Elastic instabilities in monoholar and biholar patterned networks

THESIS

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Abstract

In this Bachelor thesis we study the elastic instabilities in monoholar and biholar elastic patterned sheets. We can think of such an object as it consists of rigid regions connected by beams. In trying to understand the monoholar and biholar sheets, we first want to fully understand how these beams behave. First we give a general introduction to the properties of these monoholar and biholar sheets and explain how we can model this by 1-dimensional beams. These models will all be extensions of the classic Euler-Bernoulli model. It is already known that such models can describe buckling of beams, but we ask ourselves if these models can also account for other effects we observe in the monoholar and/or biholar sheets, such as snapping and elastic memory. Next we give the relevant definitions, notations and derive the general differential equations governing a single beam. Then we look at different specific models (e.g. inextensible or extensible, pre-curved or initially straight beams) and acquire the equations for these models. Next we solve these models both analytically and numerically. Furthermore we compare the different models and see how they differ qualitatively and give a physical interpretation. Finally we investigate if we can see effects like snapping and elastic memory in this framework.

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Chapter 1

Introduction

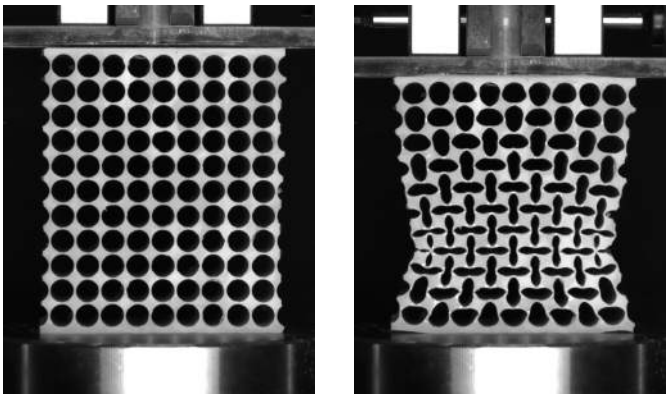
When compressing a material with a sufficiently large force, it usually deforms. Materials like glass or steel break when you apply enough force on it. This effect is of course irreversible. However, when compressing a block of rubber, it goes back to its initial shape when the load is removed. These kind of deformations are called elastic deformations and materials which allow such deformations are called elastic materials.

Now we consider a solid block of elastic material. We compress this material in one direction with a uniform force. When this material deforms as a result of the force, it clearly deforms in the direction of the force. However, the material usually also deforms in all other directions. This effect is called the Poisson effect. It is fully described by the Poisson ratio ν of a material, which tells us how strong the Poisson effect is in this material. It is given by the ratio between the strain, i.e. the relative compression ε, in the transversal direction and the strain in the axial compression:

$$\nu = -\frac{\varepsilon_{\text{transversal}}}{\varepsilon_{\text{axial}}} \tag{1.0.1}$$

Since we assume that the force is uniform, we can assume that the strains are also uniform throughout the material. So for a given solid the Poisson ratio is a well-defined quantity which is the same throughout the material. However, the Poisson ratio does not only depend on the material, but also on the geometry of the solid. For most materials, ν will be a positive number between 0 and 0.5, since a material usually expands in the directions it is not compressed. However, there are also some materials with a negative Poisson ratio between -1 and 0. These materials shrink in all directions when they are compressed, which may seem a counter-intuitive effect.

A few years ago, a new material was found with a negative Poisson ratio. This material is constructed by making a regular pattern of circular holes in a rubber-



(a) *Photo of an uncompressed monoholar sheet.* ***(b)*** *Photo of a compressed monoholar sheet.*

Figure 1.1: *In this figure we see photos of the elastic monoholar sheets. In part (b) we can clearly see that this material has a negative Poisson ratio, since the sides which are not compressed go inwards.*

like elastic material (see figure 1.1). These kind of materials are called monoholar (or holey) sheets. The negative Poisson ratio is not the only interesting property of this material. Experiments have shown that there is a strange peak in the force-compression curve of the material, which is not present in most other materials (see figure 1.3a). The question why this peak exists is as of yet unanswered and still a topic for research.

The geometry of the pattern of holes in the holey sheets plays an important role in these effects. This is best seen when we instead of a completely regular pattern make a pattern of alternating small and big circular holes (see figure 1.2). These kind of materials are called biholar sheets, because of the two different hole sizes. These materials still have a negative Poisson ratio, but experiments show that the strange peak in the force compression curve is not present for these materials (see figure 1.3b).

There is however something worth noting about both curves in figure 1.3: the relaxation curves of both the monoholar and the biholar sheets lie lower than the compression curves. This means that the force needed to compress the sheet into a certain level of compression does not only depend on the current force applied, but also on the history of forces applied. This effect is called elastic memory and is an example of the more general phenomenon hysteresis, which is the effect that the current output of a system not only depends on the current input, but also on the history of input prior to the current input. One of the goals of this project is to see if we can see some hysteresis effects in our models. There are more ways in which the biholar sheets differ from the monoholar sheets. In experiments it is

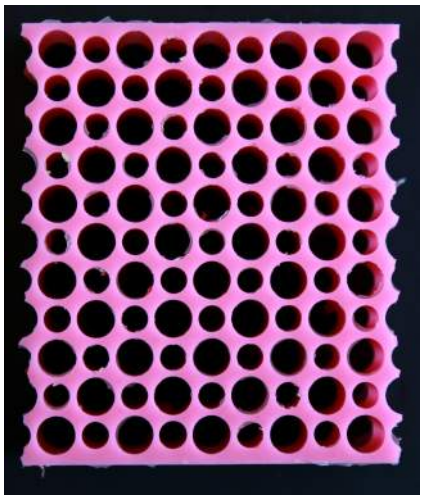
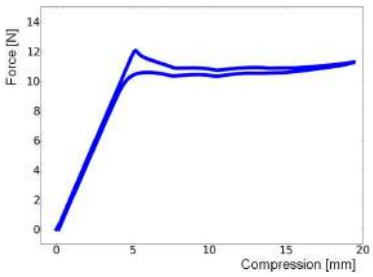


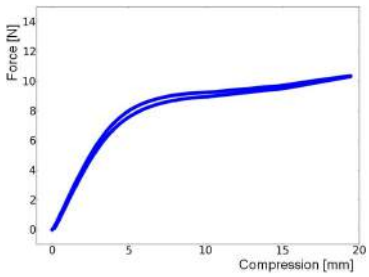
Figure 1.2: *Photo of an uncompressed biholar sheet.*

observed that if we clamp a biholar sheet in the x-direction and subsequently compress it in the y-direction, at some point it 'snaps' and makes a significant change in its shape and symmetry in a very short amount of time (see figure 1.4). However, we do not see these snapping effects for monoholar beams. Therefore the occurrence of these snapping effects must be due to the geometry of the systems. However, it is as of yet unclear what the reason is why this snapping occurs and it will be one of the goals of this research to try to model it. In trying to understand the properties of these materials, we would like to make a mathematical model which can describe their physical behavior. To model the monoholar sheets, we note that there are chunks of elastic material in between four adjacent holes. We can see these portions of elastic material as immobile nodes. Then we can see the remaining material between the beams as elastic beams, i.e. initially straight slender elastic rods. The compression of the monoholar sheet is then described by the shape of these elastic beams. See also figure 1.5 for a visual representation of this model. When trying to do the same for biholar sheets, we note that the remaining material between the nodes cannot be modeled by a straight beam. Instead, we model this by pre-curved beams, i.e. beams which are curved even without compressing them.

Single elastic beams were first described by Euler and Bernoulli in the seventeenth century. The first simple models described what happened when you compress a one-dimensional straight beam along its axis with a constant force. When we apply a sufficiently large load, the beam suddenly starts to buckle: this means that the beam suddenly goes from its initial straight state to an unstable bent state. This model assumes that the length of this beam is constant, which we call an inextensible beam. It also assumes that the deflections the beam makes are small



(a) *Experimental data showing the force versus the maximal displacement of the monoholar sheet. The curves for compression (the upper) and relaxation (the lower) are both plotted in this figure.*



(b) *Experimental data showing the force versus the maximal displacement of the biholar sheet. The curves for compression (the upper) and relaxation (the lower) are both plotted in this figure.*

Figure 1.3: *In this figure we see the force-displacement curves for the monoholar and the biholar sheets. We see that there is a strange peak in the curve for the monoholar sheet, but that is not present for the biholar sheet.*

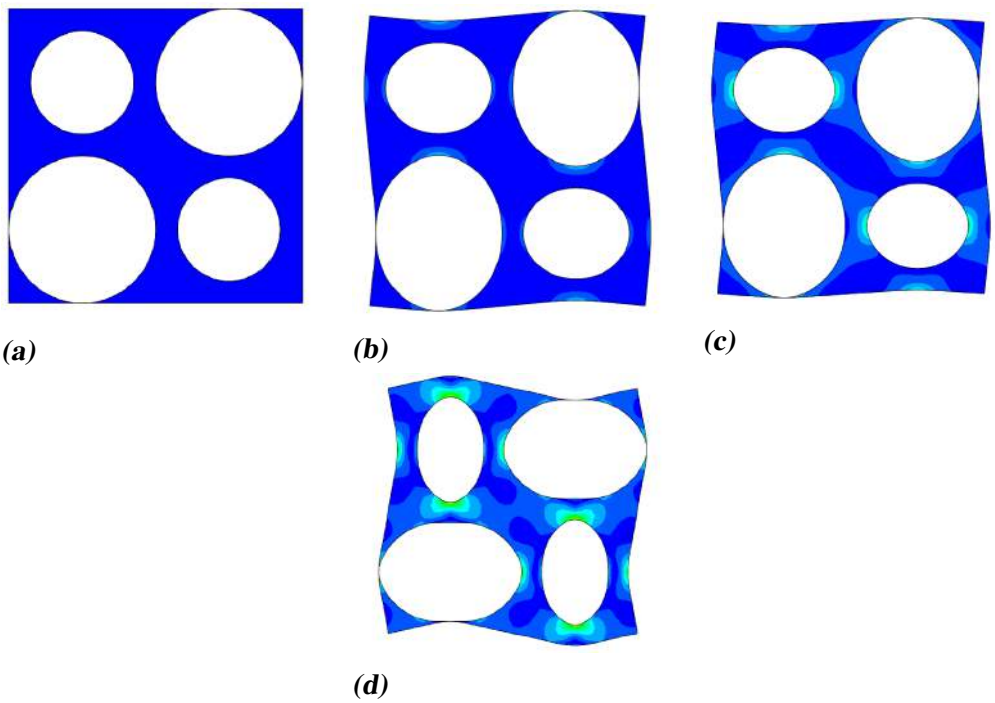
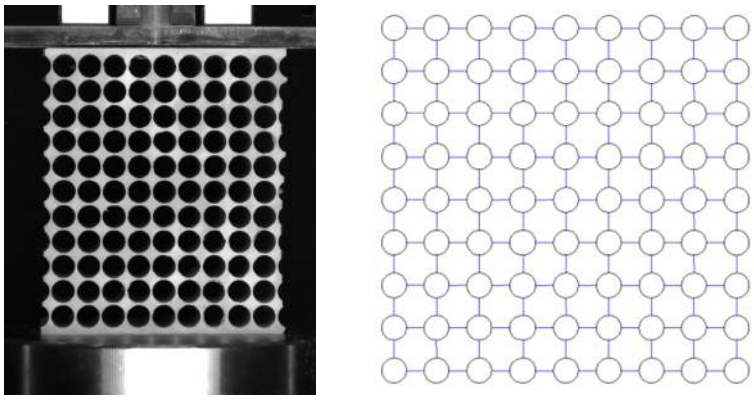


Figure 1.4: *Simulation to illustrate snapping effects. A biholar sheet (figure (a)) is clamped in the x-direction (figure (b)), then it is compressed in the y-direction (figure (c)). At a certain point the system snaps and significantly changes its shape and symmetry in a very short amount of time (figure (d)).*



(a) *Photo of an uncompressed monoholar sheet.*

(b) *A network of beams modeling the monoholar sheet.*

Figure 1.5: *These figures show a monoholar sheet and a network of beams modeling such a sheet. The nodes in part (b) represent the large chunks of elastic material in between four holes in (a), while the beams in part (b) represent the elastic material in between two holes in (a) that is left.*

in comparison to the length of the beam. In later years, many other beam models have been proposed to try to make the models more realistic. For example, it has been investigated what happens if you do not assume that the length of the beam remains constant while compressed.

In this Bachelor thesis, we will describe various beam models. The most significant addition to the existing models will be that we investigate what happens if you not only compress the beam, but also apply a moment on the endpoints of the beam. This is a natural effect to look at, since in a coupled system of beams, the beams usually apply moment on each other's endpoints. We will also investigate what the qualitative differences are between beams that are initially straight and pre-curved beams. We will see that there is great similarity between these two additions to the model. In chapter 2 we will give a general introduction to the beam models, explain the necessary physical quantities and show how to derive the general equations which will eventually describe the shape of the beams. In chapter 3 we will see what these equations look like for specific beam models. In chapter 4 we will solve most equations we derived in the previous chapter and further analyze the specific models. We are mostly interested in exact solutions to the shape of the beam and relations between the load and the moment applied and the angle the beam makes with the x-axis at the endpoints of the beams. We are particularly interested in the difference between the models with moment and those without and between the models with pre-curved beams and those with initially straight beams. In chapter 5 we will do a bifurcation analysis for one the models

and investigate its physical consequence. We are particularly interested if we can describe snapping effects and buckling in the same framework and if snapping is a 'bulk effect' of the entire system of the coupled beams in the biholar sheet of if we can model it using a single-beam model. Finally we want to investigate if hysteresis effects (or elastic memory) are also present in this model.

Chapter 2

A general introduction to the beam model

In this thesis we will look at various beam models. We will only look at single beams for two reasons: we want to look if effects like snapping can be modeled by a one-dimensional beam and we did not have time to further investigate systems of coupled beams. We will look at slender one dimensional beams in the xy-plane, where bending, axial compression and shear are taken into account. We allow the beams to have an initial curvature, which we will refer to as the pre-curvature of the beam. In this chapter we will give the necessary definitions and notation needed for the following chapters. Furthermore we will derive the most general differential equations that describe these models. This derivation will mainly be based upon energy considerations.

2.1 Notation, parameters and variables

We orient our beam such that both endpoints of the beam are positioned on the x-axis and such that the left endpoint is at the origin. Although we will not use it explicitly, we assume that the initial state of the beam is (mainly) located in the upper half of the plane. We define ℓ to be the length of the beam in its initial state. To make the notation more compact, we will mostly not use x and y as our coordinates. Instead, we will use the coordinate s, which is the arc-length parameterization of the initial (pre-curved) configuration. We let $u(s)$ be the displacement of the beam in the x-direction and $w(s)$ the displacement in the y-direction, both with respect to the initial configuration.
Derivatives will mostly be denoted by a subscript which states the variable we are

differentiating to. For example,

$$u_s := \frac{du}{ds}$$

Second derivatives are denoted in the same way, so for example we have:

$$u_{ss} := \frac{d^2u}{ds^2}$$

or

$$u_{xy} := \frac{d^2u}{dxdy}$$

To describe the shape of the initial state of the beam, we let $\theta^0(s)$ be the angle between a line tangent to the beam at the point s and a line parallel to the x-axis. For describing the shape of the beam after bending, we introduce two different angles: θ, which is the rotation angle, and χ, which is the shear angle. Note that if we ignore shear, $\theta(s)$ is the angle between a line tangent to the beam at the point s and a line parallel to the x-axis. Furthermore, we define Λ to be the length ratio between a beam element of the initial beam and of the deformed beam. See also figure 2.1.
Now we will introduce a number of quantities, which are physical properties of our system. First we define $I := I_{yy}$ to be the moment of inertia about the y-axis. Then we let κ_y be the curvature of the beam in the y-direction. Now we let A be the cross-section area of the beam. So far, these quantities could be introduced without any further physical explanation. However, for the following quantities we need to explain some of the physical properties of the system. We assume that the material is elastic, linear and isotropic. Thus we can use Hooke's law to relate the stresses and strains in the system. First we let σ be the axial stress tensor. Then, according to [4], by Hooke's law it is related to the compressive strain tensor ε by:

$$\sigma = E\varepsilon \tag{2.1.1}$$

Here E is the Young's modulus of the material. In the same way the shear stress tensor, τ, is related to the shear strain tensor γ by:

$$\tau = G\gamma \tag{2.1.2}$$

Here G is called the shear modulus. However, we look at 1d models; not at 3d models. Therefore we have to assume that all these tensors are constant over the cross-section area and thus we see these tensors as ordinary scalar quantities. However, for the shear this is not a very good approximation. Therefore we introduce the shear correction factor κ_s, which depends on the geometry of the problem

and on the Poisson ratio (see [4]).

2.2 Forces and moment

In this thesis we will mainly look at models where we apply a constant force or load P in the x-direction. We also looked at one model where the force was not constant: a model of a pre-curved beam with a spring attached between the two endpoints. However, this model turned out to be too complex to analyze. It may still be an interesting subject for further research.
We also introduce the Euler load P_e:

$$P_e = EI\left(\frac{\pi}{\ell}\right)^2 \tag{2.2.1}$$

In the classical beam theory (so when there is no moment, pre-curvature, compression and shear), this is exactly the load needed for the beam to buckle (see [2]). We will encounter this Euler load in chapter 4.
As stated in the introduction, it is natural to look at what happens if we apply moment. When looking at our one dimensional model of the coupled beams, we note that if one of the beams bends, it acts a moment on an endpoint of the beams it is coupled to. Therefore it is useful to consider beams where we externally apply a moment on the endpoints of the beam. According to [2], we have the following relation between the moment M acted on a point s and the bending angle θ at that point:

$$\theta_s(s) = \theta_s^0(s) + \frac{M}{EI} \tag{2.2.2}$$

Throughout this thesis, we will assume that the moment only acts on the two endpoints of the beam, since we do not need it to act on the other points of the beam. We also assume that the moment acting on these endpoints is constant. Moreover, we also assume that the moment on the both endpoints is equal. This is a natural assumption, since the pattern of the holes in the monoholar and biholar possess enough symmetry for these moments to be equal. It might be an interesting subject for future research to investigate what happens when we do not assume that the momenta are equal.

2.3 The relation between the displacement and the angles

In this section we follow the approach of [8]. Figure 2.1 will help us in deriving the relations we are searching for.

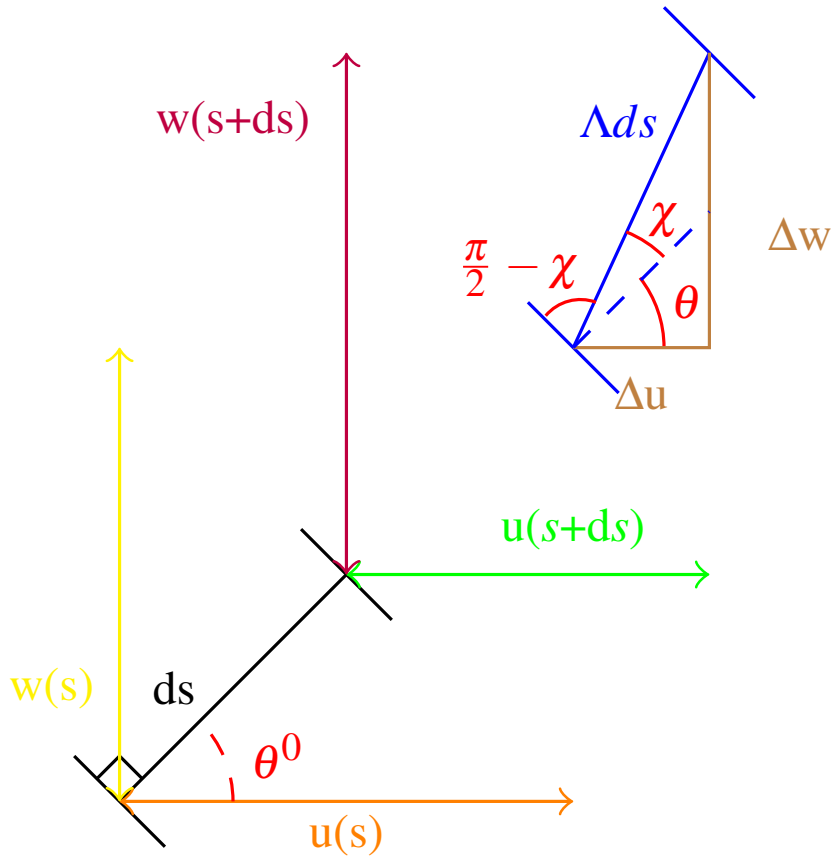


Figure 2.1: *Left: initial state of the pre-curved beam. Right: state of the beam after acting forces and moment*

Analyzing the figure, we see that:

$$\cos(\theta+\chi) = \frac{\cos(\theta^0)ds + u(s+ds) - u(s)}{\Lambda ds} \tag{2.3.1}$$

and

$$\sin(\theta+\chi) = \frac{\sin(\theta^0)ds + w(s+ds) - w(s)}{\Lambda ds} \tag{2.3.2}$$

Using Pythagoras' theorem on this infinitesimal piece of the beam, we see:

$$\Lambda ds = \sqrt{(\Delta w)^2 + (\Delta u)^2} = \\ \sqrt{(\cos(\theta^0)ds + u(s+ds) - u(s))^2 + (\sin(\theta^0)ds + w(s+ds) - w(s))^2} \tag{2.3.3}$$

So we get:

$$\cos(\theta+\chi)=\frac{\cos(\theta^0)ds+u(s+ds)-u(s)}{\sqrt{(\cos(\theta^0)ds+u(s+ds)-u(s))^2+(\sin(\theta^0)ds+w(s+ds)-w(s))^2}}=$$
$$\frac{\cos(\theta^0)+\frac{u(s+ds)-u(s)}{ds}}{\sqrt{\left(\cos(\theta^0)+\frac{u(s+ds)-u(s)}{ds}\right)^2+\left(\sin(\theta^0)+\frac{w(s+ds)-w(s)}{ds}\right)^2}} \tag{2.3.4}$$

Now we let ds go to 0 and obtain:

$$\cos(\theta+\chi)=\frac{\cos(\theta^0)+u_s}{\sqrt{(\cos(\theta^0)+u_s)^2+(\sin(\theta^0)+w_s)^2}} \tag{2.3.5}$$

In the same way we find:

$$\sin(\theta+\chi)=\frac{\sin(\theta^0)+w_s}{\sqrt{(\cos(\theta^0)+u_s)^2+(\sin(\theta^0)+w_s)^2}} \tag{2.3.6}$$

According to [8], we have:

$$\Lambda=\sqrt{(\cos(\theta^0)+u_s)^2+(\sin(\theta^0)+w_s)^2} \tag{2.3.7}$$

So we have:

$$\cos(\theta+\chi)=\frac{\cos(\theta^0)+u_s}{\Lambda} \tag{2.3.8}$$

and

$$\sin(\theta+\chi)=\frac{\sin(\theta^0)+w_s}{\Lambda} \tag{2.3.9}$$

So finally we get:

$$u_s=\Lambda\cos(\theta+\chi)-\cos(\theta^0) \tag{2.3.10}$$

and

$$w_s=\Lambda\sin(\theta+\chi)-\sin(\theta^0) \tag{2.3.11}$$

Now according to [8] we can express $\varepsilon,\kappa_y,\gamma$ in terms of Λ,θ and χ as follows:

$$\varepsilon=\Lambda\cos(\chi)-1 \tag{2.3.12}$$

$$\kappa_y=\frac{d\theta}{ds}-\frac{d\theta^0}{ds}=\theta_s-\theta_s^0 \tag{2.3.13}$$

$$\gamma = \Lambda \sin(\chi) \tag{2.3.14}$$

Now we can substitute this in our equations for u_s and w_s. With some trigonometric identities we obtain:

$$u_s = \Lambda \cos(\theta + \chi) - \cos(\theta^0) = \Lambda(\cos(\theta)\cos(\chi) - \sin(\theta)\sin(\chi)) - \cos(\theta^0) = (\varepsilon + 1)\cos(\theta) - \gamma \sin(\theta) - \cos(\theta^0) \tag{2.3.15}$$

and

$$w_s = \Lambda \sin(\theta + \chi) - \sin(\theta^0) = \Lambda(\sin(\theta)\cos(\chi) + \sin(\chi)\cos(\theta)) - \sin(\theta^0) = (\varepsilon + 1)\sin(\theta) + \gamma \cos(\theta) - \sin(\theta^0) \tag{2.3.16}$$

2.4 Elastic energy

Following [2], but using the notation as in section 2.1, we get that the total elastic energy is given by the sum of the stretching, bending and shear energies:

$$E_{el} = \int_0^\ell \left(\frac{1}{2} EA\varepsilon^2 + \frac{1}{2} EI\kappa_y^2 + \frac{\kappa_s GA}{2} \gamma^2 \right) ds \tag{2.4.1}$$

The total elastic energy is the sum of the stretching, the bending and the shear energies. Their respective lineic energies are given by:

$$\varepsilon_{\text{stretch}} = \frac{1}{2} EA\varepsilon^2, \varepsilon_{\text{bend}} = \frac{1}{2} EI\kappa_y^2, \varepsilon_{\text{shear}} = \frac{\kappa_s GA}{2} \gamma^2 \tag{2.4.2}$$

2.5 Boundary conditions

Naturally, we will use different boundary conditions in our different models. However, we always want to specify the angle of the beam at its endpoints. We will see why this is useful in section 2.6. Therefore we let our boundary conditions be:

$$\theta(0) = \theta^0(0) + \alpha \tag{2.5.1}$$

and

$$\theta(\ell) = \theta^0(\ell) + \beta \tag{2.5.2}$$

If the pre-curvature of the beam and all forces and moments are symmetric, then we obviously must have that $\beta = -\alpha$, which will be the case in most of our models.

In section 2.2 we also obtained the following boundary conditions:

$$\theta_s(0) = \theta_s^0(0) + \frac{M}{EI} \tag{2.5.3}$$

and

$$\theta_s(\ell) = \theta_s^0(\ell) + \frac{M}{EI} \tag{2.5.4}$$

θ specifies u_s and w_s completely by equations (2.3.15) and (2.3.16) respectively. One might think that we now need extra boundary conditions for u and w. However, that is not necessary. The only thing we need to do, is to specify the integration boundaries for the integrals of u_s and w_s:

$$u(\tilde{s}) = \int_0^{\tilde{s}} u_s(s)ds \tag{2.5.5}$$

and

$$w(\tilde{s}) = \int_0^{\tilde{s}} w_s(s)ds \tag{2.5.6}$$

In this way, we immediately get that $u(0) = w(0) = 0$. Since the other endpoint of the beam is allowed to move, we do need to get any boundary conditions for that.

2.6 Deriving the general equations for beam models

In this section we more or less follow the approach of [1]. It is well-known that the physical solution to a model is always the one with the lowest energy. We will use this in deriving the equations which describe our beam models. However, we only look at equilibrium solutions, since we are not particularly interested in the kinematics of the beam and since in our set-up, the beam will go to its equilibrium state very quickly. Therefore we ignore the kinetic energy in our energy considerations.

Now we consider a bent or buckled beam where the distance between the two endpoints has changed by a positive amount Δu with respect to the initial configuration (see figure 2.2). Using equation (2.3.15) we can express Δu in terms of the variables of the system, θ, ε and γ as follows:

$$\Delta u = -u(\ell) = -\int_0^{\ell} u_s(s)ds = \\ -\int_0^{\ell} (\varepsilon+1)\cos(\theta) - \gamma\sin(\theta) - \cos(\theta^0)ds \tag{2.6.1}$$

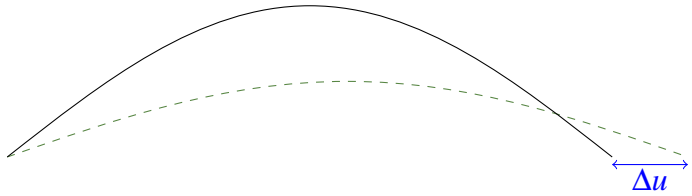


Figure 2.2: *Sketch of a buckled or bent beam. The distance between the two endpoints has changed by an amount Δu with respect to the initial configuration (the dashed dark green line).*

The minus-sign arises since we defined $u(\ell)$ to be negative, while Δu is positive. The total elastic energy in the system is given by equation (2.4.1). Since we ignore kinetic energy, we need to minimize this elastic energy to find the solution to our model. However, we need to keep in mind that we also have the additional constraint given by equation (2.6.1). Therefore we will use Lagrange multipliers to account for this constraint. The function we want to minimize then becomes, where we also use equation (2.3.13):

$$\begin{aligned}\mathscr{E} = E_{el} + \lambda(\Delta u + \int_0^\ell (\varepsilon+1)\cos(\theta) - \gamma\sin(\theta) - \cos(\theta^0)ds) = \\ \int_0^\ell \left(\frac{1}{2}EA\varepsilon^2 + \frac{1}{2}EI(\theta_s - \theta_s^0)^2 + \frac{\kappa_s GA}{2}\gamma^2\right)ds + \\ \lambda(\Delta u + \int_0^\ell (\varepsilon+1)\cos(\theta) - \gamma\sin(\theta) - \cos(\theta^0)ds) \quad (2.6.2)\end{aligned}$$

Here λ is the Lagrange multiplier. Now we can make a physical interpretation of this λ. We see in this equation that λ times the constraint has to have the unit of an energy. Since the unit of the constraint is that of a length, λ has to have the unit of a force. So we can interpret λ as the force needed to keep this displacement Δu intact. Therefore, we will later on substitute the force P for this λ since it is the force P that keeps the beam at its place.
Let $\theta, \varepsilon, \gamma$ be functions minimizing $\mathscr{E}$. Since they minimize $\mathscr{E}$, any small perturbation to these functions that preserves the boundary conditions should increase $\mathscr{E}$. Now we let $\delta(s)$ be a perturbation of θ satisfying:

$$\delta(0) = \delta(\ell) = \delta_s(0) = \delta_s(\ell) = 0 \quad (2.6.3)$$

and we define:

$$\tilde{\theta} = \theta + a\delta \quad (2.6.4)$$

Here a is a small constant. By the equation (2.6.3) the boundary conditions for $\tilde{\theta}$ are not different from those of θ. Then we can define:

$$\mathscr{E}_a = \int_0^\ell \frac{1}{2}EA\varepsilon^2 + \frac{1}{2}EI(\theta_s + a\delta_s - \theta_s^0)^2 + \frac{\kappa_s GA}{2}\gamma^2 ds + \\ \lambda(\Delta u + \int_0^\ell (\varepsilon+1)\cos(\theta + a\delta) - \gamma\sin(\theta + a\delta) - \cos(\theta^0)ds) \quad (2.6.5)$$

Now we calculate the total derivative of $\mathscr{E}_a$ with respect to a:

$$\frac{d\mathscr{E}_a}{da} = \int_0^\ell EI(\theta_s + a\delta_s - \theta_s^0)\delta_s ds + \lambda\int_0^\ell -(\varepsilon+1)\sin(\theta + a\delta)\delta ds \quad (2.6.6)$$

If we write:

$$\psi_a = \frac{1}{2}EA\varepsilon^2 + \frac{1}{2}EI(\theta_s + a\delta_s - \theta_s^0)^2 + \\ \frac{\kappa_s GA}{2}\gamma^2 + \lambda(\varepsilon+1)\cos(\theta + a\delta) - \lambda\gamma\sin(\theta + a\delta) - \lambda\cos(\theta^0) \quad (2.6.7)$$

then we see that:

$$\frac{d\mathscr{E}_a}{da} = \int_0^\ell \delta\frac{\partial\psi_a}{\partial\tilde{\theta}} + \delta_s\frac{\partial\psi_a}{\partial\tilde{\theta}_s} \quad (2.6.8)$$

Next we write:

$$\psi = \psi_0 \quad (2.6.9)$$

Then we see that if $a = 0$ we get $\tilde{\theta} = \theta$ and $\psi_a = \psi$ and furthermore we know that $\mathscr{E}_a$ has a minimum at $a = 0$, so we obtain:

$$0 = \left.\frac{d\mathscr{E}_a}{da}\right|_{a=0} = \int_0^\ell \delta\frac{\partial\psi}{\partial\theta} + \delta_s\frac{\partial\psi}{\partial\theta_s} \quad (2.6.10)$$

At this point we want to get rid of the term with δ_s, since we do not know much about that perturbation. We use integration by parts to eliminate it, and obtain:

$$0 = \int_0^\ell \delta\frac{\partial\psi}{\partial\theta} - \delta\frac{d}{ds}\left(\frac{\partial\psi}{\partial\theta_s}\right)ds + \left[\delta\frac{\partial\psi}{\partial\theta_s}\right]_0^\ell \quad (2.6.11)$$

Since we demanded that $\delta(0) = \delta(\ell) = 0$, we obtain:

$$0 = \int_0^\ell \delta\frac{\partial\psi}{\partial\theta} - \delta\frac{d}{ds}\left(\frac{\partial\psi}{\partial\theta_s}\right)ds = \int_0^\ell \delta\left(\frac{\partial\psi}{\partial\theta} - \frac{d}{ds}\left(\frac{\partial\psi}{\partial\theta_s}\right)\right)ds \quad (2.6.12)$$

Now we define for notation compactness:

$$f(s) = \frac{\partial \psi}{\partial \theta} - \frac{d}{ds}\left(\frac{\partial \psi}{\partial \theta_s}\right) \tag{2.6.13}$$

Since equation (2.6.11) holds for any function δ satisfying equation (2.6.3), we can choose our function δ a bit more specific. We define:

$$r(s) = s^2(s-\ell)^2 \tag{2.6.14}$$

Then we see immediately that $r(0) = r(\ell) = r_s(0) = r_s(\ell) = 0$, since 0 and ℓ are roots of higher multiplicity of r. Then we define:

$$\delta(s) = r(s)f(s) \tag{2.6.15}$$

Then we have:

$$\begin{aligned} \delta(0) = r(0)f(0) = 0f(0) = 0, \delta(\ell) = r(\ell)f(\ell) = 0f(\ell) = 0, \\ \delta_s(0) = r(0)f_s(0) + r_s(0)f(0) = 0f_s(0) + 0f(0) = 0, \\ \delta_s(\ell) = r(\ell)f_s(\ell) + r_s(\ell)f(\ell) = 0f_s(\ell) + 0f(\ell) = 0 \end{aligned} \tag{2.6.16}$$

So this function δ indeed satisfies equation (2.6.3). So we must have:

$$0 = \int_0^\ell \delta\left(\frac{\partial \psi}{\partial \theta} - \frac{d}{ds}\left(\frac{\partial \psi}{\partial \theta_s}\right)\right) ds = \int_0^\ell r(s)f(s)^2 ds \tag{2.6.17}$$

Since we defined r to be a non-negative function and f^2 is also non-negative, it follows that the integrand is non-negative so the only reason this integral can be 0 is for the integrand to be 0. Since r is only 0 at the endpoints, this must mean that f^2 and thus f are zero. Therefore we obtain:

$$0 = \frac{\partial \psi}{\partial \theta} - \frac{d}{ds}\left(\frac{\partial \psi}{\partial \theta_s}\right) \tag{2.6.18}$$

This equation is known as the Euler-Lagrange equation. Now we can use the definition of ψ to obtain:

$$\begin{aligned} 0 = \frac{\partial \psi}{\partial \theta} - \frac{d}{ds}\left(\frac{\partial \psi}{\partial \theta_s}\right) = \\ -\lambda(\varepsilon+1)\sin(\theta) - \lambda\gamma\cos(\theta) - EI(\theta_{ss} - \theta_{ss}^0) \end{aligned} \tag{2.6.19}$$

We can rewrite this to:

$$EI(\theta_{ss} - \theta_{ss}^0) + \lambda(\varepsilon + 1)\sin(\theta) + \lambda\gamma\cos(\theta) = 0 \tag{2.6.20}$$

In the same way we find if we perturb ε or γ with a perturbation δ:

$$0 = \frac{\partial\psi}{\partial\varepsilon} - \frac{d}{ds}\left(\frac{\partial\psi}{\partial\varepsilon_s}\right) = EA\varepsilon + \lambda\cos(\theta) \tag{2.6.21}$$

and

$$0 = \frac{\partial\psi}{\partial\gamma} - \frac{d}{ds}\left(\frac{\partial\psi}{\partial\gamma_s}\right) = \kappa_s GA\gamma - \lambda\sin(\theta) \tag{2.6.22}$$

Since the force P causes the displacement Δu, we get the following three general equations that describe the single beam model:

$$0 = EI(\theta_{ss} - \theta_{ss}^0) + P(\varepsilon + 1)\sin(\theta) + P\gamma\cos(\theta) \tag{2.6.23}$$

$$0 = EA\varepsilon + P\cos(\theta) \tag{2.6.24}$$

$$0 = \kappa_s GA\gamma - P\sin(\theta) \tag{2.6.25}$$

2.7 Adimensionalizing the equations

So now we have these three equations which fully describe our model. First, we can substitute the equations in each other to obtain one differential equation which we will try to solve in different cases (e.g. with our without extensibility). First we get from equation (2.6.24):

$$\varepsilon = -\frac{P}{EA}\cos(\theta) \tag{2.7.1}$$

So substituting this back in equation (2.6.23), this yields:

$$0 = EI(\theta_{ss} - \theta_{ss}^0) + P\sin(\theta) - \frac{P^2}{EA}\cos(\theta)\sin(\theta) + P\gamma\cos(\theta) \tag{2.7.2}$$

Next we solve equation (2.6.25) and obtain:

$$\gamma = \frac{P}{\kappa_s GA}\sin(\theta) \tag{2.7.3}$$

Substituting this in equation (2.7.2) gives:

$$0 = EI(\theta_{ss} - \theta_{ss}^0) + P\sin(\theta) + \left(\frac{P^2}{\kappa_s GA} - \frac{P^2}{EA}\right)\cos(\theta)\sin(\theta) \tag{2.7.4}$$

We first account for the length of the beam by introducing:

$$t = \frac{s}{\ell} \tag{2.7.5}$$

Rewriting equation (2.7.4) a bit, gives:

$$0 = (\theta_{tt} - \theta_{tt}^0) + \frac{P\ell^2}{EI}\sin(\theta) + \frac{\ell^2}{EI}\left(\frac{P^2}{\kappa_s GA} - \frac{P^2}{EA}\right)\cos(\theta)\sin(\theta) \tag{2.7.6}$$

Now we introduce the dimensionless force as:

$$\overline{P} = \frac{P\ell^2}{EI} \tag{2.7.7}$$

Substituting this back gives us:

$$0 = (\theta_{tt} - \theta_{tt}^0) + \overline{P}\sin(\theta) + \overline{P}^2\left(\frac{E^2I^2}{\ell^2\kappa_s GA} - \frac{EI^2}{\ell^2 A}\right)\cos(\theta)\sin(\theta) \tag{2.7.8}$$

Finally we make this expression fully dimensionless by introducing:

$$B = -\left(\frac{E^2 I^2}{\ell^2 \kappa_s G A} - \frac{E I^2}{\ell^2 A}\right) \tag{2.7.9}$$

Thus we have our dimensionless equation:

$$0 = (\theta_{tt} - \theta_{tt}^0) + \overline{P}\sin(\theta) - \overline{P}^2 B \cos(\theta)\sin(\theta) \tag{2.7.10}$$

In the next chapter we will look at different specific models and see what this equation becomes for these models.
So even though the equation describing the system is a second order differential equation, we have four boundary conditions, while one only needs two boundary conditions to solve the equation. The two equations we use for this solution are equations (2.5.1) and (2.5.3). However, by the physical interpretation of this model, there is also a relation between P, M and α. This relation is described by a third boundary condition, equation (2.5.2). The fourth boundary condition, equation (2.5.4), does not give any new information, since it automatically satisfied if the other three boundary conditions hold. We will see this in chapter 4.

Chapter 3

Obtaining equations for different specific models

In the previous chapter we derived the most general differential equations describing the single beam models. Now we will look at some specific models. We will study the following beam models, which are more or less ordered by increasing generality:

- An inextensible beam without pre-curvature and with constant force applied.
- An inextensible pre-curved beam with constant force applied.
- A pre-curved beam with zero force applied.
- An extensible pre-curved beam with constant force applied.
- An extensible pre-curved beam with both endpoints fixed.

3.1 An inextensible beam without pre-curvature and with constant force applied

First, we will look at a beam without pre-curvature, i.e. we have $\theta^0 \equiv 0$. In this model we apply a constant force P in the x-direction and a moment M on both ends of the beam. We ignore shear and assume that the beam is inextensible, so we have:

$$\varepsilon = 0, \gamma = 0 \tag{3.1.1}$$

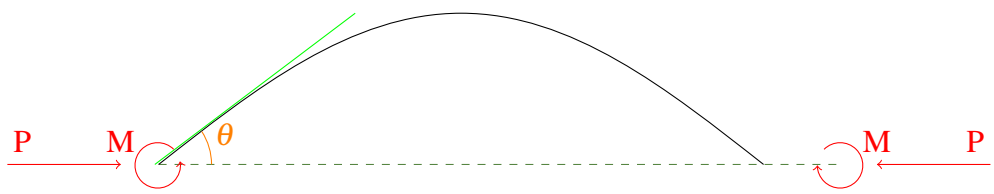


Figure 3.1: *Sketch of the model of section 3.1. The dashed line represents the initial position of the beam, while the solid line represents the beam in its bent state. The shape of the beam is described by the angle θ. The load P is applied in the x-direction. The moment M is applied on both endpoints of the beam. The moment is negative if the arrows point outwards and positive if they point inwards. Since the arrows point either both inwards or outward, the momenta at the endpoints will have the same sign.*

3.1.1 Analyzing this model

Since we have that $\varepsilon = \gamma = \theta^0 = 0$, we can substitute this in equation (2.7.10) and obtain:

$$\theta_{tt} + \overline{P}\sin(\theta) = 0 \tag{3.1.2}$$

3.1.2 Boundary conditions

Now we have to specify the boundary conditions for this model. We will use the boundary conditions in section 2.5. However, they will be used in their adimensionalized form, i.e. the variable $t = \frac{s}{\ell}$ is used instead of s. Because the forces, the moment and the pre-curvature are symmetric, this gives:

$$\theta(0) = \alpha, \theta(1) = -\alpha \tag{3.1.3}$$

Since we have to account for the moment, we obtain the following conditions:

$$\theta_t(0) = \frac{M\ell}{EI}, \theta_t(1) = \frac{M\ell}{EI} \tag{3.1.4}$$

Now we will also adimensionalize these conditions by introducing:

$$\overline{M} = \frac{M\ell}{EI} \tag{3.1.5}$$

Then this results in the conditions:

$$\theta_t(0) = \overline{M}, \theta_t(1) = \overline{M} \tag{3.1.6}$$

3.2 An inextensible pre-curved beam with constant force applied

Now we will add pre-curvature to our system. However, we still assume that the beam is inextensible and we ignore shear. Therefore we have like in section 3.1 that $\varepsilon = 0, \gamma = 0$. We also assume that the pre-curvature is symmetric in the line parallel to the y-axis which goes through the middle of the beam.

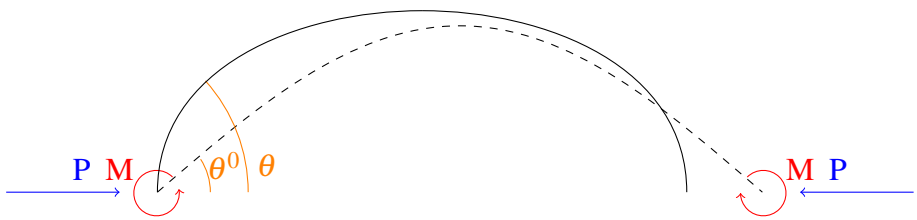


Figure 3.2: *Sketch of the model of section 3.2. The dashed line represents the initial position of the beam, while the solid line represents the beam in its bent state. The shape of the beam in the bent state is described by the angle θ, while in the initial state, it is described by the angle θ^0. The load P is applied in the x-direction. The moment M is applied on both endpoints of the beam.*

3.2.1 Analyzing this model

Analogous to section 3.1.1, we get no information from the shear and compression equations, so we can substitute $\varepsilon = \gamma = 0$ in equation (2.7.10) and obtain:

$$\theta_{tt} - \theta^0_{tt} + \overline{P}\sin(\theta) = 0 \tag{3.2.1}$$

3.2.2 Boundary conditions

The boundary conditions are the same as those derived in section 3.1.2, except that we now have pre-curvature. Therefore they are given by:

$$\theta(0) = \alpha + \theta^0(0), \theta(1) = -\alpha + \theta^0(1) \tag{3.2.2}$$

and

$$\theta_t(0) = \overline{M} + \theta^0_t(0), \theta_t(1) = \overline{M} + \theta^0_t(1) \tag{3.2.3}$$

3.3 A pre-curved beam with zero force applied

The following model is probably the simplest model we can look at. This is even the case if we include shear and extensibility, since we will see later that they do not contribute to the solution in this model. We will look at a pre-curved beam and we will apply moment M on both endpoints. However, we do not apply force, i.e. in this model we assume that $P \equiv 0$. In contrast to previous models, we do not make any assumptions on the symmetry of the pre-curvature.

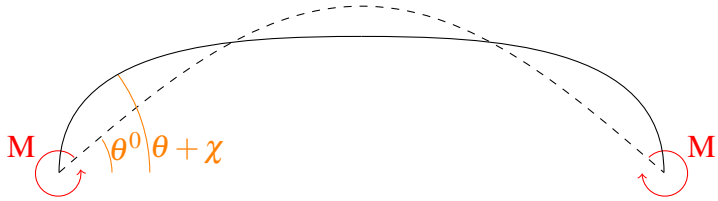


Figure 3.3: *Sketch of the model of section 3.3. The dashed line represents the initial position of the beam, while the solid line represents the beam in its bent state. The shape of the beam in the bent state is described by the rotation angle θ and the shear angle χ, while in the initial state, it is described by the angle θ^0. The moment M is applied on both endpoints of the beam.*

3.3.1 Analyzing the model

Since we have no force, we can substitute $P = 0$ in equations (2.6.23), (2.6.24) and (2.6.25) and hence we obtain the set of equations, where we use the variable t instead of s:

$$\theta_{tt} - \theta^0_{tt} = 0 \tag{3.3.1}$$

$$EA\varepsilon = 0 \tag{3.3.2}$$

$$\kappa_s GA\gamma = 0 \tag{3.3.3}$$

3.3.2 Boundary conditions

Since our model now lacks symmetry in the endpoints, we get the following boundary conditions as stated in section 2.5:

$$\theta(0) = \alpha + \theta^0(0) \tag{3.3.4}$$

$$\theta(1) = \beta + \theta^0(1) \tag{3.3.5}$$

Since we apply moment on both ends, we also get:

$$\theta_t(0) = \theta_t^0(0) + \overline{M} \tag{3.3.6}$$

$$\theta_t(1) = \theta_t^0(1) + \overline{M} \tag{3.3.7}$$

3.4 Including extensibility and shear in the model for the pre-curved beam with constant force applied

As stated in the title, we will look at our model in section 3.2, but we will include extensibility and shear in our calculations. As a reference, the following figure describes the model we are analyzing.

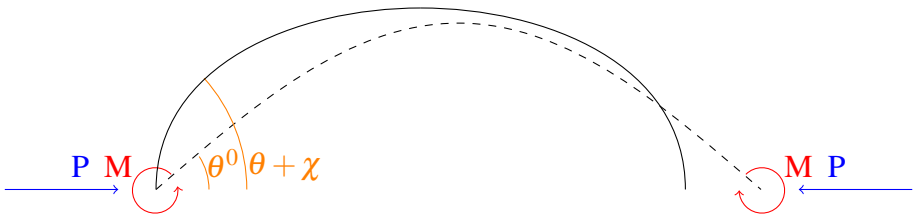


Figure 3.4: *Sketch of the model of section 3.4. The dashed line represents the initial position of the beam, while the solid line represents the beam in its bent state. The shape of the beam in the bent state is described by the rotation angle θ and the shear angle θ^0, while in the initial state, it is described by the angle θ^0. The load P is applied in the x-direction. The moment M is applied on both endpoints of the beam.*

3.4.1 Analyzing the model

Since this is the most general model, we get the most general equation. So the equation fully describing this model is equation (2.7.10):

$$0 = \theta_{tt} - \theta^0_{tt} + \overline{P}\sin(\theta) - \overline{P}^2 B\cos(\theta)\sin(\theta) \tag{3.4.1}$$

3.4.2 Boundary conditions

The boundary conditions are identical to those in section 3.2.2 and hence they are given by:

$$\theta(0) = \alpha + \theta^0(0), \theta(1) = -\alpha + \theta^0(1) \tag{3.4.2}$$

and

$$\theta_t(0) = \overline{M} + \theta^0_t(0), \theta_t(1) = \overline{M} + \theta^0_t(1) \tag{3.4.3}$$

3.5 An extensible pre-curved beam with both ends fixed

In this section we look at a model similar to that of section 3.3. Again we look at a pre-curved beam and we include shear and extensibility. There is no force applied. However, in this model we do assume that the moment on both ends is equal and we assume that the pre-curvature is symmetric. Furthermore, we apply an extra condition for this system: we assume that the endpoint of the beam is fixed, i.e. that $u(1) = 0$.

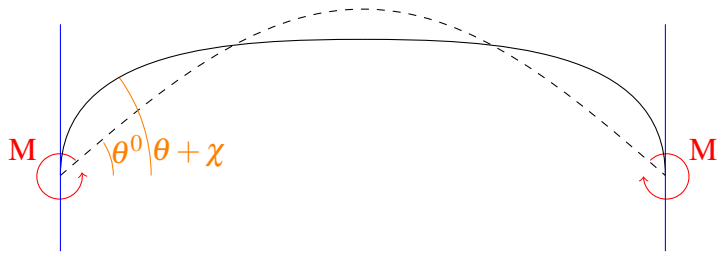


Figure 3.5: *Sketch of the model of section 3.5. The dashed line represents the initial position of the beam, while the solid line represents the beam in its bent state. The shape of the beam in the bent state is described by the angle θ, while in the initial state, it is described by the angle θ^0. Both endpoints are fixed. The moment M is applied on both endpoints of the beam.*

3.5.1 Analyzing this model

Since there is an additional constraint in this model, we cannot immediately use our results of section 2.6. First we will rewrite the condition that $u(1) = 0$. This gives:

$$0 = u(1) = u(1) - u(0) = \int_0^1 u_t dt = \int_0^1 (\varepsilon + 1)\cos(\theta) - \gamma \sin(\theta) - \cos(\theta^0) dt \quad (3.5.1)$$

This equation is the same as the constraint that we used in section 2.6. Therefore we can introduce the Lagrange multiplier μ. Then we conclude that $-\mu$ is an effective horizontal force that the system acts upon itself and which arises from the impossibility of the endpoints to move. Therefore we now can use the equations

from section 2.6 and obtain the following differential equation:

$$0 = (\theta_{tt} - \theta_{tt}^0) - \overline{\mu}\sin(\theta) - \overline{\mu}^2 B\cos(\theta)\sin(\theta) \tag{3.5.2}$$

For this reason we will not do a separate analysis of this model in the next chapter. Everything we say for the 'normal' extensible model will also apply for this model.

3.5.2 Boundary conditions

The boundary conditions are as usual identical to those in section 3.2.2 and hence they are given by:

$$\theta(0) = \alpha + \theta^0(0), \theta(1) = -\alpha + \theta^0(1) \tag{3.5.3}$$

and

$$\theta_t(0) = \overline{M} + \theta_t^0(0), \theta_t(1) = \overline{M} + \theta_t^0(1) \tag{3.5.4}$$

3.6 Summary

The following table summarizes the different models, which were described in this section.
In this table we use the adimensionalized variables and parameters:

$$t = \frac{s}{\ell} \tag{3.6.1}$$

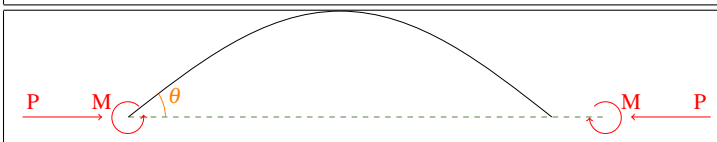
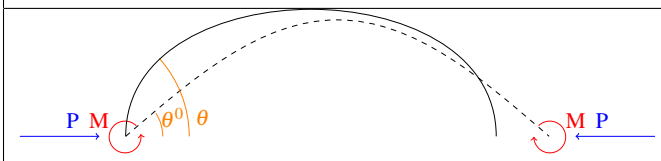
$$\overline{P} = \frac{P\ell^2}{EI} \tag{3.6.2}$$

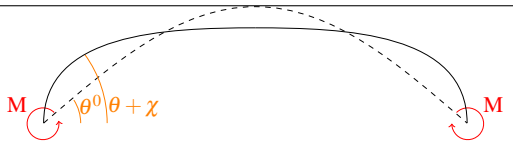
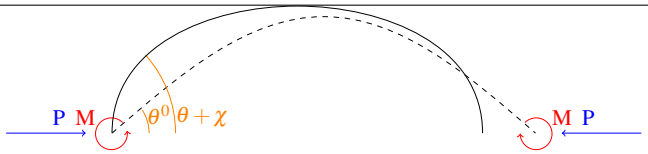
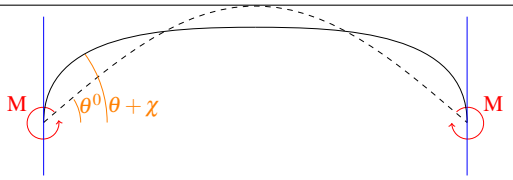
$$\overline{M} = \frac{M\ell}{EI} \tag{3.6.3}$$

$$\overline{\kappa} = \frac{\kappa\ell^3}{EI} \tag{3.6.4}$$

$$\overline{\mu} = \frac{\mu\ell^2}{EI} \tag{3.6.5}$$

$$B = -\frac{E^2I^2}{\ell^2\kappa_s GA} + \frac{EI^2}{\ell^2 A} \tag{3.6.6}$$

Sketch	Equation and boundary conditions	Specifications
	$\theta_{tt} + \overline{P}\sin(\theta) = 0$ $\theta_t(0) = \overline{M}, \theta_t(1) = \overline{M}$ $\theta(0) = \alpha, \theta(1) = -\alpha$	No shear; inextensible; no pre-curvature; moment on both ends; constant horizontal force
	$(\theta_{tt} - \theta^0_{tt}) + \overline{P}\sin(\theta) = 0$ $\theta_t(0) = \theta^0_t(0) + \overline{M}, \theta_t(1) = \theta^0_t(1) + \overline{M}$ $\theta(0) = \theta^0(0) + \alpha, \theta(1) = \theta^0(1) - \alpha$	No shear; inextensible; pre-curvature; moment on both ends; constant horizontal force

	$\theta_{tt}-\theta^0_{tt}=0$ $\theta_t(0) = \theta^0_t(0) + \overline{M}, \theta_t(1) = \theta^0_t(1)+\overline{M}$ $\theta(0) = \theta^0(0) + \alpha, \theta(1) = \theta^0(1)-\beta$	Shear; extensible; pre-curvature; moment on both ends; no horizontal force
	$0 = \theta_{tt} - \theta^0_{tt} + \overline{P}\sin(\theta) - \overline{P}^2 B\cos(\theta)\sin(\theta)$ $\theta_t(0) = \theta^0_t(0) + \overline{M}, \theta_t(1) = \theta^0_t(1)+\overline{M}$ $\theta(0) = \theta^0(0) + \alpha, \theta(1) = \theta^0(1)-\alpha$	Shear; extensible; pre-curvature; moment on both ends; constant horizontal force
	$0 = \theta_{tt} - \theta^0_{tt} - \overline{\mu}\sin(\theta) - \overline{\mu}^2 B\cos(\theta)\sin(\theta)$ $\theta_t(0) = \theta^0_t(0) + \overline{M}, \theta_t(1) = \theta^0_t(1)+\overline{M}$ $\theta(0) = \theta^0(0) + \alpha, \theta(1) = \theta^0(1)-\alpha$	Shear; extensible; pre-curvature; moment on both ends; no horizontal force; fixed endpoints

Chapter 4

Solutions and force-angle-moment relations

In this chapter we will further analyze the models described in chapter 3. We start with a small analysis of the model of chapter 3.3. The most substantial part of this chapter will go to the analysis of the model of section 3.1. This is because this model is simple enough to find exact solutions and other analytic relation, while it is still complex enough to answer a lot of questions we asked ourselves in chapter 1. Then we analyze the model of section 3.2 and see that there are many similarities with the previous model. We end with a small analysis of the model of section 3.4, since we did not have time to fully investigate this model.

4.1 Solving the model without force

We will first look at the model from section 3.3. As stated in that section, this is by far the easiest model to solve, since $\overline{P} = 0$. However, that does not necessarily imply that it is not interesting. The equations describing this system are, as derived in section 3.3:

$$\theta_{tt} - \theta^0_{tt} = 0 \tag{4.1.1}$$

$$EA\varepsilon = 0 \tag{4.1.2}$$

$$\kappa_s GA\gamma = 0 \tag{4.1.3}$$

We immediately see that the solutions of the respective equations are:

$$\theta(t) = \theta^0(t) + At + B \tag{4.1.4}$$

$$\varepsilon = 0 \tag{4.1.5}$$

$$\gamma = 0 \tag{4.1.6}$$

Here A and B are constants which follow from the boundary conditions. First we note, that even though we included shear and compression in our analysis, there are no shear and compression effects in this model. Now we will use the boundary conditions to find the values of the constants A and B. First we substitute boundary condition (3.3.4) to obtain:

$$\alpha + \theta^0(0) = B + \theta^0(0) \tag{4.1.7}$$

Thus we conclude that $B = \alpha$. Now we substitute boundary condition (3.3.6) and that gives:

$$\theta_t^0(0) + \overline{M} = \theta_t^0(0) + A \tag{4.1.8}$$

This gives that $A = \overline{M}$. So we have:

$$\theta = \theta^0 + \overline{M}t + \alpha \tag{4.1.9}$$

Now we have two unused boundary conditions. These give relations between the angles α, β and the momentum M. Substituting boundary condition (3.3.5) in equation (4.1.9) gives:

$$\theta^0(1) + \beta = \theta^0(1) + \overline{M} + \alpha \tag{4.1.10}$$

So we can express β in terms of M and α as:

$$\beta = \overline{M} + \alpha \tag{4.1.11}$$

When we use boundary condition (3.3.7) and substitute it in equation (4.1.9), that gives:

$$\theta_t^0(1) + \overline{M} = \theta_t^0(1) + \overline{M} \tag{4.1.12}$$

Or equivalently:

$$\overline{M} = \overline{M} \tag{4.1.13}$$

So this boundary condition does not give any new information, just like we said at the end of chapter 2. We can now conclude that only applying moment does not do that much to a beam: it adds a constant curvature to the beam, i.e. at each point of the beam the curvature of the beam is increased (or decreased) by a constant number, as can be seen in equation (4.1.9). It also changes the angle at zero to make sure the y-value of the endpoint of the beam remains zero. In

the following sections we will see that adding moment to a model with force is much less trivial than just adding a constant curvature and consequently, these models can be analyzed into much greater detail than this model. Now suppose that our pre-curvature is symmetric. As before, we can then conclude that $\beta = -\alpha$. Combining this with equation (4.1.11) gives us:

$$-\alpha = \overline{M} + \alpha \tag{4.1.14}$$

Rewriting this gives:

$$\alpha = -\frac{\overline{M}}{2} \tag{4.1.15}$$

Using this we can plot θ for a few different values of α, since α now fully determines θ:

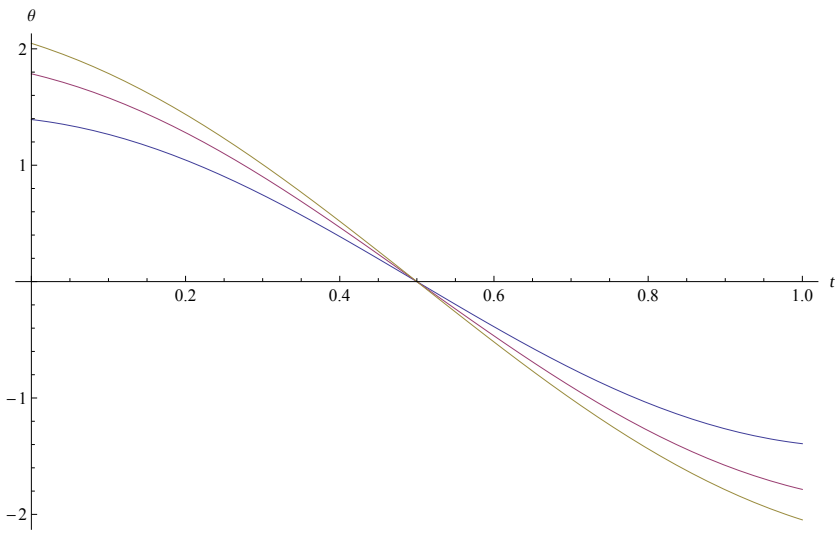


Figure 4.1: *Exact solution of the model of section 3.3. The used pre-curvature is given by: $\theta^0(t) = cos(\pi t)$. The orange curve is given by $\alpha = \frac{\pi}{3}$, the purple by $\alpha = \frac{\pi}{4}$ and the blue one by $\alpha = \frac{\pi}{8}$.*

The quantity $\frac{2EI}{\ell}$ is called the flexural stiffness of the beam. This quantity can be seen as the resistance offered by the beam while undergoing bending. So we note that in this model, since $\overline{M} = \frac{M\ell}{EI}$, the angle α is the quotient of the applied moment and the flexural stiffness of the beam. In particular we see that there is a linear relation between the moment we apply and the angle α. We will see later that when we also apply a small constant force, this relation is only approximately linear and when the force is large enough (larger than the Euler load) the relation will be fully non-linear.

4.2 Analysis of the model for an inextensible initially straight beam with a constant force applied

In this section we will look at the model of section 3.1. First we will solve this model directly and then we will derive a relation between the force, the moment and the angle α under certain assumptions. We will use this to make different plots of beams. Finally we will plot a full bifurcation diagram of the relation between the force, the moment and the angle α. We will analyze this diagram further in chapter 5.

4.2.1 Solving the model

In this section, we follow the approach of [3]
We have the equation:

$$\theta_{tt} + \overline{P}\sin(\theta) = 0 \tag{4.2.1}$$

We note that this is the equation for a non-linear classical harmonic oscillator (see [3]). Multiplying this equation with θ_t and integrating gives:

$$\frac{1}{2}\theta_t^2 - \overline{P}\cos(\theta) = C \tag{4.2.2}$$

Here C is a constant. Using the boundary conditions, we obtain:

$$\frac{1}{2}\overline{M}^2 - \overline{P}\cos(\alpha) = C \tag{4.2.3}$$

So this yields the equation:

$$\frac{1}{2}\theta_t^2 - \overline{P}\cos(\theta) = \frac{1}{2}\overline{M}^2 - \overline{P}\cos(\alpha) \tag{4.2.4}$$

We let $\omega_0 = \sqrt{\overline{P}}$ and $k = \frac{1}{2}\left(\frac{\overline{M}^2}{2\overline{P}} - \cos(\alpha) + 1\right)$. Then this results in the equation:

$$(\theta_t)^2 = 2\omega_0^2(\cos(\theta) + \frac{\overline{M}^2}{2\overline{P}} - \cos(\alpha)) \tag{4.2.5}$$

This can be written as:

$$(\theta_t)^2 = 4\omega_0^2\left(\frac{1}{2}\left(\frac{\overline{M}^2}{2\overline{P}} - \cos(\alpha) + 1\right) - \sin^2\left(\frac{\theta}{2}\right)\right) \tag{4.2.6}$$

And thus:

$$(\theta_t)^2 = 4\omega_0^2\left(k - \sin^2\left(\frac{\theta}{2}\right)\right) \tag{4.2.7}$$

We define a new variable y by:

$$y = \sin\frac{\theta}{2} \tag{4.2.8}$$

We will now write the equation in terms of y. This gives:

$$\frac{dy}{dt} = \frac{dy}{d\theta}\frac{d\theta}{dt} = \frac{1}{2}\frac{d\theta}{dt}\cos\frac{\theta}{2} \tag{4.2.9}$$

Therefore:

$$\left(\frac{dy}{dt}\right)^2 = \frac{1}{4}\left(\frac{d\theta}{dt}\right)^2\cos^2\frac{\theta}{2} = \frac{1}{4}\left(1-\sin^2\frac{\theta}{2}\right)\left(\frac{d\theta}{dt}\right)^2 = \frac{1}{4}(1-y^2)\left(\frac{d\theta}{dt}\right)^2 \tag{4.2.10}$$

Thus this yields:

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{4}{1-y^2}\left(\frac{dy}{dt}\right)^2 \tag{4.2.11}$$

Now we substitute this in equation (4.2.7) and obtain:

$$\frac{4}{1-y^2}\left(\frac{dy}{dt}\right)^2 = 4\omega_0^2(k-y^2) \tag{4.2.12}$$

Next we rewrite this equation to obtain:

$$\left(\frac{dy}{dt}\right)^2 = \omega_0^2 k(1-y^2)\left(1-\frac{y^2}{k}\right) \tag{4.2.13}$$

We now introduce $\tau = \omega_0 t$ and this gives:

$$\left(\frac{dy}{d\tau}\right)^2 = k(1-y^2)\left(1-\frac{y^2}{k}\right) \tag{4.2.14}$$

It holds that $\theta(0) = \alpha$, so $y(0) = \sin\frac{\alpha}{2}$. So we introduce:

$$z = \frac{y}{\sqrt{k}} \tag{4.2.15}$$

such that:

$$z(0) = \frac{\sin\frac{\alpha}{2}}{\sqrt{k}} \tag{4.2.16}$$

Then this results in:

$$\left(\frac{dz}{d\tau}\right)^2 = (1-z^2)\left(1-kz^2\right) \tag{4.2.17}$$

Now we will solve this equation. We get:

$$\frac{d\tau}{dz} = \pm\frac{1}{\sqrt{(1-z^2)(1-kz^2)}} \tag{4.2.18}$$

Next we define the function Sign by:

$$\text{Sign}(\zeta) = \begin{cases} -1, \text{ if } \zeta \leq 0 \\ 1, \text{ if } \zeta > 0 \end{cases} \tag{4.2.19}$$

So this gives:

$$\text{Sign}\left(\theta_t(0)\right) = \text{Sign}(M) \tag{4.2.20}$$

and thus, since all constants in the change of variables are positive:

$$\text{Sign}\left(\frac{dz}{d\tau}(0)\right) = \text{Sign}(M) \tag{4.2.21}$$

Therefore this yields:

$$\text{Sign}(M)\frac{dz}{d\tau} = \frac{1}{\sqrt{(1-z^2)(1-kz^2)}} \tag{4.2.22}$$

Now we integrate this system to obtain τ as a function of z:

$$-\text{Sign}(M)\tau = -\int_{\frac{\sin\frac{\alpha}{2}}{\sqrt{k}}}^{z} \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k\zeta^2)}} \tag{4.2.23}$$

We split this integral in two separate integrals to obtain:

$$-\text{Sign}(M)\tau = \int_0^{\frac{\sin\frac{\alpha}{2}}{\sqrt{k}}} \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k\zeta^2)}} - \int_0^{z} \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k\zeta^2)}} \tag{4.2.24}$$

Now the incomplete elliptic integral of the first kind, $F(\phi;m)$, is given by:

$$F(\phi;m) = \int_0^{\phi} \frac{dw}{\sqrt{(1-w^2)(1-m^2w^2)}} \tag{4.2.25}$$

Therefore we have:

$$-\mathrm{Sign}(M)\tau = F\left(\frac{\sin\frac{\alpha}{2}}{\sqrt{k}};\sqrt{k}\right) - F(z;\sqrt{k}) \tag{4.2.26}$$

Now we rewrite this to:

$$F(z;\sqrt{k}) = F\left(\frac{\sin\frac{\alpha}{2}}{\sqrt{k}};\sqrt{k}\right) + \mathrm{Sign}(M)\tau \tag{4.2.27}$$

If a function u is given by:

$$u = F(x;m) \tag{4.2.28}$$

then the so-called Jacobi elliptic function sn is given by:

$$sn(u;m) = x \tag{4.2.29}$$

Therefore we now obtain:

$$z = sn\left(F\left(\frac{\sin\frac{\alpha}{2}}{\sqrt{k}};\sqrt{k}\right) + \mathrm{Sign}(M)\tau;\sqrt{k}\right) \tag{4.2.30}$$

Now we substitute back to the original variables and obtain:

$$\sin\frac{\theta}{2} = \sqrt{k}sn\left(F\left(\frac{\sin\frac{\alpha}{2}}{\sqrt{k}};\sqrt{k}\right) + \mathrm{Sign}(M)\omega_0 t;\sqrt{k}\right) \tag{4.2.31}$$

with

$$k = \frac{1}{2}\left(\frac{\overline{M}^2}{2\overline{P}} - \cos(\alpha) + 1\right), \omega_0 = \sqrt{\overline{P}} \tag{4.2.32}$$

So this finally yields:

$$\theta = 2\arcsin\left(\sqrt{k}sn\left(F\left(\frac{\sin\frac{\alpha}{2}}{\sqrt{k}};\sqrt{k}\right) + \mathrm{Sign}(M)\omega_0 t;\sqrt{k}\right)\right) \tag{4.2.33}$$

Now we will plot this function for several values of $k, \alpha, \overline{P}$. Note that there is a relation between $\alpha, \overline{M}$ and $\overline{P}$ and therefore we cannot choose just any three values of these variables independently. We used the result of section 4.2.2 to choose these values correctly. The plots are made using Mathematica.

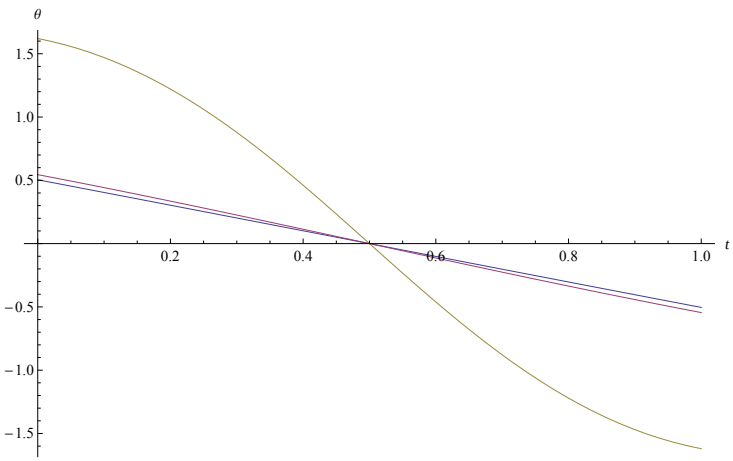


Figure 4.2: *Exact solution of the model of section 3.2. The used parameters are:* $\overline{M} = -1$ *and from the one with the highest value of* $\theta(0)$ *to the lowest:* $\overline{P} = 10, 1, 0.1$. *The value of* α *is chosen using section 4.2.2*

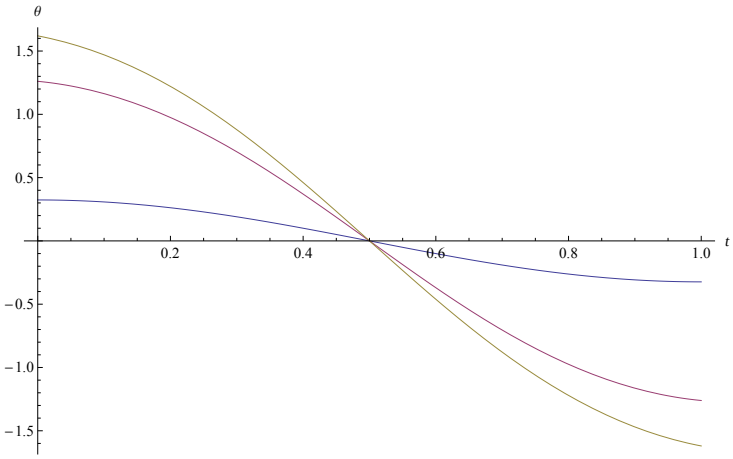


Figure 4.3: *Exact solution of the model of section 3.2. The used parameters are:* $\overline{P} = 10$ *and from the one with the highest value of* $\theta(0)$ *to the lowest:* $\overline{M} = -1, -0.5, 0$. *The value of* α *is chosen using section 4.2.2*

We note that each graph goes through 0 when $t = \frac{1}{2}$. This is of course a result of the symmetry of our problem. In the same way we see that indeed for each beam it holds that $\theta(1) = -\theta(0)$. Even more general, we can say that $\theta(t)$ is a symmetric function with respect to $t = \frac{1}{2}$. That is also why the boundary condition given by equation (2.5.4) does not give us any new information. Although these might seem simple observations, we will need these results later on in section 4.2.2. Therefore it is important that we keep these results in mind.

4.2.2 A force-angle-moment relation

In this section we follow the approach of [2].
We will assume that θ is a monotonically increasing or decreasing function of t. This is not an unreasonable assumption: when the values of $\overline{P}$ and $\overline{M}$ are small enough this will generically be the case. For example, in figures 4.2 and 4.3 all solutions that were plotted are monotonic functions. More general, if $P < P_e$ then θ will be monotonic as we will see later on. Equation (4.2.4) tells us that:

$$\frac{1}{2}\theta_t^2 - \overline{P}\cos(\theta) = \frac{1}{2}\overline{M}^2 - \overline{P}\cos(\alpha) \tag{4.2.34}$$

Now we will find a relation between α, $\overline{P}$ and $\overline{M}$. We get:

$$\theta_t^2 = 2\overline{P}\cos(\theta) + \overline{M}^2 - 2\overline{P}\cos(\alpha) \tag{4.2.35}$$

Writing $\theta_t = \frac{d\theta}{dt}$ this gives:

$$(d\theta)^2 = \left(2\overline{P}\cos(\theta) + \overline{M}^2 - 2\overline{P}\cos(\alpha)\right)(dt)^2 \tag{4.2.36}$$

Now we use the function Sign from section 4.2.1 and obtain:

$$dt = \mathrm{Sign}(M)\frac{1}{\sqrt{2\overline{P}\cos(\theta) + \overline{M}^2 - 2\overline{P}\cos(\alpha)}}d\theta \tag{4.2.37}$$

We clearly have:

$$\int_0^1 dt = 1 \tag{4.2.38}$$

We now want to express this as an integral over θ. When looking at the boundary conditions, we notice that θ goes from α to $-\alpha$. Since θ is a monotonic function, this gives:

$$1 = \int_0^1 dt = \mathrm{Sign}(M)\int_\alpha^{-\alpha} \frac{1}{\sqrt{2\overline{P}\cos(\theta) + \overline{M}^2 - 2\overline{P}\cos(\alpha)}}d\theta \tag{4.2.39}$$

However, this integral is very difficult to evaluate. Therefore we make a change of variables by introducing the variable ϕ such that:

$$\sin\left(\frac{\alpha}{2}\right)\sin(\phi) = \sin\left(\frac{\theta}{2}\right) \tag{4.2.40}$$

Then we find:

$$d\theta = 2\frac{d}{d\phi}\left(\arcsin\left(\sin\left(\frac{\alpha}{2}\right)\sin(\phi)\right)\right)d\phi = 2\frac{\sin\left(\frac{\alpha}{2}\right)\cos(\phi)}{\sqrt{1-\sin^2\left(\frac{\alpha}{2}\right)\sin^2(\phi)}}d\phi \quad (4.2.41)$$

Now we have to rewrite the original equation in terms of $\sin\left(\frac{\theta}{2}\right)$. This gives using some trigonometric identities:

$$\begin{aligned}\cos(\theta)-\cos(\alpha) = -2\sin\left(\frac{\theta-\alpha}{2}\right)\sin\left(\frac{\theta+\alpha}{2}\right) = \\ -2\left(\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\alpha}{2}\right)-\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)\cdot \\ \left(\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\alpha}{2}\right)+\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\theta}{2}\right)\right) = \\ -2\left(\sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\alpha}{2}\right)-\sin^2\left(\frac{\alpha}{2}\right)\cos^2\left(\frac{\theta}{2}\right)\right) = \\ -2\left(\sin^2\left(\frac{\theta}{2}\right)-\sin^2\left(\frac{\alpha}{2}\right)\right) = 2\sin^2\left(\frac{\alpha}{2}\right)\left(1-\frac{\sin^2\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\alpha}{2}\right)}\right) \quad (4.2.42)\end{aligned}$$

So now this yields:

$$\begin{aligned}1 = \mathrm{Sign}(M)\int_{\alpha}^{-\alpha}\frac{1}{\sqrt{2\overline{P}\cos(\theta)+\overline{M}^2-2\overline{P}\cos(\alpha)}}d\theta = \\ \mathrm{Sign}(M)\int_{\alpha}^{-\alpha}\frac{1}{\sqrt{4\overline{P}\sin^2\left(\frac{\alpha}{2}\right)\left(1-\frac{\sin^2\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\alpha}{2}\right)}\right)+\overline{M}^2}}d\theta = \\ \mathrm{Sign}(M)\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}}\frac{1}{\sqrt{4\overline{P}\sin^2\left(\frac{\alpha}{2}\right)\left(1-\sin^2(\phi)\right)+\overline{M}^2}}2\frac{\sin\left(\frac{\alpha}{2}\right)\cos(\phi)}{\sqrt{1-\sin^2\left(\frac{\alpha}{2}\right)\sin^2(\phi)}}d\phi = \\ -\mathrm{Sign}(M)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{\sqrt{4\overline{P}\sin^2\left(\frac{\alpha}{2}\right)\left(1-\sin^2(\phi)\right)+\overline{M}^2}}2\frac{\sin\left(\frac{\alpha}{2}\right)\cos(\phi)}{\sqrt{1-\sin^2\left(\frac{\alpha}{2}\right)\sin^2(\phi)}}d\phi \quad (4.2.43)\end{aligned}$$

Now we introduce yet another change of variables:

$$u = \sin(\phi)\sin\left(\frac{\alpha}{2}\right) \tag{4.2.44}$$

Then we have:

$$du = \cos(\phi)\sin\left(\frac{\alpha}{2}\right)d\phi \tag{4.2.45}$$

So substituting this gives:

$$\begin{aligned}
&-\operatorname{Sign}(M) = \\
&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\sqrt{4\overline{P}\sin^2\left(\frac{\alpha}{2}\right)\left(1-\sin^2(\phi)\right)+\overline{M}^2}} 2\frac{\sin\left(\frac{\alpha}{2}\right)\cos(\phi)}{\sqrt{1-\sin^2\left(\frac{\alpha}{2}\right)\sin^2(\phi)}}d\phi = \\
&\int_{-\sin\left(\frac{\alpha}{2}\right)}^{\sin\left(\frac{\alpha}{2}\right)} \frac{2}{\sqrt{4\overline{P}\sin^2\left(\frac{\alpha}{2}\right)+\overline{M}^2-4\overline{P}u^2}}\frac{du}{\sqrt{1-u^2}} = \\
&\int_0^{\sin\left(\frac{\alpha}{2}\right)} \frac{4}{\sqrt{4\overline{P}\sin^2\left(\frac{\alpha}{2}\right)+\overline{M}^2}\sqrt{1-\frac{4\overline{P}}{4\overline{P}\sin^2\left(\frac{\alpha}{2}\right)+\overline{M}^2}u^2}}\frac{du}{\sqrt{1-u^2}}
\end{aligned} \tag{4.2.46}$$

Here we use that the integral is symmetric in $u = 0$.
The incomplete integral of the first kind is given by:

$$F(x;k) = \int_0^x \frac{dt}{\sqrt{1-k^2t^2}\sqrt{1-t^2}} \tag{4.2.47}$$

The integral we are interested in is given by:

$$-\operatorname{Sign}(M) = \int_0^{\sin\left(\frac{\alpha}{2}\right)} \frac{4}{\sqrt{4\overline{P}\sin^2\left(\frac{\alpha}{2}\right)+\overline{M}^2}\sqrt{1-\frac{4\overline{P}}{4\overline{P}\sin^2\left(\frac{\alpha}{2}\right)+\overline{M}^2}u^2}}\frac{du}{\sqrt{1-u^2}}$$

(4.2.48)

Even though we can describe this integral as an incomplete elliptic integral of the first kind, it will turn out to be useful to make another change of variables in order to see if we get the results of [2] back for the case $M = 0$. Therefore we introduce

our final variable to be:

$$v = \sqrt{\frac{4\overline{P}}{4\overline{P}\sin^2\left(\frac{\alpha}{2}\right)+\overline{M}^2}}u \tag{4.2.49}$$

Then

$$dv = \sqrt{\frac{4\overline{P}}{4\overline{P}\sin^2\left(\frac{\alpha}{2}\right)+\overline{M}^2}}du \tag{4.2.50}$$

This gives:

$$\begin{aligned}-\operatorname{Sign}(M) &= \int_0^{\sin\left(\frac{\alpha}{2}\right)} \frac{4}{\sqrt{4\overline{P}\sin^2\left(\frac{\alpha}{2}\right)+\overline{M}^2}\sqrt{1-\frac{4\overline{P}}{4\overline{P}\sin^2\left(\frac{\alpha}{2}\right)+\overline{M}^2}u^2}}\frac{du}{\sqrt{1-u^2}} = \\ &\int_0^b \frac{2}{\sqrt{\overline{P}}\sqrt{1-\frac{4\overline{P}\sin^2\left(\frac{\alpha}{2}\right)+\overline{M}^2}{4\overline{P}}v^2}}\frac{dv}{\sqrt{1-v^2}} = \\ &\frac{2}{\sqrt{\overline{P}}}F\left(b;\sqrt{\frac{4\overline{P}\sin^2\left(\frac{\alpha}{2}\right)+\overline{M}^2}{4\overline{P}}}\right)\end{aligned} \tag{4.2.51}$$

where

$$b = \sin\left(\frac{\alpha}{2}\right)\sqrt{\frac{4\overline{P}}{4\overline{P}\sin^2\left(\frac{\alpha}{2}\right)+\overline{M}^2}} \tag{4.2.52}$$

So, we get the following exact relation between the load $\overline{P}$, the moment $\overline{M}$ and the angle α:

$$\frac{2}{\sqrt{\overline{P}}}F\left(\sin\left(\frac{\alpha}{2}\right)\sqrt{\frac{4\overline{P}}{4\overline{P}\sin^2\left(\frac{\alpha}{2}\right)+\overline{M}^2}};\sqrt{\frac{4\overline{P}\sin^2\left(\frac{\alpha}{2}\right)+\overline{M}^2}{4\overline{P}}}\right) = -\operatorname{Sign}(M) \tag{4.2.53}$$

4.2.3 Plotting and analyzing the force-angle-moment relation

Before we do any further calculations, we will first plot this exact force-angle-moment relation. Since this relation is implicit, it is impossible to write one of the three variables as a function of the other two. Therefore, we use Mathematica to make these plots. In the first plot, we plot $\frac{P}{P_e}$ versus α for different values of $\overline{M}$. This gives the following plot:

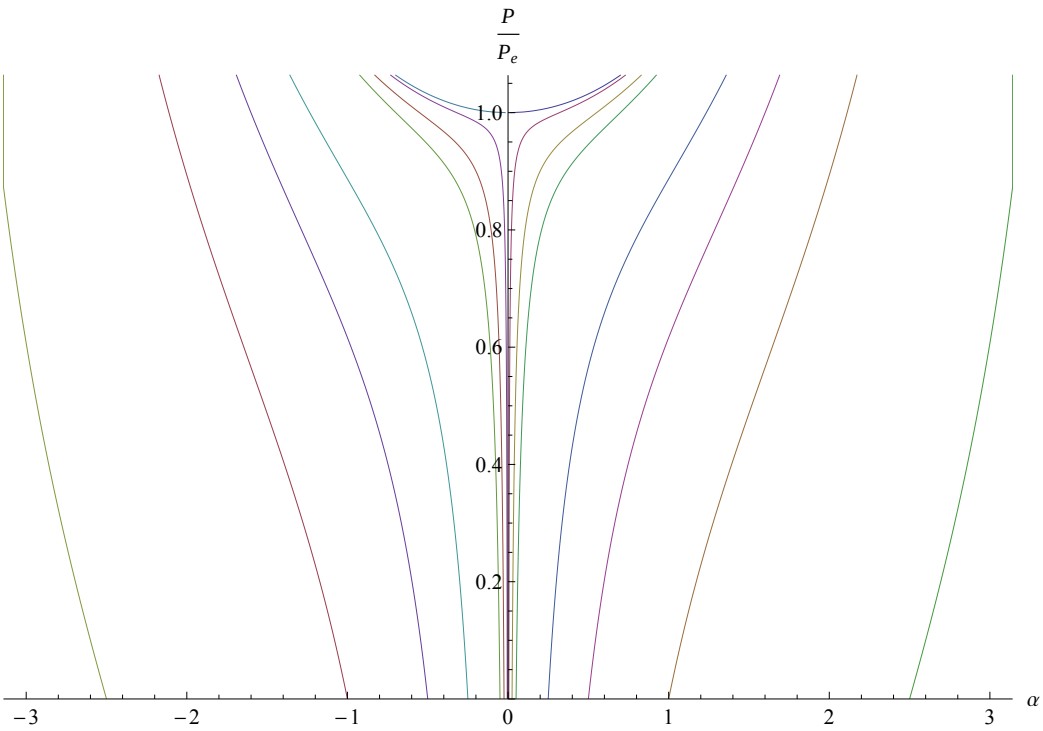


Figure 4.4: *Plot of force-angle-moment for different parameters. We plotted* $\frac{P}{P_e} = \frac{\overline{P}}{\pi^2}$ *versus* α *for different values of* $\overline{M}$*. The blue parabola-like line at the top is* $\overline{M} = 0$*. From left to right we have:* $\overline{M} = 5, \overline{M} = 2, \overline{M} = 1, \overline{M} = 0.5, \overline{M} = 0.1, \overline{M} = 0.05, \overline{M} = 0.01, \overline{M} = -0.01, \overline{M} = -0.05, \overline{M} = -0.1, \overline{M} = -0.5, \overline{M} = -1, \overline{M} = -2, \overline{M} = -5$.

Analyzing this plot, we immediately see some interesting results. First we see that for $P < P_e$ we indeed get solutions satisfying our force-angle-moment relation, so the assumption that θ is monotonic was indeed a good assumption, as can be seen by plotting θ for these values of P. For $\overline{M} = 0$ we clearly see that there is a non-zero lower bound for $\frac{P}{P_e}$ for α to be non-zero. This is of course the classical result for Euler beams: for $P < P_e$ the beam does not buckle and in that case we must have that $\alpha = 0$. So this result matches the known theory for Euler beams. For $\overline{M} > 0$ we also conclude some physical properties of the system from

the figure. For example, we see that in that case, α has a non-zero lower bound, no matter how small we choose our value of $\overline{P}$. This result makes sense when we look at the model of section 3.3. In that model we apply zero force, but non-zero momentum and there we also obtain the result that $\alpha \neq 0$. From this figure, we can also conclude that, in contrary to the Euler model, the beam bends for *any* value of $\overline{P}$. So there is no equivalent to the Euler load in the models with non-zero moment. Next we conclude that if $\overline{M}$ goes to 0, that the curves also converge to the curve of $\overline{M} = 0$, except of course that their behavior around $\alpha = 0$ is very different. Now we look at equation 4.2.1 and to the boundary conditions of our system. Then we immediately see that $\theta \equiv 0$ is a solution if and only if $\alpha = \overline{M} = 0$. So for $\overline{M} = 0$ the $\alpha = 0$ axis is also a viable solution for any value of $\overline{P}$. So in the case $\overline{M} = 0$ there is a pitchfork bifurcation at the point $\left(\alpha, \frac{P}{P_e}\right) = (0,1)$. Now we see that this pitchfork is not present in the case that $\overline{M} \neq 0$. So we can conclude that adding the parameter $\overline{M}$, which we can see as an imperfection parameter, causes an unfolding of this pitchfork bifurcation. In section 4.2.6 we will construct an expanded version of figure 4.4 and we will discuss the unfolding of the pitchfork bifurcation further using that expanded plot in chapter 5.

When we want to plot $\overline{M}$ versus α, we are interested what happens for any value of $\overline{P}$ and $\overline{M}$. Unfortunately, we do not have an exact relation for all values, since θ is not necessarily a monotonic function. Therefore we use a numerical trick to determine the correct value of $\overline{M}$, positive or negative, when $\overline{P}$ and α are given. Like we saw in section 4.2.1, the function $\theta(t)$ is symmetric with respect to $t = \frac{1}{2}$. Therefore it should hold that $\theta\left(\frac{1}{2}\right) = 0$ and that $\theta(1) = -\theta(0)$. Since we know what θ is for *any* set of values of $\alpha, \overline{P}$ and $\overline{M}$, we can use this to numerically determine the value of $\overline{M}$ for given α and $\overline{P}$ such that the corresponding solution satisfies $\theta\left(\frac{1}{2}\right) = 0$ and that $\theta(1) = -\theta(0)$. We will use this numerical trick in the following sections any time the exact force-angle-moment relation cannot help us. Applying this trick gives the following plot, where we plotted $\overline{M}$ versus α for different values of $\overline{P}$.

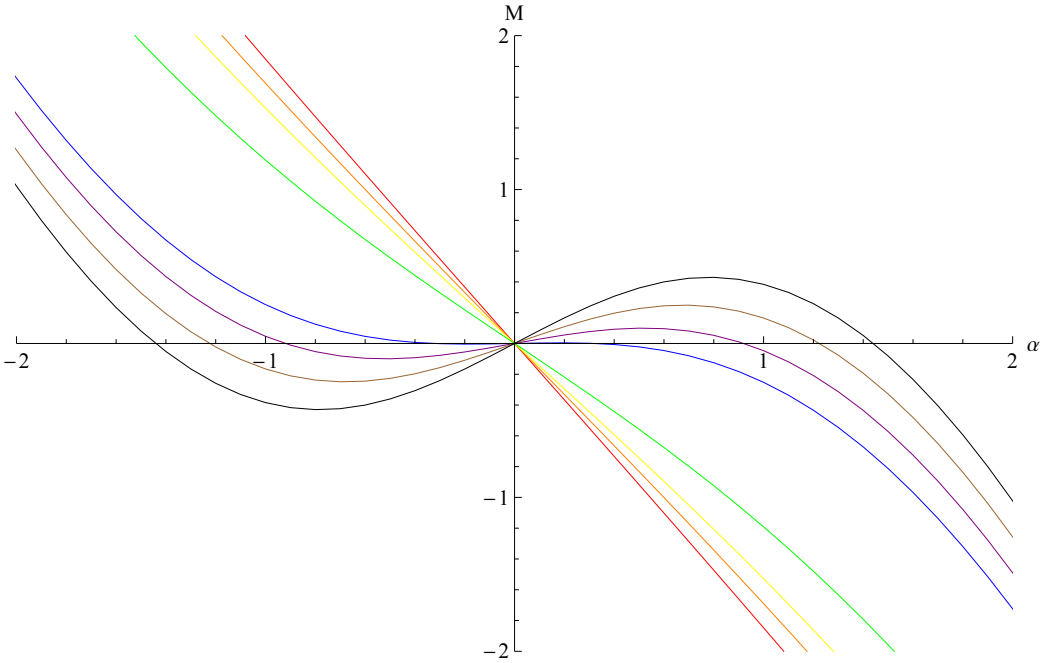


Figure 4.5: *Plot of force-angle-moment for different parameters. We plotted $\overline{M}$ versus α for different values of $\overline{P}$. From the initially most left to the initially most right we have: $\overline{P} = 13, \overline{P} = 12, \overline{P} = 11, \overline{P} = 10, \overline{P} = 5, \overline{P} = 3, \overline{P} = 2, \overline{P} = 1$.*

Now we will analyze figure 4.5. For $\overline{P} < \pi^2$, i.e. $P < P_e$, we see that there is an approximately linear relation between $\overline{M}$ and α, just like in section 4.1, although we see that when $\overline{P}$ gets larger, the relation becomes less and less linear. Since $P < P_e$ we know that the beam will not buckle if $\overline{M} = 0$. Therefore we see indeed that α goes to 0 when $\overline{M}$ goes to 0. For $\overline{P} \geq \pi^2$ the relation between $\overline{M}$ and α is clearly non-linear. We also note that if $P > P_e$, fixed values of P and $\overline{M}$ do not necessarily give a unique solution for α. In the next section we will plot beams with the same values of P and $\overline{M}$, but different values of α (see figures 4.12, 4.13 and 4.14). We note that if P is bigger than the Euler load and if $|\overline{M}|$ is large enough, there is a unique solution. We can see this physically as follows: if $\overline{M} = 0$ then we will see that we have three solutions, a positive solution which is located in the upper-half plane, a negative solution in the lower-half plane and an unstable solution positioned on the x-axis. If we now apply a very large moment on the endpoints, we see that either the negative or the positive solutions cannot exist anymore, since they are all pushed towards the positive or negative solution respectively. Therefore the different solutions will coincide and there will be one unique solution left. So we see that the unfolding of the pitchfork bifurcation causes the solution to be unique if the imperfection parameter $\overline{M}$ is large enough. In chapter 5 we will see that this has some interesting consequences.

Now we will see if substituting $M = 0$ retrieves the results of [2]. Substituting this in equation (4.2.52) gives:

$$b = \sin\left(\frac{\alpha}{2}\right)\sqrt{\frac{4\overline{P}}{4\overline{P}\sin^2\left(\frac{\alpha}{2}\right)}} = 1 \tag{4.2.54}$$

So we get by substituting this in equation (4.2.51):

$$1 = Sign(M) = \frac{2}{\sqrt{\overline{P}}}F\left(1;\sqrt{\frac{4\overline{P}\sin^2\left(\frac{\alpha}{2}\right)}{4\overline{P}}}\right) = \frac{2}{\sqrt{\overline{P}}}F\left(1;\sin\left(\frac{\alpha}{2}\right)\right) = \frac{2}{\sqrt{\overline{P}}}K\left(\sin^2\left(\frac{\alpha}{2}\right)\right) \tag{4.2.55}$$

Here $K(m)$ is the complete elliptic integral of the first kind, which is defined by:

$$K(m) = F(1;\sqrt{m}) \tag{4.2.56}$$

Now we can look up an approximation of $K(m)$ for small values of m. This gives:

$$K(m) \approx \frac{\pi}{2}\left(1 + \frac{1}{4}\frac{m}{1-m}\right) \tag{4.2.57}$$

So we get:

$$\overline{P} = \pi^2\left(\frac{K\left(\sin^2\left(\frac{\alpha}{2}\right)\right)}{\frac{\pi}{2}}\right)^2 \approx \pi^2\left(1 + \frac{1}{4}\frac{\sin^2\left(\frac{\alpha}{2}\right)}{1-\sin^2\left(\frac{\alpha}{2}\right)}\right) \approx \pi^2(1 + \frac{1}{8}\alpha^2) \tag{4.2.58}$$

We have that:

$$\overline{P} = \frac{P\ell^2}{EI} \tag{4.2.59}$$

The Euler load P_e is given by:

$$P_e = EI\frac{\pi^2}{\ell^2} \tag{4.2.60}$$

So we see that:

$$\frac{\overline{P}}{\pi^2} = \frac{P}{P_e} \tag{4.2.61}$$

Therefore we obtain:

$$\frac{P}{P_e} \approx 1 + \frac{1}{8}\alpha^2 \tag{4.2.62}$$

This is exactly equation (3.134) of [2]. So we indeed recover the results for $M = 0$. Now we include this approximation in the plot of the force-angle-moment relation. In figure 4.6 we can see that the approximation is in very good agreement with the exact relation.

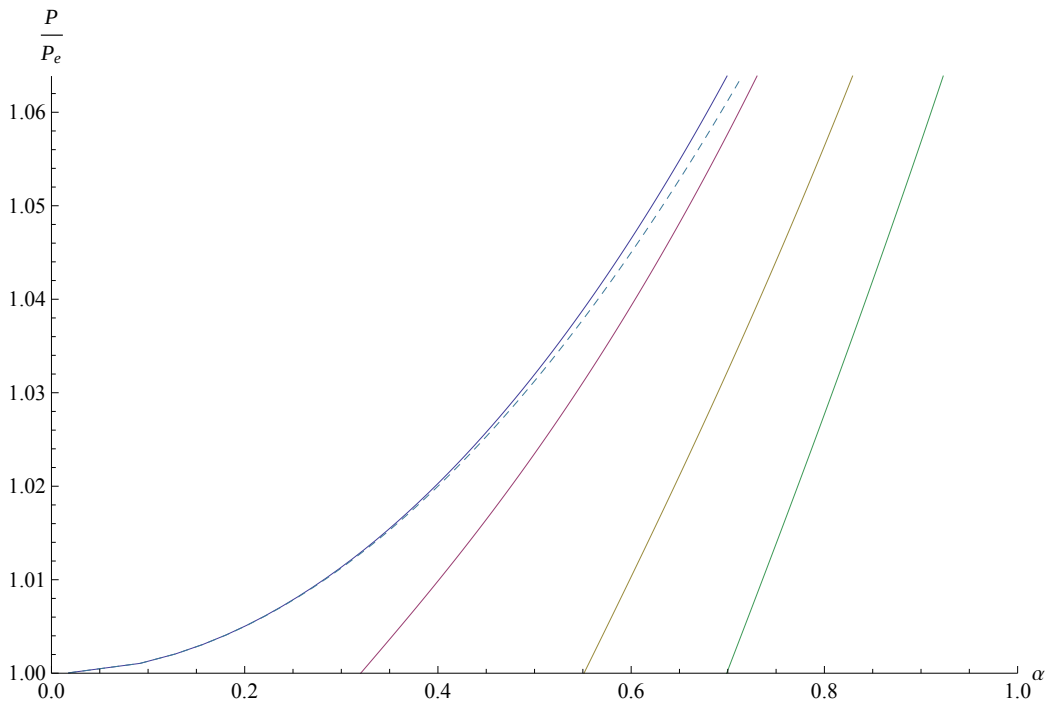


Figure 4.6: *Plot of force-angle-moment for different parameters. We plotted* $\frac{P}{P_e}$ *versus* α *for different values of* $\overline{M}$*. From left to right we have:* $\overline{M} = 0, \overline{M} = -0.01, \overline{M} = -0.05, \overline{M} = -0.1$*. The dashed line is the approximation for the case* $\overline{M} = 0$

4.2.4 Plotting the shape of the beam

In this section we assume that $\ell = 1$ and thus that the initial position of the beam is given by:

$$x_0(t) = t, y_0(t) = 0 \tag{4.2.63}$$

Now we can combine the results of section 4.2.2 and section 4.2.1 to make (numerical) plots of the actual beams. If necessary we use the numerical trick that utilizes the symmetry of $\theta(t)$ in $t = \frac{1}{2}$. All numerical calculations have been done by Mathematica.
To start we fix the values of $\overline{P}$ and $\overline{M}$. Then we use equation (4.2.53) to find a matching value of α. Since this relation is non-linear and implicit, we cannot do this exactly, so we determine the right value of α numerically. Now we can plug these values in the exact solution of θ (equation (4.2.33)). Since we are interested in the shape of the beam, we need to calculate u and w. As θ is known, this can be done using equations (2.3.15) and (2.3.16). Because we look at an inextensible beam without pre-curvature and we ignore shear effects, we get $\Lambda = 1$, $\chi = \theta^0 = 0$. So these equations become:

$$u_t = \cos(\theta) - 1 \tag{4.2.64}$$

and

$$w_t = \sin(\theta) \tag{4.2.65}$$

We now calculate u and w at N+1 equidistant points of the beam (including the endpoints) by integrating the above equations numerically for a certain large number N. However, we are interested in the position of the beam, so knowing u and w is not enough. We let x_0 and y_0 be the initial positions of the beam. Then we have for a point t that:

$$x_0(t) = t \tag{4.2.66}$$

and

$$y_0(t) = 0 \tag{4.2.67}$$

Now we finally obtain the positions x and y of the beam at a point t by calculating:

$$x(t) = x_0(t) + u(t) \tag{4.2.68}$$

and

$$y(t) = y_0(t) + w(t) \tag{4.2.69}$$

In figure 4.7 we give a few plots using this script. In the first three plots we keep $\overline{M}$ constant and plot beams for three different values of $\overline{P}$.

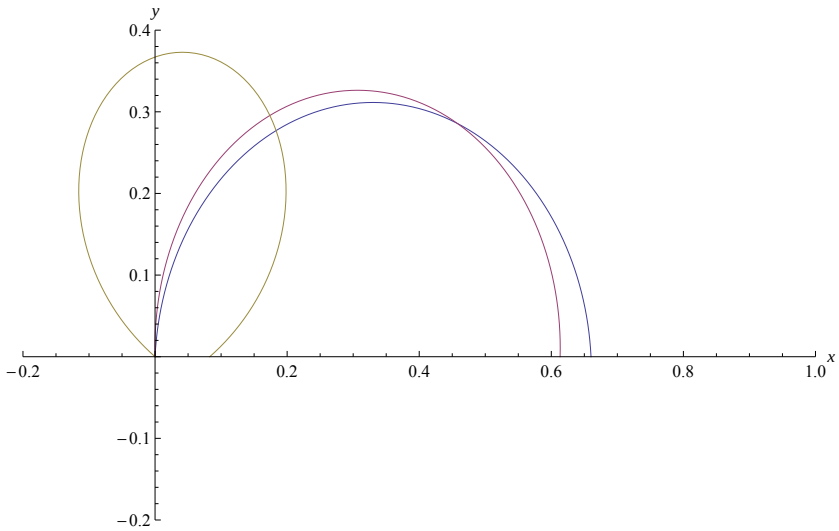


Figure 4.7: *Plot of beams for different parameters. Here* $\overline{M} = -3$*. The blue line is* $\overline{P} = 0.1$*, the purple line* $\overline{P} = 1$ *and the yellow line* $\overline{P} = 10$*.*

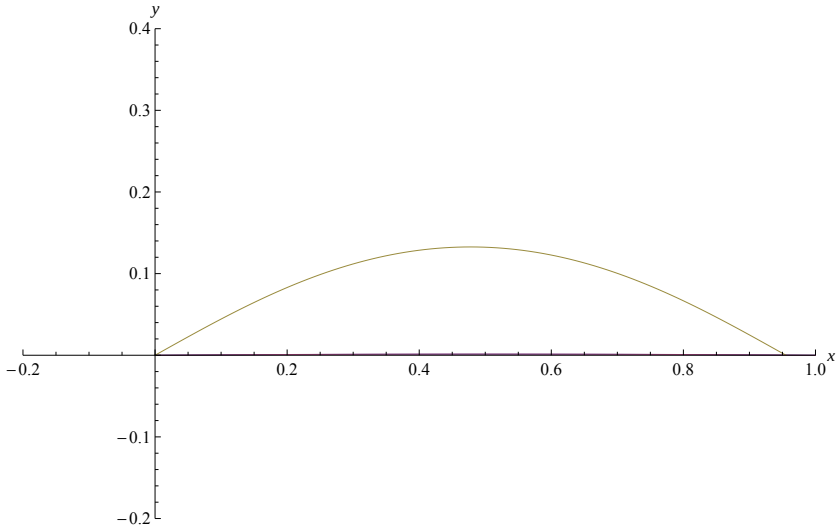


Figure 4.8: *Plot of beams for different parameters. Here* $\overline{M} = -0.01$*. The blue line is* $\overline{P} = 0.1$*, the purple line* $\overline{P} = 1$ *and the yellow line* $\overline{P} = 10$*. Even though it is not entirely visible, the blue and purple lines are non-zero.*

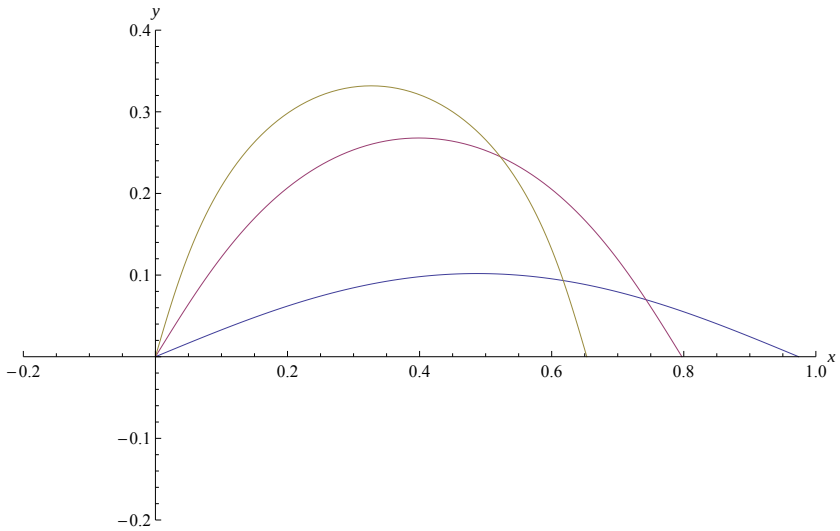


Figure 4.9: *Plot of beams for different parameters. Here $\overline{M} = 0$. The blue line is $\overline{P} = 10$, the purple line $\overline{P} = 11$ and the yellow line $\overline{P} = 12$.*

For $\overline{P} < \pi^2$ we did not get any solutions other than the trivial $\theta \equiv 0$-solution in the case $\overline{M} = 0$. This is because we have $\frac{P}{P_e} = \frac{\overline{P}}{\pi^2}$ and the beam does not buckle for $P < P_e$. Therefore this result is in good agreement with the reality. However, if $\overline{M} \neq 0$, then we get a solution no matter how small the value of $\overline{P}$ is. So this proves that if $\overline{M} \neq 0$ that the beam bends even if $P < P_e$.
In the next two plots we keep $\overline{P}$ constant and plot beams for three different values of $\overline{M}$.

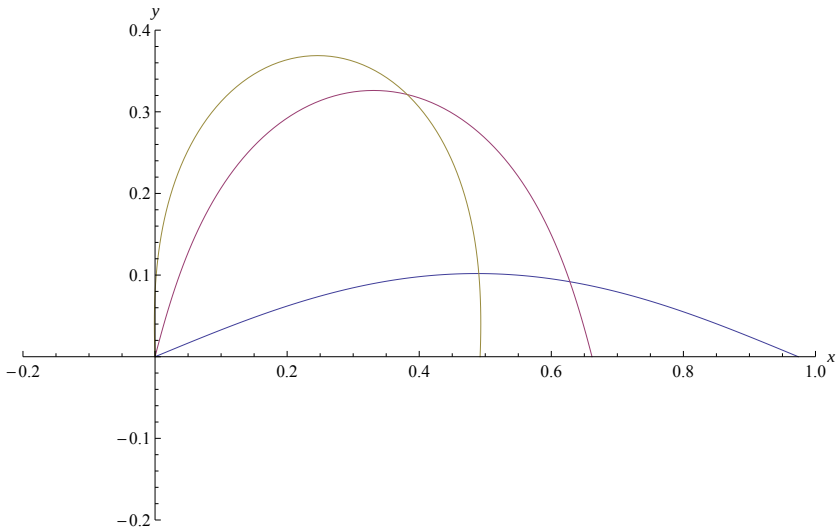


Figure 4.10: *Plot of beams for different parameters. Here* $\overline{P} = 10$*. The blue line is* $\overline{M} = 0$*, the purple line* $\overline{M} = -0.5$ *and the yellow line* $\overline{M} = -1$*.*

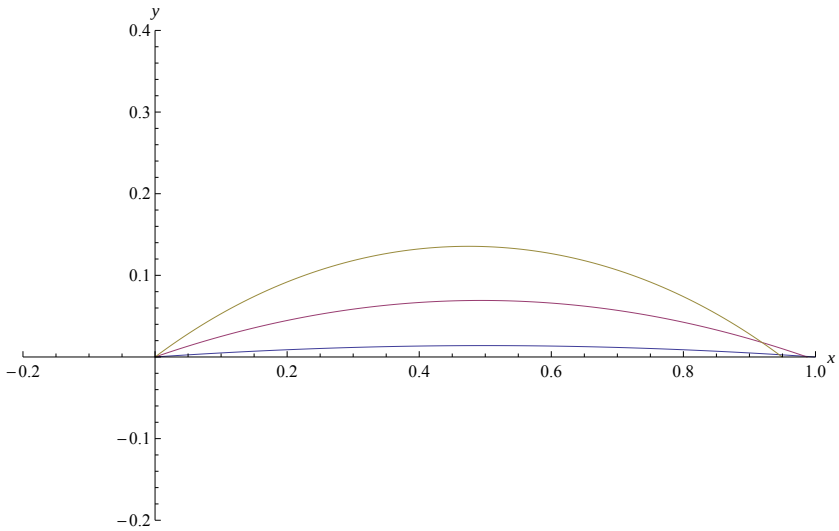


Figure 4.11: *Plot of beams for different parameters. Here* $\overline{P} = 1$*. The blue line is* $\overline{M} = -0.1$*, the purple line* $\overline{M} = -0.5$ *and the yellow line* $\overline{M} = -1$*.*

In figure 4.5 in the previous section we saw that it is possible that even though P and $\overline{M}$ are known, we cannot uniquely determine α. Therefore for three different values of α, we numerically determined the three different solutions, which are all valid according to our force-angle-moment relation.

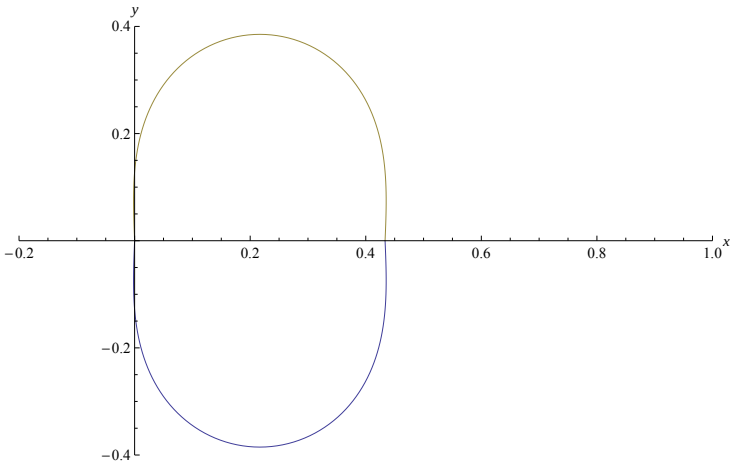


Figure 4.12: *Plot of three different beams with the same values of $\overline{M}$ and $\overline{P}$. In this plot we have $\overline{M} = 0$ and $\overline{P} = 15$. One of the solutions is the $\theta \equiv 0$ solution and is thus not visible.*

In figure 4.12 we can clearly distinguish the three different solutions. Firstly we have the solution $\theta \equiv 0$, i.e. the solution where the beam does not buckle. Of course, this solution is not stable, since the beam tends to buckle when applying a load larger than the Euler load. The other two solutions are entirely symmetric in the x-axis. Since there is no moment applied, there will be no natural preference for the system to be in either one of these two states and thus they are both stable solutions to our model. Now we will show two figures with the three solutions to a model with positive moment and a model with negative moment.

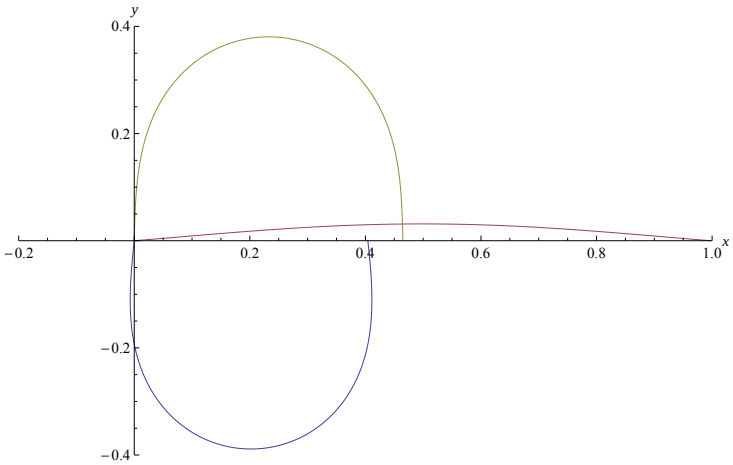


Figure 4.13: *Plot of three different beams with the same values of $\overline{M}$ and $\overline{P}$. In this plot we have $\overline{M} = 0.1$ and $\overline{P} = 15$.*

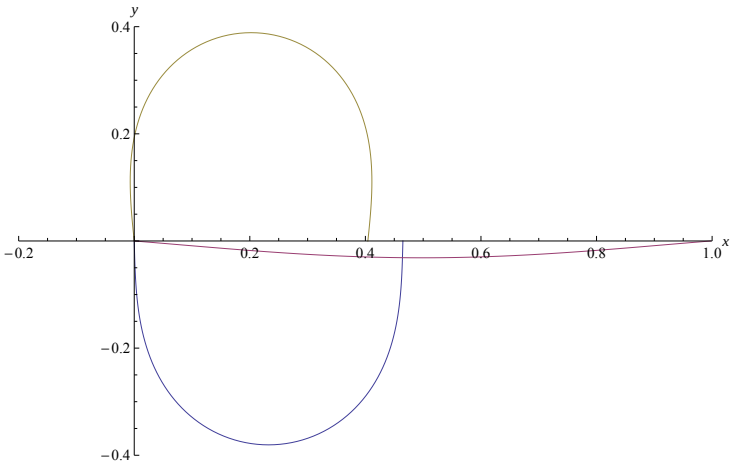


Figure 4.14: *Plot of three different beams with the same values of $\overline{M}$ and $\overline{P}$. In this plot we have $\overline{M} = -0.1$ and $\overline{P} = 15$.*

We see that we get figures 4.13 and 4.14 by applying a small moment to the three solutions of figure 4.12. Since we apply moment, there is no symmetry in the different solutions within one figure. However, we see that there is some symmetry between the solutions when we compare both figures with each other. This is because the absolute value of the moment we apply in the two figures is equal.

4.2.5 Higher order modes

We will now try to find higher order modes of the beam in the case that $\overline{M} = 0$. For this, we have to make a change in our force-angle-moment relation. More precisely, we have to change equation (4.2.39). If $n > 1$ is a natural number, then we have that θ runs from α to $-\alpha$ in $\frac{1}{n}^{\text{th}}$ of the length of the beam in the nth mode. Thus equation (4.2.39) becomes:

$$\frac{1}{n} = \int_{\alpha}^{-\alpha} \frac{1}{\sqrt{2\overline{P}\cos(\theta) + \overline{M}^2 - 2\overline{P}\cos(\alpha)}} d\theta \tag{4.2.70}$$

Then equation (4.2.53) becomes:

$$\frac{2}{\sqrt{\overline{P}}} F\left(\sin\left(\frac{\alpha}{2}\right)\sqrt{\frac{4\overline{P}}{4\overline{P}\sin^2\left(\frac{\alpha}{2}\right) + \overline{M}^2}}; \sqrt{\frac{4\overline{P}\sin^2\left(\frac{\alpha}{2}\right) + \overline{M}^2}{4\overline{P}}}\right) = \frac{1}{n} \tag{4.2.71}$$

Substituting $\overline{M} = 0$ gives:

$$\frac{2}{\sqrt{\overline{P}}} K\left(\sin^2\left(\frac{\alpha}{2}\right)\right) = \frac{1}{n} \tag{4.2.72}$$

Similar to the calculations in section 4.2.2, we find:

$$\frac{P}{P_e} \approx n^2 + \frac{n^2}{8}\alpha^2 \tag{4.2.73}$$

Now we can make plots of beams in these higher modes:

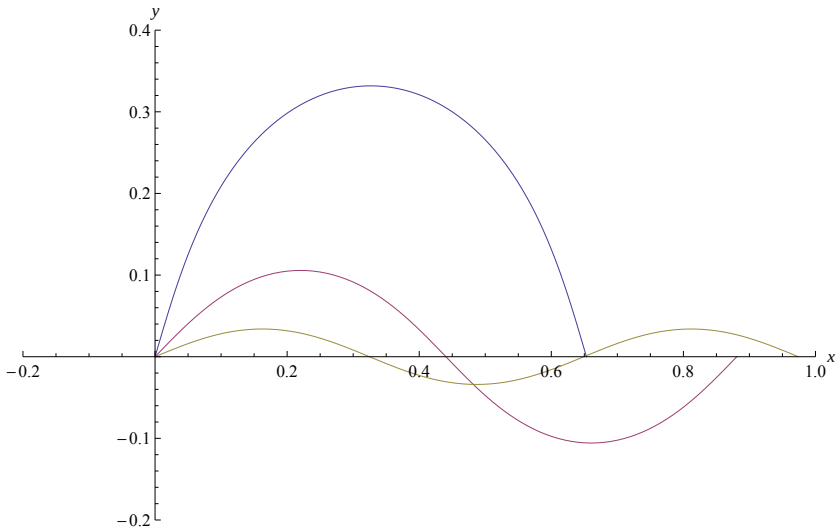


Figure 4.15: *Plot of beams in higher modes. For all beams we have $\overline{M} = 0$. For the beam in the first mode, we have $\overline{P} = 12$, for the beam in the second mode $\overline{P} = 42$ and for the beam in the third mode $\overline{P} = 92$.*

It should be noted however, that for n even, the solution does not satisfy all boundary conditions anymore: for example in the case where $n = 2$ we see that $\theta(1) = \alpha$ instead of the usual condition $\theta(1) = -\alpha$. So the only reason we included them is that they are physically realizable states, even though they do not satisfy the imposed boundary conditions. In the next section we will see that this second mode also has some behavior that does not fit into the general model which describes the solutions that do satisfy all boundary conditions.

4.2.6 Expanded force-angle-moment relation

In figure 4.4 we plotted the bifurcation diagram for beams of which $\theta(t)$ is a monotonically increasing or decreasing function. However, in section 4.2.5 we saw that there are beams for which this assumption is incorrect. The curves of these higher modes lie above the 'normal' $\overline{M} = 0$ curve, i.e. the standard pitchfork bifurcation. So therefore it might be possible that for $\overline{M} \neq 0$ there are also curves above this bound. Figure 1.4 suggests that the beam goes into a higher mode while snapping, so we need to take these higher modes into account if we want to describe effects like snapping. Unfortunately, we do not have an exact relation that describes the curves for these higher modes. Therefore we numerically expanded out force-angle-moment relation. This gave the following result:

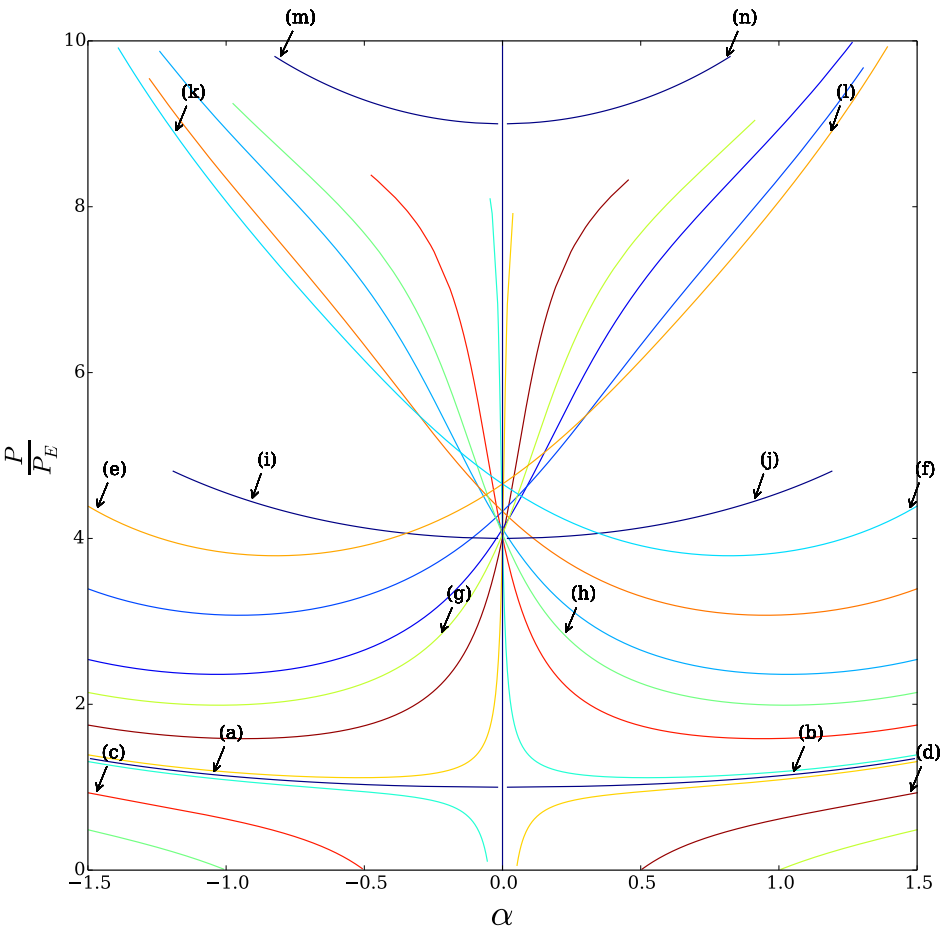


Figure 4.16: *Bifurcation diagram for the force-angle-moment relation. The dark blue/almost black lines all correspond to* $\overline{M} = 0$ *and represent the first three modes of the system and to the trivial solution where* $\theta \equiv 0$*. In between these first and second modes in the right-half of the plane, we have in order of increasing height:* $\overline{M} = 0.1$, $\overline{M} = 1$, $\overline{M} = 2$, $\overline{M} = 3$, $\overline{M} = 5$ *and* $\overline{M} = 7$*. In between the first and second modes in the left-half of the plane, we have in order of increasing height:* $\overline{M} = -0.1$, $\overline{M} = -1$, $\overline{M} = -2$, $\overline{M} = -3$, $\overline{M} = -5$ *and* $\overline{M} = -7$*. Although some lines might seem to end around* $\frac{P}{P_e} = 8$*, they do not end there and they continue in the way that the other lines do. Numerical convergence issues hampers the computation of the curves for high values of* P *and unfortunately we had no time to fix this. The labels in this plot will be explained in figure 4.17.*

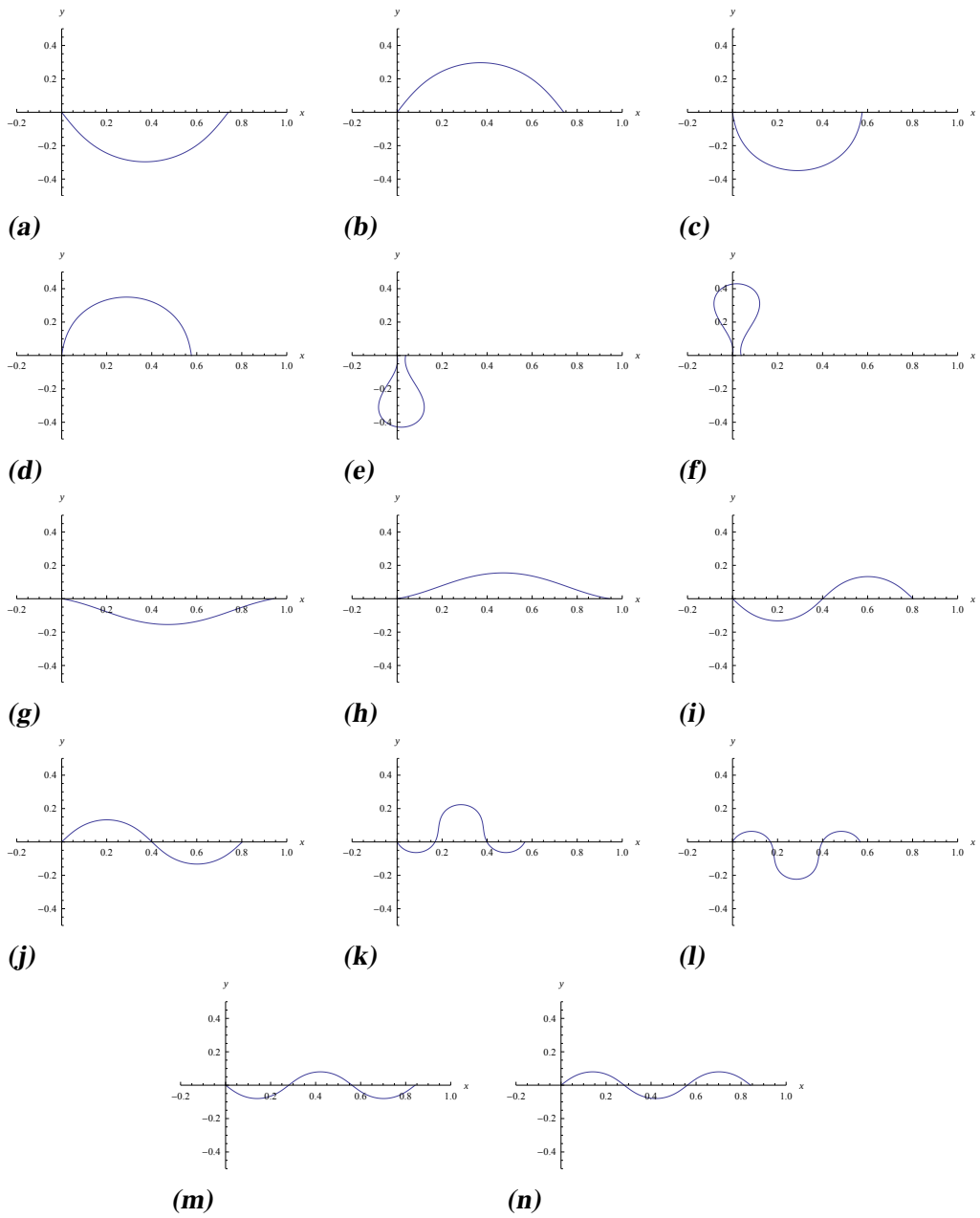


Figure 4.17: *Snapshots of qualitatively different beams. The numbering of the snapshots corresponds to the numbering in figure 4.16.*

We will analyze this bifurcation diagram in chapter 5 and show its physical consequences.

4.3 Analysis of the model for an inextensible pre-curved beam with a constant force applied

In this section we will analyze the model of section 3.2. First we will derive an exact solution to the homogeneous problem and then we will look at one specific pre-curvatures and try to find exact solutions to our equations.

4.3.1 The homogeneous equation

The equation describing this model is given by equation (3.2.1):

$$\theta_{tt} - \theta^0_{tt} + \overline{P}\sin(\theta) = 0 \tag{4.3.1}$$

The homogeneous equation then becomes, where ϑ is the homogeneous solution:

$$\vartheta_{tt} + \overline{P}\sin(\vartheta) = 0 \tag{4.3.2}$$

We notice that this is the same equation as equation (3.1.2). Therefore we can use our analysis of section 4.2.1. To solve this we need to specify the boundary conditions. Since this approach asks the boundary conditions to be general, we define:

$$A = \vartheta(0), B = \vartheta_s(0) \tag{4.3.3}$$

Then we obtain by following exactly the same approach as in section 4.2.1:

$$\vartheta = 2\arcsin\left(\sqrt{k}sn\left(F\left(\frac{\sin\frac{A}{2}}{\sqrt{k}}\right);\sqrt{k}\right) + \mathrm{Sign}(B)\omega_0 t;\sqrt{k}\right) \tag{4.3.4}$$

where:

$$k = \frac{1}{2}\left(\frac{B^2}{2\overline{P}} - \cos(A) + 1\right), \omega_0 = \sqrt{\overline{P}} \tag{4.3.5}$$

4.3.2 A linear pre-curvature

We now suppose the pre-curvature is linear, so $\theta^0(t) = \kappa t + \alpha_0$ for some constants κ, α_0. We also assume that $\theta^0(0) = -\theta^0(1)$. Since the derivative of θ^0 with respect to s is the curvature of the beam, this means that we look at a beam with constant curvature $\frac{\kappa}{\ell}$. It is natural to look at this sort of beams, since these are the beams that are used most in experiments. Since $\theta^0(0) = -\theta^0(1)$, we get the following relation between κ and α_0:

$$\kappa = -2\alpha_0 \tag{4.3.6}$$

Exact solution

Since θ^0 is linear, we get $\theta^0_{tt} = 0$, so we note that we re-obtain the homogeneous equation. Therefore we obtain:

$$\theta = 2\arcsin\left(\sqrt{k}sn\left(F\left(\frac{\sin\frac{A}{2}}{\sqrt{k}}\right);\sqrt{k}\right) + \mathrm{Sign}(B)\omega_0 t;\sqrt{k}\right) \tag{4.3.7}$$

Here $A = \theta(0) = \alpha + \theta^0(0) = \alpha + \alpha_0$ and $B = \theta_t(0) = \overline{M} + \theta^0_s(0) = \overline{M} + \kappa$. Thus we have:

$$\theta = 2\arcsin\left(\sqrt{k}sn\left(F\left(\frac{\sin\frac{\alpha+\alpha_0}{2}}{\sqrt{k}}\right);\sqrt{k}\right) + \mathrm{Sign}(\overline{M}+\kappa)\omega_0 t;\sqrt{k}\right) \tag{4.3.8}$$

with

$$k = \frac{1}{2}\left(\frac{(\overline{M}+\kappa)^2}{2\overline{P}} - \cos(\alpha+\alpha_0) + 1\right), \omega_0 = \sqrt{\overline{P}} \tag{4.3.9}$$

A force-angle-moment relation

For this model, we can also find a force-angle-moment relation. Just like before we assume that θ is a monotonic function of t. Since $\theta^0_{tt} = 0$, we can integrate equation (4.3.1) to obtain:

$$\frac{1}{2}\theta_t^2 - \overline{P}\cos(\theta) = C \tag{4.3.10}$$

Here C is a constant. Using our boundary conditions, we obtain:

$$\frac{1}{2}(\overline{M}+\kappa)^2 - \overline{P}\cos(\alpha+\alpha_0) = C \tag{4.3.11}$$

So this gives the equation:

$$\frac{1}{2}\theta_t^2 - \overline{P}\cos(\theta) = \frac{1}{2}(\overline{M}+\kappa)^2 - \overline{P}\cos(\alpha+\alpha_0) \tag{4.3.12}$$

Now we define:

$$\tilde{M} := \overline{M} + \kappa \tag{4.3.13}$$

and

$$\tilde{\alpha} := \alpha + \alpha_0 \tag{4.3.14}$$

Thus we get:

$$\theta_t^2 = 2\overline{P}\cos(\theta) + \tilde{M}^2 - 2\overline{P}\cos(\tilde{\alpha}) \tag{4.3.15}$$

Writing $\theta_t = \frac{d\theta}{dt}$ this yields:

$$(d\theta)^2 = \left(2\overline{P}\cos(\theta) + \tilde{M}^2 - 2\overline{P}\cos(\tilde{\alpha})\right)(dt)^2 \tag{4.3.16}$$

and thus:

$$\text{Sign}(\tilde{M})dt = \frac{1}{\sqrt{2\overline{P}\cos(\theta) + \tilde{M}^2 - 2\overline{P}\cos(\tilde{\alpha})}}d\theta \tag{4.3.17}$$

We clearly have:

$$\int_0^1 \text{Sign}(\tilde{M})dt = \text{Sign}(\tilde{M}) \tag{4.3.18}$$

We now want to express this as an integral over θ. When looking at the boundary conditions, we notice that θ goes from $\alpha + \alpha_0$ to $-\alpha - \alpha_0$, i.e. from $\tilde{\alpha}$ to $-\tilde{\alpha}$. So this gives:

$$\text{Sign}(\tilde{M}) = \int_0^1 dt = \int_{\tilde{\alpha}}^{-\tilde{\alpha}} \frac{1}{\sqrt{2\overline{P}\cos(\theta) + \tilde{M}^2 - 2\overline{P}\cos(\tilde{\alpha})}}d\theta \tag{4.3.19}$$

We saw in section 4.2.2 that we can rewrite this equation to (equation (4.2.53)):

$$\frac{2}{\sqrt{\overline{P}}}F\left(\sin\left(\frac{\tilde{\alpha}}{2}\right)\sqrt{\frac{4\overline{P}}{4\overline{P}\sin^2\left(\frac{\tilde{\alpha}}{2}\right) + \tilde{M}^2}}; \sqrt{\frac{4\overline{P}\sin^2\left(\frac{\tilde{\alpha}}{2}\right) + \tilde{M}^2}{4\overline{P}}}\right) = -\text{Sign}(\tilde{M}) \tag{4.3.20}$$

When looking at equations (4.3.8) and (4.3.20) and comparing them with equations (4.2.33) and (4.2.53) we see that this system is equivalent to a model where we apply constant force P, (adimensionalized) moment $\overline{M} + \kappa$ and where the angle at zero is given by $\alpha + \alpha_0$. This is even more clear when we look at the plot of the force-angle-moment relation:

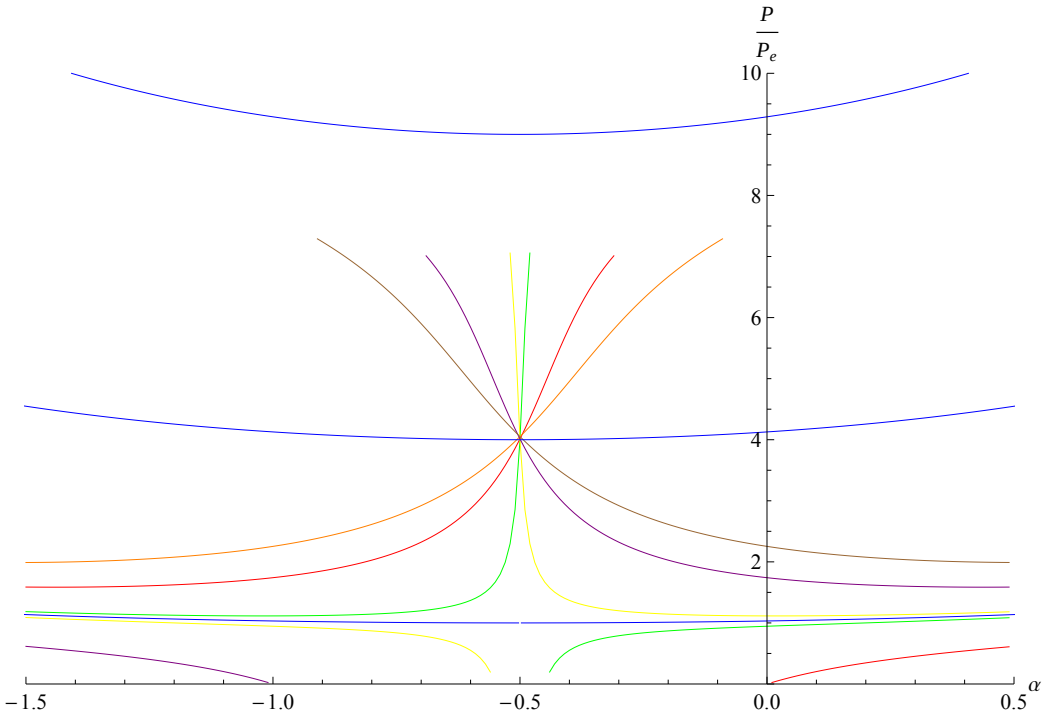


Figure 4.18: *Expanded plot of the Force-angle-moment relation for a pre-curved beam. We plotted $\frac{P}{P_e}$ versus α for different values of $\overline{M}$. Here we have chosen $\alpha_0 = 0.5$ and with equation (4.3.6) this gives $\kappa = -1$. The blue lines all correspond to $\overline{M} = 1$ (i.e. $\tilde{M} = 0$) and represent the first three modes of the system. The green line is $\overline{M} = 0.9$, the red line is $\overline{M} = 0$, the orange line is $\overline{M} = -1$, the yellow line is $\overline{M} = 1.1$, the purple line is $\overline{M} = 2$ and the brown line is $\overline{M} = 3$.*

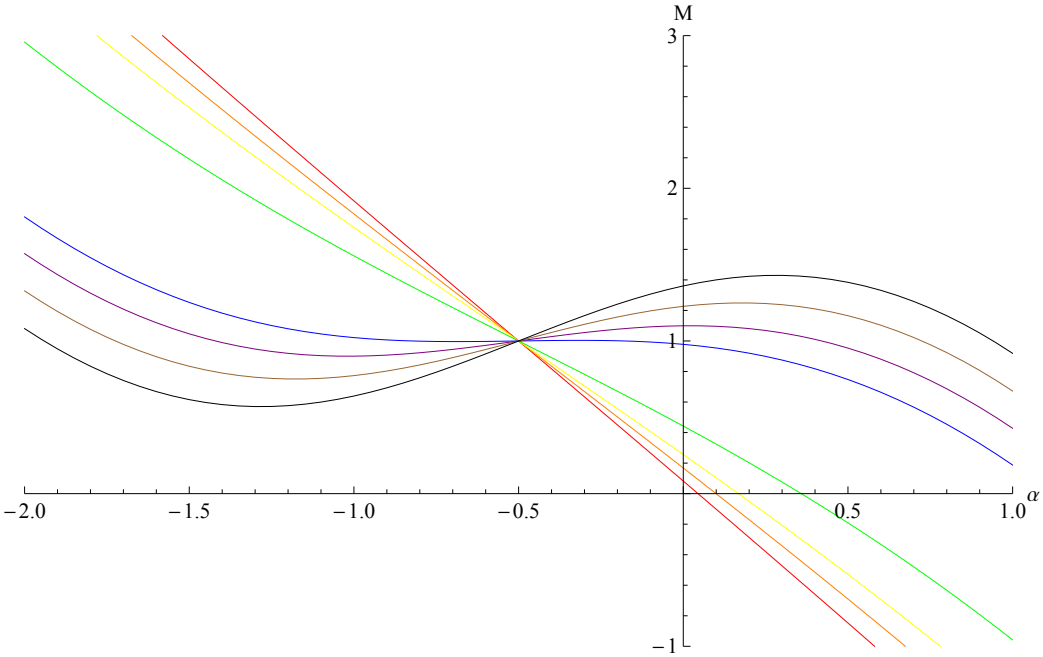


Figure 4.19: *Expanded plot of the force-angle-moment relation for a pre-curved beam. We plotted $\overline{M}$ versus α for different values of $\overline{P}$. Here we have chosen $\alpha_0 = 0.5$ and thus $\kappa = -1$. From the initially most left to the initially most right we have: $\overline{P} = 13, \overline{P} = 12, \overline{P} = 11, \overline{P} = 10, \overline{P} = 5, \overline{P} = 3, \overline{P} = 2, \overline{P} = 1$.*

In these figures we see that the pre-curvature causes an offset for the moment and the angle at zero: we see that the figures are shifted over an angle α_0 and a moment κ when comparing them to figures 4.16 and 4.5. This means that effectively there is hardly any difference between the model where we apply moment and the model with pre-curvature. Because of this similarity between the models, we will not analyze this model any further, since there is no new behavior we have not seen yet in the model for the initially straight beam.
Similar to the calculations in section 4.2.2, we find in the case that $\tilde{M} = 0$, i.e. that $\overline{M} = -\kappa$ that equation (4.3.20) reduces to:

$$\frac{P}{P_e} \approx 1 + \frac{1}{8}(\alpha + \beta)^2 \tag{4.3.21}$$

Plotting pre-curved beams

Now we can use these equations to make plots of the beams, similarly to how we made plots in section 4.2.4.

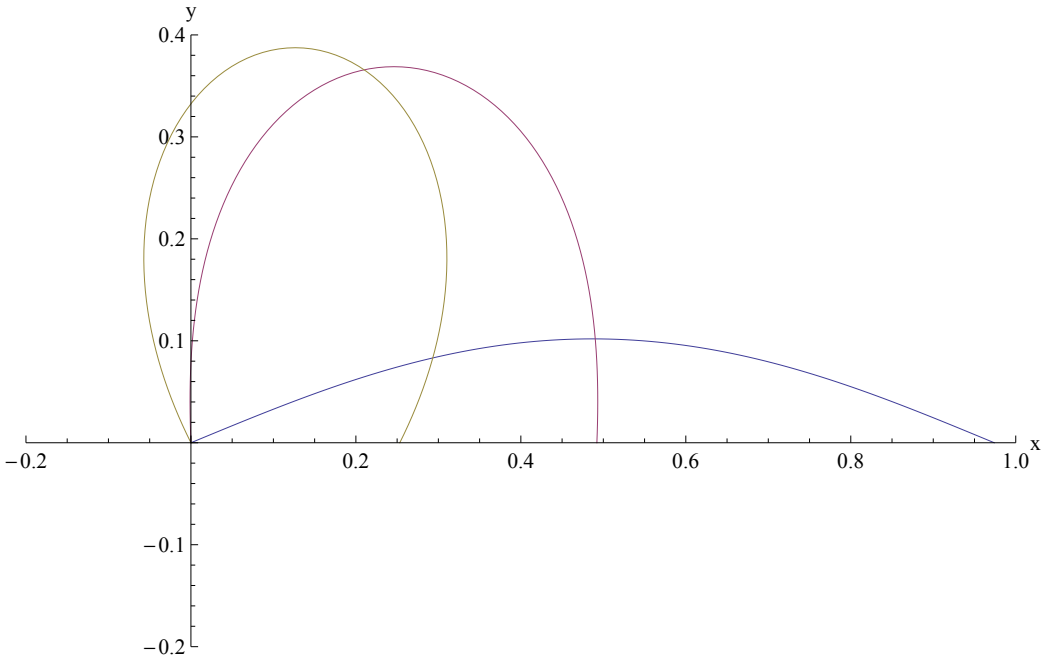


Figure 4.20: *Plots of different pre-curved beams. For all beams we have that* $\overline{P} = 10$ *and* $\overline{M} = 0$*. The blue line corresponds to* $\theta^0 \equiv 0$*, the purple line to* $\theta^0(t) = -t + \frac{1}{2}$ *and the yellow line to* $\theta^0(t) = -2t + 1$*.*

4.4 An analysis of the model for an extensible pre-curved beam with a constant force applied

In this section we will analyze the model of section 3.4, i.e. we add extensibility to the model of section 4.3. Just like in section 4.3.2 we will assume that the pre-curvature is of the following form:

$$\theta^0(t) = \kappa t + \alpha_0 \tag{4.4.1}$$

We will not look at any other pre-curvatures, since the model is complex already, even for this 'easy' pre-curvature. We also assume again that $\theta^0(1) = -\alpha_0$. One further assumption that we make is that there is no shear. This is because it makes the numerics even harder than it already is and we believe that it is more insightful to focus on extensibility.
For initially straight beams where no moment is applied, it is possible to find an exact solution and a force-angle relation (see for example [7]). However, we did not succeed in finding exact solutions or relations in case there is pre-curvature or moment. Therefore we have to analyze this model entirely numerically. Just as in the previous section we will use the symmetry of our problem to demand that $\theta(\frac{1}{2}) = 0$ and that $\theta(1) = -\theta(0)$. However, in the previous models we had an exact solution for any combination of $\overline{P}, \overline{M}$ and α and we could use this exact solution to determine the relation between the three parameters. In this model we do not have an exact solution for most sets of parameters. Luckily, Mathematica can determine a numerical solution to the differential equation for any set of parameters. To find a 'good' set of parameters, we now do the following: we give the values of $\overline{P}$ and $\overline{M}$ and let Mathematica determine a solution to the differential equation for 20 000 values of α between 0 and 2. For each solution, we check if $\theta(\frac{1}{2})$ is really close to 0 (up to the same number of decimals that we specify α) and if $\theta(1)$ is also really close to $-\alpha$. For most parameters, we will then indeed find a solution. Then we can determine Λ with equations (2.7.1) and (2.3.12). This we can plug into equations (2.3.15) and (2.3.16). Then all is left to do is to numerically integrate these equations. This gives for example the following plot:

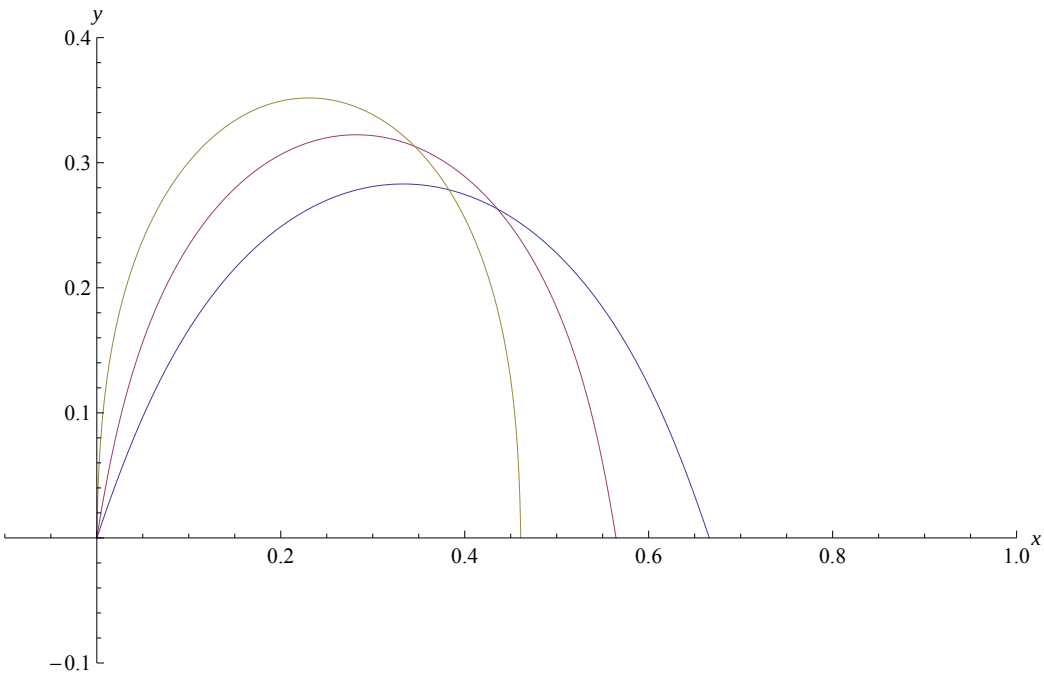


Figure 4.21: *Plots of different extensible beams. We assumed that the beams were initially straight and that $B = 0.01$. We have used the parameter $\overline{M} = -0.5$. The blue line is $\overline{P} = 10$, the purple one is $\overline{P} = 11$ and the orange one is $\overline{P} = 12$.*

Now we want to compare these plots to the inextensible plots. This gives the following result:

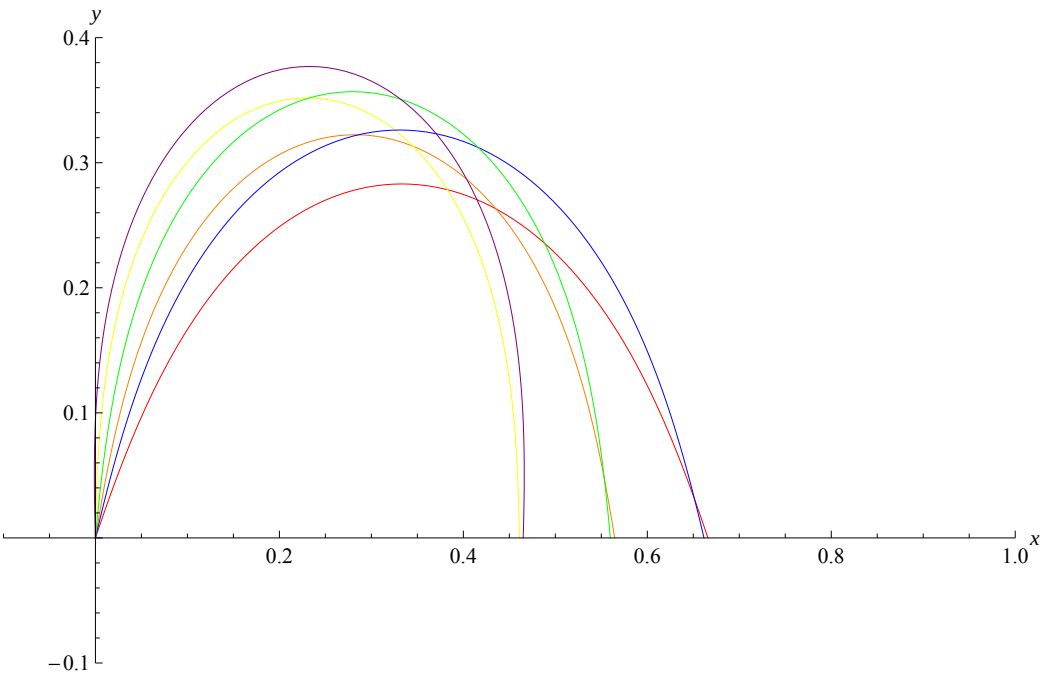


Figure 4.22: *Plots of different inextensible and extensible beams. We assumed that the beams were initially straight and that $B = 0.01$ (for the extensible beams). We have used the parameter $\overline{M} = -0.5$. The red and the blue line are $\overline{P} = 10$, the orange and the green one are $\overline{P} = 11$ and the yellow and the purple one are $\overline{P} = 12$. The lower one is always the one that is extensible.*

In figure 4.22 we see that there is quite some difference between the extensible and the inextensible beams. In the plots we see that when comparing the inextensible and the extensible beams under the same circumstances, the main difference between the two is that the height of the extensible beam is lower than that of the inextensible beam. This is because there goes less energy to bending, since the compression energy is taken into account for extensible beams. There is hardly any difference between the position of the endpoint of the beam.

Unfortunately, we did not succeed in making a bifurcation diagram just like figure 4.16. Therefore we cannot analyze this model any further, but that will be interesting for further research, since the mechanics might be very different from the inextensible model.

Chapter 5

Bifurcation analysis

In this chapter we will do a bifurcation analysis of figure 4.16 and show some of its physical consequences. Then we will do a few 'thought experiments' using Mathematica. This means that we imagined a possible experiment (for example one where we keep $\overline{M}$ constant and increase P) and used Mathematica to make a series of snapshots of the results. These snapshots will clarify the physical consequences of figure 4.16. This force-angle-moment relation has a lot in common with an example about thermodynamics in section 3.1 of [6]. Therefore we will use a lot of the terminology from that example. As a reference, we will plot figure 4.16 again:

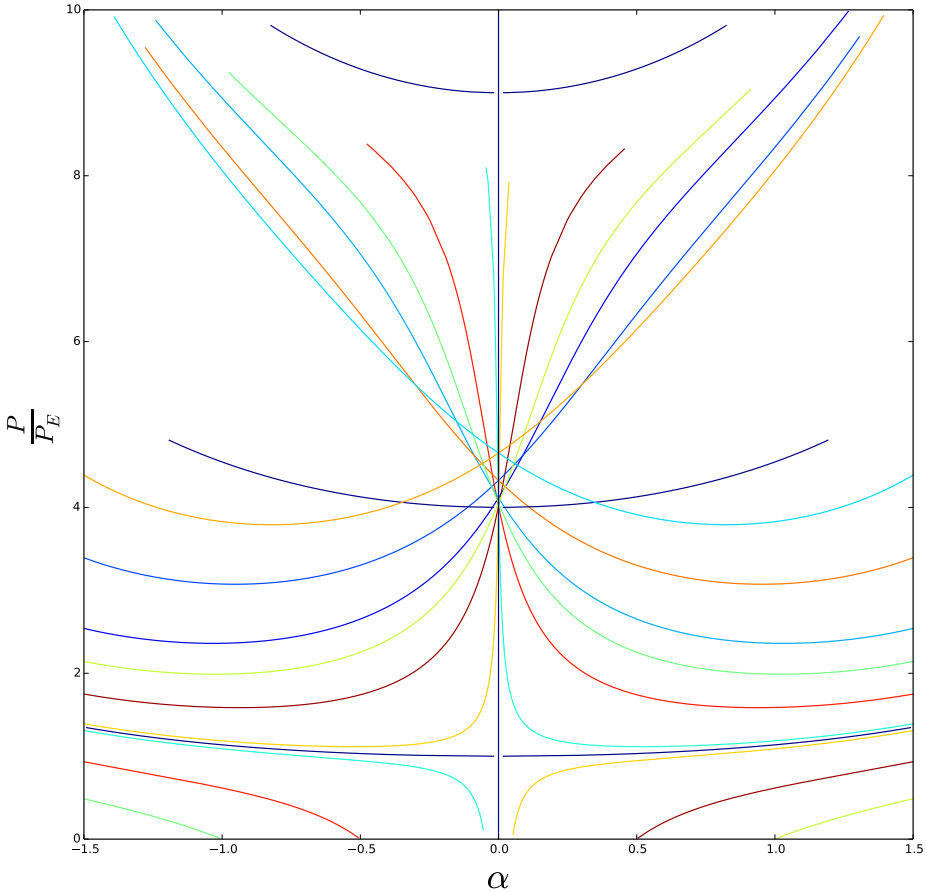


Figure 5.1: *Bifurcation diagram for the force-angle-moment relation.*

5.1 Classification of the bifurcation as a cusp catastrophe

Unfortunately time did not permit us to make a 3d-plot of the force-angle-moment relation where $\alpha, \frac{P}{P_e}$ and $\overline{M}$ are used as parameters. Luckily there is so much similarity between our problem and that of [6] that we can use his plots to interpret our situation. In his example he has the three parameters x, μ and θ which correspond to $\alpha, \frac{P}{P_e}$ and $\overline{M}$ respectively. In figure 10 of this article, which we included as figure 5.2, one of the plots is a 3d plot for not too large values of μ (i.e. $\frac{P}{P_e}$). We

see that this means that our system is governed by a so called *cusp catastrophe* which is a specific kind of bifurcation (see for example chapter 7.1 of [5]). We see in figure 10 of [6] that for values of $\frac{P}{P_e}$ smaller than 9 our system is well described by this cusp catastrophe. The only exception is the second mode of the case where $\overline{M} = 0$. However, as we noted in the previous chapter, this mode does not satisfy all boundary conditions and therefore it makes sense that it does not fit all too well into this cusp catastrophe. It is our belief that such a cusp catastrophe occurs each time $\frac{P}{P_e}$ reaches the square of an odd number, since we appear to see another cusp catastrophe at $\frac{P}{P_e} = 3^2 = 9$. However, we did not have time to fully investigate this further and we leave this as an interesting topic for further research. In the following sections we will discuss the consequences of this cusp catastrophe. We will see that it rules the mechanics of the model close to this bifurcation point and we will study these mechanics through several thought experiments.

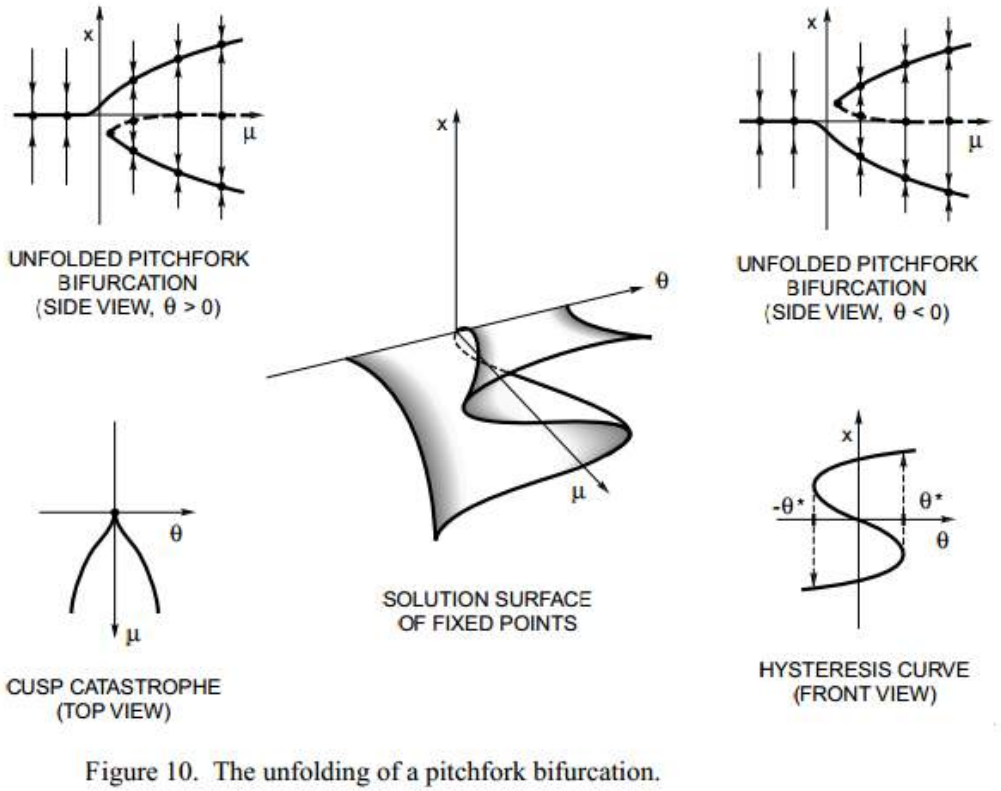


Figure 10. The unfolding of a pitchfork bifurcation.

Figure 5.2: *Figure 10 of [6].*

5.2 Experiment 1: constant force and increasing moment

In the first experiment, we want to show that we can see snapping effects and elastic memory in this model. First we will return to figure 4.5. Suppose we choose $P > P_e$ and $\alpha < 0$ and $\overline{M}$ are chosen such that we are close to the local minimum of the (α, M)-curve. Let $\overline{M}^*$ be the value of this local minimum. If we then decrease $\overline{M}$ to a value lower than $\overline{M}^*$ and keep P constant, the only thing that can happen is that the solution suddenly jumps to the solution where $\alpha > 0$ since there is no solution for $\overline{M} < \overline{M}^*$, this value of P and $\alpha < 0$. This sudden change in qualitative behavior has a lot in common with snapping and in the following experiment we will show that this effect is indeed snapping. The curves in figure 4.5 are called hysteresis curves (see [6]), since the jump between the two different states we saw, depends on the prior history of the states the beam was in, which is a hysteresis effect. We will investigate this further in the following experiment. For the first experiment, we kept $\overline{P}$ fixed at $\overline{P} = 14$ (such that $4P_e > P > P_e$) and increased $\overline{M}$ from 0 to 0.8 in steps of 0.1. This gave the following results:

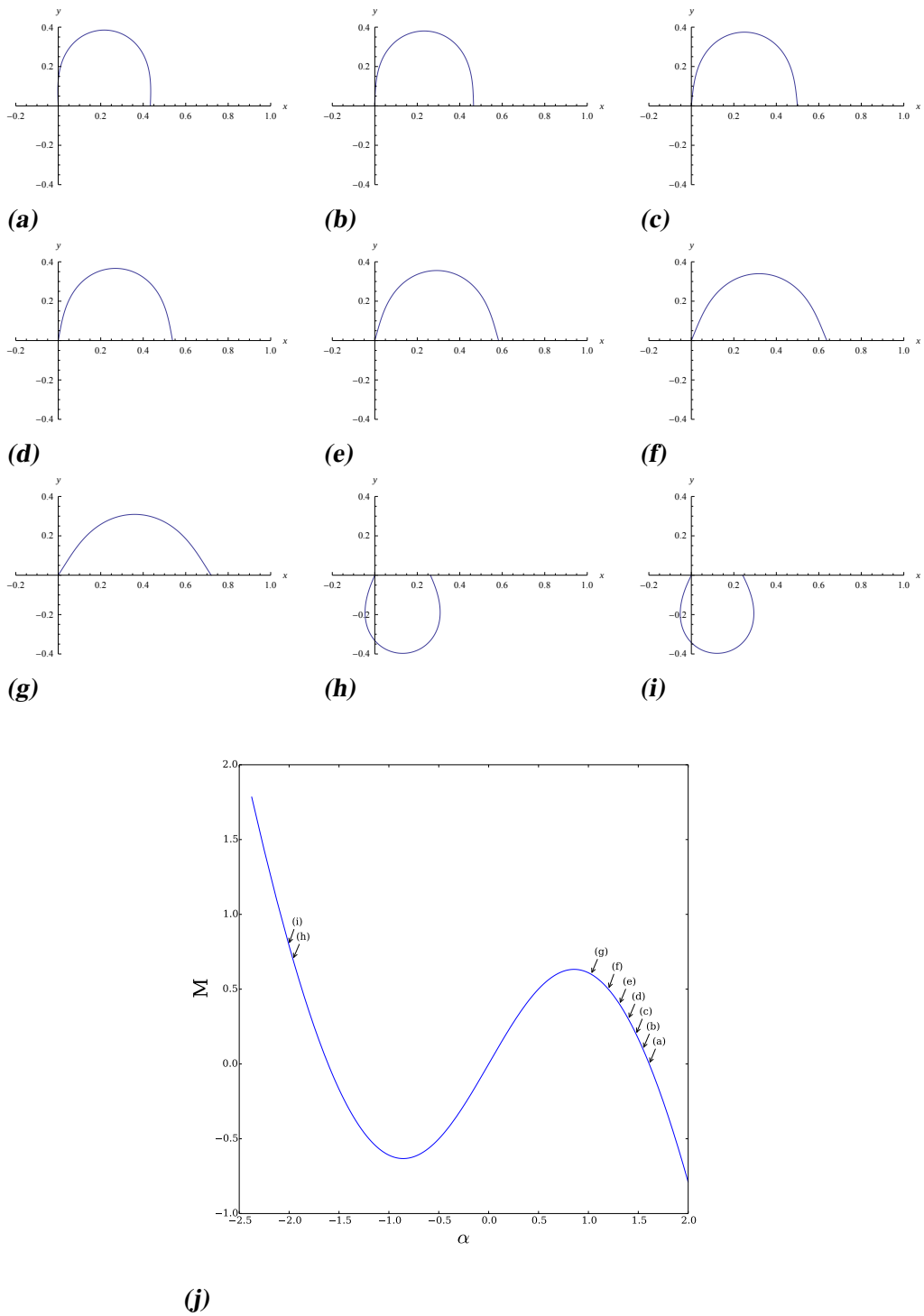


Figure 5.3: *Snapshots of a 'thought experiment' executed with Mathematica. In this plot we kept $\overline{P}$ constant at $\overline{P} = 14$ and increased $\overline{M}$ from 0 to 0.8 in steps of 0.1. Figure 5.3j shows the force-moment relation to clarify the experiment.*

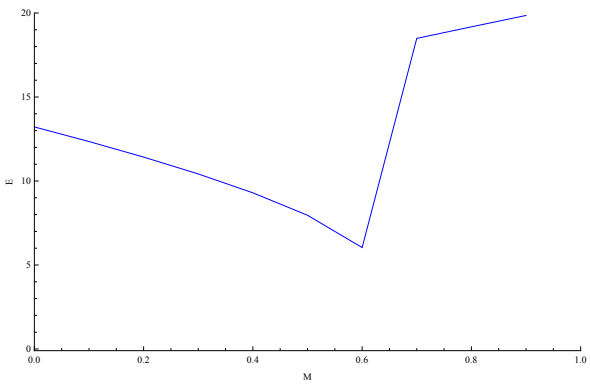


Figure 5.4: *This plot shows the moment versus the bending energy for this experiment.*

In figure 5.3 we indeed observe snapping: in the first 7 snapshots, the beam is located in the upper-half plane. However, between figures 5.3g and 5.3h the moment becomes so large in absolute value that the beam can no longer remain in this apparently unstable configuration and it snaps to a more stable state, which is located in the lower-half plane. This simple experiment therefore shows that both snapping and buckling can be described in this framework of with a single beam model. Therefore we can conclude that snapping is an effect that is not necessarily caused by the coupling of the different beams, because we see it if we apply moment on a single beam. However, in most real experiments we do not manually apply moment on the beams. In the chapter 4.3 we saw that it can be caused by the pre-curvature of the beam instead, since we saw that applying moment and having a (constant) pre-curvature are not different in their qualitative and even quantitative behavior. It may also be that the coupling of the beams causes the beams to exert moment on each other. However, we did not have time to investigate this further.
Now to show that there we can describe elastic memory, we continue first experiment. Now, after $\overline{M}$ reaches 0.8, we let it decrease again to 0 in steps of 0.2. This gives the following result:

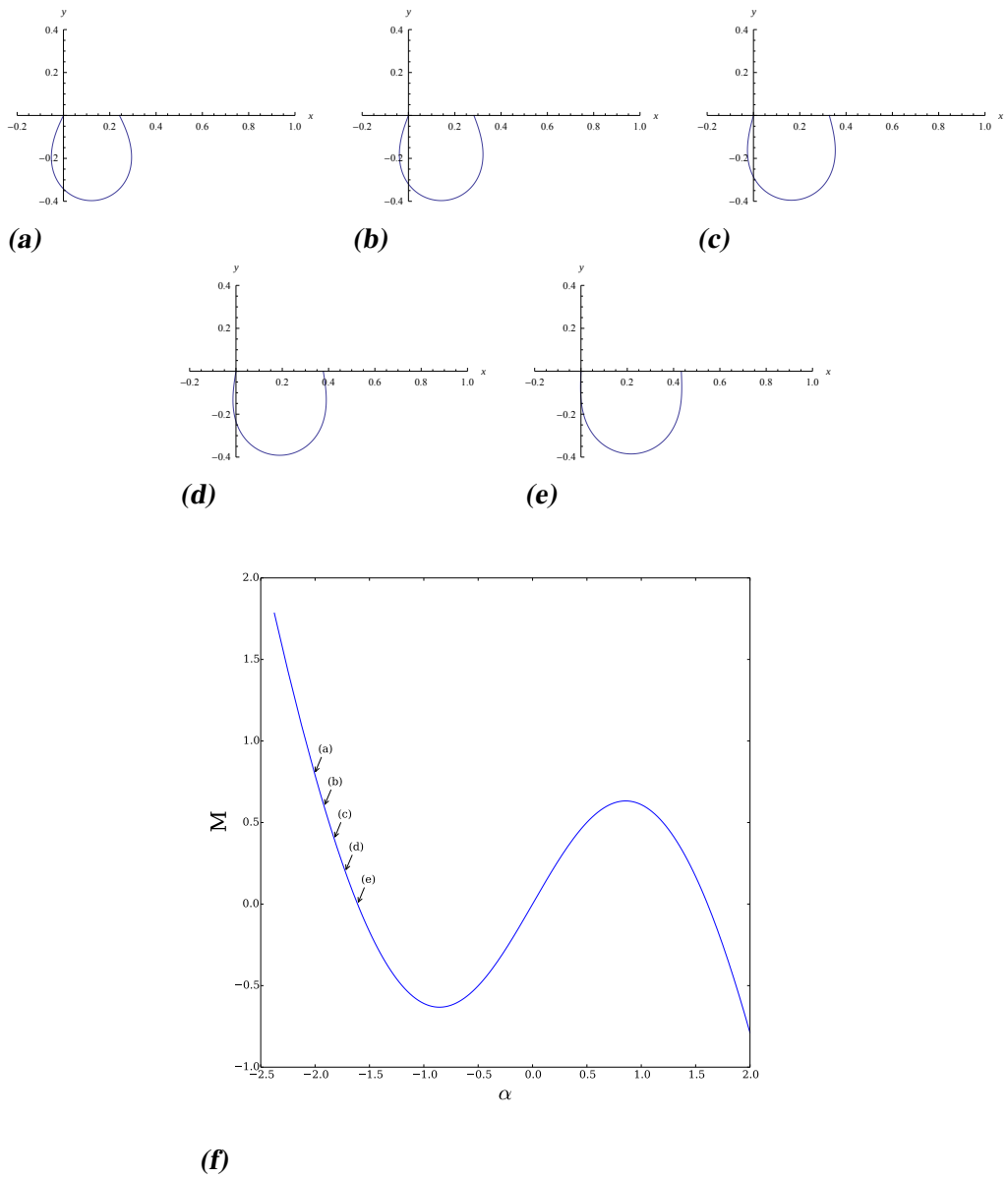


Figure 5.5: *Snapshots of a 'thought experiment' executed with Mathematica. In this plot we kept $\overline{P}$ constant at $\overline{P} = 14$ and decrease $\overline{M}$ from 0.8 to 0 in steps of 0.2. Figure 5.5f shows the force-moment relation to clarify the experiment.*

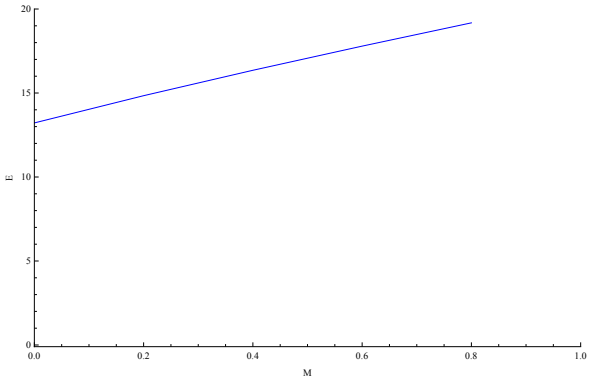


Figure 5.6: *This plotshows the moment versus the bending energy for this experiment.*

When we compare figure 5.5 with figure 5.3, we see that even though we do the same experiment, but then in reversed order, there are significant differences between the two experiments, for example because in figure 5.5 we do not see any snapping effects. So here we see that the current state of the beam, given the parameters $\overline{P}$,$\overline{M}$ and α really depends on the prior input. Therefore these experiments show that our models can also account for hysteresis effects and elastic memory.

5.3 Experiment 2: constant moment and increasing force

In figure 4.16 we observe that for any value of $\overline{M}$ one of the branches for this values of $\overline{M}$ crosses $\alpha = 0$ at some point above $\frac{P}{P_e} = 4$, i.e. above the point where the second mode of the beam starts to exist. To comprehend this result, we did another 'thought experiment'. We chose a fixed value of $\overline{M}$ (in this case $\overline{M} = -0.3$) and let $\overline{P}$ increase from a value between P_e and $4P_e$ to a value between $4P_e$ and $9P_e$. In this case we let it increase from $\overline{P} = 15$ to $\overline{P} = 55$ in steps of $\overline{P} = 5$. This gave the following result:

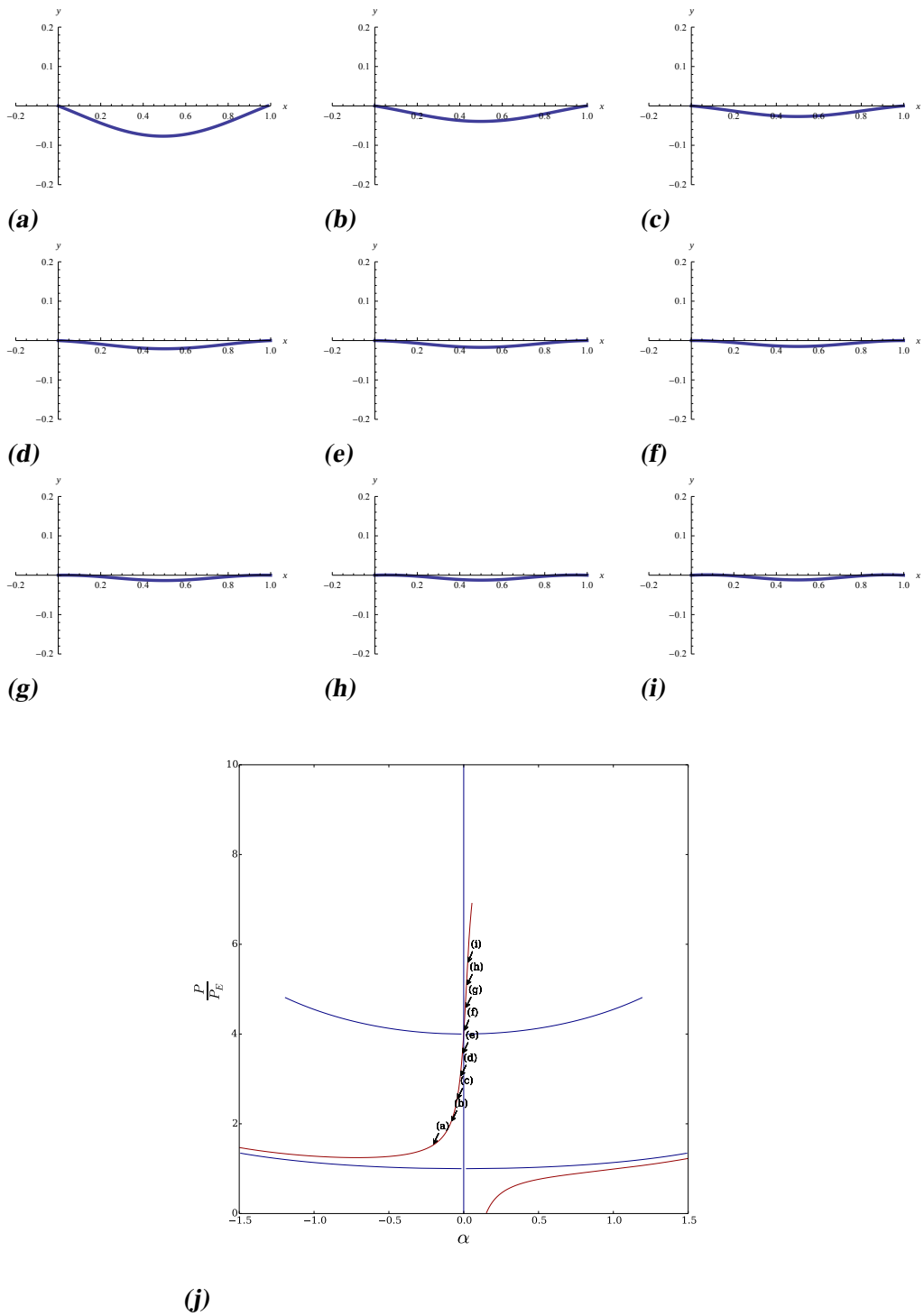


Figure 5.7: *Snapshots of a 'thought experiment' executed with Mathematica. In this plot we kept $\overline{M}$ constant at $\overline{M} = -0.3$ and increased $\overline{P}$ from 15 to 55 in steps of 5. Figure 5.7j shows the force-angle relation to clarify the experiment.*

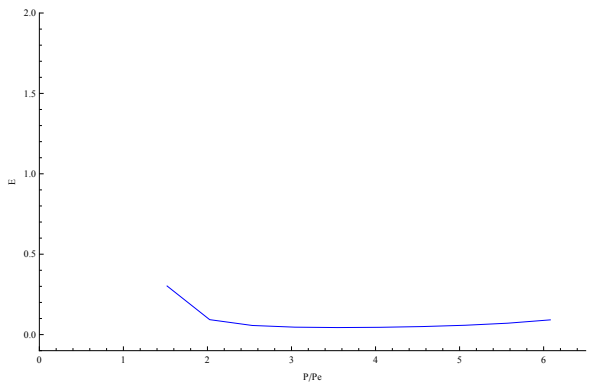


Figure 5.8: *Plot of $\frac{P}{P_e}$ versus the bending energy for this experiment.*

In the first five plots of figure 5.7 we see how we can interpret the state this beam is in: the moment is pushing the endpoints of the beam towards the upper-half plane, while in contrast the force pushes it towards the lower-half plane as $\alpha < 0$. In this configuration the two keep each other in balance. We see however that α requires to go to zero as the force increases. To keep balance when α reaches zero, the beam has to go into a higher mode, since otherwise the moment and the force would both push it towards the upper-half plane and the beam would immediately go to the stable branch (the one which start below the first mode of the case $\overline{M} = 0$ in figure 4.16) where α is large. For it to go into a higher mode, the force needs to be at least four times the Euler load and that is exactly what we see: from figure 5.7f and onwards, the beam is in a higher mode since it has three extreme values instead of the usual one. If the moment is larger in absolute value it can sustain a larger force in balance without jumping to a higher mode. Therefore we indeed see in figure 4.16 that if the moment is larger in absolute value, the point where it crosses $\alpha = 0$ is higher than that of a moment with lower absolute value. So this experiment shows that there are also higher modes for non-zero momentum and it reaches these states at a higher force than the zero-moment beam, proportional to the absolute value of the moment.

5.4 Experiment 3: constant α and increasing force

As a final 'thought experiment' to help interpret figure 4.16, we let α be constant and increase $\overline{P}$. We start at a value smaller than P_e and increase $\overline{P}$ until it is larger than $9P_e$. This gave the following results:

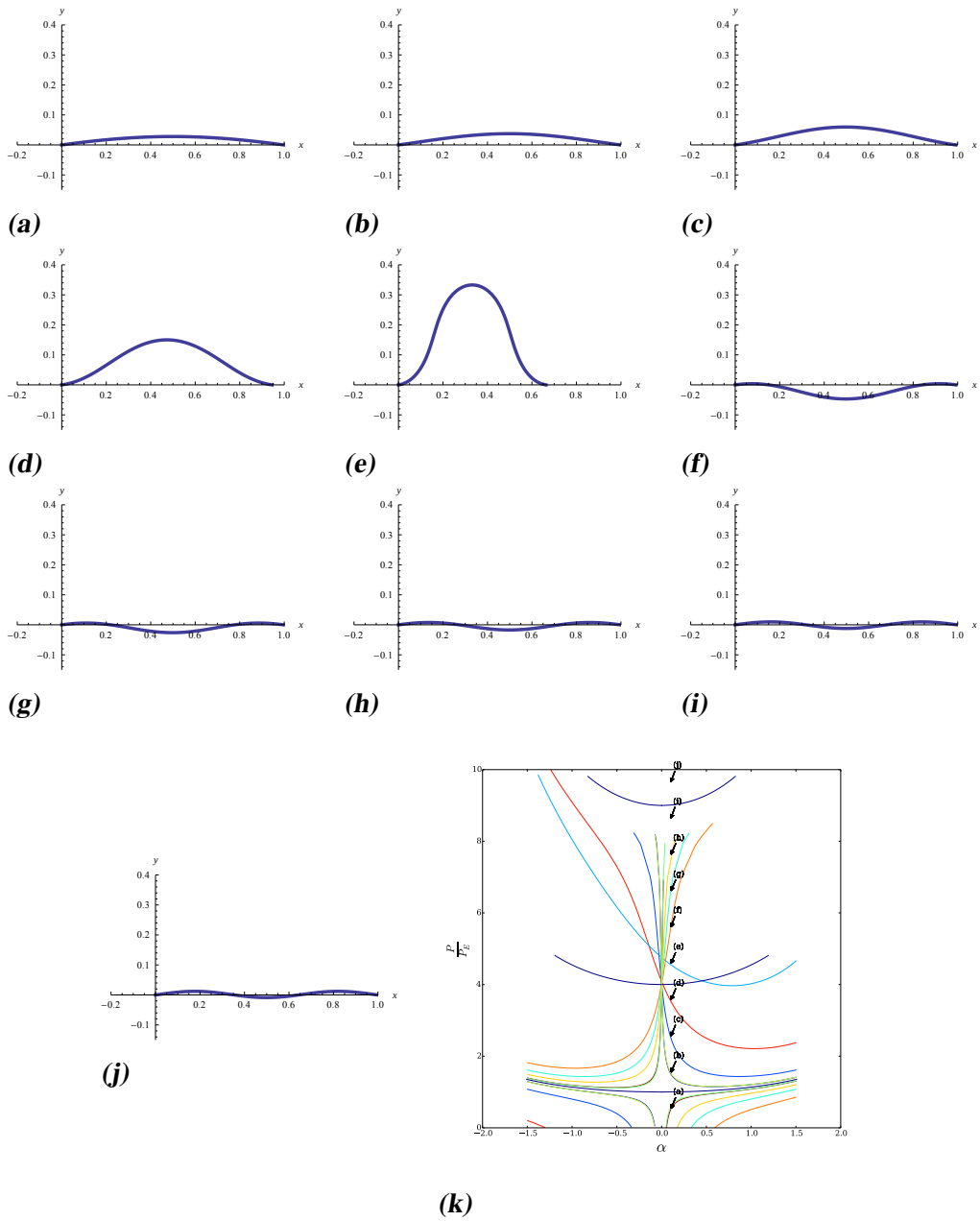


Figure 5.9: *Snapshots of a 'thought experiment' executed with Mathematica. In this plot we kept α constant at $\alpha = 0.1$ and increased $\overline{P}$ from 5 to 95 in steps of 10 so can include all qualitatively different beams in this experiment. Figure 5.9k shows the force-angle relation to clarify the experiment.*

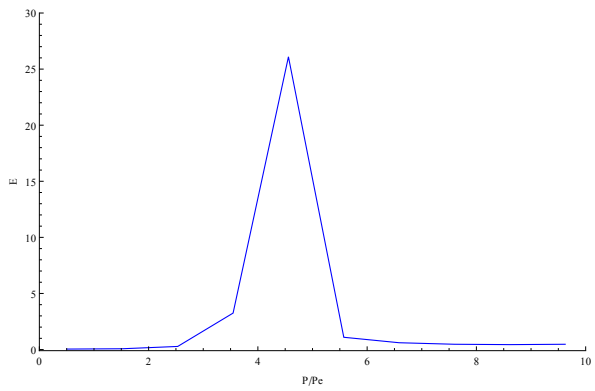


Figure 5.10: *This plot shows $\frac{P}{P_e}$ versus the bending energy in this experiment.*

In figure 5.9 we see that there a lot of qualitative differences between the different beams which have the same value of α in common. For $P < P_e$, i.e. figure 5.9a, we see that there is outward (i.e. negative) moment to achieve this value of α, because the force it too low to make the beam buckle. In figures 5.9b-5.9e we see that the beam is still in its first mode and that the force is so large that we need an inward (i.e. positive) moment to obtain this value of α. Indeed, in these figures we see that the beam is curved inwards as a result of the moment in that direction. We also see that the maximal deflection of the beam increases in these figures. At a certain point, where $P > 4P_e$ it becomes energetically more favorable to be in a higher mode (see figure 5.10) and that is what we see in figures 5.9f-5.9i. In these figures we see that the beam has three local extrema, which do not have the same height in absolute values: the absolute value of the middle top is larger than that of the other two. As P increases even more we now see that absolute value of the height of the three tops becomes more and more equal. Since $\overline{M}$ is driving the height difference between the tops, this suggests that less $\overline{M}$ is needed to sustain α. This is reflected by figure 4.16 from which we see that for $P > 4P_e$ we require less $\overline{M}$ to sustain α. Therefore the force will increase its influence on the shape of the beam. Since we saw that when there is no moment that the absolute value of the height of the tops is equal, this means that the height of these tops will become more and more the same when the influence of the force increases. Finally, in figure 5.9j we have a beam of which the force is higher than $9P_e$. In this case we have that the middle top has the smallest absolute value instead of the other two. This is because now the moment is directed in the other direction than in figures 5.9f-5.9i, which is since we now have that $P > 9P_e$.

Chapter 6

Conclusion + Outlook

In this bachelor thesis we looked at one-dimensional elastic beams. We described different beam models where we apply a horizontal force P on the beam and a moment M on the endpoints of the beam. We have looked at both models where the beam was initially straight and models where the beam was pre-curved. We mainly considered beams of which the length is constant, but we also did some analysis for an extensible beam. The questions we asked ourselves were if we could describe effects like snapping and elastic memory using this one-dimensional model and in what way applying moment to a beam, or letting the beam be pre-curved, changes the qualitative behavior of beams. Furthermore we tried to find exact solutions to this models so we could plot the shape of the beam and increase our physical understanding of these models.

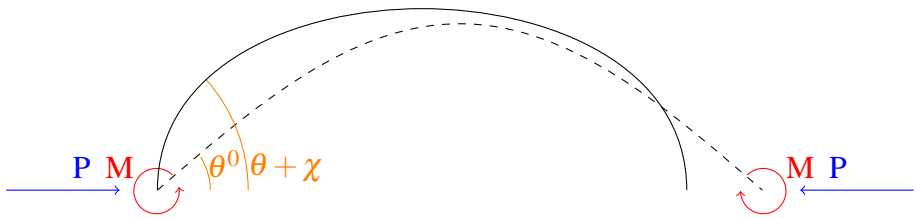


Figure 6.1: *General setup for our model. We apply a constant force P to an initially straight or pre-curved beam and we apply a moment M to the endpoints of the beam. The angle θ is the rotation angle and χ is the shear angle. If we assume that there are no shear effects, then θ is the angle between a line tangent to the beam and the x-axis. θ^0 describes the shape of the initial state of the beam.*

In chapter 4.1 we saw that if we apply moment, but the force is zero, to any beam, then the moment adds a constant amount of curvature to each point of the beam and it changes the angle at $t = 0$. This gave a linear relation between the

applied moment M and the angle at zero α.

After that, we combined the concepts of moment and force in chapter 4.2 for an initially straight beam. For this model we found most of our main results. We found an exact solution for any value of the three parameters P, M and α. For beams of which the angle θ is monotonically increasing or decreasing, we also found an exact, but implicit, relation between the parameters P, M and α.

In chapter 4.3 we took pre-curvature into account. For beams with a constant curvature (other pre-curvatures were too complex), we saw great similarity with the model of chapter 4.2. Again, we found an exact solution for any set of parameters and we found an exact relation between P, M and α in case that the beam is monotonic. If the pre-curvature is given by $\theta^0(t) = \kappa t + \alpha_0$ then we saw that we re-obtained the results of the previous chapter for an 'effective moment' $\tilde{M} = \overline{M} + \kappa$ where all results were shifted over an angle α_0. Therefore we can conclude that there is no qualitative difference between the models of pre-curved beams with a constant curvature and the models of initially straight beams where we apply moment on the endpoints of the beam.

In chapter 5 we did a bifurcation analysis for an initially straight beam. It was already known that for the case that $M = 0$, there is a pitchfork bifurcation in the $P - \alpha-$diagram at the point where the beam starts to buckle (see figure 4.4). We discovered that this pitchfork bifurcation unfolds into a cusp catastrophe in case there is a non-zero moment (see figure 4.16). The presence of this cusp catastrophe creates interesting phenomena: we saw that there are examples of hysteresis in our model, which confirms that elastic memory, at least in some form, is present is this model. Furthermore, by doing some 'thought experiments' using Mathematica, we clearly saw that snapping is indeed described by this model. Figure 4.16 contained all this information and can therefore be considered the main result of this research.

As mentioned throughout this thesis, there a quite a few aspects of this project that will trigger future research. It is our belief that it should be possible to find an exact solution and a force-angle-moment relation for an extensible beam in case there is moment present, since we know that it is possible in case there is no moment. If that is possible indeed, then it is very interesting to make a diagram similar to figure 4.16 and see what the differences are to the inextensible model. One might also want to try analyze other, more complicated pre-curvatures than we did and see if there is still similarity with the model in section 4.2 or if we get significantly different results. Furthermore, it could be interesting to look at models of which the applied force is not constant, such as a model of a pre-

curved beam with a spring attached between the endpoints. In some preliminary calculations we have seen that the differential equations describing such a model are significantly different from the equations used in this thesis and solving them could potentially give very interesting results. Next it might be interesting to look at systems of coupled beams and for example see if we can still see snapping effects when we do not apply moment manually in a system of pre-curved beams. Finally it might also be interesting to acquire relations between the applied force, the moment and the strain in the different models we described, since such relations are easier to measure in experiments than the force-angle-moment relations we described.

Chapter 7

Mathematica codes

In this chapter we will show the most relevant codes to makes the various plots used throughout this thesis. They were all made using Wolfram Mathematica 9. Text between the following signs: $(*\cdot*)$ is comment meant to make the code more readable.

7.1 Code for plotting $\frac{P}{P_e}$ versus α

```
ClearAll["Global`*"]
S[M_] := 1 /; 0 ≤ M
S[M_] := -1 /; M < 0
(*Here we define the Sign function*)
constraint[P_, M_, α_] :=
 2/√P EllipticF[Re[ArcSin[Sin[α/2] √(4 P / (4 P Sin[α/2]^2 + M^2))]], (4 P Sin[α/2]^2 + M^2)/(4 P)]
(*This is the exact force-angle-moment relation for monotonic beams*)
Np = 100.;
M = {0., -0.01, -0.05, -0.1, -0.5,
   -1., -2., -5., 0., 0.01, 0.05, 0.1, 0.5, 1., 2., 5.};
(*These are the values of M we use. In the calculations below we
 calculate α for an array of values of P for every of this value of M*)
sarray1 = N[Range[9.87, 10.5, (1. / Np)]];
P1 = sarray1;
sol1 = FindRoot[constraint[#, M[[1]], α1] - 1, {α1, 1.1}] & /@ P1;
res1 = α1 /. sol1;
α1 = Re[res1];
forceangle1 = Transpose[{α1, P1 / Pi^2}];

sarray2 = N[Range[1. / Np, 10.5, (1. / Np)]];
P2 = sarray2;
sol2 = FindRoot[constraint[#, M[[2]], α2] + S[M[[2]]], {α2, 0.01}] & /@ P2;
res2 = α2 /. sol2;
α2 = Re[res2];
forceangle2 = Transpose[{α2, P2 / Pi^2}];

sarray3 = N[Range[1. / Np, 10.5, (1. / Np)]];
P3 = sarray3;
sol3 = FindRoot[constraint[#, M[[3]], α3] + S[M[[3]]], {α3, 0.1}] & /@ P3;
res3 = α3 /. sol3;
α3 = Re[res3];
forceangle3 = Transpose[{α3, P3 / Pi^2}];

sarray4 = N[Range[1. / Np, 10.5, (1. / Np)]];
P4 = sarray4;
sol4 = FindRoot[constraint[#, M[[4]], a4] + S[M[[4]]], {a4, 0.1}] & /@ P4;
res4 = a4 /. sol4;
a4 = Re[res4];
forceangle4 = Transpose[{a4, P4 / Pi^2}];

sarray5 = N[Range[1. / Np, 10.5, (1. / Np)]];
P5 = sarray5;
sol5 = FindRoot[constraint[#, M[[5]], a5] + S[M[[5]]], {a5, 0.5}] & /@ P5;
res5 = a5 /. sol5;
a5 = Re[res5];
forceangle5 = Transpose[{a5, P5 / Pi^2}];

sarray6 = N[Range[1. / Np, 10.5, (1. / Np)]];
P6 = sarray6;
sol6 = FindRoot[constraint[#, M[[6]], a6] + S[M[[6]]], {a6, 0.55}] & /@ P6;
```

2 | *forceanglerelation.nb*

```
res6 = a6 /. sol6;
a6 = Re[res6];
forceangle6 = Transpose[{a6, P6 / Pi^2}];

sarray7 = N[Range[1. / Np, 10.5, (1. / Np)]];
P7 = sarray7;
sol7 = FindRoot[constraint[#, M[[7]], a7] + S[M[[7]]], {a7, 0.4}] & /@ P7;
res7 = a7 /. sol7;
a7 = Re[res7];
forceangle7 = Transpose[{a7, P7 / Pi^2}];

sarray8 = N[Range[1. / Np, 10.5, (1. / Np)]];
P8 = sarray8;
sol8 = FindRoot[constraint[#, M[[8]], a8] + S[M[[8]]], {a8, 3.14159}] & /@ P8;
res8 = a8 /. sol8;
a8 = Re[res8];
forceangle8 = Transpose[{a8, P8 / Pi^2}];

sarray9 = N[Range[9.87, 10.5, (1. / Np)]];
P9 = sarray9;
α9 = -α1;
forceangle9 = Transpose[{α9, P9 / Pi^2}];

sarray10 = N[Range[1. / Np, 10.5, (1. / Np)]];
P10 = sarray10;
sol10 =
  FindRoot[constraint[#, M[[10]], α10] + S[M[[10]]], {α10, -0.01}] & /@ P10;
res10 = α10 /. sol10;
α10 = Re[res10];
forceangle10 = Transpose[{α10, P10 / Pi^2}];

sarray11 = N[Range[1. / Np, 10.5, (1. / Np)]];
P11 = sarray11;
sol11 =
  FindRoot[constraint[#, M[[11]], α11] + S[M[[11]]], {α11, -0.1}] & /@ P11;
res11 = α11 /. sol11;
α11 = Re[res11];
forceangle11 = Transpose[{α11, P11 / Pi^2}];

sarray12 = N[Range[1. / Np, 10.5, (1. / Np)]];
P12 = sarray12;
sol12 = FindRoot[constraint[#, M[[12]], a12] + S[M[[12]]], {a12, -0.1}] & /@ P12;
res12 = a12 /. sol12;
a12 = Re[res12];
forceangle12 = Transpose[{a12, P12 / Pi^2}];

sarray13 = N[Range[1. / Np, 10.5, (1. / Np)]];
P13 = sarray13;
sol13 = FindRoot[constraint[#, M[[13]], a13] + S[M[[13]]], {a13, -0.5}] & /@ P13;
res13 = a13 /. sol13;
a13 = Re[res13];
forceangle13 = Transpose[{a13, P13 / Pi^2}];
```

```
sarray14 = N[Range[1. / Np, 10.5, (1. / Np)]];
P14 = sarray14;
sol14 =
  FindRoot[constraint[#, M[[14]], a14] + S[M[[14]]], {a14, -0.55}] & /@ P14;
res14 = a14 /. sol14;
a14 = Re[res14];
forceangle14 = Transpose[{a14, P14 / Pi^2}];

sarray15 = N[Range[1. / Np, 10.5, (1. / Np)]];
P15 = sarray15;
sol15 = FindRoot[constraint[#, M[[15]], a15] + S[M[[15]]], {a15, -0.4}] & /@ P15;
res15 = a15 /. sol15;
a15 = Re[res15];
forceangle15 = Transpose[{a15, P15 / Pi^2}];

sarray16 = N[Range[1. / Np, 10.5, (1. / Np)]];
P16 = sarray16;
sol16 =
  FindRoot[constraint[#, M[[16]], a16] + S[M[[16]]], {a16, -3.14159}] & /@ P16;
res16 = a16 /. sol16;
a16 = Re[res16];
forceangle16 = Transpose[{a16, P16 / Pi^2}];
(*Here we plot these results in one figure*)
pl1 = ListLinePlot[
  {forceangle1, forceangle2, forceangle3, forceangle4, forceangle5, forceangle6,
   forceangle7, forceangle8, forceangle9, forceangle10, forceangle11,
   forceangle12, forceangle13, forceangle14, forceangle15, forceangle16},
  PlotRange → {{-π, π}, {0, 10.5 / Pi^2}}, ImageSize → Large,
  AxesLabel → {α, P / Pe}, AxesStyle → Medium, PlotStyle → Boxed]
Export[NotebookDirectory[] <> "forceanglerelation.pdf",
 pl1, "PDF", "CompressionLevel" → 0]
```

7.2 Code for plotting $\overline{M}$ versus α

```
ClearAll["Global`*"]
constraint[P_, M_, α_] :=
 2/Sqrt[P] EllipticF[Re[ArcSin[Sin[α/2] Sqrt[4 P/(4 P Sin[α/2]^2 + M^2)]]], (4 P Sin[α/2]^2 + M^2)/(4 P)]
(*This function is the Force-Moment-Angle relation*)
Np = 8;
sarray = Join[N[Range[-2.0, -(1. / Np), (1. / Np)]],
   {0.0001}, N[Range[(1. / Np), 2.0, (1. / Np)]]];
α = sarray;
P = {1., 2., 3., 5., 10., 11., 12., 13.};
S[M_] := 1 /; M > 0
S[M_] := -1 /; M ≤ 0
(*Here we define the Sign function.*)
θ1[t_, M_, P_, α_] := 2 ArcSin[Sqrt[1/2 (M^2/(2 P) - Cos[α] + 1.)]
    Re[Sin[JacobiAmplitude[EllipticF[Re[ArcSin[Sin[α/2]/Sqrt[1/2 (M^2/(2 P) - Cos[α] + 1.)]]],
        1/2 (M^2/(2 P) - Cos[α] + 1.)] + S[M] Sqrt[P] t, 1/2 (M^2/(2 P) - Cos[α] + 1.)]]]];
(*This function is the exact solution for any set of parameters*)
Np = Length[sarray];
(*In the following calculations we calculate M for an array of values for α
  for all values of P using that we know that θ(1/2)=0 and θ(1)=-α*)
M1 = ConstantArray[0, Np];
For[j = 1, j < Np + 1, j++, Guess = 0;
 For[i = 1, i < 100, i++, Guess = S[α[[j]]] (-2 + i / 25);
  sol = FindRoot[θ1[(1 / 2), Mj, P[[1]], α[[j]]], {Mj, Guess}];
  res = Mj /. sol;
  M1[[j]] = Re[res];
  If[Abs[Constraint[P[[1]], M1[[j]], α[[j]]] + S[M1[[j]] < 0.000001,
   Break[], Guess = 0]]]

M2 = ConstantArray[0, Np];
For[j = 1, j < Np + 1, j++, Guess = 0;
 For[i = 1, i < 100, i++, Guess = S[α[[j]]] (-2 + i / 25);
  sol = FindRoot[θ1[(1 / 2), Mj, P[[2]], α[[j]]], {Mj, Guess}];
  res = Mj /. sol;
  M2[[j]] = Re[res];
  If[Abs[Constraint[P[[2]], M2[[j]], α[[j]]] + S[M2[[j]] < 0.000001,
   Break[], Guess = 0]]]

M3 = ConstantArray[0, Np];
For[j = 1, j < Np + 1, j++, Guess = 0;
 For[i = 1, i < 100, i++, Guess = S[α[[j]]] (-2 + i / 25);
  sol = FindRoot[θ1[(1 / 2), Mj, P[[3]], α[[j]]], {Mj, Guess}];
  res = Mj /. sol;
```

```
    M3[[j]] = Re[res];
    If[Abs[Constraint[P[[3]], M3[[j]], α[[j]]]] + S[M3[[j]]] < 0.000001,
     Break[], Guess = 0]]]

M4 = ConstantArray[0, Np];
For[j = 1, j < Np + 1, j++, Guess = 0;
 For[i = 1, i < 100, i++, Guess = S[α[[j]]] (-2 + i / 25);
  sol = FindRoot[θ1[(1 / 2), Mj, P[[4]], α[[j]]], {Mj, Guess}];
  res = Mj /. sol;
  M4[[j]] = Re[res];
  If[Abs[Constraint[P[[4]], M4[[j]], α[[j]]]] + S[M4[[j]]] < 0.000001,
   Break[], Guess = 0]]]

M5 = ConstantArray[0, Np];
For[j = 1, j < Np + 1, j++, Guess = 0;
 For[i = 1, i < 100, i++, Guess = S[α[[j]]] (-2 + i / 25);
  sol = FindRoot[θ1[(1 / 2), Mj, P[[5]], α[[j]]], {Mj, Guess}];
  res = Mj /. sol;
  M5[[j]] = Re[res];
  If[Abs[Constraint[P[[5]], M5[[j]], α[[j]]]] + S[M5[[j]]] < 0.000000001,
   Break[], Guess = 0]]]

M6 = ConstantArray[0, Np];
For[j = 1, j < Np + 1, j++, Guess = 0;
 For[i = 1, i < 100, i++, Guess = S[α[[j]]] (-2 + i / 25);
  sol = FindRoot[θ1[(1 / 2), Mj, P[[6]], α[[j]]], {Mj, Guess}];
  res = Mj /. sol;
  M6[[j]] = Re[res];
  If[Abs[Constraint[P[[6]], M6[[j]], α[[j]]]] + S[M6[[j]]] < 0.000000001,
   Break[], Guess = 0]]]

M7 = ConstantArray[0, Np];
For[j = 1, j < Np + 1, j++, Guess = 0;
    For[i = 1, i < 100, i++, Guess = S[α[[j]]] (-2 + i / 25);
     sol = FindRoot[θ1[(1 / 2), Mj, P[[7]], α[[j]]], {Mj, Guess}];
     res = Mj /. sol;
     M7[[j]] = Re[res];
     If[Abs[Constraint[P[[7]], M7[[j]], α[[j]]]] + S[M7[[j]]] < 0.000000001,
      Break[], Guess = 0]]] M7 = ConstantArray[0, Np];

M8 = ConstantArray[0, Np];
For[j = 1, j < Np + 1, j++, Guess = 0;
 For[i = 1, i < 100, i++, Guess = S[α[[j]]] (-2 + i / 25);
  sol = FindRoot[θ1[(1 / 2), Mj, P[[8]], α[[j]]], {Mj, Guess}];
  res = Mj /. sol;
  M8[[j]] = Re[res];
  If[Abs[Constraint[P[[8]], M8[[j]], α[[j]]]] + S[M8[[j]]] < 0.000000001,
   Break[], Guess = 0]]]

forceangle1 = Transpose[{α, M1}];
forceangle2 = Transpose[{α, M2}];
forceangle3 = Transpose[{α, M3}];
forceangle4 = Transpose[{α, M4}];
```

```
forceangle5 = Transpose[{α, M5}];
forceangle6 = Transpose[{α, M6}];
forceangle7 = Transpose[{α, M7}];
forceangle8 = Transpose[{α, M8}];
pl1 =
 ListLinePlot[{forceangle1, forceangle2, forceangle3, forceangle4, forceangle5,
   forceangle6, forceangle7, forceangle8}, PlotRange → {{-2, 2}, {-2, 2}},
  ImageSize → Large, AxesLabel → {"α", "M"}, AxesStyle → Medium,
  PlotStyle → {Red, Orange, Yellow, Green, Blue, Purple, Brown, Black}]
Export[NotebookDirectory[] <> "momentanglerelation.pdf",
 pl1, "PDF", "CompressionLevel" → 0]
```

7.3 Code for plotting inextensible initially straight beams

```
ClearAll["Global`*"]
S[M_] := 1 /; M > 0
S[M_] := -1 /; M ≤ 0
(*This defines the Sign function*)
constraint[P_, M_, α_] :=
 2/√P EllipticF[Re[ArcSin[Sin[α/2] √(4 P / (4 P Sin[α/2]^2 + M^2))]], (4 P Sin[α/2]^2 + M^2)/(4 P)]
(*This function is the Force-Moment-Angle relation*)
M = -0.5;
P = {10., 10., 10.};
(*Here you can define M and P. You have to either give M 3
  values and P 1 or vice-versa. With this we get three beams.*)
α = ConstantArray[0, 3];
sol = FindRoot[constraint[P[[1]], M, α1] + S[M], {α1, 1}];
res = α1 /. sol;
α[[1]] = Re[res];
sol = FindRoot[constraint[P[[2]], M, α2] + S[M], {α2, 1}];
res = α2 /. sol;
α[[2]] = Re[res];
sol = FindRoot[constraint[P[[3]], M, α3] + S[M], {α3, 1}];
res = α3 /. sol;
α[[3]] = Re[res];
(*This gives the value of alpha that satisfies the Force-
 Moment-Angle relation given the values of P and M*)
k = 1/2 (M^2/(2 P) - Cos[α] + 1.);

ω0 = √P;
Np = 100;
sarray = N[Range[0, 1, 1 / Np]];
u1 = ConstantArray[0, Np + 1];
u2 = ConstantArray[0, Np + 1];
u3 = ConstantArray[0, Np + 1];
w1 = ConstantArray[0, Np + 1];
w2 = ConstantArray[0, Np + 1];
w3 = ConstantArray[0, Np + 1];
(*The variables u are the horizontal displacements of the three beams,
while the variables w are the vertical displacements.*)
index = 1;

x0 = sarray;
y0 = ConstantArray[0, Np + 1];
(*x0 and y0 are the initial positions of the beams.*)
θ[t_] := 2 ArcSin[
    √k Re[Sin[JacobiAmplitude[EllipticF[Re[ArcSin[Sin[α/2]/√k]], k] - ω0 t, k]]]];
(*This is the exact solution for the beam, given P,M and alpha*)
Do[
```

```
 u1[[index]] = NIntegrate[Cos[θ[t][[1]]] - 1, {t, 0, s}];
 w1[[index]] = NIntegrate[Sin[θ[t][[1]]], {t, 0, s}];
 index = index + 1;
 , {s, sarray}
]
index = 1;
Do[
 u2[[index]] = NIntegrate[Cos[θ[t][[2]]] - 1, {t, 0, s}];
 w2[[index]] = NIntegrate[Sin[θ[t][[2]]], {t, 0, s}];
 index = index + 1;
 , {s, sarray}
]
index = 1;
Do[
 u3[[index]] = NIntegrate[Cos[θ[t][[3]]] - 1, {t, 0, s}];
 w3[[index]] = NIntegrate[Sin[θ[t][[3]]], {t, 0, s}];
 index = index + 1;
 , {s, sarray}
]
(*Here we find u and w at 101 point by
 numerically integrating the appropriate equations*)
x1 = Re[x0 + u1];
y1 = Re[y0 + w1];
x2 = Re[x0 + u2];
y2 = Re[y0 + w2];
x3 = Re[x0 + u3];
y3 = Re[y0 + w3];
(*Here we calculate the actual positions. By numerical
 inaccuracy it may be possible that the displacements are imaginary,
while they need to be real. Therefore we take the real part.*)
beamshape1 = Transpose[{x1, y1}];
beamshape2 = Transpose[{x2, y2}];
beamshape3 = Transpose[{x3, y3}];
pl1 = ListLinePlot[{beamshape1, beamshape2, beamshape3},
  PlotRange → {{-0.2, 1}, {-0.2, 0.4}}, ImageSize → Large,
  PlotStyle → Boxed, AxesLabel → {x, y}, AxesStyle → Medium]
Export[NotebookDirectory[] <> "Plots1.pdf", pl1, "PDF", "CompressionLevel" → 0]
```

7.4 Code for plotting the expanded force-angle-moment relation

```
ClearAll["Global`*"]
constraint[P_, α_] := 2/Sqrt[P] EllipticK[Sin[α/2]^2]
(*This is the force-angle relation for the case M=0*)
S[M_] := 1 /; M > 0
S[M_] := -1 /; M ≤ 0
(*Here we define the Sing function*)
θ1[t_, M_, P_, α_] := 2 ArcSin[Sqrt[1/2 (M^2/(2 P) - Cos[α] + 1.)]
    Re[Sin[JacobiAmplitude[EllipticF[Re[ArcSin[Sin[α/2]/Sqrt[1/2 (M^2/(2 P) - Cos[α] + 1.)]]],
        1/2 (M^2/(2 P) - Cos[α] + 1.)] + S[M] Sqrt[P] t, 1/2 (M^2/(2 P) - Cos[α] + 1.)]]]];
(*This is the exact solution for any value of the parameters P,M and α*)

Np = 400.;
M = {0., -1., -3., 0., -5., 1., 3., 5., 0, 0, 0, 0, 0.1, -0.1, 2., -2., 7., -7.};
(*Here we give the values of M we use. In the calculations below we
  calculate for given M the values of P for an array of values for α
  if M is not 0 and we calculate α for an array of values for P if M=
 0. In this calculations we use that θ(1/2)=0 and that θ(1)=-α*)
sarray1 = N[Range[9.87, 17.87, 0.1]];
P1 = sarray1;
α1 = sarray1;
Pp = Length[sarray1];
For[j = 1, j < Pp + 1, j++, Guess = 0;
 For[i = 1, i < 50, i++, Guess = i / 50;
  sol = FindRoot[constraint[P1[[j]], αj] - 1, {αj, Guess}];
  res = αj /. sol;
  α1[[j]] = Re[res];
  If[
   Abs[θ1[1, 0., P1[[j]], α1[[j]]] + α1[[j]]] < 0.000000001, Break[], Guess = 0]]
]
forceangle1 = Transpose[{α1, P1 / Pi^2}];

sarray2 =
  Join[N[Range[-1.5, -1. / Np, (1. / Np)]], N[Range[1. / Np, 1.5, (1. / Np)]]];
P2 = sarray2;
α2 = sarray2;
Pp = Length[sarray2];
For[j = 1, j < Pp + 1, j++, Guess = 0;
 For[i = 1, i < 50, i++, Guess = 2 i;
  sol = FindRoot[θ1[(1 / 2), M[[2]], Pj, α2[[j]]], {Pj, Guess}];
  res = Pj /. sol;
  P2[[j]] = Re[res];
  If[
   Abs[θ1[1, M[[2]], P2[[j]], α2[[j]]] + α2[[j]]] < 0.0000001, Break[], Guess = 0]]
```

```
]
forceangle2 = Transpose[{α2, P2 / Pi^2}];


sarray3 =
  Join[N[Range[-1.5, -1. / Np, (1. / Np)]], N[Range[1. / Np, 1.5, (1. / Np)]]];
P3 = sarray3;
α3 = sarray3;
For[j = 1, j < Pp + 1, j++, Guess = 0;
 For[i = 1, i < 50, i++, Guess = 2 i;
  sol = FindRoot[θ1[(1 / 2), M[[3]], Pj, α3[[j]]], {Pj, Guess}];
  res = Pj /. sol;
  P3[[j]] = Re[res];
  If[
   Abs[θ1[1, M[[3]], P3[[j]], α3[[j]]] + α3[[j]]] < 0.0000001, Break[], Guess = 0]]
]
forceangle3 = Transpose[{α3, P3 / Pi^2}];

sarray4 = sarray1;
P4 = sarray1;
a4 = -α1;
forceangle4 = Transpose[{a4, P4 / Pi^2}];

sarray5 =
  Join[N[Range[-1.5, -1. / Np, (1. / Np)]], N[Range[1. / Np, 1.5, (1. / Np)]]];
P5 = sarray5;
α5 = sarray5;
For[j = 1, j < Pp + 1, j++, Guess = 0;
 For[i = 1, i < 50, i++, Guess = 2 i;
  sol = FindRoot[θ1[(1 / 2), M[[5]], Pj, α5[[j]]], {Pj, Guess}];
  res = Pj /. sol;
  P5[[j]] = Re[res];
  If[
   Abs[θ1[1, M[[5]], P5[[j]], α5[[j]]] + α5[[j]]] < 0.0000001, Break[], Guess = 0]]
]
forceangle5 = Transpose[{α5, P5 / Pi^2}];

sarray6 =
  Join[N[Range[-1.5, -1. / Np, (1. / Np)]], N[Range[1. / Np, 1.5, (1. / Np)]]];
P6 = sarray6;
α6 = sarray6;
For[j = 1, j < Pp + 1, j++, Guess = 0;
 For[i = 1, i < 50, i++, Guess = 2 i;
  sol = FindRoot[θ1[(1 / 2), M[[6]], Pj, α6[[j]]], {Pj, Guess}];
  res = Pj /. sol;
  P6[[j]] = Re[res];
  If[
   Abs[θ1[1, M[[6]], P6[[j]], α6[[j]]] + α6[[j]]] < 0.0000001, Break[], Guess = 0]]
]
forceangle6 = Transpose[{α6, P6 / Pi^2}];

sarray7 =
  Join[N[Range[-1.5, -1. / Np, (1. / Np)]], N[Range[1. / Np, 1.5, (1. / Np)]]];
```

```
P7 = sarray7;
α7 = sarray7;
For[j = 1, j < Pp + 1, j++, Guess = 0;
 For[i = 1, i < 50, i++, Guess = 2 i;
  sol = FindRoot[θ1[(1 / 2), M[[7]], Pj, α7[[j]]], {Pj, Guess}];
  res = Pj /. sol;
  P7[[j]] = Re[res];
  If[
   Abs[θ1[1, M[[7]], P7[[j]], α7[[j]]] + α7[[j]]] < 0.0000001, Break[], Guess = 0]]
]
forceangle7 = Transpose[{α7, P7 / Pi^2}];

sarray8 =
  Join[N[Range[-1.5, -1. / Np, (1. / Np)]], N[Range[1. / Np, 1.5, (1. / Np)]]];
P8 = sarray8;
α8 = sarray8;
For[j = 1, j < Pp + 1, j++, Guess = 0;
 For[i = 1, i < 50, i++, Guess = 2 i;
  sol = FindRoot[θ1[(1 / 2), M[[8]], Pj, α8[[j]]], {Pj, Guess}];
  res = Pj /. sol;
  P8[[j]] = Re[res];
  If[
   Abs[θ1[1, M[[8]], P8[[j]], α8[[j]]] + α8[[j]]] < 0.0000001, Break[], Guess = 0]]
]
forceangle8 = Transpose[{α8, P8 / Pi^2}];

sarray9 = N[Range[39.48, 79.87, 0.1]];
P9 = sarray9;
α9 = sarray9;
Pp = Length[sarray1];
For[j = 1, j < Pp + 1, j++, Guess = 0;
 For[i = 1, i < 50, i++, Guess = i / 50;
  sol = FindRoot[constraint[P9[[j]], αj] - 1 / 2, {αj, Guess}];
  res = αj /. sol;
  α9[[j]] = Re[res];
  If[
   Abs[θ1[1, 0., P9[[j]], α9[[j]]] + α9[[j]]] < 0.000000001, Break[], Guess = 0]]
]
forceangle9 = Transpose[{α9, P9 / Pi^2}];

sarray10 = N[Range[88.83, 138.83, 0.1]];
P10 = sarray10;
α10 = sarray10;
Pp = Length[sarray1];
For[j = 1, j < Pp + 1, j++, Guess = 0;
 For[i = 1, i < 50, i++, Guess = i / 50;
  sol = FindRoot[constraint[P10[[j]], αj] - 1 / 3, {αj, Guess}];
  res = αj /. sol;
  α10[[j]] = Re[res];
  If[Abs[θ1[1, 0., P10[[j]], α10[[j]]] + α10[[j]]] < 0.000000001,
   Break[], Guess = 0]]
]
forceangle10 = Transpose[{α10, P10 / Pi^2}];
```

```
sarray11 = sarray10;
P11 = sarray10;
a11 = -α10;
forceangle11 = Transpose[{a11, P11 / Pi^2}];

sarray12 = sarray9;
P12 = sarray9;
a12 = -α9;
forceangle12 = Transpose[{a12, P12 / Pi^2}];

sarray13 =
  Join[N[Range[-1.5, -1. / Np, (1. / Np)]], N[Range[1. / Np, 1.5, (1. / Np)]]];
P13 = sarray13;
α13 = sarray13;
Pp = Length[sarray13];
For[j = 1, j < Pp + 1, j++, Guess = 0;
 For[i = 1, i < 50, i++, Guess = 2 i;
  sol = FindRoot[θ1[(1 / 2), M[[13]], Pj, α13[[j]]], {Pj, Guess}];
  res = Pj /. sol;
  P13[[j]] = Re[res];
  If[Abs[θ1[1, M[[13]], P13[[j]], α13[[j]]] + α13[[j]]] < 0.0000001,
   Break[], Guess = 0]]
]
forceangle13 = Transpose[{α13, P13 / Pi^2}];

sarray14 =
  Join[N[Range[-1.5, -1. / Np, (1. / Np)]], N[Range[1. / Np, 1.5, (1. / Np)]]];
P14 = sarray14;
α14 = sarray14;
For[j = 1, j < Pp + 1, j++, Guess = 0;
 For[i = 1, i < 50, i++, Guess = 2 i;
  sol = FindRoot[θ1[(1 / 2), M[[14]], Pj, α14[[j]]], {Pj, Guess}];
  res = Pj /. sol;
  P14[[j]] = Re[res];
  If[Abs[θ1[1, M[[14]], P14[[j]], α14[[j]]] + α14[[j]]] < 0.0000001,
   Break[], Guess = 0]]]
forceangle14 = Transpose[{α14, P14 / Pi^2}];

sarray15 =
  Join[N[Range[-1.5, -1. / Np, (1. / Np)]], N[Range[1. / Np, 1.5, (1. / Np)]]];
P15 = sarray15;
α15 = sarray15;
For[j = 1, j < Pp + 1, j++, Guess = 0;
 For[i = 1, i < 50, i++, Guess = 2 i;
  sol = FindRoot[θ1[(1 / 2), M[[15]], Pj, α15[[j]]], {Pj, Guess}];
  res = Pj /. sol;
  P15[[j]] = Re[res];
  If[Abs[θ1[1, M[[15]], P15[[j]], α15[[j]]] + α15[[j]]] < 0.0000001,
   Break[], Guess = 0]]]
forceangle15 = Transpose[{α15, P15 / Pi^2}];

sarray16 =
```

```
  Join[N[Range[-1.5, -1. / Np, (1. / Np)]], N[Range[1. / Np, 1.5, (1. / Np)]]];
P16 = sarray16;
α16 = sarray16;
For[j = 1, j < Pp + 1, j++, Guess = 0;
 For[i = 1, i < 50, i++, Guess = 2 i;
  sol = FindRoot[θ1[(1 / 2), M[[16]], Pj, α16[[j]]], {Pj, Guess}];
  res = Pj /. sol;
  P16[[j]] = Re[res];
  If[Abs[θ1[1, M[[16]], P16[[j]], α16[[j]]] + α16[[j]]] < 0.0000001,
   Break[], Guess = 0]]]
forceangle16 = Transpose[{α16, P16 / Pi^2}];

sarray17 =
  Join[N[Range[-1.5, -1. / Np, (1. / Np)]], N[Range[1. / Np, 1.5, (1. / Np)]]];
P17 = sarray17;
α17 = sarray17;
For[j = 1, j < Pp + 1, j++, Guess = 0;
 For[i = 1, i < 50, i++, Guess = 2 i;
  sol = FindRoot[θ1[(1 / 2), M[[17]], Pj, α17[[j]]], {Pj, Guess}];
  res = Pj /. sol;
  P17[[j]] = Re[res];
  If[Abs[θ1[1, M[[17]], P17[[j]], α17[[j]]] + α17[[j]]] < 0.0000001,
   Break[], Guess = 0]]]
forceangle17 = Transpose[{α17, P17 / Pi^2}];

sarray18 =
  Join[N[Range[-1.5, -1. / Np, (1. / Np)]], N[Range[1. / Np, 1.5, (1. / Np)]]];
P18 = sarray18;
α18 = sarray18;
For[j = 1, j < Pp + 1, j++, Guess = 0;
 For[i = 1, i < 50, i++, Guess = 2 i;
  sol = FindRoot[θ1[(1 / 2), M[[18]], Pj, α18[[j]]], {Pj, Guess}];
  res = Pj /. sol;
  P18[[j]] = Re[res];
  If[Abs[θ1[1, M[[18]], P18[[j]], α18[[j]]] + α18[[j]]] < 0.0000001,
   Break[], Guess = 0]]]
forceangle18 = Transpose[{α18, P18 / Pi^2}];

P19 = N[Range[0, 100, 1]];
Np = Length[P19];
α19 = ConstantArray[0, P19];
forceangle19 = Transpose[{α19, P19 / Pi^2}];

(*Now we combine these results into a plot. At
 some points the numerics have not given a correct result,
so we delete those cases by assuming that P/Pe does not exceed 10*)

pl2 = ListLinePlot[{forceangle1 /. y_ /; Abs@y > 10 → None,
   forceangle2 /. y_ /; Abs@y > 10 → None, forceangle3 /. y_ /; Abs@y > 10 → None,
   forceangle4 /. y_ /; Abs@y > 10 → None, forceangle5 /. y_ /; Abs@y > 10 → None,
   forceangle6 /. y_ /; Abs@y > 10 → None, forceangle7 /. y_ /; Abs@y > 10 → None,
   forceangle8 /. y_ /; Abs@y > 10 → None, forceangle9 /. y_ /; Abs@y > 10 → None,
   forceangle10 /. y_ /; Abs@y > 10 → None, forceangle11 /. y_ /; Abs@y > 10 → None,
```

6 | *forceanglemomentexpanded.nb*

```
    forceangle12 /. y_ /; Abs@y > 10 → None, forceangle13 /. y_ /; Abs@y > 10 → None,
    forceangle14 /. y_ /; Abs@y > 10 → None, forceangle15 /. y_ /; Abs@y > 10 → None,
    forceangle16 /. y_ /; Abs@y > 10 → None, forceangle17 /. y_ /; Abs@y > 10 → None,
    forceangle18 /. y_ /; Abs@y > 10 → None, forceangle19},
   PlotRange → {{-1.5, 1.5}, {0, 10}}, ImageSize → Large,
   AxesLabel → {α, P / Pe}, AxesStyle → Medium,
   PlotStyle → {Blue, Green, Red, Blue, Orange, Yellow, Purple, Brown, Blue,
     Blue, Blue, Blue, Cyan, Magenta, Gray, Black, Pink, LightRed, Blue}]
Export[NotebookDirectory[] <> "expandedforceanglemoment.pdf",
 pl2, "PDF", "CompressionLevel" → 0]
```

7.5 Code for plotting inextensible pre-curved beams

```
ClearAll["Global`*"]
β = 0.5;
M0 = -2 β;
γ = 1;
M1 = -2 γ;
(*These values describe the pre-curvature of the beam*)
constraint[P_, α_] := 2/Sqrt[P] EllipticK[Sin[(α + β)/2]^2]
S[M_] := 1 /; M > 0
S[M_] := -1 /; M ≤ 0
(*The following functions define the
  exact solutions for the different pre-curvature*)
θ2[t_, M_, P_, α_] := 2 ArcSin[Sqrt[1/2 ((M + M0)^2/(2 P) - Cos[α + β] + 1.)]
    Re[Sin[JacobiAmplitude[EllipticF[Re[ArcSin[Sin[(α+β)/2]/Sqrt[1/2 ((M+M0)^2/(2 P) - Cos[α + β] + 1.)]]],
        1/2 ((M + M0)^2/(2 P) - Cos[α + β] + 1.)] +
       S[(M + M0)] Sqrt[P] t, 1/2 ((M + M0)^2/(2 P) - Cos[α + β] + 1.)]]]];
θ1[t_, M_, P_, α_] := 2 ArcSin[Sqrt[1/2 (M^2/(2 P) - Cos[α] + 1.)]
    Re[Sin[JacobiAmplitude[EllipticF[Re[ArcSin[Sin[α/2]/Sqrt[1/2 (M^2/(2 P) - Cos[α] + 1.)]]],
        1/2 (M^2/(2 P) - Cos[α] + 1.)] + S[M] Sqrt[P] t, 1/2 (M^2/(2 P) - Cos[α] + 1.)]]]];
θ3[t_, M_, P_, α_] := 2 ArcSin[Sqrt[1/2 ((M + M1)^2/(2 P) - Cos[α + γ] + 1.)]
    Re[Sin[JacobiAmplitude[EllipticF[Re[ArcSin[Sin[(α+γ)/2]/Sqrt[1/2 ((M+M1)^2/(2 P) - Cos[α + γ] + 1.)]]],
        1/2 ((M + M1)^2/(2 P) - Cos[α + γ] + 1.)] +
       S[(M + M1)] Sqrt[P] t, 1/2 ((M + M1)^2/(2 P) - Cos[α + γ] + 1.)]]]];
M = 0.;
P = {10., 10., 10.};
```

```
(*Here you can define M and P. You have to either give M 3
  values and P 1 or vice-versa. With this we get three beams.*)
α = ConstantArray[0, 3];
Guess = 0;
For[i = 1, i < 50, i++, Guess = 1 - i / 50;
 sol = FindRoot[θ1[1 / 2, M, P[[1]], αj], {αj, Guess}];
 res = αj /. sol;
 α[[1]] = Re[res];
 If[Abs[θ1[1, M, P[[1]], α[[1]]] + α[[1]]] < 0.000000001, Break[], Guess = 0]]
For[i = 1, i < 50, i++, Guess = β + i / 50;
 sol = FindRoot[θ2[1 / 2, M, P[[2]], αj], {αj, Guess}];
 res = αj /. sol;
 α[[2]] = Re[res];
 If[Abs[θ2[1, M, P[[2]], α[[2]]] + α[[2]] + β] < 0.000000001, Break[], Guess = 0]]
For[i = 1, i < 50, i++, Guess = γ + i / 50;
 sol = FindRoot[θ3[1 / 2, M, P[[3]], αj], {αj, Guess}];
 res = αj /. sol;
 α[[3]] = Re[res];
 If[Abs[θ3[1, M, P[[3]], α[[3]]] + α[[3]] + γ] < 0.000000001, Break[], Guess = 0]]

(*This gives the value of alpha that satisfies θ(1/2)=
 0 and θ(1)=α+θ0(1) given the values of P and M*)

Np = 100;
sarray  = N[Range[0, 1, 1 / Np]];
u1 = ConstantArray[0, Np + 1];
u2 = ConstantArray[0, Np + 1];
u3 = ConstantArray[0, Np + 1];
w1 = ConstantArray[0, Np + 1];
w2 = ConstantArray[0, Np + 1];
w3 = ConstantArray[0, Np + 1];
(*The variables u are the horizontal displacements of the three beams,
while the variables w are the vertical displacements.*)
index = 1;

x0 = sarray;
y0 = ConstantArray[0, Np + 1];
a0 = sarray;
b0 = ConstantArray[0, Np + 1];
c0 = sarray;
d0 = ConstantArray[0, Np + 1];
a1 = sarray;
b1 = ConstantArray[0, Np + 1];
c1 = sarray;
d1 = ConstantArray[0, Np + 1];
(*x0 and y0 are the initial positions of the beams.*)
(*This is the exact solution for the beam, given P,M and alpha*)
Do[
 u1[[index]] = NIntegrate[Cos[θ1[t, M, P[[1]], α[[1]]]] - 1, {t, 0, s}];
 w1[[index]] = NIntegrate[Sin[θ1[t, M, P[[1]], α[[1]]]], {t, 0, s}];
 index = index + 1;
 , {s, sarray}
]
```

```
x1 = Re[x0 + u1];
y1 = Re[y0 + w1];

index = 1;
Do[
 a0[[index]] = NIntegrate[Cos[M0 t + β] - 1, {t, 0, s}];
 b0[[index]] = NIntegrate[Sin[M0 t + β], {t, 0, s}];
 index = index + 1;
 , {s, sarray}
]
c0 = Re[a0 + x0];
d0 = Re[b0 + y0];
index = 1;
Do[
 u2[[index]] = NIntegrate[Cos[θ2[t, M, P[[2]], α[[2]]]] - Cos[M0 t + β], {t, 0, s}];
 w2[[index]] = NIntegrate[Sin[θ2[t, M, P[[2]], α[[2]]]] - Sin[M0 t + β], {t, 0, s}];
 index = index + 1;
 , {s, sarray}
]
x2 = Re[c0 + u2];
y2 = Re[d0 + w2];

index = 1;
Do[
 a1[[index]] = NIntegrate[Cos[M1 t + γ] - 1, {t, 0, s}];
 b1[[index]] = NIntegrate[Sin[M1 t + γ], {t, 0, s}];
 index = index + 1;
 , {s, sarray}
]
c1 = Re[a1 + x0];
d1 = Re[b1 + y0];
index = 1;
Do[
 u3[[index]] = NIntegrate[Cos[θ3[t, M, P[[3]], α[[3]]]] - Cos[M1 t + γ], {t, 0, s}];
 w3[[index]] = NIntegrate[Sin[θ3[t, M, P[[3]], α[[3]]]] - Sin[M1 t + γ], {t, 0, s}];
 index = index + 1;
 , {s, sarray}
]
(*Here we find u and w at 101 point by
 numerically integrating the appropriate equations*)


x3 = Re[c1 + u3];
y3 = Re[d1 + w3];
(*Here we calculate the actual positions. By numerical
 inaccuracy it may be possible that the displacements are imaginary,
while they need to be real. Therefore we take the real part.*)
beamshape1 = Transpose[{x1, y1}];
beamshape2 = Transpose[{x2, y2}];
beamshape3 = Transpose[{x3, y3}];
plot1 = ListLinePlot[{beamshape1, beamshape2, beamshape3},
  PlotRange → {{-0.2, 1}, {-0.2, 0.4}}, ImageSize → Large,
  PlotStyle → Boxed, AxesLabel → {"x", "y"}, AxesStyle → Medium]
```

7.6 Code for plotting extensible pre-curved beams

```
ClearAll["Global`*"]
S[M_] := 1 /; M > 0
S[M_] := -1 /; M ≤ 0
B = 0.01;
(*Here we set the parameters of our system*)
α = 1;
P = {10., 11., 12.};


Np = 100;
sarray = N[Range[0, 1, 1 / Np]];
u1 = ConstantArray[0, Np + 1];
u2 = ConstantArray[0, Np + 1];
u3 = ConstantArray[0, Np + 1];
w1 = ConstantArray[0, Np + 1];
w2 = ConstantArray[0, Np + 1];
w3 = ConstantArray[0, Np + 1];
(*The variables u are the horizontal displacements of the three beams,
while the variables w are the vertical displacements.*)
index = 1;

x0 = sarray;
y0 = ConstantArray[0, Np + 1];
(*Here we numerically solve the differential
 equation and try to find the right set of parameters*)
Guess = 0;
For[i = 1, i < 10 000, i++; Guess = 1 + i / 10 000;
 s1 = NDSolve[{y''[x] + P[[1]] Sin[y[x]] - P[[1]]^2 B Cos[y[x]] Sin[y[x]] == 0,
    y[0] == Guess, y'[0.000000000000001] == -0.5}, y, {x, -1, 2}];
 θ1[t_] := (y[t] /. s1)[[1]];
 If[Abs[θ1[1 / 2]] < 0.00001, Break[], index = 1]]
Guess
Guess = 0;
For[i = 1, i < 10 000, i++; Guess = 1 + i / 10 000;
 s2 = NDSolve[{y''[x] + P[[2]] Sin[y[x]] - P[[2]]^2 B Cos[y[x]] Sin[y[x]] == 0,
    y[0] == Guess, y'[0.000000000000001] == -0.5}, y, {x, -1, 2}];
 θ2[t_] := (y[t] /. s2)[[1]];
 If[Abs[θ2[1 / 2]] < 0.0001, Break[], index = 1]]
Guess
Guess = 0;
For[i = 1, i < 10 000, i++; Guess = 1 + i / 10 000;
 s3 = NDSolve[{y''[x] + P[[3]] Sin[y[x]] - P[[3]]^2 B Cos[y[x]] Sin[y[x]] == 0,
    y[0] == Guess, y'[0.000000000000001] == -0.5}, y, {x, -1, 2}];
 θ3[t_] := (y[t] /. s3)[[1]];
 If[Abs[θ3[1 / 2]] < 0.00001, Break[], index = 1]]
Λ1[t_] := -P[[1]] B Cos[θ1[t]] + 1
Λ2[t_] := -P[[2]] B Cos[θ2[t]] + 1
Λ3[t_] := -P[[3]] B Cos[θ3[t]] + 1
Guess


Do[
 u1[[index]] = NIntegrate[Λ1[t] Cos[θ1[t]] - 1, {t, 0, s}];
```

```
 w1[[index]] = NIntegrate[Δ1[t] Sin[θ1[t]], {t, 0, s}];
 index = index + 1;
 , {s, sarray}
]
index = 1;
Do[
 u2[[index]] = NIntegrate[Δ2[t] Cos[θ2[t]] - 1, {t, 0, s}];
 w2[[index]] = NIntegrate[Δ2[t] Sin[θ2[t]], {t, 0, s}];
 index = index + 1;
 , {s, sarray}
]
index = 1;
Do[
 u3[[index]] = NIntegrate[Δ3[t] Cos[θ3[t]] - 1, {t, 0, s}];
 w3[[index]] = NIntegrate[Δ3[t] Sin[θ3[t]], {t, 0, s}];
 index = index + 1;
 , {s, sarray}
]
(*Here we find u and w at 101 point by
 numerically integrating the appropriate equations*)
x1 = Re[x0 + u1];
y1 = Re[y0 + w1];
x2 = Re[x0 + u2];
y2 = Re[y0 + w2];
x3 = Re[x0 + u3];
y3 = Re[y0 + w3];
(*Here we calculate the actual positions. By numerical
 inaccuracy it may be possible that the displacements are imaginary,
while they need to be real. Therefore we take the real part.*)
beamshape1 = Transpose[{x1, y1}];
beamshape2 = Transpose[{x2, y2}];
beamshape3 = Transpose[{x3, y3}];
pl1 = ListLinePlot[{beamshape1, beamshape2, beamshape3},
  PlotRange → {{-0.1, 1}, {-0.1, 0.4}}, ImageSize → Large,
  PlotStyle → Boxed, AxesLabel → {x, y}, AxesStyle → Medium]
Export[NotebookDirectory[] <> "Extensible1.pdf",
 pl1, "PDF", "CompressionLevel" → 0]
(*In this part we find the inextensible solutions
 to compare them with the extensible ones*)
S[M_] := 1 /; M > 0
S[M_] := -1 /; M ≤ 0
constraint[P_, M_, α_] :=
```

$$\frac{2}{\sqrt{P}}\,\text{EllipticF}\left[\text{Re}\left[\text{ArcSin}\left[\text{Sin}\left[\frac{\alpha}{2}\right]\sqrt{\frac{4\,P}{4\,P\,\text{Sin}\left[\frac{\alpha}{2}\right]^2+M\text{^}2}}\right]\right],\ \frac{4\,P\,\text{Sin}\left[\frac{\alpha}{2}\right]^2+M\text{^}2}{4\,P}\right]$$

```
(*This function is the Force-Moment-Angle relation*)
M = -0.5;
P = {10., 11., 12.};
(*Here you can define M and P. You have to either give M 3
  values and P 1 or vice-versa. With this we get three beams.*)
α = ConstantArray[0, 3];
sol = FindRoot[constraint[P[[1]], M, α1] - (1. / 1.), {α1, 1}];
```

```
res = α1 /. sol;
α[[1]] = Re[res];
sol = FindRoot[constraint[P[[2]], M, α2] - (1. / 1.), {α2, 1}];
res = α2 /. sol;
α[[2]] = Re[res];
sol = FindRoot[constraint[P[[3]], M, α3] - (1. / 1.), {α3, 1}];
res = α3 /. sol;
α[[3]] = Re[res];
(*This gives the value of alpha that satisfies the Force-
 Moment-Angle relation given the values of P and M*)
k = 1/2 (M^2/(2 P) - Cos[α] + 1.);

ω0 = Sqrt[P];
Np = 100;
sarray = N[Range[0, 1, 1 / Np]];
u1 = ConstantArray[0, Np + 1];
u2 = ConstantArray[0, Np + 1];
u3 = ConstantArray[0, Np + 1];
w1 = ConstantArray[0, Np + 1];
w2 = ConstantArray[0, Np + 1];
w3 = ConstantArray[0, Np + 1];
(*The variables u are the horizontal displacements of the three beams,
while the variables w are the vertical displacements.*)
index = 1;

x0 = sarray;
y0 = ConstantArray[0, Np + 1];
(*x0 and y0 are the initial positions of the beams.*)
θ[t_] := 2 ArcSin[
  Sqrt[k] Re[Sin[JacobiAmplitude[EllipticF[Re[ArcSin[Sin[α/2]/Sqrt[k]]], k] - ω0 t, k]]]];
(*This is the exact solution for the beam, given P,M and alpha*)
Do[
 u1[[index]] = NIntegrate[Cos[θ[t][[1]]] - 1, {t, 0, s}];
 w1[[index]] = NIntegrate[Sin[θ[t][[1]]], {t, 0, s}];
 index = index + 1;
 , {s, sarray}
]
index = 1;
Do[
 u2[[index]] = NIntegrate[Cos[θ[t][[2]]] - 1, {t, 0, s}];
 w2[[index]] = NIntegrate[Sin[θ[t][[2]]], {t, 0, s}];
 index = index + 1;
 , {s, sarray}
]
index = 1;
Do[
 u3[[index]] = NIntegrate[Cos[θ[t][[3]]] - 1, {t, 0, s}];
 w3[[index]] = NIntegrate[Sin[θ[t][[3]]], {t, 0, s}];
 index = index + 1;
```

```
 , {s, sarray}
]
(*Here we find u and w at 101 point by
 numerically integrating the appropriate equations*)
x1 = Re[x0 + u1];
y1 = Re[y0 + w1];
x2 = Re[x0 + u2];
y2 = Re[y0 + w2];
x3 = Re[x0 + u3];
y3 = Re[y0 + w3];
(*Here we calculate the actual positions. By numerical
 inaccuracy it may be possible that the displacements are imaginary,
while they need to be real. Therefore we take the real part.*)
beamshape4 = Transpose[{x1, y1}];
beamshape5 = Transpose[{x2, y2}];
beamshape6 = Transpose[{x3, y3}];
pl1 = ListLinePlot[
  {beamshape1, beamshape2, beamshape3, beamshape4, beamshape5, beamshape6},
  PlotRange → {{-0.1, 1}, {-0.1, 0.4}}, ImageSize → Large,
  PlotStyle → {Red, Orange, Yellow, Blue, Green, Purple},
  AxesLabel → {x, y}, AxesStyle → Medium]
Export[NotebookDirectory[] <> "Extensible2.pdf",
 pl1, "PDF", "CompressionLevel" → 0]
```

Chapter 8

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Chapter 9

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Chapter 10

List of notation

Symbol	Description
ν	Poisson ratio of a material
ℓ	Length of a beam
s	Arc-length parameterization coordinate of the initial configuration of the beam
$u(s)$	Displacement of beam in x-direction with respect to initial configuration as a function of s
$w(s)$	Displacement of beam in y-direction with respect to initial configuration as a function of s
θ^0	Angle the initial configuration of the beam makes with the x-axis
θ	Rotation angle of the beam; also angle beam makes with x-axis when no shear is present
χ	Shear angle
Λ	Length ratio between a beam element of the initial beam and of the deformed beam
I	Moment of inertia about the y-axis
κ_y	Curvature of the beam in the y-direction
A	Cross-section area of the beam
ε	Compressive strain tensor
E	Young's modulus of the material
γ	Shear strain tensor
G	Shear modulus
κ_s	Shear correction factor
P	Force or load applied in x-direction
M	Moment applied on endpoints of the beam
E_{el}	Total elastic energy

α	Difference between the angle at zero of the initial configuration and that of the final configuration
β	Difference between the angle at the right endpoint of the initial configuration and that of the final configuration
$\mathscr{E}$	Total energy in the system
t	Normalized arc-length parameterization of the beam: $t = \frac{s}{\ell}$
$\overline{P}$	Normalized force applied: $\overline{P} = \frac{P}{EI}$
$\overline{M}$	Normalized moment applied: $\overline{M} = \frac{M\ell}{EI}$
B	Constant to make equations for extensible beams dimensionless
P_e	Euler load: minimal load needed to make a beam buckle
κ	Curvature of initial configuration of a pre-curved beam with constant curvature
α_0	Angle at zero of initial configuration of a pre-curved beam with constant curvature
$\tilde{M}$	$\overline{M} + \kappa$
$\tilde{\alpha}$	$\alpha + \alpha_0$