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The Lefschetz fixed point theorem

Bachelorscriptie

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Introduction

Given a continuous map $f : X \to X$ from a topological space to itself it is natural to ask whether the map has fixed points, and how many. For a certain class of spaces, which includes compact manifolds, the Lefschetz fixed point theorem answers these questions. This class of spaces are the simplicial complexes, which are roughly speaking topological spaces built up from triangles and their higher dimensional analogues. We start in the first section by constructing singular homology for any space and simplicial homology for simplicial complexes. Both are functors from topological spaces to graded groups, so of the form $H^*(X) = \bigoplus_i H_i(X)$. Thus we can assign algebraic invariants to a map f. In the case of simplicial homology computing these amounts purely to linear algebra. By employing simplicial approximation of continuous maps by simplicial maps we prove the following.

Theorem 1 (Lefschetz fixed point theorem)**.** *Let $f : X \to X$ be a continuous map of a simplicial complex to itself. Define the Lefschetz number*

$$\Lambda(f) = \sum_i (-1)^i tr(f_*|_{H_i(X)}).$$

If $\Lambda(f) \neq 0$ then f has a fixed point.

In fact more is true, if the number of fixed points is finite, then $\Lambda(f)$ counts the number of fixed points with certain multiplicities. One would prove this using cohomology, which is a contravariant functor instead of the covariant homology functor. This has an added ring structure and some nice theorems hold, like a Künneth formula and Poincaré duality.

In the second section we look at smooth projective varieties. Even though we can not use the same constructions from the first section, because the Zariski topology is too coarse, we can look at would happen if there was a well-behaved cohomology theory. We look at Weil cohomology, which is a contravariant functor from smooth, projective varieties to graded commutative K-algebras, for some field K of characteristic 0, satisfying a list of axioms. Part of the axioms are inspired by the situation in algebraic topology, we have a Künneth formula and Poincaré duality holds, in fact for varieties over $\mathbb{C}$ singular cohomology is a Weil cohomology. The other part is so that we get a ring homomorphism from the Chow ring to the cohomology ring. From these axioms we are able to deduce an analogous theorem to the one before.

Theorem 2 (Lefschetz fixed point theorem)**.** *Let $f : X \to X$ be an endomorphism of an irreducible, smooth, projective variety. Let $\Delta, \Gamma_f \subset X \times X$ be the diagonal and the graph of f respectively. Then we have*

$$(\Delta \cdot \Gamma_f) = \sum_i (-1)^i tr(f^*|_{H^i(X)}).$$

Here $(\Delta \cdot \Gamma_f)$ is the intersection number of the graph and the diagonal, we recognize the right hand side as the Lefschetz number. So again if f has finitely many fixed points then the Lefschetz number counts this with certain multiplicities.

In the final section we assume a Weil cohomology theory exists for varieties over a field with non-zero characteristic. Assuming this we can prove some of the Weil conjectures. Let X be a geometrically connected, smooth, projective variety defined over $\mathbb{F}_q$, then we are interested in counting the K-valued points, where $\mathbb{F}_q \subset K$ is a finite extension of fields. We let N_m be the number of $\mathbb{F}_{q^m}$-valued points. Then the zeta function of X is defined as follows.

$$Z(X,T) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} T^m\right).$$

The N_m turn out to be exactly the number of fixed points of the q^m-Frobenius morphism acting on $\overline{X} = X \times_{\mathrm{Spec}\ \mathbb{F}_q} \mathrm{Spec}\ \overline{\mathbb{F}_q}$. By the Lefschetz fixed point theorem we can thus express them as an alternating sum of traces. It follows that $Z(X,T) \in \mathbb{Q}(T)$ is a rational function. Further we show using Poincaré duality that it satisfies the functional equation

$$Z(X, \frac{1}{q^n T}) = \pm q^{nE/2} T^E Z(X,T)$$

where $E = (\Delta \cdot \Delta)$ is the self intersection number of the diagonal in $\overline{X} \times \overline{X}$ and $n = \dim(X)$.

1 The classical Lefschetz fixed point theorem

1.1 Singular homology

Definition 1. *The standard n-simplex, denoted Δ_n, is the convex hull of the vectors $e_0, ..., e_n$ in $\mathbb{R}^n$,* [1] *that is the set*

$$\{(\lambda_1, ..., \lambda_n) \in \mathbb{R}^n \textit{ such that } \sum_{i=1}^{n} \lambda_i \leq 1 \textit{ and } \lambda_i \geq 0 \textit{ for } i = 1, ..., n\}.$$

In general if we have $k+1$ affinely independent points $v_0, ..., v_k$ in $\mathbb{R}^n$ their convex hull is a k-simplex and is denoted $[v_0, ..., v_k]$.

If we omit one of the vectors e_i from the set $\{e_0, ..., e_n\}$, then the convex hull of $\{e_0, ..., e_n\} \setminus \{e_i\}$ is an $(n-1)$-simplex contained in Δ_n, these are called the faces of Δ_n. We can identify a face with Δ_{n-1} via the linear map f_i that sends e_j to e_{j+1} if $j \geq i$, and e_j to e_j if $j < i$. Then $f_i(\Delta_{n-1})$ is precisely the i-th face of Δ_n. The union of the faces of Δ_n is called the boundary denoted $\partial\Delta_n$, and $\Delta_n \setminus \partial\Delta_n$ is called the interior of Δ_n and is denoted $\mathring{\Delta}_n$.

For a topological space X we can consider all continuous maps $\sigma : \Delta_n \to X$ called n-chains. By considering formal sums of n-chains with coefficients in an abelian group G we get the chain groups $C_n(X; G)$, or just $C_n(X)$ if the coefficients are clear from the context. By extending the face maps linearly we get maps $F_i^n : C_n(X) \to C_{n-1}(X)$, given by $\sum_j a_j\sigma_j \mapsto \sum_j a_j(\sigma_j \circ f_i^n)$. Next we define the boundary operator $\partial_n : C_n(X) \to C_{n-1}(X)$ as $\partial_n := \sum_{i=1}^{n}(-1)^i F_i^n$. One then checks that $\partial_n \circ \partial_{n+1} = 0$, so we have inclusions

$$B_n(X) := \text{im }(\partial_{n+1}) \subset \ker(\partial_n) =: Z_n(X),$$

hence we can consider the quotients $H_n(X) := Z_n(X)/B_n(X)$, these are called the homology groups of X (with coefficients in G).

Given a continuous map $f : X \to Y$ we get induced group homomorphisms $f_* : H_n(X) \to H_n(Y)$ given by $\sum_i a_i[\sigma_i] \mapsto \sum_i a_i[f \circ \sigma_i]$.

Proposition 1. *Let $f, g : X \to Y$ and $h : Y \to Z$ be continuous maps then:*

i. $(h \circ f)_* = h_* \circ f_*$,

ii. if f is the identity then f_* is the identity,

iii. if f and g are homotopic maps then $f_* = g_*$.

Proof. The first two are straightforward. For a proof of iii see [1] theorem 2.10. $\square$

[1] e_0 here is the origin and the e_i are the standard basis vectors

From (ii) it follows that if f is a homeomorphism then f_* is an isomorphism, combining this with (iii) we also see that homotopy equivalences induce isomorphisms. In other words the assignment $X \to H_n(X)$ is a homotopy-invariant functor.

1.2 Simplicial homology

We now consider a class of spaces that can be built from n-simplices, the so called simplicial complexes on which we will define simplicial homology.

Definition 2. *A finite simplicial complex is a finite collection K of simplices in $\mathbb{R}^n$ such that:*

i. If $\sigma \in K$ then every face of σ is in K.

ii. The intersection of two simplices is either empty or a face of each.

Note that in (ii) the definition of face is taken to be more general, i.e. if $\sigma = [v_0, ..., v_n]$ is an n-simplex then any of the simplices $[v_{i_1}, ..., v_{i_k}]$ with $v_{i_j} \in \{v_0, ..., v_n\}$ is a face of σ.

Given a simplicial complex K we define the polyhedron $|K|$ of K as the union of the simplices in K with their inherited topology from $\mathbb{R}^n$. If X is a topological space homeomorphic to the polyhedron of some simplicial complex K, then we call X triangulable.

Definition 3. *A map $f : |K| \to |L|$ of simplicial complexes is called a simplicial map if for every $\sigma \in K$ we have $f(\sigma) \in L$.*

A map between $f : X \to Y$ of triangulable spaces is called simplicial if the induced map on the polyhedra is simplicial. Note that for a simplicial map f the 0-simplices are sent to 0-simplices.

Note that if σ is a k-simplex given by points $v_0, ..., v_k$ we have a homeomorphism $\Delta_k \to \sigma$ given by $e_i \mapsto v_i$ and extending linearly. Thus we can consider K as a finite collection of n-chains with the properties inherited from definition 3. Note that the maps $\Delta_k \to \sigma$ define an ordering on $\{v_0, ..., v_k\}$. To keep things consistent we need the same ordering on the faces of σ so that the face maps preserve the ordering. It is always possible to choose consistent orderings since we can take an ordering on all the 0-simplices of K. Now we can consider the smaller chain groups $C_n^{simpl.}(K) \subset C_n(|K|)$ which are formal sums of the ordered n-chains in K. As before we have boundary operators so we get homology groups $H_n^{simpl.}(X)$ which are called the simplicial homology groups of K.

If $f : |K| \to |L|$ is a simplicial map then we get an induced map f_* on homology as follows. If σ is an n-chain then $f(\sigma)$ is either an n-chain or a k-chain for $k < n$. In the first case we define $f_*[\sigma] = [f(\sigma)]$ in the second case we define $f_*[\sigma] = 0$. We extend this map linearly to $f_* : H_n^{simpl.}(K) \to H_n^{simpl.}(L)$.

We now have two homology theories on simplicial complexes and one is led to wonder if they are the same.

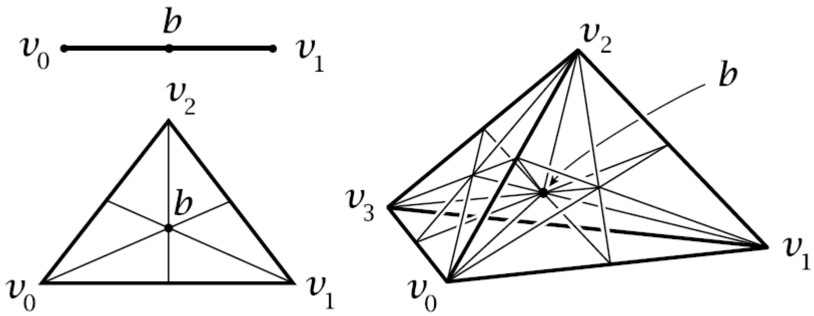

Figure 1: barycentric subdivision of low-dimensional n-simplices

Theorem 1. *For K a simplicial complex we have $H_n^{simpl.}(K) \cong H_n(|K|)$ for all n. If $f : |K| \to |L|$ is a simplicial map then the induced maps on homology f_* are the same.*

For a proof, see [1] theorem 2.27.

1.3 Simplicial approximation

Given triangulable spaces X, Y and a map $f : X \to Y$, then we know that if f is simplicial then we get an induced map on simplicial homology. If f is not simplicial then we still get a map on homology but since the singular and simplicial homology groups coincide we again get a map on simplicial homology. If it were possible to find a simplicial map g homotopic to f then we could compute invariants of f_* like traces and determinants. In general it is not always possible to find such a g, however we know that homology does not depend on the simplicial structure of X. So we might be able to find a simplicial complex $|L| \cong |K| \cong X$, such that there is a simplicial map $g : |L| \to Y$.

First to get such a complex L we look at the notion of barycentric subdivision of a complex K. Since each point in an n-simplex σ is the sum $\sum_i t_i v_i$ of some $v_0, ..., v_n$ affinely independent, we can consider the barycenter $b = \sum_i \frac{v_i}{n+1}$. Then we can define the barycentric subdivision of σ to be the collection of all n-simplices $[w_{i_1}, ..., w_{i_{n-1}}, b]$ inductively, where the w_i are the 0-simplices in the barycentric subdivision of a face of σ.

Since $|K| \subset \mathbb{R}^n$ has a metric we have for each σ the diameter

$$\mathrm{diam}(\sigma) = \max_{x,y \in \sigma} d(x, y).$$

An important consequence of barycentric subdivision is that each of the n-simplices σ_i in the barycentric subdivision of σ have

$$\mathrm{diam}(\sigma_i) \leq \frac{n}{n+1} \mathrm{diam}(\sigma) < \mathrm{diam}(\sigma).$$

We note that by the triangle inequality for $x, y \in \sigma$ writing $y = \sum_i t_i v_i$ we have

$$|x - y| \leq \sum_i |x - t_i v_i| \leq \sum_i t_i |x - v_i| \leq \max_i |x - v_i|.$$

So if $[w_1, ..., w_n, b]$ is an n-simplex in the barycentric subdivision of σ_n, then by the preceding we only need to consider the distances between w_i, w_j or b and some w_i. In the first case both w_i, w_j lie on a face of σ hence we are done by induction. In the second case we may take w_i to be some v_i. Letting b_i be the barycenter of $f_i^n(\sigma)$ we have $b = \frac{v_i}{n+1} + \frac{n}{n+1} b_i$. So b lies on the line from v_i to b_i which has length bounded by the diameter of σ. Since $|b - v_i| = \frac{n}{n+1}|b_i - v_i|$ the result follows.

Theorem 2 (Simplicial approximation)**.** *If $f : |K| \to |L|$ is a continuous map then there exists a map $g : |K| \to |L|$ which is homotopic to f and simplicial with respect to a barycentric subdivision of K.*

We will need a lemma to prove this theorem. First we define the star $\mathrm{St}\ \sigma$ of a simplex σ in a simplicial complex K to be the subcomplex of all simplices containing σ. We also define the open star $\mathrm{st}\ \sigma$ to be the union of the interiors of all simplices containing σ.

Lemma 1. *For 0-simplices $v_1, ..., v_n$ the intersection of the open stars $\cap_i st\ v_i$ is empty unless $\sigma = [v_1, ..., v_n]$ is a simplex in K. In this case we have $\cap_i st\ v_i = st\ \sigma$.*

Proof. The intersection $\cap_i \mathrm{st}\ v_i$ is the union of the interiors of all the simplices τ such that $v_i \in \tau$ for all i. If the intersection is non-empty then such a τ exists, but then after applying a couple of face maps we see that σ must be a simplex of K. Since each τ in the union contains σ it follows that $\cap_i \mathrm{st}\ v_i = \mathrm{st}\ \sigma$. □

Using this we are able to prove theorem 2. First we note that a complex as a subset of $\mathbb{R}^n$ has an induced metric and is compact as the union of finitely many simplices, which are in turn compact as closed and bounded subsets of $\mathbb{R}^n$.

proof of theorem. Note that $\{f^{-1}(\mathrm{st}\ w) : [w] \in L\}$ is an open cover of $|K|$. Since $|K|$ is a compact metric space there is a Lebesgue number $\epsilon > 0$ for this cover. After barycentric subdivision of K we may assume that each simplex of K has diameter less than $\frac{\epsilon}{2}$. Now the closed star $\mathrm{St}\ v$ of each 0-simplex in K has diameter less than ϵ. Hence we find that $f(\mathrm{St}\ v) \subset \mathrm{st}\ w$ for a 0-simplex w of L. We get a map $g : K^0 \to L^0$, from the collection of 0-simplices of K to the collection of 0-simplices of L satisfying $f(\mathrm{St}\ v) \subset \mathrm{st}\ g(v)$. Now we can extend g to a simplicial map $g : K \to L$. For this we look at how to extend g on an n-simplex σ in K which is the convex hull of $v_0, ..., v_n$. An interior point p of σ lies in $\mathrm{st}\ v_i$ for each i hence $f(p) \in \mathrm{st}\ g(v_i)$ for each i. By the lemma we then know that $g(v_0), ..., g(v_n)$ define an n-simplex in L. So we can extend g linearly on σ.

Since for $x \in |K|$ the images $f(x)$ and $g(x)$ lie on the same simplex of L we know that $(1-t)f(x) + tg(x)$ also lies on the same simplex of L for $0 \leq t \leq 1$. So we see that $(1-t)f + tg$ gives a homotopy between f and g. □

1.4 Lefschetz fixed point theorem

First we define the Lefschetz number as follows. We assume for simplicity that G is a field, in this case the homology groups are vector spaces. If $f : X \to X$ is a continuous map and $H_n(X)$ are finite dimensional such that for some $m > 0$ all groups $H_n(X)$ with $n \geq m$ are 0, then the f_* are linear maps between finite dimensional vector spaces and we can consider

$$\Lambda(f) = \sum_n (-1)^n \mathrm{tr}(f_* : H_n(X) \to H_n(X)).$$

In slightly more generality, when $G = \mathbb{Z}$, then the homology groups are $\mathbb{Z}$-modules, the f_* induce maps on the torsion-free part of each $H_n(X)$ and these have well defined traces.

Theorem 3 (Lefschetz fixed point theorem)**.** *If X is a triangulable space, or a retract of a simplicial complex and if $f : X \to X$ is continuous, then if $\Lambda(f) \neq 0$, f has a fixed point.*

First we reduce to the case X is a simplicial complex. Suppose $r : |K| \to X$ is a retraction and $f : X \to X$ is a continuous map. Then $f \circ r : |K| \to |K|$ has the same fixed points as f. The group $H_n(|K|)$ splits as a direct sum and r_* is the projection onto $H_n(X)$. Hence we find that $\mathrm{tr}(f_*) = \mathrm{tr}(f_* \circ r_*)$, so $\Lambda(f) = \Lambda(f \circ r)$.

Next we want to show that if a map has no fixed points then the Lefschetz number must be zero. We do this by calculating the traces of induced maps on the cellular chain complex and relating these to the traces on the homology groups.

For a simplicial complex K let K_n be the union of all the m-simplices, $m \leq n$, this is called the n-skeleton of K. Then we have an exact sequence of chain groups

$$0 \to C_m(K_{n-1}) \to C_m(K_n) \to C_m(K_n)/C_m(K_{n-1}) \to 0$$

for all m, n. Using the zig-zag lemma, see [1] theorem 2.16, we get a natural long exact sequence

$$\ldots H_{m+1}(K_n, K_{n-1}) \to H_m(K_{n-1}) \to H_m(K_n) \to H_m(K_n, K_{n-1}) \to H_{m-1}(K_{n-1}) \ldots$$

Since K_n is an n-dimensional simplicial complex the homology groups are 0 for $m > n$ and equal $H_m(K)$ for $m < n$ so the n-th part reads

$$0 \to H_n(K_n) \to H_n(K_n, K_{n-1}) \to H_{n-1}(K_{n-1}) \to H_{n-1}(K) \to 0.$$

By gluing these sequences at the n-th part for every n we get the cellular chain complex

$$\dots H_{n+1}(K_{n+1}, K_n) \to H_n(K_n, K_{n-1}) \to H_{n-1}(K_{n-1}, K_{n-2}) \dots$$

We have $H_n(K_n, K_{n-1}) \cong C_n^{simp}(K)$ and the homology groups of the cellular chain complex equal $H_n(K)$ for all n.

Now we are ready to prove the fixed point theorem. So suppose f has no fixed points, since $|K|$ is a compact metric space there is an $\epsilon > 0$ such that $d(x, f(x)) > \epsilon$ for all $x \in |K|$. Now we let L be a subdivision of K such that the star of each simplex has diameter less than $\epsilon/2$. We then can find a further subdivision L' and a simplicial approximation $g : |L'| \to |L|$ of f. Since g is a simplicial approximation the image $f(\sigma)$ is contained in the star of $g(\sigma)$ for each simplex σ. We thus have $g(\sigma) \cap \sigma = \emptyset$ for every $\sigma \in L'$ since for $x \in \sigma$ we have $d(x, f(x)) > \epsilon$ but $g(\sigma)$ lies within $\epsilon/2$ of $f(x)$ and σ lies within $\epsilon/2$ of x. Since g maps L'_n into L_n for each n we get induced maps $H_n(L'_n, L'_{n-1}) \to H_n(L_n, L_{n-1})$ for each n. Further since L' is a subdivision of L we get induced maps $g_{C_n} : H_n(L'_n, L'_{n-1}) \to H_n(L_n, L_{n-1})$.

If we denote the boundary maps in the cellular chain complex as follows $d_n : H_n(L'_n, L'_{n-1}) \to H_{n-1}(L'_{n-1}, L'_{n-2})$ then we have two exact sequences for each n namely

$$0 \to \ker d_n \to H_n(L'_n, L'_{n-1}) \to \text{im } d_n \to 0 \tag{1}$$

$$0 \to \text{im } d_{n+1} \to \ker d_n \to H_n(L') \to 0 \tag{2}$$

Now let $Z_n = \ker d_n, B_{n-1} = \text{im } d_n, C_n = H_n(L'_n, L'_{n-1})$. Denote the induced maps by g on these groups by g_{Z_n}, g_{B_n} and g_{C_n}. By additivity of traces over short exact sequences we find

$$\text{tr} g_{C_n} = \text{tr} g_{Z_n} + \text{tr} g_{B_{n-1}} \tag{3}$$

$$\text{tr} g_{Z_n} = \text{tr} g_{B_n} + \text{tr} g_* \tag{4}$$

Substituting the second into the first and taking the alternating sum we find

$$\sum_i (-1)^i \text{tr} g_{C_n} = \sum_i (-1)^i \text{tr} g_* = \Lambda(g) = \Lambda(f),$$

since the terms $\text{tr} g_{B_n}$ cancel in pairs and are 0 for $n = 0$ or n greater than the dimension of K. The last inequality follows because g is a simplicial approximation of f. Since $g(\sigma) \cap \sigma = \emptyset$ for every σ in L', we find that $\text{tr} g_{C_n} = 0$ since the n-simplices form a basis for C_n. Thus we conclude that $\Lambda(f) = 0$ if f has no fixed points.

As a consequence of the Lefschetz fixed point theorem we can now easily deduce the famous Brouwer fixed point theorem which is as follows:

Theorem 4. *Any continuous map $f : D_n \to D_n$ has a fixed point, where D_n is the unit ball in $\mathbb{R}^n$.*

Proof. Note that D_n retracts onto a point, a point has 0-th homology group equal to $\mathbb{Z}$ and the rest equal to 0. Each map from a point to itself is a homeomorphism hence the induced map on homology has non-zero trace, so by the Lefschetz fixed point theorem it follows that any map $f : D_n \to D_n$ must have a fixed point. □

Using singular cohomology instead of singular homology it is also possible to prove a stronger version of the Lefschetz fixed point theorem for smooth compact manifolds. In this stronger version the number $\Lambda(f)$ actually equals the amount of fixed points counted with certain multiplicities, if f has finitely many fixed points. We will not pursue this here and instead move on to the category of non-singular projective varieties, where assuming a suitable cohomology theory we can also prove an analogue of this strong Lefschetz fixed point theorem.

2 Lefschetz fixed point theorem for smooth projective varieties

2.1 Intersection theory

In this section we develop a bit of the intersection theory needed for our discussion on Weil cohomology. For proofs and more details see [2] chapter 4 and 5.

For an algebraic variety X over an algebraically closed field we define the cycle group $Z(X)$ as the free abelian group generated by symbols $\langle V \rangle$, where V is an irreducible closed subvariety of X. It is graded by dimension

$$Z(X) = \bigoplus_d Z_d(X)$$

where $Z_d(X)$ is the group of formal sums of d-dimensional subvarieties, its elements are called d-cycles.

Consider now any closed subscheme $Y \subset X$. We can look at the finitely many irreducible components $Y_1, ..., Y_r$ and the local rings $\mathcal{O}_{Y,Y_i}$. If these have lengths l_i respectively then we define $\langle Y \rangle = \sum_i l_i \langle Y_i \rangle$.

We let $Rat(X)$ be the subgroup of $Z(X)$ generated by cycles of the form $\langle V(0) \rangle - \langle V(\infty) \rangle$ where V is a subvariety of $X \times \mathbb{P}^1$ such that the projection onto the second coordinate is a dominant morphism. We let $Rat_d(X) = Rat(X) \cap Z_d(X)$. We are now able to define the Chow groups and the Chow ring.

Definition 4. *Suppose* $\dim(X) = n$*, the chow groups of* X *are the groups*

$$A_d(X) = Z_d(X)/Rat_d(X).$$

Put $A^i(X) = A_{n-i}(X)$ *the chow ring is*

$$A^*(X) = \bigoplus_d A^d(X) = Z(X)/Rat(X).$$

For a cycle $A \in Z(X)$ we denote the image in $A^*(X)$ as $[A]$, in particular if Y is a subscheme of X then we denote $[\langle Y \rangle]$ as $[Y]$.

Now we move on to the ring structure of $A^*(X)$.

Definition 5.

1. *Irreducible subschemes* A, B *of* X *are called dimensionally transverse if for every component* C *of* $A \cap B$ *we have codim* $C =$ *codim* $A +$ *codim* B.

2. *Subvarieties* A, B *of* X *are transverse at a point* $p \in X$ *if* X, A *and* B *are smooth at* p *and their tangent spaces at* p *satisfy* $T_pA + T_pB = T_pX$.

3. *Subvarieties* A, B *of* X *are generically transverse if every irreducible component of* $A \cap B$ *contains a point at which* A *and* B *are transverse.*

For cycles $A = \sum_i n_i \langle A_i \rangle, B = \sum_j m_j \langle B_j \rangle$ we say they are dimensionally transverse respectively generically transverse if each of the pairs A_i, B_j is. Further if they are dimensionally transverse then we define

$$A \cap B = \sum_{i,j} n_i m_j \langle A_i \cap B_j \rangle.$$

Proposition 2. *Subvarieties A, B of X are generically transverse if and only if they are dimensionally transverse and each irreducible component of $A \cap B$ is reduced and contains a smooth point of X.*

Theorem 5 (moving lemma)**.** *Let X be a smooth (quasi)projective variety.*

1. *If $\alpha \in A^*(X)$, $B \in Z(X)$, then there exists a cycle $A \in Z(X)$ such that $[A] = \alpha$ and A, B are generically transverse.*
2. *If $A, B \in Z(X)$ intersect generically transversely then $[A \cap B]$ only depends on $[A]$ and $[B]$.*

A consequence of this lemma is the intersection product structure on the Chow ring.

Theorem 6. *If X is a smooth (quasi)projective variety then there is a unique bilinear product structure on $A^*(X)$ that satisfies*

$$[A][B] = [A \cap B]$$

whenever A, B are generically transverse subvarieties. The Chow ring with this product is an associative, commutative graded ring.

For $\alpha_1, ..., \alpha_r \in A^*(X)$ we define the intersection number $(\alpha_1 \cdot ... \cdot \alpha_r)$ to be the degree of the grade $\dim(X)$ part of $\alpha_1 \cdot ... \cdot \alpha_r$, that is, the part corresponding to dimension 0 cycles, points. If A, B are generically transverse such that $A \cap B$ is 0-dimensional then $([A] \cdot [B])$ is the cardinality of $A \cap B$.

The assignment $X \to A^*(X)$ is functorial. We can define a pushforward as follows. If $f : X \to Y$ is a proper morphism and $A \subset X$ a closed subvariety, then $f(A)$ is closed in Y. We define $f_*[A] = \deg(A/f(A))[f(A)]$.

Definition 6. *Let $f : X \to Y$ be a morphism. A subvariety $Z \subset Y$ of codimension c is generically transverse to f if:*

1. *$f^{-1}(Z)$ is generically reduced of codimension c.*
2. *X is smooth at a point q of each irreducible component of $f^{-1}(Z)$, and Y is smooth in $f(q)$.*

Analogous as before we have a moving lemma, and it implies we can define a pull-back $f^* : A(Y) \to A(X)$ such that $f^*[Z] = [f^{-1}(Z)]$ whenever $Z \subset Y$ is generically transverse to f.

2.2 Weil cohomology

In this section we will discuss Weil cohomology mostly following [3]. We let $\bar{k}$ be an algebraically closed field and suppose all varieties are defined over $\bar{k}$. Let K be a field of characteristic zero then a Weil cohomology is defined as follows. It is given by the following data:

D1 A contravariant functor

$$H^* : \{\text{non-singular, connected, projective varieties}\} \to \{\text{graded commutative } K\text{-algebras}\}.$$

So for X a variety we have $H^*(X) = \bigoplus_i H^i(X)$. The product of two elements $\alpha, \beta \in H^*(X)$ is denoted $\alpha \cup \beta$, and $H^*(X)$ being graded commutative means that $\alpha \cup \beta = (-1)^{\deg(\alpha)\deg(\beta)} \beta \cup \alpha$ for α, β homogeneous.

D2 For every X there is a linear trace map $tr_X : H^{2\dim(X)}(X) \to K$.

D3 For every X and every irreducible closed subvariety $Z \subset X$ of codimension c there is a cohomology class $cl(Z) \in H^{2c}(X)$.

Note that since $H^*(X)$ is a K-algebra it comes with a ring homomorphism $K \to H^*(X)$. Because K is a field, and because of the graded structure, this is an inclusion $K \to H^0(X)$ called the structural morphism, it makes the $H^n(X)$ into K-vector spaces.

The data must satisfy the following axioms:

A1 For each X all the $H^i(X)$ are finite dimensional over K and are 0 unless $0 \le i \le 2\dim(X)$.

A2 (Künneth formula) For every X and Y, if we let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the canonical projections. Then we have an isomorphism of K-algebras

$$H^*(X) \otimes_K H^*(Y) \to H^*(X \times Y), \alpha \otimes \beta \to p_X^*(\alpha) \cup p_Y^*(\beta).$$

A3 (Poincaré duality) For every X, the trace map in (D2) is an isomorphism and for every $0 \le i \le 2\dim(X)$ we have a perfect pairing

$$\psi_i : H^i(X) \otimes_K H^{2\dim(X)-i}(X) \to K, \alpha \otimes \beta \to tr_X(\alpha \cup \beta).$$

A4 For every X, Y we have

$$tr_{X\times Y}(p_X^*(\alpha) \cup p_Y^*(\beta)) = tr_X(\alpha) tr_Y(\beta),$$

for every $\alpha \in H^{2\dim(X)}(X), \beta \in H^{2\dim(Y)}(Y)$.

A5 For every X, Y and every irreducible closed subvarieties $Z \subset X, W \subset Y$ we have

$$cl(Z \times W) = p_X^*(cl(Z)) \cup p_Y^*(cl(W)).$$

A6 (push-forward of cohomology classes) For every morphism $f : X \to Y$, and for every closed irreducible subvariety $Z \subset X$, we have for every $\alpha \in H^{2\dim(Z)}(Y)$ that

$$tr_X(cl(Z) \cup f^*(\alpha)) = \deg(Z/f(Z)) \cdot tr_Y(cl(f(Z)) \cup \alpha).$$

A7 (pull-back of cohomology classes) For every morphism $f : X \to Y$ and every closed subvariety $Z \subset Y$ such that

(a) all the irreducible components $W_1, ..., W_r$ of $f^{-1}(Z)$ have dimension $\dim(Z) + \dim(X) - \dim(Y)$;

(b) either f is flat in a neighbourhood of Z or Z is generically transverse to f,

if $\langle f^{-1}(Z)\rangle = \sum_i m_i \langle W_i\rangle$ as a cycle, then $f^*(cl(Z)) = \sum_{i=1}^r m_i cl(W_i)$. Note that in the generically transverse case all the $m_i = 1$.

A8 If $x = \text{Spec }(\bar{k})$, then $cl(x) = 1$ and $tr_x(1) = 1$.

Now that we have defined what a Weil cohomology is, it is important to note some examples. The simplest example is the case when $\bar{k} = \mathbb{C}$ and we may take $K = \mathbb{Q}$ or $\mathbb{R}$. In this case we can look at the euclidian topology on the closed points of a smooth, projective $\mathbb{C}$ variety, which make it into a compact, smooth $\mathbb{C}$ manifold. This again is a real manifold of twice the dimension. Hence we have the singular and de Rham cohomology theories. In this case the first three axioms are well known and classical theorems. For the class map we may take the Poincaré dual of submanifolds, see [5] page 50 onwards on how to define this.

For varieties over a field with positive characteristic we can not do the same as above. But there do exist Weil cohomologies for these fields as well, for example we have ℓ-adic cohomology with $K = \mathbb{Q}_\ell$ for $\ell \neq p = \text{char } \bar{k}$, for a definition and some properties see [4] appendix C.

Now assuming a weil cohomology theory over $\bar{k}$ with coefficients in K we will deduce some consequences which we will apply to prove the fixed point theorem.

Proposition 3. *Let X be a smooth, connected n-dimensional projective variety.*

1. *The structural morphism $K \to H^0(X)$ is an isomorphism.*
2. *We have $cl(X) = 1 \in H^0(X)$.*
3. *If $x \in X$ is a closed point, then $tr_X(cl(x)) = 1$.*
4. *If $f : X \to Y$ is a generically finite surjective morphism of degree d to a smooth projective variety Y, then*

$$tr_X(f^*(\alpha)) = d \cdot tr_Y(\alpha)$$

for all $\alpha \in H^{2n}(Y)$. In particular for $Y = X$ it follows that f^* is multiplication by d on $H^{2n}(X)$.

Proof. 1. From Poincaré duality (A3) with $i = 0$ it immediately follows that $\dim_K(H^0(X)) = 1$.

2. If we apply (A7) to the morphism $X \to \mathrm{Spec}\ \bar{k}$, combining this with (A8) proves (2).

3. Given $x \in X$ a closed point, then we apply (A6) to the morphism $X \to \mathrm{Spec}\ \bar{k}$, by taking $Z = \{x\}$ and $\alpha = 1 \in H^0(\mathrm{Spec}\ \bar{k})$. We find that $tr_X(cl(x)) = tr_{\mathrm{Spec}\ \bar{k}}(1)$ which equals 1 by (A8).

4. Let $f : X \to Y$ as in the proposition, take a general point Q in Y. Then as a cycle we have $\langle f^{-1}(Q)\rangle = \sum_{i=1}^{r} m_i \langle P_i \rangle$, where P_i are the reduced points of the fiber over Q and $\sum_{i=1}^{r} m_i = d$. By generic flatness f is flat around Q so that (A7) and (A6) imply

$$tr_X(f^*(cl(Q))) = tr_X(\sum_{i=1}^{r} m_i \cdot cl(P_i)) = d \cdot tr_Y(cl(Q)).$$

Since $cl(Q)$ generates $H^{2\dim(Y)}(Y)$ by (3), the assertion in (4) follows.

□

Definition 7. *Given a morphism $f : X \to Y$ between smooth, connected, projective varieties with $\dim(X) = m, \dim(Y) = n$. We use Poincaré duality to define a push-forward $f_* : H^*(X) \to H^*(Y)$ as follows. Given $\alpha \in H^i(X)$, there is a unique $f_*(\alpha) \in H^{2n-2m+i}(Y)$ such that*

$$tr_Y(f_*(\alpha) \cup \beta) = tr_X(\alpha \cup f^*(\beta))$$

for every $\beta \in H^{2m-i}(Y)$.

Proposition 4. *Let $f : X \to Y$ be a morphism as in the definition. The push-forward of f has the following properties:*

1. *(projection formula) $f_*(\alpha \cup f^*(\gamma)) = f_*(\alpha) \cup \gamma$.*
2. *If $g : Y \to Z$ is another morphism as above, then $(g \circ f)_* = g_* \circ f_*$ on $H^*(X)$.*
3. *If $Z \subset X$ is an irreducible closed subvariety then*

$$f_*(cl(Z)) = \deg(Z/f(Z))cl(f(Z)).$$

Proof.

1. Note that

$$\begin{aligned} tr_Y(f_*(\alpha \cup f^*(\gamma))) &= tr_Y(f_*(\alpha \cup f^*(\gamma)) \cup 1) \\ &= tr_X(\alpha \cup f^*(\gamma) \cup f^*(1)) \\ &= tr_X(\alpha \cup f^*(\gamma)) \\ &= tr_Y(f_*(\alpha) \cup \gamma), \end{aligned}$$

hence $f_*(\alpha \cup f^*(\gamma)) = f_*(\alpha) \cup \gamma$.

2. Note that

$$\begin{aligned} tr_Z((g\circ f)_*(\alpha)\cup\beta) &= tr_X(\alpha\cup(g\circ f)^*(\beta)) \\ &= tr_X(\alpha\cup f^*(g^*(\beta))) \\ &= tr_Y(f_*(\alpha)\cup g^*(\beta)) \\ &= tr_Z(g_*(f_*(\alpha))\cup\beta), \end{aligned}$$

hence it follows that $(g\circ f)_* = g_*\circ f_*$.

3. Using (A6) we have

$$tr_Y(f_*(cl(Z))\cup\alpha) = tr_X(cl(Z)\cup f^*(\alpha)) = \deg(Z/f(Z))\cdot tr_Y(cl(f(Z))\cup\alpha),$$

hence by uniqueness it follows that $f_*(cl(Z)) = \deg(Z/f(Z))\cdot cl(f(Z))$. □

Proposition 5. *For X and Y smooth, connected, projective varieties, let p_X, p_Y be the projections of $X\times Y$ onto X, Y respectively. If $\alpha\in H^i(Y)$ then $p_{X*}(p_Y^*(\alpha)) = tr_Y(\alpha)$ if $i = 2\dim(Y)$ and 0 otherwise.*

Proof. Since $p_{X*}(p_Y^*(\alpha))\in H^{i-2\dim(Y)}(X)$ it is clear that this is 0 when $i\neq 2\dim(Y)$. If $i = 2\dim(Y)$ and if $\beta\in H^{2\dim(X)}(X)$ then by (A4) we have

$$tr_X(p_{X*}(p_Y^*(\alpha))\cup\beta) = tr_{X\times Y}(p_Y^*(\alpha)\cup p_X^*(\beta)) = tr_Y(\alpha)\cdot tr_X(\beta),$$

from which it follows that $p_{X*}(p_Y^*(\alpha)) = tr_Y(\alpha)$. □

Lemma 2. *Given $\alpha = \sum_{i=1}^r n_i\langle V_i\rangle$ rationally equivalent to zero, then also $\sum_{i=1}^r n_i cl(V_i) = 0$ in $H^*(X)$.*

For a proof see lemma 4.4 in [3]. Using this lemma we see that cl induces a cycle class map

$$cc_i : A^i(X)\to H^{2i}(X)$$

from the chow group of codimension i cycles.

Proposition 6. *Putting the cc_i together gives a ring homomorphism*

$$cc : A^*(X)\to H^{2*}(X)$$

compatible with f_* and f^*.

Proof. Compatibility with f_* follows from proposition 4.3.

For compatibility with f^* we factor $f : X\to Y$ as $pr_Y\circ j$ where $j : X\to X\times Y$ is the embedding of X onto the graph of f and $pr_Y : X\times Y\to Y$ is the projection onto the second coordinate. In this case we have

$$f^*([Z]) = j^*(pr_Y^*([Z])) = j^*([X\times Z]).$$

By the moving lemma $[X\times Z]$ is rationally equivalent to a sum $\sum_i n_i[W_i]$ where each W_i is generically transverse to j. By lemma 4.4 we then have $cc([X\times Z]) =$

$\sum_i n_i cc([W_i])$. From (A7) it then follows that $f^*(cc([Z])) = cc(f^*([Z]))$.

To see that it is a ring homomorphism, let V, W be irreducible subvarieties of X and let $\Delta : X \to X \times X$ be the embedding onto the diagonal. Let $p, q : X \times X \to X$ be the projections onto the first and second coordinate respectively. Compatibility with pull-back and (A5) gives

$$\begin{aligned}
cc([V][W]) &= cc(\Delta^*([V \times W])) \\
&= \Delta^*(cc([V \times W])) \\
&= \Delta^*(cl(V \times W)) \\
&= \Delta^*(p^*(cl(V)) \cup q^*(cl(W))) \\
&= \Delta^*(p^*(cl(V)) \cup \Delta^*(q^*(cl(W)) \\
&= cl(V) \cup cl(W) \\
&= cc([V]) \cup cc([W]).
\end{aligned}$$

□

As a consequence of cc being a ring homomorphism we have the following.

Lemma 3. *Let X be a smooth projective variety, let $\alpha_i \in A^{m_i}(X)$, $i = 1, ..., r$ such that $\sum_i m_i = \dim(X)$. Then*

$$(\alpha_1 \cdots \alpha_r) = tr_X(cc(\alpha_1) \cup ... \cup cc(\alpha_r)).$$

Proof. Since cc is a ring homomorphism it is enough to prove that $\deg(\alpha) = tr_X(cc(\alpha))$ for $\alpha \in Z_0(X)$. By additivity we can assume that α is a point, so it follows from proposition 3.3. □

2.3 Lefschetz fixed point theorem

Theorem 7. *If $f : X \to X$ is an endomorphism, let $\Gamma_f, \Delta \subset X \times X$ be the graph of f and the diagonal respectively, then*

$$(\Gamma_f \cdot \Delta) = \sum_{i=0}^{2\dim(X)} (-1)^i trace(f^*|_{H^i(X)}).$$

We will need two final lemmas to prove the theorem. To simplify notation let $n = \dim(X)$ and let $p, q : X \times X \to X$ be the projection on the first and second coordinate respectively.

Lemma 4. *If $\alpha \in H^*(X)$ then $p_*(cl(\Gamma_f) \cup q^*(\alpha)) = f^*(\alpha)$.*

Proof. Let $j : X \to X \times X$ be the embedding of X onto Γ_f. We then have $p \circ j = id_X$ and $q \circ j = f$. Note that $j_*(cl(X)) = cl(\Gamma_f)$, from the projection formula it then follows that

$$\begin{aligned}
p_*(cl(\Gamma_f) \cup q^*(\alpha)) &= p_*(j_*(cl(X)) \cup q^*(\alpha)) \\
&= p_*(j_*(cl(X) \cup j^*(q^*(\alpha))) \\
&= p_*(j_*(f^*(\alpha))) \\
&= f^*(\alpha).
\end{aligned}$$

□

Lemma 5. *Let (e_i^r) be a basis of $H^r(X)$ and (f_i^{2n-r}) be the dual basis of $H^{2n-r}(X)$ with respect to Poincaré duality (A3), so that $tr_X(f_l^{2n-r} \cup e_i^r) = \delta_{i,l}$. We then have*

$$cl(\Gamma_f) = \sum_{i,r} p^*(f^*(e_i^r)) \cup q^*(f_i^{2n-r}) \in H^{2n}(X \times X).$$

Proof. By the Künneth property (A2) we can write

$$cl(\Gamma_f) = \sum_{l,s} p^*(a_{l,s}) \cup q^*(f_l^{2n-s})$$

for unique $a_{l,s} \in H^s(X)$. From the preceding lemma and the projection formula it follows that

$$\begin{aligned}
f^*(e_i^r) &= \sum_{l,s} p_*(p^*(a_{l,s}) \cup q^*(f_l^{2n-s}) \cup q^*(e_i^r)) \\
&= \sum_{l,s} a_{l,s} \cup p_*(q^*(f_l^{2n-s} \cup e_i^r)).
\end{aligned}$$

By proposition 5 we have that $p_*(q^*(f_l^{2n-s} \cup e_i^r)) = 0$, unless $r = s$, in which case it equals $tr_X(f_l^{2n-r} \cup e_i^r)$. This again is zero unless $i = l$, in which case it equals 1. Hence it follows that $f^*(e_i^r) = a_{i,r}$. □

proof of theorem. From the previous lemma it follows that

$$cl(\Gamma_f) = \sum_{i,r} p^*(f^*(e_i^r)) \cup q^*(f_i^{2n-r}).$$

Applying the same lemma to the identity morphism and the dual bases (f_l^s) and $((-1)^s e_l^{2n-s})$, we get

$$cl(\Delta) = \sum_{l,s} (-1)^s p^*(f_l^s) \cup q^*(e_l^{2n-s}).$$

Hence it now follows that

$$
\begin{aligned}
(\Gamma_f \cdot \Delta) &= tr_{X\times X}(cl(\Gamma_f) \cup cl(\Delta)) \\
&= tr_{X\times X}\left(\sum_{i,j,r,s} (-1)^{s+s(2n-r)} p^*(f^*(e_i^r) \cup f_l^s) \cup q^*(f_i^{2n-r} \cup e_l^{2n-s}) \right) \\
&= \sum_{i,r} tr_X(f^*(e_i^r) \cup f_i^{2n-r}) \cdot tr_X(f_i^{2n-r} \cup e_i^r) \\
&= \sum_r (-1)^r \mathrm{trace}(f^*|_{H^r(X)}).
\end{aligned}
$$

□

3 Weil conjectures

3.1 Statement

In this section we consider varieties over a finite field $k = \mathbb{F}_q$. Let X be a variety over k and let $x \in X$ be a closed point with local ring $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$. Its residue field $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ is a finite extension of k. We define the degree of x to be

$$\deg(x) = [k(x) : k].$$

Let $k \subset K$ be a field extension then a K-valued point of X is an element of

$$X(K) = \hom_{\mathrm{Spec}\ k}(\mathrm{Spec}\ K, X) = \bigcup_{x \in X} \hom_k(k(x), K).$$

Lemma 6. *If X is defined over k, and $K = \mathbb{F}_{q^m}$ is an extension of degree m, then*

$$|X(K)| = \sum_{d|m} d \cdot |\{x \in X | x \text{ a closed point of degree } d\}.$$

Proof. Let x be the image of a K-valued point. Then $k(x)$ embeds into K, since $k \subset K$ is algebraic then also $k \subset k(x)$ is algebraic. So x is a closed point because

$$\dim \overline{\{x\}} = \mathrm{trdeg}(k(x)/k) = 0.$$

Letting $r = [k(x) : k]$ then $r|m$ and there are exactly r embeddings of $k(x) \to K$ fixing k. $\square$

If $k \subset K$ is a finite extension there are only finitely many points in $X(K)$. We can see this by taking a finite affine open cover of X. So it is enough to look at the affine case. If $X \subset \mathbb{A}^n_k$ is defined by equations $f_1, ..., f_n$ then $X(K)$ is the set of common solutions in K^n. The preceding lemma then implies that for each m there are only finitely many points with degree dividing m.

Letting $N_m = |X(\mathbb{F}_{q^m})|$ we define the zeta function of X.

Definition 8. *The zeta function $Z(X, T)$ of X is*

$$Z(X, T) = \exp(\sum_{m=1}^{\infty} \frac{N_m \cdot T^m}{m}) \in \mathbb{Q}[[T]].$$

We are now ready to state the Weil conjectures, which asserts that this Zeta function has some nice properties.

Theorem 8 (Weil conjectures)**.** *Suppose that X is an n-dimensional, geometrically connected, smooth, projective variety defined over k, then the zeta function $Z(X, T)$ has the following properties:*

1. *It is a rational function of T, that is $Z(X, T) \in \mathbb{Q}(T)$.*

2. *Let E be the self-intersection number of the diagonal Δ of $X \times X$. Then $Z(X,T)$ satisfies the following functional equation:*

$$Z(X, \frac{1}{q^n T}) = \pm q^{nE/2} T^E Z(X,T).$$

3. *It is possible to write*

$$Z(X,T) = \frac{P_1(T)P_3(T)\cdots P_{2n-1}(T)}{P_0(T)P_2(T)\cdots P_{2n}(T)}$$

where $P_0(T) = 1 - t, P_{2n}(T) = 1 - q^n T$ and for each $1 \leq i \leq 2n-1$ we have

$$P_i(T) = \prod(1 - \alpha_{ij} T)$$

where the α_{ij} are algebraic integers with absolute value $q^{i/2}$.

We will prove the first two of these in the final section, the third property is beyond the scope of this thesis. It was the last to be proved which P. Deligne did in 1974 for which he was awarded the Fields medal.

3.2 The Frobenius morphism

The idea to prove the first two is to use the Lefschetz fixed point theorem. For a suitable f we then have $(\Gamma_{f^m} \cdot \Delta) = N_m$ for each m. The f we need is the Frobenius morphism which we will define now.

Definition 9. *The Frobenius morphism for X over $k = \mathbb{F}_q$ is the morphism of ringed spaces*

$$Frob_{X,q} : X \to X$$

which is the identity on the topological space of X, and the Frobenius morphism $a \mapsto a^q$ on the sheaf of rings. It is a morphism of $\mathbb{F}_q$-schemes because $a^q = a$ for any $a \in \mathbb{F}_q$.

Now let $\bar{k}$ be an algebraic closure and let

$$\overline{X} = X \times_{\text{Spec } \mathbb{F}_q} \text{Spec } \bar{k}.$$

Then $\overline{X}$ is a variety over $\bar{k}$ and $\overline{X}(K) = X(K)$.

We get an induced morphism of schemes over K:

$$Frob_{\overline{X},q} = Frob_{X,q} \times id : \overline{X} \to \overline{X}.$$

If X is affine, let $X \to \mathbb{A}^n_k$ be a closed immersion. Then $Frob_{\overline{X},q}$ is induced by $Frob_{\mathbb{A}^n_k,q}$. This corresponds to the morphism of K-algebras

$$K[X_1, ..., X_n] \to K[X_1, ..., X_n] : X_i \mapsto X_i^q.$$

On K-rational points this is given by

$$(x_1, ..., x_n) \mapsto (x_1^q, ..., x_n^q).$$

It follows that $X(\mathbb{F}_{q^m})$ can be identified with the fixed points of $Frob^m_{\overline{X},q}$. So if we let Δ, Γ_m be the diagonal and the graph of the m-th power of Frobenius respectively in $\overline{X} \times \overline{X}$, then we have a bijection between $\Delta \cap \Gamma_m$ and $X(\mathbb{F}_{q^m})$.

Proposition 7. *If X is smooth, then Δ, Γ_m intersect generically transverse.*

Proof. (idea) We consider the affine case $X = \mathbb{A}^n_{\mathbb{F}_q}$. If $R = \bar{k}[X_1, ..., X_n, Y_1, ..., Y_n]$, then Δ is defined by the ideal $(Y_1 - X_1, ..., Y_n - X_n)$ and Γ_m is defined by the ideal $(Y_1 - X_1^q, ..., Y_n - X_n^q)$. Therefore their intersection is isomorphic to $\prod_{i=1}^n \operatorname{Spec} \bar{k}[X_i]/(X_i - X_i^q)$, which is reduced because all of the polynomials $(X_i - X_i^q)$ have no multiple roots. Because $X(\mathbb{F}_{q^m})$ is finite we thus see that $\Delta \cap \Gamma_m$ is a reduced set of points. Generic transversality then follows from proposition 2. $\square$

The general case follows from the affine case, for details see [3] proposition 2.4. In a similar way, as we have seen in the affine case that the $Frob_{\mathbb{A}^n_{\bar{k}},q}$ corresponds to the Frobenius morphism on $\bar{k}$-algebras, we also deduce that $Frob^m_{\overline{X},q}$ is finite of degree q^{nm}.

3.3 Proof

Let X be a geometrically connected, smooth, projective variety defined over $k = \mathbb{F}_q$, we keep some of the notation from the previous section. Suppose we have a Weil cohomology for varieties over $\bar{k}$ with coefficients in some K (here K refers to a characteristic 0 field as in the definition of Weil cohomology and not a finite extension of k). Since the graph of the Frobenius endomorphism intersects the diagonal transversely we get from the Lefschetz fixed point formula that

$$N_m = (\Gamma_m \cdot \Delta) = \sum_i (-1)^i \operatorname{trace}((F^*)^m|_{H^i(\overline{X})}),$$

where F^* is the pull-back of the Frobenius. It follows that

$$Z(X, T) = \prod_{i=0}^{2n} \left[\exp\left(\sum_{m=1}^{\infty} \operatorname{trace}((F^*)^m|_{H^i(\overline{X})}) \cdot \frac{T^m}{m} \right) \right]^{(-1)^i}.$$

From the following lemma it will follow that $Z(X, T)$ is rational over K.

Lemma 7. *Suppose $\varphi : V \to V$ is an endomorphism of a finite dimensional vector space over some field K. Then the following identity holds*

$$\exp\left(\sum_{m=1}^{\infty} tr(\varphi^m) \cdot \frac{T^m}{m} \right) = \det(1 - \varphi T)^{-1}.$$

Proof. Recall that the trace of φ is the sum of its eigenvalues which lie in some algebraic closure of K. Then the trace of φ^m is the sum of the m-th powers of those eigenvalues so we get

$$\exp\left(\sum_{m=1}^{\infty} \mathrm{tr}(\varphi^m) \cdot \frac{T^m}{m}\right) = \prod_{j=1}^{\dim(V)} \exp\left(\sum_{m=1}^{\infty} \lambda_j^m \cdot \frac{T^m}{m}\right).$$

By the identity of formal power series $\sum_{m=1}^{\infty} \lambda_j^m \cdot \frac{T^m}{m} = \log(1-\lambda_j T)$ it follows that

$$\exp\left(\sum_{m=1}^{\infty} \mathrm{tr}(\varphi^m) \cdot \frac{T^m}{m}\right) = \prod_{j=1}^{\dim(V)} \frac{1}{1-\lambda_j T} = \det(1-\varphi T)^{-1}.$$

□

From the lemma it follows that

$$Z(X,T) = \frac{P_1(T)P_3(T)\cdots P_{2n-1}(T)}{P_0(T)P_2(T)\cdots P_{2n}(T)}$$

where $P_i(T) = \det(1 - F^*T|_{H^i(\overline{X})})$. Now that we know that $Z(X,T) \in K(T)$ we still need to show that it is actually in $\mathbb{Q}(T)$. Since we know that it is in $\mathbb{Q}[[T]]$, it follows from the following proposition that $Z(X,T) \in \mathbb{Q}(T)$.

Proposition 8. *Let K be a field and $f = \sum_{m\geq 0} a_m T^m \in K[[T]]$. Then $f \in K(T)$ if and only if there are $M, N \in \mathbb{Z}_{\geq 0}$ such that*

$$\mathit{span}\{(a_i, a_{i+1}, ..., a_{i+N}) \in K^{\oplus(N+1)} | i \geq M\} \subsetneq K^{\oplus(N+1)}.$$

If L/K is a field extension, then $f \in L(T)$ if and only if $f \in K(T)$.

Proof. If $f \in K(T)$ then there are M, N and $c_0, ..., c_N \in K$ not all zero such that

$$f(T) \cdot \sum_{i=0}^{N} c_i T^i \in K[T]$$

of degree at most $M + N$. This means that $c_N a_i + c_{N-1} a_{i+1} + ... + c_0 a_{i+N} = 0$ for all $i \geq M$. Thus the span of all the vectors $(a_i, ..., a_{i+N}), i \geq M$ is contained in the kernel of the linear map $(x_0, ..., x_N) \mapsto \sum_{i=0}^{N} c_{N-i} x_i$. But since not all the c_i are zero this is a proper subspace of $K^{\oplus(N+1)}$. Conversely we can find such a non-zero linear map for any proper subspace, so the converse is also clear.

Now if L/K is a field extension, and V is a vector space over K, then $v_1, ..., v_m$ are linearly independent if and only if $v_1 \otimes 1, ..., v_m \otimes 1$ are linearly independent over L in $V \otimes_K L$. So the second statement also follows. □

To show that the functional equation holds we use Poincaré duality and the following lemma.

Lemma 8. *Let $\phi : V \times W \to K$ be a perfect pairing of vector spaces of dimension r over K. Let $x \in K^*$ and let $\varphi : V \to V, \psi : W \to W$ be endomorphisms such that*

$$\phi(\varphi v, \psi w) = x\phi(v, w)$$

for all $v \in V, w \in W$. Then

$$\det(1 - \psi T) = \frac{(-1)^r x^r T^r}{\det(\varphi)} \det(1 - \frac{\varphi}{xT}) \tag{5}$$

and

$$\det(\psi) = \frac{x^r}{\det(\varphi)}. \tag{6}$$

Proof. After extending scalars we may assume K is algebraically closed. Let $e_1, ..., e_r$ be a basis of V such that φ is represented by a lower triangular matrix. And let $e'_1, ..., e'_r$ be a dual basis of W with respect to ϕ. So $\phi(e_i, e'_j) = \delta_{ij}$. Note that since x is non-zero, ψ must be invertible. Since $\phi(\varphi(e_i), e'_j) = 0$ for $j < i$, we find that $\phi(e_i, \psi^{-1}(e'_j)) = 0$ for $j < i$. It follows that the matrix representing ψ^{-1} is upper triangular. Now let $\lambda_1, ..., \lambda_r$ be the eigenvalues of φ ordered with respect to the basis. Similarly let $\mu_1, ..., \mu_r$ be the eigenvalues of ψ^{-1} (by the preceding these are just the values on the diagonal). We then have

$$\lambda_j = \phi(\varphi(e_j), e'_j) = x\phi(e_j, \psi^{-1}(e'_j)) = x\mu_j.$$

Note that $\det(\varphi) = \prod_{j=1}^r \lambda_j$ and $\det(\psi) = \prod_{j=1}^r \mu_j^{-1}$ so (6) follows. We also have

$$\begin{aligned}
\det(1 - T\psi) &= \det(\psi)\det(\psi^{-1} - T) \\
&= \frac{x^r}{\det(\varphi)} \cdot \prod_{j=1}^r (\frac{\lambda_j}{x} - T) \\
&= \frac{(-1)^r x^r T^r}{\det(\varphi)} \prod_{j=1}^r (1 - \frac{\lambda_j}{xT}) \\
&= \frac{(-1)^r x^r T^r}{\det(\varphi)} \det(1 - \frac{\varphi}{xT}),
\end{aligned}$$

so (5) follows. □

We now apply this to the perfect pairings ϕ_i in (A3) to prove the functional equation.

Proof of the functional equation. Since the Frobenius morphism is finite of degree q^n, proposition 4.3 implies that F^* is given by multiplication by q^n on $H^{2n}(\overline{X})$. So it follows that

$$\phi_i(F^*(\alpha), F^*(\beta)) = \mathrm{tr}_{\overline{X}}(F^*(\alpha \cup \beta)) = tr_{\overline{X}}(q^n \alpha \cup \beta) = q^n \phi_i(\alpha, \beta),$$

for every $\alpha \in H^i(\overline{X}), \beta \in H^{2n-i}(\overline{X})$. The lemma implies that if $B_i = \dim_K H^i(\overline{X})$ and $P_i(T) = \det(1 - TF^*|_{H^i(\overline{X})})$ then

$$\det(F^*|_{H^{2n-i}(\overline{X})}) = \frac{q^{nB_i}}{\det(F^*|_{H^i(\overline{X})})}$$

and

$$P_{2n-i}(T) = \frac{(-1)^{B_i} q^{nB_i} T^{B_i}}{\det(F^*|_{H^i(\overline{X})})} \cdot P_i(\frac{1}{q^n T}).$$

Applying the Lefschetz fixed point formula to the identity morphism it follows that $E := \sum_{i=0}^{2n} (-1)^i B_i = (\Delta \cdot \Delta)$. Using the two formulas above it now follows that

$$\begin{aligned}
Z(X, \frac{1}{q^n T}) &= \frac{P_1(\frac{1}{q^n T}) \cdots P_{2n-1}(\frac{1}{q^n T})}{P_0(\frac{1}{q^n T}) \cdots P_{2n}(\frac{1}{q^n T})} \\
&= \frac{P_{2n-1}(T) \cdots P_1(T)}{P_{2n}(T) \cdots P_0(T)} \cdot \frac{(-1)^E q^{nE} T^E}{\prod_{i=1}^{2n} \det(F^*|_{H^i(\overline{X})})^{(-1)^i}} \\
&= \pm Z(X, T) \cdot \frac{q^{nE} T^E}{q^{nE/2}} \\
&= \pm q^{nE/2} T^E Z(X, T).
\end{aligned}$$

□

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