for two objects (G,A) and (H,B) of $\mathcal{C}$ an element of $\mathrm{Mor}_{\mathcal{C}}((G,A),(H,B))$ is a compatible pair (φ,ψ) where $\varphi\colon H \longrightarrow G$ and $\psi\colon A \longrightarrow B$. Moreover, composition of two morphisms $(\varphi\colon H \longrightarrow G, \psi\colon A \longrightarrow B)$ and $(\varphi'\colon I \longrightarrow H, \psi'\colon B \longrightarrow C)$ is given by

$$(\varphi',\psi')\circ(\varphi,\psi) = (\varphi\circ\varphi', \psi'\circ\psi),$$

which is well-defined by the previous lemma and clearly an associative operation.

Note that for every $n \in \mathbf{Z}_{\geq 0}$ and for every object (G,A) of $\mathcal{C}$, the identity element $(\mathrm{id}_G, \mathrm{id}_A)$ of $\mathrm{Mor}_{\mathcal{C}}((G,A),(G,A))$ induces the identity on the groups $\mathrm{C}^n(G,A)$ and $\mathrm{H}^n(G,A)$.

Proposition 1.3. *Let $n \in \mathbf{Z}_{\geq 0}$. Then*

$$\mathrm{C}^n(\cdot,\cdot)\colon \mathcal{C} \longrightarrow \mathbf{Ab} \text{ and } \mathrm{H}^n(\cdot,\cdot)\colon \mathcal{C} \longrightarrow \mathbf{Ab}$$

are covariant functors from the category $\mathcal{C}$ to the category **Ab** *of abelian groups.*

Proof. This follows from Lemma 1.1 and Lemma 1.2. ∎

Let $\varphi\colon H \longrightarrow G$ be a continuous group homomorphism, and let $(\varphi, \psi\colon A \longrightarrow B)$ and $(\varphi, \psi'\colon A \longrightarrow B)$ be two compatible pairs. One easily checks that the identities

$$\mathrm{C}^n(\varphi, \psi + \psi') = \mathrm{C}^n(\varphi, \psi) + \mathrm{C}^n(\varphi, \psi')$$

and

$$\mathrm{H}^n(\varphi, \psi + \psi') = \mathrm{H}^n(\varphi, \psi) + \mathrm{H}^n(\varphi, \psi')$$

hold, where the addition on the right-hand sides is done in the category **Ab** of abelian groups. In particular, if we choose $G = H$ and $\varphi = \mathrm{id}_G$, we have a subcategory $\mathcal{C}_G$ of $\mathcal{C}$ consisting of the pairs (G, A) with A a topological G-module, and with morphisms all compatible pairs (id_G, ψ) where ψ is a continuous G-module homomorphism. One easily sees that the subcategory $\mathcal{C}_G$ of $\mathcal{C}$ can be canonically identified with the category G-**TMod** of topological G-modules. The following proposition is immediate.

Proposition 1.4. *Let G be a topological group, and let $n \in \mathbf{Z}_{\geq 0}$. For a morphism ψ of topological G-modules, let $\mathrm{C}^n(G, \psi) = \mathrm{C}^n(\mathrm{id}_G, \psi)$ and $\mathrm{H}^n(G, \psi) = \mathrm{H}^n(\mathrm{id}_G, \psi)$. Then*

$$\mathrm{C}^n(G,\cdot)\colon G\text{-}\mathbf{TMod} \longrightarrow \mathbf{Ab} \text{ and } \mathrm{H}^n(G,\cdot)\colon G\text{-}\mathbf{TMod} \longrightarrow \mathbf{Ab}$$

are additive covariant functors. ∎

Notation 1.5. When the group G is understood, we will for $n \in \mathbf{Z}_{\geq 0}$ sometimes write $\mathrm{C}^n(\cdot)$ and $\mathrm{H}^n(\cdot)$ for the functors $\mathrm{C}^n(G,\cdot)$ and $\mathrm{H}^n(G,\cdot)$, respectively.

Proposition 1.6. *Let G be a topological group. Then the functors $\mathrm{C}^n(G,\cdot)$ and $\mathrm{H}^n(G,\cdot)$ commute with arbitrary products.*

Proof. Let $\{A_i\}_{i\in I}$ be a collection of topological G-modules. By the universal property of topological products, we have for any $n \in \mathbf{Z}_{\geq 0}$ that $\mathrm{C}^n(G, \prod_{i\in I} A_i)$ is isomorphic to $\prod_{i\in I} \mathrm{C}^n(G, A_i)$ as a group. One easily checks that for every $n \in \mathbf{Z}_{\geq 0}$ the diagram

$$\begin{tikzcd}
\mathrm{C}^n(G, \prod_{i\in I} A_i) \arrow[r, "d_n"] \arrow[d] & \mathrm{C}^{n+1}(G, \prod_{i\in I} A_i) \arrow[d] \\
\prod_{i\in I} \mathrm{C}^n(G, A_i) \arrow[r, "(d_n)_{i\in I}"'] & \prod_{i\in I} \mathrm{C}^{n+1}(G, A_i)
\end{tikzcd}$$

is commutative, where the vertical arrows are the isomorphisms mentioned above. Then we have $\mathrm{H}^n(G, \prod_{i\in I} A_i) = \prod_{i\in I} \mathrm{H}^n(G, A_i)$. ∎

3. Long exact sequence

In this section we will construct the long exact sequence of cohomology for special types of short exact sequences of topological modules. Moreover, we show that if we slightly generalize these sequences, we have the long exact sequence until the first cohomology groups.

Proposition 1.7. *Let*

$$1 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 1$$

be a short exact sequence of not necessarily abelian topological groups. Then the following are equivalent.

(a) *The map f induces a homeomorphism from A to its image, and g admits a continuous set-theoretic section.*

(b) *There is a homeomorphism $\varphi\colon B \longrightarrow A \times C$, where $A \times C$ has the product topology, such that the diagram*

$$\begin{tikzcd}
 & & B \arrow[dr, "g"] \arrow[dd, "\varphi"] & & \\
1 \arrow[r] & A \arrow[ur, "f"] \arrow[dr, "\iota_A"'] & & C \arrow[r] & 1 \\
 & & A \times C \arrow[ur, "\pi_C"'] & &
\end{tikzcd}$$

commutes, where ι_A sends $a \in A$ to $(a, 1)$ and π_C sends $(a, c) \in A \times C$ to c.

Proof. Assume (a), that is, suppose that f induces a homeomorphism from A to its image, and g admits a continuous section, say s. Without loss of generalization, assume that $s(1) = 1$, since otherwise we may compose s with the homeomorphism $B \longrightarrow B$ sending b to $bs(1)^{-1}$, which gives a continuous section of g sending $1 \in C$ to $1 \in B$.

Let $f^\dagger\colon f(A) \longrightarrow A$ be the continuous map such that $f^\dagger \circ f = \mathrm{id}_A$. Let $b \in B$, and note that

$$bs(g(b))^{-1} \in \ker g = \operatorname{im} f.$$

Define the map $\varphi\colon B \longrightarrow A \times C$ by sending $b \in B$ to $(f^\dagger(bs(g(b))^{-1}), g(b))$. Note that φ is continuous, since it is equal to the composition of the continuous map $B \longrightarrow \ker g \times \operatorname{im} s$ sending $b \in B$ to $(bs(g(b))^{-1}, s(g(b)))$ and the continuous map $\ker g \times \operatorname{im} s \longrightarrow A \times C$ sending $(b, b') \in \ker g \times \operatorname{im} s$ to $(f^\dagger(b), g(b))$.

One easily checks that the map $\varphi^{-1}\colon A \times C \longrightarrow B$ given by $(a, c) \mapsto f(a)s(c)$ is a continuous map that is inverse to φ. Hence, the map φ is a homeomorphism.

At last, it is easy to see that $\pi_C \circ \varphi = g$, and moreover, using $s(1) = 1$, one easily sees that $\varphi \circ f = \iota_A$. Hence, the desired diagram commutes.

Conversely, assume (b). Then one easily shows that $\pi_A \circ \varphi$ induces a homeomorphism from $\operatorname{im} f$ to A, and that $\varphi^{-1} \circ \iota_C$ is a continuous section of g. ■

Definition 1.8. A short exact sequence

$$1 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 1$$

of not necessarily abelian topological groups is called *well-adjusted* if it satisfies either one of the equivalent conditions 1.7(a) and 1.7(b) above.

Note that all short exact sequences of discrete groups are well-adjusted, as are all short exact sequences of profinite groups, cf. [Wil98, Lemma 0.1.2] and [Wil98, Proposition 1.3.3].

The following proposition defines a large class of short exact sequences of topological groups that are well-adjusted, which will be useful later.

Proposition 1.9. *Let G be a topological group, and let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of topological G-modules. Then the following are equivalent.

(a) *The map f admits a retraction that is a continuous G-module homomorphism, and g is open.*

(b) *The map g admits a section that is a continuous G-module homomorphism, and f induces a homeomorphism from A to its image $f(A)$.*

(c) *There is an isomorphism $\varphi\colon B \longrightarrow A \times C$ of topological G-modules, where $A \times C$ has the product topology, such that the diagram*

$$\begin{tikzcd}
 & & B \arrow[dd, "\varphi"] \arrow[dr, "g"] & & \\
0 \arrow[r] & A \arrow[ur, "f"] \arrow[dr, "\iota_A"'] & & C \arrow[r] & 0 \\
 & & A \times C \arrow[ur, "\pi_C"'] & &
\end{tikzcd}$$

commutes, where ι_A sends $a \in A$ to $(a, 0)$ and π_C sends $(a, c) \in A \times C$ to c.

Proof. One simply imitates the proof of the splitting lemma for abelian groups, see [Ste12, Theorem 9.3]. ■

Definition 1.10. Let G be a topological group, and let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of topological G-modules. Then this sequence is called *split* if it satisfies either one of the equivalent conditions 1.9(a), 1.9(b) and 1.9(c) above. A short exact sequence of abelian topological groups is called *split* if it is split as short exact sequence of topological modules over the trivial group.

Proposition 1.11. *Any split short exact sequence of topological groups is well-adjusted.*

Proof. This is immediate from the definition. ■

Lemma 1.12. *Let G be a topological group,and let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a well-adjusted short exact sequence of topological G-modules. Then for each $n \in \mathbf{Z}_{\geq 0}$ the sequences

$$\text{(i)} \qquad 0 \longrightarrow \mathrm{C}^n(G,A) \xrightarrow{\mathrm{C}^n(f)} \mathrm{C}^n(G,B) \xrightarrow{\mathrm{C}^n(g)} \mathrm{C}^n(G,C) \longrightarrow 0$$

$$\text{(ii)} \qquad \mathrm{H}^n(G,A) \xrightarrow{\mathrm{H}^n(f)} \mathrm{H}^n(G,B) \xrightarrow{\mathrm{H}^n(g)} \mathrm{H}^n(G,C)$$

of abelian groups are exact.

Proof. We first prove the exactness of (i). To this end, note that injectivity of $\mathrm{C}^n(f)$ follows immediately from the injectivity of f. Moreover, for every $a_n \in \mathrm{C}^n(G,A)$ we have $g \circ f \circ a_n = 0$, so that $\operatorname{im} \mathrm{C}^n(f) \subset \ker \mathrm{C}^n(g)$.

Conversely, let $b_n \in \ker \mathrm{C}^n(g)$ and observe that $g \circ b_n = 0$. Then $\operatorname{im} b_n \subset \ker g$, where $\ker g = \operatorname{im} f$. Let $\iota\colon \operatorname{im} f \longrightarrow A$ be a continuous map such that $f \circ \iota = \mathrm{id}_{\operatorname{im} f}$, which exists by assumption. Then $\iota \circ b_n$ is a continuous map from G to A, that is, we have $\iota \circ b_n \in \mathrm{C}^n(G,A)$. Moreover,

$$\mathrm{C}^n(f)(\iota \circ b_n) = f \circ \iota \circ b_n = b_n,$$

which shows that $\ker \mathrm{C}^n(g) \subset \operatorname{im} \mathrm{C}^n(f)$. Hence, the sequence is exact at $\mathrm{C}^n(G,B)$.

Now, let $c_n \in \mathrm{C}^n(G,C)$, and let s be a continuous map from C to B such that $g \circ s = \mathrm{id}_C$, which exists by assumption. Note that $s \circ c_n$ is a continuous map from G to C, that is, we have $s \circ c_n \in \mathrm{C}^n(G,B)$. Moreover,

$$\mathrm{C}^n(g)(s \circ c_n) = g \circ s \circ c_n = c_n,$$

which shows that the sequence is exact at $\mathrm{C}^n(G,C)$. Thus, sequence (i) is exact.

Now, we prove exactness of (ii), that is, we show that $\operatorname{im} \mathrm{H}^n(f) = \ker \mathrm{H}^n(g)$. To this end, note that $\mathrm{H}^n(g) \circ \mathrm{H}^n(f) = 0$, so that $\operatorname{im} \mathrm{H}^n(f) \subset \ker \mathrm{H}^n(g)$.

Conversely, let $[b_n] \in \mathrm{H}^n(G,B)$ be such that $\mathrm{H}^n(g)([b_n]) = [g \circ b_n] = 0$. Then $g \circ b_n \in \mathrm{B}^n(G,C)$, so there is $c_{n-1} \in \mathrm{C}^{n-1}(G,C)$ such that

$$d_{n-1}(c_{n-1}) = g \circ b_n.$$

Using (i) in dimension $n-1$, there is a $b_{n-1} \in \mathrm{C}^{n-1}(G,B)$ such that

$$\mathrm{C}^{n-1}(g)(b_{n-1}) = c_{n-1}.$$

Moreover, by Lemma 1.1, we have

$$\mathrm{C}^n(g)(b_n) = d_{n-1}(\mathrm{C}^{n-1}(g)(b_{n-1})) = \mathrm{C}^n(g)(d_{n-1}(b_{n-1})),$$

so that

$$\mathrm{C}^n(g)(b_n - d_{n-1}(b_{n-1})) = 0.$$

Using (i) in dimension n, there is $a_n \in \mathrm{C}^n(G, A)$ such that

$$\mathrm{C}^n(f)(a_n) = b_n - d_{n-1}(b_{n-1}).$$

Note that

$$\mathrm{C}^n(f)(d_n(a_n)) = d_n(\mathrm{C}^n(f)(a_n) = d_n(b_n - d_{n-1}(b_{n-1}) = 0,$$

by Lemma 1.1 and the fact that $b_n \in \mathrm{Z}^n(G, B)$. By injectivity of $\mathrm{C}^n(f)$, we have $d_n(a_n) = 0$, so $a_n \in \mathrm{Z}^n(G, A)$. As $\mathrm{C}^n(f)(a_n) = b_n - d_{n-1}(b_{n-1})$ and $d_{n-1}(b_{n-1}) \in \mathrm{B}^n(G, B)$, the element $[a_n] \in \mathrm{H}^n(G, A)$ maps to $[b_n]$. This shows that (ii) is exact for all $n \in \mathbf{Z}_{\geq 1}$.

For $n = 0$, set $\mathrm{C}^{-1}(G, A) = \mathrm{C}^{-1}(G, B) = \mathrm{C}^{-1}(G, C) = 0$ and do the same as above to show that (ii) is exact. ■

Proposition 1.13. *Let G be a topological group, and let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a well-adjusted short exact sequence of topological G-modules. Then for each $n \geq 0$ there is a unique group homomorphism

$$\delta_n \colon \mathrm{H}^n(G, C) \longrightarrow \mathrm{H}^{n+1}(G, A)$$

such that for every $c \in \mathrm{Z}^n(G, C)$ and for every $a \in \mathrm{Z}^{n+1}(G, A)$ and $b \in \mathrm{C}^n(G, B)$ satisfying $\mathrm{C}^n(g)(b) = c$ and $\mathrm{C}^{n+1}(f)(a) = d_n(b)$, we have $\delta_n([c]) = [a]$.

Proof. First, we define δ_n. To this end, let $[c_n] \in \mathrm{H}^n(G, C)$, and observe that there is $b_n \in \mathrm{C}^n(G, B)$ such that $\mathrm{C}^n(g)(b_n) = c_n$. Then by Lemma 1.1

$$0 = d_n(c_n) = \mathrm{C}^{n+1}(g)(d_n(b_n)),$$

so there is $a_{n+1} \in \mathrm{C}^{n+1}(G, A)$ such that $\mathrm{C}^{n+1}(f)(a_{n+1}) = d_n(b_n)$. Observe that

$$\mathrm{C}^{n+2}(f)(d_{n+1}(a_{n+1})) = d_{n+1}(\mathrm{C}^{n+1}(a_{n+1})) = d_{n+1}(d_n(b_n)) = 0,$$

so by injectivity of $\mathrm{C}^{n+2}(f)$ we have $a_{n+1} \in \ker d_{n+1}$. Now, we define $\delta_n([c_n])$ to be the element $[a_{n+1}] \in \mathrm{H}^{n+1}(G, A)$.

To see that δ_n is well-defined for $n \in \mathbf{Z}_{>0}$, let $c'_n \in [c_n]$. Then $b'_n \in \mathrm{C}^n(G, B)$ and $a'_{n+1} \in \mathrm{C}^{n+1}(G, A)$ such that

$$\mathrm{C}^n(g)(b'_n) = c'_n \text{ and } \mathrm{C}^{n+1}(f)(d_n(b'_n)) = a'_{n+1}.$$

Note that $c'_n - c_n \in \operatorname{im} d_{n-1}$, hence there is $c_{n-1} \in \mathrm{C}^{n-1}(G, C)$ such that

$$d_{n-1}(c_{n-1}) = c'_n - c_n.$$

Moreover, there is $b_{n-1} \in \mathrm{C}^{n-1}(G, B)$ such that $\mathrm{C}^{n-1}(g)(b_{n-1}) = c_{n-1}$. Observe that

$$\mathrm{C}^n(g)(b'_n - b_n) = d_{n-1}(c_{n-1}) = \mathrm{C}^n(g)(d_{n-1}(b_{n-1}),$$

so $b'_n - b_n - d_{n-1}(b_{n-1}) \in \ker \mathrm{C}^n(g)$, where $\ker \mathrm{C}^n(g) = \operatorname{im} \mathrm{C}^n(f)$. Hence, we can write

$$b'_n - b_n - d_{n-1}(b_{n-1}) = \mathrm{C}^n(f)(a_n)$$

for some $a_n \in \mathrm{C}^n(G, A)$. Now, note that

$$\mathrm{C}^{n+1}(f)(a'_{n+1} - a_{n+1}) = d_n(b'_n - b_n) = d_n(\mathrm{C}^n(f)(a_n)) = \mathrm{C}^{n+1}(f)(d_n(a_n)),$$

so that by injectivity of $\mathrm{C}^{n+1}(f)$ we have $a'_{n+1} - a_{n+1} = d_n(a_n)$. This shows that $[a'_{n+1}] = [a_{n+1}]$ in $\mathrm{H}^{n+1}(G, A)$, and so δ_n is well-defined for $n \in \mathbf{Z}_{>0}$. To see that δ_0 is well-defined, set $\mathrm{C}^{-1}(G, A) = \mathrm{C}^{-1}(G, B) = \mathrm{C}^{-1}(G, C) = 0$ and do the same as above.

It is clear that for $n \in \mathbf{Z}_{\geq 0}$ the map δ_n is the unique group homomorphism having the desired property. ■

Proposition 1.14. *Let G be a topological group, and let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a well-adjusted short exact sequence of topological G-modules. Then for each $n \in \mathbf{Z}_{\geq 0}$, the sequence

$$\mathrm{H}^n(G, B) \xrightarrow{\mathrm{H}^n(g)} \mathrm{H}^n(G, C) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(G, A) \xrightarrow{\mathrm{H}^n(f)} \mathrm{H}^{n+1}(G, B)$$

is exact.

Proof. First we check exactness at $\mathrm{H}^n(G, C)$. By Lemma 1.1

$$(\delta_n \circ \mathrm{H}^n(g))([b_n]) = 0$$

for all $[b_n] \in \mathrm{H}^n(G, B)$. Thus, we have $\operatorname{im} \mathrm{H}^n(g) \subset \ker \delta_n$.

Conversely, suppose $[c_n] \in \ker \delta_n$, and let $a_{n+1} \in \mathrm{Z}^{n+1}(G, A)$ and $b_n \in \mathrm{C}^n(G, B)$ such that $\mathrm{C}^{n+1}(f)(a_{n+1}) = d_n(b_n)$ and $\mathrm{C}^n(g)(b_n) = c_n$. As $[0] = \delta_n([c_n]) = [a_{n+1}]$,

we have $a_{n+1} \in \mathrm{B}^{n+1}(G, A)$. Let $a_n \in \mathrm{C}^n(G, A)$ be such that $d_n(a_n) = a_{n+1}$. By Lemma 1.1

$$d_n(b_n) = \mathrm{C}^{n+1}(f)(d_n(a_n)) = d_n(\mathrm{C}^n(f)(a_n)),$$

so that $b_n - \mathrm{C}^n(f)(a_n) \in \mathrm{Z}^n(G, B)$. Observe that

$$\mathrm{C}^n(g)(b_n - \mathrm{C}^n(f)(a_n)) = \mathrm{C}^n(g)(b_n) = c_n,$$

so that we may conclude that $\operatorname{im} \mathrm{H}^n(g) = \ker \delta_n$.

Now, let $[c_n] \in \mathrm{H}^n(G, C)$, and suppose that $\delta_n([c_n]) = [a_{n+1}]$. Then there is $b_n \in \mathrm{C}^n(G, B)$ such that $\mathrm{C}^{n+1}(f)(a_{n+1}) = d_n(b_n)$. Hence, the equality

$$\mathrm{H}^{n+1}(f)([a_n]) = [\mathrm{C}^{n+1}(f)(a_n)] = [d_n(b_n)] = [0]$$

holds, so that $\operatorname{im} \delta_n \subset \ker \mathrm{H}^{n+1}(f)$.

Conversely, let $[a_{n+1}] \in \ker \mathrm{H}^{n+1}(f)$. Then there is $b_n \in \mathrm{C}^n(G, B)$ such that

$$\mathrm{C}^{n+1}(f)(a_{n+1}) = d_n(b_n),$$

since $\mathrm{C}^{n+1}(a_{n+1}) \in \mathrm{B}^{n+1}(G, B)$. Hence, the element $\mathrm{C}^n(g)(b_n) \in \mathrm{C}^n(G, C)$ satisfies $\delta_n([\mathrm{C}^n(g)(b_n)]) = [a_{n+1}]$, so that we may conclude that $\operatorname{im} \delta_n = \ker \mathrm{H}^{n+1}(f)$. ■

Theorem 1.15. *Let G be a topological group, and let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a well-adjusted short exact sequence of topological G-modules. Then the sequence

$$0 \longrightarrow \mathrm{H}^0(G, A) \xrightarrow{\mathrm{H}^0(f)} \mathrm{H}^0(G, B) \xrightarrow{\mathrm{H}^0(g)} \mathrm{H}^0(G, C) \xrightarrow{\delta_0} \mathrm{H}^1(G, A) \xrightarrow{\mathrm{H}^1(f)} \dots$$

$$\dots \xrightarrow{\delta_{n-1}} \mathrm{H}^n(G, A) \xrightarrow{\mathrm{H}^n(f)} \mathrm{H}^n(G, B) \xrightarrow{\mathrm{H}^n(g)} \mathrm{H}^n(G, C) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(G, A) \xrightarrow{\mathrm{H}^{n+1}(f)} \dots$$

is exact.

Proof. The injectivity of $\mathrm{H}^0(f)$ follows from the injectivity of $\mathrm{C}^0(f)$. The rest follows from Lemma 1.12 and Proposition 1.14. ■

Theorem 1.16. *Let $\varphi\colon H \longrightarrow G$ be a continuous group homomorphism, and let*

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r] \arrow[d, "\psi_1"'] & A_2 \arrow[r] \arrow[d, "\psi_2"'] & A_3 \arrow[r] \arrow[d, "\psi_3"'] & 0 \\
0 \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram where the upper row is a well-adjusted sequence of topological G-modules, and the lower row is a well-adjusted sequence of topological H-modules. Suppose that for all $i = 1, 2, 3$ the pairs (φ, ψ_i) are compatible. Then the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^0(G, A_1) \arrow[r] \arrow[d, "\mathrm{H}^0(\varphi,\psi_1)"'] & \mathrm{H}^0(G, A_2) \arrow[r] \arrow[d, "\mathrm{H}^0(\varphi,\psi_2)"'] & \mathrm{H}^0(G, A_3) \arrow[r, "\delta_0"] \arrow[d, "\mathrm{H}^0(\varphi,\psi_3)"'] & \mathrm{H}^1(G, A_1) \arrow[r] \arrow[d, "\mathrm{H}^1(\varphi,\psi_1)"'] & \dots \\
0 \arrow[r] & \mathrm{H}^0(H, B_1) \arrow[r] & \mathrm{H}^0(H, B_2) \arrow[r] & \mathrm{H}^0(H, B_3) \arrow[r, "\delta_0"] & \mathrm{H}^1(H, B_1) \arrow[r] & \dots
\end{tikzcd}$$

$$\begin{tikzcd}
\dots \arrow[r, "\delta_{n-1}"] & \mathrm{H}^n(G, A_1) \arrow[r] \arrow[d, "\mathrm{H}^n(\varphi,\psi_1)"'] & \mathrm{H}^n(G, A_2) \arrow[r] \arrow[d, "\mathrm{H}^n(\varphi,\psi_2)"'] & \mathrm{H}^n(G, A_3) \arrow[r, "\delta_n"] \arrow[d, "\mathrm{H}^n(\varphi,\psi_3)"'] & \mathrm{H}^{n+1}(G, A_1) \arrow[r] \arrow[d, "\mathrm{H}^{n+1}(\varphi,\psi_1)"'] & \dots \\
\dots \arrow[r, "\delta_{n-1}"] & \mathrm{H}^n(H, B_1) \arrow[r] & \mathrm{H}^n(H, B_2) \arrow[r] & \mathrm{H}^n(H, B_3) \arrow[r, "\delta_n"] & \mathrm{H}^{n+1}(H, B_1) \arrow[r] & \dots
\end{tikzcd}$$

with rows given by Theorem 1.15*, is commutative.*

Proof. See [Wil98, Theorem 9.3.4]. ∎

Proposition 1.17. *Let G be a topological group, and let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of topological G-modules such that f induces a homeomorphism from A to its image. Then the sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^0(G, A) \arrow[r, "\mathrm{H}^0(f)"] & \mathrm{H}^0(G, B) \arrow[r, "\mathrm{H}^0(g)"] & \mathrm{H}^0(G, C) \arrow[dll, rounded corners, "\delta_0"'] \\
 & \mathrm{H}^1(G, A) \arrow[r, "\mathrm{H}^1(f)"] & \mathrm{H}^1(G, B) \arrow[r, "\mathrm{H}^1(g)"] & \mathrm{H}^1(G, C)
\end{tikzcd}$$

is exact.

Proof. As for every topological G-module A we have $\mathrm{C}^0(G, A) \cong A$ as group, the given exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

gives the short exact sequence

$$0 \longrightarrow \mathrm{C}^0(G, A) \xrightarrow{\mathrm{C}^0(f)} \mathrm{C}^0(G, B) \xrightarrow{\mathrm{C}^0(g)} \mathrm{C}^0(G, C) \longrightarrow 0.$$

Moreover, since there is a continuous map $i\colon B \longrightarrow A$ such that $i \circ f = \mathrm{id}_A$, we have the exact sequence

$$0 \longrightarrow \mathrm{C}^1(G, A) \xrightarrow{\mathrm{C}^1(f)} \mathrm{C}^1(G, B) \xrightarrow{\mathrm{C}^1(g)} \mathrm{C}^1(G, C).$$

Now, following the proofs of Lemma 1.12 and Proposition 1.14, the proposition follows. ∎

The following example shows that we cannot continue this long exact sequence in general.

Example 1.18. Let $\mathbf{R}$ be the additive group of real numbers endowed with the usual topology, and let $\mathbf{R}/\mathbf{Z}$ be the quotient group endowed with the quotient topology. One easily checks that the topological group $\mathbf{R}/\mathbf{Z}$ is compact and connected. Now, let $\mathbf{R}/\mathbf{Z}$ act trivially on $\mathbf{R}$, and observe that the inclusion $\mathbf{Z} \subset \mathbf{R}$ induces a short exact sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{R} \longrightarrow \mathbf{R}/\mathbf{Z} \longrightarrow 0$$

of topological $\mathbf{R}/\mathbf{Z}$-modules with trivial $\mathbf{R}/\mathbf{Z}$-action. This sequence is not well-adjusted, since $\mathbf{R}$ is not homeomorphic to $\mathbf{Z}\times\mathbf{R}/\mathbf{Z}$. However, as $\mathbf{Z}$ is homeomorphic to its image in $\mathbf{R}$, we have the long exact sequence of Proposition 1.17. Using the fact that $\mathbf{R}/\mathbf{Z}$ acts trivially on $\mathbf{R}$, the sequence is as follows

$$0 \to \mathbf{Z} \to \mathbf{R} \to \mathbf{R}/\mathbf{Z} \to \mathrm{CHom}(\mathbf{R}/\mathbf{Z}, \mathbf{Z}) \to \mathrm{CHom}(\mathbf{R}/\mathbf{Z}, \mathbf{R}) \to \mathrm{CHom}(\mathbf{R}/\mathbf{Z}, \mathbf{R}/\mathbf{Z}).$$

As $\mathbf{R}/\mathbf{Z}$ is connected and the only connected components of the discrete group $\mathbf{Z}$ are the singletons, it follows that $\mathrm{CHom}(\mathbf{R}/\mathbf{Z}, \mathbf{Z}) = 0$. Moreover, since $\mathbf{R}/\mathbf{Z}$ is compact and the only compact subgroup of $\mathbf{R}$ is the trivial subgroup $\{0\}$, we have $\mathrm{CHom}(\mathbf{R}/\mathbf{Z}, \mathbf{R}) = 0$.

As the product of connected spaces is again connected, for every $n \in \mathbf{Z}_{\geq 0}$ all n-cochains are constant and in particular

$$\mathrm{C}^n(\mathbf{R}/\mathbf{Z}, \mathbf{Z}) \cong \mathbf{Z}$$

as group. It follows that for $n \in \mathbf{Z}_{\geq 0}$ we have the equality $\delta_n = \mathrm{id}_{\mathbf{Z}}$ if n is odd and $\delta_n = 0$ if n is even. Hence

$$\mathrm{H}^2(\mathbf{R}/\mathbf{Z}, \mathbf{Z}) = \ker \delta_2 / \operatorname{im} \delta_1 = \mathbf{Z}/\mathbf{Z} = 0.$$

We see that the exact sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{R} \longrightarrow \mathbf{R}/\mathbf{Z} \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathrm{CHom}(\mathbf{R}/\mathbf{Z}, \mathbf{R}/\mathbf{Z})$$

cannot be continued to $\mathrm{H}^2(\mathbf{R}/\mathbf{Z}, \mathbf{Z}) = 0$, since the latter term $\mathrm{CHom}(\mathbf{R}/\mathbf{Z}, \mathbf{R}/\mathbf{Z})$, which is the Pontryagin dual of $\mathbf{R}/\mathbf{Z}$, is isomorphic to $\mathbf{Z}$ as a discrete group (cf. [RZ09] or [HR79]) and this is nonzero. ■

4. Dimension shifting

Let X be a topological space, and let A be an abelian topological group. Let $\mathrm{C}(X, A)$ denote the set of continuous functions from X to A. Define addition on $\mathrm{C}(X, A)$ as in §1.1 and note that $\mathrm{C}(X, A)$ is an abelian group under this operation.

Endow $\mathrm{C}(X, A)$ with the compact-open topology, that is, the topology on $\mathrm{C}(X, A)$ that has a subbase consisting of the subsets

$$\mathrm{K}(V, U) = \{f \in \mathrm{C}(X, A) : f(V) \subset U\}$$

with V a compact subset of X, and U an open subset of A.

Recall that if $f\colon X \longrightarrow Y$ is a map between topological spaces, and $\mathcal{B}$ is a subbase for Y, then f is continuous if and only if for every $B \in \mathcal{B}$ the set $f^{-1}(B)$ is open in X.

Lemma 1.19. *Let G be a topological group. Let $\mathcal{U}$ be the neighbourhood system of the identity element 1 of G. Then the following hold.*

(a) *For every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \cdot V \subset U$.*

(b) *For every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V^{-1} \subset U$.*

(c) *For every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $\overline{V} \subset U$.*

Proof. See [HR79, Theorem 4.7, Corollary 4.7]. ∎

Lemma 1.20. *Let G be a topological group. Let C be a compact subset of G, and let U be an open subset of G containing C. Then there is an open neighbourhood U_1 of $1 \in G$ such that $CU_1 \subset U$.*

Proof. Let $c \in C$. As $c \in U$ and U is open, there is an open neighbourhood U_c of c contained in U. Note that $U_c = cV_c$ for some open neighbourhood V_c of $1 \in G$, since multiplication by c defines a homeomorphism from G to G. Using the previous lemma, choose an open neighbourhood W_c of $1 \in G$ such that $W_cW_c \subset V_c$, and note that $cW_c \subset cV_c \subset U$.

Note that $\{cW_c\}_{c\in C}$ is an open cover for C, hence by compactness of C there is a finite subset $S = \{c_1, \dots, c_n\}$ of C such that

$$C \subset \bigcup_{i=1}^{n} c_i W_{c_i}.$$

Then

$$C \cdot \bigcap_{j=1}^{n} W_{c_j} \subset \left(\bigcup_{i=1}^{n} c_i W_{c_i}\right) \cdot \bigcap_{j=1}^{n} W_{c_j} \subset \bigcup_{i=1}^{n} \left(c_i W_{c_i} \cdot \bigcap_{j=1}^{n} W_{c_j}\right) \subset \bigcup_{i=1}^{n} c_i V_{c_i} \subset U,$$

which shows that the open neighbourhood $\bigcap_{j=1}^{n} W_{c_j}$ of $1 \in G$, satisfies the desired condition. ∎

Proposition 1.21. *Let X be a topological space, and let A be an abelian topological group. Then the abelian group $\mathrm{C}(X, A)$ endowed with the compact-open topology is a topological group.*

Proof. For ease of notation write C for $\mathrm{C}(X, A)$. Observe that $C \longrightarrow C$ given by $\varphi \mapsto -\varphi$ is continuous, since $\mathrm{K}(V, U)$ has inverse image $\mathrm{K}(V, -U)$ under this map. Next, consider the addition map $\pi\colon C \times C \longrightarrow C$ given by $(\varphi, \psi) \mapsto \varphi + \psi$. Let

$(\varphi, \psi) \in \pi^{-1}(\mathrm{K}(V, U))$ for some $\mathrm{K}(V, U)$ in the subbase of C. We will find an open neighbourhood of φ in C and an open neighbourhood of ψ in C such that their product, which is an open neighbourhood of (φ, ψ) in $C \times C$, is inside $\pi^{-1}(\mathrm{K}(V, U))$.

To this end, observe that $(\varphi + \psi)(V) \subset U$, that is, for every $v \in V$ we have $\varphi(v) + \psi(v) \in U$. As $(\varphi + \psi)(V)$ is compact and contained in the open U, there is an open neighbourhood U_0 of $0 \in A$ such that $(\varphi + \psi)(V) + U_0 \subset U$. Using the previous lemma, choose an open neighbourhood T_0 of $0 \in A$ such that $T_0 + T_0 \subset U_0$, and an open neighbourhood S_0 of 0 such that $S_0 + S_0 \subset T_0$ and $\overline{S_0} \subset T_0$. For $v \in V$, let

$$V_v = \varphi^{-1}(\varphi(v) + \overline{S_0}) \cap V \text{ and } W_v = \psi^{-1}(\psi(v) + \overline{S_0}) \cap V$$

and observe that V_v and W_v are compact subsets of X containing v, since they are closed subsets of a compact set. Note that $\varphi(V_v) \subset \varphi(v) + T_0$ and $\varphi(W_v) \subset \psi(v) + T_0$, hence $\varphi \in \mathrm{K}(V_v, \varphi(v) + T_0)$ and $\psi \in \mathrm{K}(W_v, \psi(v) + T_0)$. Moreover, let

$$U_v = \varphi^{-1}(\varphi(v) + S_0) \cap \psi^{-1}(\psi(v) + S_0)$$

and note that it is an open neighbourhood of v satisfying $U_v \cap V \subset V_v \cap W_v$. One may note that for every $f \in \mathrm{K}(V_v, \varphi(v) + T_0)$ and $g \in \mathrm{K}(W_v, \psi(v) + T_0)$, we have

$$(f + g)(U_v \cap V) \subset f(V_v) + g(W_v) \subset \varphi(v) + \psi(v) + T_0 + T_0 \subset (\varphi + \psi)(v) + U_0 \subset U.$$

As V is compact and $\{U_v\}_{v \in V}$ is an open cover of V, there is a finite subset S of V such that $V \subset \bigcup_{v \in S} U_v$. Then clearly

$$\varphi \in \bigcap_{v \in S} \mathrm{K}(V_v, \varphi(v) + T_0) \text{ and } \psi \in \bigcap_{v \in S} \mathrm{K}(W_v, \psi(v) + T_0).$$

Let $f \in \bigcap_{v \in S} \mathrm{K}(V_v, \varphi(v) + T_0)$ and $g \in \bigcap_{v \in S} \mathrm{K}(W_v, \psi(v) + T_0)$ and let $v \in V$. Then $v \in U_{v'}$ for some $v' \in S$, hence

$$(f + g)(v) = f(v) + g(v) \in f(U_{v'} \cap V) + g(U_{v'} \cap V) \subset f(V_{v'}) + g(W_{v'}).$$

As

$$f(V_{v'}) + g(W_{v'}) \subset \varphi(v') + \psi(v') + T_0 + T_0 \subset U,$$

we have $(f + g)(V) \subset U$. Hence, the open subset

$$\left(\bigcap_{v \in S} \mathrm{K}(V_v, \varphi(v) + T_0) \right) \times \left(\bigcap_{v \in S} \mathrm{K}(W_v, \psi(v) + T_0) \right)$$

of $C \times C$ is an open neighbourhood of (φ, ψ) contained in $\pi^{-1}(\mathrm{K}(V, U))$. It follows that the map π is continuous, which finishes the proof. ■

Let G be a topological group, and let A be a topological G-module. For $g \in G$ and $\varphi \in \mathrm{C}(G, A)$, define $g\varphi$ to be the map sending h to $\varphi(hg)$. The latter map

is continuous, since it is the composition of two continuous maps, namely right-multiplication by g on G followed by φ. Moreover, it is easy to see that this action defines a G-module structure on $\mathrm{C}(G, A)$. However, the following example shows that this action is not always continuous.

Example 1.22. Let G be the topological group $\mathbf{R}^\infty$ endowed with the product topology. Moreover, let A be the topological group $\mathbf{R}$, and let A be a topological G-module via the trivial action of G on A. We will show that the map

$$\pi\colon G \times \mathrm{C}(G, A) \longrightarrow \mathrm{C}(G, A)$$

given by $(g, \varphi) \mapsto g\varphi$ is not continuous.

To this end, let $\overline{0}$ be the neutral element of $\mathbf{R}^\infty$, and consider the open $\mathrm{K}(\{\overline{0}\}, (-1, 1))$ of $\mathrm{C}(G, A)$. Let $\mathbf{0}$ be the neutral element of $\mathrm{C}(G, A)$, which sends $g \in G$ to $0 \in A$, and note that

$$(\overline{0}, \mathbf{0}) \in \pi^{-1}(\mathrm{K}(\{\overline{0}\}, (-1, 1))),$$

since we have $\overline{0} \cdot \mathbf{0}(\overline{0}) = \mathbf{0}(\overline{0}) = 0$.

Now, suppose that $\pi^{-1}(\mathrm{K}(\{\overline{0}\}, (-1, 1)))$ is open in $G \times \mathrm{C}(G, A)$. Then there are, for some $n \in \mathbf{Z}_{\geq 1}$, open sets $\mathrm{K}(V_1, U_1), \ldots, \mathrm{K}(V_n, U_n)$ in $\mathrm{C}(G, A)$ such that

$$\mathbf{0} \in \bigcap_{i=1}^{n} \mathrm{K}(V_i, U_i),$$

and an open neighbourhood $W_{\overline{0}}$ of $\overline{0} \in G$ such that

$$W_{\overline{0}} \times \bigcap_{i=1}^{n} \mathrm{K}(V_i, U_i) \subset \pi^{-1}(\mathrm{K}(\{\overline{0}\}, (-1, 1))).$$

We may assume without loss of generality that the V_i are non-empty. Note that for all $i = 1, \ldots, n$ the open sets U_i are open neighbourhoods of $0 \in A$, since $\mathbf{0} \in \mathrm{K}(V_i, U_i)$. Hence, there is an open interval $(-\varepsilon, \varepsilon)$ that is contained in every U_i for $i = 1, \ldots, n$.

By definition of the product topology, we have

$$W_{\overline{0}} = \prod_{i=1}^{\infty} W_i,$$

where for all $i \in \mathbf{Z}_{\geq 1}$, the set W_i is an open subset of $\mathbf{R}$, and $W_i = \mathbf{R}$ for all but finitely many $i \in \mathbf{Z}_{\geq 1}$. Choose $i_0 \in \mathbf{Z}_{\geq 1}$ such that $W_{i_0} = \mathbf{R}$, and consider the continuous map $\pi_{i_0}\colon G \longrightarrow A$ that projects onto the i_0th coordinate, that is, the map sending $g = (g_i)_{i=1}^{\infty} \in G$ to g_{i_0}.

Since for $i = 1, \ldots, n$ the set V_i is compact, its image under π_{i_0} is compact in $\mathbf{R}$. Hence, for $i = 1, \ldots, n$, we may choose $N_i \in \mathbf{R}_{>0}$ such that $\pi_{i_0}(V_i) \subset [-N_i, N_i]$.

Let $N = \max\{N_1, \ldots, N_n\}$ and let $t \in \mathbf{R}_{>0}$ be such that $|t| < \frac{\varepsilon}{N}$. Let $\psi_t \colon A \longrightarrow A$ be the continuous map sending $a \in A$ to ta. Then observe that for all $i = 1, \ldots, n$ we have

$$(\psi_t \circ \pi_{i_0})(V_i) \subset (-\varepsilon, \varepsilon),$$

from which it follows that

$$\psi_t \circ \pi_{i_0} \in \bigcap_{i=1}^{n} \mathrm{K}(V_i, U_i).$$

However, we do not have $(\psi_t \circ \pi_{i_0})(W_{\overline{0}}) \subset (-1, 1)$. Indeed, consider the element $(g_i)_{i=1}^{\infty}$ of $W_{\overline{0}}$ having i_0th coordinate equal to $\frac{1}{t}$, and note that it maps to $1 \in \mathbf{R}$ under $\psi_t \circ \pi_{i_0}$, which is not an element of $(-1, 1)$. This contradicts

$$W_{\overline{0}} \times \bigcap_{i=1}^{n} \mathrm{K}(V_i, U_i \subset \pi^{-1}(\mathrm{K}(\{\overline{0}\}, (-1, 1))),$$

hence the map π is not continuous. ■

One may note that the topological group $\mathbf{R}^{\infty}$ of the example above is not locally compact. This is in fact the only obstacle preventing the action from being continuous, as the following proposition shows.

Recall that a locally compact topological space is a topological space such that every point has a compact neighbourhood. Equivalently, every point has a neighbourhood whose closure is compact.

Proposition 1.23. *Let G be a locally compact topological group, and let A be an abelian topological group. Then $\mathrm{C}(G, A)$ is a topological G-module.*

Proof. For ease of notation write C for $\mathrm{C}(G, A)$. We have to show that

$$\pi \colon G \times C \longrightarrow C$$

defined by $(g, \varphi) \longrightarrow g \cdot \varphi$ is continuous. To this end, let $(g, \varphi) \in \pi^{-1}(\mathrm{K}(V, U))$ for some $\mathrm{K}(V, U)$ in the subbase of C. We will find an open neighbourhood of g in G and an open neighbourhood of φ in C such that their product, which is an open neighbourhood of (g, φ) in $G \times C$, is inside $\pi^{-1}(K(V, U))$.

Note that $Vg \subset \varphi^{-1}(U)$. As Vg is compact, there exists an open neighbourhood U_1 of $1 \in G$ satisfying $Vg \cdot U_1 \subset \varphi^{-1}(U)$. Choose an open neighbourhood S_1 of $1 \in G$ such that $S_1 \subset U_1$ and $\overline{S_1} \subset U_1$. Observe that $Vg \cdot \overline{S_1} \subset \varphi^{-1}(U)$.

As G is locally compact, there exists an open neighbourhood V_1 of $1 \in G$ such that $\overline{V_1}$ is compact. Note that $\overline{S_1 \cap V_1}$ is compact, since it is closed in the compact set $\overline{V_1}$. Moreover

$$Vg \cdot \left(\overline{S_1 \cap V_1}\right) \subset \varphi^{-1}(U), \text{ since } \overline{S_1 \cap V_1} \subset \overline{S_1} \subset U_1.$$

It follows that $\varphi(Vg \cdot \left(\overline{S_1 \cap V_1}\right)) \subset U$, and since the product of the compact sets Vg and $\overline{S_1 \cap V_1}$ is again compact, we have $\varphi \in \mathrm{K}(Vg(\overline{S_1 \cap V_1}), U)$. As $g(S_1 \cap V_1)$ is clearly an open neighbourhood of g, we have

$$(g, \varphi) \in g(S_1 \cap V_1) \times \mathrm{K}(Vg(\overline{S_1 \cap V_1}), U).$$

Let $h \in g(S_1 \cap V_1)$ and $\psi \in \mathrm{K}(Vg(\overline{S_1 \cap V_1}), U)$. Note that

$$h\psi(V) = \psi(Vh) \in \psi(Vg(\overline{S_1 \cap V_1})) \subset U,$$

so that $(h, \psi) \in \pi^{-1}(\mathrm{K}(V, U))$. It follows that

$$(g(S_1 \cap V_1)) \times \mathrm{K}(Vg(\overline{S_1 \cap V_1}), U)$$

is an open neighbourhood of (g, φ) contained in $\pi^{-1}(\mathrm{K}(V, U))$, hence π is continuous.

■

Lemma 1.24. *Let G be a topological group, and let A be an abelian group. Then the evaluation map $\mathrm{ev}_1 \colon \mathrm{C}(G, A) \longrightarrow A$ defined by $\varphi \mapsto \varphi(1)$ is a continuous group homomorphism.*

Proof. It is clear that ev_1 is a group homomorphism. Let $U \subset A$ be open, and note that $\mathrm{ev}_1^{-1}(U) = \mathrm{K}(\{1\}, U)$. It follows that ev_1 is a continuous group homomorphism.

■

Proposition 1.25. *Let G be a locally compact topological group, and let A be an abelian group. Then for every $n \in \mathbf{Z}_{\geq 1}$ we have $\mathrm{H}^n(G, \mathrm{C}(G, A)) = 0$.*

Proof. Let $n \in \mathbf{Z}_{>0}$, and let $\varphi \colon G^n \longrightarrow \mathrm{C}(G, A)$ be an element of $\mathrm{Z}^n(G, \mathrm{C}(G, A))$. We will show that φ is an n-coboundary by constructing $\psi \in \mathrm{C}^{n-1}(G, A)$ that maps to φ via the boundary map d_{n-1}. Then it follows that $[\varphi] = [0] \in \mathrm{H}^n(G, A)$.

To this end, define $\psi \colon G^{n-1} \longrightarrow \mathrm{C}(G, A)$ as the map sending $(g_1, \ldots, g_{n-1})$ to the map sending g to $\varphi(g, g_1, \ldots, g_{n-1})(1)$, that is, for $g_1, \ldots, g_{n-1}, g \in G$ we have

$$\psi(g_1, \ldots, g_{n-1})(g) = \varphi(g, g_1, \ldots, g_{n-1})(1).$$

Note that for $g_1, \ldots, g_{n-1} \in G$ the map $\psi(g_1, \ldots, g_{n-1})$ is continuous, as it is the composition of the continuous map $\varphi|_{G \times \{g_1\} \times \ldots \times \{g_{n-1}\}}$ and the evaluation map $\mathrm{ev}_1 \colon \mathrm{C}(G, A) \longrightarrow A$. Hence, the map ψ is well-defined.

To show that ψ is continuous, let $\mathrm{K}(V, U)$ be an open of $\mathrm{C}(G, A)$, and let

$$\sigma = (g_1, \ldots, g_{n-1}) \in \psi^{-1}(\mathrm{K}(V, U)).$$

Write W for the open set $\varphi^{-1}(\mathrm{ev}_1^{-1}(U))$, and note that $\varphi(V, \sigma)(1) \subset U$ is equivalent to $(V, \sigma) \subset \varphi^{-1}(\mathrm{ev}_1^{-1}(U)) = W$. Since W is open, we have for any $v \in V$ an open neighbourhood of (v, σ) contained in W, that is, there is an open neighbourhood U_v

of $v \in G$ and an open neighbourhood $U_{(v,\sigma)}$ of σ in G^{n-1} such that $U_v \times U_{(v,\sigma)}$ is an open neighbourhood of (v, σ) inside of W.

As the U_v for $v \in V$ form an open cover of V and V is compact, there exists a finite subset S of V such that $V \subset \bigcup_{v \in S} U_v$ by compactness of V. Clearly the set $\bigcap_{v \in S} U_{(v,\sigma)}$ is an open neighbourhood of (V, σ) contained in W. This shows that ψ is continuous.

Now, note that for $g_1, \ldots, g_n, g \in G$ we have

$$(g_1\psi)(g_2, \ldots, g_n)(g) = \psi(g_2, \ldots, g_n)(gg_1) = \varphi(gg_1, g_2, \ldots, g_n)(1).$$

Moreover, note that for $g_1, \ldots, g_n, g \in G$

$$\begin{aligned} d_{n-1}(\psi)(g_1, \ldots, g_n)(g) = & \\ = \ & (g_1\psi)(g_2, \ldots, g_n)(g) + \sum_{i=1}^{n-1} (-1)^i \psi(g_1, \ldots, g_i g_{i+1}, \ldots, g_n)(g) + \\ & \qquad\qquad + (-1)^n \psi(g_1, \ldots, g_{n-1})(g) \\ = \ & \varphi(gg_1, g_2, \ldots, g_n)(1) + \sum_{i=1}^{n-1} (-1)^i \varphi(g, g_1, \ldots, g_i g_{i+1}, \ldots, g_n)(1) + \\ & \qquad\qquad + (-1)^n \varphi(g, g_1, \ldots, g_{n-1})(1) \\ = \ & (\varphi(gg_1, g_2, \ldots, g_n) + \sum_{i=1}^{n-1} (-1)^i \varphi(g, g_1, \ldots, g_i g_{i+1}, \ldots, g_n) + \\ & \qquad\qquad + (-1)^n \varphi(g, g_1, \ldots, g_{n-1}))(1). \end{aligned}$$

Rename g to g_0 and note that

$$(\varphi(g_0 g_1, g_2, \ldots, g_n) + \sum_{i=1}^{n-1} (-1)^i \varphi(g_0, g_1, \ldots, g_i g_{i+1}, \ldots, g_n) + (-1)^n \varphi(g_0, g_1, \ldots, g_{n-1}))(1)$$

is equal to

$$\left(-\sum_{i=0}^{n-1} (-1)^{i+1} \varphi(g_0, g_1, \ldots, g_i g_{i+1}, \ldots, g_n) - (-1)^{n+1} \varphi(g_0, \ldots, g_{n-1}) \right)(1).$$

As $d_n \varphi = 0$, it follows that

$$-\sum_{i=0}^{n-1} (-1)^{i+1} \varphi(g_0, g_1, \ldots, g_i g_{i+1}, \ldots, g_n) - (-1)^{n+1} \varphi(g_0, \ldots, g_{n-1}) = g_0 \cdot \varphi(g_1, \ldots, g_n),$$

so that

$$(d_{n-1}\psi)(g_1, \ldots, g_n)(g_0) = (g_0\varphi)(g_1, \ldots, g_n)(1) = \varphi(g_1, \ldots, g_n)(g_0).$$

Hence we have $d_{n-1}\psi = \varphi$. ■

Lemma 1.26. *Let G be a locally compact topological group, and let A be a topological G-module. Then the map $i_A \colon A \longrightarrow \mathrm{C}(G, A)$ sending $a \in A$ to the map π_a that sends $g \in G$ to ga, is an injective continuous G-module homomorphism that induces a continuous group isomorphism from A to $i_A(A)$.*

Proof. Let $a \in A$ and note that $\pi_a \colon G \longrightarrow A$ given by $g \mapsto ga$ is continuous, since it is the composition of the continuous map $G \longrightarrow G \times A$ given by $h \mapsto (h, a)$ and the map defining the action of G on A. Hence, the map i_A is well defined. Moreover, it is easy to check that i_A is an injective G-module homomorphism.

To see continuity, let $\mathrm{K}(V, U)$ be an open of $\mathrm{C}(G, A)$ and let $a \in i_A^{-1}(\mathrm{K}(V, U))$. Note that $\pi_a(V) = Va \subset U$. By continuity of the action of G on A, there exists for $v \in V$ an open neighbourhood V_v of v in G and an open neighbourhood U_v of a in A, such that $V_v U_v \subset U$. Note that $\{V_v\}_{v \in V}$ forms an open cover of V. As V is compact, there is a finite subset S of V such that $V \subset \bigcup_{v \in S} V_v$.

Note that the open set $\bigcap_{v \in S} U_v$ is a neighbourhood of a in A that satisfies

$$V \cdot \left(\bigcap_{v \in S} U_v \right) \subset U.$$

Hence, it is an open neighbourhood of a in A contained in $i_A^{-1}(\mathrm{K}(V, U))$, from which it follows that i_A is continuous.

Now, consider the evaluation map $\mathrm{ev}_1 \colon \mathrm{C}(G, A) \longrightarrow A$ given by $\varphi \mapsto \varphi(1)$, and note that it is a continuous group homomorphism. For any $a \in A$ we have

$$\mathrm{ev}_1(i_A(a)) = \pi_a(1) = a,$$

hence ev_1 induces an inverse of $i_A \colon A \longrightarrow i_A(A)$. It follows that A is isomorphic to $i_A(A)$ as topological groups. ■

Proposition 1.27. *Let G be a locally compact topological group, and let A be a topological G-module. Then the sequence*

$$0 \longrightarrow A \xrightarrow{i_A} \mathrm{C}(G, A) \xrightarrow{\pi} \mathrm{C}(G, A)/i_A(A) \longrightarrow 0$$

where $\mathrm{C}(G, A)/i_A(A)$ is endowed with the quotient topology and π is the quotient map, is a well-adjusted short exact sequence of topological G-modules that is split as a sequence of abelian topological groups.

Proof. It is clear that it is a short exact sequence of topological G-modules. Now, as sequence of abelian topological groups, the map ev_1 is a retraction of i_A, by the previous lemma. Hence, the sequence splits as sequence of abelian topological groups. By Proposition 1.11, the sequence is well-adjusted. ■

5. Conjugation on cohomology groups

Let G be a topological group, and let A be a topological G-module. For $g \in G$, let φ_g be the endomorphism of G given by $h \mapsto g^{-1}hg$, and let ψ_g be the endomorphism of the abelian group A given by $a \mapsto ga$. It is clear that these are continuous group homomorphisms. Moreover, for $a \in A$ and $h \in G$ we have

$$\psi_g(\varphi_g(h)a) = \psi_g(g^{-1}hga) = hga = h\psi_g(a),$$

so for any $g \in G$ the pair (φ_g, ψ_g) is compatible.

Theorem 1.28. *Let G be a locally compact topological group, and let A be a topological G-module. Then for every $g \in G$ the compatible pair (φ_g, ψ_g) induces the identity on cohomology groups, i.e., for every $n \in \mathbf{Z}_{\geq 0}$ the map*

$$\mathrm{H}^n(\varphi_g, \psi_g)\colon \mathrm{H}^n(G, A) \longrightarrow \mathrm{H}^n(G, A)$$

is the identity.

Proof. Let $g \in G$. We prove the statement by induction on n. Let $n = 0$, and note that $\mathrm{H}^0(\varphi_g, \psi_g)\colon A^G \longrightarrow A^G$ is given by $a \mapsto ga$. As g acts trivially on A^G, it follows that $\mathrm{H}^0(\varphi_g, \psi_g)$ is the identity.

Now, suppose $n \in \mathbf{Z}_{>0}$ is such that for every topological G-module A the map $\mathrm{H}^{n-1}(\varphi_g, \psi_g)$ is the identity. Note that (φ_g, ψ_g) is a compatible pair for every topological G-module A, hence we have a pair for $\mathrm{C}(G, A)$ and for $\mathrm{C}(G, A)/i_A(A)$ too, which we will by abuse of notation still denote by (φ_g, ψ_g). One easily checks that the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i_A"] \arrow[d, "\psi_g"'] & \mathrm{C}(G, A) \arrow[r] \arrow[d, "\psi_g"'] & \mathrm{C}(G, A)/i_A(A) \arrow[r] \arrow[d, "\psi_g"'] & 0 \\
0 \arrow[r] & A \arrow[r, "i_A"] & \mathrm{C}(G, A) \arrow[r] & \mathrm{C}(G, A)/i_A(A) \arrow[r] & 0
\end{tikzcd}$$

is commutative, hence Theorem 1.16 gives the following commutative diagram of long exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A^G \arrow[r] \arrow[d, "\mathrm{H}^0(\varphi_g,\psi_g)"'] & \mathrm{C}(G, A)^G \arrow[r] \arrow[d, "\mathrm{H}^0(\varphi_g,\psi_g)"'] & (\mathrm{C}(G, A)/i_A(A))^G \arrow[r, "\delta_0"] \arrow[d, "\mathrm{H}^0(\varphi_g,\psi_g)"'] & \dots \\
0 \arrow[r] & A^G \arrow[r] & \mathrm{C}(G, A)^G \arrow[r] & (\mathrm{C}(G, A)/i_A(A))^G \arrow[r, "\delta_0"] & \dots
\end{tikzcd}$$

of cohomology groups. Note that Proposition 1.25 implies that

$$\mathrm{H}^n(G, \mathrm{C}(G, A)) = 0,$$

so that

$$\begin{tikzcd}
\mathrm{H}^{n-1}(G, \mathrm{C}(G, A)/i_A(A)) \arrow[r, "\delta_{n-1}"] \arrow[d, "\mathrm{H}^{n-1}(\varphi_g,\psi_g)"'] & \mathrm{H}^n(G, A) \arrow[r] \arrow[d, "\mathrm{H}^n(\varphi_g,\psi_g)"'] & 0 \arrow[r] & \dots \\
\mathrm{H}^{n-1}(G, \mathrm{C}(G, A)/i_A(A)) \arrow[r, "\delta_{n-1}"] & \mathrm{H}^n(G, A) \arrow[r] & 0 \arrow[r] & \dots .
\end{tikzcd}$$

By the induction hypotheses $\mathrm{H}^{n-1}(\varphi_g, \psi_g)$ is the identity, so that the commutativity of the diagram implies that $\mathrm{H}^n(\varphi_g, \psi_g)\colon \mathrm{H}^n(G, A) \longrightarrow \mathrm{H}^n(G, A)$ is also the identity, which concludes the proof. ∎

We will now prove the above statement without the assumption of local compactness. For G a topological group, A a topological G-module, $g \in G$ and $n \in \mathbf{Z}_{\geq 0}$, the proof of the more general theorem below is based on giving a chain homotopy between the conjugation map $\mathrm{C}^n(\varphi_g, \psi_g)\colon \mathrm{C}^n(G, A) \longrightarrow \mathrm{C}^n(G, A)$ and the identity map $\mathrm{id}_n\colon \mathrm{C}^n(G, A) \longrightarrow \mathrm{C}^n(G, A)$. Recall that a chain homotopy between these maps is a set of maps

$$\{s_n\colon \mathrm{C}^{n+1}(G, A) \longrightarrow \mathrm{C}^n(G, A)\}_{n \in \mathbf{Z}_{\geq 0}}$$

such that for each $n \in \mathbf{Z}_{\geq 0}$

$$\mathrm{C}^n(\varphi_g, \psi_g) - \mathrm{id}_n = d_{n-1} \circ s_n + s_{n+1} \circ d_n,$$

where $d_{-1} = 0$. Then the map $\mathrm{H}^n(\varphi_g, \psi_g)\colon \mathrm{H}^n(G, A) \longrightarrow \mathrm{H}^n(G, A)$ sends $[f] \in \mathrm{H}^n(G, A)$ to $[d_{n-1}(s_n(f)) + s_{n+1}(d_n(f)) + \mathrm{id}_n(f)]$, which is equal to $[\mathrm{id}_n(f)]$, since $d_{n-1}(s_n(f)) \in \mathrm{im}(d_{n-1})$ and $d_n(f) = 0$. Hence, the maps $\mathrm{C}^n(\varphi_g, \psi_g)$ and id_n induce the same maps on cohomology groups.

Theorem 1.29. *Let G be a topological group, and let A be a topological G-module. Then for every $g \in G$ the compatible pair (φ_g, ψ_g) induces the identity on cohomology groups.*

Proof. For ease of notation and the sake of clarity, we write σ for g, and for $h \in G$ we write h^σ for $\varphi_\sigma(h)$. Moreover, for $n \in \mathbf{Z}_{\geq 0}$ we write the coboundary map d_n as a sum of three maps $d_{n,1}, d_{n,2}$ and $d_{n,3}$, which for $F \in \mathrm{C}^n(G, A)$ and $g_1, \ldots, g_n \in G$ are given by

$$\begin{aligned}
d_{n,1}(F)(g_1, \ldots, g_{n+1}) &= g_1 F(g_2, \ldots, g_{n+1}) \\
d_{n,2}(F)(g_1, \ldots, g_{n+1}) &= \sum_{i=1}^{n} (-1)^i F(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \\
d_{n,3}(F)(g_1, \ldots, g_{n+1}) &= (-1)^{n+1} F(g_1, \ldots, g_n).
\end{aligned}$$

We will show that for $n \in \mathbf{Z}_{\geq 1}$ a chain homotopy s_n between $\mathrm{C}^n(\varphi_\sigma, \psi_\sigma)$ and id_n is given by sending $F \in \mathrm{C}^n(G, A)$ to the map sending $(g_1, \ldots, g_{n-1}) \in G^{n-1}$ to

$$F(\sigma, g_1^\sigma, \ldots, g_{n-1}^\sigma) - F(g_1, \sigma, g_2^\sigma, \ldots, g_{n-1}^\sigma) + F(g_1, g_2, \sigma, g_3^\sigma, \ldots, g_{n-1}^\sigma) - \ldots$$

$$\ldots + (-1)^n F(g_1, g_2, \ldots, g_{n-2}, \sigma, g_{n-1}^\sigma) + (-1)^{n+1} F(g_1, \ldots, g_{n-1}, \sigma).$$

Moreover, for $n = 0$ the chain homotopy s_0 is just the zero map. It is clear that for $n \in \mathbf{Z}_{\geq 0}$, the map $s_n(f)$ is continuous and that $s_n(f + g) = s_n(f) + s_n(g)$, hence s_n is a well-defined group homomorphism. One easily checks that $\mathrm{C}^0(\varphi_\sigma, \psi_\sigma) - \mathrm{id}_0 = d_{-1} \circ s_0 + s_1 \circ d_0$.

Now, let $n \in \mathbf{Z}_{\geq 1}$. For $F \in \mathrm{C}^n(G, A)$ and $i \in \{1, \ldots, n\}$, we let

$$F_{i,\sigma}\colon G^{n-1} \longrightarrow A$$

be the map sending $(g_1, \ldots, g_{n-1})$ to $F(g_1, \ldots, g_{i-1}, \sigma, g_i^\sigma, \ldots, g_{n-1}^\sigma)$. Note that

$$F_{1,\sigma}(g_1, \ldots, g_{n-1}) = F(\sigma, g_1^\sigma, \ldots, g_{n-1}^\sigma) \text{ and } F_{n,\sigma}(g_1, \ldots, g_{n-1}) = F(g_1, \ldots, g_{n-1}, \sigma).$$

Then clearly for $(g_1, \ldots, g_{n-1}) \in G^{n-1}$, we have

$$s_n(F)(g_1, \ldots, g_{n-1}) = \sum_{i=1}^{n} (-1)^{i+1} F_{i,\sigma}(g_1, \ldots, g_{n-1}).$$

Now, fix $F \in \mathrm{C}^n(G, A)$ and $\overline{g} = (g_1, \ldots, g_n) \in G^n$. Then

$$s_{n+1} d_n F = s_{n+1} d_{n,1} F + s_{n+1} d_{n,2} F + s_{n+1} d_{n,3} F,$$

and observe that

$$\begin{aligned}(s_{n+1} d_{n,1} F)(\overline{g}) &= \sum_{i=1}^{n+1} (-1)^{i+1} (d_{n,1} F)_{i,\sigma}(\overline{g}) \\ &= \sigma F(g_1^g, \ldots, g_n^g) + \sum_{i=1}^{n} (-1)^i g_1 F_{i,\sigma}(g_2, \ldots, g_n)\end{aligned}$$

and

$$\begin{aligned}(s_{n+1} d_{n,3} F)(\overline{g}) &= \sum_{i=1}^{n+1} (-1)^{i+1} (d_{n,3} F)_{i,\sigma}(\overline{g}) \\ &= (-1)^n (-1)^{n+1} F(g_1, \ldots, g_n) + (-1)^{n+1} \sum_{i=1}^{n} (-1)^{i+1} F_{i,g}(g_1, \ldots, g_{n-1}) \\ &= -F(g_1, \ldots, g_n) + \sum_{i=1}^{n} (-1)^{n+i} F_{i,g}(g_1, \ldots, g_{n-1}).\end{aligned}$$

Moreover

$$\begin{aligned}(s_{n+1} d_{n,2} F)(\overline{g}) &= \sum_{i=1}^{n+1} (-1)^{i+1} (d_{n,2} F)_{i,\sigma}(\overline{g}) \\ &= \sum_{i=1}^{n+1} (-1)^{i+1} (d_{n,2} F)(g_1, \ldots, g_{i-1}, \sigma, g_i^\sigma, \ldots, g_n^\sigma)\end{aligned}$$

where for $i \in \{1, \ldots, n+1\}$ we have

$$\begin{aligned}&(d_{n,2} F)(g_1, \ldots, g_{i-1}, \sigma, g_i^\sigma, \ldots, g_n^\sigma) = \\ &\quad = \sum_{j=1}^{i-2} (-1)^j F(g_1, \ldots, g_j g_{j+1}, \ldots, g_{i-1}, \sigma, g_i^\sigma, \ldots, g_n^\sigma) + \\ &\quad + (-1)^{i-1} F(g_1, \ldots, g_{i-1}\sigma, g_i^\sigma, \ldots, g_n^\sigma) + (-1)^i F(g_1, \ldots, g_{i-1}, g_i\sigma, \ldots, g_n^\sigma) + \\ &\quad + \sum_{j=i}^{n-1} (-1)^{j+1} F(g_1, \ldots, g_{i-1}, \sigma, g_i^\sigma, \ldots, (g_j g_{j+1})^\sigma, \ldots, g_n^\sigma).\end{aligned} \tag{1}$$

Note that the two terms of (1) cancel when summed over i, so that

$$(s_{n+1}d_{n,2}F)(\overline{g}) = \sum_{i=1}^{n}(-1)^{i+1}\sum_{j=1}^{n-1}(-1)^{j+1}F_{i,\sigma}(g_1,\ldots,g_jg_{j+1},\ldots,g_n).$$

On the other hand, we have

$$(d_{n-1}s_nF)(\overline{g}) = (d_{n-1,1}s_nF + d_{n-1,2}s_nF + d_{n-1,3}s_nF)(\overline{g}),$$

where

$$\begin{aligned}
(d_{n-1,1}s_nF)(\overline{g}) &= g_1\cdot\sum_{i=1}^{n}(-1)^{i+1}F_{i,g}(g_2,\ldots,g_n)\\
(d_{n-1,2}s_nF)(\overline{g}) &= \sum_{j=1}^{n-1}(-1)^{j}\sum_{i=1}^{n}(-1)^{i+1}F_{i,g}(g_1,\ldots,g_jg_{j+1},\ldots,g_n)\\
(d_{n-1,3}s_nF)(\overline{g}) &= (-1)^n\sum_{i=1}^{n}(-1)^{i+1}F_{i,g}(g_1,\ldots,g_{n-1}).
\end{aligned}$$

As

$$\begin{aligned}
s_{n+1}d_nF + d_{n-1}s_nF &= s_{n+1}d_{n,1}F + s_{n+1}d_{n,2}F + s_{n+1}d_{n,3}F +\\
&\quad + d_{n-1,1}s_nF + d_{n-1,2}s_nF + d_{n-1,3}s_nF,
\end{aligned}$$

one easily sees from the above that

$$(s_{n+1}d_nF + d_{n-1}s_nF)(\overline{g}) = \sigma F(g_1^\sigma,\ldots,g_n^\sigma) - F(g_1,\ldots,g_n) = \mathrm{C}^n(\varphi_\sigma,\psi_\sigma) - \mathrm{id}_n,$$

which shows that the s_n indeed form a chain homotopy between $\mathrm{C}^n(\varphi_\sigma,\psi_\sigma)$ and id_n.

■

Let G be a topological group, and let A be a topological G-module. Note that for every $n \in \mathbf{Z}_{\geq 0}$ there is a natural action of the center $\mathrm{Z}(G)$ of G on $\mathrm{H}^n(G,A)$ via the action of G on A. Indeed, the functor $\mathrm{H}^n(G,\cdot)$ gives a group homomorphism $\mathrm{End}_G(A) \longrightarrow \mathrm{End}(\mathrm{H}^n(G,A))$. As the functor is additive, the latter map is a ring homomorphism, which, in turn, induces a group homomorphism

$$g\colon\ \mathrm{Aut}_G(A) \longrightarrow \mathrm{Aut}(\mathrm{H}^n(G,A))$$

on the units. Observe that the image of $\mathrm{Z}(G)$ under the homomorphism $f\colon G \longrightarrow \mathrm{Aut}(A)$ defining the action of G on A is clearly inside $\mathrm{Aut}_G(A)$. Hence, we can compose f and g. This gives the homomorphism

$$g\circ f\colon\ \mathrm{Z}(G) \longrightarrow \mathrm{Aut}(\mathrm{H}^n(G,A))$$

defining the action of $\mathrm{Z}(G)$ on $\mathrm{Aut}(\mathrm{H}^n(G,A))$.

Proposition 1.30. *Let G be a topological group, and let A be a topological G-module. Then for every $g \in \mathrm{Z}(G)$ and every $n \in \mathbf{Z}_{\geq 0}$, the endomorphism*

$$\mathrm{H}^n(G, \psi_g)\colon \mathrm{H}^n(G, A) \longrightarrow \mathrm{H}^n(G, A)$$

is equal to the identity on $\mathrm{H}^n(G, A)$.

Proof. We have $\mathrm{H}^n(G, \psi_g) = \mathrm{H}^n(\mathrm{id}_G, \psi_g) = \mathrm{H}^n(\varphi_g, \psi_g)$, where the latter equality holds since $g \in \mathrm{Z}(G)$. The proposition now follows from Theorem 1.29. ■

6. Topological group extensions

In this section we briefly address the extension theory of topological groups by only introducing the necessary definitions and the relation to cohomology. Although our definitions are slightly more general than in [Hu52], the arguments are exactly the same. Therefore, we do refer to [Hu52] for more details.

Throughout this chapter, let G be a topological group and let A be a topological G-module.

Definition 1.31. A *topological group extension of G by A* is a triple (E, f, g) consisting of a topological group E together with a well-adjusted short exact sequence

$$0 \longrightarrow A \xrightarrow{f} E \xrightarrow{g} G \longrightarrow 1$$

of topological groups, such that for all $a \in A$ and $x \in E$, we have $xf(a)x^{-1} = f(g(x) \cdot a)$.

Notation 1.32. We will often denote the extension (E, f, g) by the well-adjusted short exact sequence that is associated with it, or just by E when the maps f and g are understood.

Definition 1.33. Let (E, f, g) and (E', f', g') be two topological extensions of G by A. Then (E, f, g) and (E', f', g') are said to be *equivalent* if there exists an isomorphism $\varphi\colon E \longrightarrow E'$ of topological groups such that the diagram

$$\begin{tikzcd}
 & & E \arrow[dd, "\varphi"] \arrow[dr, "g"] & & \\
0 \arrow[r] & A \arrow[ur, "f"] \arrow[dr, "f'"'] & & G \arrow[r] & 1 \\
 & & E' \arrow[ur, "g'"'] & &
\end{tikzcd}$$

commutes.

It is easy to check that being equivalent as in the definition above is an equivalence relation on the class of all topological extensions of G by A. For convenience, let X denote the set of all equivalence classes of topological extensions of G by A.

Let (E, f, g) be a topological extension of G by A, and let s be a continuous section of g. Then associating to (E, f, g), the map $G^2 \longrightarrow A$ given by

$$(g_1, g_2) \mapsto s(g_1)s(g_2)s(g_1g_2)^{-1},$$

induces a well-defined map $\varphi\colon X \longrightarrow \mathrm{H}^2(G, A)$, cf. [Hu52].

Theorem 1.34. *The map φ above is a bijection of sets.*

Proof. See [Hu52]. ■

The theorem above enables us to identify elements of $\mathrm{H}^2(G, A)$ with equivalence classes of topological extensions of G by A, and vice versa.

CHAPTER 2

On the cohomology of cyclotomic Galois groups

1. The profinite completion of the ring of integers

Let $n \in \mathbf{Z}_{\geq 1}$, and observe that for every $m \in \mathbf{Z}_{\geq 1}$ dividing n, there is a unique ring homomorphism

$$\mathbf{Z}/n\mathbf{Z} \longrightarrow \mathbf{Z}/m\mathbf{Z}.$$

The rings $\mathbf{Z}/n\mathbf{Z}$, $n \in \mathbf{Z}_{\geq 1}$, together with these ring homomorphisms form a projective system, and we let $\widehat{\mathbf{Z}}$ denote its projective limit, that is, we have

$$\widehat{\mathbf{Z}} = \varprojlim_{n \geq 1} \mathbf{Z}/n\mathbf{Z}.$$

It is the profinite completion of the ring $\mathbf{Z}$. In particular, it is a profinite ring containing $\mathbf{Z}$ as a dense subring. Its group of units $\widehat{\mathbf{Z}}^*$ is equal to the projective limit $\varprojlim_{n \geq 1} (\mathbf{Z}/n\mathbf{Z})^*$ with the obvious maps. Recall that for a prime number p, the ring of p-adic integers $\mathbf{Z}_p$ is the projective limit $\varprojlim_{i \geq 0} \mathbf{Z}/p^i\mathbf{Z}$ with the obvious maps. As projective limits commute with products, it follows by the Chinese remainder theorem that

$$\widehat{\mathbf{Z}} \cong \prod_{p \text{ prime}} \mathbf{Z}_p$$

as topological rings. Hence, we also have

$$\widehat{\mathbf{Z}}^* \cong \prod_{p \text{ prime}} \mathbf{Z}_p^*.$$

One easily checks that for $n \in \mathbf{Z}_{\geq 0}$ the ring $\widehat{\mathbf{Z}}/n\widehat{\mathbf{Z}}$ is isomorphic to $\mathbf{Z}/n\mathbf{Z}$.

Definition 2.1. Let R be a topological ring. An R-module M is called *topological* if M is a topological group and the scalar multiplication of R on M is continuous.

Lemma 2.2. (a) *Let A be a discrete abelian torsion group. Then there is a unique $\widehat{\mathbf{Z}}$-module structure on A. Moreover, this structure is topological.*

(b) *Any group homomorphism of discrete abelian torsion groups is $\widehat{\mathbf{Z}}$-linear.*

Proof. Let A be an abelian torsion group, and note that

$$A = \bigcup_{n=1}^{\infty} A[n] = \varinjlim_{n \geq 1} A[n],$$

where $A[n] = \{a \in A : na = 0\}$ is the n-torsion subgroup of A. The endomorphism ring $\mathrm{End}(A)$ is equal to the projective limit of the projective system

$$\{\mathrm{End}(A[n]), f_{n,m} : n, m \in \mathbf{Z}_{\geq 1}, m|n\},$$

where $f_{n,m}\colon \operatorname{End}(A[n]) \longrightarrow \operatorname{End}(A[m])$ is given by restriction. There is a unique ring homomorphism $\mathbf{Z}/n\mathbf{Z} \longrightarrow \operatorname{End}(A[n])$ for each $n \in \mathbf{Z}_{\geq 1}$, where $\mathbf{Z}/n\mathbf{Z}$ is isomorphic to $\widehat{\mathbf{Z}}/n\widehat{\mathbf{Z}}$. Composing with the quotient map $\widehat{\mathbf{Z}} \longrightarrow \widehat{\mathbf{Z}}/n\widehat{\mathbf{Z}}$, we get unique ring homomorphisms $\widehat{\mathbf{Z}} \longrightarrow \operatorname{End}(A[n])$, which form a compatible system by uniqueness of the maps. Thus, there is a unique ring homomorphism $\widehat{\mathbf{Z}} \longrightarrow \operatorname{End}(A)$ defining the $\widehat{\mathbf{Z}}$-module structure on A.

Give A the discrete topology, and observe that A is a topological $\widehat{\mathbf{Z}}$-module if the annihilator $\operatorname{Ann}_{\widehat{\mathbf{Z}}}(a) = \{r \in \widehat{\mathbf{Z}} : ra = 0\}$ of each $a \in A$ is an open ideal of $\widehat{\mathbf{Z}}$. As the $\widehat{\mathbf{Z}}$-annihilator of $a \in A$ is the open ideal $n\widehat{\mathbf{Z}}$, where n is the order of a, it follows that A is a topological $\widehat{\mathbf{Z}}$-module.

Part (b) follows from the fact that group homomorphisms map n-torsion to n-torsion, for $n \in \mathbf{Z}_{\geq 1}$. ■

Lemma 2.3. (a) *Let A be a profinite abelian group. Then there is a unique $\widehat{\mathbf{Z}}$-module structure on A. Moreover, this structure is topological.*

(b) *Any continuous group homomorphism of profinite abelian groups is $\widehat{\mathbf{Z}}$-linear.*

Proof. As profinite abelian groups are Pontryagin dual to discrete abelian torsion groups and $\widehat{\mathbf{Z}}$ is commutative, we obtain a proof of this lemma by dualizing the proof of Lemma 2.2. We sketch a proof of this and leave the details to the reader. First, consider the projective system $\{A/nA, f_{n,m} : n, m \in \mathbf{Z}_{\geq 0}, m|n\}$ where $f_{n,m}\colon A/nA \longrightarrow A/mA$ sends $a + nA \in A/nA$ to $a + mA$. One checks that $A \cong \varprojlim_{n\geq 1} A/nA$ as topological groups. Then

$$\operatorname{End}(A) \cong \varprojlim_{n\geq 1} \operatorname{End}(A/nA),$$

where the maps $\operatorname{End}(A/nA) \longrightarrow \operatorname{End}(A/mA)$ are the natural maps, for $m, n \in \mathbf{Z}_{\geq 1}$ with m dividing n. Hence, a ring homomorphism $\widehat{\mathbf{Z}} \longrightarrow \operatorname{End}(A)$ is a compatible system of ring homomorphisms $\widehat{\mathbf{Z}} \longrightarrow \operatorname{End}(A/nA)$, $n \geq 1$. As $n\mathbf{Z}$ annihilates A/nA and $\widehat{\mathbf{Z}}/n\widehat{\mathbf{Z}} \cong \mathbf{Z}/n\mathbf{Z}$, these homomorphisms exist. Moreover, since they are unique, we have a unique $\widehat{\mathbf{Z}}$-module structure on A.

To see that this action is topological, let A be the projective limit of a projective system $\{A_i, f_{i,j} : i, j \in I, i \leq j\}$ of finite abelian groups. As the A_i are discrete abelian torsion groups, the ring $\widehat{\mathbf{Z}}$ acts continuously on them. It follows that $\widehat{\mathbf{Z}}$ acts continuously on the projective limit A of the system. By uniqueness, this action coincides with the action above. Hence, the $\widehat{\mathbf{Z}}$-module structure on A is topological.

Part (b) follows from the fact that a homomorphism $A \longrightarrow B$ of profinite groups maps nA to nB for every $n \in \mathbf{Z}_{\geq 1}$. ■

Convention 2.4. Let A be a profinite abelian group. By Lemma 2.3 there is a unique $\widehat{\mathbf{Z}}$-module structure on A. Thus, by restriction A is a $\widehat{\mathbf{Z}}^*$-module. We call this the *natural action* of $\widehat{\mathbf{Z}}^*$ on A.

Now, we describe the closed ideals of $\widehat{\mathbf{Z}}$.

Definition 2.5. A *Steinitz number* m is a formal expression

$$m = \prod_{p \text{ prime}} p^{m(p)}$$

with each $m(p) \in \{0, 1, 2, \ldots, \infty\}$.

Note that each positive integer can be identified with a Steinitz number. Moreover, given a family $\{m_i\}_{i \in I}$ of Steinitz numbers, one can form their product $\prod_{i \in I} m_i$, their greatest common divisor $\gcd_{i \in I}(m_i)$ and their least common multiple $\operatorname{lcm}_{i \in I}(m_i)$ in an obvious manner. One also defines when a Steinitz number divides another in an obvious way. See [RZ09, p. 33] for more details.

Proposition 2.6. *Let the set of Steinitz numbers be ordered by divisibility, and let the set of closed ideals of* $\widehat{\mathbf{Z}}$ *be ordered by inclusion. Then there is an order-reversing bijection*

$$f\colon \{\text{Steinitz numbers}\} \longrightarrow \{\text{closed ideals of } \widehat{\mathbf{Z}}\}$$

of sets, given by sending the Steinitz number m *to the closed ideal* $\bigcap_{n|m} n\widehat{\mathbf{Z}}$*, where* n *ranges over the positive integers dividing* m*. Moreover, its inverse is order-reversing and sends a closed ideal* W *of* $\widehat{\mathbf{Z}}$ *to the least common multiple of all* $n \in \mathbf{Z}_{\geq 1}$ *such that* $W \subset n\widehat{\mathbf{Z}}$*.*

Proof. Let m be a Steinitz number. For $n \in \mathbf{Z}_{\geq 1}$, multiplication by n defines a continuous group endomorphism of $\widehat{\mathbf{Z}}$, so by compactness of $\widehat{\mathbf{Z}}$ we have that $n\widehat{\mathbf{Z}}$ is compact. Moreover, since $\widehat{\mathbf{Z}}$ is Hausdorff, the ideal $n\widehat{\mathbf{Z}}$ is closed. It follows that $\bigcap_{n|m, n<\infty} n\widehat{\mathbf{Z}}$ is a closed ideal of $\widehat{\mathbf{Z}}$. This shows that f is well-defined. It is clear that f is order-reversing.

Observe that an ideal of a product $\prod_{i \in I} R_i$ of topological rings is closed if and only if it is of the form $\prod_{i \in I} J_i$, where J_i is a closed ideal of R_i for $i \in I$. Recall that for a prime number p, the closed ideals of $\mathbf{Z}_p$ are the ideals $p^i\mathbf{Z}_p$ with $i \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$, where we put $p^\infty \mathbf{Z}_p = 0$.

Let W be a closed ideal of $\widehat{\mathbf{Z}}$. As $\widehat{\mathbf{Z}} \cong \prod_p \mathbf{Z}_p$ as topological rings, the ideal W corresponds to an infinite product of closed ideals $p^{m(p)}\mathbf{Z}_p$ of $\mathbf{Z}_p$ for each prime p. Write $m_W = \prod_p p^{m(p)}$ and observe that $W = \bigcap_{n|m_W, n<\infty} n\widehat{\mathbf{Z}}$. It is clear that this defines the inverse of f, and that it is order-reversing. ∎

Notation 2.7. For a Steinitz number m, we denote the closed ideal $\bigcap_{n|m, n<\infty} n\widehat{\mathbf{Z}}$ of $\widehat{\mathbf{Z}}$ by $m\widehat{\mathbf{Z}}$.

Let K be a field of characteristic 0, and let $\overline{K}$ be an algebraic closure of K. For $n \in \mathbf{Z}_{\geq 1}$ and any algebraic extension L of K contained in $\overline{K}$, let

$$\mu_n(L) = \{\xi \in L^* : \xi^n = 1\}$$

be the subgroup of $\overline{K}^*$ consisting of the nth roots of unity in L, and let $\mu_n = \mu_n(\overline{K})$ denote the subgroup of $\overline{K}^*$ consisting of all nth roots of unity in $\overline{K}$. Moreover, let

$$\mu(L) = \bigcup_{n=1}^{\infty} \mu_n(L)$$

be the subgroup of $\overline{K}^*$ consisting of all roots of unity of L, and let

$$\mu = \mu(\overline{K}) = \bigcup_{n=1}^{\infty} \mu_n$$

be the subgroup of $\overline{K}^*$ consisting of all roots of unity in $\overline{K}$.

We describe the subgroups of μ by closed ideals of $\widehat{\mathbf{Z}}$ in the following way. Give μ the discrete topology, and consider the topological $\widehat{\mathbf{Z}}$-module structure on it (cf. Lemma 2.2). As μ is multiplicatively written, we will use exponential notation for the $\widehat{\mathbf{Z}}$-module structure on μ, that is, for $r \in \widehat{\mathbf{Z}}$ and $\xi \in \mu$ we write ξ^r for the action of r on ξ.

Proposition 2.8. *Let K, $\overline{K}$ and μ be as above. Then there is an inclusion-reversing bijection*

$$g\colon \{\text{closed ideals of } \widehat{\mathbf{Z}}\} \longrightarrow \{\text{subgroups of } \mu\}$$

of sets, given by sending a closed ideal W to the W-torsion subgroup

$$\mu[W] = \{\xi \in \mu : \xi^W = 1\}$$

of μ. Moreover, its inverse is also inclusion-reversing, sending a subgroup ν of μ to the $\widehat{\mathbf{Z}}$-annihilator $\mathrm{Ann}_{\widehat{\mathbf{Z}}}(\nu) = \{r \in \widehat{\mathbf{Z}} : \nu^r = 1\}$ of ν.

Proof. First, observe that for $n \in \mathbf{Z}_{\geq 1}$ the subgroup μ_n of μ is the unique subgroup of order n, and its $\widehat{\mathbf{Z}}$-annihilator is equal to $n\widehat{\mathbf{Z}}$.

Let W be a closed ideal of $\widehat{\mathbf{Z}}$, and write $W = m\widehat{\mathbf{Z}}$, where m is the Steinitz number corresponding to W via Proposition 2.6. One easily checks that $\mu[W] = \bigcup_{n|m, n<\infty} \mu_n$, so that

$$\mathrm{Ann}_{\widehat{\mathbf{Z}}}(\mu[W]) = \mathrm{Ann}_{\widehat{\mathbf{Z}}}\left(\bigcup_{n|m, n<\infty} \mu_n\right) = \bigcap_{n|m, n<\infty} \mathrm{Ann}_{\widehat{\mathbf{Z}}}(\mu_n) = \bigcap_{n|m, n<\infty} n\widehat{\mathbf{Z}} = m\widehat{\mathbf{Z}}.$$

Analogously one shows that for a subgroup ν of μ we have

$$\mu[\mathrm{Ann}_{\widehat{\mathbf{Z}}}(\nu)] = \nu.$$

Hence, the map g is a bijection, whose inverse is given by sending the subgroup ν of μ to $\mathrm{Ann}_{\widehat{\mathbf{Z}}}(\nu)$. ∎

Now, let K and μ be as above, and consider the topological group $\mathrm{End}(\mu)$, endowed with the compact-open topology. Observe that

$$\mathrm{End}(\mu) = \mathrm{End}\left(\varinjlim_{n\geq 1} \mu_n\right) \cong \varprojlim_{n\geq 1} \mathrm{Hom}\left(\mu_n, \mu\right) \cong \varprojlim_{n\geq 1} \mathbf{Z}/n\mathbf{Z} = \widehat{\mathbf{Z}}.$$

Thus, the topological ring $\mathrm{End}(\mu)$ is isomorphic to $\widehat{\mathbf{Z}}$. In particular, the topological group $\mathrm{Aut}(\mu) = \mathrm{End}(\mu)^*$ is isomorphic to $\widehat{\mathbf{Z}}^*$.

Let $n \in \mathbf{Z}_{\geq 1}$, and note that for every $m \in \mathbf{Z}_{\geq 1}$ dividing n, there is a group homomorphism $\mu_n \longrightarrow \mu_m$ sending $\xi \in \mu_n$ to $\xi^{n/m}$. This defines a projective system, and we define *the Tate module $\hat{\mu}$ of the multiplicative group* as the projective limit of this system. By Lemma 2.3 it is a profinite module over $\widehat{\mathbf{Z}}$. Moreover, there is a compatible system of group isomorphisms $\mu_n \longrightarrow \mathbf{Z}/n\mathbf{Z}$, so that $\hat{\mu}$ is isomorphic to $\widehat{\mathbf{Z}}$ as a $\widehat{\mathbf{Z}}$-module, that is, the Tate module $\hat{\mu}$ is a free $\widehat{\mathbf{Z}}$-module of rank 1.

2. A maximal radical extension

In this section we summarize the exposition done in §2 of [LMS13], and we refer to this article for the details.

Let $\overline{\mathbf{Q}}^*$ be an algebraic closure of $\mathbf{Q}$. For $r \in \mathbf{Q}^*$, we let $F_{\infty,r}$ be the field obtained by adjoining to $\mathbf{Q}$ the group of all radicals

$$R_{\infty,r} = \{x \in \overline{\mathbf{Q}}^* : x^n \in \langle r\rangle \text{ for some } n \in \mathbf{Z}_{\geq 1}\}$$

of r in $\overline{\mathbf{Q}}^*$, that is, the field $F_{\infty,r}$ is the infinite Galois extension $F_{\infty,r} = \mathbf{Q}(R_{\infty,r})$ of $\mathbf{Q}$. Observe that $\mu = \mu(\overline{\mathbf{Q}}) \subset R_\infty$. We give $R_{\infty,r}$ the discrete topology, which in particular gives μ the discrete topology.

Now we fix a rational number $r \in \mathbf{Q}^* \setminus \{\pm 1\}$ and drop the subscript 'r' in the definitions above. There is a rational number $r_0 \in \mathbf{Q}^* \setminus \{\pm 1\}$ uniquely determined up to two signs, such that

$$R_\infty \cap \mathbf{Q}^* = \langle r_0\rangle \times \langle -1\rangle.$$

Let e be the index of the subgroup $\langle r\rangle \times \langle -1\rangle$ inside $\langle r_0\rangle \times \langle -1\rangle$, and observe that we have $r = \pm r_0^e$ or $r = \pm r_0^{-e}$.

As $\overline{\mathbf{Q}}^*$ is a divisible group, we may extend the injective group homomorphism $\mathbf{Z} \longrightarrow \overline{\mathbf{Q}}^*$ given by sending $1 \in \mathbf{Z}$ to r_0 to an injective group homomorphism $\mathbf{Q} \longrightarrow \overline{\mathbf{Q}}^*$, which we denote as

$$q \mapsto r_0^q.$$

We fix such an embedding, so that we can write $r_0^{\mathbf{Q}}$ for its image in $\overline{\mathbf{Q}}^*$. With this notation, we have

$$R_\infty = r_0^{\mathbf{Q}} \times \mu.$$

Let $A = \mathrm{Aut}_{R_\infty \cap \mathbf{Q}^*}(R_\infty)$ be the group of automorphisms of R_∞ that restrict to the identity on $R_\infty \cap \mathbf{Q}^*$. It is the projective limit, over $n \in \mathbf{Z}_{\geq 1}$, of the finite group $\mathrm{Aut}_{R_{n,r} \cap \mathbf{Q}^*}(R_{n,r})$ of those automorphisms of the group of nth radicals $R_{n,r} = \{x \in \overline{\mathbf{Q}}^* : x^n \in \langle r \rangle\}$ of r in $\overline{\mathbf{Q}}$ that restrict to the identity on $R_{n,r} \cap \mathbf{Q}^*$, and as such it is a profinite group.

As $\hat{\mu}$ is a multiplicative group, we use exponential notation for the natural action of $\widehat{\mathbf{Z}}^*$ on $\hat{\mu}$.

Proposition 2.9. *Consider the short exact sequence*

$$1 \longrightarrow \hat{\mu} \longrightarrow A \longrightarrow \widehat{\mathbf{Z}}^* \longrightarrow 1,$$

where the first map sends $(\xi_n)_{n \geq 1} \in \hat{\mu}$ to the map sending any nth root $\sqrt[n]{r_0}$ of r_0 to $\xi_n \sqrt[n]{r_0}$, for all $n \in \mathbf{Z}_{\geq 1}$, and the second map is the composition of the restriction map $A \longrightarrow \mathrm{Aut}(\mu)$ and the isomorphism $\mathrm{Aut}(\mu) \cong \widehat{\mathbf{Z}}^*$. Then the induced action of $\widehat{\mathbf{Z}}^*$ on $\hat{\mu}$ is the natural action, and the map $s\colon \widehat{\mathbf{Z}}^* \longrightarrow A$ extending the action of $\widehat{\mathbf{Z}}^*$ on μ to the identity on $r_0^{\mathbf{Q}}$, is a continuous section of the exact sequence. Moreover, there is an isomorphism

$$A \cong \hat{\mu} \rtimes_s \widehat{\mathbf{Z}}^*$$

of topological groups.

Proof. See [LMS13]. ∎

Let $G = \mathrm{Gal}(F_\infty/\mathbf{Q})$, and observe that by Galois theory we have a short exact sequence

$$1 \longrightarrow \mathrm{Gal}(F_\infty/\mathbf{Q}(\mu)) \longrightarrow G \longrightarrow \mathrm{Gal}(\mathbf{Q}(\mu)/\mathbf{Q}) \longrightarrow 1$$

of Galois groups. Since it is a sequence of profinite groups, it is well-adjusted. Recall the canonical isomorphisms $\mathrm{Gal}(\mathbf{Q}(\mu)/\mathbf{Q}) \cong \mathrm{Aut}(\mu)$ and $\mathrm{Aut}(\mu) \cong \widehat{\mathbf{Z}}^*$ of topological groups, hence we have the well-adjusted short exact sequence

$$1 \longrightarrow \mathrm{Gal}(F_\infty/\mathbf{Q}(\mu)) \longrightarrow G \longrightarrow \widehat{\mathbf{Z}}^* \longrightarrow 1.$$

As $\mathbf{Q}(R_\infty) = F_\infty$, there is an injective homomorphism

$$\iota\colon G \longrightarrow A$$

of profinite groups, sending $\sigma \in G$ to $\sigma|_{R_\infty}$. Furthermore, there is an injective homomorphism

$$\theta\colon \mathrm{Gal}(F_\infty/\mathbf{Q}(\mu)) \longrightarrow \hat{\mu}$$

of profinite groups, sending $\sigma \in \mathrm{Gal}(F_\infty/\mathbf{Q}(\mu))$ to $\left(\frac{\sigma(r_0^{1/n})}{r_0^{1/n}}\right)_{n \geq 1} \in \hat{\mu}$.

Proposition 2.10. *The map θ identifies* $\mathrm{Gal}(F_\infty/\mathbf{Q}(\mu)$ *with $\hat{\mu}^2$, and the diagram*

$$\begin{tikzcd}
1 \arrow[r] & \hat{\mu}^2 \arrow[r, "f"] \arrow[d] & G \arrow[r, "g"] \arrow[d, "\iota"] & \widehat{\mathbf{Z}}^* \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\
1 \arrow[r] & \hat{\mu} \arrow[r] & A \arrow[r] & \widehat{\mathbf{Z}}^* \arrow[r] & 1
\end{tikzcd}$$

of topological groups is commutative, where the lower row is described in Proposition 2.9*, the map f is the composition of θ and the given map $\hat{\mu} \longrightarrow A$, and g is the composition of the restriction map $G \longrightarrow \mathrm{Aut}(\mu)$ and the canonical isomorphism* $\mathrm{Aut}(\mu) \cong \widehat{\mathbf{Z}}^*$*. Moreover, the induced action of $\widehat{\mathbf{Z}}^*$ on $\hat{\mu}^2$ is the natural action, the upper row is not semisplit, and ι identifies G with a subgroup of index 2 in A.*

Proof. See [LMS13]. ■

Composing f with any isomorphism $\widehat{\mathbf{Z}} \longrightarrow \hat{\mu}^2$ of profinite groups, we see that G is a non-trivial topological extension of $\widehat{\mathbf{Z}}^*$ by $\widehat{\mathbf{Z}}$ under the natural action. This proves the (a) to (b) implication of Theorem 1 of the Introduction.

We conclude this section with an explicit description of G as a subgroup of A. For $u \in \widehat{\mathbf{Z}}^*$ we let $\sigma_u \colon \mathbf{Q}(\mu) \longrightarrow \mathbf{Q}(\mu)$ be the automorphism of $\mathbf{Q}(\mu)$ corresponding to u by the canonical isomorphism $\mathrm{Gal}(\mathbf{Q}(\mu)/\mathbf{Q}) \cong \widehat{\mathbf{Z}}^*$ of topological groups.

As $R_\infty = r_0^{\mathbf{Q}} \times \mu$, the field F_∞ is the compositum of $\mathbf{Q}(r_0^{\mathbf{Q}})$ and $\mathbf{Q}(\mu)$, and the embedding $G \subset A = \hat{\mu} \rtimes \widehat{\mathbf{Z}}^*$ amounts to a description of the field automorphisms of F_∞ in terms of their action on these constituents. The index 2 of G in A reflects the fact that the intersection $\mathbf{Q}(r_0^{\mathbf{Q}}) \cap \mathbf{Q}(\mu)$ is equal to the quadratic field $K = \mathbf{Q}(\sqrt{r_0})$. This implies that an element $((\xi_n)_{n\geq 1}, u) \in A$ is in G if and only if

$$\frac{\sigma_u(\sqrt{r_0})}{\sqrt{r_0}} = \xi_2.$$

We can phrase this slightly more formally as follows.

Let $\psi_K \colon A \longrightarrow \mu_2$ be the map sending $((\xi_n)_{n\geq 1}, u) \in A$ to ξ_2 and note that it is a continuous group homomorphism, and let $\chi_K \colon A \longrightarrow \mu_2$ be the cyclotomic character on A of conductor $d = \mathrm{disc}(K)$, that is, the character sending $((\xi_n)_{n\geq 1}, u) \in A$ to $\sigma_u(\sqrt{r_0})/\sqrt{r_0}$. Then G is the kernel of the quadratic character $\psi_K \cdot \chi_K$. We summarize this in the following theorem.

Theorem 2.11. *Let $K = \mathbf{Q}(\sqrt{r_0})$ and let $\chi \colon A \longrightarrow \mu_2$ be the quadratic character $\psi_K \cdot \chi_K$. Then the sequence*

$$1 \longrightarrow G \xrightarrow{\iota} A \xrightarrow{\chi} \mu_2 \longrightarrow 1$$

of profinite groups is exact, and we can identify G with the subgroup of $A = \hat{\mu} \rtimes \widehat{\mathbf{Z}}^*$ consisting of the elements $((\xi_n)_{n\geq 1}, u) \in A$ such that

$$\frac{\sigma_u(\sqrt{r_0})}{\sqrt{r_0}} = \xi_2.$$

Proof. See [LMS13]. ∎

3. Galois groups of radical extensions

Throughout this section, the action of $\widehat{\mathbf{Z}}^*$ on $\widehat{\mathbf{Z}}$ is the natural one.

Lemma 2.12. *Let $r \in \mathbf{Q}^* \setminus \{\pm 1\}$, and let $r_0 \in \mathbf{Q}^*$ such that $R_{\infty,r} \cap \mathbf{Q}^* = \langle r_0, -1\rangle$. Let G be the Galois group of $F_{\infty,r}$ over $\mathbf{Q}$, and let*

$$0 \longrightarrow \widehat{\mathbf{Z}} \xrightarrow{i} G \xrightarrow{j} \widehat{\mathbf{Z}}^* \longrightarrow 1$$

be a short exact sequence of profinite groups such that the induced action of $\widehat{\mathbf{Z}}^*$ on $\widehat{\mathbf{Z}}$ is the natural one. Then the following statements hold.

(a) *There is a unique continuous isomorphism $\eta\colon \hat{\mu}^2 \longrightarrow \widehat{\mathbf{Z}}$ such that $i \circ \eta = f$, and the map j is equal to g, where f and g are defined in* Proposition 2.10.

(b) *Let ζ be a generator of $\hat{\mu}$, and let $\alpha\colon \widehat{\mathbf{Z}}^* \longrightarrow \widehat{\mathbf{Z}}$ be the continuous map defined by*

$$\alpha(u) = \begin{cases} 0 & \text{if } \sigma_u(\sqrt{r_0}) = \sqrt{r_0}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $s\colon \widehat{\mathbf{Z}}^* \longrightarrow G$ defined by

$$s(u) = (\zeta^{\alpha(u)}, u)$$

where we consider G as a subgroup of the semidirect product $A = \hat{\mu} \rtimes \widehat{\mathbf{Z}}^*$, is a continuous set-theoretic section of the given exact sequence.

Proof. To prove (a), we first show that the images of i and f are equal to $\mathrm{C}_G([G,G])$. To this end, note that $\widehat{\mathbf{Z}}^*$ is abelian, so that $[G,G] \subset i(\widehat{\mathbf{Z}})$. As $i(\widehat{\mathbf{Z}})$ is abelian, we have

$$i(\widehat{\mathbf{Z}}) \subset \mathrm{C}_G(i(\widehat{\mathbf{Z}})) \subset \mathrm{C}_G([G,G]).$$

Conversely, let $x \in \mathrm{C}_G([G,G])$. For $u \in \widehat{\mathbf{Z}}$, observe that $i(2u) \in [G,G]$. Indeed, let $y \in G$ be such that $j(y) = -1$. Then

$$yi(-u)y^{-1} = i(j(y) \cdot -u) = i(u),$$

where the first equality comes from Definition 1.31 and the second follows from the fact that $\widehat{\mathbf{Z}}^*$ acts on $\widehat{\mathbf{Z}}$ in the natural way. Multiplying on the left by $i(u)$ gives $i(u)yi(u)^{-1}y^{-1} = i(2u)$ in G.

As x commutes with every element of $[G,G]$ and $i(2) \in [G,G]$, the equality

$$xi(2)x^{-1} = i(j(x) \cdot 2)$$

shows that $i(2) = i(j(x) \cdot 2)$. By injectivity of i, we have $2 = j(x) \cdot 2$, which is equivalent to $(j(x) - 1)2 = 0$. As 2 is not a zero divisor in $\widehat{\mathbf{Z}}$, it follows that $j(x) = 1$. Hence $x \in \ker j = i(\widehat{\mathbf{Z}})$, so we may conclude that $i(\widehat{\mathbf{Z}}) = \mathrm{C}_G([G, G])$. In the same way one shows that $f(\hat{\mu}^2) = \mathrm{C}_G([G, G])$.

As i and f have the same image, there exists a unique isomorphism $\eta\colon \hat{\mu}^2 \to \widehat{\mathbf{Z}}$ of profinite groups such that the diagram

$$\begin{tikzcd} \hat{\mu}^2 \arrow[r, "f"] \arrow[d, "\eta"'] & G \\ \widehat{\mathbf{Z}} \arrow[ur, "i"'] & \end{tikzcd}$$

is commutative.

The maps f and i give rise to the same map $G \longrightarrow \mathrm{Aut}(\mathrm{C}_G([G, G]))$, where $\mathrm{Aut}(\mathrm{C}_G([G, G]))$ is canonically isomorphic to $\widehat{\mathbf{Z}}^*$. This is exactly equal to the map g and the map j, implying that g and j are equal. This proves (a).

To prove (b), observe that s is a section for the short exact sequence

$$1 \longrightarrow \hat{\mu}^2 \xrightarrow{f} G \xrightarrow{g} \widehat{\mathbf{Z}}^* \longrightarrow 1$$

described in Proposition 2.10. As g and j are equal, the map s is also a section for the given exact sequence. ■

Lemma 2.13. *Let M be a topological $\widehat{\mathbf{Z}}^*$-module such that $-1 \cdot m = -m$ for each $m \in M$. Then for each $n \in \mathbf{Z}_{\geq 0}$, the group $\mathrm{H}^n(\widehat{\mathbf{Z}}^*, M)$ has exponent dividing 2.*

Proof. Let $n \in \mathbf{Z}_{\geq 0}$. Since the map ψ_{-1} is multiplication by -1 on M, the map $\mathrm{H}^n(\widehat{\mathbf{Z}}^*, \psi_{-1})$ is multiplication by -1 on $\mathrm{H}^n(\widehat{\mathbf{Z}}^*, M)$. On the other hand, Proposition 1.30 implies that the map $\mathrm{H}^n(\widehat{\mathbf{Z}}^*, \psi_{-1})$ is equal to the identity on $\mathrm{H}^n(\widehat{\mathbf{Z}}^*, M)$. Hence, multiplication by -1 on $\mathrm{H}^n(\widehat{\mathbf{Z}}^*, M)$ is equal to the identity, that is, for each $x \in \mathrm{H}^n(\widehat{\mathbf{Z}}^*, M)$ we have $-1 \cdot x = x$. Consequently, for each $x \in \mathrm{H}^n(\widehat{\mathbf{Z}}^*, M)$ we have $0 = -1 \cdot x - x = -2 \cdot x$, so that $2 \cdot \mathrm{H}^n(\widehat{\mathbf{Z}}^*, M) = 0$ for each $n \in \mathbf{Z}_{\geq 0}$. ■

Theorem 2.14. *There exists a group isomorphism*

$$\varphi\colon \mathrm{H}^2(\widehat{\mathbf{Z}}^*, \widehat{\mathbf{Z}}) \longrightarrow \mathbf{Q}^*/\pm\mathbf{Q}^{*2}$$

that for every $r \in \mathbf{Q}^* \setminus \{\pm 1\}$ maps the equivalence class of any topological extension

$$0 \longrightarrow \widehat{\mathbf{Z}} \longrightarrow \mathrm{Gal}(F_{\infty,r}/\mathbf{Q}) \longrightarrow \widehat{\mathbf{Z}}^* \longrightarrow 1$$

of $\widehat{\mathbf{Z}}^*$ by $\widehat{\mathbf{Z}}$, to $\pm r_0 \mathbf{Q}^{*2}$, where $r_0 \in \mathbf{Q}^*$ is such that

$$R_{\infty,r} \cap \mathbf{Q}^* = \langle -1, r_0 \rangle.$$

Proof. By Lemma 2.13, the group $\mathrm{H}^n(\widehat{\mathbf{Z}}^*, \widehat{\mathbf{Z}})$ has exponent dividing 2, for each $n \in \mathbf{Z}_{\geq 0}$. (In fact, for $n = 2$ we know that it has exponent equal to 2, since in section 1 we saw that $\mathrm{Gal}(F_{\infty,r}/\mathbf{Q})$ is a non-trivial extension of $\widehat{\mathbf{Z}}^*$ by $\widehat{\mathbf{Z}}$.)

Multiplication by 2 is a continuous $\widehat{\mathbf{Z}}^*$-module endomorphism of $\widehat{\mathbf{Z}}$ giving rise to the short exact sequence

$$0 \longrightarrow \widehat{\mathbf{Z}} \xrightarrow{\cdot 2} \widehat{\mathbf{Z}} \xrightarrow{\pi} \mu_2 \longrightarrow 0$$

of topological $\widehat{\mathbf{Z}}^*$-modules, where $\widehat{\mathbf{Z}}^*$ acts trivially on μ_2, and π sends $r \in \widehat{\mathbf{Z}}$ to $(-1)^r$. As the groups are profinite, the sequence is well-adjusted. For each $n \in \mathbf{Z}_{\geq 0}$, the map $\mathrm{H}^n(\cdot 2) = \cdot 2$ is the zero map, since $\mathrm{H}^n(\widehat{\mathbf{Z}}^*, \widehat{\mathbf{Z}})$ has exponent dividing 2. Using the fact that $\widehat{\mathbf{Z}}$ has no non-trivial 2-torsion, one easily sees that $\mathrm{H}^0(\widehat{\mathbf{Z}}^*, \widehat{\mathbf{Z}}) = \widehat{\mathbf{Z}}^{\widehat{\mathbf{Z}}^*}$ is equal to the trivial group. By Theorem 1.15, we have the long exact sequence

$$0 \longrightarrow \mu_2 \xrightarrow{\delta_0} \mathrm{H}^1(\widehat{\mathbf{Z}}^*, \widehat{\mathbf{Z}}) \xrightarrow{0} \mathrm{H}^1(\widehat{\mathbf{Z}}^*, \widehat{\mathbf{Z}}) \xrightarrow{\mathrm{H}^1(\pi)} \mathrm{H}^1(\widehat{\mathbf{Z}}^*, \mu_2) \xrightarrow{\delta_1} \mathrm{H}^2(\widehat{\mathbf{Z}}^*, \widehat{\mathbf{Z}}) \xrightarrow{0} \dots$$

of groups. It follows immediately that δ_0 is a group isomorphism, and that

$$0 \longrightarrow \mu_2 \xrightarrow{c} \mathrm{H}^1(\widehat{\mathbf{Z}}^*, \mu_2) \xrightarrow{\delta_1} \mathrm{H}^2(\widehat{\mathbf{Z}}^*, \widehat{\mathbf{Z}}) \longrightarrow 0 \tag{2}$$

is a short exact sequence of groups, where $c = \mathrm{H}^1(\pi) \circ \delta_0$.

As $\widehat{\mathbf{Z}}^*$ acts trivially on μ_2, we have

$$\mathrm{H}^1(\widehat{\mathbf{Z}}^*, \mu_2) = \mathrm{CHom}(\widehat{\mathbf{Z}}^*, \mu_2).$$

By Kummer theory (cf. [Neu99]), there is a group isomorphism

$$\psi \colon \mathbf{Q}^*/\mathbf{Q}^{*2} \longrightarrow \mathrm{CHom}(\widehat{\mathbf{Z}}^*, \mu_2)$$

sending $a\mathbf{Q}^{*2} \in \mathbf{Q}^*/\mathbf{Q}^{*2}$ to the character sending $u \in \widehat{\mathbf{Z}}^*$ to $\sigma_u(\sqrt{a})/\sqrt{a}$.

Now, we compute the image of μ_2 under c. Clearly $c(1) = [0] \in \mathrm{H}^1(\widehat{\mathbf{Z}}^*, \widehat{\mathbf{Z}})$. Observe that $\delta_0(-1)$ sends $u \in \widehat{\mathbf{Z}}^*$ to $\frac{u-1}{2}$ in $\widehat{\mathbf{Z}}$ (cf. Proposition 1.13). Moreover, the map $\mathrm{H}^1(\pi)(\delta_0(1))$ sends $u \in \widehat{\mathbf{Z}}^*$ to $(-1)^{\frac{u-1}{2}}$.

Note that ψ sends $-\mathbf{Q}^{*2} \in \mathbf{Q}^*/\mathbf{Q}^{*2}$ to the character sending $u \in \widehat{\mathbf{Z}}^*$ to

$$\sigma_u(\sqrt{-1})/\sqrt{-1} = (-1)^{\frac{u-1}{2}}.$$

Hence ψ induces a group isomorphism

$$\psi' \colon \mathbf{Q}^*/\pm\mathbf{Q}^{*2} \longrightarrow \mathrm{CHom}(\widehat{\mathbf{Z}}^*, \mu_2)/c(\mu_2).$$

It follows that (2) induces the group isomorphism

$$\varphi \colon \mathbf{Q}^*/\pm\mathbf{Q}^{*2} \longrightarrow \mathrm{H}^2(\widehat{\mathbf{Z}}^*, \widehat{\mathbf{Z}})$$

given by sending $\pm a\mathbf{Q}^{*2}$ to $\delta_1(\psi'(\pm a\mathbf{Q}^{*2}))$, where by abuse of notation δ_1 denotes the induced isomorphism $\mathrm{H}^1(\widehat{\mathbf{Z}}^*, \mu_2)/c(\mu_2) \longrightarrow \mathrm{H}^2(\widehat{\mathbf{Z}}^*, \widehat{\mathbf{Z}})$.

Let $r \in \mathbf{Q}^* \setminus \{\pm 1\}$ and let G be the Galois group of $\mathbf{Q}(R_{\infty,r})$ over $\mathbf{Q}$. Let $r_0 \in \mathbf{Q}^* \setminus \{\pm 1\}$ be such that $R_{\infty,r} \cap \mathbf{Q}^* = \langle r_0 \rangle \times \langle -1 \rangle$. We will show that φ maps $\pm r_0\mathbf{Q}^{*2}$ to the equivalence class of any topological extension G of $\widehat{\mathbf{Z}}^*$ by $\widehat{\mathbf{Z}}$.

Using the explicit descriptions of the homomorphisms above and the explicit description of δ_n, $n \in \mathbf{Z}_{\geq 0}$, given in Proposition 1.13, one sees that

$$\varphi(\pm r_0\mathbf{Q}^{*2})\colon \widehat{\mathbf{Z}}^* \times \widehat{\mathbf{Z}}^* \longrightarrow \widehat{\mathbf{Z}}$$

is given by

$$(u, v) \mapsto \frac{u\alpha(v) - \alpha(uv) + \alpha(u)}{2},$$

where $\alpha\colon \widehat{\mathbf{Z}}^* \longrightarrow \widehat{\mathbf{Z}}$ is defined in Lemma 2.12(b).

On the other hand, consider a topological extension

$$0 \longrightarrow \widehat{\mathbf{Z}} \xrightarrow{i} G \xrightarrow{j} \widehat{\mathbf{Z}}^* \longrightarrow 1$$

of $\widehat{\mathbf{Z}}^*$ by $\widehat{\mathbf{Z}}$. By Lemma 2.12(a), the map j is equal to g, defined in Proposition 2.10, and there is an isomorphism $\eta\colon \hat{\mu}^2 \longrightarrow \widehat{\mathbf{Z}}$ of profinite groups such that the diagram

$$\begin{tikzcd} \hat{\mu}^2 \arrow[r, "f"] \arrow[d, "\eta"'] & G \\ \widehat{\mathbf{Z}} \arrow[ur, "i"'] & \end{tikzcd}$$

is commutative. Let ζ be the generator of $\hat{\mu}$ such that $\eta^{-1}(1) = \zeta^2$.

By Lemma 2.12(b), the map $s\colon \widehat{\mathbf{Z}}^* \longrightarrow G$ given by

$$s(u) = (\zeta^{\alpha(u)}, u)$$

is a continuous section of this sequence, and $\omega\colon \widehat{\mathbf{Z}}^* \times \widehat{\mathbf{Z}}^* \longrightarrow \widehat{\mathbf{Z}}$ defined by $\omega(u, v) = s(u)s(v)s(uv)^{-1}$ is the 2-cocycle corresponding to the given topological extension. Note that

$$\begin{aligned} \omega(u, v) &= \left(\zeta^{\alpha(u)}, u\right)\left(\zeta^{\alpha(v)}, v\right)\left(\zeta^{\alpha(uv)}, uv\right)^{-1} \\ &= \left(\zeta^{\alpha(u)+u\alpha(v)}, uv\right)\left(\zeta^{-(uv)^{-1}\alpha(uv)}, (uv)^{-1}\right) \\ &= \left(\zeta^{\alpha(u)+u\alpha(v)-\alpha(uv)}, 1\right), \end{aligned}$$

which defines the element $\zeta^{\alpha(u)+u\alpha(v)-\alpha(uv)}$ in $\hat{\mu}^2$. Composing with the isomorphism η, we find that the 2-cocycle ω is given by

$$(u, v) \mapsto \frac{\alpha(u) + u\alpha(v) - \alpha(uv)}{2}.$$

As this is exactly the image of $\pm r_0\mathbf{Q}^{*2}$, we are done. ∎

Proof of Theorem 1. The implication (a) to (b) was proven after Proposition 2.10. To prove the other implication, let

$$0 \longrightarrow \widehat{\mathbf{Z}} \longrightarrow G \longrightarrow \widehat{\mathbf{Z}}^* \longrightarrow 1$$

be a non-semisplit short exact sequence of profinite groups such that the induced action of $\widehat{\mathbf{Z}}^*$ on $\widehat{\mathbf{Z}}$ is the natural one. Then the group isomorphism φ of Theorem 2 maps this extension to a non-trivial element $\pm a\mathbf{Q}^{*2}$ of $\mathbf{Q}^*/\pm\mathbf{Q}^{*2}$. Choose r to be a square-free integer in the coset $\pm a\mathbf{Q}^{*2}$. Then $r \neq \pm 1$ and $R_{\infty,r} \cap \mathbf{Q}^* = \langle -1, r\rangle$. By Theorem 2, the map φ^{-1} sends $\pm r\mathbf{Q}^{*2} = \pm a\mathbf{Q}^{*2}$ to the extension

$$0 \longrightarrow \widehat{\mathbf{Z}} \longrightarrow \mathrm{Gal}(F_{\infty,r}/\mathbf{Q}) \longrightarrow \widehat{\mathbf{Z}}^* \longrightarrow 1.$$

As these extensions are equivalent, it follows that $G \cong \mathrm{Gal}(F_{\infty,r}/\mathbf{Q})$. ■

4. Roots of unity and cohomology

Let Γ be a closed subgroup $\widehat{\mathbf{Z}}^*$. Define

$$V_\Gamma = \sum_{\gamma\in\Gamma} \widehat{\mathbf{Z}}(\gamma - 1)$$

to be the $\widehat{\mathbf{Z}}$-ideal generated by $\Gamma - 1 = \{\gamma - 1 : \gamma \in \Gamma\}$ and

$$W_\Gamma = \overline{V_\Gamma}$$

to be its closure in $\widehat{\mathbf{Z}}$. For example, one has $V_{\widehat{\mathbf{Z}}^*} = W_{\widehat{\mathbf{Z}}^*} = 2\widehat{\mathbf{Z}}$.

Lemma 2.15. *Let M be a profinite abelian group, and let M_p be the unique pro-p Sylow subgroup of M. Then M is the direct product $M = \prod_{p \text{ prime}} M_p$ of its pro-p Sylow subgroups.*

Proof. See [RZ09, Proposition 2.3.8]. ■

Let M be a profinite abelian group, and consider the unique topological $\widehat{\mathbf{Z}}$-module structure on it (cf. Lemma 2.3). By restriction M is a topological Γ-module. Moreover, the action of $\widehat{\mathbf{Z}}$ on M induces an action of $\widehat{\mathbf{Z}}$ on $\mathrm{H}^n(\Gamma, M)$ for each $n \in \mathbf{Z}_{\geq 0}$.

Theorem 2.16. *For all $n \geq 0$, we have $W_\Gamma \cdot \mathrm{H}^n(\Gamma, M) = 0$.*

Proof. Let $n \in \mathbf{Z}_{\geq 0}$. By Proposition 1.30 we have for each $\gamma \in \Gamma$ and each $x \in \mathrm{H}^n(\Gamma, M)$ that $\gamma \cdot x = x$. Hence, for each $\gamma \in \Gamma$ we have

$$(\gamma - 1) \cdot \mathrm{H}^n(\Gamma, M) = 0.$$

It follows that

$$V_\Gamma \cdot \mathrm{H}^n(\Gamma, M) = 0.$$

It remains to show that the closure of V_Γ is also in the annihilator of $\mathrm{H}^n(\Gamma, M)$.

By Lemma 2.15 we have $M = \prod_p M_p$, so by Proposition 1.6 the equality $\mathrm{H}^n(\Gamma, M) = \prod_p \mathrm{H}^n(\Gamma, M_p)$ holds. As for primes p the action of $\widehat{\mathbf{Z}} = \prod_p \mathbf{Z}_p$ on M_p factors via $\mathbf{Z}_p$, the action of $\widehat{\mathbf{Z}}$ on $\mathrm{H}^n(\Gamma, M_p)$ factors via $\mathbf{Z}_p$ too. Observe that the $\mathbf{Z}_p$-annihilator of $\mathrm{H}^n(\Gamma, M_p)$ is a closed ideal, since all ideals of $\mathbf{Z}_p$ are closed. It follows that the $\widehat{\mathbf{Z}}$-annihilator of $\prod_p \mathrm{H}^n(\Gamma, M_p)$ is a product of closed ideals of $\mathbf{Z}_p$ and therefore a closed ideal of $\widehat{\mathbf{Z}}$. Now, as V_Γ is contained in the closed ideal $\mathrm{Ann}_{\widehat{\mathbf{Z}}}(\mathrm{H}^n(\Gamma, M))$, also its closure is contained in it. Thus, the ideal W_Γ annihilates $\mathrm{H}^n(\Gamma, M)$. ∎

Now, let K be a field of characteristic 0, and let $\overline{K}$ be an algebraic closure of K. Let μ be the subgroup of $\overline{K}^*$ of roots of unity as defined in §2.1, and let $K(\mu)$ be the maximal cyclotomic extension of K. Note that it is a Galois extension over K, and let Γ_K be its Galois group. There is a canonical injection

$$\Gamma_K \longrightarrow \mathrm{Aut}(\mu)$$

of profinite groups. Composing with the canonical isomorphism $\mathrm{Aut}(\mu) \cong \widehat{\mathbf{Z}}^*$ given in §2.1, we have an injection

$$\Gamma_K \longrightarrow \widehat{\mathbf{Z}}^*$$

of profinite groups. As Γ_K is compact, its image in $\widehat{\mathbf{Z}}^*$ is compact. Moreover, since $\widehat{\mathbf{Z}}^*$ is Hausdorff, it follows that we may identify Γ_K with a closed subgroup of $\widehat{\mathbf{Z}}^*$, which we again denote by Γ_K. When the field K is understood, we often write Γ for Γ_K.

Theorem 2.17. *We have* $W_\Gamma = \mathrm{Ann}_{\widehat{\mathbf{Z}}}(\mu(K))$.

Proof. By Proposition 2.8 we have that

$$W_\Gamma = \mathrm{Ann}_{\widehat{\mathbf{Z}}}(\mu(K)) \text{ if and only if } \mu[W_\Gamma] = \mu(K).$$

Let $\xi \in \mu$. Then $\xi \in \mu(K)$ if and only if for all $\gamma \in \Gamma$ we have $\xi^{\gamma-1} = 1$, which is equivalent to $\xi \in \mu[V_\Gamma]$. As μ is Hausdorff, the kernel of the map $\widehat{\mathbf{Z}} \longrightarrow \mu$ sending $u \in \widehat{\mathbf{Z}}$ to ξ^u, which is the annihilator $\mathrm{Ann}_{\widehat{\mathbf{Z}}}(\xi)$, is closed. Now, since the $\widehat{\mathbf{Z}}$-annihilator of ξ is closed, we have $\xi \in \mu[V_\Gamma]$ if and only if $\xi \in \mu[W_\Gamma]$. ∎

Corollary 2.18. *For* $\Gamma = \Gamma_K$ *and* M *as in Theorem 2.16, we have for all* $n \in \mathbf{Z}_{\geq 0}$ *that* $\mathrm{Ann}_{\widehat{\mathbf{Z}}}(\mu(K)) \cdot \mathrm{H}^n(\Gamma, M) = 0$.

Proof. This follows immediately from Theorem 2.16 and Theorem 2.17. ∎

We end this section with two examples. The first one shows that one does not generally have $V_\Gamma = W_\Gamma$.

Example 2.19. Let U be an $\mathbf{F}_2$-vector space of countably infinite dimension such that the set of nonzero elements of U coincides with the set $\mathcal{P}$ of prime numbers. The group

$$\Gamma = \mathrm{Hom}_{\mathbf{F}_2}(U, \{\pm 1\})$$

may then be viewed as a subgroup of the group $\{\pm 1\}^{\mathcal{P}}$. As

$$\{\pm 1\}^{\mathcal{P}} \subset \prod_{p \in \mathcal{P}} \mathbf{Z}_p^* \cong \widehat{\mathbf{Z}}^*,$$

we can consider Γ as a subgroup of $\widehat{\mathbf{Z}}^*$. One easily checks that Γ is in fact a closed subgroup of $\widehat{\mathbf{Z}}^*$.

Let $\overline{\mathbf{Q}}$ be an algebraic closure of $\mathbf{Q}$, and let μ be the group of all roots of unity in $\overline{\mathbf{Q}}$. By Galois theory we have $\mathrm{Gal}(\mathbf{Q}(\mu)/K) = \Gamma$ for the subfield $K = \mathbf{Q}(\mu)^\Gamma$ of $\mathbf{Q}(\mu)$. For each $p \in \mathcal{P}$ there is an $\mathbf{F}_2$-homomorphism $U \longrightarrow \{\pm 1\}$ that sends p to -1, so there is an element of Γ whose pth coordinate is -1; it maps each $\xi \in \mu$ of p-power order to ξ^{-1}, so that ξ can only be in K if $\xi^2 = 1$. We conclude that $\mu(K) = \{\pm 1\}$, so by Proposition 2.17 we have $W_\Gamma = 2\widehat{\mathbf{Z}}$.

Now, suppose $V_\Gamma = W_\Gamma$. Then we have $2 \in V_\Gamma$, so

$$2 = \sum_{\gamma \in S} a_\gamma (\gamma - 1)$$

for some finite subset S of Γ and $a_\gamma \in \widehat{\mathbf{Z}}$. For each $\gamma \in S$, the kernel $\ker \gamma$ has finite index in U. Hence, the common kernel $\bigcap_{\gamma \in S} \ker \gamma$ is of finite index in U as well, so that, in particular, it is infinite. Let $p \in \bigcap_{\gamma \in S} \ker \gamma$ be nonzero and different from 2. Then for each $\gamma \in S$, the element $\gamma - 1$ is in the kernel of the ring homomorphism $\widehat{\mathbf{Z}} \to \mathbf{Z}_p$; since 2 is not, this contradicts the identity above. We conclude $V_\Gamma \neq W_\Gamma$.

■

The second example shows that Theorem 2.16 is not generally valid for any topological $\widehat{\mathbf{Z}}$-module M, not even when M is also assumed to be Hausdorff.

Example 2.20. Let $\Gamma \subset \widehat{\mathbf{Z}}^*$ be a closed subgroup with $V_\Gamma \neq W_\Gamma$ and $\mathrm{Ann}_{\widehat{\mathbf{Z}}}(V_\Gamma) = 0$, e.g. as in Example 2.19. Let Γ act on $\widehat{\mathbf{Z}}$ by multiplication, and observe that V_Γ is a Γ-submodule of $\widehat{\mathbf{Z}}$. The Γ-invariants of the topological Γ-module $\widehat{\mathbf{Z}}/V_\Gamma$ are equal to $\widehat{\mathbf{Z}}/V_\Gamma$, since every $\gamma \in \Gamma$ acts as the identity on $\widehat{\mathbf{Z}}/V_\Gamma$. It follows that $\mathrm{H}^0(\Gamma, \widehat{\mathbf{Z}}/V_\Gamma) = \widehat{\mathbf{Z}}/V_\Gamma$. As $V_\Gamma \not\supset W_\Gamma$, the ideal W_Γ does not annihilate $\widehat{\mathbf{Z}}/V_\Gamma$. Choosing $K = \mathbf{Q}(\mu)^\Gamma$, $M = \widehat{\mathbf{Z}}/V_\Gamma$ and $n = 0$, we have an example as announced, except that M is not Hausdorff.

Consider the short exact sequence

$$0 \longrightarrow V_\Gamma \longrightarrow \widehat{\mathbf{Z}} \longrightarrow \widehat{\mathbf{Z}}/V_\Gamma \longrightarrow 0$$

of topological Γ-modules. Note that V_Γ is homeomorphic to its image in $\widehat{\mathbf{Z}}$, but the sequence is not well-adjusted. Indeed, since $\widehat{\mathbf{Z}}/V_\Gamma$ is not Hausdorff, the product $V_\Gamma \times \widehat{\mathbf{Z}}/V_\Gamma$ is not either, so that $\widehat{\mathbf{Z}}$ is not homeomorphic to $V_\Gamma \times \widehat{\mathbf{Z}}/V_\Gamma$.

By Proposition 1.17 the sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^0(\Gamma, V_\Gamma) \arrow[r] & \mathrm{H}^0(\Gamma, \widehat{\mathbf{Z}}) \arrow[r] & \mathrm{H}^0(\Gamma, \widehat{\mathbf{Z}}/V_\Gamma) \arrow[dll, "\delta_0"'] \\
& \mathrm{H}^1(\Gamma, V_\Gamma) \arrow[r] & \mathrm{H}^1(\Gamma, \widehat{\mathbf{Z}}) \arrow[r] & \mathrm{H}^1(\Gamma, \widehat{\mathbf{Z}}/V_\Gamma)
\end{tikzcd}$$

is exact. Clearly $\mathrm{H}^0(\Gamma, \widehat{\mathbf{Z}}) = 0$, so that $\mathrm{H}^0(\Gamma, V_\Gamma) = 0$ by exactness. It follows that $\delta_0\colon \mathrm{H}^0(\Gamma, \widehat{\mathbf{Z}}/V_\Gamma) \longrightarrow \mathrm{H}^1(\Gamma, V_\Gamma)$ is injective. As W_Γ does not annihilate $\mathrm{H}^0(\Gamma, \widehat{\mathbf{Z}}/V_\Gamma)$ and δ_0 is injective, the ideal W_Γ does not annihilate $\mathrm{H}^1(\Gamma, V_\Gamma)$ either. Hence, with the same K, we obtain a Hausdorff example by putting $M = V_\Gamma$ and $n = 1$.

One can in fact show that δ_0 is an isomorphism, using that $\mathrm{H}^1(\Gamma, \widehat{\mathbf{Z}}) \cong \widehat{\mathbf{Z}}/W_\Gamma$ and $\mathrm{H}^1(\Gamma, \widehat{\mathbf{Z}}/V_\Gamma) = \mathrm{CHom}(\Gamma, \widehat{\mathbf{Z}}/V_\Gamma)$. ■

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