Random walks in dynamic random environments

Proefschrift

ter verkrijging van

de graad van Doctor aan de Universiteit Leiden,

op gezag van Rector Magnificus prof. mr. P.F. van der Heijden,

volgens besluit van het College voor Promoties

te verdedigen op dinsdag 26 oktober 2010

klokke 16:15 uur

door

Luca Avena

geboren te Rome

in 1981

Samenstelling van de promotiecommissie:

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Typeset using LaTeX
Printed by Ipskamp Drukkers, Enschede.
Cover design by Dmitry Nadezhkin.
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Contents

Preface

In the past forty years, models of Random Walks in Random Environments (RWREs) have been intensively studied by the physics and the mathematics community, giving rise to an important and still lively research area that is part of the field of disordered systems. RWREs on $\mathbb{Z}^d$ are Random Walks (RWs) evolving according to a random transition kernel, i.e., their transition probabilities depend on a random field or a random process ξ on $\mathbb{Z}^d$ called Random Environment (RE). What makes these models interesting is that, depending on the RE, several unusual phenomena arise, such as sub-diffusive behavior, sub-exponential decay of probabilities of large deviations, and trapping effects. The REs can be divided into two main classes: *static* and *dynamic*. We refer to *static* RE if ξ is chosen at random at time zero and is kept fixed throughout the time evolution of the RW, while we refer to *dynamic* RE when ξ changes in time according to some stochastic dynamics. For *static* RE, in one dimension the picture is fairly well understood: recurrence criteria, laws of large numbers, invariance principles and refined large deviation estimates have been obtained in a series of papers. In higher dimensions many results have been obtained as well, but still many questions remain open. In *dynamic* RE the state of the art is poorer, even in one dimension. In this thesis we will focus on a class of RWs in dynamic REs constituted by interacting particle systems. The analysis of these models leads us to derive new results and to formulate challenging questions for the future.

The thesis is organized as follows. In Chapter 1 we review what is known in the literature, both for static and dynamic RE, and we introduce the class of models we are interested in. In Chapter 2 we prove a strong law of large numbers under a certain space-time mixing condition on the RE, both in one and in higher dimensions. Furthermore, by using a perturbation argument, we give a series expansion in the size of the drift for the asymptotic speed of RWs with small drifts in highly disordered REs. Chapter 3 focuses on the scaling limits of such processes. By adapting to our context a proof of Comets and Zeitouni [36] for multi-dimensional RWs in static REs, we show that, under a certain space-time mixing condition, an annealed invariance principle holds in any dimension. We further give an alternative proof of this invariance principle in the context of highly disordered REs under small drift assumptions. Chapter 4 deals with the large deviation analysis for the empirical speed of one-dimensional RWs in dynamic REs. We prove a quenched and an annealed large deviation principle and we exhibit some qualitative properties of the associated rate functions. In particular, we give examples of fast and slow-mixing REs for which, respectively, exponential and sub-exponential decay of large deviation probabilities occur. In Chapter 5 we prove a law of large numbers for transient RWs on top of a simple symmetric exclusion process and we conclude with a brief discussion about possible extensions to more general slow-mixing REs, which are part of an ongoing project.

Chapter 1

Introduction: Random walks in random environments (RWRE)

In Sections 1.1 and 1.2 we introduce RWs in *static* and *dynamic* REs and we present a brief overview of the known results relevant for our discussion. In Section 1.3 we define the class of models that are the core of this thesis, i.e., RWs on interacting particle systems. In Section 1.4 we briefly mention other topics related to RWRE that are not covered in this introduction.

1.1 Static RE

The first model of a static RWRE appeared in the biophysics literature (Chernov [33], Temkin [93]) as a toy model for replication of DNA chains. In the early 70' Solomon [80] began a rigorous mathematical analysis of such models by considering a RW in a static RE on the one-dimensional integer lattice. Nowadays the behavior of this random process is completely understood. An overview of the results relevant for our discussion will be presented in Section 1.1.1. In Section 1.1.2 we describe the multi-dimensional case. Most of the techniques used in one dimension cannot be applied in the multi-dimensional setting, due to a more complicated structure of hitting times. Although powerful tools have been developed in the last twenty years and many important results have been achieved, several problems are still open. We will briefly describe what is known in the literature. For detailed statements, proofs and methods we refer the reader to [89, 99].

A formal definition of RW in *static* RE on $\mathbb{Z}^d$ is as follows.

Definition 1.1. (RW in static RE)
For each site $x \in \mathbb{Z}^d$, consider a $2d$-dimensional vector $\xi(x,\cdot) = \{\xi(x,e) \in [0,1] : e \in \mathbb{Z}^d, |e| = 1\}$ such that $\sum_{e:|e|=1} \xi(x,e) = 1$. Let S be the set of all possible values of these vectors, and let $\Omega = S^{\mathbb{Z}^d}$. Given a probability measure μ on Ω, we call a *random environment* an element $\xi \in \Omega$ distributed according to μ. For each realization of $\xi \in \Omega$, we define the RW X in the environment ξ as the Markov chain $X = (X_n)_{n\in\mathbb{N}}$ with state space $\mathbb{Z}^d$ and transition probabilities

$$P^{\xi}(X_{n+1} = x + e | X_n = x) = \xi(x,e), \quad e \in \mathbb{Z}^d, |e| = 1. \tag{1.1}$$

We write P_z^{ξ} to denote the *quenched* law of the RW in the environment ξ starting from position z. We write $\mathbb{P}_z$ to denote the *annealed* law starting from z, i.e.,

$$\mathbb{P}_z(X \in \cdot) = \int_{\Omega} P_z^{\xi}(X \in \cdot)\mu(\mathrm{d}\xi). \tag{1.2}$$

We write $E_z^{\xi}, \mathbb{E}_z$ and E_{μ}, respectively, for expectation with respect to the laws $P_z^{\xi}, \mathbb{P}_z$ and μ.

Henceforth we say that a statement involving the RW X holds $\mathbb{P}_z$-a.s., if for μ-almost every ξ the statement holds P_z^{ξ}-a.s. Note that under the *quenched* law X is a space-inhomogeneous Markov chain, whereas under the *annealed* law X is space-homogeneous but not Markovian. The definition above could have been stated without the nearest-neighbor restriction. This choice was made to avoid cumbersome notations and further technicalities. In the sequel we will sometimes point out when results hold without this restriction.

1.1.1 One dimension

1.1.1.1 Ergodic behavior

The first natural problem is to determine when X is transient or recurrent, whether it admits an asymptotic deterministic speed (under the quenched and the annealed law), i.e., a Law of Large Numbers (LLN), and what can be said about this speed. The next theorem answers these questions.

Theorem 1.2. (Transience, recurrence, LLN)
Let $\xi_x = \xi(x,1)$ and $\rho_x = (1-\xi_x)/\xi_x$. Assume that

$$\mu(\xi_x \in (0,1)) = 1, \tag{1.3}$$

and that μ is stationary and ergodic under translations. Then

1. $\mathbb{P}_0$*-a.s., X is recurrent if* $\mathbb{E}_\mu[\log \rho_0] = 0$*, transient to the left if* $\mathbb{E}_\mu[\log \rho_0] > 0$*, and transient to the right if* $\mathbb{E}_\mu[\log \rho_0] < 0$*.*

2. $\mathbb{P}_0$*-a.s., there exists a deterministic $v \in (-1, 1)$ such that*

$$\lim_{n\to\infty} \frac{X_n}{n} = v \begin{cases} > 0, & \textit{if } \sum_{i=1}^{\infty} \mathbb{E}_\mu\Big[\prod_{j=0}^{i} \rho_{-j}\Big] < \infty, \\ < 0, & \textit{if } \sum_{i=1}^{\infty} \mathbb{E}_\mu\Big[\prod_{j=0}^{i} \rho_{-j}^{-1}\Big] < \infty, \\ = 0, & \textit{if both these conditions fail.} \end{cases} \tag{1.4}$$

3. *If μ is a product measure, then*

$$v = \begin{cases} (1 - \mathbb{E}_\mu[\rho_0])/(1 + \mathbb{E}_\mu[\rho_0]), & \textit{if } \mathbb{E}_\mu[\rho_0] < 1, \\ -(1 - \mathbb{E}_\mu[\rho_0^{-1}])/(1 + \mathbb{E}_\mu[\rho_0^{-1}]), & \textit{if } \mathbb{E}_\mu[\rho_0^{-1}] < 1, \\ 0, & \textit{otherwise.} \end{cases} \tag{1.5}$$

This result is mainly due to Solomon [80]. The original paper only deals with the case in which μ is a product measure. The generalization to the ergodic setup was proven later in [1].

When μ is a product measure, we can already appreciate some surprising features. For instance, if $\mathbb{E}_\mu[\log \rho_0] < 0$, then μ-a.s. $\lim_{n\to\infty} X_n = +\infty$. However, by Jensen's inequality, $\mathbb{E}_\mu[\log \rho_0] \leq \log \mathbb{E}_\mu[\rho_0]$, and if $\mathbb{E}_\mu[\rho_0] > 1$, then $v = 0$, in which case X is transient with zero speed. In other words, the RWRE moves to infinity in a sub-ballistic manner, a phenomenon that never happens for a homogeneous RW. This behavior is due to the presence of 'traps' in the environment: localized pockets in which the walk spends a long time because the transition probabilities push it towards the center of the pocket. In particular, it can be shown that if $v \geq 0$, then $v < 2\mathbb{E}_\mu[\xi_0] - 1 = \bar{v}$. By interpreting $\bar{v}$ as the speed of a homogeneous nearest-neighbor RW jumping to the right with probability $\mathbb{E}_\mu[\xi_0]$ and to the left with probability $1 - \mathbb{E}_\mu[\xi_0]$ ('average medium RW'), we see that in general the RE causes a *slow-down* with respect to the average environment.

1.1.1.2 Scaling limits

Next, we may ask whether X when properly scaled admits a limiting law. Results in this direction have been derived in a number of papers. The invariance principles are typically different under the quenched and the annealed law, and several types of scaling laws occur depending on μ. For example, in the recurrent case ($\mathbb{E}_\mu[\log \rho_0] = 0$), Sinai [79] proved that extreme sub-diffusive behavior holds, i.e. ,

$$\frac{\sigma^2 X_n}{(\log n)^2} \xrightarrow[n\to\infty]{(\mathbb{P}_0)} Z, \qquad \sigma^2 = \mathbb{E}_\mu[(\log \rho_0)^2] \in (0, \infty), \tag{1.6}$$

where Z is a functional of a standard Wiener process (independent of μ) with a non-trivial law that was later identified by Kesten [59].

1.1.1.3 Large deviations

The last item of interest for our introduction is the analysis of the large deviation behavior of the empirical speed of X. We briefly recall that a family of probability measures $(P_n)_{n\in\mathbb{N}}$ satisfies a Large Deviation Principle (LDP) with rate a_n and with rate function I if, for any measurable set A,

$$-\inf_{\theta\in\mathrm{int}(A)} I(\theta) \le \liminf_{n\to\infty} \frac{1}{a_n} \log P_n(A) \le \limsup_{n\to\infty} \frac{1}{a_n} \log P_n(A) \le -\inf_{\theta\in\bar{A}} I(\theta), \tag{1.7}$$

where $\bar{A}$ and $\mathrm{int}(A)$, are, respectively, the closure and the interior of A. If we consider the family of probability measures associated with the empirical speed of X, i.e., $P_n(\cdot) = P(X_n/n \in \cdot), n \in \mathbb{N}$, then a LDP for this family tells us how unlikely it is to observe the walk travelling at any given speed we may be interested in. The most general large deviation result for the one-dimensional RWRE is the following.

Theorem 1.3. (Quenched and Annealed LDP)
Assume that μ is stationary and ergodic. Then, for μ-a.e. realization of ξ, the family of probability measures $P_0^\xi(X_n/n \in \cdot), n \in \mathbb{N}$, satisfies a LDP with rate n and with convex deterministic rate function I_μ^{que}. Moreover, the family of probability measures $\mathbb{P}_0(X_n/n \in \cdot), n \in \mathbb{N}$, satisfies a LDP with rate n and with convex rate function

$$I_\mu^{\mathrm{ann}}(\theta) = \inf_{\nu\in\mathcal{M}_e} [h(\nu|\mu) + I_\nu^{\mathrm{que}}(\theta)], \tag{1.8}$$

where $\mathcal{M}_e$ denotes the set of stationary and ergodic measures on Ω, and $h(\nu|\mu)$ is the relative entropy of ν with respect to μ. In particular, $I_\mu^{\mathrm{ann}}(\theta) \le I_\mu^{\mathrm{que}}(\theta)$. Furthermore in some cases both rate functions are not strictly convex, and are zero in the interval $[0, v]$ (and only in this interval).

From this general statement, we can already appreciate two interesting and unusual features: the rate functions need not be strictly convex and they may vanish on $[0, v]$, indicating sub-exponential decay for the probability of slow-down. In contrast, we recall that for homogeneous RWs the corresponding rate function is strictly convex and vanishes only at the typical speed v (see e.g. [39, 40]). The quenched LDP when $(\xi_x)_{x\in\mathbb{Z}}$ is an i.i.d. sequence was derived in [50], while the annealed LDP, refined quenched estimates and the generalization to ergodic REs were obtained later in [34, 38, 47, 70, 71, 99]. In the next section we give an explicit example.

1.1.1.4 An example

Let $\xi = (\xi_x)_{x\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ be a random sequence distributed according to a Bernoulli product measure ν_ρ with parameter $\rho \in (0,1)$. When $\xi_x = 1$ we say that site x is occupied, while when $\xi_x = 0$ we say that it is vacant. In particular, ρ represents the density of the occupied sites. Conditional on ξ, let $X = (X_n)_{n\in\mathbb{N}_0}$ be the RW with local transition probabilities

$$P^\xi(X_{n+1} = x + e \mid X_n = x) = \begin{cases} p\xi_x + q(1-\xi_x), & \text{if } e = +1, \\ q\xi_x + p(1-\xi_x), & \text{if } e = -1, \end{cases} \tag{1.9}$$

where w.l.o.g. we assume that $p = 1 - q \in \left(\frac{1}{2}, 1\right)$. Note that the formulation of the model in this example is consistent with (1.1). Thus, on occupied sites the RW has a local drift to the right while on vacant sites it has a local drift to the left, of the same size. Note that for $p = \frac{1}{2}$ the model reduces to a simple RW and for $\rho = 1$(respectively 0) to a RW with drift $2p - 1$(respectively $1 - 2p$). From Theorem 1.2, we have that X is recurrent if $\rho = \frac{1}{2}$, transient to the right(left) if $\rho > \frac{1}{2}$ $\left(< \frac{1}{2}\right)$. Moreover, μ-a.s.,

$$\lim_{n\to\infty} X_n/n = v \begin{cases} = 0, & \text{if } \rho \in [q,p], \\ > 0, & \text{if } \rho \in (p,1], \\ < 0, & \text{if } \rho \in [0,q). \end{cases} \tag{1.10}$$

We thus see that if $\rho \in \left(\frac{1}{2}, p\right]$, then the walk will eventually go to the right but at zero speed. This effect is due to the presence of 'traps' in the environment. Indeed, even though occupied sites are more frequent than vacant sites, on its way to $+\infty$, X will cross arbitrarily long intervals in which the local drift is pointing to the left, which results in a displacement of X that grows sub-linearly.

When we look at the large deviations of the empirical speed of X, we see that 'trapping effects' play an important role even in the transient regime with non-zero speed. Without loss of generality we will restrict to the case $\rho \in \left[\frac{1}{2}, 1\right)$.

Theorem 1.4. (Quenched LDP)
For μ-a.e. ξ, the family of probability measures $P_0^\xi(X_n/n \in \cdot\,), n \in \mathbb{N}$, satisfies a LDP

with rate n and with deterministic rate function I^{que} that can be computed in terms of a variational problem and that has the following properties:

1. *I^{que} is continuous and convex on $[-1,1]$ and infinite elsewhere.*
2. *$I^{\text{que}}(-\theta) = I^{\text{que}}(\theta) + \theta(2\rho-1)\log\left(\frac{p}{q}\right)$ for $\theta \in (0,1]$.*
3. *I^{que} is zero on $[0,v]$ and strictly positive on $(v,1]$.*
4. *I^{que} is strictly convex and analytic on $(v,1)$.*

Here are qualitative pictures of $\theta \mapsto I^{\text{que}}(\theta)$ on $[-1,1]$ in the three respective cases:

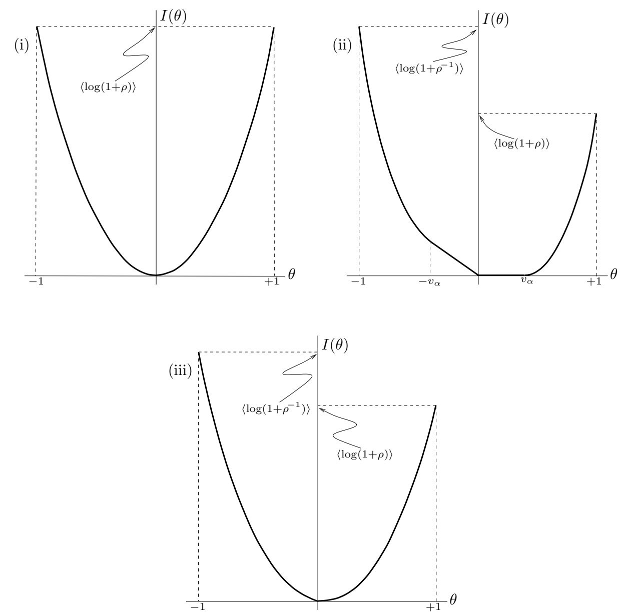

FIGURE 1.1: (i) recurrent; (ii) transient: positive speed; (iii) transient: zero speed. Permission to use the picture has been kindly granted by F. den Hollander [40]. The notations $I(\theta)$, ρ and $< \cdot >$, stand for I^{que}, $\frac{q}{p}\xi_0 + \frac{p}{q}[1-\xi_0]$ and E^{ξ}, respectively.

From Theorem 1.4 (see [34, 50]) we see that both in the recurrent case and in the transient case with zero speed the rate function has a unique zero at $\theta = 0$ and is strictly convex everywhere, while in the transient case with positive speed the rate function has two linear pieces: one horizontal piece for $\theta \in [0,v]$ and one tilted piece for $\theta \in [-v,0]$.

The flat piece means that speeds smaller than the typical speed v are not exponentially costly. This is again because the RE contains long stretches of sites where the local drifts point to the left. Between 0 and θn the longest stretches have a length of order $\log n$, and for the walker to lose a time of order n in these stretches has a cost that is sub-exponential in n.

Under the annealed measure, as stated in Theorem 1.3, an LDP is satisfied as well. The corresponding rate function I^{ann} is given by (1.8). A symmetry relation as in part 2 of Theorem 1.4 does not hold. In particular, I^{ann} and I^{que} coincide on $[0, v]$. Moreover a small linear piece can be present in the annealed rate function for some choice of the parameters, see [34].

1.1.2 Higher dimensions

In the multi-dimensional setup, even fundamental questions like recurrence vs. transience and the existence of a limiting speed remain partially open. We give here a brief summary of the main results and formulate unsolved conjectures. For a more detailed overview we refer the reader to [89, 99]. In what follows we restrict to the case where $\big(\xi(x,\cdot)\big)_{x\in\mathbb{Z}^d}$ is an i.i.d. sequence satisfying the so-called *ellipticity* condition

$$\mu\left(\inf_{|e|=1}\xi(x,e)>0\right)=1. \tag{1.11}$$

Let $\mathbb{S}^{d-1}$ be the unit sphere. Given a vector $l\in\mathbb{S}^{d-1}$, consider the event

$$A_l=\left\{\lim_{n\to\infty}X_n\cdot l=\infty\right\}, \tag{1.12}$$

where $\cdot$ denotes the vector inner product. In 1981 [58] Kalikow proved that $\mathbb{P}_0(A_l\cup A_{-l})\in\{0,1\}$ for all $l\in\mathbb{S}^{d-1}$ and $d\geq 1$, and he conjectured that if μ is *uniformly elliptic*, i.e., there exists a constant $\delta>0$ such that

$$\mu\left(\inf_{|e|=1}\xi(x,e)>\delta\right)=1, \tag{1.13}$$

then

$$\mathbb{P}_0(A_l)\in\{0,1\}\quad\forall l\in\mathbb{S}^{d-1}. \tag{1.14}$$

Furthermore, he formulated a technical condition (known as Kalikow's condition; see [58]) that ensures a strong bias in the direction l and implies (1.14).

In $d=1$, the 0-1 law in (1.14) is a simple consequence of Theorem 1.2. For $d=2$, (1.14) has been proven in [92] under the ellipticity condition in (1.11). For $d\geq 3$, (1.14) is still open and is a cornerstone to prove a LLN, as shown by the following theorem.

Theorem 1.5. (LLN)
Assume that μ is uniformly elliptic. Fix $l\in\mathbb{S}^{d-1}$. Then there exist $v^+,v^-\in[0,1]$ such that

$$\lim_{n\to\infty}\frac{X_n\cdot l}{n}=v^+1_{A_l}-v^-1_{A_{-l}}\quad\mathbb{P}_0-a.s. \tag{1.15}$$

In particular, for $d = 2$ the LLN holds.

The proof of this theorem [92, 101], and many other results in the multi-dimensional setting, are based on a construction of *regeneration times* introduced by Sznitman and Zerner [92]. Roughly speaking, a random time $\tau \in \mathbb{N}$ is a regeneration time in direction l if $X_\tau \cdot l \geq X_n \cdot l$ for all $n \leq \tau$ and $X_\tau \cdot l < X_n \cdot l$ for all $n > \tau$, i.e., $X_n \cdot l$ achieves a record at time τ and never moves backward from that record. Once these times are constructed, it is possible to show that the sequences of space and time increments between regeneration times form i.i.d. sequences, from which the LLN and the CLT can be derived.

Further results when μ is i.i.d. and uniformly elliptic were obtained in [8, 95]. In these papers it is shown that there are at most two deterministic limit points for the sequence $(X_n/n)_{n\in\mathbb{N}}$, say, v_1 and v_2. If $v_1 \neq v_2$ then there exists a constant $a \geq 0$ such that $v_2 = -av_1$. For $d \geq 5$, [8] proves that if $v_1 \neq v_2$, then at least one of them is zero.

There is no general criterion to establish when RWRE in $d \geq 2$ is transient or recurrent, although one expects transience as soon as $d \geq 3$. Moreover, when the LLN holds, no explicit formula for the limiting speed v is known. A natural question is to at least understand under which condition RWRE is *ballistic*, i.e., $v \neq 0$. Some results have been obtained in this direction in the last decade. This problem is related to the properties of the RE and the possible presence of 'traps' (i.e., regions where the walk may spend a long time with a high probability). In [100], the author considered the drift at the origin

$$d_0 = \sum_{i=1}^{d} [\xi(0, e_i) - \xi(0, -e_i)] \cdot e_i, \tag{1.16}$$

with $\{e_i\}_{i=1}^d$ the canonical basis of $\mathbb{Z}^d$, and showed that if, for some $l \in \mathbb{S}^{d-1}$, $d_0 \cdot l > 0$ for μ-a.e. environment, then X_n/n converges to a deterministic v with $v \cdot l > 0$. Such REs are called *non-nestling*. The interest is in understanding the so-called *nestling* REs, i.e., when the origin belongs to the closed convex hull of the support of d_0, for which a non-ballistic regime might be possible . Some progress has been achieved by Sznitman in [88, 89], who formulated the following conditions that guarantee ballisticity even for the nestling case. Given a direction $l \in \mathbb{S}^{d-1}$ and $b, L > 0$, define the slab $U_{b,l,L} = \{x \in \mathbb{Z}^d : -bL < x \cdot l < L\}$ and the exit time $\tau_{b,l,L} = \inf\{n \in \mathbb{N} : X_n \notin U_{b,l,L}\}$. Let $\gamma \in (0, 1]$. Condition $(T)_\gamma|l$ is said to hold relative to $l \in \mathbb{S}^{d-1}$ if, for all $l' \in \mathbb{S}^{d-1}$ in a neighborhood of l and all $b > 0$,

$$\limsup_{L\to\infty} L^{-\gamma} \log \mathbb{P}_0(X_{\tau_{b,l',L}} \cdot l' < 0) < 0. \tag{1.17}$$

In words, consider a slab in $\mathbb{Z}^d$ contained between the hyperplanes normal to l at distance L and $-bL$ in direction l. When $\gamma = 1$, the condition $(T)_1|l$ holds if the probability of exit from this slab in direction $-l$ is exponentially small in L. Condition $(T')|l$ is said to hold if condition $(T)_\gamma|l$ holds for all $\gamma \in (0, 1)$. Clearly

$$(T)_1|l \Longrightarrow (T')|l \Longrightarrow (T)_\gamma|l \text{ for } \gamma \in (0, 1), \tag{1.18}$$

and it is believed that $(T)_1|l, (T')|l$ and $(T)_\gamma|l$ are equivalent. This equivalence is still open and some recent progress can be found in [44, 45, 78]. The importance of these conditions is given by the following theorem due to Sznitman [88].

Theorem 1.6. (Ballisticity and CLT under Sznitman's $(T')|l$ condition)
Assume that μ is i.i.d. and uniformly elliptic and that condition $(T')|l$ holds relative to $l \in \mathbb{S}^{d-1}$. Then X satisfies a LLN with a deterministic limiting speed v such that $v \cdot l > 0$. Moreover, there exists a deterministic $\sigma > 0$ such that, under the annealed measure $\mathbb{P}_0$, $(X_n - nv)/\sigma\sqrt{n}$ converges in distribution to a standard Gaussian random variable.

Other recent results in the i.i.d. setting can be found in [27, 28].

When we drop the i.i.d. assumption on μ further complications arise. If the environment has a finite-range dependence, then a slight modification of the arguments for the i.i.d. situation, developed in [76, 99], shows that the LLN and the CLT carry over. If the space correlations are long-range but strong mixing in some appropriate sense, then only few results have been obtained. In this context, [35, 36] derived a LLN and a CLT via a regeneration-time argument under a uniform mixing condition. In [72] a LLN was derived by analyzing the *environment process*, i.e., the environment as seen from the point of view of the walker. [29] developed a renormalization scheme to prove a CLT when the transition probabilities of the RW are sufficiently close to those of a simple RW.

Large deviations for the empirical speed X_n/n have been studied only recently. The main result is stated in the next theorem due to Varadhan [94].

Theorem 1.7. (Quenched and Annealed LDP)
Let $d \geq 2$. Assume that μ is uniformly elliptic and ergodic. Then, for μ-a.e. realization of ξ, the family of probability measures $P_0^\xi(X_n/n \in \cdot), n \in \mathbb{N}$, satisfies a LDP with rate n and with convex deterministic rate function I^{que}. If μ is i.i.d., then also an annealed LDP is satisfied with rate n and with convex rate function I^{ann}. Furthermore, in the latter case I^{que} and I^{ann} have the same zero set, and this set is convex and consists of either a single point or a line segment.

The proof of Theorem 1.7 is based on a subadditivity argument. As for $d = 1$, the rate functions are in general not strictly convex. A relation like (1.8) is not available. It is not known under which conditions and in which region the two rate functions coincide. Partial progress and relalted results can be found in [73, 86, 87, 97].

1.2 Dynamic RE

In this section we introduce RWs in dynamic REs, which will be the main topic of this thesis. This is a variant of the problem in the previous section (see Definition 1.1) in which the environment ξ evolves in time according to a given autonomous dynamics. In other words, ξ is given by a collection of random vectors $\{\xi_n(x,\cdot) : x \in \mathbb{Z}^d, n \in \mathbb{N}_0\}$ with a prescribed joint law, and X is a RW with space-time dependent transition probabilities given by

$$P^{\xi}(X_{n+1} = x + e | X_n = x) = \xi_n(x, e), \quad e \in \mathbb{Z}^d, |e| = 1, n \in \mathbb{N}_0. \tag{1.19}$$

Under the quenched law P_0^{ξ}, X is now a space-time inhomogeneous Markov chain. Due to the dynamics of the environment, we expect different behavior than in the static situation. In particular, trapping phenomena, which played a chief role in static models, may not survive. The next sections are devoted to a brief exposition of the different types of problems that have been studied so far in the literature. We will first describe the easiest models and then move on to the more challenging ones.

1.2.1 Early work

In 1986 [64] Madras studied a one-dimensional RW that is a deterministic functional of a randomly fluctuating environment, which can be considered as a degenerate case of a RW in a dynamic RE. The model is defined as follows. For each $x \in \mathbb{Z}$, let $\xi = (\xi_t(x))_{t\geq 0}$ be an independent stationary continuous-time Markov process with state space $\{+1, -1\}$ and transition probability matrix

$$\begin{pmatrix} p_{-1,-1}(t) & p_{-1,+1}(t) \\ p_{+1,-1}(t) & p_{+1,+1}(t) \end{pmatrix} = \begin{pmatrix} q + pr^t & p - pr^t \\ q - qr^t & p + qr^t \end{pmatrix},$$

where $p = \alpha/(\alpha + \beta), q = 1 - p, r = e^{-(\alpha+\beta)}$, and $\alpha, \beta \in (0, \infty)$. Let $X = (X_t)_{t\geq 0}$ be a RW, starting from the origin ($X_0 = 0$) and moving on $\mathbb{R}$ as follows. For $i \in \mathbb{N}_0$, define

$$X_t = X_i + (t - i)\xi_i(X_i), \quad i < t \leq i + 1. \tag{1.20}$$

Thus, X represents the motion of a particle, travelling on $\mathbb{R}$ at unit speed, that at each unit of time chooses its direction according to the state of the local environment. By using the ergodic properties of ξ it can be shown that the process X has a stationary and exponentially mixing measure, which can be used to derive a recurrence criterion, a strong LLN and a CLT. In particular, X is recurrent if and only if $\alpha = \beta$. This model does not exhibit surprising behavior and, in contrast to RWRE models, has just one level of randomness. Nevertheless, the above results were obtained with the help of highly non-trivial methods and were the first in the dynamic setting.

1.2.2 Space-time i.i.d. RE

In 1992, Boldrighini et al. [16] introduced the first model of a RW in a dynamic RE. Since then this model has been studied intensively under several assumptions and using different tools. Though results like LLNs and CLTs have been derived, the general picture is far from being understood. The simplest setting is when the environment $\xi = \{\xi_n(x, \cdot) : x \in \mathbb{Z}^d, n \in \mathbb{N}\}$ is a collection of i.i.d. random variables, which we call i.i.d. space-time RE. Note that this is equivalent to a $(d+1)$-dimensional RW in a static i.i.d. RE in which, at each time step, one coordinate of the walk increases deterministically by one unit. Under the annealed measure, this RWRE becomes a simple RW in an averaged environment. Thus, the interest is in studying the quenched properties. The most general result has been derived in [74]. With the help of a martingale approach for additive functionals of a Markov chain, they obtained a quenched CLT in arbitrary dimension. In particular, they showed that the displacement of the RW in the i.i.d. space-time RE always has diffusive behavior with deterministic parameters. Similar results under a somewhat stronger condition on the RE were already found in [11], [22–24], via a cluster-expansion technique together with a small-noise assumption (see (1.21)), and in [6], with the help of generating functions.

A variant of the i.i.d. setting has been considered in [26]. Here, the environment is independent in time but has spatial correlation, i.e., at each time unit a new RE is sampled from a given distribution with dependence in space. The RW X is taken to be a perturbation of a homogeneous RW with transition kernel $p(e)$, namely,

$$P^{\xi}(X_{n+1} = x + e | X_n = x) = p(e) + \epsilon c(e; \xi_n(x)), \quad e \in \mathbb{Z}^d, |e| = 1, \tag{1.21}$$

where ϵ is a small positive parameter. The function c is such that (1.21) are transition probabilities, and represents the influence of ξ on the evolution of X. For ϵ small ('small noise regime'), a quenched CLT was proved with Brownian motion as scaling limit.

An analysis of the large deviations for RW in a space-time i.i.d. RE is presented in [98]. Except for our result in Chapter 4, this is the only paper dealing with LDPs in the dynamic setup. Indeed, as we pointed out in Section 1.1.2, even for static RE in $d \geq 2$ the large deviation analysis is difficult and is still far from being understood. Under the annealed law, RW in a space-time i.i.d. RE behaves like a homogeneous RW, for which the LDP for the empirical speed is given by Cramer's theorem [39, 40]. [98] shows that also under the quenched law a LDP for the empirical speed holds when $d \geq 3$. In particular, for speeds that are sufficiently close to the typical speed v, the quenched and annealed rate functions coincide. Furthermore, conditioned on any rare event (i.e., the empirical speed being any value different from v), the empirical process associated with *the environment process*, i.e., the environment as seen from the walker, converges to a certain stationary process, both under the quenched and the annealed law.

1.2.3 Time-dependent RE

Further complications arise when considering dynamic RE ξ in which the collection $\xi = \{\xi_n(x,\cdot) : x \in \mathbb{Z}^d, n \in \mathbb{N}\}$ is i.i.d. in space but Markovian in time, i.e., at each site x there is an independent copy of the same ergodic Markov chain. Note that, in this setup, the loss of time-independence makes even the annealed properties of X non-trivial. Such problems have been investigated in [5], [13], [42]. [13] considers the case in which the transition probabilities of the RW depend weakly on the environment ('small noise regime'; see (1.21)). By means of a cluster-expansion technique, it is proved that a quenched CLT holds a.s. for any $d \geq 3$. Similar results have been obtained in [5]. Via a probabilistic argument based on regeneration times, and under ellipticity conditions weaker than in [13], a strong LLN and a CLT were derived under the annealed law for any $d \geq 1$, and a quenched invariance principle only in high dimensions, namely, $d > 7$. Further progress was achieved with the help of an analytical approach to analyze *the environment process* in two recent papers [42, 43]. [42] deals with the case in which the transition probabilities of the RW are again weakly dependent on the environment, while the environment has a deterministic but strongly chaotic evolution. In [43], the authors consider a RW that is strongly dependent on a dynamic RE that again is assumed to be independent in space and Markovian in time. In both papers, a strong LLN and a CLT have been proven, under both the annealed and the quenched law.

1.2.4 Space-time mixing RE

A major challenge is to consider more general REs in which correlations in both space and time are allowed. Results in this direction have been obtained recently in [30, 41].

Both papers deal with finite-range RWs whose transition probabilities depend weakly on a RE whose space-time correlations decay exponentially. By means of a renormalization group technique [30], respectively, by analyzing *the environment process* via a martingale approximation [41], they proved a LLN and showed that in the scaling limit the behavior is diffusive for any $d \geq 1$. In particular, [30] does not assume a Markovian structure of the RE.

1.3 RW on an Interacting Particle System (IPS)

We are finally ready to introduce a class of RW in dynamic RE that will be the main subject of this thesis, namely, our RE will evolve as an Interacting Particle System (IPS). The main reason for this choice is that IPSs constitute a well-established research area and are natural examples of dynamic RE with space-time correlations. In the next sections we first define the class of IPSs we are interested in, providing some explicit examples, and then introduce our class of RWs. To avoid heavy notation, the definitions are stated for $d = 1$ and for nearest-neighbor RW, even though they easily extend to $d \geq 2$ and/or to more general step distributions. Such possible extensions will be pointed out in the next chapters.

1.3.1 IPS

1.3.1.1 Definition

Let $\Omega = \{0,1\}^{\mathbb{Z}^d}$. Denote by $D_\Omega[0,\infty)$ the set of paths in Ω that are right-continuous and have left limits. Let $\{P^\eta, \eta \in \Omega\}$ be a collection of probability measures on $D_\Omega[0,\infty)$ satisfying the Markov property. An IPS

$$\xi = (\xi_t)_{t\geq 0} \quad \text{with} \quad \xi_t = \{\xi_t(x)\colon\ x \in \mathbb{Z}^d\}, \tag{1.22}$$

is a Markov process on Ω with law P^η, when $\xi_0 = \eta \in \Omega$ is the starting configuration. We say that site x at time t is vacant or occupied when $\xi_t(x) = 0$ or 1.

Let $\mathcal{P}(\Omega)$ be the set of probability measures on Ω. Given $\mu \in \mathcal{P}(\Omega)$, we denote by P^μ the law of ξ when ξ_0 is drawn from $\mu \in \mathcal{P}(\Omega)$, i.e.,

$$P^\mu(\cdot) = \int_\Omega P^\eta(\cdot)\, \mu(\mathrm{d}\eta). \tag{1.23}$$

Throughout the sequel we will assume that

$$P^{\mu} \text{ is stationary and ergodic under space-time shifts.} \tag{1.24}$$

Thus, in particular, μ is a homogeneous extremal equilibrium for ξ.

Let $C(\Omega)$ be the set of continuous functions on Ω taking values in $\mathbb{R}$, viewed as a Banach space with norm

$$\|f\|_\infty = \sup_{\eta\in\Omega} |f(\eta)|. \tag{1.25}$$

The Markov semigroup associated with ξ is denoted by $S_{\mathrm{IPS}} = (S_{\mathrm{IPS}}(t))_{t\geq 0}$. This semigroup acts from the left on $C(\Omega)$ as

$$\big(S_{\mathrm{IPS}}(t)f\big)(\cdot) = E^{(\cdot)}[f(\xi_t)], \qquad f \in C(\Omega), \tag{1.26}$$

and acts from the right on $\mathcal{P}(\Omega)$ as

$$\big(\nu S_{\mathrm{IPS}}(t)\big)(\cdot) = P^{\nu}(\xi_t \in \cdot\,), \qquad \nu \in \mathcal{P}(\Omega). \tag{1.27}$$

In particular, we assume that ξ is a *Feller* process, i.e., $S_{\mathrm{IPS}}(t)f \in C(\Omega)$ for every $t \geq 0$ and $f \in C(\Omega)$.

Informally, an IPS is a collection of particles on the integer lattice evolving in a Markovian way. Depending on the specific transition rates between the different configurations, we obtain several types of IPS. Each particle may interact with the others: the evolution of each particle is defined in terms of local transition rates that may depend on the state of the system in a neighborhood of the particle. For a formal construction, we refer the reader to Liggett [63], Chapter I. Some explicit examples will be given below, and in the next chapters whenever needed.

1.3.1.2 Examples

(1) **Stochastic Ising Model (SIM)**

This model goes back to Glauber [49] and was introduced as a model for magnetism. The SIM is a Markov process on $\Omega' = \{+1,-1\}^{\mathbb{Z}^d}$, where each site represents an iron atom whose spin can be either up $(+1)$ or down (-1). In the original and easiest formulation, the dynamics can be described as follows. Let $\beta = T^{-1} \geq 0$ represent the inverse of the temperature T of the system. Given a starting configuration of spins $\eta \in \Omega'$, the spin

$\eta(x)$ at site x flips to $-\eta(x)$ at rate

$$c(x,\eta) = \exp\Big\{ -\beta \sum_{y:|y-x|=1} \eta(x)\eta(y) \Big\}, x \in \mathbb{Z}^d, \eta \in \Omega'. \tag{1.28}$$

With this choice of the rates we see that each spin tends to be aligned with its neighborhood. Indeed, the flip rate in (1.28) is higher when the spin at x differs from most of its neighbors than when it agrees with most of them. Such a monotonicity property is called *attractiveness* (see Section 2.4.2). In the language of statistical mechanics it is referred to as *ferromagnetism*. Note that, replacing the state space Ω' by Ω, we can pass from the 'spin interpretation' of the system to an interpretation of an IPS in which particles/holes flip into holes/particles.

Depending on the temperature and the dimension, the SIM shows interesting behavior. For example, when $d = 1$ it admits a unique ergodic measure for any $\beta \in \mathbb{R}^+$, while for $d \geq 2$ there exists a critical $\beta_c(d)$ such that for $\beta > \beta_c(d)$ there are at least two extremal invariant measures (which means that the system has a phase transition).

For $d \geq 1$, if $\beta = 0$, then the SIM is an example of an independent spin-flip dynamics (see Section 2.5), namely, the coordinates $\eta_t(x)$ become independent two-state Markov chains and the system has a unique ergodic measure given by the Bernoulli product measure with density $\frac{1}{2}$. The dynamics defined by the rates in (1.28) is only an example of a SIM. It is possible to also consider flip rates that depend not only on nearest-neighbor sites. For a general definition of the stochastic Ising model, see Liggett [63] Chapter 4.

(2) **Exclusion Process (EP)**

Let $p(x,y), x, y \in \mathbb{Z}^d$, be a transition kernel of a finite-range homogeneous RW on $\mathbb{Z}^d$. Given a configuration $\eta \in \Omega = \{0,1\}^{\mathbb{Z}^d}$, let $\{x \in \mathbb{Z}^d : \eta(x) = 1\}$ be the set of locations of the particles at time 0. The exclusion process is the IPS in which particles move according to the following rules:

- A particle at site x waits an exponential time with mean 1 and then chooses a site y with probability $p(x,y)$.
- The particle jumps to site y if this site is vacant, but does not jump when it is occupied.

The exclusion process is an example of a conservative IPS (i.e., the number of particles is preserved by the evolution) in which at each transition two coordinates of the system may change. It was originally introduced by Spitzer [81] as a model for a lattice gas at infinite temperature.

(3) **Contact Process (CP)**

The contact process (introduced by Harris [51]) is a toy model for the spread of an infection in a large population of individuals. Let $\Omega = \{0,1\}^{\mathbb{Z}^d}$. Each site $x \in \mathbb{Z}^d$ represents an individual. Given $\eta \in \Omega$, we say that the individual x is infected or healthy if $\eta(x)$ equals 1 respectively 0. The evolution of the system makes a healthy individual infected at rate λ times the number of infected neighbors, while infected individuals recover independently at rate 1. In other words, for each $x \in \mathbb{Z}^d$, if $\eta(x) = 1$, then $\eta(x)$ flips to 0 at rate 1, while if $\eta(x) = 0$, then it flips to 1 at rate $\lambda \sum_{y:|y-x|=1} \eta(y)$, where $\lambda \geq 0$ is a parameter representing the intensity of the infection spread. It is easy to see that the pointmass concentrated at the configuration with all 0's is a trivial invariant measure. It is possible to prove that for any $d \geq 1$ there exists a critical value $\lambda_c(d) \in (0,\infty)$ such that for $\lambda > \lambda_c(d)$ the system has at least one non-trivial invariant measure.

1.3.2 RW on IPS

Conditional on a realization of an IPS ξ, let

$$X = (X_t)_{t\geq 0} \tag{1.29}$$

be the RW with local transition rates

$$\begin{aligned} x &\to x+1 \quad \text{at rate} \quad \alpha_1\, \xi_t(x) + \alpha_0\, [1-\xi_t(x)], \\ x &\to x-1 \quad \text{at rate} \quad \beta_1\, \xi_t(x) + \beta_0\, [1-\xi_t(x)], \end{aligned} \tag{1.30}$$

where $\alpha_1, \beta_1, \alpha_0, \beta_0 \in (0,\infty)$ with $\alpha_1 + \beta_1 = \alpha_0 + \beta_0$. Thus, on occupied sites the RW has a local drift $\alpha_1 - \beta_1$, while on vacant sites it has a local drift $\alpha_0 - \beta_0$. Note that the sum of the jump rates is independent of ξ. Let P_0^ξ denote the law of X starting from $X_0 = 0$ conditional on ξ, which is the *quenched* law of X. The *annealed* law of X is

$$\mathbb{P}_{\mu,0}(\cdot) = \int_{D_\Omega[0,\infty)} P_0^\xi(\cdot)\, P^\mu(\mathrm{d}\xi). \tag{1.31}$$

Note that X is a continuous-time variant of the RW in a dynamic RE defined in (1.19).

By choosing $\alpha_1 = \beta_0, \alpha_0 = \beta_1$ with $\alpha_1 > \beta_1$, we obtain the continuous-time dynamic analogue of the static model given in Section 1.1.1.4, where $\alpha_1/(\alpha_1+\beta_1)$ takes over the role of p.

1.4 Related models

We close by listing some topics which are closely related to RWRE but not covered in this introduction.

- *Non-nearest-neighbor RWRE*: When dealing with non-nearest-neighbor finite-range RW in static and dynamic RE, some results and techniques we discussed in this chapter can be easily extended; see e.g. [41, 42, 73, 95]. Nevertheless, in dropping the nearest-neighbor assumption extra difficulties may arise and tools to analyze Lyapunov exponents associated with certain random matrices are needed (see e.g. [31, 60]).

- *RW in dynamic RE with mutual interaction*: These are models in which the dynamics of the RE is locally affected by the evolution of the RW (recall that in Section 1.2 we dealt with situations where the RE is completely independent of the RW). Under certain assumptions on the mutual interaction, LLNs, CLTs and LDPs have been obtained for such models in [11, 16, 19–21, 56].

- *RW on random graphs*: Several papers in the literature have been focusing on the asymptotic properties of RWs that evolves on a realization of a random graph. Two main classes concern random subgraphs of $\mathbb{Z}^d$ like percolation clusters (see e.g. [9, 10, 68, 90]), and random trees like Galton-Watson branching processes (see e.g. [37, 69]).

- *Random conductance model*: In these models, with each bond (x, y) of the integer lattice $\mathbb{Z}^d$ is associated a random variable $C_{x,y} \geq 0$ representing a conductance, with $C = \{C_{x,y}\}_{x,y\in\mathbb{Z}^d}$ i.i.d. Given a realization of C, the aim is to study the behavior of the RW whose transition probabilities from site x to site y are given by $C_{x,y}/\sum_{z:|z-x|=1} C_{x,z}$. Such a model is closely related to RWs on supercritical percolation clusters. Annealed and quenched CLTs for this RW were derived in [15, 62, 77].

- *Diffusion with random potential*: These models represent the natural analogue of RWRE in the theory of diffusion processes. Informally speaking, the idea is to find a 'solution' to the stochastic differential equation $\mathrm{d}X_t = -\frac{1}{2}\nabla V(X_t)\mathrm{d}t + \mathrm{d}W_t$, $X_0 = 0$, where the function $V = F + B$ is a sum of a deterministic function $F : \mathbb{R}^d \to \mathbb{R}$ plus a random field B indexed by $\mathbb{R}^d$, $B = (B(x))_{x\in\mathbb{R}^d}$, and W is a d-dimensional Brownian motion independent of V. For results on this topic we refer the reader to [32, 67, 91] and the references therein.

Chapter 2

Law of large numbers for a class of RW in dynamic RE

This chapter is based on a paper with Frank den Hollander and Frank Redig that has been submitted to Electronic Journal of Probability.

Abstract

In this paper we consider a class of one-dimensional interacting particle systems in equilibrium, constituting a dynamic random environment, together with a nearest-neighbor random walk that on occupied/vacant sites has a local drift to the right/left. We adapt a regeneration-time argument originally developed by Comets and Zeitouni [35] for static random environments to prove that, under a space-time mixing property for the dynamic random environment called cone-mixing, the random walk has an a.s. constant global speed. In addition, we show that if the dynamic random environment is exponentially mixing in space-time and the local drifts are small, then the global speed can be written as a power series in the size of the local drifts. From the first term in this series the sign of the global speed can be read off.

The results can be easily extended to higher dimensions.

Acknowledgment. The authors are grateful to R. dos Santos and V. Sidoravicius for fruitful discussions.
MSC 2000. Primary 60H25, 82C44; Secondary 60F10, 35B40.
Key words and phrases. Random walk, dynamic random environment, cone-mixing, exponentially mixing, law of large numbers, perturbation expansion.

2.1 Introduction and main result

In Section 2.1 we define the random walk in dynamic random environment, introduce a space-time mixing property for the random environment called cone-mixing, and state our law of large numbers for the random walk subject to cone-mixing. In Section 2.2 we give the proof of the law of large numbers with the help of a space-time regeneration-time argument. In Section 2.3 we assume a stronger space-time mixing property, namely, exponential mixing, and derive a series expansion for the global speed of the random walk in powers of the size of the local drifts. This series expansion converges for small enough local drifts and its first term allows us to determine the sign of the global speed. (The perturbation argument underlying the series expansion provides an alternative proof of the law of large numbers.) In Section 2.4 we give examples of random environments that are cone-mixing. In Section 2.5 we compute the first three terms in the expansion for an independent spin-flip dynamics.

2.1.1 Model

Let $\Omega = \{0,1\}^{\mathbb{Z}}$. Let $C(\Omega)$ be the set of continuous functions on Ω taking values in $\mathbb{R}$, $\mathcal{P}(\Omega)$ the set of probability measures on Ω, and $D_\Omega[0,\infty)$ the path space, i.e., the set of càdlàg functions on $[0,\infty)$ taking values in Ω. In what follows,

$$\xi = (\xi_t)_{t\geq 0} \qquad \text{with} \qquad \xi_t = \{\xi_t(x)\colon\ x \in \mathbb{Z}\} \tag{2.1}$$

is an interacting particle system taking values in Ω, with $\xi_t(x) = 0$ meaning that site x is vacant at time t and $\xi_t(x) = 1$ that it is occupied. The paths of ξ take values in $D_\Omega[0,\infty)$. The law of ξ starting from $\xi_0 = \eta$ is denoted by P^η. The law of ξ when ξ_0 is drawn from $\mu \in \mathcal{P}(\Omega)$ is denoted by P^μ, and is given by

$$P^\mu(\cdot) = \int_\Omega P^\eta(\cdot)\,\mu(\mathrm{d}\eta). \tag{2.2}$$

Through the sequel we will assume that

$$P^\mu \text{ is stationary and ergodic under space-time shifts.} \tag{2.3}$$

Thus, in particular, μ is a homogeneous extremal equilibrium for ξ. The Markov semigroup associated with ξ is denoted by $S_{\mathrm{IPS}} = (S_{\mathrm{IPS}}(t))_{t\geq 0}$. This semigroup acts from the left on $C(\Omega)$ as

$$\big(S_{\mathrm{IPS}}(t)f\big)(\cdot) = E^{(\cdot)}[f(\xi_t)], \qquad f \in C(\Omega), \tag{2.4}$$

and acts from the right on $\mathcal{P}(\Omega)$ as

$$\big(\nu S_{\mathrm{IPS}}(t)\big)(\cdot) = P^{\nu}(\xi_t \in \cdot\,), \qquad \nu \in \mathcal{P}(\Omega). \tag{2.5}$$

See Liggett [63], Chapter I, for a formal construction.

Conditional on ξ, let

$$X = (X_t)_{t \geq 0} \tag{2.6}$$

be the random walk with local transition rates

$$\begin{aligned} x \to x+1 \quad &\text{at rate} \quad \alpha\,\xi_t(x) + \beta\,[1 - \xi_t(x)], \\ x \to x-1 \quad &\text{at rate} \quad \beta\,\xi_t(x) + \alpha\,[1 - \xi_t(x)], \end{aligned} \tag{2.7}$$

where w.l.o.g.

$$0 < \beta < \alpha < \infty. \tag{2.8}$$

Thus, on occupied sites the random walk has a local drift to the right while on vacant sites it has a local drift to the left, of the same size. Note that the sum of the jump rates $\alpha + \beta$ is independent of ξ. Let P_0^{ξ} denote the law of X starting from $X_0 = 0$ conditional on ξ, which is the *quenched* law of X. The *annealed* law of X is

$$\mathbb{P}_{\mu,0}(\cdot) = \int_{D_\Omega[0,\infty)} P_0^{\xi}(\cdot)\, P^{\mu}(\mathrm{d}\xi). \tag{2.9}$$

2.1.2 Cone-mixing and law of large numbers

In what follows we will need a *mixing property* for the law P^μ of ξ. Let $\cdot$ and $\|\cdot\|$ denote the inner product, respectively, the Euclidean norm on $\mathbb{R}^2$. Put $\ell = (0,1)$. For $\theta \in (0, \frac{1}{2}\pi)$ and $t \geq 0$, let

$$C_t^{\theta} = \big\{ u \in \mathbb{Z} \times [0,\infty) \colon\ (u - t\ell) \cdot \ell \geq \|u - t\ell\| \cos\theta \big\} \tag{2.10}$$

be the cone whose tip is at $t\ell = (0,t)$ and whose wedge opens up in the direction ℓ with an angle θ on either side (see Figure 2.1). Note that if $\theta = \frac{1}{2}\pi$ ($\theta = \frac{1}{4}\pi$), then the cone is the half-plane (quarter-plane) above $t\ell$.

Definition 2.1. A probability measure P^μ on $D_\Omega[0,\infty)$ satisfying (2.3) is said to be cone-mixing if, for all $\theta \in (0, \frac{1}{2}\pi)$,

$$\lim_{t\to\infty} \sup_{\substack{A \in \mathcal{F}_0,\, B \in \mathcal{F}_t^{\theta} \\ P^\mu(A) > 0}} \Big| P^{\mu}(B \mid A) - P^{\mu}(B) \Big| = 0, \tag{2.11}$$

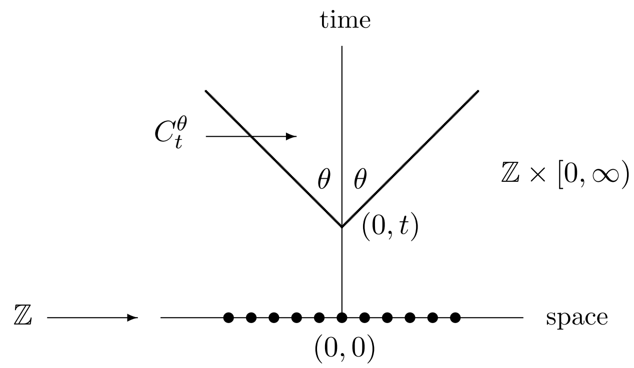

FIGURE 2.1: The cone C_t^θ.

where

$$\begin{aligned} \mathcal{F}_0 &= \sigma\{\xi_0(x) \colon\ x \in \mathbb{Z}\}, \\ \mathcal{F}_t^\theta &= \sigma\{\xi_s(x) \colon\ (x, s) \in C_t^\theta\}. \end{aligned} \tag{2.12}$$

In Section 2.4 we give examples of interacting particle systems that are cone-mixing.

We are now ready to formulate our law of large numbers (LLN).

Theorem 2.2. *Assume* (2.3). *If P^μ is cone-mixing, then there exists a $v \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} X_t/t = v \qquad \mathbb{P}_{\mu,0} - a.s. \tag{2.13}$$

The proof of Theorem 2.2 is given in Section 2.2, and is based on a *regeneration-time argument* originally developed by Comets and Zeitouni [35] for static random environments (based on earlier work by Sznitman and Zerner [92]).

We have no criterion for when $v < 0$, $v = 0$ or $v > 0$. In view of (2.8), a naive guess would be that these regimes correspond to $\rho < \frac{1}{2}$, $\rho = \frac{1}{2}$ and $\rho > \frac{1}{2}$, respectively, with $\rho = P^\mu(\xi_0(0) = 1)$ the density of occupied sites. However, $v = (2\tilde{\rho} - 1)(\alpha - \beta)$, with $\tilde{\rho}$ the asymptotic fraction of time spent by the walk on occupied sites, and the latter is a non-trivial function of P^μ, α and β. We do not (!) expect that $\tilde{\rho} = \frac{1}{2}$ when $\rho = \frac{1}{2}$ in general. Clearly, if P^μ is invariant under swapping the states 0 and 1, then $v = 0$.

2.1.3 Global speed for small local drifts

For small $\alpha - \beta$, X is a perturbation of simple symmetric random walk. In that case it is possible to derive an expansion of v in powers of $\alpha - \beta$, provided P^μ satisfies an exponential space-time mixing property referred to as $M < \epsilon$ (Liggett [63], Section I.3). Under this mixing property, μ is even uniquely ergodic.

Suppose that ξ has shift-invariant local transition rates

$$c(A,\eta), \qquad A \subset \mathbb{Z} \text{ finite}, \ \eta \in \Omega, \tag{2.14}$$

i.e., $c(A,\eta)$ is the rate in the configuration η to change the states at the sites in A, and $c(A,\eta) = c(A+x, \tau_x \eta)$ for all $x \in \mathbb{Z}$ with τ_x the shift of space over x. Define

$$\begin{aligned} M &= \sum_{A \ni 0} \sum_{x \neq 0} \sup_{\eta \in \Omega} |c(A,\eta) - c(A,\eta^x)|, \\ \epsilon &= \inf_{\eta \in \Omega} \sum_{A \ni 0} |c(A,\eta) + c(A,\eta^0)|, \end{aligned} \tag{2.15}$$

where η^x is the configuration obtained from x by changing the state at site x. The interpretation of (2.15) is that M is a measure for the *maximal* dependence of the transition rates on the states of single sites, while ϵ is a measure for the *minimal* rate at which the states of single sites change. See Liggett [63], Section I.4, for examples.

Theorem 2.3. *Assume* (2.3) *and suppose that* $M < \epsilon$. *If* $\alpha - \beta < \frac{1}{2}(\epsilon - M)$, *then*

$$v = \sum_{n \in \mathbb{N}} c_n \, (\alpha - \beta)^n \in \mathbb{R} \qquad \textit{with} \qquad c_n = c_n(\alpha + \beta; P^\mu), \tag{2.16}$$

where $c_1 = 2\rho - 1$ *and* $c_n \in \mathbb{R}$, $n \in \mathbb{N}\backslash\{1\}$, *are given by a recursive formula (see Section* 2.3.3*).*

The proof of Theorem 2.3 is given in Section 2.3, and is based on an analysis of the semigroup associated with the environment process, i.e., the environment as seen relative to the random walk. The generator of this process turns out to be a sum of a large part and a small part, which allows for a perturbation argument. In Section 2.4 we show that $M < \epsilon$ implies cone-mixing for spin-flip systems, i.e., systems for which $c(A,\eta) = 0$ when $|A| \geq 2$.

It follows from Theorem 2.3 that for $\alpha - \beta$ small enough the global speed v changes sign at $\rho = \frac{1}{2}$:

$$v = (2\rho - 1)(\alpha - \beta) + O\big((\alpha - \beta)^2\big) \text{ as } \alpha \downarrow \beta \text{ for } \rho \text{ fixed.} \tag{2.17}$$

We will see in Section 2.3.3 that $c_2 = 0$ when μ is a *reversible* equilibrium, in which case the error term in (2.17) is $O((\alpha - \beta)^3)$.

In Section 2.5 we consider an independent spin-flip dynamics such that 0 changes to 1 at rate γ and 1 changes to 0 at rate δ, where $0 < \gamma, \delta < \infty$. By reversibility, $c_2 = 0$. We show that

$$c_3 = \frac{4}{U^2} \, \rho(1-\rho)(2\rho - 1) \, f(U,V), \quad f(U,V) = \frac{2U + V}{\sqrt{V^2 + 2UV}} - \frac{2U + 2V}{\sqrt{V^2 + UV}} + 1, \tag{2.18}$$

with $U = \alpha + \beta$, $V = \gamma + \delta$ and $\rho = \gamma/(\gamma + \delta)$. Note that $f(U, V) < 0$ for all U, V and $\lim_{V\to\infty} f(U, V) = 0$ for all U. Therefore (2.18) shows that

$$\begin{array}{ll} (1) & c_3 > 0 \text{ for } \rho < \frac{1}{2},\ c_3 = 0 \text{ for } \rho = \frac{1}{2},\ c_3 < 0 \text{ for } \rho > \frac{1}{2}, \\ (2) & c_3 \to 0 \text{ as } \gamma + \delta \to \infty \text{ for fixed } \rho \neq \frac{1}{2} \text{ and fixed } \alpha + \beta. \end{array} \tag{2.19}$$

If $\rho = \frac{1}{2}$, then the dynamics is invariant under swapping the states 0 and 1, so that $v = 0$. If $\rho > \frac{1}{2}$, then $v > 0$ for $\alpha - \beta > 0$ small enough, but v is smaller in the random environment than in the average environment, for which $v = (2\rho-1)(\alpha-\beta)$ ("slow-down phenomenon"). In the limit $\gamma + \delta \to \infty$ the walk sees the average environment.

2.1.4 Discussion and outline

Three classes of dynamic random environments have been studied in the literature so far:

(1) *Independent in time*: globally updated at each unit of time ;

(2) *Independent in space*: locally updated according to independent single-site Markov chains;

(3) *Dependent in space and time.*

Our models fit into class (3), which is the most challenging and still is far from being understood. For an extended list of references we refer the reader to [4].

Many results, like a LLN, annealed and quenched invariance principles or decay of correlations, have been obtained for the above three classes under suitable extra assumptions. In particular, it is assumed either that the random enviornment has a strong space-time mixing property and/or that the transition probabilities of the walks are close to constant, i.e., small perturbation of a homogeneous random walk.

The LLN in Theorem 2.2 is a successful attempt to move away from the restrictions. Cone mixing is one of the weakest mixing conditions under which we may expect to be able to derive a LLN via regeneration times: no rate of mixing is imposed in (2.11). Still, (2.11) is not optimal because it is a *uniform* mixing condition. For instance, the simple symmetric exclusion process, which has a one-parameter family of equilibria parameterized by the particle density, is not cone-mixing.

Our expansion of the global speed in Theorem 2.3 which is a perturbation of a homogeneous random walk falls in class (3), but unlike what was done in previous works, it offers an explicit control on the coefficients and on the domain of convergence of the expansion.

Both Theorem 2.2 and 2.3 are easily extended to higher dimensions (with the obvious generalization of cone-mixing), and to random walks whose step rates are local functions of the environment, i.e., in (2.7) replace $\xi_t(x)$ by $R(\tau_x\xi_t)$, with τ_x the shift over x and R any cylinder function on Ω. It is even possible to allow for steps with a finite range. All that is needed is that the total jump rate is independent of the random environment. The reader is invited to take a look at the proofs in Sections 2.2 and 2.3 to see why.

In the context of Theorem 2.3, the LLN can be extended to a central limit theorem (CLT) with somewhat strong mixing assumptions and to a large deviation principle (LDP), issues which we plan to address in future work.

2.2 Proof of Theorem 2.2

In this section we prove Theorem 2.2 by adapting the proof of the LLN for random walks in *static* random environments developed by Comets and Zeitouni [35]. The proof proceeds in seven steps. In Section 2.2.1 we look at a discrete-time random walk X on $\mathbb{Z}$ in a *dynamic* random environment and show that it is equivalent to a discrete-time random walk Y on

$$\mathbb{H} = \mathbb{Z} \times \mathbb{N}_0 \tag{2.20}$$

in a *static* random environment that is *directed* in the vertical direction. In Section 2.2.2 we show that Y in turn is equivalent to a discrete-time random walk Z on $\mathbb{H}$ that suffers *time lapses*, i.e., random times intervals during which it does not observe the random environment and does not move in the horizontal direction. Because of the cone-mixing property of the random environment, these time lapses have the effect of *wiping out the memory.* In Section 2.2.3 we introduce *regeneration times* at which, roughly speaking, the future of Z becomes independent of its past. Because Z is directed, these regeneration times are stopping times. In Section 2.2.4 we derive a bound on the moments of the gaps between the regeneration times. In Section 2.2.5 we recall a basic coupling property for sequences of random variables that are weakly dependent. In Section 2.2.6, we collect the various ingredients and prove the LLN for Z, which will immediately imply the LLN for X. In Section 2.2.7, finally, we show how the LLN for X can be extended from *discrete* time to *continuous* time.

The main ideas in the proof all come from [35]. In fact, by exploiting the directedness we are able to simplify the argument in [35] considerably.

2.2.1 Space-time embedding

Conditional on ξ, we define a *discrete-time* random walk on $\mathbb{Z}$

$$X = (X_n)_{n\in\mathbb{N}_0} \tag{2.21}$$

with transition probabilities

$$P_0^\xi\big(X_{n+1} = x+i \mid X_n = x\big) = \begin{cases} p\,\xi_{n+1}(x) + q\,[1-\xi_{n+1}(x)] & \text{if } i = 1,\\ q\,\xi_{n+1}(x) + p\,[1-\xi_{n+1}(x)] & \text{if } i = -1,\\ 0 & \text{otherwise,}\end{cases} \tag{2.22}$$

where $x \in \mathbb{Z}$, $p \in (\frac{1}{2}, 1)$, $q = 1-p$, and P_0^ξ denotes the law of X starting from $X_0 = 0$ conditional on ξ. This is the discrete-time version of the random walk defined in (2.6–2.7), with p and q taking over the role of $\alpha/(\alpha+\beta)$ and $\beta/(\alpha+\beta)$. As in Section 2.1.1, we write P_0^ξ to denote the *quenched* law of X and $\mathbb{P}_{\mu,0}$ to denote the *annealed* law of X.

Our interacting particle system ξ is assumed to start from an equilibrium measure μ such that the path measure P^μ is stationary and ergodic under space-time shifts and is cone-mixing. Given a realization of ξ, we observe the values of ξ at integer times $n \in \mathbb{Z}$, and introduce a random walk on $\mathbb{H}$

$$Y = (Y_n)_{n\in\mathbb{N}_0} \tag{2.23}$$

with transition probabilities

$$P_{(0,0)}^\xi\big(Y_{n+1} = x+e \mid Y_n = x\big) = \begin{cases} p\,\xi_{x_2+1}(x_1) + q\,[1-\xi_{x_2+1}(x_1)] & \text{if } e = \ell^+,\\ q\,\xi_{x_2+1}(x_1) + p\,[1-\xi_{x_2+1}(x_1)] & \text{if } e = \ell^-,\\ 0 & \text{otherwise,}\end{cases} \tag{2.24}$$

where $x = (x_1, x_2) \in \mathbb{H}$, $\ell^+ = (1,1)$, $\ell^- = (-1,1)$, and $P_{(0,0)}^\xi$ denotes the law of Y given $Y_0 = (0,0)$ conditional on ξ. By construction, Y is the random walk on $\mathbb{H}$ that moves inside the cone with tip at $(0,0)$ and angle $\frac{1}{4}\pi$, and jumps in the directions either l^+ or l^-, such that

$$Y_n = (X_n, n), \qquad n \in \mathbb{N}_0. \tag{2.25}$$

We refer to $P_{(0,0)}^\xi$ as the quenched law of Y and to

$$\mathbb{P}_{\mu,(0,0)}(\cdot) = \int_{D_\Omega[0,\infty)} P_{(0,0)}^\xi(\cdot)\, P^\mu(\mathrm{d}\xi) \tag{2.26}$$

as the annealed law of Y. If we manage to prove that there exists a $u = (u_1, u_2) \in \mathbb{R}^2$ such that

$$\lim_{n\to\infty} Y_n/n = u \qquad \mathbb{P}_{\mu,(0,0)} - a.s., \tag{2.27}$$

then, by (2.25), $u_2 = 1$, and the LLN in Theorem 2.2 holds with $v = u_1$.

2.2.2 Adding time lapses

Put $\Lambda = \{0, \ell^+, \ell^-\}$. Let $\epsilon = (\epsilon_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of random variables taking values in Λ according to the product law $W = w^{\otimes \mathbb{N}}$ with marginal

$$w(\epsilon_1 = e) = \begin{cases} r & \text{if } e \in \{\ell^+, \ell^-\}, \\ p & \text{if } e = 0, \end{cases} \tag{2.28}$$

with $r = \frac{1}{2}q$. For fixed ξ and ϵ, introduce a second random walk on $\mathbb{H}$

$$Z = (Z_n)_{n\in\mathbb{N}_0} \tag{2.29}$$

with transition probabilities

$$\begin{aligned} &\bar{P}^{\xi,\epsilon}_{(0,0)}\big(Z_{n+1} = x + e \mid Z_n = x\big) \\ &\quad = 1_{\{\epsilon_{n+1}=e\}} + \frac{1}{p}\, 1_{\{\epsilon_{n+1}=0\}} \left[P^{\xi}_{(0,0)}\big(Y_{n+1} = x + e \mid Y_n = x\big) - r\right], \end{aligned} \tag{2.30}$$

where $x \in \mathbb{H}$ and $e \in \{\ell^+, \ell^-\}$, and $\bar{P}^{\xi,\epsilon}_{(0,0)}$ denotes the law of Z given $Z_0 = (0,0)$ conditional on ξ, ϵ. In words, if $\epsilon_{n+1} \in \{\ell^+, \ell^-\}$, then Z takes step ϵ_{n+1} at time $n+1$, while if $\epsilon_{n+1} = 0$, then Z copies the step of Y.

The quenched and annealed laws of Z defined by

$$\bar{P}^{\xi}_{(0,0)}(\cdot) = \int_{\Lambda^{\mathbb{N}}} \bar{P}^{\xi,\epsilon}_{(0,0)}(\cdot)\, W(\mathrm{d}\epsilon), \qquad \bar{\mathbb{P}}_{\mu,(0,0)}(\cdot) = \int_{D_\Omega[0,\infty)} \bar{P}^{\xi}_{(0,0)}(\cdot)\, P^{\mu}(\mathrm{d}\xi), \tag{2.31}$$

coincide with those of Y, i.e.,

$$\bar{P}^{\xi}_{(0,0)}(Z \in \cdot\,) = P^{\xi}_{(0,0)}(Y \in \cdot\,), \qquad \bar{\mathbb{P}}_{\mu,(0,0)}(Z \in \cdot\,) = \mathbb{P}_{\mu,(0,0)}(Y \in \cdot\,). \tag{2.32}$$

In words, Z becomes Y when the average over ϵ is taken. The importance of (2.32) is two-fold. First, to prove the LLN for Y in (2.27) it suffices to prove the LLN for Z. Second, Z suffers time lapses during which its transitions are dictated by ϵ rather than ξ. By the cone-mixing property of ξ, these time lapses will allow ξ to steadily loose memory, which will be a crucial element in the proof of the LLN for Z.

2.2.3 Regeneration times

Fix $L \in 2\mathbb{N}$ and define the L-vector

$$\epsilon^{(L)} = (\ell^+, \ell^-, \dots, \ell^+, \ell^-), \tag{2.33}$$

where the pair ℓ^+, ℓ^- is alternated $\frac{1}{2}L$ times. Given $n \in \mathbb{N}_0$ and $\epsilon \in \Lambda^{\mathbb{N}}$ with $(\epsilon_{n+1}, \dots, \epsilon_{n+L}) = \epsilon^{(L)}$, we see from (2.30) that (because $\ell^+ + \ell^- = (0,2) = 2\ell$)

$$\bar{P}^{\xi,\epsilon}_{(0,0)}\big(Z_{n+L} = x + L\ell \mid Z_n = x\big) = 1, \qquad x \in \mathbb{H}, \tag{2.34}$$

which means that the stretch of walk $Z_n, \dots, Z_{n+L}$ travels in the vertical direction ℓ irrespective of ξ.

Define *regeneration times*

$$\tau_0^{(L)} = 0, \qquad \tau_{k+1}^{(L)} = \inf\big\{n > \tau_k^{(L)} + L\colon\ (\epsilon_{n-L}, \dots, \epsilon_{n-1}) = \epsilon^{(L)}\big\}, \qquad k \in \mathbb{N}. \tag{2.35}$$

Note that these are stopping times w.r.t. the filtration $\mathcal{G} = (\mathcal{G}_n)_{n\in\mathbb{N}}$ given by

$$\mathcal{G}_n = \sigma\{\epsilon_i\colon\ 1 \le i \le n\}, \qquad n \in \mathbb{N}. \tag{2.36}$$

Also note that, by the product structure of $W = w^{\otimes\mathbb{N}}$ defined in (2.28), we have $\tau_k^{(L)} < \infty$ $\bar{\mathbb{P}}_0$-a.s. for all $k \in \mathbb{N}$.

Recall Definition 2.1 and put

$$\Phi(t) = \sup_{\substack{A\in\mathcal{F}_0,\, B\in\mathcal{F}_t^\theta \\ P^\mu(A)>0}} \Big| P^\mu(B \mid A) - P^\mu(B) \Big|. \tag{2.37}$$

Cone-mixing is the property that $\lim_{t\to\infty} \Phi(t) = 0$ (for all cone angles $\theta \in (0, \frac{1}{2}\pi)$, in particular, for $\theta = \frac{1}{4}\pi$ needed here). Let

$$\mathcal{H}_k = \sigma\Big((\tau_i^{(L)})_{i=0}^k,\ (Z_i)_{i=0}^{\tau_k^{(L)}},\ (\epsilon_i)_{i=0}^{\tau_k^{(L)}-1},\ \{\xi_t\colon\ 0 \le t \le \tau_k^{(L)} - L\} \Big), \qquad k \in \mathbb{N}. \tag{2.38}$$

This sequence of sigma-fields allows us to keep track of the walk, the time lapses and the environment up to each regeneration time. Our main result in the section is the following.

Lemma 2.4. *For all $L \in 2\mathbb{N}$ and $k \in \mathbb{N}$,*

$$\Big\| \bar{\mathbb{P}}_{\mu,(0,0)}\Big(Z^{[k]} \in \cdot \mid \mathcal{H}_k\Big) - \bar{\mathbb{P}}_{\mu,(0,0)}\big(Z \in \cdot\,\big) \Big\|_{\mathrm{tv}} \le \Phi(L), \tag{2.39}$$

where

$$Z^{[k]} = \left(Z_{\tau_k^{(L)}+n} - Z_{\tau_k^{(L)}} \right)_{n\in\mathbb{N}_0} \tag{2.40}$$

and $\|\cdot\|_{\mathrm{tv}}$ *is the total variation norm.*

Proof. We give the proof for $k = 1$. Let $A \in \sigma(\mathbb{H}^{\mathbb{N}_0})$ be arbitrary, and abbreviate $1_A = 1_{\{Z\in A\}}$. Let h be any $\mathcal{H}_1$-measurable non-negative random variable. Then, for all $x \in \mathbb{H}$ and $n \in \mathbb{N}$, there exists a random variable $h_{x,n}$, measurable w.r.t. the sigma-field

$$\sigma\left((Z_i)_{i=0}^n,\ (\epsilon_i)_{i=0}^{n-1},\ \{\xi_t\colon\ 0 \le t < n - L\} \right), \tag{2.41}$$

such that $h = h_{x,n}$ on the event $\{Z_n = x, \tau_1^{(L)} = n\}$. Let $E_{P^\mu\otimes W}$ and $\mathrm{Cov}_{P^\mu\otimes W}$ denote expectation and covariance w.r.t. $P^\mu \otimes W$, and write θ_n to denote the shift of time over n. Then

$$\begin{aligned}
\bar{\mathbb{E}}_{\mu,(0,0)}\left(h\left[1_A \circ \theta_{\tau_1^{(L)}} \right] \right) &= \sum_{x\in\mathbb{H},n\in\mathbb{N}} E_{P^\mu\otimes W}\left(\bar{E}_0^{\xi,\epsilon}\left(h_{x,n}\left[1_A\circ\theta_n\right] 1_{\left\{Z_n=x,\tau_1^{(L)}=n\right\}} \right)\right) \\
&= \sum_{x\in\mathbb{H},n\in\mathbb{N}} E_{P^\mu\otimes W}\big(f_{x,n}(\xi,\epsilon)\, g_{x,n}(\xi,\epsilon) \big) \\
&= \bar{\mathbb{E}}_{\mu,(0,0)}(h)\, \bar{\mathbb{P}}_{\mu,(0,0)}(A) + \rho_A,
\end{aligned} \tag{2.42}$$

where

$$f_{x,n}(\xi,\epsilon) = \bar{E}_{(0,0)}^{\xi,\epsilon}\left(h_{x,n}\, 1_{\left\{Z_n=x,\tau_1^{(L)}=n\right\}} \right), \quad g_{x,n}(\xi,\epsilon) = \bar{P}_x^{\theta_n\xi,\theta_n\epsilon}(A), \tag{2.43}$$

and

$$\rho_A = \sum_{x\in\mathbb{H},n\in\mathbb{N}} \mathrm{Cov}_{P^\mu\otimes W}\big(f_{x,n}(\xi,\epsilon), g_{x,n}(\xi,\epsilon) \big). \tag{2.44}$$

By (2.11), we have

$$\begin{aligned}
|\rho_A| &\le \sum_{x\in\mathbb{H},n\in\mathbb{N}} \left| \mathrm{Cov}_{P^\mu\otimes W}\big(f_{x,n}(\xi,\epsilon), g_{x,n}(\xi,\epsilon) \big) \right| \\
&\le \sum_{x\in\mathbb{H},n\in\mathbb{N}} \Phi(L)\, E_{P^\mu\otimes W}\big(f_{x,n}(\xi,\epsilon) \big) \sup_{\xi,\epsilon} g_{x,n}(\xi,\epsilon) \\
&\le \Phi(L) \sum_{x\in\mathbb{H},n\in\mathbb{N}} E_{P^\mu\otimes W}\big(f_{x,n}(\xi,\epsilon) \big) = \Phi(L)\, \bar{\mathbb{E}}_{\mu,(0,0)}(h).
\end{aligned} \tag{2.45}$$

Combining (2.42) and (2.45), we get

$$\left| \bar{\mathbb{E}}_{\mu,(0,0)}\left(h\left[1_A \circ \theta_{\tau_1^{(L)}} \right] \right) - \bar{\mathbb{E}}_{\mu,(0,0)}(h)\, \bar{\mathbb{P}}_{\mu,(0,0)}(A) \right| \le \Phi(L)\, \bar{\mathbb{E}}_{\mu,(0,0)}(h). \tag{2.46}$$

Now pick $h = 1_B$ with $B \in \mathcal{H}_1$ arbitrary. Then (2.46) yields

$$\left| \bar{\mathbb{P}}_{\mu,(0,0)} \left(Z^{[k]} \in A \mid B \right) - \bar{\mathbb{P}}_{\mu,(0,0)} \left(Z \in A \right) \right| \leq \Phi(L) \text{ for all } A \in \sigma(\mathbb{H}^{\mathbb{N}_0}),\ B \in \mathcal{H}_1. \quad (2.47)$$

There are only countably many cylinders in $\mathbb{H}^{\mathbb{N}_0}$, and so there is a subset of $\mathcal{H}_1$ with P^μ-measure 1 such that, for all B in this set, the above inequality holds simultaneously for all A. Take the supremum over A to get the claim for $k = 1$.

The extension to $k \in \mathbb{N}$ is straightforward. ∎

2.2.4 Gaps between regeneration times

Define (recall (2.35))

$$T_k^{(L)} = r^L \left(\tau_k^{(L)} - \tau_{k-1}^{(L)} \right), \qquad k \in \mathbb{N}. \quad (2.48)$$

Note that $T_k^{(L)}$, $k \in \mathbb{N}$, are i.i.d. In this section we prove two lemmas that control the moments of these increments.

Lemma 2.5. *For every $\alpha > 1$ there exists an $M(\alpha) < \infty$ such that*

$$\sup_{L \in 2\mathbb{N}} \bar{\mathbb{E}}_{\mu,(0,0)} \left([T_1^{(L)}]^\alpha \right) \leq M(\alpha). \quad (2.49)$$

Proof. Fix $\alpha > 1$. Since $T_1^{(L)}$ is independent of ξ, we have

$$\bar{\mathbb{E}}_{\mu,(0,0)} \left([T_1^{(L)}]^\alpha \right) = E_W \left([T_1^{(L)}]^\alpha \right) \leq \sup_{L \in 2\mathbb{N}} E_W \left([T_1^{(L)}]^\alpha \right), \quad (2.50)$$

where E_W is expectation w.r.t. W. Moreover, for all $a > 0$, there exists a constant $C = C(\alpha, a)$ such that

$$[aT_1^{(L)}]^\alpha \leq C\, e^{aT_1^{(L)}}, \quad (2.51)$$

and hence

$$\bar{\mathbb{E}}_{\mu,(0,0)} \left([T_1^{(L)}]^\alpha \right) \leq \frac{C}{a^\alpha} \sup_{L \in 2\mathbb{N}} E_W \left(e^{aT_1^{(L)}} \right). \quad (2.52)$$

Thus, to get the claim it suffices to show that, for a small enough,

$$\sup_{L \in 2\mathbb{N}} E_W \left(\mathrm{e}^{aT_1^{(L)}} \right) < \infty. \quad (2.53)$$

To prove (2.53), let

$$I = \inf \left\{ m \in \mathbb{N} \colon\ (\epsilon_{mL}, \ldots, \epsilon_{(m+1)L-1}) = \epsilon^{(L)} \right\}. \quad (2.54)$$

By (2.28), I is geometrically distributed with parameter r^L. Moreover, $\tau_1^{(L)} \le (I+1)L$. Therefore

$$\begin{aligned} E_W\left(\mathrm{e}^{aT_1^{(L)}}\right) &= E_W\left(\mathrm{e}^{ar^L\tau_1^{(L)}}\right) \le \mathrm{e}^{ar^L L}\, E_W\left(\mathrm{e}^{ar^L IL}\right) \\ &= \mathrm{e}^{ar^L L} \sum_{j\in\mathbb{N}} (\mathrm{e}^{ar^L L})^j\, (1-r^L)^{j-1}\, r^L = \frac{r^L \mathrm{e}^{2ar^L L}}{\mathrm{e}^{ar^L L}(1-r^L)}, \end{aligned} \tag{2.55}$$

with the sum convergent for $0 < a < (1/r^L L)\log[1/(1-r^L)]$ and tending to zero as $L \to \infty$ (because $r < 1$). Hence we can choose a small enough so that (2.53) holds. ∎

Lemma 2.6. $\liminf_{L\to\infty} \bar{\mathbb{E}}_{\mu,(0,0)}(T_1^{(L)}) > 0$.

Proof. Note that $\bar{\mathbb{E}}_{\mu,(0,0)}(T_1^{(L)}) < \infty$ by Lemma 2.5. Let $N = (N_n)_{n\in\mathbb{N}_0}$ be the Markov chain with state space $S = \{0,1,\dots,L\}$, starting from $N_0 = 0$, such that $N_n = s$ when

$$s = 0 \vee \max\left\{k \in \mathbb{N}\colon (\epsilon_{n-k},\dots,\epsilon_{n-1}) = (\epsilon_1^{(L)},\dots,\epsilon_k^{(L)})\right\} \tag{2.56}$$

(with $\max\emptyset = 0$). This Markov chain moves up one unit with probability r, drops to 0 with probability $p+r$ when it is even, and drops to 0 or 1 with probability p, respectively, r when it is odd. Since $\tau_1^{(L)} = \min\{n \in \mathbb{N}_0\colon N_n = L\}$, it follows that $\tau_1^{(L)}$ is bounded from below by a sum of independent random variables, each bounded from below by 1, whose number is geometrically distributed with parameter r^{L-1}. Hence

$$\bar{\mathbb{P}}_{\mu,(0,0)}\left(\tau_1^{(L)} \ge c\, r^{-L}\right) \ge (1-r^{L-1})^{\lfloor cr^{-L}\rfloor}. \tag{2.57}$$

Since

$$\begin{aligned} \bar{\mathbb{E}}_{\mu,(0,0)}(T_1^{(L)}) &= r^L\, \bar{\mathbb{E}}_{\mu,(0,0)}(\tau_1^{(L)}) \\ &\ge r^L\, \bar{\mathbb{E}}_{\mu,(0,0)}\left(\tau_1^{(L)}\, 1_{\{\tau_1^{(L)}\ge cr^{-L}\}}\right) \ge c\, \bar{\mathbb{P}}_{\mu,(0,0)}\left(\tau_1^{(L)} \ge cr^{-L}\right), \end{aligned} \tag{2.58}$$

it follows that

$$\liminf_{L\to\infty} \bar{\mathbb{E}}_{\mu,(0,0)}(\tau_1^{(L)}) \ge c\,\mathrm{e}^{-c/r}. \tag{2.59}$$

This proves the claim. ∎

2.2.5 A coupling property for random sequences

In this section we recall a technical lemma that will be needed in Section 2.2.6. The proof of this lemma is a standard coupling argument (see e.g. Berbee [7], Lemma 2.1).

Lemma 2.7. *Let $(U_i)_{i\in\mathbb{N}}$ be a sequence of random variables whose joint probability law P is such that, for some marginal probability law μ and $a \in [0,1]$,*

$$\Big\| P\big(U_i \in \cdot \mid \sigma\{U_j\colon\, 1 \le j < i\}\big) - \mu(\cdot) \Big\|_{\mathrm{tv}} \le a \quad a.s. \qquad \forall\, i \in \mathbb{N}. \tag{2.60}$$

Then there exists a sequence of triples of random variables $(\widetilde{U}_i, \Delta_i, \widehat{U}_i)_{i\in\mathbb{N}}$ satisfying

(a) $(\widetilde{U}_i, \Delta_i)_{i\in\mathbb{N}}$ *are i.i.d.,*
(b) $\widetilde{U}_i$ *has probability law* μ*,*
(c) $P(\Delta_i = 0) = 1 - a$*,* $P(\Delta_i = 1) = a$*,*
(d) Δ_i *is independent of* $(\widetilde{U}_j, \Delta_j)_{1\le j<i}$ *and* $\widehat{U}_i$*,*

such that for all $i \in \mathbb{N}$

$$U_i = (1-\Delta_i)\widetilde{U}_i + \Delta_i \widehat{U}_i \quad \text{ in distribution.} \tag{2.61}$$

2.2.6 LLN for Y

Similarly as in (2.48), define

$$Z_k^{(L)} = r^L \left(Z_{\tau_k^{(L)}} - Z_{\tau_{k-1}^{(L)}} \right), \qquad k \in \mathbb{N}. \tag{2.62}$$

In this section we prove the LLN for these increments and this will imply the LLN in (2.27).

Proof. By Lemma 2.4, we have

$$\Big\| \bar{\mathbb{P}}_{\mu,(0,0)}\big((T_k^{(L)}, Z_k^{(L)}) \in \cdot \mid \mathcal{H}_{k-1}\big) - \mu^{(L)}(\cdot) \Big\|_{\mathrm{tv}} \le \Phi(L) \quad a.s. \qquad \forall\, k \in \mathbb{N}, \tag{2.63}$$

where

$$\mu^{(L)}(A \times B) = \bar{\mathbb{P}}_{\mu,(0,0)}\big(T_1^{(L)} \in A, Z_1^{(L)} \in B\big) \qquad \forall\, A \subset r^L\mathbb{N},\, B \subset r^L\mathbb{H}. \tag{2.64}$$

Therefore, by Lemma 2.7, there exists an i.i.d. sequence of random variables

$$(\widetilde{T}_k^{(L)}, \widetilde{Z}_k^{(L)}, \Delta_k^{(L)})_{k\in\mathbb{N}} \tag{2.65}$$

on $r^L\mathbb{N} \times r^L\mathbb{H} \times \{0,1\}$, where $(\widetilde{T}_k^{(L)}, \widetilde{Z}_k^{(L)})$ is distributed according to $\mu^{(L)}$ and $\Delta_k^{(L)}$ is Bernoulli distributed with parameter $\Phi(L)$, and also a sequence of random variables

$$(\widehat{Z}_k^{(L)}, \widehat{Z}_k^{(L)})_{k\in\mathbb{N}}, \tag{2.66}$$

where $\Delta_k^{(L)}$ is independent of $(\widehat{Z}_k^{(L)}, \widehat{Z}_k^{(L)})$ and of

$$\widetilde{\mathcal{G}}_k = \sigma\big\{(\widetilde{T}_l^{(L)}, \widetilde{Z}_l^{(L)}, \Delta_l^{(L)}):\ 1 \le l < k\big\}, \tag{2.67}$$

such that

$$(T_k^{(L)}, Z_k^{(L)}) = (1 - \Delta_k^{(L)})\,(\widetilde{T}_k^{(L)}, \widetilde{Z}_k^{(L)}) + \Delta_k^{(L)}\,(\widehat{Z}_k^{(L)}, \widehat{Z}_k^{(L)}). \tag{2.68}$$

Let

$$z_L = \bar{\mathbb{E}}_{\mu,(0,0)}(Z_1^{(L)}), \tag{2.69}$$

which is finite by Lemma 2.5 because $|Z_1^{(L)}| \le T_1^{(L)}$.

Lemma 2.8. *There exists a sequence of numbers* $(\delta_L)_{L\in\mathbb{N}_0}$, *satisfying* $\lim_{L\to\infty}\delta_L = 0$, *such that*

$$\limsup_{n\to\infty}\left|\frac{1}{n}\sum_{k=1}^{n} Z_k^{(L)} - z_L\right| < \delta_L \quad \bar{\mathbb{P}}_{\mu,(0,0)} - a.s. \tag{2.70}$$

Proof. With the help of (2.68) we can write

$$\frac{1}{n}\sum_{k=1}^{n} Z_k^{(L)} = \frac{1}{n}\sum_{k=1}^{n}\widetilde{Z}_k^{(L)} - \frac{1}{n}\sum_{k=1}^{n}\Delta_k^{(L)}\widetilde{Z}_k^{(L)} + \frac{1}{n}\sum_{k=1}^{n}\Delta_k^{(L)}\widehat{Z}_k^{(L)}. \tag{2.71}$$

By independence, the first term in the r.h.s. of (2.71) converges $\bar{\mathbb{P}}_{\mu,(0,0)}$-a.s. to z_L as $L\to\infty$. Hölder's inequality applied to the second term gives, for $\alpha,\alpha' > 1$ with $\alpha^{-1} + \alpha'^{-1} = 1$,

$$\left|\frac{1}{n}\sum_{k=1}^{n}\Delta_k^{(L)}\widetilde{Z}_k^{(L)}\right| \le \left(\frac{1}{n}\sum_{k=1}^{n}\left|\Delta_k^{(L)}\right|^{\alpha'}\right)^{\frac{1}{\alpha'}}\left(\frac{1}{n}\sum_{k=1}^{n}\left|\widetilde{Z}_k^{(L)}\right|^{\alpha}\right)^{\frac{1}{\alpha}}. \tag{2.72}$$

Hence, by Lemma 2.5 and the inequality $|\widetilde{Z}_k^{(L)}| \le \widetilde{T}_k^{(L)}$ (compare (2.48) and (2.62)), we have

$$\limsup_{n\to\infty}\left|\frac{1}{n}\sum_{k=1}^{n}\Delta_k^{(L)}\widetilde{Z}_k^{(L)}\right| \le \Phi(L)^{\frac{1}{\alpha'}}\,M(\alpha)^{\frac{1}{\alpha}} \quad \bar{\mathbb{P}}_{\mu,(0,0)} - a.s. \tag{2.73}$$

It remains to analyze the third term in the r.h.s. of (2.71). Since $|\Delta_k^{(L)}\widehat{Z}_k^{(L)}| \le Z_k^{(L)}$, it follows from Lemma 2.5 that

$$\begin{aligned} M(\alpha) &\ge \bar{\mathbb{E}}_{\mu,(0,0)}\big(|Z_k^{(L)}|^{\alpha}\big) \\ &\ge \bar{\mathbb{E}}_{\mu,(0,0)}\Big(|\Delta_k^{(L)}\widehat{Z}_k^{(L)}|^{\alpha} \mid \widetilde{\mathcal{G}}_k\Big) = \Phi(L)\ \bar{\mathbb{E}}_{\mu,(0,0)}\Big(|\widehat{Z}_k^{(L)}|^{\alpha} \mid \widetilde{\mathcal{G}}_k\Big) \quad a.s. \end{aligned} \tag{2.74}$$

Next, put $\widehat{Z}_k^{*(L)} = \bar{\mathbb{E}}_{\mu,(0,0)}(\widehat{Z}_k^{(L)} \mid \widetilde{\mathcal{G}}_k)$ and note that

$$M_n = \frac{1}{n}\sum_{k=1}^{n} \Delta_k^{(L)} \left(\widehat{Z}_k^{(L)} - \widehat{Z}_k^{*(L)}\right) \tag{2.75}$$

is a mean-zero martingale w.r.t. the filtration $\widetilde{\mathcal{G}} = (\widetilde{\mathcal{G}}_k)_{k\in\mathbb{N}}$. By the Burkholder-Gundy maximal inequality (Williams [96], (14.18)), it follows that, for $\beta = \alpha \wedge 2$,

$$\begin{aligned} \bar{\mathbb{E}}_{\mu,(0,0)}\left(\Big|\sup_{n\in\mathbb{N}} M_n\Big|^{\beta}\right) &\le C(\beta)\,\bar{\mathbb{E}}_{\mu,(0,0)}\Big(\sum_{k\in\mathbb{N}} \frac{[\Delta_k^{(L)}(\widehat{Z}_k^{(L)} - \widehat{Z}_k^{*(L)})]^2}{k^2}\Big)^{\beta/2} \\ &\le C(\beta)\sum_{k\in\mathbb{N}} \bar{\mathbb{E}}_{\mu,(0,0)}\left(\frac{|\Delta_k^{(L)}(\widehat{Z}_k^{(L)} - \widehat{Z}_k^{*(L)})|^{\beta}}{k^{\beta}}\right) \le C'(\beta), \end{aligned} \tag{2.76}$$

for some constants $C(\beta), C'(\beta) < \infty$. Hence M_n a.s. converges to an integrable random variable as $n \to \infty$, and by Kronecker's lemma $\lim_{n\to\infty} M_n = 0$ a.s. Moreover, if $\Phi(L) > 0$, then by Jensen's inequality and (2.74) we have

$$|\widehat{Z}_k^{*(L)}| \le \left[\bar{\mathbb{E}}_{\mu,(0,0)}\left(\big|\widehat{Z}_k^{(L)}\big|^{\alpha} \mid \widetilde{\mathcal{G}}_k\right)\right]^{\frac{1}{\alpha}} \le \left(\frac{M(\alpha)}{\Phi(L)}\right)^{\frac{1}{\alpha}} \quad \bar{\mathbb{P}}_{\mu,(0,0)} - a.s. \tag{2.77}$$

Hence

$$\left|\frac{1}{n}\sum_{k=1}^{n} \Delta_k^{(L)} \widehat{Z}_k^{*(L)}\right| \le \left(\frac{M(\alpha)}{\Phi(L)}\right)^{\frac{1}{\alpha}} \frac{1}{n}\sum_{k=1}^{n} \Delta_k^{(L)}. \tag{2.78}$$

As $n \to \infty$, the r.h.s. converges $\bar{\mathbb{P}}_{\mu,(0,0)}$-a.s. to $M(\alpha)^{\frac{1}{\alpha}}\Phi(L)^{\frac{1}{\alpha'}}$. Therefore, recalling (2.78) and choosing $\delta_L = 2M(\alpha)^{\frac{1}{\alpha}}\Phi(L)^{\frac{1}{\alpha'}}$, we get the claim. ∎

Finally, since $\widetilde{Z}_k^{(L)} \ge r^L$ and

$$\frac{1}{n}\sum_{k=1}^{n} T_k^{(L)} = t_L = \bar{\mathbb{E}}_{\mu,(0,0)}(T_1^{(L)}) > 0 \quad \bar{\mathbb{P}}_{\mu,(0,0)} - a.s., \tag{2.79}$$

Lemma 2.8 yields

$$\limsup_{n\to\infty} \left|\frac{\frac{1}{n}\sum_{k=1}^{n} Z_k^{(L)}}{\frac{1}{n}\sum_{k=1}^{n} T_k^{(L)}} - \frac{z_L}{t_L}\right| < C_1\,\delta_L \quad \bar{\mathbb{P}}_{\mu,(0,0)} - a.s. \tag{2.80}$$

for some constant $C_1 < \infty$ and L large enough. By (2.48) and (2.62), the quotient of sums in the l.h.s. equals $Z_{\tau_n^{(L)}}/\tau_n^{(L)}$. It therefore follows from a standard interpolation argument that

$$\limsup_{n\to\infty} \left|\frac{Z_n}{n} - \frac{z_L}{t_L}\right| < C_2\,\delta_L \quad \bar{\mathbb{P}}_{\mu,(0,0)} - a.s. \tag{2.81}$$

for some constant $C_2 < \infty$ and L large enough. This implies the existence of the limit $\lim_{L\to\infty} z_L/t_L$, as well as the fact that $\lim_{n\to\infty} Z_n/n = u$ $\bar{\mathbb{P}}_{\mu,(0,0)}$-a.s., which in view of (2.32) is equivalent to the statement in (2.27) with $u = (v, 1)$. ∎

2.2.7 From discrete to continuous time

It remains to show that the LLN derived in Sections 2.2.1–2.2.6 for the discrete-time random walk defined in (2.21–2.22) can be extended to the continuous-time random walk defined in (2.6–2.7).

Let $\chi = (\chi_n)_{n\in\mathbb{N}_0}$ denote the jump times of the continuous-time random walk $X = (X_t)_{t\geq 0}$ (with $\chi_0 = 0$). Let Q denote the law of χ. The increments of χ are i.i.d. random variables, independent of ξ, whose distribution is exponential with mean $1/(\alpha+\beta)$. Define

$$\begin{aligned} \xi^* &= (\xi^*_n)_{n\in\mathbb{N}_0} \quad \text{with} \quad \xi^*_n = \xi_{\chi_n}, \\ X^* &= (X^*_n)_{n\in\mathbb{N}_0} \quad \text{with} \quad X^*_n = X_{\chi_n}. \end{aligned} \tag{2.82}$$

Then X^* is a discrete-time random walk in a discrete-time random environment of the type considered in Sections 2.2.1–2.2.6, with $p = \alpha/(\alpha+\beta)$ and $q = \beta/(\alpha+\beta)$. Lemma 2.9 below shows that the cone-mixing property of ξ carries over to ξ^* under the joint law $P^\mu \times Q$. Therefore we have (recall (2.9))

$$\lim_{n\to\infty} X^*_n/n = v^* \quad \text{exists } (\mathbb{P}_{\mu,0}\times Q) - a.s. \tag{2.83}$$

Since $\lim_{n\to\infty} \chi_n/n = 1/(\alpha+\beta)$ Q-a.s., it follows that

$$\lim_{n\to\infty} X_{\chi_n}/\chi_n = (\alpha+\beta)v^* \quad \text{exists } (\mathbb{P}_{\mu,0}\times Q) - a.s. \tag{2.84}$$

A standard interpolation argument now yields (2.13) with $v = (\alpha+\beta)v^*$.

Lemma 2.9. *If ξ is cone-mixing with angle $\theta > \arctan(\alpha+\beta)$, then ξ^* is cone-mixing with angle $\frac{1}{4}\pi$.*

Proof. Fix $\theta > \arctan(\alpha+\beta)$, and put $c = c(\theta) = \cot\theta < 1/(\alpha+\beta)$. Recall from (2.10) that C_t^θ is the cone with angle θ whose tip is at $(0,t)$. For $M \in \mathbb{N}$, let $C_{t,M}^\theta$ be the cone obtained from C_t^θ by extending the tip to a rectangle with base M, i.e.,

$$C_{t,M}^\theta = C_t^\theta \cup \{([-M,M]\cap\mathbb{Z}) \times [t,\infty)\}. \tag{2.85}$$

Because ξ is cone-mixing with angle θ, and

$$C_{t,M}^\theta \subset C_{t-cM}^\theta, \qquad M \in \mathbb{N}, \tag{2.86}$$

ξ is cone-mixing with angle θ and base M, i.e., (2.11) holds with C_t^θ replaced by $C_{t,M}^\theta$. This is true for every $M \in \mathbb{N}$.

Define, for $t \geq 0$ and $M \in \mathbb{N}$,

$$\begin{aligned} \mathcal{F}_t^\theta &= \sigma\big\{\xi_s(x)\colon\ (x,s) \in C_t^\theta\big\}, \\ \mathcal{F}_{t,M}^\theta &= \sigma\big\{\xi_s(x)\colon\ (x,s) \in C_{t,M}^\theta\big\}, \end{aligned} \tag{2.87}$$

and, for $n \in \mathbb{N}$,

$$\begin{aligned} &\mathcal{F}_n^* = \sigma\big\{\xi_m^*(x)\colon\ (x,m) \in C_n^{\frac{1}{4}\pi}\big\}, \\ &\mathcal{G}_n = \sigma\big\{\chi_m\colon\ m \geq n\big\}, \end{aligned} \tag{2.88}$$

where $C_n^{\frac{1}{4}\pi}$ is the discrete-time cone with tip $(0,n)$ and angle $\frac{1}{4}\pi$.

Fix $\delta > 0$. Then there exists an $M = M(\delta) \in \mathbb{N}$ such that $Q(D[M]) \geq 1 - \delta$ with $D[M] = \{\chi_n/n \geq c \;\forall\, n \geq M\}$. For $n \in \mathbb{N}$, define

$$D_n = \big\{\chi_n/n \geq c\big\} \cap \sigma^n D[M], \tag{2.89}$$

where σ is the left-shift acting on χ. Since $c < 1/(\alpha+\beta)$, we have $P(\chi_n/n \geq c) \geq 1 - \delta$ for $n \geq N = N(\delta)$, and hence $P(D_n) \geq (1-\delta)^2 \geq 1 - 2\delta$ for $n \geq N = N(\delta)$. Next, observe that

$$B \in \mathcal{F}_n^* \Longrightarrow B \cap D_n \in \mathcal{F}_{cn,M}^\theta \otimes \mathcal{G}_n \tag{2.90}$$

(the r.h.s. is the product sigma-algebra). Indeed, on the event D_n we have $\chi_m \geq cm$ for $m \geq n + M$, which implies that, for $m \geq M$,

$$(x,m) \in C_n^{\frac{1}{4}\pi} \Longrightarrow |x| + m \geq n \Longrightarrow c|x| + \chi_n \geq cn \Longrightarrow (x,\chi_m) \in C_{cn,M}^\theta. \tag{2.91}$$

Now put $\bar{P}^\mu = P^\mu \otimes Q$ and, for $A \in \mathcal{F}_0$ with $P^\mu(A) > 0$ and $B \in \mathcal{F}_n^*$ estimate

$$|\bar{P}^\mu(B \mid A) - \bar{P}^\mu(B)| \leq I + II + III \tag{2.92}$$

with

$$\begin{aligned} I &= |\bar{P}^\mu(B \mid A) - \bar{P}^\mu(B \cap D_n \mid A)|, \\ II &= |\bar{P}^\mu(B \cap D_n \mid A) - \bar{P}^\mu(B \cap D_n)|, \\ III &= |\bar{P}^\mu(B \cap D_n) - \bar{P}^\mu(B)|. \end{aligned} \tag{2.93}$$

Since D_n is independent of A, B and $P(D_n) \geq 1-2\delta$, it follows that $I \leq 2\delta$ and $III \leq 2\delta$ uniformly in A and B. To bound II, we use (2.90) to estimate

$$II \leq \sup_{\substack{A\in\mathcal{F}_0,\, B'\in\mathcal{F}^{\theta}_{cn,M}\otimes\mathcal{G}_n \\ P^{\mu}(A)>0}} |\bar{P}^{\mu}(B' \mid A) - \bar{P}^{\mu}(B')|. \tag{2.94}$$

But the r.h.s. is bounded from above by

$$\sup_{\substack{A\in\mathcal{F}_0,\, B''\in\mathcal{F}^{\theta}_{cn,M} \\ P^{\mu}(A)>0}} |P^{\mu}(B'' \mid A) - P^{\mu}(B'')| \tag{2.95}$$

because, for every $B'' \in \mathcal{F}^{\theta}_{cn,M}$ and $C \in \mathcal{G}_n$,

$$|\bar{P}^{\mu}(B''\times C \mid A)-\bar{P}^{\mu}(B''\times C)| = |[P^{\mu}(B'' \mid A)-P^{\mu}(B'')]\, Q(C)| \leq |P^{\mu}(B'' \mid A)-P^{\mu}(B'')|, \tag{2.96}$$

where we use that C is independent of A, B''.

Finally, because ξ is cone-mixing with angle θ and base M, (2.95) tends to zero as $n \to \infty$, and so by combining (2.92–2.95) we get

$$\limsup_{n\to\infty} \sup_{\substack{A\in\mathcal{F}_0,\, B\in\mathcal{F}^{*}_n \\ P^{\mu}(A)>0}} |\bar{P}^{\mu}(B \mid A) - \bar{P}^{\mu}(B)| \leq 4\delta. \tag{2.97}$$

Now let $\delta \downarrow 0$ to obtain that ξ^* is cone mixing with angle $\frac{1}{4}\pi$. ∎

2.2.8 Remarks on the cone-mixing assumption

By using the cone-mixing assumption and the auxiliary process Z introduced in Section 2.2.2, we could have followed a shorter approach to derive the strong LLN in Theorem 2.2, avoiding the technicalities of Sections 2.2.5 and 2.2.6. Indeed, it is possible to deduce that the process of the environment as seen from the walk admits a mixing equilibrium measure μ_e. Consequently, a weak law of large numbers, L^2 convergence, and an almost sure convergence with respect to μ_e can be inferred. If we could subsequently show that the equilibrium measure μ is absolutely continuous with respect to μ_e (which is not trivial in the present generality), then Theorem 2.2 would follow.

As pointed out in Section 2.1.4, cone-mixing is one of the weakest assumptions under which we may expect to get the strong LLN, since no rate of mixing is imposed in (2.11). If we strengthen (2.11) to an exponential decay of the function in (2.37), then it seems possible to adapt the proof in [36] to derive an annealed CLT in the present context.

2.3 Series expansion for $M < \epsilon$

Throughout this section we assume that the dynamic random environment ξ falls in the regime for which $M < \epsilon$ (recall (2.14)). In Section 2.3.1 we define the *environment process*, i.e., the environment as seen relative to the position of the random walk. In Section 2.3.2 we prove that this environment process has a *unique ergodic equilibrium* μ_e, and we derive a series expansion for μ_e in powers of $\alpha - \beta$ that converges when $\alpha - \beta < \frac{1}{2}(\epsilon - M)$. In Section 2.3.3 we use the latter to derive a series expansion for the global speed v of the random walk.

2.3.1 Definition of the environment process

Let $X = (X_t)_{t\geq 0}$ be the random walk defined in (2.6–2.7). For $x \in \mathbb{Z}$, let τ_x denote the shift of space over x.

Definition 2.10. The environment process is the Markov process $\zeta = (\zeta_t)_{t\geq 0}$ with state space Ω given by

$$\zeta_t = \tau_{X_t}\xi_t, \qquad t \geq 0, \tag{2.98}$$

where

$$(\tau_{X_t}\xi_t)(x) = \xi_t(x + X_t), \qquad x \in \mathbb{Z},\, t \geq 0. \tag{2.99}$$

Equivalently, if ξ has generator L_{IPS}, then ζ has generator L given by

$$(Lf)(\eta) = c^+(\eta)\big[f(\tau_1\eta) - f(\eta)\big] + c^-(\eta)\big[f(\tau_{-1}\eta) - f(\eta)\big] + (L_{\mathrm{IPS}}f)(\eta), \quad \eta \in \Omega, \tag{2.100}$$

where f is an arbitrary cylinder function on Ω and

$$\begin{aligned} c^+(\eta) &= \alpha\,\eta(0) + \beta\,[1 - \eta(0)], \\ c^-(\eta) &= \beta\,\eta(0) + \alpha\,[1 - \eta(0)]. \end{aligned} \tag{2.101}$$

Let $S = (S(t))_{t\geq 0}$ be the semigroup associated with the generator L. Suppose that we manage to prove that ζ is ergodic, i.e., there exists a unique probability measure μ_e on Ω such that, for any cylinder function f on Ω,

$$\lim_{t\to\infty}(S(t)f)(\eta) = \langle f\rangle_{\mu_e} \qquad \forall\,\eta \in \Omega, \tag{2.102}$$

where $\langle\cdot\rangle_{\mu_e}$ denotes expectation w.r.t. μ_e. Then, picking $f = \phi_0$ with $\phi_0(\eta) = \eta(0)$, $\eta \in \Omega$, we have

$$\lim_{t\to\infty}(S(t)\phi_0)(\eta) = \langle\phi_0\rangle_{\mu_e} = \widetilde{\rho} \qquad \forall\,\eta \in \Omega \tag{2.103}$$

for some $\widetilde{\rho} \in [0,1]$, which represents the limiting probability that X is on an occupied site given that $\xi_0 = \zeta_0 = \eta$ (note that $(S(t)\phi_0)(\eta) = E^\eta(\zeta_t(0)) = E^\eta(\xi_t(X_t))$).

Next, let N_t^+ and N_t^- be the number of shifts to the right, respectively, left up to time t in the environment process. Then $X_t = N_t^+ - N_t^-$. Since $M_t^j = N_t^j - \int_0^t c^j(\eta_s)\,\mathrm{d}s$, $j \in \{+,-\}$, are martingales with stationary and ergodic increments, we have

$$X_t = M_t + (\alpha - \beta)\int_0^t \big(2\eta_s(0) - 1\big)\,\mathrm{d}s \tag{2.104}$$

with $M_t = M_t^+ - M_t^-$ a martingale with stationary and ergodic increments. It follows from (2.103–2.104) that

$$\lim_{t\to\infty} X_t/t = (2\widetilde{\rho} - 1)(\alpha - \beta) \qquad \mu - a.s. \tag{2.105}$$

In Section 2.3.2 we prove the existence of μ_e, and show that it can be expanded in powers of $\alpha - \beta$ when $\alpha - \beta < \frac{1}{2}(\epsilon - M)$. In Section 2.3.3 we use this expansion to obtain an expansion of $\widetilde{\rho}$.

2.3.2 Unique ergodic equilibrium measure for the environment process

In Section 2.3.2.1 we prove four lemmas controlling the evolution of ζ. In Section 2.3.2.2 we use these lemmas to show that ζ has a unique ergodic equilibrium measure μ_e that can be expanded in powers of $\alpha - \beta$, provided $\alpha - \beta < \frac{1}{2}(\epsilon - M)$.

We need some notation. Let $\|\cdot\|_\infty$ be the sup-norm on $C(\Omega)$. Let $|||\cdot|||$ be the triple norm on Ω defined as follows. For $x \in \mathbb{Z}$ and a cylinder function f on Ω, let

$$\Delta_f(x) = \sup_{\eta\in\Omega} |f(\eta^x) - f(\eta)| \tag{2.106}$$

be the maximum variation of f at x, where η^x is the configuration obtained from η by flipping the state at site x, and put

$$|||f||| = \sum_{x\in\mathbb{Z}} \Delta_f(x). \tag{2.107}$$

It is easy to check that, for arbitrary cylinder functions f and g on Ω,

$$|||fg||| \le \|f\|_\infty\, |||g||| + \|g\|_\infty\, |||f|||. \tag{2.108}$$

2.3.2.1 Decomposition of the generator of the environment process

Lemma 2.11. *Assume* (2.3) *and suppose that* $M < \epsilon$. *Write the generator of the environment process* ζ *defined in* (2.100) *as*

$$L = L_0 + L_* = (L_{\mathrm{SRW}} + L_{\mathrm{IPS}}) + L_*, \tag{2.109}$$

where

$$\begin{aligned} (L_{\mathrm{SRW}} f)(\eta) &= \tfrac{1}{2}(\alpha+\beta)\left[f(\tau_1\eta) + f(\tau_{-1}\eta) - 2f(\eta)\right], \\ (L_* f)(\eta) &= \tfrac{1}{2}(\alpha-\beta)\left[f(\tau_1\eta) - f(\tau_{-1}\eta)\right]\big(2\eta(0)-1\big). \end{aligned} \tag{2.110}$$

Then L_0 *is the generator of a Markov process that still has* μ *as an equilibrium, and that satisfies*

$$|||S_0(t)f||| \leq \mathrm{e}^{-ct}\, |||f||| \tag{2.111}$$

and

$$\|S_0(t)f - \langle f\rangle_\mu\|_\infty \leq C\,\mathrm{e}^{-ct}\, |||f|||, \tag{2.112}$$

where $S_0 = (S_0(t))_{t\geq 0}$ *is the semigroup associated with the generator* L_0, $c = \epsilon - M$, *and* $C < \infty$ *is a positive constant.*

Proof. Note that L_{SRW} and L_{IPS} commute. Therefore, for an arbitrary cylinder function f on Ω, we have

$$|||S_0(t)f||| = |||\mathrm{e}^{tL_{\mathrm{SRW}}}\big(\mathrm{e}^{tL_{\mathrm{IPS}}}f\big)||| \leq |||\mathrm{e}^{tL_{\mathrm{IPS}}}f||| \leq \mathrm{e}^{-ct}\, |||f|||, \tag{2.113}$$

where the first inequality uses that $\mathrm{e}^{tL_{\mathrm{SRW}}}$ is a contraction semigroup, and the second inequality follows from the fact that ξ falls in the regime $M < \epsilon$ (see Liggett [63], Theorem I.3.9). The inequality in (2.112) follows by a similar argument. Indeed,

$$\|S_0(t)f - \langle f\rangle_\mu\|_\infty = \|\mathrm{e}^{tL_{\mathrm{SRW}}}\big(\mathrm{e}^{tL_{\mathrm{IPS}}}f\big) - \langle f\rangle_\mu\|_\infty \leq \|\mathrm{e}^{tL_{\mathrm{IPS}}}f - \langle f\rangle_\mu\|_\infty \leq C\,\mathrm{e}^{-ct}\, |||f|||, \tag{2.114}$$

where the last inequality again uses that ξ falls in the regime $M < \epsilon$ (see Liggett [63], Theorem I.4.1). The fact that μ is an equilibrium measure is trivial, since L_{SRW} only acts on η by shifting it. ∎

Note that L_{SRW} is the generator of simple random walk on $\mathbb{Z}$ jumping at rate $\alpha+\beta$. We view L_0 as the generator of an unperturbed Markov process and L_* as a perturbation of L_0. The following lemma gives us control of the latter.

Lemma 2.12. *For any cylinder function f on Ω,*

$$\|L_* f\|_\infty \leq (\alpha - \beta)\|f\|_\infty \tag{2.115}$$

and

$$|||L_* f||| \leq 2(\alpha - \beta)\, |||f||| \qquad \text{if } \langle f \rangle_\mu = 0. \tag{2.116}$$

Proof. To prove (2.115), estimate

$$\begin{aligned} \|L_* f\|_\infty &= \tfrac{1}{2}(\alpha - \beta)\, \big\| \big[f(\tau_1 \cdot) - f(\tau_{-1} \cdot)\big]\, \big(2\phi_0(\cdot) - 1\big) \big\|_\infty \\ &\leq \tfrac{1}{2}(\alpha - \beta)\, \|f(\tau_1 \cdot) + f(\tau_{-1} \cdot)\|_\infty \leq (\alpha - \beta)\, \|f\|_\infty. \end{aligned} \tag{2.117}$$

To prove (2.116), recall (2.110) and estimate

$$\begin{aligned} |||L_* f||| &= \tfrac{1}{2}(\alpha - \beta)\, \big|\big|\big| \big[f(\tau_1 \cdot) - f(\tau_{-1} \cdot)\big]\, \big(2\phi_0(\cdot) - 1\big) \big|\big|\big| \\ &\leq \tfrac{1}{2}(\alpha - \beta) \Big(|||f(\tau_1 \cdot)(2\phi_0(\cdot) - 1)||| + |||f(\tau_{-1} \cdot)(2\phi_0(\cdot) - 1)||| \Big) \\ &\leq (\alpha - \beta) \Big(\|f\|_\infty\, |||\, (2\phi_0 - 1)||| + |||f|||\, \|(2\phi_0 - 1)\|_\infty \Big) \\ &= (\alpha - \beta) \Big(\|f\|_\infty + |||f||| \Big) \leq 2(\alpha - \beta)|||f|||, \end{aligned} \tag{2.118}$$

where the second inequality uses (2.108) and the third inequality follows from the fact that $\|f\|_\infty \leq |||f|||$ for any f such that $\langle f \rangle_\mu = 0$. ∎

We are now ready to expand the semigroup S of ζ. Henceforth abbreviate

$$c = \epsilon - M. \tag{2.119}$$

Lemma 2.13. *Let $S_0 = (S_0(t))_{t \geq 0}$ be the semigroup associated with the generator L_0 defined in* (2.110)*. Then, for any $t \geq 0$ and any cylinder function f on Ω,*

$$S(t)f = \sum_{n \in \mathbb{N}} g_n(t, f), \tag{2.120}$$

where

$$g_1(t, f) = S_0(t)f \quad \text{and} \quad g_{n+1}(t, f) = \int_0^t S_0(t - s)\, L_*\, g_n(s, f)\, \mathrm{d}s, \qquad n \in \mathbb{N}. \tag{2.121}$$

Moreover, for all $n \in \mathbb{N}$,

$$\|g_n(t, f)\|_\infty \leq |||f|||\, \left(\frac{2(\alpha - \beta)}{c}\right)^{n-1} \tag{2.122}$$

and

$$|||g_n(t,f)||| \le \mathrm{e}^{-ct}\,\frac{[2(\alpha-\beta)t]^{n-1}}{(n-1)!}\,|||f|||, \tag{2.123}$$

where $0! = 1$. *In particular, for all* $t > 0$ *and* $\alpha - \beta < \frac{1}{2}c$ *the series in* (2.120) *converges uniformly in* η.

Proof. Since $L = L_0 + L_*$, Dyson's formula gives

$$\mathrm{e}^{tL}\,f = \mathrm{e}^{tL_0}f + \int_0^t \mathrm{e}^{(t-s)L_0}\,L_*\,\mathrm{e}^{sL}\,f\,\mathrm{d}s, \tag{2.124}$$

which, in terms of semigroups, reads

$$S(t)f = S_0(t)f + \int_0^t S_0(t-s)L_*\,S(s)f\,\mathrm{d}s. \tag{2.125}$$

The expansion in (2.120–2.121) follows from (2.125) by induction on n.

We next prove (2.123) by induction on n. For $n = 1$ the claim is immediate. Indeed, by Lemma 2.11 we have the exponential bound

$$|||g_1(t,f)||| = |||S_0(t)f||| \le \mathrm{e}^{-ct}\,|||f|||. \tag{2.126}$$

Suppose that the statement in (2.123) is true up to n. Then

$$\begin{aligned}
|||g_{n+1}(t,f)||| &= |||\int_0^t S_0(t-s)\,L_*\,g_n(s,f)\,\mathrm{d}s\,||| \\
&\le \int_0^t |||S_0(t-s)\,L_*\,g_n(s,f)|||\,\mathrm{d}s \\
&\le \int_0^t \mathrm{e}^{-c(t-s)}\,|||L_*\,g_n(s,f)|||\,\mathrm{d}s \\
&= \int_0^t \mathrm{e}^{-c(t-s)}\,|||L_*\big(g_n(s,f) - \langle g_n(s,f)\rangle_\mu\big)|||\,\mathrm{d}s \\
&\le 2(\alpha-\beta)\int_0^t \mathrm{e}^{-c(t-s)}\,|||g_n(s,f)|||\,\mathrm{d}s, \\
&\le |||f|||\,\mathrm{e}^{-ct}\,[2(\alpha-\beta)]^n \int_0^t \frac{s^{n-1}}{(n-1)!}\,\mathrm{d}s \\
&= |||f|||\,\mathrm{e}^{-ct}\,\frac{[2(\alpha-\beta)t]^n}{n!},
\end{aligned} \tag{2.127}$$

where the third inequality uses (2.116), and the fourth inequality relies on the induction hypothesis.

Using (2.123), we can now prove (2.122). Estimate

$$\begin{aligned}
\|g_{n+1}(t,f)\|_\infty &= \left\| \int_0^t S_0(t-s)\, L_*\, g_n(s,f)\mathrm{d}s \right\|_\infty \\
&\le \int_0^t \|L_*\, g_n(s,f)\|_\infty \,\mathrm{d}s \\
&= \int_0^t \left\| L_* \left(g_n(s,f) - \langle g_n(s,f)\rangle_\mu \right) \right\|_\infty \mathrm{d}s \\
&\le (\alpha-\beta) \int_0^t \left\| g_n(s,f) - \langle g_n(s,f)\rangle_\mu \right\|_\infty \mathrm{d}s \qquad (2.128) \\
&\le (\alpha-\beta) \int_0^t |||g_n(s,f)|||\,\mathrm{d}s \\
&\le (\alpha-\beta)|||f||| \int_0^t \mathrm{e}^{-cs}\, \frac{[2(\alpha-\beta)s]^{n-1}}{(n-1)!}\,\mathrm{d}s \\
&\le |||f||| \left(\frac{2(\alpha-\beta)}{c} \right)^n,
\end{aligned}$$

where the first inequality uses that $S_0(t)$ is a contraction semigroup, while the second and fourth inequality rely on (2.115) and (2.123). ∎

We next show that the functions in (2.120) are uniformly close to their average value.

Lemma 2.14. *Let*

$$h_n(t,f) = g_n(t,f) - \langle g_n(t,f)\rangle_\mu, \quad t \ge 0,\, n \in \mathbb{N}. \tag{2.129}$$

Then

$$\|h_n(t,f)\|_\infty \le C\,\mathrm{e}^{-ct}\, \frac{[2(\alpha-\beta)t]^{n-1}}{(n-1)!}\, |||f|||, \tag{2.130}$$

for some $C < \infty$.

Proof. Note that $|||h_n(t,f)||| = |||g_n(t,f)|||$ for $t \ge 0$ and $n \in \mathbb{N}$, and estimate

$$\begin{aligned}
\|h_{n+1}(t,f)\|_\infty &= \left\| \int_0^t \Big(S_0(t-s)\, L_*\, g_n(s,f) - \langle L_*\, g_n(s,f)\rangle_\mu \Big)\,\mathrm{d}s \right\|_\infty \\
&\le C \int_0^t \mathrm{e}^{-c(t-s)}\, |||L_*\, g_n(s,f)|||\,\mathrm{d}s \\
&= C \int_0^t \mathrm{e}^{-c(t-s)}\, |||L_*\, h_n(s,f)|||\,\mathrm{d}s \qquad (2.131) \\
&\le C\, 2(\alpha-\beta) \int_0^t \mathrm{e}^{-c(t-s)}\, |||h_n(s,f)|||\,\mathrm{d}s \\
&\le C\, |||f|||\, \mathrm{e}^{-ct}\, [2(\alpha-\beta)]^n \int_0^t \frac{s^{n-1}}{(n-1)!}\,\mathrm{d}s \\
&= C\, |||f|||\, \mathrm{e}^{-ct}\, \frac{[2(\alpha-\beta)t]^n}{n!},
\end{aligned}$$

where the first inequality uses (2.112), while the second and third inequality rely on (2.116) and (2.123). ∎

2.3.2.2 Expansion of the equilibrium measure of the environment process

We are finally ready to state the main result of this section.

Theorem 2.15. *For $\alpha - \beta < \frac{1}{2}c$, the environment process ζ has a unique invariant measure μ_e. In particular, for any cylinder function f on Ω,*

$$\langle f \rangle_{\mu_e} = \lim_{t\to\infty} \langle S(t)f \rangle_\mu = \sum_{n\in\mathbb{N}} \lim_{t\to\infty} \langle g_n(t,f) \rangle_\mu. \tag{2.132}$$

Proof. By Lemma 2.14, we have

$$\begin{aligned}\Big\| S(t)f - \langle S(t)f \rangle_\mu \Big\|_\infty &= \Big\| \sum_{n\in\mathbb{N}} g_n(t,f) - \langle \sum_{n\in\mathbb{N}} g_n(t,f) \rangle_\mu \Big\|_\infty = \Big\| \sum_{n\in\mathbb{N}} h_n(t,f) \Big\|_\infty \\ &\leq \sum_{n\in\mathbb{N}} \| h_n(t,f) \|_\infty \leq C\, \mathrm{e}^{-ct} \, |||f||| \sum_{n\in\mathbb{N}} \frac{[2(\alpha-\beta)t]^n}{n!} \\ &= C |||f||| \, \mathrm{e}^{-t[c-2(\alpha-\beta)]}.\end{aligned} \tag{2.133}$$

Since $\alpha - \beta < \frac{1}{2}c$, we see that the r.h.s. of (2.133) tends to zero as $t \to \infty$. Consequently, the l.h.s. tends to zero uniformly in η, and this is sufficient to conclude that the set $\mathcal{I}$ of equilibrium measures of the environment process is a singleton, i.e., $\mathcal{I} = \{\mu_e\}$. Indeed, suppose that there are two equilibrium measures $\nu, \nu' \in \mathcal{I}$. Then

$$\begin{aligned}|\langle f \rangle_\nu - \langle f \rangle_{\nu'}| &= |\langle S(t)f \rangle_\nu - \langle S(t)f \rangle_{\nu'}| \\ &\leq |\langle S(t)f \rangle_\nu - \langle S(t)f \rangle_\mu| + |\langle S(t)f \rangle_{\nu'} - \langle S(t)f \rangle_\mu| \\ &= |\langle \big[S(t)f - \langle S(t)f \rangle_\mu \big] \rangle_\nu| + |\langle \big[S(t)f - \langle S(t)f \rangle_\mu \big] \rangle_{\nu'}| \\ &\leq 2 \, \| S(t)f - \langle S(t)f \rangle_\mu \|_\infty .\end{aligned} \tag{2.134}$$

Since the l.h.s. of (2.134) does not depend on t, and the r.h.s. tends to zero as $t \to \infty$, we have $\nu = \nu' = \mu_e$. Next, μ_e is uniquely ergodic, meaning that the environment process converges to μ_e as $t \to \infty$ no matter what its starting distribution is. Indeed, for any μ',

$$|\langle S(t)f \rangle_{\mu'} - \langle S(t)f \rangle_\mu| = |\langle \big[S(t)f - \langle S(t)f \rangle_\mu \big] \rangle_{\mu'}| \leq \| S(t)f - \langle S(t)f \rangle_\mu \|_\infty , \tag{2.135}$$

and therefore

$$\begin{aligned}\langle f\rangle_{\mu_e} &= \lim_{t\to\infty} S(t)f = \lim_{t\to\infty}\langle S(t)f\rangle_\mu = \lim_{t\to\infty}\left\langle \sum_{n\in\mathbb{N}} g_n(t,f)\right\rangle_\mu \\ &= \lim_{t\to\infty}\sum_{n\in\mathbb{N}}\langle g_n(t,f)\rangle_\mu = \sum_{n\in\mathbb{N}}\lim_{t\to\infty}\langle g_n(t,f)\rangle_\mu,\end{aligned} \tag{2.136}$$

where the last equality is justified by the bound in (2.122) in combination with the dominated convergence theorem. ∎

We close this section by giving a more transparent description of μ_e, more suitable for explicit computation.

Theorem 2.16. *For $\alpha-\beta<\frac{1}{2}c$,*

$$\langle f\rangle_{\mu_e} = \sum_{n\in\mathbb{N}}\langle\Psi_n\rangle_\mu \tag{2.137}$$

with

$$\Psi_1 = f \quad \textit{and} \quad \Psi_{n+1} = L_* L_0^{-1}(\Psi_n - \langle\Psi_n\rangle_\mu), \qquad n\in\mathbb{N}, \tag{2.138}$$

where $L_0^{-1}=\int_0^\infty S_0(t)\,\mathrm{d}t$ (whose domain is the set of all $f\in C(\Omega)$ with $\langle f\rangle_\mu=0$).

Proof. By (2.136), the claim is equivalent to showing that for all $n\geq 1$

$$\lim_{t\to\infty}\langle g_n(t,f)\rangle_\mu = \langle\Psi_n\rangle_\mu. \tag{2.139}$$

First consider the case $n=2$. Then

$$\begin{aligned}\lim_{t\to\infty}\langle g_2(t,f)\rangle_\mu &= \lim_{t\to\infty}\left\langle\int_0^t \mathrm{d}s\, S_0(t-s)\, L_*\, g_1(s,f)\right\rangle_\mu \\ &= \lim_{t\to\infty}\left\langle\int_0^t \mathrm{d}s\ L_*\, g_1(s,f)\right\rangle_\mu \\ &= \lim_{t\to\infty}\left\langle\int_0^t \mathrm{d}s\ L_* S_0(s) f\right\rangle_\mu \\ &= \lim_{t\to\infty}\left\langle\int_0^t \mathrm{d}s\ L_*\big[S_0(s)(f-\langle f\rangle_\mu)\big]\right\rangle_\mu \\ &= \left\langle \lim_{t\to\infty} L_*\int_0^t \mathrm{d}s\ S_0(s)(f-\langle f\rangle_\mu)\right\rangle_\mu = \langle L_* L_0^{-1}(f-\langle f\rangle_\mu)\rangle_\mu,\end{aligned} \tag{2.140}$$

where the second equality uses that μ is invariant w.r.t. S_0, while the fifth equality uses the linearity and continuity of L_* in combination with the bound in (2.122).

For general n, the argument runs as follows. First write

$$
\begin{aligned}
&\langle g_n(t,f)\rangle_\mu \\
&= \left\langle \int_0^t \mathrm{d}s\, S_0(t-t_1)\, L_*\, g_{n-1}(t_1,f) \right\rangle_\mu \\
&= \left\langle \int_0^t \mathrm{d}t_1\; L_*\, g_{n-1}(t_1,f) \right\rangle_\mu \\
&= \left\langle \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \left[L_* S_0(t_1-t_2)\cdots L_* S_0(t_{n-1}-t_n) L_* S_0(t_n)\right] f \right\rangle_\mu \\
&= \left\langle \int_0^t \mathrm{d}t_n \int_0^{t-t_n} \mathrm{d}t_{n-1} \cdots \int_0^{t-t_2} \mathrm{d}t_1 \left[L_* S_0(t_1) L_* S_0(t_2)\cdots L_* S_0(t_{n-1}) L_* S_0(t_n)\right] f \right\rangle_\mu .
\end{aligned}
\tag{2.141}
$$

Next let $t \to \infty$ to obtain

$$
\begin{aligned}
&\lim_{t\to\infty} \langle g_n(t,f)\rangle_\mu \\
&= \left\langle \int_0^\infty \mathrm{d}t_n \int_0^\infty \mathrm{d}t_{n-1} \cdots \int_0^\infty \mathrm{d}t_1 \left[L_* S_0(t_1) L_* S_0(t_2)\cdots L_* S_0(t_{n-1}) L_* S_0(t_n)\right] f \right\rangle_\mu \\
&= \left\langle L_* \int_0^\infty \mathrm{d}t_1\, S_0(t_1)\, L_* \int_0^\infty \mathrm{d}t_2\, S_0(t_2) \cdots L_* \int_0^\infty \mathrm{d}t_n\, S_0(t_n)\, (f - \langle f\rangle_\mu) \right\rangle_\mu \\
&= \left\langle L_* \int_0^\infty \mathrm{d}t_1\, S_0(t_1)\, L_* \int_0^\infty \mathrm{d}t_2\, S_0(t_2) \cdots L_* L_0^{-1} (f - \langle f\rangle_\mu) \right\rangle_\mu \\
&= \left\langle L_* \int_0^\infty \mathrm{d}t_1\, S_0(t_1)\, L_* \int_0^\infty \mathrm{d}t_2\, S_0(t_2) \cdots L_* \int_0^\infty \mathrm{d}t_{n-1}\, S_0(t_{n-1}) \Psi_2 \right\rangle_\mu ,
\end{aligned}
\tag{2.142}
$$

where we insert $L_* L_0^{-1}(f - \langle f\rangle_\mu) = \Psi_2$. Iteration shows that the latter expression is equal to

$$
\begin{aligned}
\left\langle L_* \int_0^\infty \mathrm{d}t_1\, S_0(t_1)\Psi_{n-1} \right\rangle_\mu &= \left\langle L_* \int_0^\infty \mathrm{d}t_1\, S_0(t_1)(\Psi_{n-1} - \langle \Psi_{n-1}\rangle_\mu) \right\rangle_\mu \\
&= \left\langle L_* L_0^{-1} (\Psi_{n-1} - \langle \Psi_{n-1}\rangle_\mu) \right\rangle_\mu = \langle \Psi_n\rangle_\mu .
\end{aligned}
\tag{2.143}
$$

∎

2.3.3 Expansion of the global speed

As we argued in (2.105), the global speed of X is given by

$$
v = (2\widetilde{\rho} - 1)(\alpha - \beta) \tag{2.144}
$$

with $\widetilde{\rho} = \langle \phi_0 \rangle_{\mu_e}$. By using Theorem 2.16, we can now expand $\widetilde{\rho}$.

First, if $\langle \phi_0 \rangle_\mu = \rho$ is the particle density, then

$$\widetilde{\rho} = \langle \phi_0 \rangle_{\mu_e} = \rho + \sum_{n=2}^{\infty} \langle \Psi_n \rangle_\mu, \tag{2.145}$$

where Ψ_n is constructed recursively via (2.138) with $f = \phi_0$. We have

$$\langle \Psi_n \rangle_\mu = d_n \, (\alpha - \beta)^{n-1}, \quad n \in \mathbb{N}, \tag{2.146}$$

where $d_n = d_n(\alpha + \beta; P^\mu)$, and the factor $(\alpha - \beta)^{n-1}$ comes from the fact that the operator L_* is applied $n-1$ times to compute Ψ_n, as is seen from (2.138). Recall that, in (2.110), L_{SRW} caries the prefactor $\alpha + \beta$, while L_* carries the prefactor $\alpha - \beta$. Combining (2.144–2.145), we have

$$v = \sum_{n \in \mathbb{N}} c_n \, (\alpha - \beta)^n, \tag{2.147}$$

with $c_1 = 2\rho - 1$ and $c_n = 2d_n$, $n \in \mathbb{N} \backslash \{1\}$.

For $n = 2, 3$ we have

$$\begin{aligned} c_2 &= 2 \big\langle \phi_0 L_0^{-1} \big(\phi_1 - \phi_{-1} \big) \big\rangle_\mu \\ c_3 &= \tfrac{1}{2} \left\langle \psi_0 L_0^{-1} \big[\psi_{-1} L_0^{-1} \bar{\phi}_{-2} - \psi_1 L_0^{-1} \bar{\phi}_0 - \psi_{-1} L_0^{-1} \bar{\phi}_0 + \psi_1 L_0^{-1} \bar{\phi}_2 \big] \right\rangle_\mu , \end{aligned} \tag{2.148}$$

where $\phi_i(\eta) = \eta(i)$, $\eta \in \Omega$, $\bar{\phi}_i = \phi_i - \langle \phi_i \rangle_\mu$ and $\psi_i = 2\phi_i - 1$. It is possible to compute c_2 and c_3 for appropriate choices of ξ.

If the law of ξ is invariant under reflection w.r.t. the origin, then ξ has the same distribution as ξ' defined by $\xi'(x) = \xi(-x)$, $x \in \mathbb{Z}$. In that case $c_2 = 0$, and consequently $v = (2\rho - 1)(\alpha - \beta) + O((\alpha - \beta)^3)$. For examples of interacting particle systems with $M < \epsilon$, see Liggett [63], Section I.4. Some of these examples have the reflection symmetry property.

An alternative formula for c_2 is (recall (2.110))

$$c_2 = 2 \int_0^\infty \mathrm{d}t \, \Big(E_{\mathrm{SRW},1}[K(Y_t, t)] - E_{\mathrm{SRW},-1}[K(Y_t, t)] \Big), \tag{2.149}$$

where

$$K(i, t) = E_{P^\mu}[\xi_0(0) \xi_t(i)] = \langle \phi_0 \, (S_{\mathrm{IPS}}(t) \phi_i) \rangle_\mu, \qquad i \in \mathbb{Z}, \, t \geq 0, \tag{2.150}$$

is the space-time correlation function of the interacting particle system (with generator L_{IPS}), and $E_{\mathrm{SRW},i}$ is the expectation over simple random walk $Y = (Y_t)_{t \geq 0}$ jumping at rate $\alpha + \beta$ (with generator L_{SRW}) starting from i. If μ is a *reversible* equilibrium, then (recall (2.3))

$$K(i,t) = \langle \phi_0\, (S_{\mathrm{IPS}}(t)\phi_i)\rangle_\mu = \langle (S_{\mathrm{IPS}}(t)\,\phi_0)\phi_i\rangle_\mu = \langle (S_{\mathrm{IPS}}(t)\phi_{-i})\,\phi_0\rangle_\mu = K(-i,t), \tag{2.151}$$

implying that $c_2 = 0$.

In Section 2.5 we compute c_3 for the independent spin-flip dynamics, for which $c_2 = 0$.

2.4 Examples of cone-mixing

2.4.1 Spin-flip systems in the regime $M < \epsilon$

Let ξ be a spin-flip system for which $M < \epsilon$. We recall that in a spin-flip system only one coordinate changes in a single transition. The rate to flip the spin at site $x \in \mathbb{Z}$ in configuration $\eta \in \Omega$ is $c(x,\eta)$. As shown in Steif [85] and in Maes and Shlosman [65], two copies ξ, ξ' of the spin-flip system starting from configurations η, η' can be coupled such that, uniformly in t and η, η',

$$\widehat{P}_{\eta,\eta'}\big(\exists\, s \geq t\colon\ \xi_s(x) \neq \xi'_s(x)\big) \leq \sum_{\substack{y\in\mathbb{Z}:\\ \eta(y)\neq\eta'(y)}} \mathrm{e}^{-\epsilon t}\left(\mathrm{e}^{\Gamma t}\right)(y,x) \leq \mathrm{e}^{-(\epsilon-M)t}, \tag{2.152}$$

where $\widehat{P}_{\eta,\eta'}$ is the Vasershtein coupling (or basic coupling), and Γ is the matrix $\Gamma = (\gamma(u,v))_{u,v\in\mathbb{Z}}$ with elements

$$\gamma(u,v) = \sup_{\eta\in\Omega} |c(u,\eta) - c(u,\eta^v)|. \tag{2.153}$$

Recall (2.15) to see that Γ is a bounded operator on $\ell_1(\mathbb{Z})$ with norm M (see also Liggett [63], Section I.3).

Define

$$\rho(t) = \sup_{\eta,\eta'\in\Omega} \widehat{P}_{\eta,\eta'}\big(\exists\, s \geq t\colon\ \xi_s(0) \neq \xi'_s(0)\big), \qquad t \geq 0. \tag{2.154}$$

Recall Definition 2.1, fix $\theta \in (0, \frac{1}{2}\pi)$ and put $c = c(\theta) = \cot\theta$. For $B \in \mathcal{F}_t^\theta$, estimate

$$\begin{aligned}
|P_\eta(B) - P_{\eta'}(B)| &\leq \widehat{P}_{\eta,\eta'}\big(\exists\, x \in \mathbb{Z}\ \exists\, s \geq t + c|x| \colon\ \xi_s(x) \neq \xi'_s(x)\big) \\
&\leq \sum_{x\in\mathbb{Z}} \widehat{P}_{\eta,\eta'}\big(\exists\, s \geq t + c|x| \colon\ \xi_s(x) \neq \xi'_s(x)\big) \\
&\leq \sum_{x\in\mathbb{Z}} \rho(t + c|x|) \\
&\leq \rho(t) + 2\int_0^\infty \rho(t + cu)\,\mathrm{d}u \\
&= \rho(t) + \frac{2}{c}\int_0^\infty \rho(t+v)\,\mathrm{d}v.
\end{aligned} \tag{2.155}$$

Since this estimate is uniform in B and η, η', it follows that for the cone mixing property to hold it suffices that

$$\int_0^\infty \rho(v)\,\mathrm{d}v < \infty. \tag{2.156}$$

It follows from (2.152) that $\rho(t) \leq \mathrm{e}^{-(\epsilon-M)t}$, which indeed is integrable.

Note that if the supremum in (2.154) is attained at the same pair of starting configurations η, η' for all $t \geq 0$, then (2.156) amounts to the condition that the average coupling time at the origin for this pair is finite.

2.4.2 Attractive spin-flip dynamics

An attractive spin-flip system ξ has rates $c(x, \eta)$ satisfying

$$\begin{aligned}
c(x,\eta) &\leq c(x,\eta') \quad \text{if } \eta(x) = \eta'(x) = 0, \\
c(x,\eta) &\geq c(x,\eta') \quad \text{if } \eta(x) = \eta'(x) = 1,
\end{aligned} \tag{2.157}$$

whenever $\eta \leq \eta'$ (see Liggett [63], Chapter III). If $c(x, \eta) = c(x + y, \tau_y\eta)$ for all $y \in \mathbb{Z}$, then attractivity implies that, for any pair of configurations η, η',

$$\widehat{P}_{\eta,\eta'}\big(\exists\, s \geq t \colon\ \xi_s(x) \neq \xi'_s(x)\big) \leq \widehat{P}_{[0],[1]}\big(\exists\, s \geq t \colon\ \xi_s(0) \neq \xi'_s(0)\big), \tag{2.158}$$

where [0] and [1] are the configurations with all 0's and all 1's, respectively. Proceeding as in (2.155), we find that for the cone-mixing property to hold it suffices that

$$\int_0^\infty \rho^*(v)\,\mathrm{d}v < \infty, \qquad \rho^*(t) = \widehat{P}_{[0],[1]}\big(\exists\, s \geq t \colon\ \xi_s(0) \neq \xi'_s(0)\big). \tag{2.159}$$

Examples of attractive spin-flip systems are the (ferromagnetic) Stochastic Ising Model, the Contact Process, the Voter Model, and the Majority Vote Process (see Liggett [63], Chapter III). For the one-dimensional Stochastic Ising Model, $t \mapsto \rho^*(t)$ decays exponentially fast at any temperature (see Holley [53]). The same is true for the one-dimensional Majority Vote Process (Liggett [63], Example III.2.12). Hence both are cone-mixing. The one-dimensional Voter Model has equilibria $p\delta_{[0]}+(1-p)\delta_{[1]}$, $p \in [0,1]$, and therefore is not interesting for us. The Contact Process has equilibria $p\delta_{[0]} + (1-p)\nu$, $p \in [0,1]$, but ν is not cone-mixing.

In view of the remark made at the end of Section 2.1.4, we note the following. For the Stochastic Ising Model in dimensions $d \geq 2$ exponentially fast decay occurs only at high enough temperature (Martinelli [66], Theorem 4.1). The Voter Model in dimensions $d \geq 3$ has non-trivial ergodic equilibria, but none of these is cone-mixing. The same is true for the Contact Process in dimensions $d \geq 2$.

2.4.3 Space-time Gibbs measures

We next give an example of a discrete-time dynamic random environment that is cone-mixing but not Markovian. Accordingly, in (2.12) we must replace $\mathcal{F}_0$ by $\mathcal{F}_{-\mathbb{N}_0} = \{\xi_t(x)\colon\, x \in \mathbb{Z},\, t \in (-\mathbb{N}_0)\}$. Let $\sigma = \{\sigma(x,y)\colon\, (x,y) \in \mathbb{Z}^2\}$ be a two-dimensional Gibbsian random field in the Dobrushin regime (see Georgii [48], Section 8.2). We can define a discrete-time dynamic random environment ξ on Ω by putting

$$\xi_t(x) = \sigma(x,t) \qquad (x,t) \in \mathbb{Z}^2. \tag{2.160}$$

The cone-mixing condition for ξ follows from the mixing condition of σ in the Dobrushin regime. In particular, the decay of the mixing function Φ in (2.37) is like the decay of the Dobrushin matrix, which can be polynomial.

2.5 Independent spin-flips

Let ξ be the Markov process with generator L_{ISF} given by

$$(L_{\mathrm{ISF}} f)(\eta) = \sum_{x\in\mathbb{Z}} c(x,\eta)\,\big[f(\eta^x) - f(\eta)\big], \qquad \eta \in \Omega, \tag{2.161}$$

where

$$c(x,\eta) = \gamma[1-\eta(x)] + \delta\eta(x), \tag{2.162}$$

i.e., 0's flip to 1's at rate γ and 1's flip to 0's at rate δ, independently of each other. Such a ξ is an example of a dynamics with $M < \epsilon$, for which Theorem 2.16 holds. From the expansion of the global speed in (2.147) we see that $c_2 = 0$, because the dynamics is invariant under reflection in the origin. We explain the main ingredients that are needed to compute c_3 in (2.18).

The equilibrium measure of ξ is the Bernoulli product measure ν_ρ with parameter $\rho = \gamma/(\gamma+\delta)$. We therefore see from (2.148) that we must compute expressions of the form

$$I(j,i) = \big\langle (2\eta(0)-1)L_0^{-1}\big[(2\eta(j)-1)L_0^{-1}(\eta(i)-\rho)\big]\big\rangle_{\nu_\rho}, \tag{2.163}$$

where η is a typical configuration of the environment process $\zeta = (\zeta_t)_{t\geq 0} = (\tau_{X_t}\xi_t)_{t\geq 0}$ (recall Definition 2.10), and

$$(j,i) \in A = \{(-1,-2),(-1,0),(1,0),(1,2)\}. \tag{2.164}$$

By Lemma 2.11 we have $L_0 = L_{\mathrm{SRW}} + L_{\mathrm{ISF}}$, with L_{SRW} the generator of simple random walk on $\mathbb{Z}$ jumping at rate $U = \alpha+\beta$. Hence

$$(S_0(t)\eta)(i) = E_R^\eta[\eta_t(i)] = \sum_{y\in\mathbb{Z}} p_{Ut}(0,y)\, E_{\mathrm{ISF}}^{\tau_y\eta}[\eta_t(i)] = \sum_{y\in\mathbb{Z}} p_{Ut}(0,y)\, E_{\mathrm{ISF}}^{\eta}[\eta_t(i-y)], \tag{2.165}$$

where τ_y is the shift of space over y, and

$$E_{\mathrm{ISF}}^{\eta}[\eta_t(i)] = \eta(i)\,\mathrm{e}^{-Vt} + \rho(1-\mathrm{e}^{-Vt}) \tag{2.166}$$

with $V = \gamma+\delta$, and $p_t(0,y)$ is the transition kernel of simple random walk on $\mathbb{Z}$ jumping at rate 1. Therefore, by (2.165–2.166), we have

$$L_0^{-1}(\eta(i)-\rho) = \int_0^\infty S_0(t)(\eta(i)-\rho)\,\mathrm{d}t = \sum_{y\in\mathbb{Z}} \eta(i-y)\,G_V(y) - \rho\,\frac{1}{V} \tag{2.167}$$

with

$$G_V(y) = \int_0^\infty \mathrm{e}^{-Vt}\,p_{Ut}(0,y)\,\mathrm{d}t. \tag{2.168}$$

With these ingredients we can compute (2.163), ending up with

$$c_3 = \sum_{(j,i)\in A} I(j,i) = \frac{4}{U}\,\rho(2\rho-1)(1-\rho)\left[\frac{2U+V}{U}\,G_V(0) - \frac{3U+2V}{U}\,G_{2V}(0) - G_{2V}(1)\right]. \tag{2.169}$$

The expression between square brackets can be worked out, because

$$G_V(0) = \int_0^\infty \mathrm{e}^{-Vt}\,p_{Ut}(0,0)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{(U+V)-U\cos\theta} = \frac{1}{\sqrt{(U+V)^2-U^2}} \tag{2.170}$$

and

$$G_V(1) = \frac{U+V}{U}\,G_V(0) - \frac{1}{U}, \tag{2.171}$$

where the latter is derived by using that

$$\frac{\partial}{\partial t} p_{Ut}(0,0) = \tfrac{1}{2}U\left[p_{Ut}(0,1) + p_{Ut}(0,-1) - 2p_{Ut}(0,0)\right] \tag{2.172}$$

and $p_{Ut}(0,1) = p_{Ut}(0,-1)$. This leads to (2.18).

Chapter 3

Annealed central limit theorem for RW in mixing dynamic RE

3.1 Introduction and main result

In this chapter we continue to investigate the model in Section 2.1.1. We show that under a certain strong-mixing assumption on the RE ξ, called *n-cone-mixing* (see Definition 3.1), the RW X satisfies an annealed invariance principle with a Brownian motion as scaling limit. The proof of this functional CLT relies on a direct adaptation of a technique used in [36] for static REs. The *n-cone-mixing* property is a technical assumption directly connected with the machinery used in the proof. In Section 3.2.4 we will exhibit examples of dynamic REs satisfying this assumption.

We first need some definitions. Recall (2.20), and let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^2$. Put $\ell = (0,1)$. For $x = (z,m) \in \mathbb{H} = \mathbb{Z} \times \mathbb{N}_0$, let

$$C_N(x) = \Big\{u \in \mathbb{R} \times [0,\infty) \colon \sqrt{2}/2\|u-x\| \leq (u-x)\cdot \ell \leq N\Big\} \tag{3.1}$$

be the cone of angle $\pi/2$ with tip in (z,m) truncated at time $N+m$.

For fixed $L \geq 0$, let $\{C_{N_i}(x_i) \colon x_i = (z_i, m_i) \in \mathbb{H}\}_{i=1}^n$ be a set of n truncated cones such that, for $1 \leq i < n$,

$$m_1 \geq L, \quad m_{i+1} = N_i + m_i + L, \quad | z_{i+1} - z_i | \leq N_i. \tag{3.2}$$

We call these *nested-cones*. In words, we are considering n space-time truncated cones separated in time by a distance L such that the $(i+1)$-st cone is contained in the i-th extended cone.

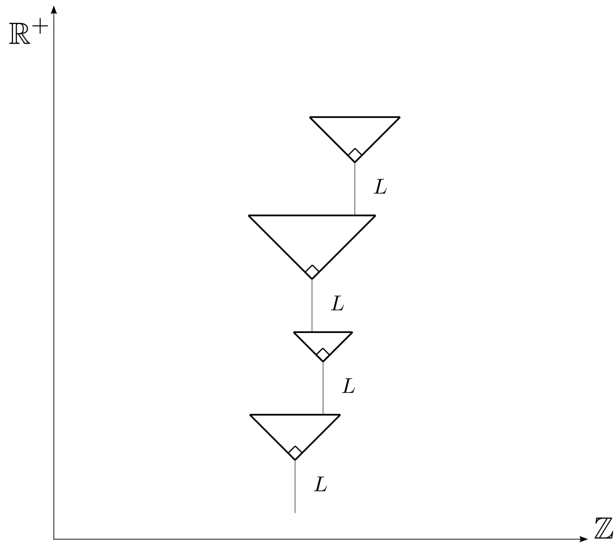

FIGURE 3.1: Example of 4 nested-cones.

Definition 3.1. Fix $L \geq 0$ and any set of n *nested-cones* $\{C_{N_i}(x_i)\colon\ x_i = (z_i, m_i) \in \mathbb{H}\}_{i=1}^n$. A dynamic RE ξ on $\Omega = \{0,1\}^{\mathbb{Z}}$ is said to be *n-cone-mixing* if for any $n \in \mathbb{N}$ there exists a function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$, with

$$\int_0^\infty \Psi(t)\mathrm{d}t < \infty, \tag{3.3}$$

such that

$$\sup_{\substack{A \in \mathcal{G}_n, B \in \mathcal{G}^{<n} \\ \eta, \eta' \in \Omega}} \Big| P_\eta(A \mid B) - P_{\eta'}(A \mid B) \Big| \leq \Psi(nL), \tag{3.4}$$

where

$$\begin{aligned} \mathcal{G}_n &= \sigma\Big\{\xi_s(z)\colon\ (z,s) \in C_{N_n}(x_n)\Big\}, \\ \mathcal{G}^{<n} &= \sigma\left\{\xi_s(z)\colon\ (z,s) \in \bigcup_{i=1}^{n-1} C_{N_i}(x_i)\right\}. \end{aligned} \tag{3.5}$$

Note that if a dynamic RE is *n-cone-mixing*, then the associated path measure P^μ in (1.23) satisfies the *cone-mixing* property in Definition 2.1. Indeed, (2.11) follows easily from (3.4) with $n = 1$. Therefore, by Theorem 2.2, X satisfies a strong LLN with asymptotic speed v. We are now ready to state the main result of this chapter.

Theorem 3.2. *Assume* (2.3) *and suppose that* ξ *is n-cone-mixing. Then there exists a deterministic* $\sigma^2 \in (0, \infty)$ *such that, under the annealed measure* $\mathbb{P}_{\mu,0}$*, the path* $(S_t(s))_{s \geq 0}$*, with*

$$S_t(s) = \frac{X_{ts} - vts}{\sqrt{t}} \tag{3.6}$$

and taking values in the space of right-continuous functions with left limits, converges weakly to a Brownian motion with variance σ^2 *as* $t \to \infty$*.*

The proof of Theorem 3.2 will be given in Section 3.2. In Section 3.3 we give an alternative proof in the context of the perturbative regime introduced in Section 2.3. Indeed, in the latter regime, the strong control on the environment process allows for a much simpler proof than in the general case, and the claim can be easily obtained via a martingale approximation in the spirit of Kipnis-Varhadan [61].

3.2 Proof of Theorem 3.2

In this section we prove Theorem 3.2 by adapting the proof of the CLT for random walks in *static* random environments developed by Comets and Zeitouni [36]. The proof heavily uses the regeneration scheme introduced in Section 2.2.3 and is based on the following steps. In Section 3.2.1 we show that the path of the RW Z in (2.29), together with the evolution of the RE ξ between regeneration times, can be encoded into a *chain with complete connections* for which the dependence of the future on the past can be controlled by the *n-cone-mixing* condition. Chains with complete connections are natural extensions of Markov chains when the transitions of the associated stochastic process depend on its full past. For details we refer the reader to [46, 57]. In Section 3.2.2, using standard results from the theory of such chains, we prove an invariance principle. In Section 3.2.3, we show how Theorem 3.2 follows from the latter.

3.2.1 A chain with complete connections

We construct a chain with complete connections that carries the necessary information relative to the evolution of the path of the RW Z in (2.29), together with the states of the RE ξ inside the truncated cones visited by the path between regeneration times. Lemma 3.3 below uses the *n-cone-mixing* property to control the dependence of the future evolution of the chain on its past. In particular, we will see that the influence of the past decays as fast as the correlations in the RE.

We start by defining the relevant state space. Recall (3.1) and for $N \in \mathbb{N}$ let

$$\mathcal{P}_N = \left\{ \begin{array}{c} \underline{x} = (x(0), x(1), \ldots, x(N)) \in C_N(0)^N : \\ x(0) = 0,\ x(i+1) \sim x(i),\ i = 0, 1, \ldots, N-1 \end{array} \right\} \tag{3.7}$$

be the set of possible paths of the process Z within the truncated cone $C_N(0)$, where $x(i+1) \sim x(i)$ stands for $|x_1(i+1) - x_1(i)| = 1, x_2(i+1) - x_2(i) = 1$. Define

$$\mathcal{T} = \bigcup_{N \in \mathbb{N}} \{N\} \times \mathcal{P}_N \times \mathcal{E}_N, \tag{3.8}$$

where $\mathcal{E}_N = \{\xi_t(z)\colon (z,t) \in C_N(0)\}$ is the set of possible values of the environment ξ in the cone $C_N(0)$. Let

$$\mathcal{W} = \big\{\mathcal{T} \cup \{s\}\big\}^{\mathbb{N}} \tag{3.9}$$

be the set of infinite vectors with components either in $\mathcal{T}$ or equal to the stopping symbol s, with the restriction that if $w_k = s$ then $w_i = s$ for $i \geq k$. Note that, for fixed $L \in \mathbb{N}$, the sequence of regeneration times $\left(\tau_k^{(L)}\right)_{k\in\mathbb{N}}$ in (2.35), together with the path Z, determine an infinite sequence $\underline{r} = (r_1, r_2, \dots) \in \mathcal{W}$ given by

$$r_k = \left(S_{k+1,L},\, \left(Z_{\tau_k^{(L)}+j}\right)_{j=1}^{S_{k+1,L}},\, \left\{\xi_t(z)\colon (z,t) \in C_{S_{k+1,L}}\left(Z_{\tau_k^{(L)}}\right)\right\}\right) \in \mathcal{T}, \quad k \in \mathbb{N}, \tag{3.10}$$

with

$$S_{k,L} = \tau_k^{(L)} - \tau_{k-1}^{(L)} - L, \quad k \in \mathbb{N}. \tag{3.11}$$

Observe that the sequence $\underline{r} = (r_1, r_2, \dots) \in \mathcal{W}$ encodes the information relative to the environment and the path of the walker just after time $S_{1,L} = \tau_1^{(L)} - L$.

Next, we define a set in which we can gather the information prior to time $S_{1,L}$, i.e.,

$$\mathcal{U} = \left\{\begin{array}{c} u = \big(M, y(1), y(2), \dots, y(M), \underline{\xi}(u)\big) : \\ M \in \mathbb{N},\ y(i) \in \mathbb{H},\ y(i+1) \sim y(i),\ i = 0, 1, \dots, M-1 \end{array}\right\} \tag{3.12}$$

with $\underline{\xi}(u) = \{\xi_t : t \leq M\}$.

Recall the sigma-fields in (2.38). For $A \in \mathcal{H}_1$, write

$$A = \bigcup_{(z,n)\in\mathbb{H}} A_{z,n}, \quad A_{z,n} = A \cap \big\{S_{1,L} = n, Z_{S_{1,L}} = (z,n)\big\} \in \mathcal{U}. \tag{3.13}$$

Then the law $\bar{\mathbb{P}}_{\mu,(0,0)}$ induces a probability measure $\mathbb{Q}$ on $\mathcal{U}$ such that

$$\mathbb{Q}(A_{z,n}) = \bar{\mathbb{P}}_{\mu,(0,0)}\Big(\big(S_{1,L}, Z_1, \dots, Z_n, \{\xi_t\colon t \leq n\}\big) \in A_{z,n}\Big), \quad (z,n) \in \mathbb{H}. \tag{3.14}$$

Furthermore, the law $\bar{\mathbb{P}}_{\mu,(0,0)}(\cdot \mid \mathcal{H}_1)$ induces a probability distribution on the sequence $\underline{r} = (r_1, r_2, \dots) \in \mathcal{W}$ in (3.10). Indeed, for fixed $k \in \mathbb{N}$, note that $\bar{\mathbb{P}}_{\mu,(0,0)}(r_k \in \cdot \mid \mathcal{H}_k)$ defines a measurable function $h_k(\cdot \mid w_{k-1}, \dots, w_1, u)$ on $\mathcal{U} \times \mathcal{T}^{k-1}$ such that

$$\begin{aligned} \bar{\mathbb{E}}_{\mu,(0,0)}[\mathbb{1}_A \mathbb{1}_B] &= \bar{\mathbb{E}}_{\mu,(0,0)}[\mathbb{1}_A \bar{\mathbb{P}}_{\mu,(0,0)}((r_1, \dots, r_k) \in B \mid \mathcal{H}_1)] \\ &= \int_A \mathbb{Q}(\mathrm{d}u) \int_{\mathcal{T}} \cdots \int_{\mathcal{T}} \mathbb{1}_B \prod_{i=1}^{k} h_i(\mathrm{d}w_i \mid w_{i-1}, \dots, w_1, u), \end{aligned} \tag{3.15}$$

with $B \subset \mathcal{T}^k$.

In the following lemma we provide an estimate to control the dependence on the past in the sequence $\underline{r}$ whose law is governed by the random kernels $(h_k)_{k\in\mathbb{N}}$. In particular, we show that the influence of the past decays as fast as the correlations in the environment controlled by the Ψ-function in Definition 3.1.

Lemma 3.3. *Let $j \geq i \geq k$, $\underline{w}^{(i)} = (w_i, \dots, w_1)$ and $\underline{w'}^{(j)} = (w'_j, \dots, w'_1)$ be such that $w_{i-l} = w'_{j-l}$ for $l = 0, 1, \dots, k$. Then*

$$\sup_{u,u'\in\mathcal{U}} \left\| h_{i+1}(\cdot \mid \underline{w}^{(i)}, u) - h_{j+1}(\cdot \mid \underline{w'}^{(j)}, u') \right\|_{\text{tv}} \leq \Psi(kL). \tag{3.16}$$

Proof. Observe that the maximum in the left-hand side of (3.16) is attained for $i = j = k$. Therefore, we restrict the proof to this case.

For $u = (M, y(1), y(2), \dots, y(M), \underline{\xi}(u)) \in \mathcal{U}$ and $w_i = (N_i, x(1), x(2), \dots, x(N_i), \underline{\xi}(C_i)) \in \mathcal{T}$, where $\underline{\xi}(C_i)$ denotes the state of the environment in a certain truncated cone C_i, let π be the projection on $\mathcal{U}$ and $\mathcal{T}$, given by, respectively, $\pi(u) = (M, y(1), y(2), \dots, y(M))$ and $\pi(w_i) = (N_i, x(1), x(2), \dots, x(N_i))$. Thus, the first i regeneration points and regeneration times can be reconstructed from $u, \underline{w}^{(i)}$ as follows:

$$\overline{x}_0 = Z_{\tau_1^{(L)}} = y(M) + (0, L), \quad \overline{x}_i = Z_{\tau_{i+1}^{(L)}} = \overline{x}_{i-1} + x(N_i) + (0, L), \tag{3.17}$$

$$\overline{t}_i = \tau_{i+1}^{(L)} = M + L + \sum_{j=1}^{i} (N_j + L). \tag{3.18}$$

Note that the entire path of Z up to time $\overline{t}_i$ is also encoded in $(\pi(u), \pi(w_1), \dots, \pi(w_i))$. Hereafter we denote this path by $\tilde{x} = \tilde{x}\big(\pi(u), \pi(w_1), \dots, \pi(w_i)\big)$, and its k-th component by $\tilde{x}[k]$. In particular, $\tilde{x}[\overline{t}_j] = \overline{x}_j$.

Next, consider a non-negative bounded random variable F measurable w.r.t. $\mathcal{H}_1$. For any given $\pi_0 \in \pi(\mathcal{T})$, there exists a non-negative bounded random variable F_{π_0}, measurable w.r.t. $\sigma\left(\underline{\xi}(u), \{\epsilon_k \colon k = 1, \dots, M\}\right)$, such that $F = F_{\pi_0}$ on the event $\{\pi(r_0) = \pi_0\}$.

Similarly, let G be a non-negative bounded random variable measurable w.r.t. $\sigma(r_1, \dots, r_i)$. For all $\underline{\pi}(i) \in \pi(\mathcal{T})^i$, there exists a random variable $G_{\underline{\pi}(i)}$ measurable w.r.t. $\sigma(\Lambda_\xi(i))$, with

$$\Lambda_\xi(i) = \left\{ \xi_t(z) \colon (z, t) \in \bigcup_{j=1}^{i} (C_j + \overline{x}_{j-1}) \right\}, \tag{3.19}$$

such that $G = G_{\underline{\pi}(i)}$ on the event $\{\pi(r_k) = \pi_k \colon k = 1, \dots, i\}$.

Next, define the events

$$B(\pi_0) = \{Z_k = \tilde{x}[k]\colon\ k = 0, \dots, \bar{t}_0\}, \tag{3.20}$$

and

$$B(\underline{\pi}(i)) = \{Z_{k+\bar{t}_0} - Z_{\bar{t}_0} = \tilde{x}[k+\bar{t}_0] - \tilde{x}[\bar{t}_0]\colon k = 0, \dots, \bar{t}_i - \bar{t}_0\}, \tag{3.21}$$

and the random variable

$$G'_{\pi_0,\underline{\pi}(i)} = G_{\underline{\pi}(i)} \bar{P}_0^{\xi,\epsilon}\left(B\left(\underline{\pi}\left(i\right)\right) \mid Z_n, n \le \bar{t}_0, Y_{\bar{t}_0} = \overline{x}_0\right), \tag{3.22}$$

which is measurable w.r.t. the σ-algebra generated by $\Lambda_\xi(i)$. Abbreviate $\mathbb{1}_A = \mathbb{1}_{\{r_0 \in A\}}$ for a measurable subset $A \subset \mathcal{T}$, and write θ_n to denote the shift of time over n.

By using the above notations and the Markov property, we can write

$$\begin{aligned}
&\bar{\mathbb{E}}_{\mu,(0,0)}\left(FG\left[\mathbb{1}_A \circ \theta_{\tau^{(L)}_{i+1}}\right]\right) \\
&= \sum_{\pi_0,\underline{\pi}(i)} E_{P^\mu \otimes W}\left(\bar{E}_0^{\xi,\epsilon}\left(F_{\pi_0}\,\mathbb{1}_{B(\pi_0)}\,G_{\underline{\pi}(i)}\mathbb{1}_{B(\underline{\pi}(i))}\left[1_A \circ \theta_{\bar{t}_i}\right]\right)\right) \\
&= \sum_{\pi_0,\underline{\pi}(i)} E_{P^\mu \otimes W}\left(\bar{E}_0^{\xi,\epsilon}\left(F_{\pi_0}\,\mathbb{1}_{B(\pi_0)}\,G_{\underline{\pi}(i)}\mathbb{1}_{B(\underline{\pi}(i))}\right)\bar{P}_{\overline{x}_i}^{\theta_{\bar{t}_i}(\xi,\epsilon)}(A)\right) \\
&= \sum_{\pi_0,\underline{\pi}(i)} E_{P^\mu \otimes W}\left(E_{P^\mu \otimes W}\left(\bar{E}_0^{\xi,\epsilon}\left(F_{\pi_0}\,\mathbb{1}_{B(\pi_0)}\,G_{\underline{\pi}(i)}\mathbb{1}_{B(\underline{\pi}(i))}\right)\bar{P}_{\overline{x}_i}^{\theta_{\bar{t}_i}(\xi,\epsilon)}(A)\,\middle|\,\Lambda_\xi(i)\right)\right) \\
&= \sum_{\pi_0,\underline{\pi}(i)} E_{P^\mu \otimes W}\left(G'_{\pi_0,\underline{\pi}(i)} E_{P^\mu \otimes W}\left(F_{\pi_0}\bar{P}_0^{\xi,\epsilon}\left(B(\pi_0)\right)\bar{P}_{\overline{x}_i}^{\theta_{\bar{t}_i}(\xi,\epsilon)}(A)\,\middle|\,\Lambda_\xi(i)\right)\right),
\end{aligned} \tag{3.23}$$

where the sum on $\pi_0, \underline{\pi}(i)$ runs over $\pi(\mathcal{T})^{i+1}$. Define

$$\rho_A = \mathrm{Cov}_{P^\mu \otimes W(\cdot|\Lambda_\xi(i))}\left[\bar{P}_{\overline{x}_i}^{\theta_{\bar{t}_i}(\xi,\epsilon)}(A); F_{\pi_0}\bar{P}_0^{\xi,\epsilon}\left(B(\pi_0)\right)\right], \tag{3.24}$$

and

$$\widetilde{\rho_A} = \sum_{\pi_0,\underline{\pi}(i)} E_{P^\mu \otimes W}\left(G'_{\pi_0,\underline{\pi}(i)}\rho_A\right). \tag{3.25}$$

Write $\widetilde{h_{i+1}}(\cdot \mid \underline{w}^{(i)})$ for the conditional law of r_{i+1} given $r^{(i)} = (r_1, \dots, r_i)$, and note that

$$\widetilde{h_{i+1}}(A \mid \underline{w}^{(i)}) = E_{P^\mu \otimes W}\left(\bar{P}_{\overline{x}_i}^{\theta_{\bar{t}_i}(\xi,\epsilon)}(A)\,\middle|\,\Lambda_\xi(i)\right) \tag{3.26}$$

on the event $B(\underline{\pi}(i)) \cap B(\pi_0)$. Combining (3.23), (3.25) and (3.26), we have

$$
\begin{aligned}
&\bar{\mathbb{E}}_{\mu,(0,0)}\left(FG\left[\mathbb{1}_A\circ\theta_{\tau^{(L)}_{i+1}}\right]\right)=\widetilde{\rho_A}+\\
&\sum_{\pi_0,\underline{\pi}(i)}E_{P^\mu\otimes W}\left(G'_{\pi_0,\underline{\pi}(i)}E_{P^\mu\otimes W}\left(F_{\pi_0}\bar{P}_0^{\xi,\epsilon}\left(B(\pi_0)\right)\,\Big|\,\Lambda_\xi(i)\right)E_{P^\mu\otimes W}\left(\bar{P}_{\overline{x}_i}^{\theta_{\bar{t}_i}(\xi,\epsilon)}(A)\,\Big|\,\Lambda_\xi(i)\right)\right)\\
&=\widetilde{\rho_A}+\sum_{\pi_0,\underline{\pi}(i)}E_{P^\mu\otimes W}\left(G'_{\pi_0,\underline{\pi}(i)}E_{P^\mu\otimes W}\left(F_{\pi_0}\bar{P}_0^{\xi,\epsilon}\left(B(\pi_0)\right)\,\Big|\,\Lambda_\xi(i)\right)\widetilde{h_{i+1}}(A\mid\underline{w}^{(i)})\right)\\
&=\widetilde{\rho_A}+\sum_{\pi_0,\underline{\pi}(i)}\bar{\mathbb{E}}_{\mu,(0,0)}\left(F_{\pi_0}\,\mathbb{1}_{B(\pi_0)}\,G_{\underline{\pi}(i)}\mathbb{1}_{B(\underline{\pi}(i))}\widetilde{h_{i+1}}(A\mid\underline{w}^{(i)})\right)\\
&=\widetilde{\rho_A}+\bar{\mathbb{E}}_{\mu,(0,0)}\left(FG\,\widetilde{h_{i+1}}(A\mid r^{(i)})\right).
\end{aligned}
\tag{3.27}
$$

Observe at this point that, for g measurable w.r.t. $\sigma\big(\xi_t(z)\colon (z,t)\in C_{i+1}+\overline{x}_i\big)$, the n-cone-mixing in (3.4), together with the Markovian nature of the RE ξ, imply that, $\bar{\mathbb{P}}_{\mu,(0,0)}$-a.s.

$$
\left|E_\mu\left[g\mid\underline{\xi}(u)\cup\Lambda_\xi(i)\right]-E_\mu\left[g\mid\Lambda_\xi(i)\right]\right|\le\Psi(iL)\|g\|_\infty. \tag{3.28}
$$

Consequently, for f measurable w.r.t. $\sigma\big(\underline{\xi}(u)\big)$, we have

$$
\begin{aligned}
&\left|E_\mu\left[fg\mid\Lambda_\xi(i)\right]-E_\mu\left[f\mid\Lambda_\xi(i)\right]E_\mu\left[g\mid\Lambda_\xi(i)\right]\right|\\
&=\left|E_\mu\left[fE_\mu\left[g\,|\underline{\xi}(u)\cup\Lambda_\xi(i)\right]\mid\Lambda_\xi(i)\right]-E_\mu\left[f\mid\Lambda_\xi(i)\right]E_\mu\left[g\mid\Lambda_\xi(i)\right]\right|\\
&\le\Psi(iL)\|g\|_\infty E_\mu\left[|f|\mid\Lambda_\xi(i)\right].
\end{aligned}
\tag{3.29}
$$

By estimating (3.24) with the help of (3.29), we obtain from (3.25) that

$$
|\widetilde{\rho_A}|\le\Psi(iL)\bar{\mathbb{E}}_{\mu,(0,0)}\left(FG\right). \tag{3.30}
$$

Finally, combining (3.27) and (3.30), we get

$$
\left|\bar{\mathbb{E}}_{\mu,(0,0)}\left(FG\left[\mathbb{1}_A\circ\theta_{\tau^{(L)}_{i+1}}\right]\right)-\bar{\mathbb{E}}_{\mu,(0,0)}\left(FG\,\widetilde{h_{i+1}}(A\mid r^{(i)})\right)\right|\le\Psi(iL)\bar{\mathbb{E}}_{\mu,(0,0)}\left(FG\right), \tag{3.31}
$$

which, in view of (3.15), implies(3.16). ∎

With the help of Lemma 3.3, we show in the following lemma that the kernel h_k converges as $k\to\infty$ to a kernel h that is independent of $u\in\mathcal{U}$.

Lemma 3.4. *Let $d(w,w') = 2^{-\min\{i\in\mathbb{N}:\ w_i\neq w_i'\}}$ be the lexicographic distance on the space $\mathcal{W}$ defined in (3.9), and let $M(\mathcal{T})$ be the set of probability measures on $\mathcal{T}$. For $\underline{w}^{(k)} = (w_k, w_{k-1},\dots,w_1) \in \mathcal{T}^k$, define $\underline{w} = (w_k, w_{k-1},\dots,w_1,s,s,\dots) \in \mathcal{W}$. Then, there exists a measurable kernel*

$$h : \mathcal{W} \longrightarrow M(\mathcal{T}) \tag{3.32}$$

such that

$$\sup_{\substack{k\geq i,\, u\in\mathcal{U},\, \underline{w}^{(k-1)}\in\mathcal{T}^{k-1},\\ w'\in\mathcal{W}:\, d(\underline{w},w')<2^{-i}}} \left\| h_k(\cdot \mid \underline{w}^{(k-1)}, u) - h(\cdot \mid w') \right\|_{\mathrm{tv}} \leq \Psi(iL) \tag{3.33}$$

and

$$\sup_{w,w'\in\mathcal{W}:\, d(w,w')<2^{-k}} \left\| h(\cdot \mid w) - h(\cdot \mid w') \right\|_{\mathrm{tv}} \leq 2\Psi(kL). \tag{3.34}$$

Proof. Fix $u \in \mathcal{U}$ and $w = (w_1, w_2, \dots) \in \mathcal{W}$, and put $w^{(k)} = (w_1,\dots,w_k) \in \mathcal{T}^k$. By Lemma 3.3, we have that

$$\sup_{u,u'\in\mathcal{U},\, w\in\mathcal{W}} \left\| h_k(\cdot \mid w^{(k-1)}, u) - h_{k'}(\cdot \mid w^{(k'-1)}, u') \right\|_{\mathrm{tv}} \leq \Psi((k\wedge k')L). \tag{3.35}$$

Therefore the sequence $\left(h_k(\cdot \mid w^{(k-1)}, u)\right)_{k\in\mathbb{N}}$ of kernels in $M(\mathcal{T})$ forms a Cauchy sequence w.r.t. the total variation distance, and the completeness of $M(\mathcal{T})$ ensures the existence of a limit $h(\cdot \mid w, u)$. Furthermore, from (3.35) we have that

$$\sup_{u,u'\in\mathcal{U},\, w\in\mathcal{W}} \left\| h_k(\cdot \mid w^{(k-1)}, u') - h(\cdot \mid w, u) \right\|_{\mathrm{tv}} \leq \sum_{i\geq k} \Psi(iL),$$

which, in view of (3.3), implies that $h(\cdot \mid w, u) = h(\cdot \mid w)$ does not depend on $u \in \mathcal{U}$. In particular, the estimates in (3.33) and (3.34) follow easily from (3.35).

∎

3.2.2 Invariance principle for the chain with complete connections

In Section 3.2.1 we constructed a chain with complete connections on $\mathcal{W}$ defined via the kernel h. From the latter we next construct a Markov chain $(w(n))_{n\in\mathbb{N}}$ with state space $\mathcal{W}$ for which we can use standard results from the theory of chains with complete connections.

Let $w(n) = (w_1(n), w_2(n), \dots) \in \mathcal{W}$, and let $y(n+1) \in \mathcal{T}$ be a random variable distributed according to $h(\cdot \mid w(n))$. The next state of the chain, $w(n+1)$, is obtained by setting

$$w_1(n+1) = y(n+1), \quad w_i(n+1) = w_{i-1}(n), \quad i \geq 2. \tag{3.36}$$

In particular, Lemma 3.4 implies that the chain $(w(n))_{n\in\mathbb{N}}$ satisfies conditions $FLS(\mathcal{T}, 1)$ and $M(1)$ in [57], pages 47 and 51. Thus, by Theorem 2.2.7 in [57], it is uniformly ergodic with a unique invariant measure P^w. Next, given $y = (N, x(1), x(2), \dots, x(N), \underline{\xi}(C)) \in \mathcal{T}$, set $f(y) = x(N)$ and $g(y) = N$. The integrability condition (2.49) in Lemma 2.5 implies that

$$\sup_{w\in\mathcal{W}} \int_{\mathcal{T}} |f(y)|^\alpha h(dy|w) < \infty, \quad \alpha > 1. \tag{3.37}$$

Therefore, by Proposition 4.1.1 and Theorem 4.1.2 in [57], we have that, P^w-a.s.,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} g(w_1(i)) = E_{P^w}[g(w_1)] = C_1, \quad \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(w_1(i)) = E_{P^w}[f(w_1)] = C_2. \tag{3.38}$$

Furthermore, by the ϕ mixing property (see [57]) of $f(w_1(i))$ given by Theorem 2.1.5 in [57], together with (3.37) and Theorem 4.1.5 in [57], the following invariance principle holds. Let $c = C_2/C_1$, and

$$\Upsilon_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \big[f(w_1(i)) - c g(w_1(i)) \big], \quad n \in \mathbb{N}, t \geq 0. \tag{3.39}$$

Then, under P^w, the path $\Upsilon_n(t)$, converges weakly to a Brownian motion with a non-degenerate deterministic variance that is independent of the initial condition w.

3.2.3 Invariance principle for the random walk

It remains to show that the invariance principle in Section 3.2.2 for the chain $(w(n))_{n\in\mathbb{N}_0}$ implies the invariance principle of Theorem 3.2. To this aim, consider the random process

$$\left(\widetilde{S}_n(k)\right)_{k\in\mathbb{N}} \quad \text{with} \quad \widetilde{S}_n(k) = \frac{Z_{\tau_k^{(L)}} - c\tau_k^{(L)}}{\sqrt{n}}. \tag{3.40}$$

We first construct a coupling that allows us to compare $\widetilde{S}_n$ with Υ_n. After that we pass from $\widetilde{S}_n$ to S_t defined in (3.6).

Fix $w \in \mathcal{W}$ and $\epsilon \in (0,1)$. Consider an enlarged probability space, with law $P_{\epsilon,w}$, on which there exist a sequence $(r_k)_{k\in\mathbb{N}}$ distributed according to $\bar{\mathbb{P}}_{\mu,(0,0)}(\underline{r} \in \cdot \mid \mathcal{H}_1)$, with $\underline{r}$ as in (3.10), and a sequence $(w(k))_{k\in\mathbb{N}}$ distributed according to P^w. On this enlarged probability space, by using (3.33), we can couple $(r_k)_{k\in\mathbb{N}}$ and $(w(k))_{k\in\mathbb{N}}$ in a recursive manner such that

$$P_{\epsilon,w}\big(r_{i+1} = w_1(i+1) \mid r_1,\dots,r_i, w_1(1),\dots,w_1(i)\big) \geq 1 - \Psi(kL) \tag{3.41}$$

on the event $\{r_l = w_1(l), i-k+1 \leq l \leq i\}$ for any $k \in \{1,\dots,i\}$. Hence, by (3.41) and the fact that $\sum_{k\in\mathbb{N}} \Psi(kL) < \infty$, we have a sequence $k_0(\epsilon) < \infty$, with $k_0(\epsilon) \to \infty$ as $\epsilon \to 0$, such that

$$P_{\epsilon,w}\big(\exists k \geq k_0(\epsilon)\colon\ r_k \neq w_1(k)\big) \leq \epsilon. \tag{3.42}$$

Next, recall Lemma 2.6, fix $T > 0$, and let

$$I_T = 2(T+1)/(Jr^{-L}) \quad \text{with} \quad J = \liminf_{L\to\infty} \bar{\mathbb{E}}_{\mu,(0,0)}(T_1^{(L)}). \tag{3.43}$$

From (3.42), we have that

$$P_{\epsilon,w}\left(\sup_{k_0(\epsilon)\leq k\leq nI_T} \|\widetilde{S}_n(k) - \widetilde{S}_n(k_0(\epsilon)) - \Upsilon_n(k/n) - \Upsilon_n(k_0(\epsilon)/n)\|_1 > 0\right) \leq \epsilon. \tag{3.44}$$

Moreover, for any $\delta > 0$,

$$\lim_{n\to\infty} \bar{\mathbb{P}}_{\mu,(0,0)}\left(\sup_{t\leq \tau_1^{(L)}} \|Z_t\|_1 > \delta\sqrt{n}\right) \leq \lim_{n\to\infty} \bar{\mathbb{P}}_{\mu,(0,0)}\left(\tau_1^{(L)} > \delta\sqrt{n}\right) = 0, \tag{3.45}$$

and, by using (2.49) with $\alpha > 1$, we get

$$\begin{aligned}
&\bar{\mathbb{P}}_{\mu,(0,0)}\left(\sup_{1\leq k\leq n}\left\{\sup_{\tau_k^{(L)}\leq t\leq \tau_{k+1}^{(L)}} \left\{\|Z_t - Z_{\tau_k^{(L)}}\|_1 + (t-\tau_k^{(L)})\right\}\right\} > 3\delta\sqrt{n} \mid \mathcal{H}_1\right)\\
&\leq \bar{\mathbb{P}}_{\mu,(0,0)}\left(\sup_{1\leq k\leq n}\left\{\tau_{k+1}^{(L)} - \tau_k^{(L)}\right\} > \delta\sqrt{n}\right) = 1 - \left[1 - \bar{\mathbb{P}}_{\mu,(0,0)}\left(\tau_1^{(L)} > \delta\sqrt{n}\right)\right]^n. \\
&\leq 1 - \left[1 - \frac{\bar{\mathbb{E}}_{\mu,(0,0)}\left[\left(\tau_1^{(L)}\right)^\alpha\right]}{(\delta\sqrt{n})^\alpha}\right]^n \leq 1 - \left[1 - \frac{M(\alpha)r^{-\alpha L}}{(\delta\sqrt{n})^\alpha}\right]^n,
\end{aligned} \tag{3.46}$$

The r.h.s. of (3.46) tends to zero as $n \to \infty$. Therefore, in view of (3.45) and (3.46), taking first $n \to \infty$ and then $\epsilon \to 0$ in (3.44), we see that the invariance principle for Υ_n in (3.39) can be transferred to an invariance principle for $\widetilde{S}_n(\lfloor tn\rfloor)$ under $\bar{\mathbb{P}}_{\mu,(0,0)}$, on the interval $[0, I_T]$, with the same covariance.

To return to the original process Z, note that by (3.38) and (3.44) we have that

$$\begin{aligned}&\limsup_{n\to\infty}\bar{\mathbb{P}}_{\mu,(0,0)}\left(\sup_{k\le nI_T}\mid\frac{\tau_k^{(L)}}{n}-C_1\frac{k}{n}\mid>\delta\right)\\&\le\limsup_{\epsilon\to 0}\limsup_{n\to\infty}P_{\epsilon,w}\left(\sup_{k\le nI_T}|\frac{\tau_k^{(L)}}{n}-C_1\frac{k}{n}|>\delta\right)=0\end{aligned}. \tag{3.47}$$

On the other hand, by (3.42), we have that

$$\limsup_{n\to\infty}\bar{\mathbb{P}}_{\mu,(0,0)}\left(\tau_{nI_T}^{(L)}<Tn\right)\le\limsup_{\epsilon\to 0}\limsup_{n\to\infty}P_{\epsilon,w}\left(\tau_{nI_T}^{(L)}<Tn\right)=0. \tag{3.48}$$

Thus, by (3.44) and the stability of the invariance principle under random time changes (see [14]) we obtain the invariance principle under $\bar{\mathbb{P}}_{\mu,(0,0)}$, for

$$\left(\frac{Z_{\lfloor nt\rfloor}-vnt}{\sqrt{n}}\right)_{n\in\mathbb{N}},$$

which due to (2.32) carries over to Y, and in particular to its first component (see (2.25)). To pass to continuous time, note that the jump times of X in (2.6) are distributed according to a Poisson process with parameter $\alpha+\beta$ independently of the environment. Therefore, again by the stability of the invariance principle under random time changes, Theorem 3.2 holds.

3.2.4 Examples of mixing dynamic RE

We give here some example of *n-cone-mixing* dynamic RE according to Definition 3.1.
(1) **Independent spin-flip dynamics**
Let $\xi=(\xi_t)_{t\ge 0}$ be an independent spin-flip dynamics (see Section 2.5). Recall the notations of Section 3.1. Fix a set of n *nested-cones* $\{C_{N_i}(x_i)\colon\, x_i=(z_i,m_i)\in\mathbb{H}\}_{i=1}^n$. Define

$$R_n=\{y\in\mathbb{Z}\colon\;\mid y-z_n\mid\le N_n\}$$

to be the set of sites in $\mathbb{Z}$ belonging to the n-th cone, and

$$R^{<n}=\{y\in\mathbb{Z}\colon\,(y,s)\in C_{N_i}(x_i)\text{ for some }i\le n-1\}$$

to be the set of sites belonging to the first $n-1$ cones. For any subsets $A \in \mathcal{G}_n, B \in \mathcal{G}^{<n}$, and any two starting configurations $\eta, \eta' \in \Omega$, in the spirit of Section 2.4.1, estimate

$$\begin{aligned} \Big|P_\eta(A \mid B) - P_{\eta'}(A \mid B)\Big| &\leq \widehat{P}_{\eta,\eta'}\big(\exists\,(z,s) \in C_{N_n}(x_n):\ \xi_s(z) \neq \xi'_s(z) \mid B\big) \\ &\leq \sum_{z \in R_n \setminus R^{<n}} \widehat{P}\big(\exists\, s \geq m_n + |z_n - z|:\ \xi_s(0) \neq \xi'_s(0)\big) \\ &\leq c_1 \mathrm{e}^{-c_2 m_n} \leq c_1 \mathrm{e}^{-c_2 n L}, \end{aligned} \tag{3.49}$$

for some constants $c_1, c_2 > 0$. In the second inequality we have used the independence in space and $\widehat{P}$ stands for the single-site basic coupling measure. In the third inequality we used the exponential convergence to equilibrium and in the fourth inequality that $m_n \geq nL$.

(2) **Space-time strong-mixing Gibbsian field and IPS in the regime** $M < \epsilon$ Consider a dynamic RE ξ constituted by a space-time Gibbsian field as in the example of Section 2.4.3. As shown in [36] (see just after Eq. (2.7) therein), by requiring that the Gibbsian field ξ is strong-mixing in the sense of Definition 1.7 in [36] (see Eq. (1.9) therein), it follows that ξ is an *n-cone-mixing* dynamic RE.

If ξ is a spin-flip system in the regime $M < \epsilon$ (see Section 2.4.1), then due to spatial correlations the argument used in (3.49) does not hold. Nevertheless, such systems are equivalent, in terms of mixing properties, to a Gibbsian field in the uniqueness regime at high temperature (see e.g. [65, 66]), and are therefore expected to satisfy the *n-cone-mixing* property in Definition 3.1. We plan to settle this technical issue in the future.

3.3 CLT in the perturbative regime

In the context of Section 2.3, the proof of Theorem 3.2 does not need the machinery of the previous section. Indeed, as we pointed out in (2.104), $X_t - vt$ can be decomposed as a sum of a martingale $(M_t)_{t\geq 0}$ and an additive functional of the environment process $(\eta_t)_{t\geq 0}$, i.e.,

$$X_t - vt = M_t + (\alpha-\beta)\int_0^t \big(2\eta_s(0) - 1\big)\,\mathrm{d}s - vt = M_t + \int_0^t f(\eta_s)\mathrm{d}s. \tag{3.50}$$

In the spirit of Kipnis-Varadhan [61], we would like to write the additive functional in (3.50) as the sum of a martingale $(M'_t)_{t\geq 0}$ plus a term $(\epsilon_t)_{t\geq 0}$ that is negligible when we divide by $\sqrt{t}$, i.e.,

$$\int_0^t f(\eta_s)\mathrm{d}s = M'_t + \epsilon_t, \quad \epsilon_t = o(\sqrt{t}). \tag{3.51}$$

Since the environment process is not in general a reversible Markov process in $L_2(\mu_e)$, we cannot directly apply the theorem stated in [61]. Nevertheless, several refinements of the Kipnis-Varadhan approach have been obtained for non-reversible Markov processes, e.g. [54], Corollary 3.2, gives a sufficient condition for a martingale approximation, namely,

$$\int_1^\infty t^{-1/2} \left\|S(t)f\right\|_2 \mathrm{d}t < \infty, \tag{3.52}$$

where $(S(t))_{t\geq 0}$ is the semigroup associated with $(\eta_t)_{t\geq 0}$ and $\|\cdot\|_2$ denote the $L_2(\mu_e)$-norm. From (2.133) we easily see that (3.52) holds. Indeed,

$$\left\|S(t)f\right\|_2 \leq \left\|S(t)f\right\|_\infty \leq Ce^{-[c-2(\alpha-\beta)]t}. \tag{3.53}$$

Hence, (3.50) holds, and we can write

$$X_t - vt = M_t + \int_0^t f(\eta_s)\mathrm{d}s = M_t + M'_t + \epsilon_t = M''_t + \epsilon_t. \tag{3.54}$$

The invariance principle for $(X_t)_{t\geq 0}$ then follows from the standard invariance principle for martingales (see e.g. [14]).

Chapter 4

Large deviation principle for one-dimensional RW in dynamic RE: attractive spin-flips and simple symmetric exclusion

This chapter appeared in the form of a paper [4] and is based on joint work with Frank den Hollander and Frank Redig.

Abstract

Consider a one-dimensional shift-invariant attractive spin-flip system in equilibrium, constituting a dynamic random environment, together with a nearest-neighbor random walk that on occupied sites has a local drift to the right but on vacant sites has a local drift to the left. In [3] we proved a law of large numbers for dynamic random environments satisfying a space-time mixing property called cone-mixing. If an attractive spin-flip system has a finite average coupling time at the origin for two copies starting from the all-occupied and the all-vacant configuration, respectively, then it is cone-mixing.

In the present paper we prove a large deviation principle for the empirical speed of the random walk, both quenched and annealed, and exhibit some properties of the associated rate functions. Under an exponential space-time mixing condition for the spin-flip system, which is stronger than cone-mixing, the two rate functions have a unique zero, i.e., the slow-down phenomenon known to be possible in a static random environment does not survive in a fast mixing dynamic random environment. In contrast,

we show that for the simple symmetric exclusion dynamics, which is not cone-mixing (and which is not a spin-flip system either), slow-down does occur.

4.1 Introduction and main results

4.1.1 Random walk in dynamic random environment: attractive spin-flips

Let

$$\xi = (\xi_t)_{t\geq 0} \quad \text{with} \quad \xi_t = \{\xi_t(x)\colon\ x \in \mathbb{Z}\} \tag{4.1}$$

denote a one-dimensional *spin-flip system*, i.e., a Markov process on state space $\Omega = \{0,1\}^{\mathbb{Z}}$ with generator L given by

$$(Lf)(\eta) = \sum_{x\in\mathbb{Z}} c(x,\eta)[f(\eta^x) - f(\eta)], \qquad \eta \in \Omega, \tag{4.2}$$

where f is any cylinder function on Ω, $c(x,\eta)$ is the local rate to flip the spin at site x in the configuration η, and η^x is the configuration obtained from η by flipping the spin at site x. We think of $\xi_t(x) = 1$ ($\xi_t(x) = 0$) as meaning that site x is occupied (vacant) at time t. We assume that ξ is *shift-invariant*, i.e., for all $x \in \mathbb{Z}$ and $\eta \in \Omega$,

$$c(x,\eta) = c(x+y, \tau_y\eta), \qquad y \in \mathbb{Z}, \tag{4.3}$$

where $(\tau_y\eta)(z) = \eta(z-y)$, $z \in \mathbb{Z}$, and also that ξ is *attractive*, i.e., if $\eta \leq \zeta$, then, for all $x \in \mathbb{Z}$,

$$\begin{aligned} c(x,\eta) &\leq c(x,\zeta) \quad \text{if} \quad \eta(x) = \zeta(x) = 0,\\ c(x,\eta) &\geq c(x,\zeta) \quad \text{if} \quad \eta(x) = \zeta(x) = 1. \end{aligned} \tag{4.4}$$

For more on shift-invariant attractive spin-flip systems we refer to [63], Chapter III. Examples are the (ferromagnetic) Stochastic Ising Model, the Voter Model, the Majority Vote Process and the Contact Process.

We assume that

$$\xi \text{ has an equilibrium } \mu \text{ that is shift-invariant and shift-ergodic.} \tag{4.5}$$

For $\eta \in \Omega$, we write P^η to denote the law of ξ starting from $\xi(0) = \eta$, which is a probability measure on the path space $D_\Omega[0,\infty)$, i.e., the set of trajectories in Ω that are right-continuous and have left limits (see [63], Section I.1). We denote by

$$P^{\mu}(\cdot) = \int_{\Omega} P^{\eta}(\cdot)\,\mu(\mathrm{d}\eta) \tag{4.6}$$

the law of ξ when $\xi(0)$ is drawn from μ. We further assume that

$$P^{\mu} \text{ is tail trivial,} \tag{4.7}$$

i.e., all events in the tail σ-algebra $\mathcal{T} = \cap_{s\geq 0}\,\sigma\{\xi_t\colon t \geq s\}$ have probability 0 or 1 under P^{μ}. Conditional on ξ, let

$$X = (X_t)_{t\geq 0} \tag{4.8}$$

be the random walk with local transition rates

$$\begin{aligned} x \to x+1 \quad &\text{at rate} \quad \alpha\,\xi_t(x) + \beta\,[1-\xi_t(x)],\\ x \to x-1 \quad &\text{at rate} \quad \beta\,\xi_t(x) + \alpha\,[1-\xi_t(x)], \end{aligned} \tag{4.9}$$

where w.l.o.g.

$$0 < \beta < \alpha < \infty. \tag{4.10}$$

In words, on occupied sites the random walk jumps to the right at rate α and to the left at rate β, while at vacant sites it does the opposite. Note that, by (4.10), on occupied sites the drift is positive, while on vacant sites it is negative. Also note that the sum of the jump rates is $\alpha+\beta$ and is independent of ξ. For $x \in \mathbb{Z}$, we write P_0^{ξ} to denote the law of X starting from $X_0 = 0$ conditional on ξ, and

$$\mathbb{P}_{\mu,0}(\cdot) = \int_{D_{\Omega}[0,\infty)} P_0^{\xi}(\cdot)\,P^{\mu}(\mathrm{d}\xi) \tag{4.11}$$

to denote the law of X averaged over ξ. We refer to P_0^{ξ} as the *quenched* law and to $\mathbb{P}_{\mu,0}$ as the *annealed* law.

4.1.2 Large deviation principles

Let $\cdot$ and $\|\cdot\|$ denote the inner product, respectively, the Euclidean norm on $\mathbb{R}^2$. Put $\ell = (0,1)$. For $\theta \in (0,\pi/2)$ and $t \geq 0$, let

$$C_t^{\theta} = \big\{u \in \mathbb{Z}\times[0,\infty)\colon\ (u-t\ell)\cdot\ell \geq \|u-t\ell\|\cos\theta\big\} \tag{4.12}$$

be the cone whose tip is at $t\ell = (0,t)$ and whose wedge opens up in the direction ℓ with an angle θ on either side. Note that if $\theta = \pi/2$, then the cone is the half-plane above $t\ell$.

Definition 4.1. An attractive spin-flip system ξ satisfying (4.5) is said to be cone-mixing if, for all $\theta \in (0, \pi/2)$,

$$\lim_{t\to\infty} \sup_{\substack{A\in\mathcal{F}_0,\, B\in\mathcal{F}_t^\theta \\ \mu(A)>0}} \Big| P^\mu(B \mid A) - P^\mu(B) \Big| = 0, \tag{4.13}$$

where

$$\mathcal{F}_0 = \sigma\big\{\xi_0(x)\colon\ x \in \mathbb{Z}\big\}, \quad \mathcal{F}_t^\theta = \sigma\big\{\xi_s(x)\colon\ (x,s) \in C_t^\theta\big\}. \tag{4.14}$$

In [3] we proved that if ξ is cone-mixing, then X satisfies a law of large numbers (LLN), i.e., there exists a $v \in \mathbb{R}$ such that

$$\lim_{t\to\infty} t^{-1} X_t = v \qquad \mathbb{P}_{\mu,0}\text{-a.s.} \tag{4.15}$$

In particular, we showed that all attractive spin-flip systems for which the coupling time at the origin, starting from the configurations $\eta \equiv 1$ and $\eta \equiv 0$, has finite mean are cone-mixing. Theorems 4.2–4.3 below state that X satisfies both an annealed and a quenched large deviation principle (LDP).

Define

$$\begin{aligned} M &= \sum_{x\neq 0} \sup_{\eta\in\Omega} |c(0,\eta) - c(0,\eta^x)|, \\ \epsilon &= \inf_{\eta\in\Omega} |c(0,\eta) + c(0,\eta^0)|. \end{aligned} \tag{4.16}$$

The interpretation of (4.16) is that M is a measure for the *maximal* dependence of the transition rates on the states of single sites, while ϵ is a measure for the *minimal* rate at which the states of single sites change. See [63], Section I.4, for examples. In [3] we showed that if $M < \epsilon$ then ξ is cone-mixing.

Theorem 4.2. (Annealed LDP)
Assume (4.3)–(4.5)*, and let v be as in (4.15).*

(a) *There exists a convex rate function $I^{\rm ann}\colon \mathbb{R} \to [0,\infty)$, satisfying*

$$I^{\rm ann}(\theta) \begin{cases} = 0, & \text{if } \theta \in [v_-^{\rm ann}, v_+^{\rm ann}], \\ > 0, & \text{if } \theta \in \mathbb{R}\backslash[v_-^{\rm ann}, v_+^{\rm ann}], \end{cases} \tag{4.17}$$

for some $-(\alpha-\beta) \le v_-^{\rm ann} \le v \le v_+^{\rm ann} \le \alpha - \beta$, such that

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\mu,0}\big(t^{-1}X_t \in K\big) = -\inf_{\theta\in K} I^{\rm ann}(\theta) \tag{4.18}$$

for all intervals K such that either $K \not\subseteq [v_-^{\rm ann}, v_+^{\rm ann}]$ or $\mathrm{int}(K) \ni v$.

(b) $\lim_{|\theta|\to\infty} I^{\mathrm{ann}}(\theta)/|\theta| = \infty$.

(c) *If* $M < \epsilon$ *and* $\alpha - \beta < \frac{1}{2}(\epsilon - M)$*, then*

$$v_-^{\mathrm{ann}} = v = v_+^{\mathrm{ann}}. \tag{4.19}$$

Theorem 4.3. (Quenched LDP)
Assume (4.3)–(4.5) and (4.7).

(a) *There exists a convex rate function* $I^{\mathrm{que}}\colon \mathbb{R} \to [0,\infty)$*, satisfying*

$$I^{\mathrm{que}}(\theta) \begin{cases} = 0, & \text{if } \theta \in [v_-^{\mathrm{que}}, v_+^{\mathrm{que}}], \\ > 0, & \text{if } \theta \in \mathbb{R}\backslash[v_-^{\mathrm{que}}, v_+^{\mathrm{que}}], \end{cases} \tag{4.20}$$

for some $-(\alpha-\beta) \le v_-^{\mathrm{que}} \le v \le v_+^{\mathrm{que}} \le \alpha - \beta$*, such that*

$$\lim_{t\to\infty} \frac{1}{t} \log P^{\xi}\big(t^{-1}X_t \in K\big) = -\inf_{\theta\in K} I^{\mathrm{que}}(\theta) \qquad \xi\text{-a.s.} \tag{4.21}$$

for all intervals K.

(b) $\lim_{|\theta|\to\infty} I^{\mathrm{que}}(\theta)/|\theta| = \infty$ *and*

$$I^{\mathrm{que}}(-\theta) = I^{\mathrm{que}}(\theta) + \theta(2\rho - 1)\log(\alpha/\beta), \qquad \theta \ge 0. \tag{4.22}$$

(c) *If* $M < \epsilon$ *and* $\alpha - \beta < \frac{1}{2}(\epsilon - M)$*, then*

$$v_-^{\mathrm{que}} = v = v_+^{\mathrm{que}}. \tag{4.23}$$

The v in Theorems 4.2 and 4.3 is the speed in the LLN in (4.15). In [3] we have only proved (4.15) under the additional assumption that ξ is cone-mixing. Theorems 4.2 and 4.3 are proved in Sections 4.2 and 4.3, respectively. The interval K in (4.18) and (4.21) can be open, closed, half-open or half-closed. We are not able to show that (4.18) holds for all intervals K, although we expect this to be true in general.

Because

$$I^{\mathrm{que}} \ge I^{\mathrm{ann}}, \tag{4.24}$$

Theorems 4.3(b)–(c) follow from Theorems 4.2(b)–(c), with the exception of the symmetry relation (4.22). There is no symmetry relation analogous to (4.22) for I^{ann}. It follows from (4.24) that

$$v_-^{\mathrm{ann}} \le v_-^{\mathrm{que}} \le v \le v_+^{\mathrm{que}} \le v_+^{\mathrm{ann}}. \tag{4.25}$$

4.1.3 Random walk in dynamic random environment: simple symmetric exclusion

It is natural to ask whether in a dynamic random environment the rate functions always have a unique zero. The answer is no. In this section we show that when ξ is the *simple symmetric exclusion process* in equilibrium with an arbitrary density of occupied sites $\rho \in (0,1)$, then for any $0 < \beta < \alpha < \infty$ the probability that X_t is near the origin decays slower than exponential in t. Thus, slow-down is possible not only in a static random environment (see Section 4.1.4), but also in a dynamic random environment, provided it is not fast mixing. Indeed, the simple symmetric exclusion process is not even cone-mixing.

The one-dimensional simple symmetric exclusion process

$$\xi = \{\xi_t(x)\colon\, x \in \mathbb{Z},\, t \geq 0\} \tag{4.26}$$

is the Markov process on state space $\Omega = \{0,1\}^{\mathbb{Z}}$ with generator L given by

$$(Lf)(\eta) = \sum_{\substack{x,y\in\mathbb{Z}\\ x\sim y}} [f(\eta^{xy}) - f(\eta)], \qquad \eta \in \Omega, \tag{4.27}$$

where f is any cylinder function on $\mathbb{R}$, the sum runs over unordered neighboring pairs of sites in $\mathbb{Z}$, and η^{xy} is the configuration obtained from η by interchanging the states at sites x and y. We will assume that ξ starts from the Bernoulli product measure with density $\rho \in (0,1)$, i.e., at time $t = 0$ each site is occupied with probability ρ and vacant with probability $1-\rho$. This measure, which we denote by ν_ρ, is an equilibrium for the dynamics (see [63], Theorem VIII.1.44).

Conditional on ξ, the random walk

$$X = (X_t)_{t\geq 0} \tag{4.28}$$

has the same local transition rates as in (4.9)–(4.10). We also retain the definition of the quenched law P_0^ξ and the annealed law $\mathbb{P}_{\nu_\rho,0}$, as in (4.11) with $\mu = \nu_\rho$.

Since the simple symmetric exclusion process is *not* cone-mixing (the space-time mixing property assumed in [3]), we do not have the LLN. Since it is *not* an attractive spin-flip system either, we also do not have the LDP. We plan to address these issues in future work. Our main result here is the following.

Theorem 4.4. *For all* $\rho \in (0,1)$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\nu_\rho,0}\big(|X_t| \leq 2\sqrt{t\log t}\big) = 0. \tag{4.29}$$

Theorem 4.4 is proved in Section 4.4.

4.1.4 Discussion

Literature. Random walk in *static* random environment has been an intensive research area since the early 1970's. One-dimensional models are well understood. In particular, recurrence vs. transience criteria, laws of large numbers and central theorems have been derived, as well as quenched and annealed large deviation principles. In higher dimensions a lot is known as well, but some important questions still remain open. For an overview of these results, we refer the reader to [89, 99].

For random walk in *dynamic* random environment the state of the art is rather more modest, even in one dimension. Early work was done in [64], which considers a one-dimensional environment consisting of spins flipping independently between -1 and $+1$, and a walk that at integer times jumps left or right according to the spin it sees at that time. A necessary and sufficient criterion for recurrence is derived, as well as a law of large numbers.

Three classes of dynamic random environments have been studied in the literature so far:

(1) *Independent in time*: globally updated at each unit of time [6, 11–13, 22, 24, 26, 74, 83, 98];

(2) *Independent in space*: locally updated according to independent single-site Markov chains [5, 16–21, 23, 25, 42, 43, 55, 56];

(3) *Dependent in space and time*: [30, 41].

The focus of these references is: transience vs. recurrence [55, 64], law of large numbers and central limit theorem [5, 6, 11–13, 16, 19, 22–26, 30, 41–43, 74, 83], decay of correlations in space and time [17, 18, 20], convergence of the law of the environment as seen from the walk [21], large deviations [56, 98]. Some papers allow for a mutual interaction between the walk and the environment [11, 16, 19–21, 56].

Classes (2) and (3) are the most challenging. Most papers require additional assumptions, e.g. a strong decay of the time, respectively, space-time correlations in the random environment, or the transition probabilities of the random walk depend only weakly on the random environment (i.e., a small perturbation of a homogeneous random walk). In [3] we improved on this situation by proving a law of large numbers for a class of dynamic random environments in class (3) satisfying only a mild space-time mixing condition, called *cone-mixing* (see Definition 4.1). We showed that a large class of uniquely ergodic attractive spin-flip systems is cone-mixing.

Consider a *static* random environment η with law ν_ρ, the Bernoulli product measure with density $\rho \in (0,1)$, and a random walk $X = (X_t)_{t\geq 0}$ with transition rates (compare with (4.9))

$$\begin{aligned} x \to x+1 \quad &\text{at rate} \quad \alpha\eta(x) + \beta[1-\eta(x)], \\ x \to x-1 \quad &\text{at rate} \quad \beta\eta(x) + \alpha[1-\eta(x)], \end{aligned} \tag{4.30}$$

where $0 < \beta < \alpha < \infty$. In [80] it is shown that X is recurrent when $\rho = 1/2$ and transient to the right when $\rho > 1/2$. In the transient case both ballistic and non-ballistic behavior occur, i.e., $\lim_{t\to\infty} X_t/t = v$ for $\mathbb{P}_{\nu_\rho}$-a.e. ξ, and

$$v \begin{cases} = 0 & \text{if } \rho \in [1/2, \rho_c], \\ > 0 & \text{if } \rho \in (\rho_c, 1], \end{cases} \tag{4.31}$$

where

$$\rho_c = \frac{\alpha}{\alpha+\beta} \in (\tfrac{1}{2}, 1), \tag{4.32}$$

and, for $\rho \in (\rho_c, 1]$,

$$\begin{aligned} v = v(\rho,\alpha,\beta) &= (\alpha+\beta)\,\frac{\alpha\beta + \rho(\alpha^2-\beta^2) - \alpha^2}{\alpha\beta - \rho(\alpha^2-\beta^2) + \alpha^2} \\ &= (\alpha-\beta)\,\frac{\rho-\rho_c}{\rho(1-\rho_c) + \rho_c(1-\rho)}. \end{aligned} \tag{4.33}$$

Attractive spin flips. The analogues of (4.18) and (4.21) in the *static* random environment (with no restriction on the interval K in the annealed case) were proved in [50] (quenched) and [34] (quenched and annealed). Both I^{ann} and I^{que} are zero on the interval $[0, v]$ and are strictly positive outside ("slow-down phenomenon"). For I^{que} the same symmetry property as in (4.22) holds. Moreover, an explicit formula for I^{que} is known in terms of random continued fractions.

We do not have explicit expressions for I^{ann} and I^{que} in the *dynamic* random environment. Even the characterization of their zero sets remains open, although under the stronger assumptions that $M < \epsilon$ and $\alpha - \beta < (\epsilon - M)/2$ we know that both have a unique zero at v.

Theorems 4.2–4.3 can be generalized beyond spin-flip systems, i.e., systems where more than one site can flip state at a time. We will see in Sections 4.2–4.3 that what really matters is that the system has positive correlations in space and time. As shown in [52], this holds for monotone systems (see [63], Definition II.2.3) if and only if all transitions are such that they make the configuration either larger or smaller in the partial order induced by inclusion.

Simple symmetric exclusion. What Theorem 4.4 says is that, for all choices of the parameters, the annealed rate function (if it exists) is zero at 0, and so there is a slow-down phenomenon similar to what happens in the static random environment. We will see in Section 4.4 that this slow-down comes from the fact that the simple symmetric exclusion process suffers “traffic jams”, i.e., long strings of occupied and vacant sites have an appreciable probability to survive for a long time.

To test the validity of the LLN for the simple symmetric exclusion process, we performed a simulation the outcome of which is drawn in Figs. 4.1–4.2. For each point in these figures, we drew 10^3 initial configurations according to the Bernoulli product measure with density ρ, and from each of these configurations ran a discrete-time exclusion process with parallel updating for 10^4 steps. Given the latter, we ran a discrete-time random walk for 10^4 steps, both in the static environment (ignoring the updating) and in the dynamic environment (respecting the updating), and afterwards averaged the displacement of the walk over the 10^3 initial configurations. The probability to jump to the right was taken to be p on an occupied site and $q = 1 - p$ on a vacant site, where p replaces $\alpha/(\alpha + \beta)$ in the continuous-time model. In Figs. 4.1–4.2, the speeds resulting from these simulations are plotted as a function of p for $\rho = 0.8$, respectively, as a function of ρ for $p = 0.7$. In each figure we plot four curves: (1) the theoretical speed in the static case (as described by (4.33)); (2) the simulated speed in the static case; (3) the simulated speed in the dynamic case; (4) the speed for the average environment, i.e., $(2\rho - 1)(2p - 1)$. The order in which these curves appear in the figures is from bottom to top.

Fig. 4.1 shows that, in the static case with ρ fixed, as p increases the speed first goes up (because there are more occupied than vacant sites), and then goes down (because the vacant sites become more efficient to act as a barrier). In the dynamic case, however, the speed is an increasing function of p: the vacant sites are not frozen but move around and make way for the walk. It is clear from Fig. 4.2 that the only value of ρ for which there is a zero speed in the dynamic case is $\rho = 1/2$, for which the random walk is recurrent. Thus, the simulation suggests that there is no (!) non-ballistic behavior in the transient case. In view of Theorem 4.4, this in turn suggests that the annealed rate function (if it exists) has zero set $[0, v]$.

In both pictures the two curves at the bottom should coincide. Indeed, they almost coincide, except for values of the parameters that are close to the transition between ballistic and non-ballistic behavior, for which fluctuations are to be expected. Note that

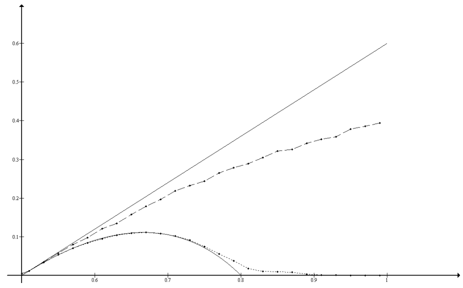

FIGURE 4.1: Speeds as a function of p for $\rho = 0.8$.

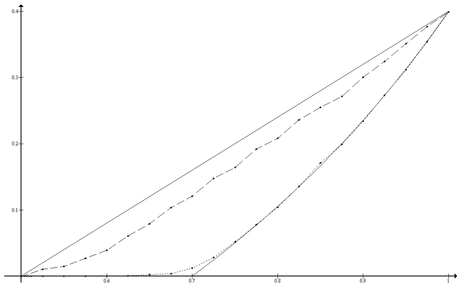

FIGURE 4.2: Speeds as a function of ρ for $p = 0.7$.

the simulated speed in the dynamic environment lies in between the speed for the static environment and the speed for the average environment. We may think of the latter two as corresponding to a simple symmetric exclusion process running at rate 0, respectively, ∞ rather than at rate 1 as in (4.27).

4.2 Proof of Theorem 4.2

In Section 4.2.1 we prove three lemmas for the probability that the empirical speed is above a given threshold. These lemmas will be used in Section 4.2.2 to prove Theorems 4.2(a)–(b). In Section 4.2.3 we prove Theorems 4.2(c).

4.2.1 Three lemmas

Lemma 4.5. *For all $\theta \in \mathbb{R}$,*

$$J^+(\theta) = -\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\mu,0}(X_t \geq \theta t) \quad \textit{exists and is finite.} \tag{4.34}$$

Proof. For $z \in \mathbb{Z}$ and $u \geq 0$, let $\sigma_{z,u}$ denote the operator acting on ξ as

$$(\sigma_{z,u}\xi)(x,t) = \xi(z+x, u+t), \qquad x \in \mathbb{Z},\, t \geq 0. \tag{4.35}$$

Fix $\theta \neq 0$, and let $G_\theta = \{t \geq 0\colon\, \theta t \in \mathbb{Z}\}$ be the non-negative grid of width $1/|\theta|$. For any $s,t \in G_\theta$, we have

$$\begin{aligned}
\mathbb{P}_{\mu,0}\big(X_{s+t} \geq \theta(s+t)\big) &= E^\mu\big[P_0^\xi\big(X_{s+t} \geq \theta(s+t)\big)\big] \\
&= \sum_{y\in\mathbb{Z}} E^\mu\big[P_0^\xi(X_s = y)\, P_y^{\sigma_{0,s}\xi}\big(X_t \geq \theta(s+t)\big)\big] \\
&\geq \sum_{y\geq\theta s} E^\mu\big[P_0^\xi(X_s = y)\, P_{\theta s}^{\sigma_{0,s}\xi}\big(X_t \geq \theta(s+t)\big)\big] \\
&= E^\mu\big[P_0^\xi(X_s \geq \theta s)\, P_0^{\sigma_{\theta s,s}\xi}\big(X_t \geq \theta t\big)\big] \\
&\geq E^\mu\big[P_0^\xi(X_s \geq \theta s)\big]\, E^\mu\big[P_0^{\sigma_{\theta s,s}\xi}\big(X_t \geq \theta t\big)\big] \\
&= \mathbb{P}_{\mu,0}(X_s \geq \theta s)\, \mathbb{P}_{\mu,0}(X_t \geq \theta t).
\end{aligned} \tag{4.36}$$

The first inequality holds because two copies of the random walk running on the same realization of the random environment can be coupled so that they remain ordered. The second inequality uses that

$$\xi \mapsto P_0^\xi(X_s \geq \theta s) \quad \text{and} \quad \xi \mapsto P_0^{\sigma_{\theta s,s}\xi}\big(X_t \geq \theta t\big) \tag{4.37}$$

are non-decreasing and that the law P^μ of an attractive spin-flip system has the FKG-property in space-time (see [63], Corollary II.2.12). Let

$$g(t) = -\log \mathbb{P}_{\mu,0}(X_t \geq \theta t). \tag{4.38}$$

Then it follows from (4.36) that $(g(t))_{t\geq 0}$ is subadditive along G_θ, i.e., $g(s+t) \leq g(s) + g(t)$ for all $s,t \in G_\theta$. Since $\mathbb{P}_{\mu,0}(X_t \geq \theta t) > 0$ for all $t \geq 0$, it therefore follows that

$$J^+(\theta) = -\lim_{\substack{t\to\infty \\ t\in G_\theta}} \frac{1}{t} \log \mathbb{P}_{\mu,0}(X_t \geq \theta t) \quad \text{exists and is finite.} \tag{4.39}$$

Because X takes values in $\mathbb{Z}$, the restriction $t \in G_\theta$ can be removed. This proves the claim for $\theta \neq 0$. The claim easily extends to $\theta = 0$, because the transition rates of the random walk are bounded away from 0 and ∞ uniformly in ξ (recall (4.9)). ∎

Lemma 4.6. *$\theta \mapsto J^+(\theta)$ is non-decreasing and convex on $\mathbb{R}$.*

Proof. We follow an argument similar to that in the proof of Proposition 4.5. Fix $\theta, \gamma \in \mathbb{R}$ and $p \in [0,1]$ such that $p\gamma, (1-p)\theta \in \mathbb{Z}$. Estimate

$$
\begin{aligned}
&\mathbb{P}_{\mu,0}\big(X_t \geq [p\gamma + (1-p)\theta]\, t\big) \\
&\quad = E^{\mu}\big[P_0^{\xi}\big(X_t \geq [p\gamma + (1-p)\theta]\, t\big)\big] \\
&\quad = \sum_{y\in\mathbb{Z}} E^{\mu}\big[P_0^{\xi}(X_{pt} = y)\, P_y^{\sigma_{0,pt}\xi}\big(X_{t(1-p)} \geq [p\gamma + (1-p)\theta]\, t\big)\big] \\
&\quad \geq \sum_{y \geq p\gamma t} E^{\mu}\big[P_0^{\xi}(X_{pt} = y)\, P_{p\gamma t}^{\sigma_{0,pt}\xi}\big(X_{t(1-p)} \geq [p\gamma + (1-p)\theta]\, t\big)\big] \\
&\quad = E^{\mu}\big[P_0^{\xi}(X_{pt} \geq p\gamma t)\, P_0^{\sigma_{p\gamma t,pt}\xi}\big(X_{t(1-p)} \geq (1-p)\theta t\big)\big] \qquad (4.40)\\
&\quad \geq E^{\mu}\big[P_0^{\xi}(X_{pt} \geq p\gamma t)\big]\, E^{\mu}\big[P_{p\gamma\, t}^{\sigma_{p\gamma t,pt}\xi}\big(X_{t(1-p)} \geq (1-p)\theta t\big)\big] \\
&\quad = \mathbb{P}_{\mu,0}(X_{pt} \geq p\gamma t)\, \mathbb{P}_{\mu,0}\big(X_{t(1-p)} \geq (1-p)\theta t\big).
\end{aligned}
$$

It follows from (4.40) and the remark below (4.39) that

$$
- J^{+}\big(p\gamma + (1-p)\theta\big) \geq -pJ^{+}(\gamma) - (1-p)J^{+}(\theta), \qquad (4.41)
$$

which settles the convexity. ∎

Lemma 4.7. *$J^{+}(\theta) > 0$ for $\theta > \alpha - \beta$ and $\lim_{\theta\to\infty} J^{+}(\theta)/\theta = \infty$.*

Proof. Let $(Y_t)_{t\geq 0}$ be the nearest-neighbor random walk on $\mathbb{Z}$ that jumps to the right at rate α and to the left at rate β. Write $\mathbb{P}_0^{\mathrm{RW}}$ to denote its law starting from $Y(0) = 0$. Clearly,

$$
\mathbb{P}_{\mu,0}(X_t \geq \theta t) \leq \mathbb{P}_0^{\mathrm{RW}}(Y_t \geq \theta t) \qquad \forall\, \theta \in \mathbb{R}. \qquad (4.42)
$$

Moreover,

$$
J^{\mathrm{RW}}(\theta) = -\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_0^{\mathrm{RW}}(Y_t \geq \theta t) \qquad (4.43)
$$

exists, is finite and satisfies

$$
J^{\mathrm{RW}}(\alpha - \beta) = 0, \quad J^{\mathrm{RW}}(\theta) > 0 \text{ for } \theta > \alpha - \beta, \qquad \lim_{\theta\to\infty} J^{\mathrm{RW}}(\theta)/\theta = \infty. \qquad (4.44)
$$

Combining (4.42)–(4.44), we get the claim. ∎

Lemmas 4.5–4.7 imply that an *upward* annealed LPD holds with a rate function J^{+} whose qualitative shape is given in Fig. 4.3.

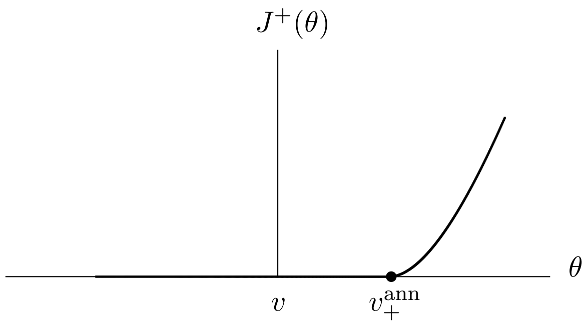

FIGURE 4.3: Shape of $\theta \mapsto J^+(\theta)$.

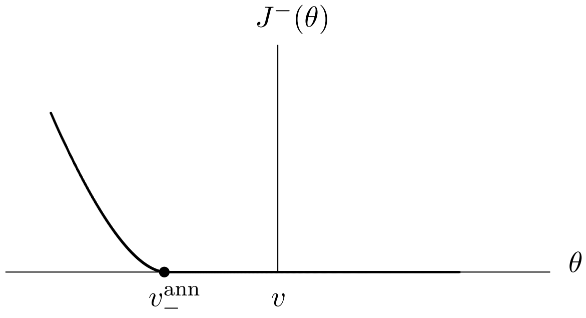

FIGURE 4.4: Shape of $\theta \mapsto J^-(\theta)$.

4.2.2 Annealed LDP

Clearly, J^+ depends on P^μ, α and β. Write

$$J^+ = J_{P^\mu,\alpha,\beta} \tag{4.45}$$

to exhibit this dependence. So far we have not used the restriction $\alpha > \beta$ in (4.10). By noting that $-X_t$ is equal in distribution to X_t when α and β are swapped and P^μ is replaced by $\bar{P}^\mu$, the image of P^μ under reflection in the origin (recall (4.9)), we see that the *upward* annealed LDP proved in Section 4.2.1 also yields a *downward* annealed LDP

$$J^-(\theta) = -\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\mu,0}(X_t \le \theta t), \qquad \theta \in \mathbb{R}, \tag{4.46}$$

with

$$J^- = J_{\bar{P}^\mu,\beta,\alpha}, \tag{4.47}$$

whose qualitative shape is given in Fig. 4.4. Note that

$$v_-^{\text{ann}} \le v \le v_+^{\text{ann}}, \tag{4.48}$$

because v, the speed in the LLN proved in [3], must lie in the zero set of both J^+ and J^-.

Our task is to turn the upward and downward annealed LDP's into the annealed LDP of Theorem 4.2.

Proposition 4.8. *Let*

$$I^{\mathrm{ann}}(\theta) = \begin{cases} J_{P^\mu,\alpha,\beta}(\theta) & \text{if } \theta \geq v, \\ J_{\bar{P}^\mu,\beta,\alpha}(-\theta) & \text{if } \theta \leq v. \end{cases} \tag{4.49}$$

Then

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\mu,0}\big(t^{-1}X_t \in K\big) = -\inf_{\theta\in K} I^{\mathrm{ann}}(\theta) \tag{4.50}$$

for all closed intervals such that either $K \not\subseteq [v_-^{\mathrm{ann}}, v_+^{\mathrm{ann}}]$ *or* $\mathrm{int}(K) \ni v$.

Proof. We distinguish three cases.

(1) $K \subset [v,\infty)$, $K \not\subseteq [v, v_+^{\mathrm{ann}}]$: Let $\mathrm{cl}(K) = [a,b]$. Then, because J^+ is continuous,

$$\frac{1}{t} \log \mathbb{P}_{\mu,0}\big(t^{-1}X_t \in K\big) = \frac{1}{t} \log\left[\mathrm{e}^{-tJ^+(a)+o(t)} - \mathrm{e}^{-tJ^+(b)+o(t)}\right]. \tag{4.51}$$

By Lemma 4.6, J^+ is strictly increasing on $[v_+^{\mathrm{ann}},\infty)$, and so $J^+(b) > J^+(a)$. Letting $t\to\infty$ in (4.51), we therefore see that

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\mu,0}\big(t^{-1}X_t \in K\big) = -J^+(a) = -\inf_{\theta\in K} I^{\mathrm{ann}}(\theta). \tag{4.52}$$

(2) $K \subset (-\infty, v]$, $K \not\subseteq [v_-^{\mathrm{ann}}, v]$: Same as for (1) with J^- replacing J^+.

(3) $\mathrm{int}(K) \ni v$: In this case (4.50) is an immediate consequence of the LLN in (4.15). ∎

Proposition 4.8 completes the proof of Theorems 4.2(a)–(b). Recall (4.45) and (4.47). The restriction on K comes from the fact that the difference of two terms that are both $\exp[o(t)]$ may itself not be $\exp[o(t)]$.

4.2.3 Unique zero of I^{ann} when $M < \epsilon$

In [3] we showed that if $M < \epsilon$ and $\alpha - \beta < (\epsilon - M)/2$, then a proof of the LLN can be given that is based on a perturbation argument for the generator of the *environment process*

$$\zeta = (\zeta_t)_{t\geq 0}, \qquad \zeta_t = \tau_{X_t}\xi_t, \tag{4.53}$$

i.e., the random environment as seen relative to the random walk. In particular, it is shown that ζ is uniquely ergodic with equilibrium μ_e. This leads to a series expansion for v in powers of $\alpha-\beta$, with coefficients that are functions of P^μ and $\alpha+\beta$ and that are computable via a recursive scheme. The speed in the LLN is given by

$$v = (2\widetilde{\rho}-1)(\alpha-\beta) \tag{4.54}$$

with $\widetilde{\rho} = \langle \eta(0) \rangle_{\mu_e}$, where $\langle \cdot \rangle_{\mu_e}$ denotes expectation over μ_e ($\widetilde{\rho}$ is the fraction of time X spends on occupied sites).

Proposition 4.9. *Let ξ be an attractive spin-flip system with $M < \epsilon$. If $\alpha - \beta < (\epsilon - M)/2$, then the rate function I^{ann} in (4.51) has a unique zero at v.*

Proof. It suffices to show that

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\mu,0}\big(|t^{-1}X_t - v| \geq 2\delta\big) < 0 \qquad \forall\, \delta > 0. \tag{4.55}$$

To that end, put $\gamma = \delta/2(\alpha-\beta) > 0$. Then, by (4.54), $v \pm \delta = [2(\widetilde{\rho} \pm \gamma) - 1](\alpha-\beta)$. Let

$$A_t = \int_0^t \xi_s(X_s)\,\mathrm{d}s \tag{4.56}$$

be the time X spends on occupied sites up to time t, and define

$$E_t = \big\{|t^{-1}A_t - \widetilde{\rho}| \geq \gamma\big\}. \tag{4.57}$$

Estimate

$$\mathbb{P}_{\mu,0}(|t^{-1}X_t - v| \geq 2\delta\big) \leq \mathbb{P}_{\mu,0}(E_t) + \mathbb{P}_{\mu,0}\big(|t^{-1}X_t - v| \geq 2\delta \mid E_t^c\big). \tag{4.58}$$

Conditional on E_t^c, X behaves like a homogeneous random walk with speed in $[v-\delta, v+\delta]$. Therefore the second term in the r.h.s. of (4.58) vanishes exponentially fast in t. In [3], Lemma 3.4, Eq. (3.26) and Eq. (3.36), we proved that

$$|||S(t)f||| \leq \mathrm{e}^{-c_1 t}\, |||f||| \quad \text{ and } \quad \big\|S(t)f - \langle f \rangle_{\mu_e}\big\|_\infty \leq c_2\, \mathrm{e}^{-(\epsilon-M)t}\, |||f||| \tag{4.59}$$

for some $c_1, c_2 \in (0,\infty)$, where $S = (S(t))_{t\geq 0}$ denotes the semigroup associated with the environment process ζ, and $|||f|||$ denotes the triple norm of f. As shown in [75], (4.59) implies a Gaussian concentration bound for additive functionals, namely,

$$\mathbb{P}_{\mu,0}\left(\left|t^{-1}\int_0^t f(\zeta_s) - \langle f \rangle_\mu\right| \geq \gamma\right) \leq c_3\, \exp\{-\gamma^2 t/c_4\, |||f|||^2\} \tag{4.60}$$

for some $c_3, c_4 \in (0,\infty)$, uniformly in $t > 0$, f with $|||f||| < \infty$ and $\gamma > 0$. By picking $f(\eta) = \eta(0)$, $\eta \in \Omega$, we get

$$\mathbb{P}_{\mu,0}\big(E_t\big) \leq c_5 \exp\{-c_6 t\} \tag{4.61}$$

for some $c_5, c_6 \in (0,\infty)$. Therefore also the first term in the r.h.s. of (4.58) vanishes exponentially fast in t. ∎

Proposition 4.9 completes the proof of Theorems 4.2(c).

4.3 Proof of Theorem 4.3

In Section 4.3.1 we prove three lemmas for the probability that the empirical speed equals a given value. These lemmas will be used in Section 4.3.2 to prove Theorems 4.3(a)–(b). In Section 4.3.3 we prove Theorem 4.3(c). Theorem 4.3(d) follows from Theorem 4.2(c) because $I^{\text{que}} \geq I^{\text{ann}}$.

4.3.1 Three lemmas

In this section we state three lemmas that are the analogues of Lemmas 4.5–4.7.

Lemma 4.10. *For all $\theta \in \mathbb{R}$,*

$$I^{\text{que}}(\theta) = -\lim_{t\to\infty} \frac{1}{t} \log P_0^{\xi}(X_t = \lfloor \theta t \rfloor) \tag{4.62}$$

exists, is finite and is constant ξ-a.s.

Proof. Fix $\theta \neq 0$, and recall that $G_\theta = \{t \geq 0\colon\ \theta t \in \mathbb{Z}\}$ is the non-negative grid of width $1/|\theta|$. For any $s, t \in G_\theta$, we have

$$\begin{aligned} P_0^{\xi}\big(X_{s+t} = \theta(s+t)\big) &\geq P_0^{\xi}\big(X_s = \theta s\big)\, P_0^{\xi}\big(X_{s+t} = \theta(s+t) \mid X_s = \theta s\big) \\ &= P_0^{\xi}\big(X_s = \theta s\big)\, P_0^{T_s\xi}(X_t = \theta t), \end{aligned} \tag{4.63}$$

where $T_s = \sigma_{\theta s, s}$. Let

$$g_t(\xi) = -\log P_0^{\xi}(X_t = \theta t). \tag{4.64}$$

Then it follows from (4.63) that $(g_t(\xi))_{t\geq 0}$ is a subadditive random process along G_θ, i.e., $g_{s+t}(\xi) \leq g_s(\xi) + g_t(T_s\xi)$ for all $s, t \in G_\theta$. From Kingman's subadditive ergodic theorem (see e.g. [84]) it therefore follows that

$$\lim_{\substack{t\to\infty \\ t\in G_\theta}} \frac{1}{t} \log P_0^{\xi}(X_t = \theta t) = -I^{\text{que}}(\theta) \tag{4.65}$$

exists, is finite ξ-a.s, and is T_s-invariant for every $s \in G_\theta$. Moreover, since ξ is ergodic under space-time shifts (recall (4.5) and (4.7)), this limit is constant ξ-a.s. Because the transition rates of the random walk are bounded away from 0 and ∞ uniformly in ξ (recall (4.9)), the restriction $t \in G_\theta$ may be removed after $X_t = \theta t$ is replaced by $X_t = \lfloor \theta t \rfloor$ in (4.65). This proves the claim for $\theta \neq 0$. By the boundedness of the transition rates, the claim easily extends to $\theta = 0$. ∎

Lemma 4.11. *$\theta \mapsto I^{\text{que}}(\theta)$ is convex on $\mathbb{R}$.*

Proof. The proof is similar to that of Proposition 4.5. Fix $\theta, \zeta \in \mathbb{R}$ and $p \in [0,1]$. For any $t \geq 0$ such that $p\zeta t, (1-p)\theta t \in \mathbb{Z}$, we have

$$\begin{aligned} &P_0^\xi\big(X_t \geq [p\zeta + (1-p)\theta]\, t\big) \\ &\quad\geq P_0^\xi\big(X_{pt} = p\zeta t\big)\, P_0^\xi\big(X_t = [p\zeta + (1-p)\theta]\, t \mid X_{pt} = p\zeta t\big) \\ &\quad= P_0^\xi\big(X_{pt} = p\zeta t\big)\, P_0^{\sigma_{p\zeta t, pt}\xi}\big(X_{(1-p)t} = (1-p)\theta t\big). \end{aligned} \tag{4.66}$$

It follows from (4.66) and the remark below (4.39) that

$$- I^{\text{que}}\big(p\zeta + (1-p)\theta\big) \geq -pI^{\text{que}}(\zeta) - (1-p)I^{\text{que}}(\theta), \tag{4.67}$$

which settles the convexity. ∎

Lemma 4.12. *$I^{\text{que}}(\theta) > 0$ for $|\theta| > \alpha - \beta$ and*

$$\lim_{\theta \to \infty} I^{\text{que}}(\theta)/|\theta| = \infty.$$

Proof. Same as Lemma 4.7. ∎

4.3.2 Quenched LDP

We are now ready to prove the quenched LDP.

Proposition 4.13. *For P^μ-a.e. ξ, the family of probability measures*

$$P_0^\xi(X_t/t \in \cdot), \quad t > 0,$$

satisfies the LDP with rate t and with deterministic rate function I^{que}.

Proof. Use Lemmas 4.10–4.12. ∎

Proposition 4.13 completes the proof of Theorems 4.3, except for the symmetry relation in (4.22), which will be proved in Section 4.3.3. Recall (4.24) and the remark below it.

4.3.3 A quenched symmetry relation

Proposition 4.14. *For all $\theta \in \mathbb{R}$, the rate function in Theorem* 4.13 *satisfies the symmetry relation*

$$I^{\text{que}}(-\theta) = I^{\text{que}}(\theta) + \theta(2\rho - 1)\log(\alpha/\beta). \tag{4.68}$$

Proof. We first consider a discrete-time random walk, i.e., a random walk that observes the random environment and jumps at integer times. Afterwards we will extend the argument to the continuous-time random walk defined in (4.8)–(4.10).

1. *Path probabilities*

Let

$$X = (X_n)_{n\in\mathbb{N}_0} \tag{4.69}$$

be the random walk with transition probabilities

$$\begin{aligned} x &\to x+1 \quad \text{with probability} \quad p\,\xi_n(x) + q\,[1-\xi_n(x)],\\ x &\to x-1 \quad \text{with probability} \quad q\,\xi_n(x) + p\,[1-\xi_n(x)], \end{aligned} \tag{4.70}$$

where w.l.o.g. $p > q$. For an oriented edge $e = (i, i \pm 1)$, $i \in \mathbb{Z}$, write $\overline{e} = (i \pm 1, i)$ to denote the reverse edge. Let $p_n(e)$ denote the probability for the walk to jump along the edge e at time n. Note that in the *static* random environment these probabilities are time-independent, i.e., $p_n(e) = p_0(e)$ for all $n \in \mathbb{N}$.

We will be interested in n-step paths $\omega = (\omega_0, \dots, \omega_n) \in \mathbb{Z}^n$ with $\omega_0 = 0$ and $\omega_n = \lfloor \theta n \rfloor$ for a given $\theta \neq 0$. Write $\Theta\omega$ to denote the time-reversed path, i.e., $\Theta\omega = (\omega_n, \dots, \omega_0)$. Let $N_e(\omega)$ denote the number of times the edge e is crossed by ω, and write $t_e^j(\omega)$, $j = 1, \dots, N_e(\omega)$, to denote the successive times at which the edge e is crossed. Let $E(\omega)$ denote the set of edges in the path ω, and $E^+(\omega)$ the subset of forward edges, i.e., edges of the form $(i, i+1)$. Then we have

$$N_e(\Theta\omega) = N_{\overline{e}}(\omega) \tag{4.71}$$

and

$$t_e^j(\Theta\omega) = n + 1 - t_{\overline{e}}^{N_{\overline{e}}(\omega)+1-j}(\omega), \qquad j = 1, \dots, N_e(\Theta\omega) = N_{\overline{e}}(\omega). \tag{4.72}$$

Given a realization of ξ, the probability that the walk follows the path ω equals

$$
\begin{aligned}
P^{\xi}(\omega) &= \prod_{e\in E(\omega)} \prod_{j=1}^{N_e(\omega)} p_{t_e^j(\omega)}(e) \\
&= \prod_{e\in E^+(\omega)} \prod_{j=1}^{N_e(\omega)} p_{t_e^j(\omega)}(e) \prod_{j=1}^{N_{\overline{e}}(\omega)} p_{t_{\overline{e}}^j(\omega)}(\overline{e}).
\end{aligned}
\tag{4.73}
$$

The probability of the reversed path is, by (4.71)–(4.72),

$$
\begin{aligned}
P^{\xi}(\Theta\omega) &= \prod_{e\in E(\omega)} \prod_{j=1}^{N_e(\Theta\omega)} p_{t_e^j(\Theta\omega)}(e) \\
&= \prod_{e\in E(\omega)} \prod_{j=1}^{N_{\overline{e}}(\omega)} p_{n+1-t_{\overline{e}}^{N_{\overline{e}}(\omega)+1-j}(\omega)}(e) \\
&= \prod_{e\in E(\omega)} \prod_{j=1}^{N_{\overline{e}}(\omega)} p_{n+1-t_{\overline{e}}^j(\omega)}(e) \\
&= \prod_{e\in E^+(\omega)} \prod_{j=1}^{N_e(\omega)} p_{n+1-t_e^j(\omega)}(\overline{e}) \prod_{j=1}^{N_{\overline{e}}(\omega)} p_{n+1-t_{\overline{e}}^j(\omega)}(e).
\end{aligned}
\tag{4.74}
$$

Given a path going from ω_0 to ω_n, all the edges e in between ω_0 and ω_n pointing in the direction of ω_n, which we denote by $\mathcal{E}(\omega_0,\omega_n)$, are traversed one time more than their reverse edges, while all other edges are traversed as often as their reverse edges. Therefore we obtain, assuming w.l.o.g. that $\omega_n > \omega_0$ (or $\theta > 0$),

$$
\begin{aligned}
\log\frac{P^{\xi}(\Theta\omega)}{P^{\xi}(\omega)} &= \sum_{e\in\mathcal{E}(\omega_0,\omega_n)} \log\frac{p_{n+1-t_e^{N_e(\omega)}(\omega)}(\overline{e})}{p_{t_e^{N_e(\omega)}(\omega)}(e)} \\
&\quad + \sum_{e\in E^+(\omega)} \sum_{j=1}^{N_{\overline{e}}(\omega)} \log\Big(\frac{p_{n+1-t_{\overline{e}}^j(\omega)}(e)p_{n+1-t_e^j(\omega)}(\overline{e})}{p_{t_e^j(\omega)}(e)p_{t_{\overline{e}}^j(\omega)}(\overline{e})}\Big).
\end{aligned}
\tag{4.75}
$$

In the *static* random environment we have $p_n(e) = p_0(e)$ for all $n \in \mathbb{N}$ and $e \in E(\omega)$, and hence the second sum in (4.75) is identically zero, while by the ergodic theorem the first sum equals

$$
\begin{aligned}
&(\omega_n - \omega_0)\langle\log[p_0(1,0)/p_0(0,1)]\rangle_{\nu_\rho} + o(n) \\
&= (\omega_n - \omega_0)(2\rho-1)\log(p/q) + o(n), \qquad n\to\infty,
\end{aligned}
\tag{4.76}
$$

where ν_ρ is the Bernoulli product measure on Ω with density ρ (which is the law that is typically chosen for the static random environment). In the *dynamic* random environment, both sums in (4.75) still look like ergodic sums, but since in general

$$p_{t_e^j(\omega)}(e) \neq p_{t_e^i(\omega)}(e), \qquad i \neq j, \tag{4.77}$$

we have to use space-time ergodicity.

2. *Space-time ergodicity*

Rewrite (4.75) as

$$\begin{aligned}\log\frac{P^\xi(\Theta\omega)}{P^\xi(\omega)} = &\sum_{e\in\mathcal{E}(\omega_0,\omega_n)} \log p_{n+1-t_e^{N_e(\omega)}(\omega)}(\overline{e}) - \sum_{e\in\mathcal{E}(\omega_0,\omega_n)} \log p_{t_e^{N_e(\omega)}(\omega)}(e)\\ &+ \sum_{e\in E^+(\omega)} \log p_{n+1-t_{\overline{e}}^1(\omega)}(e) + \sum_{e\in E^+(\omega)} \log p_{n+1-t_e^1(\omega)}(\overline{e})\\ &- \sum_{e\in E^+(\omega)} \log p_{t_e^1(\omega)}(e) - \sum_{e\in E^+(\omega)} \log p_{t_{\overline{e}}^1(\omega)}(\overline{e})\\ &+ \sum_{e\in E^+(\omega)} \sum_{j=2}^{N_{\overline{e}}(\omega)} \log\Big(\frac{p_{n+1-t_{\overline{e}}^j(\omega)}(e)p_{n+1-t_e^j(\omega)}(\overline{e})}{p_{t_e^j(\omega)}(e)p_{t_{\overline{e}}^j(\omega)}(\overline{e})}\Big),\end{aligned} \tag{4.78}$$

and note that all the sums in (4.78) are of the form

$$\sum_{i=1}^N \log p_{t(i)}(\omega_0+i) = \begin{cases} (\log p)\displaystyle\sum_{i=1}^N \xi_{t_i}(\omega_0+i) + (\log q)\sum_{i=1}^N [1-\xi_{t_i}(\omega_0+i)],\\ (\log q)\displaystyle\sum_{i=1}^N \xi_{t_i}(\omega_0+i) + (\log p)\sum_{i=1}^N [1-\xi_{t_i}(\omega_0+i)],\end{cases} \tag{4.79}$$

where $t_i = t((i,i+1))$, with $t = t(\omega)\colon \{0,1,\dots,N\} \to \{0,1,\dots,n\}$ either strictly increasing or strictly decreasing with image set $I_n(t) \subset \{0,1,\dots,n\}$ such that $|I_n(t)|$ is of order n. Note that $N = N(\omega) = |\mathcal{E}(\omega_0,\omega_n)| = \omega_n - \omega_0 = \lfloor\theta n\rfloor$ in the first two sums in (4.78), $N = N(\omega) = |E^+(\omega)| \geq \omega_n - \omega_0 = \lfloor\theta n\rfloor$ in the remaining sums, and

$$|t_j - t_i| \geq j - i, \qquad j > i. \tag{4.80}$$

The aim is to show that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N \log p_{t_i}(i) = \langle \log p_0(0)\rangle_\mu = \rho\log p + (1-\rho)\log q \quad \xi\text{-a.s. for all } \omega \tag{4.81}$$

or, equivalently,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \xi_{t_i}(i) = \langle \xi_0(0)\rangle_\mu = \rho \quad \xi\text{-a.s. for all } \omega, \tag{4.82}$$

where, since we take the limit $N \to \infty$, we think of ω as an infinite path in which the n-step path $(\omega_0, \ldots, \omega_n)$ with $\omega_0 = 0$ and $\omega_n = \lfloor \theta n \rfloor$ is embedded. Because P^μ is tail trivial (recall (4.7)) and $\lim_{i\to\infty} t_i = \infty$ for all ω by (4.80), the limit exists ξ-a.s. for all ω. To prove that the limit equals ρ we argue as follows. Write

$$\mathrm{Var}^{P^\mu}\left(\frac{1}{N}\sum_{i=1}^{N}\xi_{t_i}(i)\right) = \frac{\rho(1-\rho)}{N} + \frac{2}{N^2}\sum_{i=1}^{N}\sum_{j>i}\mathrm{Cov}^{P^\mu}\big(\xi_{t_i}(i), \xi_{t_j}(j)\big). \tag{4.83}$$

By (4.5), we have

$$\mathrm{Cov}^{P^\mu}\big(\xi_{t_i}(i), \xi_{t_j}(j)\big) = \mathrm{Cov}^{P^\mu}\big(\xi_0(0), \xi_{|t_j - t_i|}(j-i)\big). \tag{4.84}$$

In view of (4.80), it therefore follows that

$$\lim_{k\to\infty}\sup_{l\geq k}\mathrm{Cov}^{P^\mu}\big(\xi_0(0),\xi_l(k)\big) = 0 \quad \Longrightarrow \quad \lim_{N\to\infty}\mathrm{Var}^{P^\mu}\left(\frac{1}{N}\sum_{i=1}^{N}\xi_{t_i}(i)\right) = 0. \tag{4.85}$$

But the l.h.s. of (4.85) is true by the tail triviality of P^μ.

3. *Implication for the rate function*

Having proved (4.81) holds, we can now use (4.78)–(4.79) and (4.81)–(4.82) to obtain

$$\frac{P^\xi(\Theta\omega)}{P^\xi(\omega)} = \exp\{A(\omega_n - \omega_0) + o(n)\} \quad \text{with} \quad A = (2\rho - 1)\log(p/q). \tag{4.86}$$

Thus, the probability that the walk moves from 0 to $\lfloor\theta n\rfloor$ in n steps is given by

$$\begin{aligned}
P^\xi(\omega_n = \lfloor\theta n\rfloor \mid \omega_0 = 0) &= \sum_{\substack{\omega\colon\ |\omega|=n\\ \omega_0=0,\omega_n=\lfloor\theta n\rfloor}} P^\xi(\omega)\\
&= \sum_{\substack{\omega\colon\ |\omega|=n\\ \omega_0=0,\omega_n=\lfloor\theta n\rfloor}} P^\xi(\Theta\omega)\,\mathrm{e}^{-A\lfloor\theta n\rfloor + o(n)}\\
&= \mathrm{e}^{-A\lfloor\theta n\rfloor + o(n)} \sum_{\substack{\omega\colon\ |\omega|=n\\ \omega_0=\lfloor\theta n\rfloor,\omega_n=0}} P^\xi(\omega)\\
&= \mathrm{e}^{-A\lfloor\theta n\rfloor + o(n)}\, P^\xi(\omega_n = 0 \mid \omega_0 = \lfloor\theta n\rfloor).
\end{aligned} \tag{4.87}$$

Since the quenched rate function is ξ-a.s. constant, we have

$$\begin{aligned} P^{\xi}(\omega_n = \lfloor \theta n \rfloor \mid \omega_0 = 0) &= P_0^{\xi}(X_n = \lfloor \theta n \rfloor) = \mathrm{e}^{-nI^{\mathrm{que}}(\theta)+o(n)}, \\ P^{\xi}(\omega_n = 0 \mid \omega_0 = \lfloor \theta n \rfloor) &= P_0^{\tau_{\lfloor \theta n \rfloor}\xi}(X_n = -\lfloor \theta n \rfloor) = \mathrm{e}^{-nI^{\mathrm{que}}(-\theta)+o(n)}, \end{aligned} \tag{4.88}$$

and hence

$$\frac{1}{n}\log\left(\frac{P^{\xi}(\omega_n = \lfloor \theta n \rfloor \mid \omega_0 = 0)}{P^{\xi}(\omega_n = 0 \mid \omega_0 = \lfloor \theta n \rfloor)}\right) \to -I^{\mathrm{que}}(\theta) + I^{\mathrm{que}}(-\theta). \tag{4.89}$$

Together with (4.87), this leads to the symmetry relation

$$-I^{\mathrm{que}}(\theta) + I^{\mathrm{que}}(-\theta) = -A\theta. \tag{4.90}$$

4. *From discrete to continuous time*

Let $\chi = (\chi_n)_{n\in\mathbb{N}_0}$ denote the jump times of the continuous-time random walk $X = (X_t)_{t\geq 0}$ (with $\chi_0 = 0$). Let Q denote the law of χ. The increments of χ are i.i.d. random variables, independent of ξ, whose distribution is exponential with mean $1/(\alpha+\beta)$. Define

$$\begin{aligned} \xi^* &= (\xi_n^*)_{n\in\mathbb{N}_0} \quad \text{with} \quad \xi_n^* = \xi_{\chi_n}, \\ X^* &= (X_n^*)_{n\in\mathbb{N}_0} \quad \text{with} \quad X_n^* = X_{\chi_n}. \end{aligned} \tag{4.91}$$

Then X^* is a discrete-time random walk in a random environment ξ^* of the type considered in Steps 1–3, with $p = \alpha/(\alpha+\beta)$ and $q = \beta/(\alpha+\beta)$. The analogue of (4.82) reads

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \xi_{\chi_{t_i}}(i) = \rho \qquad \xi,\chi\text{-a.s. for all } \omega, \tag{4.92}$$

where we use that the law of χ is invariant under permutations of its increments. All we have to do is to show that

$$\lim_{N\to\infty} E^{Q}\left(\mathrm{Var}^{P^{\mu}}\left(\frac{1}{N}\sum_{i=1}^{N}\xi_{\chi_{t_i}}(i)\right)\right) = 0. \tag{4.93}$$

But

$$E^{Q}\big(\mathrm{Cov}^{P^{\mu}}\big(\xi_{\chi_{t_i}}(i), \xi_{\chi_{t_j}}(j)\big)\big) = E^{Q}\big(\mathrm{Cov}^{P^{\mu}}\big(\xi_0(0), \xi_{|\chi_{t_j}-\chi_{t_i}|}(j-i)\big)\big), \tag{4.94}$$

while (4.80) ensures that $\lim_{j\to\infty}|\chi_{t_j}-\chi_{t_i}| \to \infty$ χ-a.s. for all ω as $j-i\to\infty$. Together with the tail triviality of P^{μ} assumed in (4.7), this proves (4.93). ∎

4.4 Proof of Theorem 4.4

In Section 4.4.1 we show that the simple symmetric exclusion process suffers traffic jams. In Section 4.4.2 we prove that these traffic jams cause the slow-down of the random walk.

4.4.1 Traffic jams

In this section we derive two lemmas stating that long strings of occupied and vacant sites have an appreciable probability to survive for a long time under the simple symmetric exclusion dynamics, both when they are alone (Lemma 4.15) and when they are together but sufficiently separated from each other (Lemma 4.16). These lemmas, which are proved with the help of the *graphical representation*, are in the spirit of [2].

In the *graphical representation* of the simple symmetric exclusion process, space is drawn sidewards, time is drawn upwards, and for each pair of nearest-neighbor sites $x, y \in \mathbb{Z}$ links are drawn between x and y at Poisson rate 1. The configuration at time t is obtained from the one at time 0 by transporting the local states along paths that move upwards with time and sidewards along links (see Fig. 4.5).

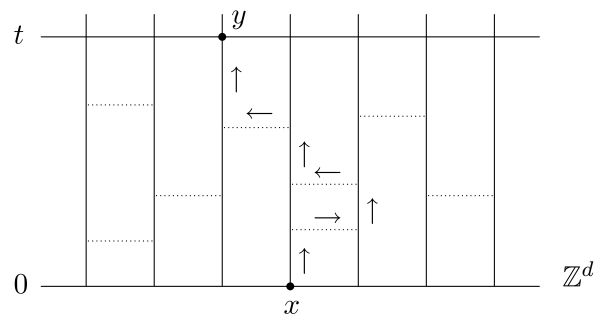

FIGURE 4.5: Graphical representation. The dashed lines are links. The arrows represent a path from $(x, 0)$ to (y, t).

Lemma 4.15. *There exists a $C = C(\rho) > 0$ such that, for all $Q \subset \mathbb{Z}$ and all $t \geq 1$,*

$$P^{\nu_\rho}\big(\xi_s(x) = 0\ \forall\, x \in Q\ \forall\, s \in [0,t]\big) \geq \mathrm{e}^{-C|Q|\sqrt{t}}. \tag{4.95}$$

Proof. Denote by $\mathcal{G}$ the graphical representation. Let

$$H_t^Q = \big\{x \in \mathbb{Z}\colon\ \exists \text{ path in } \mathcal{G} \text{ from } (x,0) \text{ to } Q \times [0,t]\big\}. \tag{4.96}$$

Note that $H_0^Q = Q$ and that $t \mapsto H_t^Q$ is non-decreasing. Denote by $\mathcal{P}$ and $\mathcal{E}$, respectively, probability and expectation w.r.t. $\mathcal{G}$. Let $V_0 = \{x \in \mathbb{Z}\colon\ \xi_0(x) = 0\}$ be the set of initial locations of the vacancies. Then

$$P^{\nu_\rho}\big(\xi_s(x) = 0\ \forall\, x \in Q\ \forall\, s \in [0,t]\big) = (\mathcal{P} \otimes \nu_\rho)\big(H_t^Q \subset V_0\big). \tag{4.97}$$

Indeed, if $\xi_0(x) = 1$ for some $x \in H_t^Q$, then this 1 will propagate into Q prior to time t (see Fig. 4.6).

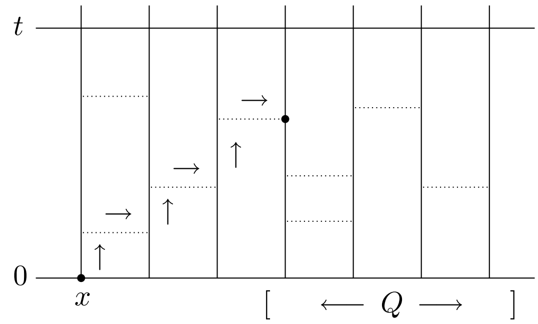

FIGURE 4.6: A path from $(x,0)$ to $Q \times [0,t]$.

By Jensen's inequality,

$$(\mathcal{P} \otimes \nu_\rho)\big(H_t^Q \subset V_0\big) = \mathcal{E}\big((1-\rho)^{|H_t^Q|}\big) \geq (1-\rho)^{\mathcal{E}(|H_t^Q|)}. \tag{4.98}$$

Moreover, since $H_t^Q = \cup_{x \in Q} H_t^x$ and $\mathcal{E}(|H_t^x|)$ does not depend on x, we have

$$\mathcal{E}(|H_t^Q|) \leq |Q|\, \mathcal{E}(|H_t^0|), \tag{4.99}$$

and, by time reversal, we see that

$$\begin{aligned}\mathcal{E}(|H_t^0|) &= \sum_{x \in \mathbb{Z}} \mathcal{P}\big(\exists \text{ path in } \mathcal{G} \text{ from } (x,0) \text{ to } \{0\} \times [0,t]\big) \\ &= \sum_{x \in \mathbb{Z}} \mathbb{P}_0^{\mathrm{SRW}}(\tau_x \leq t) = \mathbb{E}_0^{\mathrm{SRW}}(|R_t|),\end{aligned} \tag{4.100}$$

where $\mathbb{P}_0^{\mathrm{SRW}}$ is the law of simple symmetric random walk jumping at rate 1 starting from 0, R_t is the range (= number of distinct sites visited) at time t and τ_x is the first hitting time of x. Combining (4.97)–(4.100), we get

$$P^{\nu_\rho}\big(\xi_s(x) = 0\ \forall\, x \in Q\ \forall\, s \in [0,t]\big) \geq (1-\rho)^{|Q|\, \mathbb{E}_0^{\mathrm{SRW}}(|R_t|)}. \tag{4.101}$$

The claim now follows from the fact that $R_0 = 1$ and $\mathbb{E}_0^{\mathrm{SRW}}(|R_t|) \sim C'\sqrt{t}$ as $t \to \infty$ for some $C' > 0$ (see [82], Section 1). ∎

Lemma 4.16. *There exist $C = C(\rho) > 0$ and $\delta > 0$ such that, for all intervals $Q, Q' \subset \mathbb{Z}$ separated by a distance at least $2\sqrt{t \log t}$ and all $t \geq 1$,*

$$P^{\nu_\rho}\big\{\xi_s(x) = 1,\, \xi_s(y) = 0 \;\forall\, x \in Q \;\forall\, y \in Q' \;\forall\, s \in [0,t]\big\} \geq \delta\, \mathrm{e}^{-C(|Q|+|Q'|)\sqrt{t}}. \tag{4.102}$$

Proof. Recall (4.96) and abbreviate $A_t = \{H_t^Q \cap H_t^{Q'} = \emptyset\}$. Similarly as in (4.97)–(4.98), we have

$$\text{l.h.s.}(4.102) = (\mathcal{P} \otimes \nu_\rho)(A_t) = \mathcal{E}\big(1_{A_t}\, \rho^{|H_t^Q|}\, (1-\rho)^{|H_t^{Q'}|}\big). \tag{4.103}$$

Both $|H_t^Q|$ and $|H_t^{Q'}|$ are non-decreasing in the number of arrows in $\mathcal{G}$, while 1_{A_t} is non-increasing in the number of arrows in $\mathcal{G}$. Therefore, by the FKG-inequality ([63], Chapter II), we have

$$\mathcal{E}\big(1_{A_t}\, \rho^{|H_t^Q|}(1-\rho)^{|H_t^{Q'}|}\big) \geq \mathcal{P}(A_t)\, \mathcal{E}\big(\rho^{|H_t^Q|}\big)\, \mathcal{E}\big((1-\rho)^{|H_t^{Q'}|}\big). \tag{4.104}$$

We saw in the proof of Lemma 4.15 that, for $t \geq 1$ and some $C > 0$,

$$\mathcal{E}\big(\rho^{|H_t^Q|}\big)\, \mathcal{E}\big((1-\rho)^{|H_t^{Q'}|}\big) \geq \mathrm{e}^{-C(|Q|+|Q'|)\sqrt{t}}. \tag{4.105}$$

Thus, to complete the proof it suffices to show that there exists a $\delta > 0$ such that

$$\mathcal{P}(A_t) \geq \delta \text{ for } t \geq 1. \tag{4.106}$$

To that end, let $q = \max\{x \in Q\}$, $q' = \min\{x' \in Q'\}$ (where without loss of generality we assume that Q lies to the left of Q'). Then, using that Q, Q' are intervals, we may estimate (see Fig. 4.6)

$$\begin{aligned}
\mathcal{P}([A_t]^c) &= \mathcal{P}\big(\exists\, z \in \mathbb{Z}\colon\ (z,0) \to \partial Q \times [0,t],\, (z,0) \to \partial Q' \times [0,t]\big) \\
&\leq \sum_{\substack{x \in \partial Q \\ x' \in \partial Q'}} \int_0^t \big[\mathcal{P}\big(\exists\, z \in \mathbb{Z}\colon\ (z,0) \to x \times [s, s+\mathrm{d}s],\, (x,s) \to x' \times [s,t]\big) \\
&\qquad + \mathcal{P}\big(\exists\, z \in \mathbb{Z}\colon\ (z,0) \to x' \times [s, s+ds],\, (x',s) \to x \times [s,t]\big)\big] \\
&= \sum_{\substack{x \in \partial Q \\ x' \in \partial Q'}} \int_0^t \big[\mathcal{P}\big(\exists\, z \in \mathbb{Z}\colon\ (z,0) \to x \times [s, s+\mathrm{d}s]\big)\, \mathcal{P}\big((x,s) \to x' \times [s,t]\big) \\
&\qquad + \mathcal{P}\big(\exists\, z \in \mathbb{Z}\colon\ (z,0) \to x' \times [s, s+\mathrm{d}s]\big)\, \mathcal{P}\big((x',s) \to x \times [s,t]\big)\big] \\
&\leq 4 \int_0^t \mathcal{P}\big(\exists\, z \in \mathbb{Z}\colon (z,0) \to 0 \times [s, s+\mathrm{d}s]\big)\, \mathcal{P}\big((0,0) \to q'-q \times [0, t-s]\big) \\
&\leq 4\, \mathbb{E}_0^{\mathrm{SRW}}(|R_t|)\, \mathbb{P}_0^{\mathrm{SRW}}(\tau_{q'-q} \leq t),
\end{aligned} \tag{4.107}$$

where the last inequality uses (4.100). We already saw that $\mathbb{E}_0^{\mathrm{SRW}}(|R_t|) \sim C'\sqrt{t}$ as $t \to \infty$. By using, respectively, the reflection principle, the fact that $q' - q \geq 2\sqrt{t \log t}$, and the Azuma-Hoeffding inequality (see [96], (E14.2)), we get

$$\begin{aligned} \mathbb{P}_0^{\mathrm{SRW}}(\tau_{q'-q} \leq t) &= 2\mathbb{P}_0^{\mathrm{SRW}}(S_t \geq q' - q) \leq 2\mathbb{P}_0^{\mathrm{SRW}}(S_t \geq 2\sqrt{t \log t}) \\ &\leq 2 \exp\Big\{-\frac{4t \log t}{2t}\Big\} = \frac{2}{t^2}. \end{aligned} \tag{4.108}$$

Combining (4.107)–(4.108), we get $\mathcal{P}([A_t]^c) \leq 2C'/t^{3/2}$, which tends to zero as $t \to \infty$. This proves the claim in (4.106), because $\mathcal{P}(A_t) > 0$ for all $t \geq 0$. ∎

4.4.2 Slow-down

We are now ready to prove Theorem 4.4. The proof comes in two lemmas.

Lemma 4.17. *For all $\rho \in (0,1)$ and $C > 1/\log(\alpha/\beta)$,*

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\nu_\rho,0}(X_t \leq C \log t) = 0, \\ \lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\nu_\rho,0}(X_t \geq -C \log t) = 0. \end{aligned} \tag{4.109}$$

Proof. To prove the first half of (4.109), the idea is to force ξ to vacate an interval of length $C \log t$ to the right of 0 up to time t and to show that, with probability tending to 1 as $t \to \infty$, X does not manage to cross this interval up to time t when C is large enough.

For $t > 0$, let $L_t = C \log t$ and

$$E_t = \big\{\xi_s(x) = 0 \; \forall\, x \in [0, L_t] \cap \mathbb{Z} \; \forall\, s \in [0,t]\big\}. \tag{4.110}$$

By Lemma 4.15 we have, for some $C' > 0$ and t large enough,

$$P^{\nu_\rho}(E_t) \geq \mathrm{e}^{-C'\sqrt{t}\log t}. \tag{4.111}$$

Hence

$$\begin{aligned} \mathbb{P}_{\nu_\rho,0}(X_t \leq L_t) &\geq \mathbb{P}_{\nu_\rho,0}(X_t \leq L_t \mid E_t)\, P^{\nu_\rho}(E_t) \\ &\geq \mathbb{P}_{\nu_\rho,0}(X_t \leq L_t \mid E_t)\, \mathrm{e}^{-C'\sqrt{t}\log t}. \end{aligned} \tag{4.112}$$

To complete the proof it therefore suffices to show that

$$\lim_{t\to\infty} \mathbb{P}_{\nu_\rho,0}(X_t \leq L_t \mid E_t) = 1. \tag{4.113}$$

Let $\tau_{L_t} = \inf\{t \geq 0\colon\, X_t > L_t\}$. Then $\{X_t \leq L_t \mid E_t\} \supset \{\tau_{L_t} > t \mid E_t\}$, and so it suffices to show that

$$\lim_{t\to\infty} \mathbb{P}_{\nu_\rho,0}(\tau_{L_t} > t \mid E_t) = 1. \tag{4.114}$$

We say that X starts a *trial* when it enters the interval $[0, L_t] \cap \mathbb{Z}$ from the left prior. We say that the trial is successful when X hits L_t before returning to 0. Let $M(t)$ be the number of trials prior to time t, and let A_n be the event that the n-th trial is successful. Since

$$\{\tau_{L_t} \leq t\} \subset \bigcup_{n=1}^{M(t)} A_n, \tag{4.115}$$

we have

$$\begin{aligned}
\mathbb{P}_{\nu_\rho,0}\left(\tau_{L_t} \leq t \mid E_t\right) &\leq \mathbb{P}_{\nu_\rho,0}\left(\bigcup_{n=1}^{M(t)} A_n \,\middle|\, E_t\right)\\
&\leq \mathbb{P}_{\nu_\rho,0}\left(\bigcup_{n=1}^{2(\alpha+\beta)t} A_n,\, M(t) \leq 2(\alpha+\beta)t \,\middle|\, E_t\right)\\
&\quad + \mathbb{P}_{\nu_\rho,0}\left(M(t) > 2(\alpha+\beta)t \mid E_t\right).
\end{aligned} \tag{4.116}$$

We will show that both terms in the r.h.s. tend to zero as $t \to \infty$.

To estimate the second term in (4.116), let $N(t)$ be the number of jumps by X prior to time t, which is Poisson distributed with mean $(\alpha+\beta)t$ and is independent of ξ. Since $N(t) \geq M(t)$, it follows that

$$\mathbb{P}_{\nu_\rho,0}\big(M(t) > 2(\alpha+\beta)t \mid E_t\big) \leq \mathrm{Poi}\big(N(t) > 2(\alpha+\beta)t\big), \tag{4.117}$$

which tends to zero as $t \to \infty$. To estimate the first term in (4.116), note that, since $\mathbb{P}_{\nu_\rho,0}(A_n \mid E_t)$ is independent of n, we have

$$\begin{aligned}
&\mathbb{P}_{\nu_\rho,0}\left(\bigcup_{n=1}^{2(\alpha+\beta)t} A_n,\, M(t) \leq 2(\alpha+\beta)t \,\middle|\, E_t\right)\\
&\leq \mathbb{P}_{\nu_\rho,0}\left(\bigcup_{n=1}^{2(\alpha+\beta)t} A_n \,\middle|\, E_t\right) \leq 2(\alpha+\beta)t\, \mathbb{P}_{\nu_\rho,0}\left(A_1 \mid E_t\right).
\end{aligned} \tag{4.118}$$

But $\mathbb{P}_{\nu_\rho,0}\big(A_1 \mid E_t\big)$ is the probability that the random walk on $\mathbb{Z}$ that jumps to the right with probability $\beta/(\alpha+\beta)$ and to the left with probability $\alpha/(\alpha+\beta)$ hits L_t before 0 when it starts from 1. Consequently,

$$2(\alpha+\beta)t\, \mathbb{P}_{\nu_\rho,0}\big(A_1 \mid E_t\big) = 2(\alpha+\beta)t\, \frac{(\alpha/\beta)-1}{(\alpha/\beta)^{L_t}-1}, \tag{4.119}$$

which tends to zero as $t \to \infty$ when $L_t > C \log t$ with $C > 1/\log(\alpha/\beta)$. This completes the proof of the first half of (4.109).

To get the second half of (4.109), note that $-X_t$ is equal in distribution to X_t when ρ is replaced by $1-\rho$. ∎

Lemma 4.18. *For all* $\rho \in (0,1)$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\nu_\rho,0}(|X_t| \le 2\sqrt{t \log t}) = 0. \tag{4.120}$$

Proof. The idea is to create a trap around 0 by forcing ξ up to time t to vacate an interval to the right of 0 and occupy an interval to the left of 0, separated by a suitable distance.

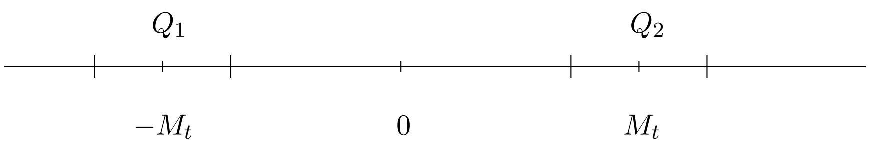

FIGURE 4.7: Location of the intervals Q_1 and Q_2. The width of Q_1, Q_2 is $2L_t$. The interval spanning Q_1, Q_2 and the space in between is I_t.

For $t > 0$, let $L_t = C \log t$ with $C > \log(\alpha/\beta)$, $M_t = \sqrt{t \log t}$,

$$Q_1 = \big(-M_t + [-L_t, L_t]\big) \cap \mathbb{Z}, \qquad Q_2 = \big(M_t + [-L_t, L_t]\big) \cap \mathbb{Z}, \tag{4.121}$$

and $I_t = [-M_t - L_t, M_t + L_t] \cap \mathbb{Z}$ (see Fig. 4.7). For $i = 1, 2$ and $j = 0, 1$, define the event

$$E_i^j = \big\{\xi_s(x) = j \ \forall\, x \in Q_i,\ \forall\, s \in [0,t]\big\}. \tag{4.122}$$

Estimate, noting that $L_t \le M_t$ for t large enough,

$$\begin{aligned} &\mathbb{P}_{\nu_\rho,0}\big(|X_t| \le 2M_t\big) \ge \mathbb{P}_{\nu_\rho,0}\big(X_t \in I_t\big) \\ &\quad \ge \mathbb{P}_{\nu_\rho,0}\big(X_t \in I_t,\ E_1^1, E_2^0\big) = \mathbb{P}_{\nu_\rho,0}\big(X_t \in I_t \mid E_1^1, E_2^0\big)\, \mathbb{P}_{\nu_\rho,0}\big(E_1^1, E_2^0\big). \end{aligned} \tag{4.123}$$

Since $\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\nu_\rho,0}\big(E_1^1, E_2^0\big) = 0$ by Lemma 4.16, it suffices to show that

$$\lim_{t\to\infty} \mathbb{P}_{\nu_\rho,0}\big(X_t \in I_t \mid E_1^1, E_2^0\big) = 1. \tag{4.124}$$

To that end, estimate

$$\begin{aligned} \mathbb{P}_{\nu_\rho,0}\big(X_t \in I_t \mid E_1^1, E_2^0\big) &\ge \mathbb{P}_{\nu_\rho,0}\big(X_t \le M_t + L_t \mid E_1^1, E_2^0\big) \\ &\quad + \mathbb{P}_{\nu_\rho,0}\big(X_t \ge -M_t - L_t \mid E_1^1, E_2^0\big) - 1. \end{aligned} \tag{4.125}$$

Now, irrespective of what ξ does in between Q_1 and Q_2 up to time t, the same argument as in the proof of Lemma 4.17 shows that

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{P}_{\nu_\rho,0}\big(X_t \leq M_t + L_t \mid E_1^1, E_2^0\big) = 1,\\
&\lim_{t\to\infty} \mathbb{P}_{\nu_\rho,0}\big(X_t \geq -M_t - L_t \mid E_1^1, E_2^0\big) = 1.
\end{aligned} \tag{4.126}$$

Combine this with (4.125) to obtain (4.124). ∎

Chapter 5

Law of large numbers for one-dimensional transient RW on the exclusion process

5.1 Introduction and result

In this chapter we present some results from an ongoing project with R.S. dos Santos and F. Völlering.

5.1.1 Slow-mixing REs and the exclusion process

In Chapter 2 we derived a LLN for the RW in (2.6) when the dynamic RE has the cone-mixing property in Definition 2.1. In particular, Theorem 2.2 holds for the more general model in Section 1.3.2 in which the RW X has two different (not only opposite) drifts $\alpha_0 - \beta_0$ and $\alpha_1 - \beta_1$ on top of holes and particles, respectively. The weak point of Theorem 2.2 is that many natural and interesting examples of dynamic REs are not cone-mixing, e.g., conservative dynamics like the exclusion process or, more generally, Kawasaki dynamics.

It is worthwhile to investigate examples of slow-mixing REs, because significantly different behavior may occur compared to fast-mixing REs, such as cone-mixing REs. Indeed, in Chapter 4 we have already met the case of a RW X on the one-dimensional simple symmetric exclusion (SSE) with opposite drifts on top of particles and holes (i.e., $\alpha_1 - \beta_1 = \beta_0 - \alpha_0$). In particular, in Section 4.1.4 we presented the results of some simulations for the asymptotic speed of X, which suggest that X is recurrent if and only if $\rho = \frac{1}{2}$, and that X is ballistic as soon as it is transient. Thus, the transient regime with zero speed, which is known to occur for static REs (see Section 1.1.1.1), does not survive in the dynamic setup, because even a 'slow' motion of the particles in the RE makes it hard for a 'trap' to survive. Nevertheless, similarly to the one-dimensional static RE and in contrast to the fast-mixing dynamic RE, Proposition 4.4 shows that when we look at large deviation estimates for the empirical speed of X, the slow-mixing properties of the exclusion process allow for a 'trap' to persist up to time t with a probability that is decaying sub-exponentially in t. Furthermore, similarly to the static RE (see Section 1.1.1.2), we may expect a sub-diffusive scaling limit for X to occur at least in the recurrent case, i.e., for $\rho = \frac{1}{2}$.

These results and observations motivate the interest in slow-mixing REs. In this chapter we prove a LLN under a somewhat strong drift condition, which represents a small step forward. At the end we mention some further extensions that are still part of a work in progress.

5.1.2 Model and main theorem

Consider a dynamic RE ξ constituted by a SSE (see Section 4.1.3) starting from a Bernoulli product measure ν_ρ of density ρ. Let

$$X = (X)_{t\geq 0} \tag{5.1}$$

be the RW in dynamic RE defined in Section 1.3.2, under the the following drift conditions:

$$\alpha_1 > \alpha_0 > \beta_0 > \beta_1 > 0, \quad \alpha_1 + \beta_1 = \alpha_0 + \beta_0, \quad \alpha_0 - \beta_0 > 1. \tag{5.2}$$

Note that the jump rate of the SSE equals 1 and that the latter condition implies

$$\liminf_{t\to\infty} X_t/t \geq \alpha_0 - \beta_0 > 1 \qquad \mathbb{P}_{\nu_\rho,0} - a.s. \tag{5.3}$$

Theorem 5.1. *Assume (5.2). Then, for any $\rho \in (0,1)$, there exists a constant $v > 1$ such that*

$$\lim_{t\to\infty} X_t/t = v \qquad \mathbb{P}_{\nu_\rho,0} - a.s. \tag{5.4}$$

5.2 Proof of Theorem 5.1

The main idea in the proof is that, under the third condition in (5.2), X travels to the right faster than the 'information' in the RE. As a consequence, it is possible to construct certain regeneration times at which the RE to the right of X is *freshly* sampled from its equilibrium distribution.

5.2.1 Coupling and minimal walker

In this section we show that the RW X defined in (5.1) can be constructed from an independent homogeneous RW and the RE. In particular, the following construction is valid for any general dynamic RE constituted by an IPS $\xi = (\xi_t)_{t\geq 0}$.

Let $M = (M_t)_{t\geq 0}$ be a homogeneous continuous-time RW with jump rates α_0 and β_0, to the right and to the left, respectively. Let $(b_n)_{n\in\mathbb{N}}$ an i.i.d. sequence of Bernoulli random variables with parameter $(\alpha_1 - \alpha_0)/\beta_0$. The path of the RW X in (5.1) can be constructed as a function of

$$(M, (b_n)_{n\in\mathbb{N}}, \xi) \tag{5.5}$$

by using the following rules:

1. $M_0 = X_0 = 0$.
2. X jumps only when M jumps.
3. If M_t jumps to the right at time t, then so does X_t.
4. If M_t jumps to the left at time t and X_t is on top of a hole, i.e., $\xi_t(X_t) = 0$, then X_t jumps to the left too.
5. If M_t jumps to the left at time t and X is on top of a particle, i.e., $\xi_t(X_t) = 1$, then X_t jumps to the right when an independent Bernoulli trial with parameter $(\alpha_1 - \alpha_0)/\beta_0$ succeeds, and jumps to the left otherwise.

Denote by

$$\left(\widetilde{P}, \Gamma, \mathcal{F}_t\right) \tag{5.6}$$

the probability space associated with (5.5), with

$$\mathcal{F}_t = \sigma\left(\{M_s\}_{s\leq t}, \{b_n\}_{n\leq m_t}, \{\xi_s\}_{s\leq t}\right), \tag{5.7}$$

where m_t is the number of jumps of M up to time t, which is distributed according to a Poisson random variable with parameter $(\alpha_0 + \beta_0)t$.

By construction, for any $t \geq 0$,

$$M_t \leq X_t \qquad \widetilde{P} - a.s. \tag{5.8}$$

We are therefore justified to call M the *minimal walker*.

5.2.2 Graphical representation: symmetric exclusion as an interchange process

The interchange process $\gamma = (\gamma_t)_{t\geq 0}$ on $\mathbb{Z}$ is a process, taking values on the permutations of $\mathbb{Z}$, that can be defined through a graphical representation as follows. Start with a permutation γ_0. We call the state of the coordinates of γ 'agents'. We take γ_0 to be the identity, i.e., the agents are $(\ldots, -2, -1, 0, 1, 2, \ldots)$. Associate to each non-directed nearest-neighbor edge $(x, x+1)$ in $\mathbb{Z}$ an independent Poisson clock $I^{x,x+1} = (I_t^{x,x+1})_{t\geq 0}$ ticking at rate 1. Denote by

$$I = \{I^{x,x+1} \colon\ x \in \mathbb{Z}\} \tag{5.9}$$

the set of all those clocks. Then γ_t is obtained from γ_0 by exchanging the labels of x and $x+1$ each time the Poisson clock $I^{x,x+1}$ rings. In particular, $\gamma_t(x) \in \mathbb{Z}$ represents the starting position of the agent who at time t is at site x.

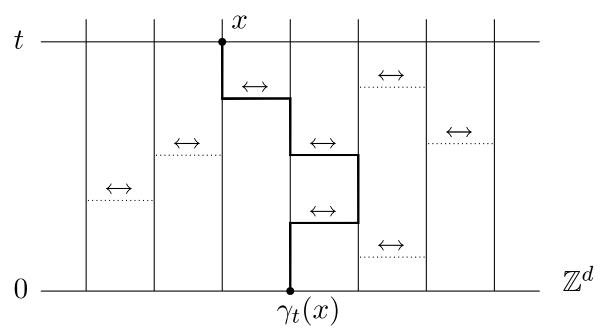

FIGURE 5.1: Graphical representation. The dashed lines are the links given by the realization of I. The thick line represents the path of the agent $\gamma_t(x)$.

Given the interchange process γ, the simple symmetric exclusion process (SSE) (see Section 4.1.3) $\xi = (\xi_t)_{t\geq 0}$ on $\mathbb{Z}$ starting from a configuration $\eta \in \Omega = \{0,1\}^{\mathbb{Z}}$ can be obtained from γ by putting $\xi_t(x) = \eta(\gamma_t(x))$.

The interpretation is that, in the interchange process, the agents move around in the lattice by exchanging their places with their nearest neighbors. For exclusion, we choose one of two states for these agents at the start (1 or 0, which we refer to as 'particle' and 'hole') and assign the state of a site at a later time as the initial state of the agent who is there at this time.

Next, recall (5.5). By the coupling with the *minimal walker* M of the previous section, we have that, for any starting configuration $\eta \in \Omega$, X is a function of

$$(M, (b_n)_{n\in\mathbb{N}}, I) \text{ and } \eta, \tag{5.10}$$

where, in the coupling space (5.6), $\mathcal{F}_t$ is given by

$$\mathcal{F}_t = \sigma\left(\{M_s\}_{s\le t}, \{b_n\}_{n\le m_t}, \{I_s\}_{s\le t}\right). \tag{5.11}$$

In particular, if we consider $\zeta, \eta \in \Omega$ such that $\zeta \succeq \eta$ (where $\succeq$ denotes the partial order on Ω), then for any $t \ge 0$ we have by construction

$$M_t \le X_t(\eta) \le X_t(\zeta) \qquad \widetilde{P} - a.s., \tag{5.12}$$

where $X_t(\eta)$ and $X_t(\zeta)$ represent the RW starting from η and ζ, respectively.

5.2.3 Marked agents set

As the RW X moves, it will meet the agents of the interchange process. Sometimes, due to the coupling with the *minimal walker*, it will not need to know their state in order to proceed, i.e., when the *minimal walker* M goes to the right. If M goes to the left, then X will have to 'ask' the agent at its current position what is its state to know how to move. We say that at this time X and the agent 'meet', and we call an agent *marked* at time t if it has met X at some time $s \le t$. For any $t \ge 0$, we can define A_t to be the set of *marked* agents up to time t. For reasons that will become clear at the end of this section, we add to this *marked* agents set all the agents $x \le 0$.

Formally, define $A_0 = \{x \in \mathbb{Z}\colon\ x \le 0\}$, let t be a time at which M_t jumps to the left, and put

$$A_t = A_{t^-} \cup \{\gamma_t(X_t)\}. \tag{5.13}$$

Next, let

$$U_1 = \inf\{t > 0\colon\ M_t \ne 0\} = \inf\{t > 0\colon\ X_t \ne 0\} \tag{5.14}$$

and define

$$\tau_0 = \inf\Big\{t \ge U_1\colon\ X_t > \max\{x \in \mathbb{Z}\colon\ \gamma_t(x) \in A_t\}\Big\}, \tag{5.15}$$

i.e., the first time such that all the sites with *marked* agents are to the left of X_t.

Lemma 5.2. *Let τ_0 be as in (5.15) and denote by $\widetilde{E}$ the expectation w.r.t. $\widetilde{P}$. Then*

$$\widetilde{E}[\tau_0^2] < \infty.$$

Proof. Let

$$Y = (Y_t)_{t\ge 0} \tag{5.16}$$

be a path starting from 0 that jumps to the right according to the realization of the process I in (5.9), see Figure 5.2.

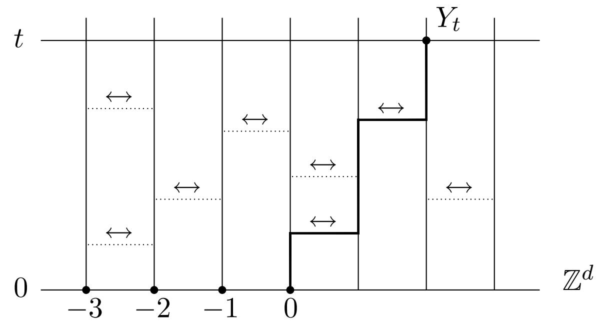

FIGURE 5.2: As in Figure 5.1, the dashed lines are links given by the realization of I. The path Y starts at the origin and goes only to the right following the links determined by I.

Then Y is distributed according to a Poisson process with rate 1.

Denote by $\gamma^{-1}(x) = \left(\gamma_t^{-1}(x)\right)_{t\geq 0}$ the path of the agent x. By construction, for any $x \leq 0$ and $t \geq 0$, $\gamma_t^{-1}(x) \leq Y_t$. Furthermore, let $S_1 = \inf\{t > 0\colon\ M_t - Y_t > 0\}$, and note that

$$\tau_0 \leq S_1. \tag{5.17}$$

Recalling that the *minimal walker* M is independent of the RE, while Y is a function of the RE, we have that $Z = (Z_t)_{t\geq 0}$ with $Z_t = M_t - Y_t$ is a continuous-time homogeneous RW, starting from the origin, that jumps to the right at rate α_0 and to the left at rate $\beta_0 + 1$. Since $\alpha_0 - \beta_0 > 1$ by the third condition in (5.2), Z is transient to the right with positive speed $\alpha_0 - \beta_0 - 1 > 0$. Thus, $\mathbb{E}[S_1^2] < \infty$, and the claim follows from (5.17). ■

The crucial point, which we state in the next proposition, is that if we start from a configuration $\eta \in \Omega$ sampled from ν_ρ to the right of the origin, then, no matter what is η to the left of the origin, the RW X at time τ_0 will still see to its right a configuration that is *freshly* sampled from ν_ρ. Such a fact is related to the nature of the SSE and its construction from the interchange process, and it is the main ingredient for the proof of the LLN.

Let $\mathbb{Z}_{>0} = \{x \in \mathbb{Z}\colon\ x > 0\}$, and put $\mathbb{Z}_{\leq 0} = \mathbb{Z} \setminus \mathbb{Z}_{>0}$. Given $\zeta \in \{0,1\}^{\mathbb{Z}_{\leq 0}}$, let $\nu_\rho^{(\zeta)}$ be the product measure of single site measures on Ω given by

$$\begin{aligned} &\nu_\rho^{(\zeta)}\left(\eta(x) = \zeta(x)\right) = 1, \text{ if } x \in \mathbb{Z}_{\leq 0}, \\ &\nu_\rho^{(\zeta)}\left(\eta(x) = 1\right) = \rho, \text{ otherwise,} \end{aligned} \tag{5.18}$$

i.e., $\nu_\rho^{(\zeta)}$ coincides with ν_ρ on $\{0,1\}^{\mathbb{Z}_{>0}}$, and is the delta measure δ_ζ on $\{0,1\}^{\mathbb{Z}_{\leq 0}}$.

Proposition 5.3. *For any $\zeta \in \{0,1\}^{\mathbb{Z}_{\leq 0}}$, let ξ be the SSE starting from $\nu_\rho^{(\zeta)}$, and denote by $\mathbb{P}_{\nu_\rho^{(\zeta)},0}$ the law $\widetilde{P}$ when the starting configuration η is sampled from $\nu_\rho^{(\zeta)}$. Then, for any finite $B \subset \{0,1\}^{\mathbb{Z}_{>0}}$,*

$$\mathbb{P}_{\nu_\rho^{(\zeta)},0}\Big(\xi_{\tau_0}(X_{\tau_0} + \cdot) \in B \mid (X_t)_{t \leq \tau_0}\Big) = \nu_\rho(B), \tag{5.19}$$

i.e., the SSE at time τ_0 to the right of X_{τ_0} is independent of $(X_t)_{t\leq\tau_0}$, and is distributed according to ν_ρ.

Proof. $(X_t)_{t\leq\tau_0}$ is a function of $(M_t)_{t\leq\tau_0}$, $\{b_n\}_{n\leq m_{\tau_0}}$, $(I_t)_{t\leq\tau_0}$ (see (5.10)), and the state of the agents belonging to A_{τ_0}. Therefore $(X_t)_{t\leq\tau_0}$ is independent of $\{\xi_0(x)\colon\ x \in \mathbb{Z}\backslash A_{\tau_0}\}$. By the definition of τ_0, $\gamma_{\tau_0}(x) \in \mathbb{Z}_{>0}\backslash A_{\tau_0}$, for all $x > X_{\tau_0}$. Therefore, since $\nu_\rho^{(\zeta)}$ coincides with ν_ρ on $\{0,1\}^{\mathbb{Z}_{>0}}$, it follows that $\xi_{\tau_0}(x)$ is a Bernoulli random variable with parameter ρ for $x > X_{\tau_0}$. ∎

5.2.4 Right walker and a sub-additivity argument

Denote by $\underline{1} \in \{0,1\}^{\mathbb{Z}_{\leq 0}}$ the configuration with all coordinates equal to 1. Let

$$R = (R_t)_{t\geq 0} \tag{5.20}$$

be the RW X starting from $\nu_\rho^{(\underline{1})}$. For any $\zeta \in \{0,1\}^{\mathbb{Z}_{\leq 0}}$, if we denote by $X(\zeta)$ a RW starting from $\nu_\rho^{(\zeta)}$, then, as a consequence of (5.12), for any $t \geq 0$ we have that

$$M_t \leq X_t(\zeta) \leq R_t \qquad \widetilde{P} - a.s. \tag{5.21}$$

We call R the *right walker*. We anticipate that in the sequel we first prove that R satisfies a LLN, and then Theorem 5.1 follows by showing that the limiting speed of the *right walker* does not depend on the configuration $\underline{1}$.

We next construct a renewal structure in the coupling space (5.6). The idea of this construction is that, starting from R and from the τ_0 associated to R, we have that, by Proposition 5.3, at time τ_0 the states of the SSE ξ to the right of $R_{\tau_0}^{(0)}$ are distributed according to ν_ρ. At time τ_0 we define a new configuration $\eta^{(1)}$ of the SSE from ξ_{τ_0}, by replacing all its states to the left of R_{τ_0} by 1 (i.e., put $\xi_{\tau_0}(x) = 1$ for $x \leq R_{\tau_0}$), and we define $R^{(1)}$ to be the RW evolving as X in Section 5.2.1 starting at time τ_0 at position R_{τ_0} from this new configuration of the SSE. In particular, such $R^{(1)}$ has the following properties:

1. $R^{(1)}$ is a function of $\big(\{M_t\}_{t\geq\tau_0}, \{b_n\}_{n\geq m_{\tau_0}}, \{I\}_{t\geq\tau_0}\big)$ and $\eta^{(1)}$.

2. By (5.12), $R^{(1)}$ is coupled to R in such a way that $R_{t+\tau_0} - R_{\tau_0} \leq R_t^{(1)}$ for $t \geq 0$.
3. R and $\left(R_t^{(1)} - R_0^{(1)}\right)_{t\geq 0}$ have the same distribution.

We can then repeat the same argument to construct a new RW $R^{(n)}$ from $R^{(n-1)}$ for any $n \in \mathbb{N}$.

More precisely, let $\eta^{(0)} \in \Omega$ be a configuration sampled from $\nu_\rho^{(1)}$, set $R^{(0)} = R$, and construct inductively the random vector-sequence

$$\left\{\left(\eta^{(n)}, R^{(n)}, \tau_n\right)\right\}_{n\in\mathbb{N}}, \quad R^{(n)} = \left(R_t^{(n)}\right)_{t\geq 0}, \tag{5.22}$$

as follows. For $n \in \mathbb{N}$, let $\eta^{(n)} \in \Omega$ given by

$$\eta^{(n)}(x) = \begin{cases} \eta^{(n-1)}\left(\gamma_{\tau_{n-1}}(x)\right), & \text{if } x > R_{\tau_{n-1}}^{(n-1)}, \\ 1, & \text{otherwise.} \end{cases} \tag{5.23}$$

For $t \geq \tau_{n-1}$, let $R^{(n)} = \left(R_t^{(n)}\right)_{t\geq 0}$ be the RW evolving according to the rules given for X in Section 5.2.1, starting from $R_{\tau_{n-1}}^{(n-1)}$ with initial states of the RE given by $\eta^{(n)}$.

Let $A_t^{(n)}$ be the marked agents set constructed from $R_t^{(n)}$ as in (5.13), namely, set $A_0^{(n)} = \left\{x \in \mathbb{Z}\colon\ x \leq R_0^{(n)} = R_{\tau_{n-1}}^{(n-1)}\right\}$, let t be a time at which $M_{\tau_{n-1}+t}$ jumps to the left, and put

$$A_t^{(n)} = A_{t^-}^{(n)} \cup \left\{\gamma_t\left(R_t^{(n)}\right)\right\}. \tag{5.24}$$

Define

$$\tau_n = \inf\left\{t \geq U_1\colon\ R_t^{(n)} > \max\left\{x \in \mathbb{Z}\colon\ \gamma_t(x) \in A_t^{(n)}\right\}\right\}. \tag{5.25}$$

As a consequence of this construction, it follows from (5.12) that

$$R_{t+\tau_n}^{(n)} - R_{\tau_n}^{(n)} \leq R_t^{(n+1)} \qquad \widetilde{P} - a.s. \tag{5.26}$$

The main advantage is now that, by Proposition 5.3, $\left\{\left(\eta^{(n)}, R^{(n)}, \tau_n\right)\right\}_{n\in\mathbb{N}}$ is a stationary sequence.

Lemma 5.4. *Let $T_n = \sum_{i=1}^n \tau_n$. For integers $0 \leq m < n$, define the double indexed random variables*

$$\bar{R}_{m,n} = R_{T_n - T_m}^{(m)}. \tag{5.27}$$

Then there exists a constant $c(R) \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \frac{\bar{R}_{0,n}}{n} = \lim_{n\to\infty} \frac{R_{T_n}}{n} = c(R) \qquad \widetilde{P} - a.s. \tag{5.28}$$

Proof. The proof relies on the subadditive ergodic theorem of Liggett (see Theorem 1.10 in [63]). By (5.26), for any $0 \leq m < n$, we have

$$\bar{R}_{0,n} = \bar{R}_{0,m} + (\bar{R}_{0,n} - \bar{R}_{0,m}) \leq \bar{R}_{0,m} + \bar{R}_{m,n} \qquad \widetilde{P} - a.s. \tag{5.29}$$

Moreover, by construction and since $\left\{(R^{(n)}, \tau_n)\right\}_{n\in\mathbb{N}}$ is a stationary sequence, for every $n > m$, $\{\bar{R}_{m+k,n+k}\}_{k\in\mathbb{N}_0}$ is a sequence of i.i.d. random variables. Therefore, for each $m \in \mathbb{N}_0$, the joint distribution of $\left\{\bar{R}_{m+1,m+k+1}\right\}_{k\in\mathbb{N}}$ is the same as that of $\left\{\bar{R}_{m,m+k}\right\}_{k\in\mathbb{N}}$. Furthermore, for each $k \in \mathbb{N}$, we have that $\left\{\bar{R}_{nk,(n+1)k}\right\}_{n\in\mathbb{N}}$ is a stationary and ergodic process. Therefore the assumptions of Theorem 1.10 in [63] are satisfied, and the claim follows. ∎

5.2.5 LLN

Lemma 5.5. *There exists a constant $v(R) > 1$ such that*

$$\lim_{t\to\infty} \frac{R_t}{t} = v(R) \qquad \mathbb{P}_{\nu_\rho^{(1)},0} - a.s. \tag{5.30}$$

Proof. For $t \geq 0$, let $n(t)$ be such that

$$T_{n(t)} \leq t < T_{n(t)+1}. \tag{5.31}$$

Denote by $\mathbb{E}_{\nu_\rho^{(1)},0}$ the expectation associated to $\mathbb{P}_{\nu_\rho^{(1)},0}$. By Lemma 5.2, $\mathbb{E}_{\nu_\rho^{(1)},0}[\tau_0] < \infty$. Since $T_{n(t)}/n(t) \to \mathbb{E}_{\nu_\rho^{(1)},0}[\tau_0]$ as $n \to \infty$, dividing by $n(t)$ and taking $t \to \infty$ in (5.31), we have

$$\lim_{t\to\infty} \frac{n(t)}{t} = \frac{1}{\mathbb{E}_{\nu_\rho^{(1)},0}[\tau_0]} \qquad \mathbb{P}_{\nu_\rho^{(1)},0} - a.s. \tag{5.32}$$

By Lemma 5.4 and (5.32), we get

$$\lim_{t\to\infty} \frac{R^{(0)}_{T_{n(t)}}}{t} = \lim_{t\to\infty} \frac{R_{T_{n(t)}}}{n(t)} \frac{n(t)}{t} = \frac{c(R)}{\mathbb{E}_{\nu_\rho^{(1)},0}[\tau_0]} =: v(R). \tag{5.33}$$

Since

$$\frac{R_t}{t} = \frac{|R_t - R_{T_{n(t)}}|}{t} + \frac{R_{T_{n(t)}}}{t}, \tag{5.34}$$

the claim follows by combining (5.33) and (5.34), and observing that

$$\limsup_{t\to\infty} \frac{|R_t - R_{T_{n(t)}}|}{t} = 0 \qquad \mathbb{P}_{\nu_\rho^{(1)},0} - a.s. \tag{5.35}$$

To show (5.35) we argue as follows. Note first that R can be coupled with a Poisson process $N = (N_t)_{t\geq 0}$ of rate $\alpha_0 + \beta_0$ such that

$$R_t \leq N_t \quad \text{for any } t \geq 0. \tag{5.36}$$

In particular, it follows from Lemma 5.2 that $\mathbb{E}_{\nu_\rho^{(1)},0}\left[N_{\tau_0}^2\right] < \infty$, which together with (5.36) ensures that there exists a constant $C \in (0,\infty)$ such that

$$\mathbb{E}_{\nu_\rho^{(1)},0}\left[\max_{t\leq\tau_0} |R_t|^2\right] \leq C. \tag{5.37}$$

By the Markov inequality and (5.37), for any $\epsilon > 0$ we have

$$\begin{aligned}\mathbb{P}_{\nu_\rho^{(1)},0}\Big(|R_t - R_{T_{n(t)}}| \geq \epsilon t\Big) &\leq \mathbb{P}_{\nu_\rho^{(1)},0}\left(\max_{T_{n(t)}\leq t\leq T_{n(t)+1}} |R_t - R_{T_{n(t)}}| \geq \epsilon t\right) \\ &= \mathbb{P}_{\nu_\rho^{(1)},0}\left(\max_{t\leq\tau_0} |R_t| \geq \epsilon t\right) \leq \frac{\mathbb{E}_{\nu_\rho^{(1)},0}\left[\max_{t\leq\tau_0} |R_t|^2\right]}{(\epsilon t)^2} \\ &\leq C(\epsilon t)^{-2}.\end{aligned} \tag{5.38}$$

Finally, (5.35) follows from (5.38) and the Borel-Cantelli lemma. ∎

Next, let $\underline{0} \in \{0,1\}^{\mathbb{Z}_{\leq 0}}$ be the configuration with all coordinates equal to 0. Let

$$L = (L_t)_{t\geq 0} \tag{5.39}$$

be the RW X starting from $\nu_\rho^{(\underline{0})}$.

For any $\zeta \in \{0,1\}^{\mathbb{Z}_{\leq 0}}$, if we denote by $X(\zeta)$ the RW starting from $\nu_\rho^{(\zeta)}$, then, as a consequence of (5.12), for any $t \geq 0$ we have that

$$M_t \leq L_t \leq X_t(\zeta) \leq R_t \qquad \widetilde{P} - a.s. \tag{5.40}$$

Note that by repeating the same argument as in Section 5.2.4 and in the proof of Lemma 5.5 for the *left walker* L, we get that there exists a constant $v(L) > 1$ such that

$$\lim_{t\to\infty} \frac{L_t}{t} = v(L) \qquad \mathbb{P}_{\nu_\rho^{(\underline{0})},0} - a.s.$$

The only difference is that in Lemma 5.5 we obtain super-additivity instead of sub-additivity. Finally, by observing that $v(R) = v(L)$, Theorem 5.1 follows from (5.40).

To see this latter observation we argue as follows. As both L and R are identical if they do not encounter an agent from the left of the origin, the associated speeds are the same on this event. Thus, if this event is not a null-set, then, as the speeds are a.s. constants on the whole space, we obtain $v(R) = v(L)$. To show that the latter event has positive probability, recall the RW Y in (5.16) and observe that the event that L and R do not encounter an agent from the left of the origin includes the event $\{M_t > Y_t, \quad \forall t \geq 0\}$, which has positive probability due to the third drift condition in (5.2).

5.3 Concluding remarks

The assumption that the total jump rates of X are the same on top of particles or holes (i.e., $\alpha_0 + \beta_0 = \alpha_1 + \beta_1$ in (5.2)) is not relevant for the proof and can be easily dropped by constructing a different coupling with the minimal walker in Section 5.2.1, and essentially keeping the rest of the proof unchanged. We made this assumption just to avoid cumbersome notations.

The proof of Theorem 5.1 is simple and uses the specific nature of the SSE. Indeed, we exploited the graphical representation of the SSE, in particular, its construction from the interchange process, to ensure the integrability of the time τ_0 in (5.15) and to ensure that the sequence in (5.22) is stationary.

We are currently working on extensions of Theorem 2.2 for a larger class of dynamic RE under strong drift assumptions as in (5.3), namely, for dynamic RE in which, intuitively, the 'information' travels to the right slower than the minimal drift of X. If we consider other dynamic RE, like e.g. an asymmetric exclusion process or a Poissonian field of independent RWs, then we cannot a priori be sure of the existence of a non-degenerate integrable time at which X observes to its right a RE in equilibrium. A heavier regeneration scheme in the spirit of Chapter 2 seems to be needed. We plan to treat such cases in future works.

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Samenvatting

Gedurende de afgelopen veertig jaar zijn modellen voor “Random Wandelingen in Random Omgevingen” (RWRO) intensief bestudeerd, zowel in de natuurkundige als in de wiskundige gemeenschap. Dit heeft geleid tot een zeer levendig onderzoeksgebied, dat een onderdeel is van het grotere onderzoeksgebied van wanordelijke systemen. RWROs in $\mathbb{Z}^d$ zijn Random Wandelingen (RWs) die evolueren volgens een random overgangsmatrix, d.w.z. hun overgangskansen hangen af van een stochastisch veld of proces ξ op $\mathbb{Z}^d$, genaamd Random Omgeving (RO). Wat deze modellen zo interessant maakt is dat zich, afhankelijk van de RO, verschillende typen verschijnselen kunnen voordoen: sub-diffusief gedrag, sub-exponentieel vervalvan correlaties of van kansen op grote afwijkingen, en trap-effecten. De ROs kunnen worden onderverdeeld in twee hoofdklassen: statisch en dynamisch. In een statische RO wordt ξ willekeurig gekozen op tijstip 0 en wordt vervolgens constant gehouden gedurende de tijdsevolutie van de RW. In een dynamische RO, daarentegen, verandert ξ in de loop van de tijd volgens een van te voren gekozen stochastisch process.

Statische RO’s in 1 dimensie zijn goed begrepen: recurrentie criteria, wetten van grote aantallen, invariantie-principes en schattingen voor grote afwijkingen zijn uitgebreid bestudeerd in een lange reeks van artikelen. Ook in hogere dimensies zijn er vele fraaie resultaten verkregen, maar tegelijk zijn er nog vele open vragen.

Dynamische RO’s zijn, zelfs in 1 dimensie, nog niet zo ver ontwikkeld. In dit proefschrift richten we onze aandacht op een klasse van RWs in dynamische ROs bestaande uit een systeem van deeltjes die onderling met elkaar wisselwerken. De analyse van dit soort modellen leidt niet alleen tot interessante nieuwe resultaten, maar geeft ook aanleiding tot het formuleren van uitdagende open vragen voor de toekomst.

Dit proefschrift heeft de volgende opbouw. In hoofdstuk 1 geven we een samenvatting van de bestaande literatuur voor zowel statische als dynamische ROs. Tevens introduceren we de klasse van modellen waarin we in dit proefschrift geinteresseerd zijn. In hoofdstuk 1 bewijzen we, onder bepaalde ruimte-tijd-mengingsvoorwaarden, een sterke wet van de grote aantallen voor ROs in zowel 1 als meer dimensies. Bovendien leiden we, met behulp van een verstoringsargument, een reeksontwikkeling af, in termen van de grootte van de drift, voor de asymptotische snelheid van RWs met een kleine drift in sterk wanordelijke ROs. Hoofdstuk 3 richt zich op de schalingslimieten van dergelijke processen. Door een bewijs van Comets en Zeitouni [36] voor statische ROs in hogere dimensies aan te passen en te vereenvoudigen, bewijzen we, onder een bepaalde ruimte-tijd-mengingsvoowaarde, een annealed invariantie principe voor iedere dimensie.

Verder geven we een alternatief bewijs voor dit invariantieprincipe in de context van sterk wanordelijke ROs.

Hoofdstuk 4 behandelt grote afwijkingen voor de empirische snelheid van 1-dimensionale RWs in dynamische ROs. We bewijzen een quenched en een annealed grote afwijkingen principe en we leiden een aantal kwalitatieve eigenschappen van de geassocieerde rate-functies af. In het bijzonder geven we voorbeelden van snelle en langzaam mengende ROs, die exponentieel respectievelijk sub-exponentieel gedrag van de grote afwijkingen kansen vertonen. In hoofdstuk 5 bewijzen we een wet van de grote aantallen voor transiente RWs voor een RO een symmetrisch exclusieproces is, en sluiten we af met een korte discussie over mogelijke uitbreidingen naar meer algemene langzaam-mengende ROs. Het laatste maakt deel uit van een nog lopend project.

Acknowledgements

The completion of my Ph.D. could not have been accomplished without the support and the help of many people. First, I would like to thank my supervisor Prof. Dr. Frank den Hollander. Frank was a great mentor and I really enjoyed the time we spent working together. From him I have learned a lot about mathematics, its presentation and life in the academic world. He always found the right words to encourage me when I was stuck in my research and he has been a guide even from a personal point of view.
After Frank another Frank: Prof. Dr. Frank Redig, for whom I repeat the same words used for my supervisor. Moreover, the common love both Franks have for Italy helped with my integration in The Netherlands. I sincerely hope to keep in touch with them.
In these four years I had the pleasure to interact with several mathematicians. A special thanks goes to Prof. Dr. Vladas Sidoravicius for the fruitful discussions and for introducing me to the Brazilian mathematical community.
I am very grateful to Prof. Dr. Fabio Martinelli, Prof. Dr. Elisabetta Scoppola and Dr. Pietro Caputo who instilled in me the passion for the "random world" during my master. They always give me a warm welcome whenever I pass by Rome.
I would also like to thank all the committee members of my Ph.D. defense for their time, remarks and attention.

When I arrived in Leiden, I was a bit sad for being the only Ph.D. student in the new probability group. Now, I am sad to leave a really friendly, cooperative and productive group of Ph.D. students: Alessio, Alex (post-doc), Feija, Florian, Julian, Kiamars, Stefan and Renato. I am sure we will keep in touch professionally and more, I wish the best for all of you.
I would also like to thank all the scientific and supporting staff of the mathematics department in Leiden who contributed in creating a nice and stimulating environment.

My old and new friends, who always gave me the chance to stop thinking about mathematics, have been essential for my survival. The same comments and much more go to my family in Rome, Castelluccio Inferiore and Naples. Last but not least, thanks to Giulia: she has been the one who had to stand and understand me more... I would not be in her shoes.

Sincere thanks to all of you.

Curriculum Vitae

Luca Avena was born in Rome on February 13, 1981. After finishing his high school studies at Liceo Scientifico Statale Plinio Seniore in 2000 in Rome, he started his bachelor programme in mathematics at ROMATRE University. In the meantime he continued his studies of classical guitar started in a private music academy a few years before, and in 2001 he obtained a 3-year Diploma in musical theory at the Conservatorio di Musica Licino Refice (Frosinone, Italy). During his bachelor programme he spent one year in Spain through the Erasmus exchange programme, studying at the mathematics department of the University of Granada. In 2004 he obtained the bachelor degree in Rome and started the master programme, during which he became interested in Probability Theory. In 2006 he graduated (cum laude) at ROMATRE University under the supervision of Prof. Dr. Fabio Martinelli and Dr. Pietro Caputo with the master thesis "On the threshold of the random k-sat". In the fall of 2006 he moved to The Netherlands to start a Ph.D. programme at Leiden University under the supervision of Prof. Dr. Frank den Hollander. His Ph.D. research project has been focusing on models of random walks in dynamic random environments. On November 1, 2010, he will move to Switzerland to work at the University of Zürich as a post-doctoral researcher under the guidance of Prof. Dr. Erwin Bolthausen.