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Robustness of graphs

Bachelor thesis

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Contents

Chapter 1

Introduction

In our world there are many networks, for example railroad networks. It is important that these networks keep functioning, even if parts of it stop working. Therefore, it is important to be able to quantify how resistant a network or graph is to damage. The resistance to damage of a graph is called the *robustness* of a graph. For the most part, we will look at simple graphs. These are undirected graphs without loops or paralel edges. We will study adequate measures for robustness on these graphs. We will also try to find ways to extend the definitions of these measures to directed graphs.

Before we will study some measures for robustness we will first define the properties a good measure for robustness should have. Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. Then, in our opinion, an appropriate measure for robustness r should satisfy the following rules.

$G = (V, E)$ is a simple graph with vertex set V and edge set E.

1. $r(G)$ should be defined for all simple graphs G.

2. For every graph $G = (V, E)$ and edge $e \in E$ we should have $r(G) \geq r(G')$, where $G' = (V, E \backslash e)$. In other words, by adding an edge the robustness of a graph should not decrease.

3. The robustness of a graph G should not contradict out intuitive idea of the robustness.

We study two measures in this project. In Chapter 2 we study the so-called assortativity [3] of a graph. It measures how well vertices with a high degree are connected to other vertices of high degree. However, as explained in Chapter 3, the assortativity is not a good measure for robustness. It satisfies none of the three rules above. In Chapter 4 we define the effective graph resistance. For the effective graph resistance we view the graph as an electronic circuit, and we take the sum of the resistances between every pair of vertices. Because in [1] it has already been explained why the effective graph resistance is a good measure for robustness, we decide not to do this. Instead, we study how to extend the definition of the effective graph resistance to directed graphs in Chapter 5. To do this, we discuss equivalent definitions of the effective graph resistance that are defined on directed graphs. However, as it turns out, neither way of extending the definition provides us with a measure that satisfies Rule 2. There may be another way of extending the definition of the effective graph resistance to directed graphs. Further research may find a measure on directed graphs that does satisfy Rule 2.

Throughout this paper we use the notation $\mathbb{Z}_+ = \{1, 2, \dots\}$.

Chapter 2

Assortativity

While doing research on measures for robustness we came across an article written by Newman [3]. It was about a fairly new measure called the *assortativity* of a graph. In the article it was claimed that graphs with high assortativity tended to be more resistant to the removal of vertices and edges. We therefore decided to study this measure. The idea is as follows.

If vertices of a high degree are mostly connected with other vertices of a high degree a graph is said to show signs of assortative mixing. A graph shows disassortative mixing if vertices of a high degree are connected to vertices of a low degree. We will define a way to quantify this.

Let p_k be the probability that a randomly chosen vertex has degree $k, k \in \mathbb{Z}_+$. That is, if there are n_k vertices with degree k, $k \in \mathbb{Z}_+$ then $p_k = \frac{n_k}{N}$. Here $N = \sum_{v \in V} \delta(v)$, where $\delta(v)$ is the degree of vertex v.

Now follow a randomly chosen edge to one of the two vertices it connects, and look at the degree of this vertex. This way there is a higher probability to arrive at a vertex with high degree, because there are more edges that lead to this vertex. The probability that a vertex has degree k is now $\frac{kp_k}{\sum_j jp_k}, k \in \mathbb{Z}_+$.

However, from the point of view of the edge it is more useful to look at the *remaining* degree of a vertex, which is equal to the number of edges leaving the vertex other than the one we arrived along. The probability that the remaining degree is k is now $q_k = \frac{(k+1)p_{k+1}}{\sum_j jp_j}, k \in \mathbb{Z}_+$. We now define the quantity $e_{jk}, j, l \in \mathbb{Z}_+$ to be the joint probability distribution of the remaining degrees j and k of the two vertices at either end of a randomly chosen edge, in random order. Therefore, we necessarily have $e_{jk} = e_{kj}, j, k \in \mathbb{Z}_+$. Of course we must have $\sum_j \sum_k e_{jk} = 1$ and $\sum_j e_{jk} = q_k$ for every k.

If the remaining degrees of the two vertices are independent of each other, then e_{jk} will be equal to $q_j q_k$ for $j, k \in \mathbb{Z}_+$. Such a network exhibits neither assortative nor disassortative mixing. If there is assortative mixing e_{jk} will differ from $q_j q_k$.

We can use this to quantify to what extent vertices of high degree are connected to other vertices of high degree. We do this by taking the covariance between two random variables X and Y. Let X and Y be the values we get by choosing a random edge and taking for X and Y the remaining degrees of the vertices on both ends of the edge. X has a probability of $\frac{1}{2}$ to be equal to the remaining degree at either end. Then for the covariance we have

$$\sigma(X, Y) = E(XY) - E(X)E(Y) = \sum_{jk} jk(e_{jk} - q_j q_k).$$

If we want to compare this quantity for different networks, it is convenient to divide by the value it achieves on a perfectly assortative network, on which an edge can only run between two vertices if the vertices have the same degree, so that $e_{jk} = q_k \delta_{jk}$. In this case we have

$$\sigma(X, Y) = \sum_k k^2 q_k - \sum_{jk} jk q_j q_k = \sum_k k^2 q_k - [\sum_k k q_k]^2.$$

This is equal to the variance. Now we can define assortativity.

Definition 2.1 The assortativity $r(G)$ of a simple graph G is

$$r(G) = \frac{\sigma(X,Y)}{\sigma(X)\sigma(Y)} = \frac{\sum_{jk} jk(e_{jk} - q_j q_k)}{\sum_k k^2 q_k - [\sum_k k q_k]^2}.$$

where $\sigma(X)$ is the variance of X. This is the correlation coefficient between X and Y.

The following theorem is found in [3], but without a proof. Therefore we give a proof of our own.

Theorem 2.1 *For the assortativity $r(G)$ of a graph $G = (V, E)$ we have*

$$r(G) = \frac{\sum_{j,k} jk(e_{jk} - q_j q_k)}{\sum_k k^2 q_k - [\sum_k k q_k]^2} = \frac{M^{-1}\sum_{i\in E} j_i k_i - [M^{-1}\sum_{i\in E}\frac{1}{2}(j_i + k_i)]^2}{M^{-1}\sum_{i\in E}\frac{1}{2}(j_i^2 + k_i^2) - [M^{-1}\sum_{i\in E}\frac{1}{2}(j_i + k_i)]^2}.$$

where j_i and k_i are the degrees of the vertices at the ends of edge $i \in E$, and $M = |E|$.

Proof Let m_{jk} be the number of combinations of vertices v and w such that $d(v) = j + 1$, $d(w) = k + 1$ and $v, w \in E$. This way, if $d(v) = d(w) = k + 1$, then v, w and w, v are both counted seperately in m_{kk}. Then e_{jk} is equal to $\frac{m_{jk}}{2M}$. We have $q_k = \frac{(k+1)n_{k+1}}{\sum_j j n_j} = \frac{(k+1)n_{k+1}}{2M}$.

Let v_i, v_i' be the degrees at the ends of edge i, for $i \in E$. Then

$$\sum_{k\in\mathbb{Z}_+} k^2 n_k = \sum_{v\in V}(\delta(v))^2 = \sum_{i\in E}(\delta(v_i) + \delta(v_i')) = \sum_{i\in E}(j_i + k_i).$$

Equivalently we find

$$\sum_{k\in\mathbb{Z}_+} k n_k = 2M$$

and

$$\sum_{k\in\mathbb{Z}_+} k^3 n_k = \sum_{i\in E}\frac{1}{2}(j_i + k_i).$$

Therefore

$$\begin{aligned}
\sum_{jk} jk \cdot e_{jk} &= (2M)^{-1}\sum_{jk} jk m_{jk} = M^{-1}\sum_{i\in E}(j_i - 1)(k_i - 1) = M^{-1}(\sum_{i\in E}(j_i k_i) + \sum_{i\in E}(-j_i - k_i)) + 1. \\
\sum_k k q_k &= (2M)^{-1}\sum_k k(k+1)n_{k+1} = (2M)^{-1}\sum_k k(k-1)n_k = (2M)^{-1}\sum_k k^2 n_k - 1 \\
&= M^{-1}\sum_{i\in E}\frac{1}{2}(j_i + k_i) - 1. \\
\sum_k k^2 q_k &= (2M)^{-1}\sum_k k^2(k+1)n_{k+1} = (2M)^{-1}\sum_k (k-1)^2 k n_k \\
&= (2M)^{-1}\sum_k (k^3 n_k - 2k^2 n_k + k n_k) = M^{-1}(\sum_{i\in E}\frac{1}{2}(j_i^2 + k_i^2) + \sum_{i\in E}(-j_i - k_i)) + 1.
\end{aligned}$$

So

$$\begin{aligned}
\sum_{j,k\in\mathbb{Z}_+} jk(e_{jk}-q_jq_k) &= \sum_{j,k\in\mathbb{Z}_+} jke_{jk} - [\sum_{k\in\mathbb{Z}_+} kq_k]^2 \\
&= M^{-1}(\sum_{i\in E}(j_ik_i) + \sum_{i\in E}(-j_i-k_i)) + 1 - [M^{-1}(\sum_{i\in E}\frac{1}{2}(j_i+k_i)) - 1]^2 \\
&= M^{-1}\sum_{i\in E}(j_ik_i) + M^{-1}\sum_{i\in E}(-j_i-k_i) + 1 \\
&- [M^{-1}\sum_{i\in E}\frac{1}{2}(j_i+k_i)]^2 + M^{-1}\sum_{i\in E}(j_i+k_i) - 1 \\
&= M^{-1}\sum_{i\in E} j_ik_i - [M^{-1}\sum_{i\in E}\frac{1}{2}(j_i+k_i)]^2
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k\in\mathbb{Z}_+} k^2q_k - [\sum_{k\in\mathbb{Z}_+} kq_k]^2 &= M^{-1}(\sum_{i\in E}\frac{1}{2}(j_i^2+k_i^2) + \sum_{i\in E}(-j_i-k_i)) + 1 - [M^{-1}(\sum_{i\in E}\frac{1}{2}(j_i+k_i)) - 1]^2 \\
&= M^{-1}\sum_{i\in E}\frac{1}{2}(j_i^2+k_i^2) + M^{-1}\sum_{i\in E}(-j_i-k_i) + 1 \\
&- [M^{-1}\sum_{i\in E}\frac{1}{2}(j_i+k_i)]^2 + M^{-1}\sum_{i\in E}(j_i+k_i) - 1 \\
&= M^{-1}\sum_{i\in E}\frac{1}{2}(j_i^2+k_i^2) - [M^{-1}\sum_{i\in E}\frac{1}{2}(j_i+k_i)]^2.
\end{aligned}$$

We therefore conclude

$$r(G) = \frac{\sum_{j,k\in\mathbb{Z}_+} jk(e_{jk}-q_jq_k)}{\sum_{k\in\mathbb{Z}_+} k^2q_k - [\sum_{k\in\mathbb{Z}_+} kq_k]^2} = \frac{M^{-1}\sum_{i\in E} j_ik_i - [M^{-1}\sum_{i\in E}\frac{1}{2}(j_i+k_i)]^2}{M^{-1}\sum_{i\in E}\frac{1}{2}(j_i^2+k_i^2) - [M^{-1}\sum_{i\in E}\frac{1}{2}(j_i+k_i)]^2}.$$

□

Definition 2.2 Let G be a simple graph. There is

(i) assortative mixing if $r(G) > 0$,

(ii) disassortative mixing if $r(G) < 0$, and

(iii) no assortative mixing if $r(G) = 0$.

To understand assortativity better, we have calculated the assortativity for different types of graphs.

Complete bipartite graphs. A graph $G = (V, E)$ is a bipartite graph if there exists a partition (W, W') of V such that $(i, j) \in E$ implies $i \in W, j \in W'$ or $i \in W', j \in W$. A complete bipartite graph is a bipartite graph for which $(i, j) \in E$ for all $i \in W, j \in W', i \neq j$. Figure 2.1 is an example of a complete bipartite graph.

If $|W| = n \geq 1$, $|W'| = m \geq 1$ and $n \neq m$ then

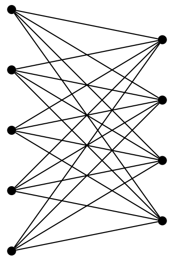


Figure 2.1. $r = -1$ for the graph $K_{5,4}$

$$\begin{aligned}
r(G) &= \frac{M^{-1}\sum_{i\in E} j_i k_i - [M^{-1}\sum_{i\in E}\frac{1}{2}(j_i+k_i)]^2}{M^{-1}\sum_{i\in E}\frac{1}{2}(j_i^2+k_i^2) - [M^{-1}\sum_{i\in E}\frac{1}{2}(j_i+k_i)]^2} \\
&= \frac{(nm)^{-1}\sum_{i\in E} nm - [(nm)^{-1}\sum_{i\in E}\frac{1}{2}(n+m)]^2}{(nm)^{-1}\sum_{i\in E}\frac{1}{2}(n^2+m^2) - [(nm)^{-1}\sum_{i\in E}\frac{1}{2}(n+m)]^2} \\
&= \frac{(nm)^{-1}\cdot nm\cdot nm - [(nm)^{-1}\cdot nm\cdot\frac{1}{2}(n+m)]^2}{(nm)^{-1}\cdot nm\cdot\frac{1}{2}(n^2+m^2) - [(nm)^{-1}\cdot nm\cdot\frac{1}{2}(n+m)]^2} \\
&= \frac{nm - \frac{1}{4}(n+m)^2}{\frac{1}{2}(n^2+m^2) - \frac{1}{4}(n+m)^2} = \frac{\frac{1}{2}nm - \frac{1}{4}(n^2+m^2)}{-\frac{1}{2}nm + \frac{1}{4}(n^2+m^2)} = -1.
\end{aligned}$$

It seems odd that the robustness is so low, because in a complete bipartite graph there are many paths between every pair of vertices. If $n = m$, the graph is regular. We will now show that the assortativity is undefined for regular graphs.

Regular graphs. A graph G is regular if every vertex in the graph has the same degree. An example of a regular graph is the complete graph K_5, pictured in Figure 2.2.

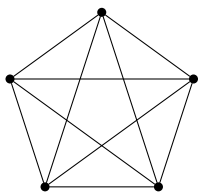


Figure 2.2. The assortativity is undefined, but the graph is very robust.

If a simple graph G is regular, then there is $l \in \mathbb{Z}_+$ such that $j_i = k_i = l$ for every edge i. We find that

$$\begin{aligned}
r(G) &= \frac{M^{-1}\sum_i j_i k_i - [M^{-1}\sum_i\frac{1}{2}(j_i+k_i)]^2}{M^{-1}\sum_i\frac{1}{2}(j_i^2+k_i^2) - [M^{-1}\sum_i\frac{1}{2}(j_i+k_i)]^2} \\
&= \frac{M^{-1}\sum_i k^2 - [M^{-1}\sum_i\frac{1}{2}(2k)]^2}{M^{-1}\sum_i\frac{1}{2}(2k^2) - [M^{-1}\sum_i\frac{1}{2}(2k)]^2} = \frac{k-k}{k-k} = \frac{0}{0}.
\end{aligned}$$

So the assortativity for these graphs is not defined.

There is another way to show this. Let X and Y once more be the degrees of the vertices at the ends of a random edge. Then X and Y will always be equal to l. Therefore $\delta(X,Y) = 0$. Also, $\delta(X) = \delta(Y) = 0$. Therefore $r(G) = \frac{\delta(X,Y)}{\delta(X)\delta(Y)} = \frac{0}{0}$ which is not defined.

Wheel graphs. A graph G with $n+1$ vertices is a wheel graph if n of the vertices are connected in a circle and the last vertice is connected to all the other vertices. An example of a wheel graph for $n = 6$ can be found in Figure 2.3.

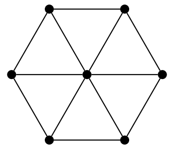


Figure 2.3. a wheelgraph

For these graphs there are n edges for which the endpoints have degrees 3 and n, and n edges for which the ends both have degree 3. Then, for a wheelgraph G_n with $n+1$ vertices we get

$$\begin{aligned} r(G_n) &= \frac{M^{-1}\sum_i j_i k_i - [M^{-1}\sum_i \frac{1}{2}(j_i + k_i)]^2}{M^{-1}\sum_i \frac{1}{2}(j_i^2 + k_i^2) - [M^{-1}\sum_i \frac{1}{2}(j_i + k_i)]^2} \\ &= \frac{(2n)^{-1}(n \cdot 3 \cdot n + n \cdot 3 \cdot 3) - [(2n)^{-1}\frac{1}{2}(n \cdot (3+n) + n \cdot (3+3))]^2}{(2n)^{-1}\frac{1}{2}(n(3^2+n^2) + n(3^2+3^2)) - [(2n)^{-1}\frac{1}{2}(n \cdot (3+n) + n \cdot (3+3))]^2} \\ &= \frac{\frac{1}{2}(3n+9) - \frac{1}{16}(n^2+18n+81)}{\frac{1}{4}(n^2+27) - \frac{1}{16}(n^2+18n+81)}. \end{aligned}$$

Then

$$\lim_{n\to\infty} r(G_n) = \frac{-\frac{1}{16}n^2}{\frac{1}{4}n^2 - \frac{1}{16}n^2} = \frac{-1}{3}.$$

So large wheelgraphs have an assortativity close to $\frac{-1}{3}$ and are therefore disassortative.

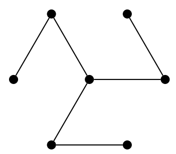
Chapter 3

Is assortativity a reasonable measure for robustness?

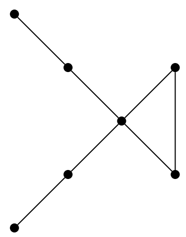
Assortativity is seen as an acceptable way to calculate robustness, because when a graph has high assortativity, the vertices with a high degree tend to be tightly connected to each other. Even if a few are removed, the others will still be connected to each other.

However, we have seen that $r(K_5)$ is not defined. This is in conflict with the first rule we imposed. One could artificially define the assortativity of regular graphs, of course. It is however not clear what the assortativity of a regular graph should be. On one hand, the edges of a regular graph always connect vertices of the same degree, so we could define the assortativity to be equal to 1. On the other hand, complete bipartite graphs that are not regular have an assortativity of -1, so we would expect regular complete bipartite graphs to have an assortativity of -1 as well.

We will now investigate whether the second rule holds or not. Consider Figure 3.1.



(a) A graph with $r = 0$



(b) A graph with $r = \frac{-1}{6}$

Figure 3.1. By adding an edge the assortativity goes down.

We can obtain Figure 3.1(b) by adding an edge to the graph in Figure 3.1(a), but the first graph has an assortativity of 0, and the second has an assortativity of $\frac{-1}{6}$. Therefore, the second rule does not hold for these graphs.

Lastly, we will check whether the assortativity of a graph always intuitively fits. As we have seen earlier, a complete bipartite graph has an assortativity of -1.

As we can see in Figure 2.1, there are many paths of length two between two vertices on the same side in a complete bipartite graph. There is also a path of length one and many paths of length 3 between vertices on opposite sides. We would therefore expect the assortativity to be very high, but it is equal to -1.

We can also look at unconnected graphs. See for example Figure 3.2. In this graph there are only edges between vertices of the same degree. This graph is therefore perfectly assortative. Because it is not regular the assortativity is defined on this graph and will be equal to 1.

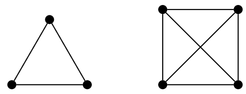


Figure 3.2. The assortativity is equal to 1, but the graph is not connected.

However, the graph is unconnected. Therefore, intuitively we would expect the assortativity to be very low.

If we allow parallel edges we can double each edge in a graph. We would expect the robustness to increase when we do this. This is not the case for assortativity. We can prove that if we double each edge of a graph G_1 the assortativity will remain the same.

Theorem 3.1 *Let G_2 be the graph we get when we double each edge in a graph $G_1 = (V, E)$. Then $r(G_1) = r(G_2)$.*

Proof Let j_i, k_i be the degrees of the vertices at the ends of edge i, for $i \in E$. If we double each edge the degree of every vertex is doubled and there are twice as many edges.

Therefore

$$\begin{aligned} r(G_2) &= \frac{(2M)^{-1}2\sum_i 2j_i 2k_i - [(2M)^{-1}2\sum_i \frac{1}{2}(2j_i + 2k_i)]^2}{(2M)^{-1}2\sum_i \frac{1}{2}((2j_i)^2 + (2k_i)^2) - [(2M)^{-1}2\sum_i \frac{1}{2}(2j_i + 2k_i)]^2} \\ &= \frac{4M^{-1}\sum_i j_i k_i - 4[M^{-1}\sum_i \frac{1}{2}(j_i + k_i)]^2}{4M^{-1}\sum_i \frac{1}{2}(j_i^2 k_i^2) - 4[M^{-1}\sum_i \frac{1}{2}(j_i + k_i)]^2} = r(G_1). \end{aligned}$$

□

As we have shown, assortativity satisfies none of the rules we expect a measure for robustness to have. There are graphs for which the assortativity is not defined. Adding an edge can lower the assortativity of a graph and there are graphs that seem very robust yet have low assortativity. Lastly, an unconnected graph can have high assortativity.

We therefore conclude that assortativity is not a good measure for robustness.

Chapter 4

Effective graph resistance

A well-known robustness measure is the *effective graph resistance* [1, 2]. To define the effective graph resistance we must first define the *effective resistance* of two vertices in a graph.

To calculate the effective resistance between vertices a and b in a graph G we must regard the graph as an electrical circuit, where an edge (i, j) corresponds to a resistor of $r_{ij} = 1$. Connect a voltage between vertices a and b and let I be the current out of a and into b.

Kirchhoff's current law states that for the current y_{ij} between vertices i and j it holds that

$$\sum_{j \in N(i)} y_{ij} = \begin{cases} I & \text{if } y = a \\ -I & \text{if } y = b \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where $N(i)$ is the set of vertices adjacent to vertex i. In other words, the flow that goes into a vertex is equal to the flow out of the vertex. We can now use Ohm's law, which states that a potential v can be associated with any vertex i such that for all edges (i, j) we have

$$y_{ij} r_{ij} = v_i - v_j. \tag{4.2}$$

With this, we can define the effective resistance.

Definition 4.1 The *effective resistance* R_{ab} between vertices a and b is defined as

$$R_{ab} = \frac{v_a - v_b}{I}.$$

We will later show that this exists and is uniquely defined. For our measure for robustness we need the *effective graph resistance.* This is equal to the sum of the effective resistance over every pair of vertices.

Definition 4.2 The *effective graph resistance* is defined as

$$R^{\text{tot}} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} R_{ij}.$$

In Section 4.6 of [1] Ellens explains why the effective graph resistance can be used to measure robustness. Among other things, the effective graph resistance will always decrease when an edge is added to a graph. Because of this property, the inverse of the effective graph resistance will always increase when an edge is added. Therefore, if we use this inverse as a measure for robustness it will satisfy our second rule.

Definition 4.3 The robustness $r(G)$ of a simple graph G is

$$r(G) = \frac{1}{R^{\text{tot}}(G)}$$

Chapter 5

Effective graph resistance for directed graphs

Because the effective graph resistance seems to be a good measure for robustness, we would like to extend the definition to other types of graphs. It is easy to extend the definition to graphs in which each edge has a weight. To do this, we simply define the resistance r_{ij} over the edge between vertices i and j to be $r_{ij} = \frac{1}{w_{ij}}$, where w_{ij} is the weight of edge (i,j). We would also like to extend the definition to directed graphs without loops and at most one arrow from vertex i to vertex j, for $i,j \in A$. Here we run into a problem, because the effective resistance has no meaning on directed graphs. Therefore we must first find a different, equivalent definition of the effective graph resistance. We can do this by means of the Laplacian.

Definition 5.1 For a simple graph $G = (V, E)$ the Laplacian L^W is the difference of the degree matrix Δ and the adjacency matrix A.

So:

$$L^W_{ij} = \begin{cases} \delta_i & \text{if } i = j \\ -1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise,} \end{cases}$$

where δ_i is the degree of vertex i. We can also define the Laplacian for weighted graphs. In this case, the weighted Laplacian is $L^W = S - W$, where w_{ij} is the weight of edge (i,j) if it exists, and $S_{ii} = \sum_{j=1}^{N} w_{ij}$, where $N = |V|$.

The Laplacian is symmetric and positive semidefinite, so all the eigenvalues are real and greater than or equal to zero. Also, the rows of the Laplacian sum up to zero, so the smallest eigenvalue of the Laplacian is equal to zero. Therefore, we can order the eigenvalues. Call the eigenvalues λ_i^W for $i = 1, \ldots, N$ such that $0 = \lambda_1^W \leq \lambda_2^W \leq \cdots \leq \lambda_N^W$. We will use this later.

Definition 5.2 The Laplacian pseudoinverse $(L^W)^+$ is the unique matrix satisfying

$$(L^W)^+ \mathbf{1} = \mathbf{0}$$

and for every $\boldsymbol{w} \perp \mathbf{1}$

$$(L^W)^+ \boldsymbol{w} = \boldsymbol{v}$$

such that $L^W \boldsymbol{v} = \boldsymbol{w}$ and $\boldsymbol{v} \perp \mathbf{1}$.

Theorem 5.1 *For the effective resistance R_{ab} between vertices a en b we have*

$$R_{ab} = (e_a - e_b)^T (L^W)^+ (e_a - e_b) = (L^W)^+_{aa} - (L^W)^+_{ab} - (L^W)^+_{ba} + (L^W)^+_{bb}.$$

Proof We will give the proof for graphs that are not weighted. The proof for the weighted case is very similar, but not necessary in order to define the Laplacian for directed graphs.

If we substitute equation (4.2) in (4.1) we get:

$$\sum_{j\in N(i)} (v_i - v_j) = \begin{cases} I & \text{if } i = a \\ -I & \text{if } i = b \\ 0 & \text{else} \end{cases},$$

where $j \in N(i)$ if and only if $(i,j) \in E$, for $i,j \in V$. This is equal to :

$$\delta_i v_i - \sum_{j\in N(i)} v_j = \begin{cases} I & \text{if } i = a \\ -I & \text{if } i = b \\ 0 & \text{else} \end{cases}.$$

We can write this in vector notation. If we do this, the equation becomes

$$L^W \boldsymbol{v} = I(e_a - e_b),$$

where e_i is the i'th unit vector.

$I(e_a - e_b)$ is perpendicular to $\mathbf{1}$ so we can invert it by the pseudoinverse. The set of solutions to the equation is therefore

$$\{\boldsymbol{v} = (L^W)^+(I(e_a - e_b)) + c\mathbf{1}, c \in \mathbb{R}\}.$$

So the vector of potentials is defined up to a constant vector. Therefore, for any (i,j) we have

$$\boldsymbol{v}_i - \boldsymbol{v}_j = e_i^T (L^W)^+ I(e_a - e_b) - e_j (L^W)^+ I(e_a - e_b) = I(e_i - e_j)^T (L^W)^+ (e_a - e_b).$$

As a consequence for the effective resistance between vertices a and b we get

$$R_{ab} = \frac{\boldsymbol{v_a} - \boldsymbol{v_b}}{I} = (e_i - e_j)^T (L^W)^+ (e_a - e_b) = (L^W)^+_a a - (L^W)^+_{ab} - (L^W)^+_{ba} + (L^W)^+_{bb}.$$

□

Theorem 5.2 *The total effective resistance satisfies*

$$R^{tot} = n \sum_{i=2}^{n} \frac{1}{\lambda_i^W}.$$

Proof $R_{ii} = 0$ for all i, because $R_{ii} = \frac{\boldsymbol{v}_i - \boldsymbol{v}_i}{I} = 0$. We also know that $(L^W)^+\mathbf{1} = 0$. Therefore

$$\begin{aligned} R^{\text{tot}} &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} R_{ij} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=i+1}^{n} R_{ij} + \frac{1}{2} \sum_{j=1}^{n} \sum_{i=j+1}^{n} R_{ij} \\ &= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} R_{ij} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} ((L^W)^+_{ii} - (L^W)^+_{ij} - (L^W)^+_{ji} + (L^W)^+_{jj}) \\ &= n \sum_{i=1}^{n} (L^W)^+_{ii} - \mathbf{1}^T (L^W)^+ \mathbf{1} = n\text{tr}((L^W)^+). \end{aligned}$$

Because the Laplacian is symmetric it has an orthonormal basis of eigenvectors, $\{u_1, \ldots, u_n\}$, such that

$$u_1 = \begin{pmatrix} \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \end{pmatrix}, \qquad u_2, \ldots, u_n \perp u_1,$$

and such that the eigenvector u_i corresponds to eigenvalue λ_i^W.

Let U be the matrix with these eigenvectors as its columns and let D be the matrix with the eigenvalues on the diagonal. Then $L^W = UDU^{-1} = UDU^T$. L^W is therefore given by D when all vectors are written with respect to the orthogonal basis of eigenvectors. It is easy to see that the pseudoinverse of L^W with respect to this basis is

$$D^+ = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & (\lambda_2^W)^{-1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & (\lambda_n^W)^{-1} \end{pmatrix}.$$

So $(L^W)^+ = UD^+U^{-1} = UD^+U^T$.

Writing a matrix with respect to a different basis does not change the eigenvalues of the matrix. So $(L^W)^+$ has eigenvalues $0, (\lambda_2^W)^{-1}, \ldots, (\lambda_n^W)^{-1}$.

Because the trace is invariant when writing a matrix with respect to a different basis we get

$$R^{\text{tot}} = n \sum_{i=2}^{n} \frac{1}{\lambda_i^W}.$$

□

We now have written the total effective resistance in terms of the Laplacian. Therefore, if we can define the Laplacian for directed graphs, we can extend the definition of the total effective resistance to these graphs.

Let $G = (V, A)$ be a directed graph. We define the Laplacian of G to be the matrix with:

$$L_{ij}^W = \begin{cases} \delta_i & \text{if } i = j \\ -1 & \text{if } (i,j) \in A \\ 0 & \text{otherwise} \end{cases},$$

Where δ_i is the out degree of vertex i and (i,j) is an arrow from vertex i to vertex j. Because this Laplacian does not have to be positive semidefinite, it may have complex eigenvalues, and we must therefore order the eigenvalues by their absolute value.

Definition 5.3 We define the total effective resistance in a directed graph to be

$$R^{\text{tot}} = n\text{tr}((L^W)^+) = n\text{tr}(D^+) = n \sum_{i=2}^{n} \frac{1}{\lambda_i^W}.$$

If an eigenvalue z is complex, its complex conjugate $\overline{z}$ must also be an eigenvalue of L^W. Also, $\frac{1}{\overline{z}} = \overline{(\frac{1}{z})}$ for every complex number z, so $\frac{1}{z} + \frac{1}{\overline{z}} = 2\text{Re}(\frac{1}{z})$ is real for every complex number z. Therefore R^{tot} will always be real.

With this definition, if we take an undirected graph G and we change it into a directed graph by substituting two arrows (i,j) and (j,i) for each edge (i,j), the Laplacian will remain the same.

Results. For each graph with two or more vertices i with $\delta_i = 0$ the total effective resistance will be undefined. This is because two or more rows of the Laplacian will consist of zeroes, which means that it will have two or more eigenvalues equal to zero. The total effective resistance is the sum of the inverse of all the eigenvalues, except one zero. If two eigenvectors λ_1 and λ_2 are equal to zero the total effective resistance would be equal to $n \sum_{i=2}^{n} \frac{1}{\lambda_i^W} = \frac{1}{0} + \sum_{i=3}^{n} \frac{1}{\lambda_i^W}$ which is undefined or equal to infinity. Because the robustness is the inverse of the total effective resistance we can define it to be zero for graphs like these.

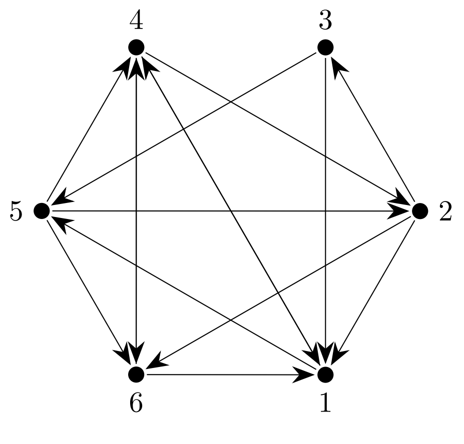
However, graphs with two or more vertices i with $\delta_i = 0$ are not the only ones for which the robustness is zero.

Theorem 5.3 *If a directed graph $G = (V, A)$ has two or more strongly connected components, the robustness will equal zero.*

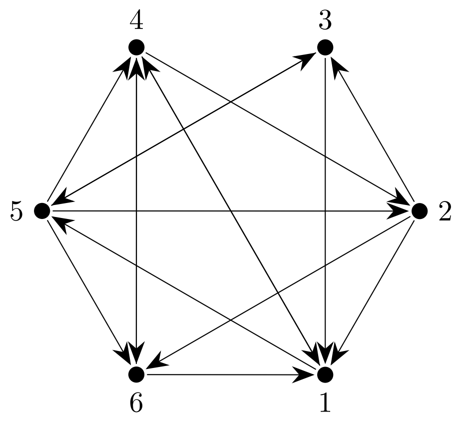
Proof Take a directed graph $G = (V, A)$ with strongly connected components. Let V_1 and V_2 be the vertices in these components. Now take the columns of the Laplacian L^W of G that correspond to the vertices in V_1. If column number i is a vertex in V_1 and it has a value $m > 0$ in place i, then it will have m outgoing arrows. The vertices these arrows point to will be in V_1. The corresponding m columns will have -1 in row i. Therefore $\sum_{j \in V_1}^{n} L_{ij}^W = 0$. This is the case for all rows i, so the sum of the columns in V_1 is the zero vector. In the same way we can prove that the columns of the vertices in V_2 sum to zero. Therefore for the rank $\mathrm{rk}(L^W)$ of L^W we have $\mathrm{rk}(L^W) \leq n - 2$, where n is the number of vertices in V. Therefore L^W has two or more eigenvalues equal to zero, so the robustness is also equal to zero. $\square$

To give graphs with two or more eigenvalues equal to zero a robustness value unequal to zero, one idea is to calculate the total effective resistance by summing over all eigenvalues unequal to zero. We decided against this, because this would mean that an eigenvalue very small could make a big difference in the total effective resistance, while an eigenvalue equal to zero would have no effect on the total effective resistance. For the same reason we do not ignore the eigenvalues equal to zero when we calculate the robustness for undirected graphs.

We tested a lot of graphs to find out whether the second property holds. Recall that this property is that by adding an arrow the robustness can not decrease. After testing quite a few graphs, we came across a pair of graphs $G_1(V, A_1)$ and $G_2(V, A_2)$ for which this does not hold:



(a) A graph with a robustness of 0.10118



(b) A graph with a robustness of 0.10020

Figure 5.1. The robustness goes down by adding an edge.

By adding the arrow $(5, 3)$ to the first graph we get the second. The first of these graphs has the following matrix as its Laplacian:

$$L^W = \begin{pmatrix} 2 & 0 & 0 & -1 & -1 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 \\ -1 & 0 & 2 & 0 & -1 & 0 \\ -1 & -1 & 0 & 3 & 0 & -1 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}.$$

The eigenvalues of this matrix are:

$$\lambda_1 = 0, \lambda_2 = 3.21508 + 1.30714i, \lambda_3 = 3.21508 - 1.30714i, \lambda_4 = 2, \lambda_5 = 3, \lambda_6 = 3.56984.$$

This gives a robustness of $r = 0.10118$. The second of these graphs has the following matrix as its Laplacian:

$$L^W = \begin{pmatrix} 2 & 0 & 0 & -1 & -1 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 \\ -1 & 0 & 2 & 0 & -1 & 0 \\ -1 & -1 & 0 & 3 & 0 & -1 \\ 0 & -1 & -1 & -1 & 4 & -1 \\ -1 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}.$$

The eigenvalues of this matrix are:

$$\lambda_1 = 0, \lambda_2 = 1.74008, \lambda_3 = 3.62996 + 1.09112i, \lambda_4 = 3.62996 - 1.09112i, \lambda_5 = 4, \lambda_6 = 3.$$

This gives a robustness of $r = 0.10020$. So the robustness of the second graph is smaller in spite of the fact that it has one arrow more.

We therefore conclude this is not a good way of extending the effective resistance to directed graphs.

We can also define the Laplacian of a directed graph $G = (V, A)$ to be the matrix with:

$$L_{ij}^W = \begin{cases} \delta_i & \text{if } i = j \\ -1 & \text{if } (i, j) \in A \\ 0 & \text{otherwise,} \end{cases}$$

where δ_i is the in degree of vertex i, instead of the out degree of i. We can use this to get a slightly different definition of the robustness of a graph. In this case we can reverse all the arrows in G_1 and G_2 to get the graphs $G_1'(V, A_1')$ and $G_2'(V, A_2')$. The in degrees of the vertices of G_1' and G_2' are now equal to the out degrees of the vertices of G_1 and G_2. Also, (i, j) is an arrow of G_1' if and only if (j, i) is an arrow of G_1 and (i, j) is an arrow of G_2' if and only if (j, i) is an arrow of G_2, for $i, j \in V$ With this new definition the Laplacian of G_1' and G_2' will be the transpose of the Laplacian G_1 and G_2 have with the old definition. Therefore, they will have the same eigenvalues. So G_2' has one arrow more and yet is less robust. Therefore, this definition of the Laplacian does not help us.

5.1 Random walks.

Defining the effective graph resistance on directed graphs with the help of the Laplacian did not turn out to be useful. However, there are other ways to define the effective resistance. One of these ways is by means of random walks.

Let $G = (V, E)$ be a simple graph. We define a random walk on G with transition probabilities $p_{ij} = \frac{1}{\delta_i}$ for $j \in N(i)$. The *stationary probability* of vertex i is the fraction of time spent in vertex i after a great amount of transitions. It is equal to $\pi_i = \frac{\delta_i}{\sum_{i=1}^{N} \delta_i}$ where N is the number of vertices in G. Let T_{ij} be the number of transitions it takes to reach vertex j if we start in vertex i. If $i = j$, this will be zero. We define T_{aa}^{+} to be the number of transitions needed from starting in a to returning to a. We have the following lemma.

Lemma 5.4 *The following relation holds for $a \neq b$.*

$$P(T_{ab} < T_{aa}^{+}) = \frac{1}{\pi_a(E(T_{ab}) + E(T_{ba}))}.$$

Proof The renewal theorem (see [4]) with cycle length S states

$$\pi_a = \frac{\boldsymbol{E}(\text{time in } a \text{ during one cycle})}{\boldsymbol{E}(S)}.$$

In this case we let S be the length of the cycle in which we start in b, visit a and then return to b. The equation then becomes

$$\pi_a = \frac{E(B_{baa}) + E(B_{aab})}{E(T_{ab}) + E(T_{ba})} = \frac{E(B_{aab})}{E(T_{ab}) + E(T_{ba})},$$

where B_{avb} is the number of visits to v given a start in a before arriving in b. Note that therefore B_{baa} will be zero for all $a, b \in V$.

The number of visits B_{aab} to a before arriving in b has a geometric distribution with probability of succes $p = P(T_{aa}^{+} < T_{ab})$. Therefore the expected value of B_{aab} is equal to

$$E(B_{aab}) = \frac{1}{p} = \frac{1}{P(T_{aa}^{+} < T_{ab})}.$$

If we substitute this in our equation for π_a we get

$$\pi_a = \frac{1}{P(T_{aa}^{+} < T_{ab})(E(T_{ab}) + E(T_{ba}))}.$$

From this we get

$$P(T_{ab} < T_{aa}^{+}) = \frac{1}{\pi_a(E(T_{ab}) + E(T_{ba}))}.$$

$\square$

Theorem 5.5 *Let a graph $G = (V, E)$ be given. For the effective graph resistance R^{tot} the following holds.*

$$R^{tot} = \frac{1}{\sum_{i=1}^{n} \delta_i} \sum_{i=1}^{N} \sum_{j=1}^{N} \boldsymbol{E}(T_{ij}).$$

Proof For $a = b$ the theorem is true, because the $T_{ab} = T_{ba} = 0$, so $R_{ab} = \frac{1}{\sum_{i=1}^{n} \delta_i}(\boldsymbol{E}(T_{ab}) + \boldsymbol{E}(T_{ba})) = 0$. Therefore assume $a \neq b$. From Lemma 3.4 it follows that

$$\pi_a(E(T_{ab}) + E(T_{ba})) = \frac{1}{P(T_{ab} < T_{aa}^{+})}.$$

Therefore

$$\frac{1}{\sum_{i=1}^n \delta_i}(\boldsymbol{E}(T_{ab}) + \boldsymbol{E}(T_{ba})) = \frac{1}{\delta_a \boldsymbol{P}(T_{ab} < T_{aa}^+)},$$

so we just need to show that

$$R_{ab} = \frac{1}{\delta_a \boldsymbol{P}(T_{ab} < T_{aa}^+)}$$

because then

$$R^{\text{tot}} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} R_{ij} = \frac{1}{\sum_{i=1}^n \delta_i} \sum_{i=1}^{N} \sum_{j=1}^{N} \boldsymbol{E}(T_{ij}).$$

Let $v_i = \boldsymbol{P}(T_{ia} < T_{ib}), y_{ij} = (v_i - v_j)$ and $I = \delta_a \boldsymbol{P}(T_{ab} < T_{aa}^+)$. Because we have defined y_{ij} to be equal to $(v_i - v_j)$, it is clear that these values for v,y and I satisfy Ohm's law (4.2). We will now prove that these values also satisfy Kirchhoff's current law (4.1). We get the following equations:

$$\begin{aligned}
\sum_{j\in N(a)} y_{aj} &= \sum_{j\in N(a)} (v_a - v_j) = \sum_{j\in N(a)} (\boldsymbol{P}(T_{aa} < T_{ab}) - \boldsymbol{P}(T_{ja} < T_{jb})) \\
&= \sum_{j\in N(a)} (1 - (1 - \boldsymbol{P}(T_{jb} < T_{ja}))) \sum_{j\in N(a)} \boldsymbol{P}(T_{jb} < T_{ja}) = \delta_a \sum_{j\in V} p_{aj} \boldsymbol{P}(T_{jb} < T_{ja}) \\
&= \delta_a \boldsymbol{P}(T_{ab} < T_{aa}^+) = I. \\
\sum_{j\in N(i)} y_{ij} &= \sum_{j\in N(i)} (v_i - v_j) = \sum_{j\in N(i)} (\boldsymbol{P}(T_{ia} < T_{ib}) - \boldsymbol{P}(T_{ja} < T_{jb})) \\
&= \delta_i \boldsymbol{P}(T_{ia} < T_{ib}) - \sum_{j\in N(i)} \boldsymbol{P}(T_{ja} < T_{jb}) = \delta_i \boldsymbol{P}(T_{ia} < T_{ib}) - \delta_i \sum_{j\in V} p_{ij} \boldsymbol{P}(T_{ja} < T_{jb}) \\
&= \delta_i \boldsymbol{P}(T_{ia} < T_{ib}) - \delta_i \boldsymbol{P}(T_{ia} < T_{ib}) = 0. \\
\sum_{j\in N(b)} y_{bj} &= \sum_{j\in N(b)} (v_b - v_j) = \sum_{j\in N(b)} (\boldsymbol{P}(T_{ba} < T_{bb}) - \boldsymbol{P}(T_{ja} < T_{jb})) = -\sum_{j\in N(b)} \boldsymbol{P}(T_{ja} < T_{jb}) \\
&= -\delta_b \sum_{j\in V} p_{bj} \boldsymbol{P}(T_{ja} < T_{jb}) = -\delta_b \boldsymbol{P}(T_{ba} < T_{bb}^+) = \frac{\delta_b \cdot \sum_{i=q}^n \delta_i}{\delta_b(\boldsymbol{E}(T_{ba}) + \boldsymbol{E}(T_{ab}))} \\
&= \frac{\delta_a \cdot \sum_{i=q}^n \delta_i}{\delta_a(\boldsymbol{E}(T_{ba}) + \boldsymbol{E}(T_{ab}))} = -\delta_b \boldsymbol{P}(T_{ab} < T_{aa}^+) = -I.
\end{aligned}$$

The next to last equality follows from Lemma 5.4.

Because these values for v,y and I satisfy Kirchhoff's current law we get

$$R_{ab} = \frac{v_a - v_b}{I} = \frac{\boldsymbol{P}(T_{aa} < T_{ab}) - \boldsymbol{P}(T_{ba} < T_{bb})}{\delta_a \boldsymbol{P}(T_{ab} < T_{aa}^+)} = \frac{1}{\delta_a \boldsymbol{P}(T_{ab} < T_{aa}^+)}$$

□

Now we can define the effective resistance for a directed graph.

Definition 5.4 The effective graph resistance R^{tot} is

$$R^{\text{tot}} = \frac{1}{\sum_{i=1}^n \delta_i} \sum_{i=1}^{n} \sum_{j=1}^{n} \boldsymbol{E}(T_{ij}).$$

The robustness is again defined as $r(G) = \frac{1}{R^{\mathrm{tot}}}$. We now have two different definitions of the effective graph resistance for directed graphs. To avoid confusion, we will call the first the eigenvalue resistance and the second the random walk resistance.

Results. Just like with the eigenvalue resistance for directed graphs, we have calculated the values of the random walk resistance for many graphs, to see wether our rules for a robustness measure hold. Clearly, $\boldsymbol{E}(T_{ij})$ is only defined for graphs in which it is possible to go from vertex i to vertex j. Therefore the random walk resistance is only defined for strongly connected graphs. This means that the random walk resistance is not the same as the eigenvalue resistance. This is because, with the eigenvalue resistance, there are graphs that are not strongly connected for which the effective resistance is still defined.

We found that even for strongly connected graph our new definition is not equivalent to the old one.

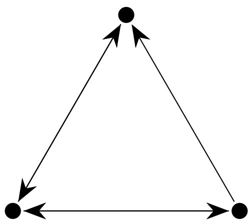


Figure 5.2. The robustness depends on the method we use.

Regard for example Figure 5.2. The Laplacian of this graph is the following matrix:

$$L^W = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

The eigenvalues of this matrix are $\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 3$. This gives an eigenvalue resistance of 2.5.

If we calculate the expected amount of transitions between every pair of vertices we get

$$E(T_{12}) = 1, E(T_{23}) = 3, E(T_{31}) = 2, E(T_{13}) = 4, E(T_{32}) = 1.5, E(T_{21}) = 2.$$

This gives us a random walks resistance resistance of 2.7. We have seen that more often than not, the two definitions give a different value for the robustness.

We hoped that for the random walk resistance Rule 2 would hold, because it did not hold for the eigenvalue resistance. However, it turned out that there are a lot of counterexamples to this rule. Regard Figure 5.3.

In the following tabels are the expected amounts of transitions for both graphs. In row i and column j is $E(T_{ij})$

	1	2	3	4
1	0	1	6.5	7
2	2	0	5.5	6
3	1	2	0	8
4	2	1.5	7	0

	1	2	3	4
1	0	3	$\frac{18}{7}$	13
2	2	0	$\frac{22}{7}$	10
3	1	4	0	14
4	2	2.5	$\frac{27}{7}$	0

In the first graph the sum of the expected amounts of transitions is 49.5. This gives a random walk resistance of $\frac{99}{14}$ and therefore a robustness of $\frac{14}{99} \approx 0.14141$.

In the second graph the sum of the expected numbers of transitions is $\frac{855}{14}$. This gives a random walk resistance of $\frac{855}{112}$ and therefore a robustness of $\frac{112}{855} \approx 0.13099$

(a) A graph with a robustness of 0.14141 (b) A graph with a robustness of 0.13101

Figure 5.3. The robustness goes down by adding an edge.

So the second graph is less robust, while it has one arrow more.

After some tests, we found that there are many more counterexamples for this definition than for the first. We therefore conclude that this definition is worse than the first one.

Chapter 6

Conclusion

In this paper we looked at two measures for robustness. First we looked at the assortativity. The proof of Theorem 2.1 is our own. We calculated the robustness for different types of graphs. While the article of Newman [3] claimed that the assortativity of a graph was a good measure of robustness, we found that it did not satisfy any of the rules we expected such a measure to have. Because we concluded that the assortativity of a graph is not a good measure of robustness, it does not seem like further research in this direction could be useful. We next looked at the effective graph resistance. Because there already are articles that show that the effective graph resistance is a good measure for robustness, we decided not to do the same. Instead we looked at ways to define the effective graph resistance for directed graphs. First we defined it with the help of the eigenvalues of the Laplacian. Next we found a way to define the effective graph resistance with random walks. Unfortunately, neither definition satisfied Rule 2. Further research on the effective graph resistance may be possible. There might be another way to define the effective graph resistance for directed graphs. It might be possible to find one which satisfies Rule 2.

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Appendix A

Programs

These are the programs we used to calculate the effective graph resistance for directed graphs. For both methods there is a program.

Using the Laplacian

```
clear
A=[2,-1,-1,0;-1,3,-1,-1;-1,-1,2,0;0,-1,0,1]
%This is the Laplacian of the graph we want to look at.
B=4;
C=A;
c=0;
d=0;
Eff=0;
Eff1=0;
Hoera =0;
Rob=0;
b=eig(A);
b
for i=1:B
    if abs(b(i)) <= 0.0001
        c=c+1;                  %We calculate the amount of eigenvalues equal to zero.
        d=i;
    end
end
if c==1                         %Precisely one eigenvalue must be equal to zero.
    for i=1:B
        if i!=d
            Eff =Eff +1/b(i)    %Here we sum the inverses of the eigenvalues.
        end
    end
    Rob =1/(B*Eff)              %This is the robustness of the graph.
else
    Rob =0                      %The robustness is zero if more than one eigenvalue is zero.
end
c=0;
for i=1:B                       %Here we calculate the robustness of every graph we can get by
    for j=1:B                   %removing one arrow of the original graph.
        if i!=j
            if A(i,j)!= 0
                C(i,j) =0;
                C(i,i) =A(i,i) + A(i,j);
                C               %C is the laplacian of the new graph
                b=eig(C);
                for k=1:B                   %We calculate the robustness of C.
                    if abs(b(k)) <= 0.0001
```

```
                c=c+1;
                d=k;
            end
        end
        c;
        if c==1
            for k=1:B
                if k!=d
                    Eff1 =Eff1 +1/b(k);
                end
            end
            Rob1 =1/(B*Eff1)
        else
            Rob1 =0
        end
        C=A;
        c=0;
        if abs(Rob1) >= abs(Rob) +0.000001
            Hoera =1;              %We check wether we have found a
            b                      %counterexample to rule two.
        end
        Eff1 =0;
      end
    end
  end
end

Hoera                    %this shows wether one of the graph we looked at is
                         %a counterexample to rule two.
```

Using random walks

The second program needs some explanation. Let G be a directed graph, and let P be the matrix such that p_{ij} is the probability to go from vertex i to vertex j, for $i, j \in V$. In a random walk we have $\boldsymbol{E}(T_{ij}) = \sum_{k \neq j} p_{ik} \cdot \boldsymbol{E}(T_{kj}) + 1$ for $i, j \in V$. Let $T(j)$ be the vector with i'th coordinate equal to $\boldsymbol{E}(T_{ij})$, and let $P(j)$ be the matrix equal to P but with row j and column j removed, for $j \in V$. Then $T(j) = P(j) \cdot T(j) + \mathbf{1}$. So

$$(I - P(j))T(j) = \mathbf{1},$$

where I is the identity matrix. Therefore

$$T(j) = (I - P(j))^{-1} \cdot \mathbf{1}$$

and therefore

$$T(j) = (\sum_{n=0}^{\infty} P(j)^n)\mathbf{1}.$$

So, if we calculate the effective resistance with random walks we get

$$R^{\text{tot}} = \frac{\sum_{j \in V} \mathbf{1}^T T(j)}{\sum_{i \in V} \delta_i} = \frac{\sum_{j \in V} \mathbf{1}^T (\sum_{n=0}^{\infty} P(j)^n)\mathbf{1}}{\sum_{i \in V} \delta_i}.$$

```

clear
A=[2,-1,-1,0;-1,3,-1,-1;-1,0,1,0;-1,-1,0,2]  %This is the Laplacian of the graph we look at.
B=4;                     %This is the amount of rows in the matrix.
robustness =0;
s=0;
```

```
for i=1:(B-1)
    C(i,1)=1;
end

for i=1:B                   %Here we calculate the transition probabilities.
    for j=1:B
        if i==j
            D(i,i)=0;
        elseif A(i,j) ==-1
            D(i,j)=1/A(i,i);  %D(i,j) is the probability to go from vertex i to vertex j.
        else
            D(i,j)=0;
        end
    end
end
D
for k=1:B
    for i=1:(k-1)
        for j=1:(k-1)
            E(i,j,k)=D(i,j);
        end
        for j=k:(B-1)
            E(i,j,k)= D(i,j+1);
        end
    end
    for i=k:(B-1)
        for j=1:(k-1)
            E(i,j,k)=D(i+1,j);
        end
        for j=k:(B-1)
            E(i,j,k)= D(i+1,j+1);       %E(;,;,k) is the probability matrix D with
        end                             %row k and column k removed.
    end
end
F=0;
for j=1:B
    for i=1:500
        F=F+ E(:,:,j)^i;
    end
    F;
    T= F*C + C;                  %Now T =  \sum_{n=0}^{\infty} E(;,;,k)^n.
    for i=1:(B-1)
        T(i)
        robustness =robustness +T(i);     %We sum the elements of all T to get
    end                                   %\sum_{j\in V} \vt{1}^T T(j).
    F=0;
end
for i=1:B
    s=s+A(i,i)
end
robustness1 =s/robustness               %This is the robustness of our first graph.
D=0;
E=0;
robustness=0;
s=0;

for m=1:B                    %Here we calculate the robustness for each graph equal to our
    for n=1:B                %first but with one arrow removed.

        if m!=n
            if A(m,n)!= 0
                A1=A;
                A1(m,n) =0;
```

```
                A1(m,m) =A(m,m) + A(m,n);
                A1;
                for i=1:B
                    for j=1:B
                        if i==j
                            D(i,i)=0;
                        elseif A1(i,j) ==-1
                            D(i,j)=1/A1(i,i);
                        else
                            D(i,j)=0;
                        end
                    end
                end
                D
                for k=1:B
                    for i=1:(k-1)
                        for j=1:(k-1)
                            E(i,j,k)=D(i,j);
                        end
                        for j=k:(B-1)
                            E(i,j,k)= D(i,j+1);
                        end
                    end
                    for i=k:(B-1)
                        for j=1:(k-1)
                            E(i,j,k)=D(i+1,j);
                        end
                        for j=k:(B-1)
                            E(i,j,k)= D(i+1,j+1);
                        end
                    end
                end
                F=0;
                for j=1:B
                    for i=1:100
                        F=F+ E(:,:,j)^i;
                    end
                    F;
                    T= F*C + C;
                    for i=1:(B-1)
                        robustness =robustness +T(i);
                    end
                    F=0;
                end
                for i=1:B
                    s=s+A1(i,i);
                end
                robustness2 =s/robustness
                D=0;
                E=0;
                s=0;
                robustness=0;
            end
        end
    end
end
%The program does not check wether the graph with one arrow removed still has one
%eigenvalue equal to zero. However, if this is the case the robustness it
%calculates is very small. These cases are therefore easy to find.
```