Leiden University

Bachelor Thesis

# Majority Judgment
## A better social choice

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# 1 Introduction

Voting is incorporated into our daily lives in more ways that we usually realise. From electing national leaders and parliaments to selecting a group leader of a collaboration project and from electing the winner of an Olympic sport event to choosing the best date for a group meeting, we use voting as a solution of many problems, some of which are daily, some of which only turn up every few years.

But are the common methods used for voting these days the best and most honest methods? A lot of people probably have heard of the problem with one of the most important elections in the world, the election of the American president. The best-known problem occurred during the 2000 election, where Al Gore got a majority of the votes, but due to the American voting system, George W. Bush jr. won the election. This is just one of the problems the traditional voting systems have.

This poses the question: *What is honest voting, and what is an honest voting system?*

In this thesis we will look at the traditional voting systems and their (dis)advantages. Next, we will look at the voting system created by Michel Balinski and Rida Laraki, called the Majority Judgment. Once again, we will look into the system itself, look at its (dis)advantages and compare it to the traditional systems. We will conclude with a small research that was conducted regarding the Dutch political voting system.

For writing this thesis, the book *"Majority Judgment - Measuring, Ranking and Electing"*[1] has been employed.

A large part of this thesis consists of transforming this 'semi-mathematical' subject into more well-defined and precise mathematical definitions and theorems.

If one does not have a background in mathematics, or just wants to have a brief overview of the various voting systems and their uses, an article about this subject can be found in the scientific magazine 'Eureka!'. The specific edition and further information can be found in the bibliography[4].

# 2 Traditional voting systems

Before we can compare the different voting systems, we will need to introduce some phrasing and definitions.

## 2.1 Set up

First of all, we define the most important aspects of any voting system, the voters, or judges, and the competitors.

**Definition 2.1.** (*Judges*) *Define* $\mathcal{J} := \{j_1, j_2, ..., j_n\}$ *to be a finite set of* $n$ *judges,* $n \in \mathbb{N}_{>0}$.

**Definition 2.2.** (*Competitors*) *Define* $\mathcal{C} := \{c_1, c_2, ..., c_m\}$ *to be a finite set of* $m$ *competitors,* $m \in \mathbb{N}_{>0}$.

Theoretically, these sets could be infinite in size, but since we are looking at real-life elections, it is only logical that we limit the size of these sets.

**Definition 2.3.** (*Ranking*) *Define* $c_1 \succ c_2$ *to mean that competitor* $c_1$ *is preferred to* $c_2$. *Define* $c_1 \succeq c_2$ *to mean that competitor* $c_1$ *might be preferred over competitor* $c_2$, *but not the other way around.*

## 2.2 Plurality voting system

The first voting system we will look at is the first-past-the-post system, or the 'winner take all' system. Simply put, this system relies on each judge to cast one vote in favour of their candidate. When all votes have been cast, the competitor with the most votes wins.

A winner of this system is the competitor $c \in \mathcal{C}$ with

$$\text{Winner FPTP } = \underset{c \in C}{\text{argmax}} \{s_c\}$$

where

$$s_c = \sum_{j=1}^{n} v_{cj}$$

with

$$v_{cj} = \begin{cases} 1 & \text{if judge } j \text{ cast his vote on competitor } c \\ 0 & \text{if judge } j \text{ did not cast his vote on competitor } c \end{cases}, \quad 0 \leq c \leq m, \ 0 \leq j \leq n.$$

This immediately presents a problem the first-past-the-post system has: a competitor can win, even though a majority of the judges would like an other competitor to win.

**Example 2.1.** Suppose there are 11 competitors, and competitor $x$ receives 10% of the votes, where the others receive 9% of the votes each. Then competitor $x$ will be declared the winner, even though 90% of the judges voted for another competitor.

Another voting system that utilises the one-vote-per-judge system is the two-round system, or instant runoff system. The voting works the same as for the first-past-the-post system, but an additional requirement to be declared the winner is that a competitor received at least fifty percent of the votes. If no such competitor exists, the two competitors with the highest number of votes will face off in a second round of voting, where the first-past-the-post system applies.

## 2.3 Multiple vote systems

The options for judging and comparing different systems are very limited when each judge only has one vote to cast. In this section, we will expand on this concept, giving each judge either multiple votes or the ability to rank competitors.
In the following systems, the judges have been given the opportunity to rank the competitors, where a lower rank represents a better place on the ranked list. Each judge can rank all of the competitors, but is

given the choice to only rank a few. If a judge only ranks $k < m$ competitors, the $m - k$ competitors that were not ranked share the place that is associated with the lowest ranking. We want to find a function that ranks these competitors in such a way that the ranking represents the combination of the ranking of all judges.

**Definition 2.4.** (*Voting profile*) *Let $r(c, j) : \mathcal{C} \times \mathcal{J} \to \{1, ..., m\}$ be a function that defines the place on the ranking list where judge c puts competitor j, so that $1 \leq r(c, j) \leq m$. Then a voting profile $\Phi$ is an $m \times n$-matrix where $\Phi_{c,j} = r(c, j)$, representing the ranking each judge voted for. $\boldsymbol{\Phi}$ is the collection of all profiles $\Phi$.*

**Definition 2.5.** (*Social ranking function*) *A social ranking function is a function $f : \mathcal{C} \to \{1, 2, ..., m\}$ that assigns a place on the ranking list to each competitor, which represents the overall ranking of the competitor according the judges.* This function is subject to the votes of the judges, but does not have to be surjective.

**Definition 2.6.** (*Condorcet-winner*) *Let the number of times competitor c is placed higher on the list than competitor d be denoted by $\zeta_{cd} = \sum_{j \in \mathcal{J}} \mathbb{1}_{r(c,j)<r(d,j)}$. Then, a Condorcet-winner is a competitor $c^* \in \mathcal{C}$ that is preferred over every other competitor:*

$$\zeta_{c^*d} > \zeta_{dc^*} \;\; \forall d \in \mathcal{C}.$$

A problem with this system is, of course, that such a winner doesn't necessarily exists. Take a look at the following example:

**Example 2.2.** Let $\mathcal{C} = \{c_1, c_2, c_3\}$ and $\mathcal{J} = \{j_1, j_2, j_3\}$. Then a possible outcome could be:

$$\Phi = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \end{array} \begin{array}{c} \begin{array}{ccc} j_1 & j_2 & j_3 \end{array} \\ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \end{array}.$$

In this case $\zeta_{12} = \zeta_{23} = \zeta_{31} = 2$, so there is no Condorcet-winner. This phenomenon is also known as the *Condorcet-paradox.*

**Definition 2.7.** (*Hare's method*) *Hare's method is a method of ranking where the competitor that is placed on the top of the ranking list the least number of times is eliminated. The competitor is removed from the ranking list, and the competitors that were ranked lower are moved up the list. This process continues until there is one competitor remaining.* Hare's method is also known as the instant runoff method. It is used in a number of countries like Australia (House of Representatives), India (Electoral college) and until 1992, New Zealand.

**Definition 2.8.** (*Borda count*) *Let $p : \{1, ..., m\} \to \mathbb{R}$ be a function that assigns a weight to every place on the ranking list. Let $p_{cj} \in \mathbb{R}$ denote the number of points a competitor c get from his ranking by judge j. Let $P_c = \sum_{j=1}^{n} p_{cj}$ be the total number of points for a competitor c. Then the winner is the competitor $c \in \mathcal{C}$ that has the highest number of points:*

$$\textit{winner } = \underset{0 \leq c \leq m}{\text{argmax}} \{P_c\}.$$

*This election method is called the Borda count.*

An *honest* Borda count is a Borda count where $p_{c\bullet} = (m + 1 - r(c, \bullet))\lambda$, with $\lambda \in \mathbb{R}_{>0}$. In other words, a Borda count where the difference in points between each rank is constant.

## 2.4 Problems and paradoxes

Now that we have established several voting methods, we can compare one to another. To do this, we have to have some requirements that we think a 'good' voting system should satisfy.

1. **Non-dictatorship** There is no judge $j^*$ so that for every possible result of ranking lists, the list of $j^*$ determines the outcome of the social ranking function. In other words, there is no judge $j^*$ with $f(c) = r(c, j^*)$ for all $c \in \mathcal{C}$ for all possible profiles $\Phi$.

2. **Independence of irrelevant alternatives (abbreviation: IIA)** If a competitor $c_i$ is added to or removed from the election, this should not result in a change of $\max\{f(c_a), f(c_b)\}$ for any $c_a, c_b \neq c_i$ and for all $\Phi \in \mathbf{\Phi}$.

3. **Unanimity** If $r(c, j) \leq r(d, j)$ for all $j \in \mathcal{J}$, then $f(c) \leq f(d)$.

4. **Transitivity** Every possible voting profile $\Phi$ is allowed. Thus, the voting mechanism must account for all individual preferences and it must do so in a manner that results in a complete ranking of preferences for society, i.e. if $r(a, j) < r(b, j)$ and $r(b, j) < r(c, j)$ for all $j\ in \mathcal{J}$, then $f(a) < f(c)$.

5. **Always a winner** Every possible profile $\Phi$ produces at least one winner.

6. **Strategy proof** It is a dominant strategy to vote honestly.

#### 2.4.1 Arrow's Impossibility Theorem

Kenneth Arrow (1921-) has proven in his impossibility theorem that it is impossible for a ranking function to satisfy the first four requirements as described in the previous paragraph. This theorem is a generalization of a voting paradox that was actually discovered by Condorcet[3].

**Theorem 2.1.** Impossibility theorem (1950) *When there are at least three competitors, there is no ranking function that satisfies unanimity, transitivity, independence of irrelevant alternatives and non-dictatorship.*

Another way of saying this is that any ranking system ranking a competition with at least three competitors that satisfies unanimity, transitivity and independence of irrelevant alternatives has to be a dictatorship.
Multiple proofs of this theorem have been conceived over the years, the one shown below is the one created by John Geanakoplos in 2005[2]. In this[1] proof Geanakoplos shows that the second, third and fourth requirements imply a dictatorship.

**Proof.** First, let there be $m \in \mathbb{N}_{>2}$ competitors and let $B \in \mathcal{C}$ be a random competitor. Suppose the preferences of the judges are so that $r(B, j) \in \{1, m\}$ for all $j \in \mathcal{J}$. Then the ranking function must satisfy $f(B) = 1$ or $f(B) = m$. If not, there are candidates $A$, $C \in \mathcal{C}$ such that $A \succ B \succ C$. Suppose all judges change their ranked lists such that $r(A, j) < r(C, j)$ for all $j \in \mathcal{J}$, while $r(B, j) = 1$ or $r(B, j) = m$ still holds for all $j \in \mathcal{J}$. Then, because of the IIA, $A \succ B$ and $B \succ C$ must be true. But for the changed profile, unanimity is also satisfied, so $C \succ A$. This is a contradiction, so $f(B)$ equals either 1 or $m$.

Second, we will prove that there is a judge $j^*$ whose vote can change $B$ position in the society's list. Without loss of generality, assume a profile where $r(B, j) = m$ for all $j \in \mathcal{J}$. Then $f(B) = m$ holds. One by one, change each judge's preference to $r(B, j) = 1$. Then there must be a judge $j^* \in \mathcal{J}$ such that the change to $r(B, j^*) = 1$ causes $f(B)$ to change. This has to happen, because when $r(B, j) = 1$ for all $j$, then, according to unanimity, $f(B)$ has to be 1. Call $\Phi$ the judges' profile before and $\Phi'$ the judges' profile after this change of $r(B, j^*)$. In this new profile, $\Phi'$, $r(B, j)$ is either 1 or $m$ for all $j$, so according to the first argument $f'(B)$ must be either 1 or $m$. We know that $f(B)$ was $m$, so that means that $f'(B) = 1$.

Third, we will prove that $j^*$ is a dictator for any pairs of competitors $A, C \in \mathcal{C}$ with $A, C \neq B$: suppose that all judges have ranked competitors $A$ and $C$ arbitrary, but judge $j^*$ moves $A$ above $B$, such that $r(A, j^*) < r(B, j^*) < r(C, j^*)$, and call this profile $\Phi''$. Now the IIA implies that $A \succ B$, because all the judges' preferences regarding $A$ and $B$ are as in profile $\Phi$, but it also implies that $B \succ C$, since all the judges' preferences regarding $B$ and $C$ are as in profile $\Phi'$. Then transitivity implies $A \succ C$. Furthermore, we now see that IIA implies that $A \succ C$ for all other profiles where judge $j^*$ has ranked $A$ and $C$ such that $r(A, j^*) < r(C, j^*)$, but where $B$ is ranked arbitrary. This holds, even when all other judges would

[1]This is the first proof of Geanakoplos' three proofs published in 2005.

rank $A$ and $C$ so that $r(C, j) < r(A, j)$ for all $j \neq j^*$. So we see that $j^*$ is a unique dictator regarding all pairs $A, C$ with $A, C \neq B$.

Fourth, we will prove that $j^*$ is a unique dictator with respect to $A$ and $B$. Let $D \in \mathcal{C}$ with $D \neq B$. Let $D$ take the role of competitor $B$ in the second argument. Then the third argument shows that there must be a judge $j^{**}$ that is a dictator over all pairs of candidates $E, F \neq D$, in particular the pair $A, B$. But we knew that only the preferences of $j^*$ regarding $A$ and $B$ matched the social ranking in both $\Phi$ and $\Phi'$, so that means that $j^* = j^{**}$. $\square$

# 3 Majority Judgment

## 3.1 Set up

**Definition 3.1.** (*Language*) *A language is a set* $\Lambda$ *of grades* $\lambda_1, \lambda_2, ...$ *that are strictly ordered. So if* $\lambda_1 \neq \lambda_2$*, that means that either* $\lambda_1 \prec \lambda_2$ *or* $\lambda_1 \succ \lambda_2$*. Let* $\lambda_{cj}$ *denote the grade that judge* $j$ *assigns to competitor* $c$*. Let* $\lambda^+ = \max_{\lambda}\{\lambda \in \Lambda\}$ *and* $\lambda^- = \min_{\lambda}\{\lambda \in \Lambda\}$*.* We define a a higher grade to be considered better.

Since we are no longer using ranks as the elements of the voting profile, we have to redefine it so that the elements are grades given to the competitors.

**Definition 3.2.** (*Voting profile*) *A profile in voting is an* $m \times n$ *matrix* $\Phi$ *of the grades* $\Phi(c, j) = \lambda_{cj} \in \Lambda$ *that all judges* $j \in J$ *assigned to each of the competitors* $c \in \mathcal{C}$*.*

**Definition 3.3.** (*Method of grading*) *A method of grading is a function* $F : \mathbf{\Phi} \to \Lambda^m$ *that assigns one single grade to each competitor, where* $\mathbf{\Phi}$ *is a collection of voting profiles* $\Phi$*.* This function does not have to be surjective.

**Definition 3.4.** (*Method of ranking*) *Let* $c_1, c_2 \in \mathcal{C}$*. A method of ranking is a relation that compares* $c_1$ *and* $c_2$ *with the following properties:*

- $c_1 \succ c_2$ *means that* $c_1$ *is ranked higher than* $c_2$*.*
- $c_1 \approx c_2$ *means that* $c_1$ *and* $c_2$ *get the same ranking.*
- $c_1 \succeq c_2$*,* $c_2 \succeq c_1 \Rightarrow c_1 \approx c_2$*.*
- $c_1 \succeq c_2$*,* $c_1 \neq c_2 \Rightarrow c_1 \succ c_2$*.*

**Definition 3.5.** (*Aggregation function*) *An aggregation function is a function* $f : \Lambda^m \to \Lambda$ *that satisfies the following properties:*

- *Anonimity:* $f(..., \lambda_{c1}, ..., \lambda_{c2}, ...) = f(..., \lambda_{c2}, ..., \lambda_{c1}, ...)$ *for all* $\lambda \in \Lambda$*.*
- *Unanimity:* $f(\lambda^*, \lambda^*, ..., \lambda^*) = \lambda^*$*.*
- *Monotonicity:* $\lambda_{cj} \preceq \lambda'_{cj}$ *for all* $j \in \mathcal{J} \Rightarrow f(..., \lambda_{cj^*}, ...) \preceq f(..., \lambda'_{cj^*}, ...)$ *and* $\lambda_{cj} \prec \lambda'_{cj}$ *for all* $j \in \mathcal{J} \Rightarrow f(\lambda_{c1}, ..., \lambda_{cn}) \prec f(\lambda'_{c1}, ..., \lambda'_{cn})$*. If* $f$ *only satisfies the first of the monotonicity properties, it will be said to be weakly monotonic.*

**Definition 3.6.** (*Monotonicity*) *A method of grading* $F$ *is monotonic if the following two properties hold:*

1. $\Phi(c^*, j) \succeq \Phi'(c^*, j)$ *for all* $j \in \mathcal{J} \;\Rightarrow\; F(\Phi)_{c^*} \succeq F(\Phi')_{c^*}$
2. $\Phi(c^*, j) \succ \Phi'(c^*, j)$ *for all* $j \in \mathcal{J} \;\Rightarrow\; F(\Phi)_{c^*} \succ F(\Phi')_{c^*}$

If $F$ satisfies only the first (respectively second) property, it is said to be weakly (respectively strictly) monotonic.
Now that we have general methods of grading and ranking, we can define certain properties of these methods.

**Property 3.1.** $F$ is neutral, i.e. for any permutation $\rho$ of the rows of $\Phi$, $F(\rho\Phi) = \rho F(\Phi)$.

**Property 3.2.** $F$ is anonymous, i.e. for any permutation $\tau$ of the columns of $\Phi$, $F(\tau\Phi) = F(\Phi)$.

**Property 3.3.** $F$ is unanimous, i.e. if $\lambda_{cj} = \lambda^*$ for all $j \in \mathcal{J}$, then $F(\Phi_c) = \lambda^*$.

**Property 3.4.** $F$ is monotonic.

**Property 3.5.** $F$ is independent of irrelevant alternatives in grading (IIAG), i.e. $\Phi(c, j) = \Phi'(c, j)$ for all $j \in \mathcal{J} \Rightarrow F(\Phi_c) = F(\Phi_c)$.

**Property 3.6.** The method of ranking is neutral, i.e. for a given $\Phi$ and any permutation $\rho$ of the rows, $c_1 \succeq c_2 \Rightarrow c_1 \succeq c_2$ for $F(\rho\Phi)_{\rho c}$.

**Property 3.7.** The method of ranking is anonymous, i.e. for a given $\Phi$ and any permutation $\tau$ of the columns, $c_1 \succeq c_2 \Rightarrow c_1 \succeq c_2$ for the profile $\tau\Phi$.

**Property 3.8.** The method of ranking is transitive, i.e. $c_1 \succeq c_2, c_2 \succeq c_3 \Rightarrow c_1 \succeq c_3$.

**Property 3.9.** The method of ranking is independent of irrelevant alternatives in ranking (IIAR), i.e. for a given $\Phi$, obtain $\Phi'$ by adding or eliminating one competitor. Then $c_1 \succeq c_2$ for the profile $\Phi$ implies $c_1 \succeq c_2$ for the profile $\Phi'$.

**Theorem 3.1.** *A method of grading $F$ satisfies Properties 3.1 through 3.5 if and only if there exists an aggregation function $f$ such that $f(\Phi_{c\bullet}) = F(\Phi)_c$.*

**Proof.** If there is an aggregation function $f$ that defines $F$ as in the theorem, then the properties are obviously met by $F$. On the other hand, suppose that $F$ satisfies the properties. Then Properties 3.1 and 3.5 imply that $F$ determines the grade of a competitor $c$ on the basis of the grades $\lambda_{cj}$. Call the function that does this $f$. Then Properties 3.2, 3.3 and 3.4 imply the three properties of an aggregation function, so $f$ must be an aggregation function. □

This means that there is an aggregation function that can be applied to each row of $\Phi$ to assign a final grade to each competitor, based on the grades it received from all judges. This is one of the fundamental differences between the Majority Judgment and the traditional voting systems: the grade a competitor receives from the method of ranking does not depend on the grades the other competitors received. Each competitor is assigned a final grade independent of one another.

Assumption: From now on, I will consider a method of grading $F$ to have an aggregation function $f$ so that the properties as described in the theorem above hold.

**Definition 3.7.** (*Social grading function*) *A social grading function (SGF) is a method of ranking that satisfies Properties 3.6 through 3.9.*

## 3.2 Strategy in voting

Strategy has always played an important part in elections and voting. Since it is highly likely that, for a given judge, the other judges do not have the same preferences that he has, it can be very lucrative to try and manipulate the outcome of the election. This can be done in both legal and illegal ways. The illegal methods involve practices such as bribing and gerrymandering, while most legal methods abuse flaws in the voting system. For example, consider the following situation which shows the manipulability of the Borda count:

**Example 3.1.** Let $\mathcal{C} = \{A, B, C\}$ and $\mathcal{J} = \{j_1, j_2, ..., j_{21}\}$. Consider the following situation, where the judges give the following rankings:

$$\Phi = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{cccccccccc} j_1 & j_2 & \cdots & j_8 & j_9 & \cdots & j_{15} & j_{16} & \cdots & j_{21} \end{array} \\ \begin{pmatrix} 1 & 1 & \dots & 1 & 3 & \dots & \dots & \dots & \dots & 3 \\ 2 & 3 & \dots & 3 & 1 & \dots & 1 & 2 & \dots & 2 \\ 3 & 2 & \dots & \dots & \dots & \dots & 2 & 1 & \dots & 1 \end{pmatrix} \end{array}.$$

Let $p : \{1, 2, 3\} \to \mathbb{R}$ be given by $p(x) = 4 - x$. Then, according to the Borda count, $C$ is the winner. But we see that if judges 9 through 15 change their votes to $r(A, i) = 2$ and $r(C, i) = 3$, $9 \le i \le 15$, $B$ is the winner, which is a result they prefer. So the judges can manipulate the outcome in their favor by not voting honestly.

For the Majority Judgment, a lot of the manipulability is taken away by abandoning the ranking system. However, it still isn't completely strategy-proof.

**Theorem 3.2.** *There is no SGF $F$ (with its aggregation function $f$) that can prevent the judges from collectively changing the final grade.*

**Proof.** Suppose the opposite, and let $x, y \in \Lambda$ be two grades with $x \neq y$. Then $f(x, x, ..., x) = x$ holds, because of unanimity. We know that the first judge can't change the grade, so $f(y, x, ..., x) = x$. The same goes for the second judge, so $f(y, y, x, ..., x) = x$. Continuing, we find that $x = f(y, y, ..., y) = y$, so we find a contradiction. □

Now that we know that it is impossible to prevent all judges from changing the final grade, we can look how we can limit the use of strategic voting.

**Definition 3.8.** (*Strategy-proof in grading*) *An SGF is strategy-proof in grading if, for a judge $j \in \mathcal{J}$ and any competitor $c \in \mathcal{C}$, the following holds:*

- $\lambda_{cj} > f(\Phi_{c\bullet}) \;\Rightarrow\; f(\Phi'_{c\bullet}) \leq f(\Phi_{c\bullet})$ *where $\Phi'$ is obtained from $\Phi$ by changing $\lambda_{cj}$ to $\lambda'_{cj} \geq \lambda_{cj}$.*
- $\lambda_{cj} < f(\Phi_{c\bullet}) \;\Rightarrow\; f(\Phi'_{c\bullet}) \geq f(\Phi_{c\bullet})$ *where $\Phi'$ is obtained from $\Phi$ by changing $\lambda_{cj}$ to $\lambda'_{cj} \leq \lambda_{cj}$.*

In other words, a judge can not give a competitor a higher (or lower) grade that results in a higher (or lower) SGF grade if the grade resulting from the SGF was lower (of higher) than the judge's grade.

There is a class of SGFs whose functions are strategy-proof in grading.

**Definition 3.9.** (*Order functions*) *The kth-order function $f^k$ is the kth highest grade given.*
It can be easily seen that $f$ is an aggregation function. We call the class of SGFs with these aggregation functions *Order SGFs*.

**Example 3.2.** Let $\Lambda = \{1, ..., 6\}$ and $\Phi_{c\bullet} = (1, 6, 3, 2, 5)$ for a competitor $c \in \mathcal{C}$. Then, the fourth order function would be $f^4$, or the fourth highest grade, which is 2, so $f^4(\Phi_{c\bullet}) = 2$.

**Theorem 3.3.** *The unique strategy-proof-in-grading SGFs are the order functions.*

**Proof.** Let $f(\Phi_{c\bullet}) = \lambda^*$ for a competitor $c \in \mathcal{C}$, then Properties 3.3 and 3.4 imply that $\min_j \lambda_{cj} \leq \lambda^* \leq \max_j \lambda_{cj}$.
<u>Claim</u>: suppose that $\lambda_{c1} \geq ... \geq \lambda_{cn}$, then there is a $k$ with $1 \leq k \leq n$ such that $f(\Phi_{c\bullet}) = \lambda_{ck}$.
We can always rearrange the grades $\lambda_{cj}$, $1 \leq j \leq n$ such that they are ranked as in the claim. The claim holds, because since $f$ is strategy-proof-in-grading, for a judge $j \in \mathcal{J}$:

$$\lambda_{cj} > \lambda^* \;\Rightarrow\; \forall \lambda'_{cj} \geq \lambda^* : f(\Phi'_{c\bullet}) = \lambda^*, \text{ where } \Phi' \text{ is obtained from } \Phi \text{ by changing } \lambda_{cj} \text{ to } \lambda'_{cj}.$$

The equality above holds, because of the monotonicity and strategy-proof-in-grading of $f$ we have:

- If $\lambda'_{cj} > \lambda_{cj}$, then we have:
  - ⋆ Strategy-proof-in-grading: $f(\Phi'_{c\bullet}) \leq \lambda^*$.
  - ⋆ Monotonicity: $f(\Phi'_{c\bullet}) \geq f(\Phi'_{c\bullet}) = \lambda^*$.

  So we see that $f(\Phi'_{c\bullet}) = \lambda^*$.
- If $\lambda^* < \lambda'_{cj} < \lambda_{cj}$, then we have:
  - ⋆ Monotonicity $f(\Phi'_{c\bullet}) \leq \lambda^*$.
  - ⋆ Strategy-proof-in-grading: $f(\Phi'_{c\bullet}) \leq \lambda^*$. Suppose that $f(\Phi'_{c\bullet}) < \lambda^*$. Then, if we raise the grade from $\lambda'_{cj}$ to $\lambda''_{cj} = \lambda_{cj}$, strategy-proof-in-grading implies $f(\Phi_{c\bullet}) = f(\Phi''_{c\bullet}) \leq f(\Phi'_{c\bullet}) < \lambda^*$, which is a contradiction.

Analogously, we also find that the following holds:

$$\lambda_{cj} < \lambda^* \;\Rightarrow\; \forall \lambda'_{cj} \leq \lambda^* : f(\Phi'_{c\bullet}) = \lambda^*, \text{ where } \Phi' \text{ is obtained from } \Phi \text{ by changing } \lambda_{cj} \text{ to } \lambda'_{cj}.$$

The equalities above hold true, because if the final grade could have been decreased (respectively increased) by changing $\lambda_{cj}$ to $\lambda'_{cj}$, then increasing (respectively decreasing) $\lambda'_{cj}$ would contradict the strategy-proofness of $f$.

So $\lambda^* = \lambda^+ \Rightarrow \lambda_{c1} = \max_j \lambda_{cj} = \lambda^+$, so $k = 1$. Similar, $\lambda^* = \lambda^- \Rightarrow \lambda_{cn} = \min_j \lambda_{cj} = \lambda^-$, so $k = n$.
Now, suppose that $\lambda^- < \lambda^* < \lambda^+$ holds. Now assume that $\lambda^* \neq \lambda_{cj}$ for all $j \in \mathcal{J}$. Then we find a contradiction, since $\lambda_{c1} \geq ...\lambda_{cn}$ implies that there is a $i \in \mathcal{J}$ such that $\lambda_{ci} > \lambda^* > \lambda_{c(i+1)}$. Therefore, for any grades $\lambda^> > \lambda^*$ and $\lambda^< < \lambda^*$, the following holds:

- For $\lambda_{cp} = \lambda^>$, $1 \leq p \leq i$ and $\lambda_{cp} = \lambda^*$, $i+1 \leq p \leq n$: $f(\Phi_{c\bullet}) = \lambda^*$.
- For $\lambda_{cp} = \lambda^*$, $1 \leq p \leq i$ and $\lambda_{cp} = \lambda^<$, $i+1 \leq p \leq n$: $f(\Phi_{c\bullet}) = \lambda^*$.

But the second part of the monotonicity of $F$ implies that the bottom final grade should be lower than the upper; we find a contradiction, so $\exists j \in \mathcal{J}$ with $\lambda^* = \lambda_{cj}$.
Combining these arguments, we find that $f(\Phi'_{c\bullet}) = \lambda_{ck}$ whenever

$$\lambda'_{c1} \geq ... \geq \lambda'_{c(k-1)} \geq \lambda'_{ck} = \lambda_{ck} \geq \lambda'_{c(k+1)} \geq ... \geq \lambda'_{cn}.$$

That is, if there are at least $k-1$ grades at least as high as $\lambda_{ck}$ and at most $n-k$ grades as high as $\lambda_{ck}$, $f(\Phi_{c\bullet})$ doesn't change: it stays the $k$-th largest of the grades.
Finally, I will show that $k$ is independent of the assigned grades $\lambda_{cj}$, $1 \leq j \leq n$. Let $g : \Lambda^n \to \mathbb{N}_{>0}$ be given by $g(\Phi_{c\bullet}) = k$ if $f(\Phi_{c\bullet}) = \lambda_{ck}$ on the open set $\lambda^+ > \lambda_{c1} > ... > \lambda_{cn} > \lambda^-$. Since $f$ is continuous, $g$ is also continuous on this set. Since $g$ assigns integers, it has to be constant on the set. So $f(\Phi_{c\bullet}) = \lambda_{ck}$ for the same constant $k$, hence everywhere by continuity. □

From the proof of Theorem 3.3, we also see that the order functions have two other interesting properties: they are reinforcing and they conform with the assigned grades.

**Definition 3.10.** (*Reinforcing*) *A social grading function $F$ with aggregation function $f$ is reinforcing if $f(\Phi_{c\bullet}) = \lambda^*$ and either $\lambda_{cj} > \lambda'_{cj} \geq \lambda^*$ or $\lambda^* \geq \lambda'_{cj} > \lambda_{cj}$ imply that $f(\Phi'_{c\bullet}) = \lambda^*$, where $\Phi'$ is obtained from $\Phi$ by changing $\lambda_{cj}$ to $\lambda'_{cj}$ for a certain $j \in \mathcal{J}$.*

**Definition 3.11.** (*Conformation*) *A social grading function $F$ with aggregation function $f$ conforms with the assigned grades if $\{\lambda_{c1}, ..., \lambda_{cn}\} := L \subseteq \Lambda \;\Rightarrow\; f(\Phi_{c\bullet}) \in L$.*

There are a lot of fascinating properties of order functions that will not be discussed here. There is one property that will be stated in the theorem below, but a proof will not be provided. The detailed proof of this property can be found in the book "Majority Judgment"[2]. The properties that we will accept as true are the following.

**Theorem 3.4.** *For any $\Phi_{c\bullet}$, the order functions are the unique SGFs where at most one judge can both increase and decrease $f(\Phi_{c\bullet})$.*

Now that we see that the order functions are strategy proof in grading, we can also look at the manipulability of this system. First we need to make manipulability a quantifiable variable.

**Definition 3.12.** (*Manipulability*) *Let $\Phi$ be an arbitrary voting profile and let $f$ be an aggregation function. Then let $\mu^+\big(f(\Phi_{c\bullet})\big)$ be the number of judges that can increase $f(\Phi_{c\bullet})$ on their own and let $\mu^-\big(f(\Phi_{c\bullet})\big)$ be the number of judges that can decrease $f(\Phi_{c\bullet})$ on their own. Now $\mu\big(f(\Phi_{c\bullet})\big) = \mu^+\big(f(\Phi_{c\bullet})\big) + \mu^-\big(f(\Phi_{c\bullet})\big)$ is the total number of judges that can change $f(\Phi_{c\bullet})$ on their own. Then the manipulability of $f$ is $\mu(f) = \max_{\Phi_{c\bullet}} \mu\big(f(\Phi_{c\bullet})\big)$.*

So a higher manipulability can be considered a bad thing. In the worst case, every judge can both increase and decrease the final grade, so $\mu(f) \leq 2n$. When $f$ is an $k$-th order function, $\mu(f) = n+1$.

We see that order functions can not eliminate manipulation, but we can prove that they do minimize it.

[2]The proof can be found in Chapter 10, paragraph 3, pages 194-196

**Theorem 3.5.** *The only SGFs that minimize manipulability are the order functions.*

**Proof.** Let $f$ be any aggregation function with $\mu(f) \le n+1$ and let $\Phi_{c\bullet}$ be any voting profile. Then define $I^+$ as

$$I^+(\Phi_{c\bullet}) := \{ j \in \mathcal{J} \mid \exists \lambda^* \in \Lambda \text{ with } \lambda^* \neq \lambda_{cj} : f(\Phi'_{c\bullet}) > f(\Phi_{c\bullet}) \},$$

where $\Phi'$ is obtained from $\Phi$ by changing $\lambda_{cj}$ to $\lambda^*$.
Likewise, $I^-$ can be defined as

$$I^-(\Phi_{c\bullet}) := \{ j \in \mathcal{J} \mid \exists \lambda^* \in \Lambda \text{ with } \lambda^* \neq \lambda_{cj} : f(\Phi'_{c\bullet}) < f(\Phi_{c\bullet}) \},$$

where $\Phi'$ is obtained from $\Phi$ by changing $\lambda_{cj}$ to $\lambda^*$.
Then

$$\left| I^+(\Phi_{c\bullet}) \cap I^-(\Phi_{c\bullet}) \right| \ge 2 \;\Rightarrow\; \mu(f) \ge n+2$$

holds true, since each judge in the intersection can both increase and decrease the final grade and every other judge can either increase or decrease the final score. But $\mu(f) \le n+1$, so that means that

$$\left| I^+(\Phi_{c\bullet}) \cap I^-(\Phi_{c\bullet}) \right| \le 1.$$

Now Theorem 3.4 implies that $f$ is an order function. □

## 3.3 Arrow's theorem revisited

Since the main paradox from the previous section was Arrow's Impossibility Theorem, it is only logical that we look at social grading functions and see how they interact with this theorem.
Firstly, if we want an SGF to be meaningful, comparisons between aggregations should be preserved when we transform each of the judges' grades. This is especially important if the judges use different languages.

**Definition 3.13.** (*Preference-consistency*) *Let $a, b \in \Lambda'$ be grades from a language $\Lambda'$ with $a < b$ and $\Lambda \neq \Lambda'$. Let for any $1 \le j \le n$ $\phi_j : [\lambda^-, \lambda^+] \to [a_j, b_j]$ be any arbitrary increasing, continuous function with $\phi_j(\lambda^-) = a_j$ and $\phi(\lambda^+) = b_j$. Then a social grading function with its aggregation function $f$ is preference-consistent if*

$$f(\Phi_{c\bullet}) \ge f(\Phi'_{c\bullet}) \;\Rightarrow\; f(\Phi^*_{c\bullet}) \ge f(\Phi'^*_{c\bullet}),$$

*where $\Phi^*$ is obtained from $\Phi$ by changing all grades $\lambda_{cj}$ to $\phi_j(\lambda_{cj})$.*

As we can see, there are no social grading functions that are preference-consistent. Because if there would be, the corresponding aggregation functions would be the order functions, which are not preference-consistent. This can best be shown using an example.

**Example 3.3.** Let $\Lambda = \{0, ..., 10\}$, $\Lambda_1 = \{0, ..., 20\}$, $\Lambda_2 = \{0, ..., 50\}$, $\Lambda_3 = \{0, ..., 70\}$ and $\Lambda_4 = \{0, ..., 100\}$ be different language, where judge $j$ uses language $\Lambda_j$ to grade. Let $A, B \in \mathcal{C}$. Then a possible profile is

$$\Phi = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cccc} j_1 & j_2 & j_3 & j_4 \end{array} \\ \begin{pmatrix} 14 & 45 & 56 & 30 \\ 10 & 30 & 28 & 80 \end{pmatrix} \end{array}.$$

Then $f^1(\Phi_{A\bullet}) = 56 < 80 = f^1(\Phi_{B\bullet})$ and $f^4(\Phi_{A\bullet}) = 14 > 10 = f^4(\Phi_{B\bullet})$ hold. But using the functions $\phi_1(\lambda) = \dfrac{\lambda}{2}$, $\phi_2(\lambda) = \dfrac{\lambda}{5}$, $\phi_3(\lambda) = \dfrac{\lambda}{7}$ and $\phi_4(\lambda) = \dfrac{\lambda}{10}$, we get the following profile, with all grades in the language $\Lambda$:

$$\Phi^* = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cccc} j_1 & j_2 & j_3 & j_4 \end{array} \\ \begin{pmatrix} 7 & 9 & 8 & 3 \\ 5 & 6 & 4 & 8 \end{pmatrix} \end{array}.$$

Then $f^1(\Phi^*_{A\bullet}) = 9 > 8 = f^1(\Phi^*_{B\bullet})$ and $f^4(\Phi^*_{A\bullet}) = 3 < 4 = f^4(\Phi^*_{B\bullet})$ hold. It shows clearly that the order functions are not preference-consistent.

So where does this leave us with Arrow's Theorem? We can modify the social grading functions into weak social grading functions (WSGFs) by dropping the necessity for anonymity among the judges.

**Theorem 3.6.** *The unique weak social grading functions that are preference-consistent are dictatorial, in the sense that for any profile $\Phi$, there is a judge $k \in \mathcal{J}$ so that $f(\Phi_{c\bullet}) = \lambda_{ck}$ for all $c$.*

**Proof.** Let $F$ be a weak social grading function with its aggregation function $f$ and let $\Lambda^* := \Lambda \backslash \{\lambda^-, \lambda^+\}$. Consider the grades $\lambda_{c1}, ..., \lambda_{cn}$ with $\lambda_{ci} \neq \lambda_{cj}$ if $i \neq j$, $\lambda_{ci} \in \Lambda^*$ for all $i \in \mathcal{J}$. Since $F$ is a WSGF, we can rank the grades:

$$\lambda^+ > \lambda_{c1} > ... > \lambda_{cn}.$$

Then $f(\Phi_{c\bullet}) = \lambda_{ck}$ for a certain $k$.
Let $i \neq k$, and suppose $\lambda_{ci} > \lambda_{ck}$. Let $\lambda^*_{ci} > \lambda_{ck}$. Since $f$ is continuous, we now claim that the following must hold:

$$f(\Phi_{c\bullet}) = f(\Phi^{\epsilon}_{c\bullet}) = \lambda_{ck},$$

where $\Phi^{\epsilon}_{c\bullet}$ is obtained from $\Phi_{c\bullet}$ by changing $\lambda_{ci}$ to $\lambda_{ci} + \epsilon$, with $\epsilon$ sufficiently small.
Now let $\phi$ be a strictly increasing function with

$$\phi(\lambda) = \begin{cases} \lambda^- & \text{if } \lambda = \lambda^-; \\ \lambda^*_{ci} & \text{if } \lambda = \lambda_{ci}; \\ \lambda_{ck} + \epsilon & \text{if } \lambda = \lambda_{ci} + \epsilon; \\ \lambda^+ & \text{if } \lambda = \lambda^+. \end{cases}$$

Now define the following profiles:

- $\Phi_1$, which is obtained from $\Phi_{c\bullet}$ by changing $\lambda_{ci}$ to $\phi(\lambda_{ci})$.
- $\Phi_2$, which is obtained from $\Phi_{c\bullet}$ by changing $\lambda_{ci}$ to $\phi(\lambda_{ci} + \epsilon)$.
- $\Phi_3$, which is obtained from $\Phi_{c\bullet}$ by changing $\lambda_{ci}$ to $\phi(\lambda^*_{ci})$.
- $\Phi_4$, which is obtained from $\Phi_{c\bullet}$ by changing $\lambda_{ci}$ to $\phi(\lambda_{ck} + \epsilon)$.

Then we see that preference consistency implies $f(\Phi_1) = f(\Phi_2)$, so that for all $\lambda^*_{ci} \leq \lambda_{ck}$ the following holds:

$$f(\Phi_3) = f(\Phi_4) = \lambda_{ck}.$$

So we see that the grade given by $f$ remains the same, regardless the grade given by $i$. However, if $k$ changes his grade, then so does $f$ change the final grade. Repeating this for all other $j \neq k$, we find that this holds for all $\lambda_{cj}$, completing this proof. □

One should not that unlike judge $j^*$ in the first theorem of Arrow, the 'dictatorial' judge $k$ is not necessarily the same judge for all possible profiles $\Phi$. The above theorem merely states that there is always a judge whose preferences are matched by the final grades. So this result is more desireable.

## 3.4 Majority judgment

The Majority Judgment gets its name from the fact that the voting results are facts that the majority of the judges agree on.

**Definition 3.14.** (*Middlemost*) *A social grading function is middlemost, if for $\lambda_{c1} \geq ... \geq \lambda_{cn}$ the following hold:*

$$f(\Phi_{c\bullet}) = \lambda_{c\frac{n+1}{2}}$$

*if $n$ is odd, and*

$$\lambda_{c\frac{n}{2}} \geq f(\Phi_{c\bullet}) \geq \lambda_{c\frac{n+2}{2}}$$

*if $n$ is even.*

Since we can not give two grades to a competitor, we will have to choose one grade if $n$ is even. The Majority Judgment utilises the lower middlemost function, given that $\lambda_{ci} \geq \lambda_{cj}$ if $i \leq j$.

**Definition 3.15.** (*Majority grade*) *The Majority grade $f^{maj}$ is the lower middlemost order function:*

$$f^{maj} = \begin{cases} f^{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ f^{\frac{n+2}{2}} & \text{if } n \text{ is even.} \end{cases}$$

Here we see the origin of the name: the majority grade is the grade that the majority of the judges agree to be the minimum grade a competitor deserves.
Now we can define how competitors are compared with eachother, given their majority grades. Let $A, B \in \mathcal{C}$.

1. If $f^{maj}(\Phi_{A\bullet}) > f^{maj}(\Phi_{B\bullet})$, then $A \succ B$.
2. If $f^{maj}(\Phi_{A\bullet}) = f^{maj}(\Phi_{B\bullet})$, then remove $\lambda_{\bullet i}$, where $i = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n+2}{2} & \text{if } n \text{ is even.} \end{cases}$

   Redetermine $f^{maj}$ and go to step 1.

This way, we see that the only way that $A \approx B$ is if $\Phi_{Aj} = \Phi_{Bj}$ for all $j$, which is what we would expect. If two competitors didn't receive exactly the same grades, one has to be preferred over the other.

**Example 3.4.** Let $\mathcal{C} = \{A, B\}, \mathcal{J} = \{j_1, ..., j_7\}, \Lambda = \{0, ..., 100\}$. Then a profile could be

$$\Phi = \begin{matrix} \\ A \\ B \end{matrix} \begin{matrix} j_1 & j_2 & j_3 & j_4 & j_5 & j_6 & j_7 \\ \end{matrix} \begin{pmatrix} 85 & 73 & 78 & 90 & 69 & 70 & 73 \\ 73 & 68 & 95 & 81 & 66 & 73 & 73 \end{pmatrix}.$$

We can arrange the votes so that they are strictly ordered:

$$\Phi = \begin{matrix} A \\ B \end{matrix} \begin{pmatrix} 90 & 85 & 78 & \mathbf{73} & 73 & 70 & 69 \\ 95 & 81 & 73 & \mathbf{73} & 73 & 68 & 66 \end{pmatrix}.$$

Now $f^{maj}$ is shown in bold. Since $f^{maj}(\Phi_{A\bullet}) = f^{maj}(\Phi_{B\bullet})$, we remove these grades and redetermine $f^{maj}$ until we find different majority grades:

$$\Phi^{(2)} = \begin{matrix} A \\ B \end{matrix} \begin{pmatrix} 90 & 85 & 78 & \mathbf{73} & 70 & 69 \\ 95 & 81 & 73 & \mathbf{73} & 68 & 66 \end{pmatrix}.$$

$$\Phi^{(3)} = \begin{matrix} A \\ B \end{matrix} \begin{pmatrix} 90 & 85 & \mathbf{78} & 70 & 69 \\ 95 & 81 & \mathbf{73} & 68 & 66 \end{pmatrix}.$$

The third majority grade differs, so we see that $A \succ B$.

This method of eliminating the majority grade suffices for when $|\mathcal{J}|$ is small. When there are a lot of judges, eg. in national elections, the profile $\Phi$ would be too big to sort and solve by hand. Therefore, we can use another method to process the votes in a simple way.

**Definition 3.16.** (*Majority gauge*) *The Majority gauge for a competitor $c$ is a 3-tuple $(p, \alpha^*, q)_c$, where*

- $p = \dfrac{\left|\{\lambda_{cj} \mid \lambda_{cj} > f^{maj}(\Phi_{c\bullet})\}\right|}{|\mathcal{J}|}$,
- $q = \dfrac{\left|\{\lambda_{cj} \mid \lambda_{cj} < f^{maj}(\Phi_{c\bullet})\}\right|}{|\mathcal{J}|}$,
- $\alpha^* = \begin{cases} \alpha^+ & \text{if } p > q \\ \alpha & \text{if } p = q \\ \alpha^- & \text{if } p \leq q \end{cases}$, *where $\alpha$ is the majority grade $f^{maj}(\Phi_{c\bullet})$.*

If $f^{maj}(\Phi_{A\bullet}) = \alpha$ and $f^{maj}(\Phi_{B\bullet}) = \beta$, we see that $A \succ B$ if

1. $\alpha > \beta$, or
2. $\alpha = \beta$ and either $\alpha^* = \alpha^+$ or $\beta^* = \alpha^-$.

A ranking created using the majority gauge is the same as a ranking created using the majority grade. The majority gauge is a simple way to compare competitors. For example, the plus or minus above the $\alpha$ doesn't hold any mathematical significance, but is simply a sign to compare two competitors that have the same majority grade.

**Example 3.5.** The majority gauges for the competitors $A, B$ from Example 3.3 would be

$$(0.43, 73^+, 0.29)_A \quad \text{and} \quad (0.29, 73, 0.29)_B.$$

However, if any one of the judges $j_1, j_6$ or $j_7$ changed their grade for $B$ to $73 < \lambda'_{Bj} \leq 77$, competitor $A$ would still win according to the majority grade, but the majority gauges of $A$ and $B$ would be equal.

So there are a rare few instances in which the majority gauge of two competitors is equal, but their majority grades would (eventually) differ. However, experiments show that this happens only about 0.1% of the time[3] and especially with large electorates, the chance of this happening is minimal.

The majority value is a good tool for comparing competitors, but it can be hard to calculate. The initial majority value can be found quite easily, but since it is highly possible that the first few are equal, we introduce a tool to quickly find all subsequent majority values.

**Definition 3.17.** (*Abbreviated majority value*) *For a competitor c, the abbreviated majority value is a vector*

$$\aleph_c = \left[\overset{\nu(\lambda_i\lambda_j)}{(\lambda_i\lambda_j)} \ldots \overset{\nu(\lambda_k\lambda_l)}{(\lambda_k\lambda_l)}\right]$$

*of ordered pairs of grades* $(\lambda_i\lambda_j)$ *where* $\lambda_i \leq \lambda_j$, *adjoined with the frequency with which it appears* $\nu(\lambda_i, \lambda_j)$ *in percentages.*

For a given competitor $c$, this vector can be obtained using the following algorithm:

1. (a) Let $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ be ordered so that $\lambda_i > \lambda_j$ if $i > j$. Calculate the vector

   $$V_c = \big(\overset{\lambda_1}{\nu(\lambda_1)}, \ldots, \overset{\lambda_r}{\nu(\lambda_r)}\big),$$

   where $\nu(\lambda_z) = \dfrac{\sum_{j \in \mathcal{J}} \mathbb{1}|_{\lambda_{cj} = \lambda_z}}{|\mathcal{J}|}$.

   (b) $i = 1$, $\aleph = [\varnothing]$ and $\Xi = \big(\nu(\lambda_1), \ldots, \nu(\lambda_t)^+ \mid\mid \nu(\lambda_t)^-, \ldots, \nu(\lambda_r)\big)$, where $\nu(\lambda_t)^+ + \nu(\lambda_t)^- = \nu(\lambda_t)$ and $\nu(\lambda_1) + \ldots + \nu(\lambda_t)^+ = 0.5 = \nu(\lambda_t)^- + \ldots + \nu(\lambda_r)$.

2. (a) Find grades $\lambda_k$ and $\lambda_l$, where

   $$\lambda_k = \min_z\{\lambda_z \mid \lambda_z \geq \lambda_t,\ \nu(\lambda_z) \neq 0\}, \text{ and}$$
   $$\lambda_l = \max_z\{\lambda_z \mid \lambda_z \leq \lambda_t,\ \nu(\lambda_z) \neq 0\}.$$

   (b) Let $\delta_i = \min\{\nu(\lambda_k), \nu(\lambda_l)\} \cdot 100$.

3. (a) $\Xi := \big(\nu(\lambda_1), \ldots, \nu(\lambda_k) - \delta_i, \ldots, \nu(\lambda_t)^+ \mid\mid \nu(\lambda_t)^-, \ldots, \nu(\lambda_l) - \delta_i, \ldots, \nu(\lambda_r)\big)$.

   (b) Add $\overset{\delta_i}{(\lambda_k\lambda_l)}$ to the right of $\aleph_c$, i.e. $\aleph_c := \left[\aleph_c \ \overset{\delta}{(\lambda_k\lambda_l)}\right]$; $i := i + 1$.

   (c) If $\Xi = (0, \ldots, 0)$: STOP. Else: go to step 2.

[3]M. Balinski and R. Laraki, *Majority Judgment - Measuring, Ranking and Electing*, page 237.

Using the abbreviated majority-value, we can easily find the subsequent majority grades, as well as the majority grade. The first grade in $\aleph_c$ is the majority-grade, and the majority gauge $(p, a^*, q)_c$ is

- $a^* = \begin{cases} a^+ & \text{if the first grade in } \aleph_c \text{ that isn't } a \text{ is above } a \\ a^- & \text{if the first grade in } \aleph_c \text{ that isn't } a \text{ is below } a \end{cases}.$

- $p = \begin{cases} 0.5 - \frac{\delta_1}{100} & \text{if } a^* = a^+ \\ 0.5 - \sum_i \left( \frac{\delta_i}{100} \middle| \lambda_c, \lambda_d \neq a \text{ for } (\overset{\delta_i}{\lambda_c \lambda_d}) \right) & \text{if } a^* = a^- \end{cases}$

- $q = \begin{cases} 0.5 - \sum_i \left( \frac{\delta_i}{100} \middle| \lambda_c, \lambda_d \neq a \text{ for } (\overset{\delta_i}{\lambda_c \lambda_d}) \right) & \text{if } a^* = a^+ \\ 0.5 - \frac{\delta_1}{100} & \text{if } a^* = a^- \end{cases}$

**Example 3.6.** Let $c \in \mathcal{C} = \{c_1, ..., c_m\}$, $\mathcal{J} = \{j_1, ..., j_{1000}\}$ and $\Lambda = \{A, B, C, D, E, F\}$. Then there is a profile $\Phi \in \mathbf{\Phi}$ such that

$$V_c = \big(\overset{A}{0.136}, \overset{B}{0.307}, \overset{C}{0.251}, \overset{D}{0.148}, \overset{E}{0.084}, \overset{F}{0.074}\big).$$

We start the algorithm with

$$\Xi_c = (0.136,\ 0.307,\ 0.057^+ \ || \ 0.194^-,\ 0.148,\ 0.087, 0.074) \ \text{ and } \ \aleph_c = [\varnothing].$$

After one iteration, we have

$$\Xi_c = (0.136,\ 0.307,\ 0^+ \ || \ 0.137^-,\ 0.148,\ 0.087, 0.074) \ \text{ and } \ \aleph_c = \big[(\overset{5.7}{CC})\big].$$

Continuing the algorithm, eventually we find

$$\Xi_c = (0,\ 0,\ 0^+ \ || \ 0^-,\ 0,\ 0,\ 0) \ \text{ and } \ \aleph_c = \big[(\overset{5.7}{CC})(\overset{13.7}{CB})(\overset{14.8}{DB})(\overset{2.2}{EB})(\overset{6.2}{EA})(\overset{7.4}{FA})\big].$$

Here we see that the first grade in $\aleph_c$ is the majority-grade, $C$. So that means that the first $5.7\% \cdot 1000 = 57$ majority grades will be $C$. After that, the majority grade will be alternating between $C$ and $B$ for $2 \cdot 137 = 274$ times.
As for the majority gauge $(p, a^*, q)_c$, we find that it is $(0.443, C^+, 0.205)_c$.

## 3.5 Objections

Although the Majority Judgment is a better voting method in the area of preventing strategic voting, it is not without its flaws. There are objections regarding the method, some of which are mathematical and some of which are psychological in nature.

**Objection 1.** The 'average' objection.
The Majority Judgment utilises the median of a series of grades, as opposed to the average. However, people are more intuitively inclined to use the average to calculate an outcome. This can lead to some interesting results, where the average and the median do not present the same grade.

**Example 3.7.** Let $\mathcal{C} = \{A, B\}$, $\mathcal{J} = \{j_1, ..., j_{21}\}$ and $\Lambda = \{0, ..., 20\}$. Then a possible voting profile is

$$\Phi = \begin{matrix} \\ A \\ B \end{matrix} \begin{matrix} j_1 & \dots & \dots & j_{10} & j_{11} & j_{12} & \dots & \dots & j_{21} \\ \end{matrix} \begin{pmatrix} 12 & \dots & \dots & 12 & 12 & 4 & \dots & \dots & 4 \\ 16 & \dots & \dots & 16 & 8 & 8 & \dots & \dots & 8 \end{pmatrix}.$$

In this scenario, we vind that the average grade for $A$ is 8.2, while the average for $B$ is 12.2. Using a variation of the Condorcet-method, we find that $B$ is preferred over $A$ 20 times, while $A$ is only preferred one time over $B$. In both cases, $B$ would be declared the winner. However, $f^{maj}(\Phi_{A\bullet}) = 12$ and $f^{maj}(\Phi_{B\bullet}) = 8$, so the Majority Judgment would favour $A$ over $B$.

While such a scenario is possible, it is highly unlikely that it will come to pass in practice. Both in the research conducted by Balinski and Laraki as well as in my own research we see that actual voting results do not resemble this highly artificial example.

**Objection 2.** The no-show paradox
The no-show paradox, also known as the 'participation criterion', is a paradox that consists of a change in the winner, resulting from changing the judges: suppose $\mathcal{C} = \{A, B\}$, and $f^{maj}(\Phi_{A\bullet}) > f^{maj}(\Phi_{B\bullet})$, so $A$ is the winner. Now $\mathcal{J}$ is augmented by adding an additional judge $j$ who grades the competitors with grades $\lambda_{Aj} > \lambda_{Bj}$, which results in $f^{maj}(\Phi_{A\bullet}) < f^{maj}(\Phi_{B\bullet})$. So while the judge ranks $A$ higher than $B$, the outcome will change at the expense of $A$.

**Example 3.8.** Let $\mathcal{C} = \{A, B\}$, $\mathcal{J} = \{j_1, ..., j_7\}$ and $\Lambda = \{0, ..., 20\}$. Then a possible voting profile is

$$\Phi = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{ccccccc} j_1 & j_2 & j_3 & j_4 & j_5 & j_6 & j_7 \end{array} \\ \begin{pmatrix} 20 & 17 & 15 & 15 & 12 & 11 & 7 \\ 18 & 17 & 16 & 14 & 13 & 10 & 5 \end{pmatrix} \end{array}.$$

We see that $f^{maj}(\Phi_{A\bullet}) = 15 > 14 = f^{maj}(\Phi_{B\bullet})$, so $A$ is the winner by the Majority Judgment. Now suppose an additional judge, $j_8$, gives grades $\lambda_{Aj_8} = 6$ and $\lambda_{Bj_8} = 4$. Then the majority grades change to $f^{maj}(\Phi_{A\bullet}) = 12 < 13 = f^{maj}(\Phi_{B\bullet})$, so now $B$ is the winner.

While this situation is more realistic, it can be partially explained using the apparent preferences of the additional judge. If he grades both competitors low like in the example, he apparently has a low opinion of both competitors and does not care who wins. If he grades them both high, he apparently has a high opinion of both competitors and once again is indifferent about the winner. If he grades one competitor a lot higher than the other he does appear to have a preference, but as long as he grades $A$ higher than $f^{maj}(\Phi_{A\bullet})$ and $B$ lower than $f^{maj}(\Phi_{B\bullet})$, the majority grades do not change. More importantly, when $|\mathcal{J}|$ is large, this paradox becomes a lot less likely to happen.

**Objection 3.** Failed reversal symmetry
Reversal symmetry is a criterion for a voting system requiring that if all judge's preferences are reversed, the initial winner of the election must not be elected. In the case of the Majority judgment, this means that each grade is changed to its grade opposite the middlemost grade.
The Borda-count is a voting system that fulfills this criterion: let $g : \{1, ..., m\} \to \{1, ..., m\}$ be a continuous function with $g(s) = |m + 1 - s|$. Now let $p$ be the same weight-assigning function, but let the domain be $\{g(1), ..., g(m)\}$. Then the points assigned with a rank are inverted.
It's easy to see that the Majority Judgment fails this criterion. The sole reason for this is that the majority-grade is calculated using the lower middlemost order functions.

**Example 3.9.** Let $\mathcal{C} = \{A, B\}$, $\mathcal{J} = \{j_1, j_2\}$, $\Lambda = \{1, 2, 3\}$. Then a possible voting profile is

$$\Phi = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} j_1 & j_2 \end{array} \\ \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} \end{array}.$$

We see that $f^{maj}(\Phi_{A\bullet}) = 1 < 2 = f^{maj}(\Phi_{B\bullet})$, so $B$ is the winner. But when we reverse the grades, the voting profile becomes

$$\Phi = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} j_1 & j_2 \end{array} \\ \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \end{array}.$$

The majority-grades don't change, so $B$ stays the winner.

Because of paradoxes and since some properties cannot co-exist in the same system, every voting system has its flaws. As shown above, it can be very easy to construct extreme examples that accentuate these flaws. But something that also should be considered is the question if these examples are realistically likely to happen. If not, and if the flaw is not a major down spike in the quality of the voting system, some of these flaws can be overlooked.

# 4 Research

Since Balinski and Laraki claimed that the Majority Judgment was better than other voting methods, I tried to find this out for myself. I created a small survey which I distributed among the university, and which was mostly taken part in by students. The survey was started by 116 participants, of which 103 completed the survey. The following results only take these 103 participants into account.
In the survey, the participants were asked to take part in a fictional election of the Dutch House of Representatives (Tweede Kamer). The participants had to vote thrice, in three difference ways, representing three different ways of voting. First the participants had to vote for their favourite party, secondly they had to create a ranked list of their top three parties, and finally the participants were asked to grade each party that took part in the election. These methods represented the First-past-the-post election, an election using ranked lists -where I used the Borda count to come up with a winner-, and the Majority Judgment.
This survey took place over the period May 2014-September 2014, with two peaks of voting in May and September. The participants were presented with a pre-made list of the major parties, but were given the option to list the party of their choice if it wasn't in the list already. The only parties that weren't in the pre-made list but did get votes were the SP and the Piratenpartij.

## 4.1 Results

The result of the elections are summed up blow in several tables and graphs. The numbers are mostly accurate, but the numbers in the pie charts can differ slightly due to rounding errors.

### 4.1.1 First-past-the-post

In the First-past-the-post election, the participants had one vote, which they could cast in favour of their favourite party.

| Party | Percentage |
|---|---|
| D66 | 43,69% |
| PvdA | 15,53% |
| GroenLinks | 10,68% |
| VVD | 10,68% |
| CDA | 7,77% |
| Christenunie | 1,94% |
| SP | 1,94% |
| Piratenpartij | 1.94% |
| PVV | 0,97% |
| PvdD | 0,00% |
| SGP | 0,00% |
| 50Plus | 0,00% |
| Blanc | 4,85% |

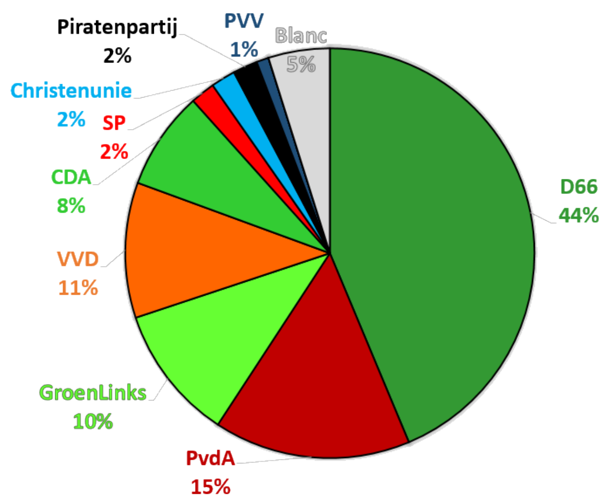


Figure 1: The results from the First-past-the-post voting

The result is quite interesting, and it is easy to see that the majority of the survey participants were students. The fact that the D66 nearly has an absolute majority of the votes is a testament to this. As can be seen in national elections, D66 is a party that always receives a high number of votes from students. The results also show this. Also interesting is that three parties, PvdD, SGP and 50Plus didn't receive any votes at all.

### 4.1.2 Borda count

The Borda count is the type of ranked election that I chose to use. The participants were given the option to list their top three parties, but they were also told that this wasn't necessary. If the participant only

wanted to list one or two parties, they were able to do so. This option resulted in the fact that 103 people listed their number one party, but only 98 and 83 participants listed their second and third party of choice, respectively.
To calculate the number of points each party received, I used the function $p : \{1, 2, 3\} \to \{1, 2, 3\}$, with $p(x) = |4 - x|$. This made this Borda count an *honest* Borda count. The results can be found in the following graphs:

| Party | 1st place | 2nd place | 3rd place | Points |
|---|---|---|---|---|
| CDA | 8 | 7 | 11 | 49 |
| Christenunie | 1 | 5 | 2 | 15 |
| D66 | 46 | 31 | 10 | 210 |
| GroenLinks | 10 | 14 | 17 | 75 |
| Piratenpartij | 2 | 1 | 0 | 8 |
| PvdA | 17 | 16 | 21 | 104 |
| PvdD | 0 | 2 | 4 | 8 |
| PVV | 1 | 0 | 1 | 4 |
| SGP | 0 | 0 | 1 | 1 |
| SP | 4 | 2 | 3 | 19 |
| VVD | 12 | 20 | 12 | 88 |
| 50Plus | 0 | 0 | 1 | 1 |
| Blanc | 2 | 0 | 0 | 6 |
| Number of Votes | 103 | 98 | 83 | 588 |

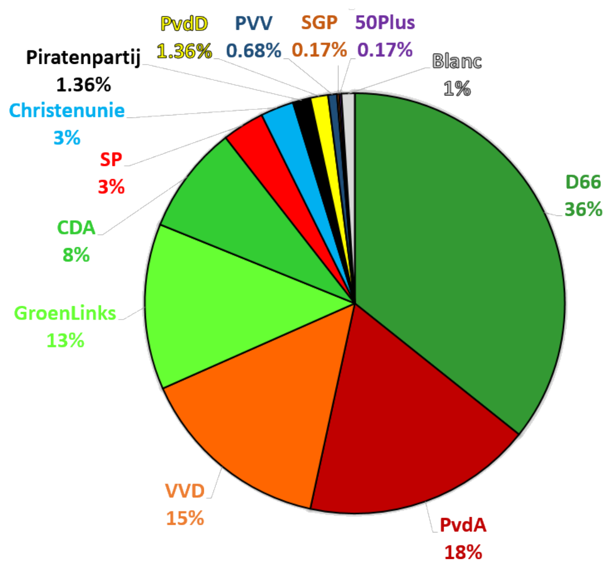


Figure 2: The results from the Borda count

We can immediately see that the fact that the participants can give more information results in a different result. For starters, the biggest party loses about 8 percent in regards to the First-past-the-post system, and every party got at least one point. It can be seen that over 80 percent of the participants listed three parties, and possibly would expand their list if it had been possible.

#### 4.1.3 Majority Judgment

For the Majority Judgment, the participants were given the pre-made list of parties and were asked to grade these with one of the grades from the following set:

| Dutch | Excellent | Zeer goed | Goed | Acceptabel | Onvoldoende | Slecht | Verwerpelijk |
|---|---|---|---|---|---|---|---|
| English | Excellent | Very good | Good | Acceptable | Inadequate | Bad | Condemnable |

The question asked to the participants was: “Om onderdeel te worden van het aanstaande kabinet, acht ik de volgende partij ...” (“To be a part of the next cabinet, I judge the following party to be ... ”). Participants were asked to grade all parties, but were informed that this wasn’t a necessity. They were told that if they chose not to grade a party, that that would receive the grade ‘Verwerpelijk’ (‘Condemnable’). The results show that the maximum number of times a listed party didn’t receive a grade was 10 times, with an overall average of 5.6 times. Of course, the parties that were not in the pre-made list received a low number of grades overall, which is the reason that their majority-grade is very low. They received an average of 6.5 grades, resulting in a majority of ‘Verwerpelijk’ grades.
After the voting was complete, I changed the grades into numerical values, with a higher value representing a better grade. After this, I calculated for each party the majority-grade, its majority gauge and its abbreviated majority value. Te results can be found in the table below.

| Party | Majority-grade | Majority gauge | Abbreviated Majority value | Rank |
|---|---|---|---|---|
| CDA | 4 | $(0.45, 4^{+}, 0.19)$ | $\overset{5.3}{(44)}\overset{25.2}{(45)}\overset{8.8}{(35)}\overset{2.9}{(36)}\overset{2.9}{(26)}\overset{4.9}{(16)}$ | 5 |
| Christenunie | 4 | $(0.18, 4^{-}, 0.43)$ | $\overset{7.2}{(44)}\overset{24.3}{(34)}\overset{0.1}{(24)}\overset{6.7}{(25)}\overset{8.8}{(15)}\overset{2.9}{(16)}$ | 6 |
| D66 | 6 | $(0.04, 6^{-}, 0.46)$ | $\overset{4.4}{(66)}\overset{28.2}{(56)}\overset{10.7}{(46)}\overset{2.8}{(36)}\overset{0.1}{(37)}\overset{1.9}{(27)}\overset{1.9}{(17)}$ | 1 |
| GroenLinks | 4 | $(0.48, 4^{+}, 0.27)$ | $\overset{2.5}{(44)}\overset{20.3}{(45)}\overset{13.6}{(35)}\overset{2}{(25)}\overset{2.9}{(26)}\overset{6.8}{(16)}\overset{1.9}{(17)}$ | 4 |
| Piratenpartij | 1 | $(0.03, 1^{+}, 0)$ | $\overset{47.1}{(11)}\overset{1.9}{(15)}\overset{1}{(17)}$ | 12 |
| PvdA | 5 | $(0.20, 5^{-}, 0.35)$ | $\overset{15.1}{(55)}\overset{14.5}{(45)}\overset{5.9}{(46)}\overset{5.8}{(36)}\overset{2.9}{(26)}\overset{4.8}{(16)}\overset{1}{(17)}$ | 2 |
| PvdD | 3 | $(0.22, 3^{-}, 0.47)$ | $\overset{3.4}{(33)}\overset{24.3}{(23)}\overset{13.6}{(14)}\overset{8.7}{(15)}$ | 7 |
| PVV | 1 | $(0.31, 1^{+}, 0)$ | $\overset{18.9}{(11)}\overset{17.3}{(12)}\overset{8.7}{(13)}\overset{3.9}{(14)}\overset{1}{(15)}$ | 10 |
| SGP | 2 | $(0.45, 2^{+}, 0.35)$ | $\overset{5.4}{(22)}\overset{9.6}{(23)}\overset{15.6}{(13)}\overset{18.4}{(14)}\overset{1}{(15)}$ | 9 |
| SP | 1 | $(0.09, 1^{+}, 0)$ | $\overset{41.2}{(11)}\overset{4.9}{(14)}\overset{2.9}{(16)}\overset{1}{(17)}$ | 11 |
| VVD | 5 | $(0.24, 5^{-}, 0.49)$ | $\overset{1.4}{(55)}\overset{24.4}{(45)}\overset{2.8}{(46)}\overset{9.7}{(36)}\overset{4.9}{(26)}\overset{4.9}{(16)}\overset{1.9}{(17)}$ | 3 |
| 50Plus | 2 | $(0.47, 2^{+}, 0.25)$ | $\overset{3.4}{(22)}\overset{21.4}{(23)}\overset{2.9}{(13)}\overset{19.4}{(14)}\overset{2.9}{(15)}$ | 8 |

Figure 3: The results from the Majority Judgment

As can be expected, the parties that were not in the pre-made list have the lowest possible majority-grade, 1. Only the PVV, a party that was in the pre-made list, received also a majority grade of 1, whilst it was given 94 grades by the participants.
The eventual ranking can be derived from the abbreviated majority values, but it is also listed in the table above. The results of the Majority Judgment election can be found in the Appendix.

## 4.2 Improvements

Of course, this survey was not very representable of a real election, and it had quite a lot of flaws:

1. The SP, one of the larger parties in national elections, was not listed in the pre-made list. If it had been, the results from all three elections would have been different, and maybe could have represented the actual national elections more.
2. There was a large time period over which the survey took place. This could also have changed the public opinions about the parties a bit, resulting in a different result than when it would have taken place on a single day.

Other than these flaws, there is a problem with the Majority Judgment overall. After ranking the parties according to their majority grades, I thought of finding a way to distribute the seats of the House of Representatives according to the result. However, I was not able to find a distribution that would mimic the results. Seeing as how both dr. Spieksma and myself were not able to find a way to distribute the seats, I contacted Rida Laraki to ask for help. However, he informed me that the Majority Judgment was not a voting method that could be used to assign seats to parties in a House of Representatives. He was working on that problem along with a PhD-student of his, but had no conclusive solution as of this day. This might imply that the Majority Judgment is a good voting method to select a single winner from a group of competitors, but is not a system that could be used in the Dutch national elections.

# 5 References

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[3] Andrew Hindmoor, *Rational Choice*, Palgrave Macmillan, Basingstoke, United Kingdom, 1st edition, 2006

[4] Bob Sleeuwenhoek, *Majority Judgment: a better way to vote*, UFB, Leiden University, Leiden, Eureka! magazine, Volume 49, Year 12 - June 2015

# Appendix A

## Sorted results Majority Judgment

| | CDA | CU | D66 | GL | PvdA | PvdD | PVV | SGP | VVD | 50+ | SP | PP |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 6 | 6 | 7 | 7 | 7 | 5 | 5 | 5 | 7 | 5 | 7 | 7 |
| 2 | 6 | 6 | 7 | 7 | 6 | 5 | 4 | 4 | 7 | 5 | 6 | 5 |
| 3 | 6 | 6 | 7 | 6 | 6 | 5 | 4 | 4 | 6 | 5 | 6 | 5 |
| 4 | 6 | 5 | 7 | 6 | 6 | 5 | 4 | 4 | 6 | 4 | 6 | 1 |
| 5 | 6 | 5 | 6 | 6 | 6 | 5 | 4 | 4 | 6 | 4 | 4 | 1 |
| 6 | 6 | 5 | 6 | 6 | 6 | 5 | 3 | 4 | 6 | 4 | 4 | 1 |
| 7 | 6 | 5 | 6 | 6 | 6 | 5 | 3 | 4 | 6 | 4 | 4 | 1 |
| 8 | 6 | 5 | 6 | 6 | 6 | 5 | 3 | 4 | 6 | 4 | 4 | 1 |
| 9 | 6 | 5 | 6 | 6 | 6 | 5 | 3 | 4 | 6 | 4 | 4 | 1 |
| 10 | 6 | 5 | 6 | 6 | 6 | 4 | 3 | 4 | 6 | 4 | 1 | 1 |
| 11 | 6 | 5 | 6 | 6 | 6 | 4 | 3 | 4 | 6 | 4 | 1 | 1 |
| 12 | 5 | 5 | 6 | 6 | 6 | 4 | 3 | 4 | 6 | 4 | 1 | 1 |
| 13 | 5 | 5 | 6 | 5 | 6 | 4 | 3 | 4 | 6 | 4 | 1 | 1 |
| 14 | 5 | 5 | 6 | 5 | 6 | 4 | 3 | 4 | 6 | 4 | 1 | 1 |
| 15 | 5 | 5 | 6 | 5 | 6 | 4 | 2 | 4 | 6 | 4 | 1 | 1 |
| 16 | 5 | 5 | 6 | 5 | 6 | 4 | 2 | 4 | 6 | 4 | 1 | 1 |
| 17 | 5 | 5 | 6 | 5 | 6 | 4 | 2 | 4 | 6 | 4 | 1 | 1 |
| 18 | 5 | 5 | 6 | 5 | 6 | 4 | 2 | 4 | 6 | 4 | 1 | 1 |
| 19 | 5 | 5 | 6 | 5 | 6 | 4 | 2 | 4 | 6 | 4 | 1 | 1 |
| 20 | 5 | 4 | 6 | 5 | 6 | 4 | 2 | 4 | 6 | 4 | 1 | 1 |
| 21 | 5 | 4 | 6 | 5 | 6 | 4 | 2 | 3 | 6 | 4 | 1 | 1 |
| 22 | 5 | 4 | 6 | 5 | 5 | 4 | 2 | 3 | 6 | 4 | 1 | 1 |
| 23 | 5 | 4 | 6 | 5 | 5 | 4 | 2 | 3 | 6 | 4 | 1 | 1 |
| 24 | 5 | 4 | 6 | 5 | 5 | 3 | 2 | 3 | 6 | 3 | 1 | 1 |
| 25 | 5 | 4 | 6 | 5 | 5 | 3 | 2 | 3 | 6 | 3 | 1 | 1 |
| 26 | 5 | 4 | 6 | 5 | 5 | 3 | 2 | 3 | 5 | 3 | 1 | 1 |
| 27 | 5 | 4 | 6 | 5 | 5 | 3 | 2 | 3 | 5 | 3 | 1 | 1 |
| 28 | 5 | 4 | 6 | 5 | 5 | 3 | 2 | 3 | 5 | 3 | 1 | 1 |
| 29 | 5 | 4 | 6 | 5 | 5 | 3 | 2 | 3 | 5 | 3 | 1 | 1 |
| 30 | 5 | 4 | 6 | 5 | 5 | 3 | 2 | 3 | 5 | 3 | 1 | 1 |
| 31 | 5 | 4 | 6 | 5 | 5 | 3 | 2 | 3 | 5 | 3 | 1 | 1 |
| 32 | 5 | 4 | 6 | 5 | 5 | 3 | 2 | 3 | 5 | 3 | 1 | 1 |
| 33 | 5 | 4 | 6 | 5 | 5 | 3 | 1 | 3 | 5 | 3 | 1 | 1 |
| 34 | 5 | 4 | 6 | 5 | 5 | 3 | 1 | 3 | 5 | 3 | 1 | 1 |
| 35 | 5 | 4 | 6 | 5 | 5 | 3 | 1 | 3 | 5 | 3 | 1 | 1 |
| 36 | 5 | 4 | 6 | 5 | 5 | 3 | 1 | 3 | 5 | 3 | 1 | 1 |
| 37 | 5 | 4 | 6 | 5 | 5 | 3 | 1 | 3 | 5 | 3 | 1 | 1 |
| 38 | 5 | 4 | 6 | 5 | 5 | 3 | 1 | 3 | 5 | 3 | 1 | 1 |
| 39 | 5 | 4 | 6 | 5 | 5 | 3 | 1 | 3 | 5 | 3 | 1 | 1 |
| 40 | 5 | 4 | 6 | 5 | 5 | 3 | 1 | 3 | 5 | 3 | 1 | 1 |
| 41 | 5 | 4 | 6 | 5 | 5 | 3 | 1 | 3 | 5 | 3 | 1 | 1 |
| 42 | 5 | 4 | 6 | 5 | 5 | 3 | 1 | 3 | 5 | 3 | 1 | 1 |
| 43 | 5 | 4 | 6 | 5 | 5 | 3 | 1 | 3 | 5 | 3 | 1 | 1 |
| 44 | 5 | 4 | 6 | 5 | 5 | 3 | 1 | 3 | 5 | 3 | 1 | 1 |
| 45 | 5 | 4 | 6 | 5 | 5 | 3 | 1 | 3 | 5 | 3 | 1 | 1 |
| 46 | 5 | 4 | 6 | 5 | 5 | 3 | 1 | 3 | 5 | 3 | 1 | 1 |
| 47 | 4 | 4 | 6 | 5 | 5 | 3 | 1 | 2 | 5 | 3 | 1 | 1 |
| 48 | 4 | 4 | 6 | 5 | 5 | 3 | 1 | 2 | 5 | 3 | 1 | 1 |
| 49 | 4 | 4 | 6 | 5 | 5 | 3 | 1 | 2 | 5 | 2 | 1 | 1 |
| 50 | 4 | 4 | 6 | 4 | 5 | 3 | 1 | 2 | 5 | 2 | 1 | 1 |

| | CDA | CU | D66 | GL | PvdA | PvdD | PVV | SGP | VVD | 50+ | SP | PP |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 51 | 4 | 4 | 6 | 4 | 5 | 3 | 1 | 2 | 5 | 2 | 1 | 1 |
| 52 | 4 | 4 | 6 | 4 | 5 | 3 | 1 | 2 | 5 | 2 | 1 | 1 |
| 53 | 4 | 4 | 6 | 4 | 5 | 3 | 1 | 2 | 5 | 2 | 1 | 1 |
| 54 | 4 | 4 | 6 | 4 | 5 | 3 | 1 | 2 | 4 | 2 | 1 | 1 |
| 55 | 4 | 4 | 6 | 4 | 5 | 3 | 1 | 2 | 4 | 2 | 1 | 1 |
| 56 | 4 | 4 | 6 | 4 | 5 | 2 | 1 | 2 | 4 | 2 | 1 | 1 |
| 57 | 4 | 4 | 5 | 4 | 5 | 2 | 1 | 2 | 4 | 2 | 1 | 1 |
| 58 | 4 | 4 | 5 | 4 | 5 | 2 | 1 | 2 | 4 | 2 | 1 | 1 |
| 59 | 4 | 4 | 5 | 4 | 5 | 2 | 1 | 2 | 4 | 2 | 1 | 1 |
| 60 | 4 | 3 | 5 | 4 | 5 | 2 | 1 | 2 | 4 | 2 | 1 | 1 |
| 61 | 4 | 3 | 5 | 4 | 5 | 2 | 1 | 2 | 4 | 2 | 1 | 1 |
| 62 | 4 | 3 | 5 | 4 | 5 | 2 | 1 | 2 | 4 | 2 | 1 | 1 |
| 63 | 4 | 3 | 5 | 4 | 5 | 2 | 1 | 2 | 4 | 2 | 1 | 1 |
| 64 | 4 | 3 | 5 | 4 | 5 | 2 | 1 | 2 | 4 | 2 | 1 | 1 |
| 65 | 4 | 3 | 5 | 4 | 5 | 2 | 1 | 2 | 4 | 2 | 1 | 1 |
| 66 | 4 | 3 | 5 | 4 | 5 | 2 | 1 | 2 | 4 | 2 | 1 | 1 |
| 67 | 4 | 3 | 5 | 4 | 5 | 2 | 1 | 2 | 4 | 2 | 1 | 1 |
| 68 | 4 | 3 | 5 | 4 | 4 | 2 | 1 | 1 | 4 | 2 | 1 | 1 |
| 69 | 4 | 3 | 5 | 4 | 4 | 2 | 1 | 1 | 4 | 2 | 1 | 1 |
| 70 | 4 | 3 | 5 | 4 | 4 | 2 | 1 | 1 | 4 | 2 | 1 | 1 |
| 71 | 4 | 3 | 5 | 4 | 4 | 2 | 1 | 1 | 4 | 2 | 1 | 1 |
| 72 | 4 | 3 | 5 | 4 | 4 | 2 | 1 | 1 | 4 | 2 | 1 | 1 |
| 73 | 4 | 3 | 5 | 4 | 4 | 2 | 1 | 1 | 4 | 2 | 1 | 1 |
| 74 | 4 | 3 | 5 | 4 | 4 | 2 | 1 | 1 | 4 | 2 | 1 | 1 |
| 75 | 4 | 3 | 5 | 4 | 4 | 2 | 1 | 1 | 4 | 2 | 1 | 1 |
| 76 | 4 | 3 | 5 | 3 | 4 | 2 | 1 | 1 | 4 | 2 | 1 | 1 |
| 77 | 4 | 3 | 5 | 3 | 4 | 2 | 1 | 1 | 4 | 2 | 1 | 1 |
| 78 | 4 | 3 | 5 | 3 | 4 | 2 | 1 | 1 | 4 | 1 | 1 | 1 |
| 79 | 4 | 3 | 5 | 3 | 4 | 2 | 1 | 1 | 4 | 1 | 1 | 1 |
| 80 | 4 | 3 | 5 | 3 | 4 | 2 | 1 | 1 | 4 | 1 | 1 | 1 |
| 81 | 4 | 3 | 5 | 3 | 4 | 1 | 1 | 1 | 4 | 1 | 1 | 1 |
| 82 | 4 | 3 | 5 | 3 | 4 | 1 | 1 | 1 | 3 | 1 | 1 | 1 |
| 83 | 4 | 3 | 5 | 3 | 4 | 1 | 1 | 1 | 3 | 1 | 1 | 1 |
| 84 | 3 | 3 | 5 | 3 | 4 | 1 | 1 | 1 | 3 | 1 | 1 | 1 |
| 85 | 3 | 2 | 5 | 3 | 4 | 1 | 1 | 1 | 3 | 1 | 1 | 1 |
| 86 | 3 | 2 | 4 | 3 | 4 | 1 | 1 | 1 | 3 | 1 | 1 | 1 |
| 87 | 3 | 2 | 4 | 3 | 4 | 1 | 1 | 1 | 3 | 1 | 1 | 1 |
| 88 | 3 | 2 | 4 | 3 | 4 | 1 | 1 | 1 | 3 | 1 | 1 | 1 |
| 89 | 3 | 2 | 4 | 3 | 3 | 1 | 1 | 1 | 3 | 1 | 1 | 1 |
| 90 | 3 | 2 | 4 | 2 | 3 | 1 | 1 | 1 | 3 | 1 | 1 | 1 |
| 91 | 3 | 2 | 4 | 2 | 3 | 1 | 1 | 1 | 3 | 1 | 1 | 1 |
| 92 | 3 | 1 | 4 | 2 | 3 | 1 | 1 | 1 | 2 | 1 | 1 | 1 |
| 93 | 3 | 1 | 4 | 2 | 3 | 1 | 1 | 1 | 2 | 1 | 1 | 1 |
| 94 | 3 | 1 | 4 | 2 | 3 | 1 | 1 | 1 | 2 | 1 | 1 | 1 |
| 95 | 3 | 1 | 4 | 1 | 2 | 1 | 1 | 1 | 2 | 1 | 1 | 1 |
| 96 | 2 | 1 | 4 | 1 | 2 | 1 | 1 | 1 | 2 | 1 | 1 | 1 |
| 97 | 2 | 1 | 3 | 1 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 98 | 2 | 1 | 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 99 | 1 | 1 | 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 100 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 101 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 102 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 103 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |