F.C. Claassens

Combination of Orlicz norms

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Scriptiebegeleider: dr. O. van Gaans

Universiteit Leiden

Mathematisch Instituut, Universiteit Leiden

1 Introduction

The dual space plays an important role in Linear Algebra. Now for $1 \leq p < \infty$, consider the vector space of p-summable sequences:

$$l^p = \left\{ x \in \mathbb{F}^{\mathbb{N}} : \|x\|_p = \left(\sum_{i=1}^{\infty} |x(i)|^p \right)^{\frac{1}{p}} < \infty \right\}, \ \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C},$$

equipped with the norm $\| \cdot \|_p$.
Its dual space turns out to be isometric isomorphic to l^q where $\frac{1}{p} + \frac{1}{q} = 1$.
Now we can create a norm by combining these so called p-norms, for example:

$$\left(\sum_{i=1}^{\infty} |x(2i-1)|^2 \right)^{\frac{1}{2}} + \left(\sum_{i=1}^{\infty} |x(2i)|^{\pi} \right)^{\frac{1}{\pi}}$$

or

$$\left(\sum_{i=1}^{\infty} \left(\left| |x(2i-1)|^2 + |x(2i)|^2 \right)^{\frac{1}{2}} \right|^3 \right)^{\frac{1}{3}}$$

or even more complex combinations.
Also, instead of taking a p-power, we can use a more general function $\varphi : [0, \infty) \to [0, \infty)$ where $\lim_{x \to \infty} \varphi(x)/x = \infty$ and $\lim_{x \downarrow 0} \varphi(x)/x = 0$, and use a measure space (X, Σ, μ) instead of sequences. The space created by the span of

$$\left\{ x \in \mathbb{F}^X : x \ \Sigma\text{-measurable and } \int \varphi(|x(i)|) d\mu < \infty \right\}$$

is called an Orlicz-space. And it can be given the Luxemburg-norm:

$$\|x\|_{\varphi} = \inf \left\{ u > 0 : \int \varphi(|\frac{x(i)}{u}|) d\mu \leq 1 \right\}$$

We will find a general formula for combinations of p-norms and find the corresponding dual space. Also we will find the dual space for Orlicz spaces and for combinations of Luxemburg-norms in the same way as for p-norms. For general terminology and results on Orlicz spaces, see [1]

2 p-norms

To begin I will state some definitions and theorems of which I assume the reader is familliar with. Therefore I will not prove any of these statements. Most proofs can be found in [2]

An infinite sequence is usually denoted by $x = (x_i)_{i \in \mathbb{N}}$, where $x_i \in \mathbb{F}$ and $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$. As Cantor showed us however, $\mathbb{F}^{\infty}$ is a very ambiguous notation since there is no indication wether ∞ is countable or not. Therefore we will denote the set of infinite sequence by $\mathbb{F}^{\mathbb{N}}$ the set of all maps from $\mathbb{N}$ to $\mathbb{F}$. An element $x \in \mathbb{F}$ will however still be denoted as a sequence, where $x(i) = x_i$.

Definition 2.1. Let $1 \leq p < \infty$ and $x \in \mathbb{F}^{\mathbb{N}}$. We call $\| \cdot \|_p$ given by $\|x\|_p = (\sum_{i \in \mathbb{N}} |x_i|^p)^{\frac{1}{p}}$ a p-norm, which is a norm on the space: $l^p := \{x \in \mathbb{F}^{\mathbb{N}} : \|x\|_p < \infty\}$.

Remark 2.2. We can also define a norm for $p = \infty$ with a corresponding space l^{∞} by taking $\|x\|_{\infty} := \max_{i \in \mathbb{N}} |x_i|$.

Proposition 2.3. *$(l^p, \| \cdot \|_p)$ is a Banach space.*

Definition 2.4. Let $p, q \in \mathbb{R}_{\geq 1} \cup \{\infty\}$. We call q the conjugate of p, or we call p and q conjugated to each other, if $\frac{1}{p} + \frac{1}{q} = 1$.
We call $\| \cdot \|_q$ the conjugate norm of $\| \cdot \|_p$.

Theorem 2.5. *(Hölder)*
Let $p, q \in \mathbb{R}_{\geq 1} \cup \{\infty\}$ conjugated to each other, let $x, y \in \mathbb{F}^{\mathbb{N}}$. Then we have:

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_p \|y\|_q.$$

Theorem 2.6. *(Young)*
Let $p, q \in \mathbb{R}_{\geq 1} \cup \{\infty\}$ conjugated to each other, let $x, y \in \mathbb{F}^{\mathbb{N}}$. Then we have:

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_p + \|y\|_q.$$

Definition 2.7. Let X be a normed space over $\mathbb{F}$ and $\|\cdot\|$ its norm. Let $f : X \to \mathbb{F}$ a linear function. f is called bounded if $\sup_{x \in X, \|x\| \leq 1} |f(x)| < \infty$.

Definition 2.8. Let X be a normed space and $\|\cdot\|$ its norm. $X' = B(X)$ is the dual space of X with the dual norm $\|\cdot\|'$ given by: $\|f\| = \sup_{x \in X, \|x\| \leq 1} |f(x)|$.

We will now prove that the dual of l^p is isometric isomrphic to l^q where p and q are conjugate, before proving the more general statement for combinations of p-norms.

Theorem 2.9. *Let $p \in \mathbb{R}_{\geq 1}$, and let q conjugated to p.*
The dual of l^p is isometric isomorphic to l^q.

Before we will prove this theorem, we will show that we can always construct a map from a dual space of infinite sequences to a space of dual sequences.

Lemma 2.10. *Let $X \subset \mathbb{F}^{\mathbb{N}}$ a normed space such that*

$$\{x \in X : \text{ the exists a } N \in \mathbb{N} \text{ such that } x_i = 0 \text{ for all } i \geq N\}$$

is dense in X. Then there exists an injective homomorphism $\phi : X' \to \mathbb{F}^{\mathbb{N}}$.

Proof. Let $x \in l^p$ given by $(x_i)_{i \in \mathbb{N}}$ and let $e_i = (e_{i,j})_{j \in \mathbb{N}} \in l^p$, for all $i \in \mathbb{N}$, given by $e_{i,j} = 0$ whenever $i \neq j$ and $e_{i,i} = 1$.
Note that the span of e_i is dense in X.
Let $f \in X'$. Note $f(x) = f(\sum_{i \in \mathbb{N}} x_i e_i) = \sum_{i \in \mathbb{N}} x_i f(e_i)$, because f is bounded and linear.
Hence we find a sequence $y_f \in \mathbb{F}^{\mathbb{N}}$ given by $y_{f,i} = f(e_i)$, such that $f(x) = \sum_{i \in \mathbb{N}} x_i y_{f,i}$.
We can now construct ϕ by $\phi : X' \to \mathbb{F}^{\mathbb{N}}, f \mapsto y_f$.

Now we will show that ϕ is a homomorphism.
Let $f, g \in (l^p)'$ and $\lambda \in \mathbb{F}$.
Then $\phi(f + g) = y_{f+g} = (y_{f+g,i})_{i \in \mathbb{N}} = ((f + g)(e_i))_{i \in \mathbb{N}} = (f(e_i) + g(e_i))_{i \in \mathbb{N}} = (f(e_i))_{i \in \mathbb{N}} + (g(e_i))_{i \in \mathbb{N}} = y_f + y_g = \phi(f) + \phi(g)$.
Also, $\phi(\lambda f) = (y_{\lambda f+g,i})_{i \in \mathbb{N}} = (\lambda f(e_i))_{i \in \mathbb{N}} = \lambda (f(e_i))_{i \in \mathbb{N}} = \lambda \phi(f)$.
So ϕ is a homomorphism.
Finally, if $\phi(f) = 0$ then $y_{f,i} = 0$ for all i, so $f(x) = 0$ for all $x \in X$. Hence ϕ is injective. □

Proof. of theorem 2.9.
Let ϕ as in lemma 2.10 where $X = l^p$. Consider $\varphi : (l^p)' \to \phi[(l^p)'], f \mapsto \phi(f)$, then φ is an isomorphism. It remains to prove that ϕ is isometric because $f \in (l^p)'$ if f is bounded, which is the same as $\|f\|' = \sup_{x \in X, \|x\| \leq 1} |f(x)| < \infty$, and if ϕ is isometric we have $\|f\|' = \|y_f\|_q < \infty$, which implies $y_f \in l^q$.
This gives us $\phi((l^p)') \subset l^q$. Since for each $i \in \mathbb{N}$ we have $f_i \in (l^p)'$ where $f_i(x) = x_i$, we have that $e_i \in \phi((l^p)')$ so l^q is the closure of span$(\{e_i\}_{i \in I})$ with respect to $\|\cdot\|_q$, which is a subset of $\phi(l^p)$ with respect to $\|\cdot\|_q$, so $l^q = \phi(l^p)$, which gives that φ is bijective.

To show that g is isometric, let $f \in (l^p)'$, consider:

$$\|f\|' = \sup_{x \in l^p, \|x\|_p \leq 1} |f(x)| = \sup_{x \in l^p, \|x\|_p \leq 1} \sum y_{f,i} x_i \overset{\text{Hölder}}{\leq} \sup_{x \in l^p, \|x\|_p \leq 1} \|y_f\|_q \cdot \|x\|_p = \|y_f\|_q.$$

We will now define an $x \in l^p$ for which $\sup_{z\in l^p, \|z\|_p \leq 1} |f(z)| = |f(x)|$. We do this by using something which looks a lot like the gradient of the q-norm,

$$x_i' = (\sum_{j\in\mathbb{N}} |y_{f,j}|^q)^{\frac{1}{q}-1} \overline{y_{f,i}} |y_{f,i}|^{q-2} \text{ or } x_i' = 0 \text{ if } y_{f,i} = 0$$

Since $(\sum_{j\in\mathbb{N}} |y_{f,j}|^q)^{\frac{1}{q}-1}$ is independent of i, we can as well take $x_i = \overline{y_{f,i}} |y_{f,i}|^{q-2}$.
Note that $\|\frac{x}{\|x\|_p}\|_p = 1$, so we find:

$$\begin{aligned}
\|y_f\|_q \cdot \|\frac{x}{\|x\|_p}\|_p &= \frac{1}{\|x\|_p} \|y_f\|_q \cdot \|x\|_p \\
&= \frac{1}{\|x\|_p} (\sum_{i\in\mathbb{N}} |y_{f,i}|^q)^{\frac{1}{q}} ((\sum_{i\in\mathbb{N}} |y_{f,i}|^{q-1})^p)^{\frac{1}{p}} \\
&= \frac{1}{\|x\|_p} (\sum_{i\in\mathbb{N}} |y_{f,i}|^q)^{\frac{1}{q}} (\sum_{i\in\mathbb{N}} |y_{f,i}|^{pq-p})^{\frac{1}{p}} \\
&= \frac{1}{\|x\|_p} (\sum_{i\in\mathbb{N}} |y_{f,i}|^q)^{\frac{1}{q}} (\sum_{i\in\mathbb{N}} |y_{f,i}^q|)^{\frac{1}{p}} \\
&= \frac{1}{\|x\|_p} \sum_{i\in\mathbb{N}} |y_{f,i}|^q \\
&= \frac{1}{\|x\|_p} \sum_{i\in\mathbb{N}} y_{f,i} \overline{y_{f,i}} |y_{f,i}|^{q-2} \\
&= \frac{1}{\|x\|_p} \sum y_{f,i} x_i \\
&\leq \sup_{x\in l^p, \|x\|_p \leq 1} \sum_{i\in\mathbb{N}} y_{f,i} x_i = \|f\|'
\end{aligned}$$

Above we use that p and q are conjugated, which gives that $pq - p = q$.
Hence we find that $\|f\|' = \|y_f\|_q$, so φ is an isometric isomorphism.

□

3 Combinations of p-norms

We will now take a look at combinations of p-norms. This can be naturally defined by considering a set of vector spaces $\mathbb{F}^{x_i},\ x_i \in \mathbb{N}$ endowed with a p_i-norm Now the combination of these vector spaces is given by $\oplus_{i\in I} \mathbb{F}^{x_i}$ with an a-norm, i.e. the norm $\|\cdot\|$ is given by:

$$\|y\| = (\sum_{i\in I} (\sum_{0\leq j\leq x_i} |y_{x_i,j}|^{p_i})^{\frac{a}{p_i}})^{\frac{1}{a}},$$

where $1 \leq a < \infty,\ 1 \leq p_i < \infty$.
More generally a combination of p-norms is given as follows.

Definition 3.1. Let $X_i \subset \mathbb{N},\ i \in I$ where $X_i \cap X_j = \emptyset$ for $i \neq j$ and $\bigcup_{i\in I} X_i = \mathbb{N}$.
Let $\{p_i\}_{i\in I} \subset \mathbb{R}_{\geq 1} \cup \{\infty\}$, $a \in \mathbb{R}_{\geq 1} \cup \{\infty\}$ and $x \in \mathbb{F}^{\mathbb{N}}$.
We call $\|\cdot\|$ given by $(\sum_{i\in I} (\sum_{j\in X_i} |x_j|^{p_i})^{\frac{a}{p_i}})^{\frac{1}{a}}$ a p-combination norm given by $\{p_i\}$ and a which belongs to the space: $l^{\|\cdot\|} := \{x \in \mathbb{F}^{\mathbb{N}} : \|x\| < \infty\}$.

As with the p-norms we want to know what the dual space of $l^{\|\cdot\|}$ looks like. Again we wil find that the dual space is isometric isomorphic to the space with a conjugated norm, where the conjugated is exactly what one would expect.

Theorem 3.2. *Let $\{X_i\}_{i\in I} \subset \mathcal{P}(\mathbb{N})$ such that for every $i, j \in I,\ i \neq j$ we have $X_i \cap X_j = \emptyset$ and that $\bigcup_{i\in I} X_i = \mathbb{N}$ and let $\{p_i\}_{i\in I} \subset \mathbb{R} \geq 1$, $\{q_i\}_{i\in I} \subset \mathbb{R} \geq 1$ the respective conjugates and let $a \geq 1$, with b the conjugate of a.*
Let $\|\cdot\|$ a p-combination norm given by $\{p_i\}_{i\in I}$ and a and $\|x\|^c$ a p-combination norm given by $\{q_i\}_{i\in I}$ and b.
Then $(l^{\|\cdot\|})'$ is isometric isomorphic to $l^{\|\cdot\|^c}$.

Proof. Let ϕ as in lemma 2.10 with $X = l^{\|\cdot\|}$. We define $\varphi : (l^{\|\cdot\|})' \to \phi[(l^{\|\cdot\|})'], f \mapsto \phi(f)$.
From lemma 2.11 we have that φ is an isomorphism. Hence we want that $\phi[(l^{\|\cdot\|})'] = l^{\|\cdot\|^c}$, which is entailed, as we have seen in theorem 2.10, by proving that φ is isometric.
Let $f \in (l^{\|\cdot\|})'$. Consider:

$$\begin{aligned}
\|f\|' &= \sup_{x\in l^{\|\cdot\|}, \|x\|\leq 1} |\sum_{i\in\mathbb{N}} y_{f,i}x_i| = \sup_{x\in l^{\|\cdot\|}, \|x\|\leq 1} |\sum_{i\in I}\sum_{j\in X_i} y_{f,j}x_j| \\
&\overset{\text{Hölder}}{\leq} \sup_{x\in l^{\|\cdot\|}, \|x\|\leq 1} \sum_{i\in I}((\sum_{j\in X_i} y_{f,j}^{q_i})^{\frac{1}{q_i}}(\sum_{j\in X_i} x_j^{p_i})^{\frac{1}{p_i}}) \\
&\overset{\text{Hölder}}{\leq} \sup_{x\in l^{\|\cdot\|}, \|x\|\leq 1} (\sum_{i\in I}(\sum_{j\in X_i} y_{f,j}^{q_i})^{\frac{b}{q_i}})^{\frac{1}{b}}(\sum_{i\in I}(\sum_{j\in X_i} x_j^{p_i})^{\frac{a}{p_i}})^{\frac{1}{a}} \\
&= \sup_{x\in l^{\|\cdot\|}, \|x\|\leq 1} \|y_f\|^c \cdot \|x\| = \|y_f\|^c.
\end{aligned}$$

Now we wil define an $x \in l^{\|\cdot\|}$ for which $\sup_{z\in l^{\|\cdot\|}, \|z\|\leq 1} |f(z)| = |f(x)|$. Again we will find a candidate by using a gradient.
Let $j \in \mathbb{N}$ and let $i \in I$ such that $j \in X_i$. Then define:

$$x'_j = \frac{\partial \|y_{f,j}\|^c}{\partial y_{f,j}} = (\sum_{n\in I}(\sum_{m\in X_n} |y_{f,m}|^{q_n})^{\frac{b}{q_n}})^{\frac{1}{b}-1} \cdot (\sum_{m\in X_i} |y_{f,m}|^{q_i})^{\frac{b}{q_i}-1} \cdot |y_{f,j}|^{q_i-1} \text{ or } 0 \text{ if } y_{f,j} = 0$$

and let $x_j = (\sum_{m\in X_i} |y_{f,m}|^{q_i})^{\frac{b}{q_i}-1} \cdot \overline{y_{f,j}}|y_{f,j}|^{q_i-2}$ or 0 if $y_{f,j} = 0$.
Now consider:

$$\begin{aligned}
\frac{1}{\|x\|}\|y_f\|^c\|x\| &= \frac{1}{\|x\|}\|y_f\|^c(\sum_{i\in I}(((\sum_{j\in X_i} |y_{f,j}|^{q_i})^{\frac{b}{q_i}-1})^{p_i}(\sum_{j\in X_i} |y_{f,j}|^{q_i-1})^{p_i})^{\frac{a}{p_i}})^{\frac{1}{a}} \\
&= \frac{1}{\|x\|}\|y_f\|^c(\sum_{i\in I}((\sum_{j\in X_i} |y_{f,j}|^{q_i})^{\frac{p_i b}{q_i}-p_i}(\sum_{j\in X_i} |y_{f,j}|^{q_i}))^{\frac{a}{p_i}})^{\frac{1}{a}} \\
&= \frac{1}{\|x\|}\|y_f\|^c(\sum_{i\in I}(\sum_{j\in X_i} |y_{f,j}|^{q_i})^{(\frac{p_i b}{q_i}-p_i+1)\frac{a}{p_i}})^{\frac{1}{a}} = (*).
\end{aligned}$$

Now we will take a closer look at the expression $(\frac{p_i b}{q_i} - p_i + 1)\frac{a}{p_i}$. First note that $1 - p_i = \frac{p_i}{q_i}$, so

$$(\frac{p_i b}{q_i} - p_i + 1)\frac{a}{p_i} = \frac{p_i}{q_i}(b-1)\frac{a}{p_i} = \frac{1}{q_i}(ab - a) = \frac{b}{q_i}.$$

So we find:

$$\begin{aligned}
(*) &= \frac{1}{\|x\|}\|y_f\|^c(\sum_{i\in I}(\sum_{j\in X_i} |y_{f,j}|^{q_i})^{\frac{b}{q_i}})^{\frac{1}{a}} \\
&= \frac{1}{\|x\|}(\sum_{i\in I}(\sum_{j\in X_i} |y_{f,j}|^{q_i})^{\frac{b}{q_i}})^{\frac{1}{b}}(\sum_{i\in I}(\sum_{j\in X_i} |y_{f,j}|^{q_i})^{\frac{b}{q_i}})^{\frac{1}{a}} \\
&= \frac{1}{\|x\|}(\sum_{i\in I}(\sum_{j\in X_i} |y_{f,j}|^{q_i})^{\frac{b}{q_i}}) \\
&= \sum_{i\in I}(\sum_{m\in X_i} |y_{f,m}|^{q_i})^{\frac{b}{q_i}-1} \cdot |y_{f,j}|^{q_i-2}\overline{y_{f,j}} \cdot y_{f,j} = \sum_{j\in\mathbb{N}} y_{f,j}x_j.
\end{aligned}$$

This gives us that φ is isometric, thus φ is an isometric isomorphism from $(l^{\|\cdot\|})'$ to $l^{\|\cdot\|^c}$.

□

4 Orlicz spaces

We want to generalize the result on dual spaces of sequence spaces with combinations of p-norms. A natural next step is to consider Orlicz spaces. What we do to define a p-norm is that we take the elements of the sequence, raise them to the pth power, and then take the inverse function over the sum. With Orlicz spaces we will make a similar construction, but than for more general functions then taking the pth powers.

Definition 4.1. Let X be a vector space over $\mathbb{F}$. $\rho : X \to [0, \infty]$ is called a convex modular if it satisfies:
(M1) $\rho(x) = 0 \Leftrightarrow x = 0$
(M2) $\rho(x) = \rho(\alpha x)$ for all $\alpha \in \mathbb{F}$, $|\alpha| = 1$.
(M3) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ for $\alpha, \beta \in \mathbb{R}_{\geq 0}$ with $\alpha + \beta = 1$.

Definition 4.2. Φ is the class of functions φ such that $\varphi : [0, \infty) \to [0, \infty)$ is strictly increasing, continuous and convex and such that $\varphi(0) = 0$ and

$$\lim_{u \to 0^+} \frac{\varphi(u)}{u} = 0 \text{ and } \lim_{u \to \infty} \frac{\varphi(u)}{u} = \infty$$

Definition 4.3. Let Ω be a non-empty set, Σ a σ-algebra of subsets of Ω and μ a nonnegative, complete measure in Σ, which does not vanish identically. If Ω is the union of a countable number of sets of finite measure, then we call (Ω, Σ, μ) a σ-finite complete measure space.

Definition 4.4. Let (Ω, Σ, μ) a σ-finite complete measure space.
Then $S(\Omega, \Sigma, \mu)$ is the space of all $\mathbb{F}$-valued, Σ-measurable and μ-a.s. finite functions on Ω, with equality μ-a.s.

Proposition 4.5. *Let (Ω, Σ, μ) a σ-finite complete measure space, $\varphi \in \Phi$, and let $X = S(\Omega, \Sigma, \mu)$. Then $\rho(x) = \rho_\varphi(x) = \int_\Omega \varphi(|x(t)|)d\mu$ is a convex modular in X.*

Proof. First note that since φ is strictly increasing and $\varphi(0) = 0$ clearly (M1) holds.
For (M2) we find:

$$\rho(x) = \rho_\varphi(x) = \int_\Omega \varphi(|x(t)|)d\mu = \int_\Omega \varphi(|\alpha x(t)|)d\mu = \rho(\alpha x), \ \alpha \in \mathbb{C}, |\alpha| = 1.$$

Finally to proof (M3), let $\alpha, \beta \in \mathbb{R}_{\geq 0}$ with $\alpha + \beta = 1$, consider:

$$\rho(\alpha x + \beta y) = \int_\Omega \varphi(|\alpha x(t) + \beta y(t)|) \leq \alpha \int_\Omega \varphi(|x(t)|) + \beta \int_\Omega \varphi(|y(t)|) = \alpha\rho(x) + \beta\rho(y).$$

□

Definition 4.6. The modular space X_ρ is called an Orlicz space. Instead of X_ρ we usually denote it as X_φ.

We have to remark that X_ρ as we have constructed it, is not the general definition of an Orlicz space, but only a subclass of the Orlicz spaces. For sake of convenience, when we call a space an Orlicz space, we refer to an Orlicz space from this specific subclass. Now we have our space, we want it to have a norm as well.

Proposition 4.7. *Let ρ a convex modular, then the Luxemburg norm:*

$$\|x\|_\rho := \inf\{u > 0 : \rho(\frac{x}{u}) \leq 1\}$$

and the Ameniya norm:

$$\|x\|_\rho^0 := \inf\{\frac{1}{t}(1 + \rho(tx)) : t > 0\}$$

are norms on X_ρ.

Proof. We will start with proving that $\|x\|_\rho = 0 \Leftrightarrow x = 0$ and $\|x\|_\rho^0 = 0 \Leftrightarrow x = 0$.
First note that from (M1) we have that if $x = 0$, then $\rho(\frac{x}{u}) = 0$ for all $u > 0$ hence

$$\|x\|_\rho = \inf\{u > 0 : \rho(\frac{x}{u}) \leq 1\} = 0$$

and

$$\|x\|_\rho^0 = \inf\{\frac{1}{t}(1 + \rho(tx)) : t > 0\} = 0$$

If $\|x\|_\rho = 0$ then there exist $u_n \downarrow 0$ such that $\rho(\frac{x}{u_n}) \leq 1$. Then (M3) gives us $\rho(x) = \rho(u_n \frac{x}{u_n}) \leq u_n \rho(\frac{x}{u_n}) \leq u_n$, so $\rho(x) = 0$, hence $x = 0$.
If $\|x\|_\rho = 0$ then there exist t_n such that $\frac{1}{t_n}(1 + \rho(t_n x)) = \frac{1}{t_n} + \frac{1}{t_n}\rho(t_n x) \downarrow 0$. So $\frac{1}{t_n} \to 0$, hence we find that $\rho(x) = \rho(t_n \frac{x}{t_n}) \leq \frac{1}{t_n}\rho(t_n x) \leq \frac{1}{t_n}(1 + \rho(t_n x))$.
So $\rho(x) = 0$, hence $x = 0$.
Let $a \in \mathbb{F}$, then

$$|a|\|x\|_\rho = \inf\{|a|{\cdot}u > 0 : \rho(\frac{x}{u}) \leq 1\} = \inf\{u > 0 : \rho(\frac{|a| \cdot x}{u}) \leq 1\} = \inf\{u > 0 : \rho(\frac{|a| \cdot x}{u}) \leq 1\} = \|a{\cdot}x\|_\rho.$$

And we also find

$$|a|{\cdot}\|x\|_\rho^0 = \inf\{\frac{|a|}{t}(1+\rho(tx)) : t > 0\} = \inf\{\frac{1}{t}(1+\rho(t|a|x)) : at > 0\} = \inf\{\frac{1}{t}(1+\rho(tax)) : t > 0\} = \|ax\|_\rho^0.$$

Finally we have to proof the triangle inequality, we will start with the Luxemburg norm.
Let $u = \|x\|_\rho$ and $v = \|y\|_\rho$. Then

$$\begin{aligned}\rho(\frac{x+y}{u+v}) &= \rho(\frac{u}{u+v}\frac{x}{u} + \frac{v}{u+v}\frac{y}{v}) \\ &\leq \frac{u}{u+v}\rho(\frac{x}{u}) + \frac{v}{u+v}\rho(\frac{y}{v}) \\ &\leq \frac{u}{u+v} + \frac{v}{u+v} = 1.\end{aligned}$$

Thus $\|x + y\|_\rho \leq u + v$. So $\|x + y\|_\rho \leq \|x\|_\rho + \|y\|_\rho$.

For the Ameniya norm let $\epsilon > 0$. Then for some $u, v > 0$ we have:
$u + u\rho(\frac{x}{u}) < \|x\|_\rho^0 + \frac{1}{2}\epsilon$ and
$v + v\rho(\frac{y}{v}) < \|y\|_\rho^0 + \frac{1}{2}\epsilon$. But then:

$$\rho(\frac{x+y}{u+v}) \leq \frac{u}{u+v}\rho(\frac{x}{u}) + \frac{v}{u+v}\rho(\frac{y}{v}) < \frac{\|x\|_\rho^0 + \|y\|_\rho^0 + \epsilon - u - v}{u+v}.$$

Hence

$$\|x + y\|_\rho^0 \leq (u + v)[1 + \rho(\frac{x+y}{u+v})] < \|x\|_\rho^0 + \|y\|_\rho^0.$$

□

Definition 4.8. Let X and ρ as in proposition 4.5. Then $X_{\|\cdot\|_\varphi} = \{x \in X : \|x\|_\varphi < \infty\}$ and $X_{\|\cdot\|_\varphi^0} = \{x \in X : \|x\|_\varphi^0 < \infty\}$ are normed Orlicz spaces.

Example 4.9. Because we want to use Orlicz spaces as a generalization of p-norms, we want that the l^p is an Orlicz space as well.
Let $\Omega = \mathbb{N}$, μ such that $\mu(\{n\}) = 1$, $p > 1$ and $\varphi_p : [0, \infty) \to [0, \infty), |x_n| \mapsto |x_n|^p$.
Notice that $\varphi \in \Phi$. So we have a Orlicz space X_ρ where:

$$\rho_\varphi(x) = \sum_{n \in \mathbb{N}} |x_n|^p.$$

Now consider the Luxemburg norm:

$$\|x\|_\varphi = \inf\{u > 0 : \rho(\frac{x}{u}) \leq 1\} = \inf\{u > 0 : \frac{1}{u^p} \sum_{n \in \mathbb{N}} |x_n|^p \leq 1\} = (\sum_{n \in \mathbb{N}} |x_n|^p)^{\frac{1}{p}} = \|x\|_p.$$

Remark 4.10. Each $\varphi \in \Phi$ has an μ-a.s. unique representation:
$\varphi(u) = \int_0^u p(s)ds$, where p is a nondecreasing, right-continuous function, i.e. p is the right derivative of φ.

Lemma 4.11. *Let $\varphi \in \Phi$ and p its representation, then $p(0) = 0$ and $\lim_{u \to \infty} p(u) = \infty$.*

Proof. Suppose p is bounded. We then find

$$\lim_{u \to \infty} \frac{\phi(u)}{u} = \lim_{u \to \infty} \frac{1}{u} \int_0^u p(s)ds \leq \lim_{u \to \infty} \frac{1}{u} \int_0^u \sup_{t \in \mathbb{R}_{\geq 0}} \{p(t)\}ds = \sup_{t \in \mathbb{R}_{\geq 0}} \{p(t)\} \neq \infty.$$

Since p is non-decreasing it follows that $\lim_{u \to \infty} p(u) = \infty$.
Note that for all $u > 0$ we have

$$p(u) \leq \frac{1}{u} \int_u^{2u} p(s)ds \leq \frac{1}{u} \int_0^{2u} p(s)ds = 2\frac{\phi(u)}{u}.$$

Hence $p(0) = 0$. □

Definition 4.12. Let $\varphi \in \Phi$ then $q(t) = \sup\{s : p(s) \leq t\}$ is the right inverse of p. We cal $\varphi^*(v) = \int_0^v q(t)dt$ the complementary function of φ.
Evidently $\varphi^* \in \Phi$.

The complemantary function will fulfill the same role in the Orlicz norms as the conjugate numbers in the p-norms. We will find that theorems such as Hölder and Young also hold with complementary functions, instead of p-norms.

Theorem 4.13. *(Young)*
Let $\varphi \in \Phi$ and φ^* be complementary to φ. Then

$$uv \leq \varphi(u) + \varphi^*(v), \;\; u, v \geq 0.$$

Moreover $uv = \varphi(u) + \varphi^*(v)$ if and only if $v = p(u)$ and $u = q(v)$.

Proof. Suppose $p(u) \leq v$. Then:

$$\varphi(u) + \varphi^*(v) = \int_0^u p(s)ds + \int_0^{p(u)} q(t)dt + \int_{p(u)}^v q(t)dt.$$

Note first that:

$$\int_{p(u)}^v q(t)dt \geq q(p(u))(v - p(u)) \geq u(v - p(u)) = uv - up(u)$$

and by integration by parts:

$$\int_0^{p(u)} q(t)dt = \int_0^u q(p(t))dp(t) = \int_0^u tdp(t) = up(u) - \int_0^u p(t)dt.$$

Hence we find:

$$\int_0^u p(s)ds + \int_0^{p(u)} q(t)dt + \int_{p(u)}^v q(t)dt \geq uv.$$

If $p(u) > v$ then $q(v) < u$ and we can use the same argument as above changing the roles of u, v and p, q. □

Up until now we didn't know whether φ is the complementary function of φ^*, but with the theorem of Young we can redefine φ^* and φ from where it follows directly.

Corollary 4.14.

$$\varphi^*(v) = \sup_{u\geq 0}[uv - \varphi(u)] \quad \textit{and} \quad \varphi(v) = \sup_{u\geq 0}[uv - \varphi^*(u)].$$

Corollary 4.15. $\varphi^{**} = \varphi$.

We will try to find the dual space of Orlicz spaces.

Lemma 4.16. *Let* $\varphi \in \Phi$, $X_1 = \{y \in X; \rho_\varphi(y) \leq 1\}$, $X_2 = \{y \in X; \|y\|_\varphi \leq 1\}$. *Then*

$$\|x\|^1_{\varphi^*} := \sup_{y\in X_1} |\int_\Omega x(t)y(t)d\mu| = \sup_{y\in X_2} |\int_\Omega x(t)y(t)d\mu| =: \|x\|^{\otimes}_{\varphi^*}.$$

Proof. Let $y \in X_\varphi$ and let $X_y := \operatorname{span}\{y\}$.
We want that the restricted function $\rho_\varphi|_{X_y} : X_y \to \mathbb{F}$ is continuous.
If $\lambda_n \to \lambda$ then $\varphi(|\lambda_n y(t)|) \to \varphi(|\lambda y(t)|)$ because φ is continuous.
Since $|\lambda_n||y(t)| \leq \sup_n |\lambda_n||y(t)| < \infty$ we have that $\varphi(|\lambda_n||y(t)) \leq \varphi(\sup_n |\lambda_n||y(t)|) < \infty$ for all n.
Then the dominated convergence theorem gives us:

$$\rho_\varphi(\lambda_n y) = \int_\Omega \varphi(|\lambda_n||y(t))d\mu \to \int_\Omega \varphi(|\lambda||y(t))d\mu = \int_\Omega \rho_\varphi(\lambda y).$$

So $\rho_\varphi|_{X_y}$ is continuous.
Note: if $u_n \downarrow 1$ then $\rho_\varphi(\frac{y}{u_n}) \to \rho_\varphi(y)$. From this follows that

$$\|y\|_\varphi = \inf\{u > 0 : \rho_\varphi(\frac{y}{u}) \leq 1\} \leq 1 \Leftrightarrow \rho_\varphi(y) \leq 1.$$

Hence $X_1 = X_2$. □

Theorem 4.17. *Let* $\varphi \in \Phi$, p *its representation. Then*

$$\|x\|^0_{\varphi^*} = \|x\|^1_{\varphi^*} = \sup_{y\in X_1} |\int_\Omega x(t)y(t)d\mu|$$

Proof. Let $x \in X$ be bounded, then $f(k) = \rho_\varphi(q(k|x|))$, $k \in [0, \infty)$ is continuous and $f(0) = 0$ and $f(k) \to \infty$ if $k \to \infty$, so there exists a $c > 0$ such that $f(c) = 1$.
Then $q(c|x|) \in X_1$, so

$$\|x\|^1_{\varphi^*} \geq \int_\Omega |x(t)|q(c|x(t)|)d\mu.$$

By Young inequality we find:

$$\begin{aligned}\|x\|^1_{\varphi^*} = \sup_{y\in X_1} |\int x(t)y(t)d\mu| &\leq \frac{1}{k} \sup_{y\in X_1} [\rho_{\varphi^*}(kx) + \rho_\varphi(y)] \\ &\leq \frac{1}{k}[\rho_{\varphi^*}(kx) + 1] = \frac{1}{k}[\rho_{\varphi^*}(kx) + \rho_\varphi(q(c|x|))].\end{aligned}$$

If we further assume that $k = c$ we find:

$$\frac{1}{k}[\rho_{\varphi^*}(kx) + \rho_\varphi(q(c|x|))] = \frac{1}{c}[\int_\Omega \varphi^*(c|x(t)|) + \varphi(q(c|x(t)|))d\mu]$$
$$= \frac{1}{c}[\int_\Omega \varphi^*(c|x(t)|) + c|x(t)|q(c|x(t)|) - \varphi^*(c|x(t)|)]$$
$$= \int_\Omega |x(t)|q(k|x(t)|)d\mu \le \|x\|^1_{\varphi^*}$$

so in particular we find: $\|x\|^1_{\varphi^*} \ge \frac{1}{c}[\rho_{\varphi^*}(cx) + 1] \ge \|x\|^0_{\varphi^*}$.
Since $\|x\|^0_{\varphi^*} = \inf\{\frac{1}{k}(1 + \rho(kx)) : k > 0\}$ we find that $\|x\|^1_{\varphi^*} \le \|x\|^0_{\varphi^*}$.

□

Corollary 4.18. *$X_{\|\cdot\|^0_{\varphi^*}}$ is isometrically isomorphic to the associate space*

$$X_{\|\cdot\|^\otimes_{\varphi^*}} := \{x \in X : \|x\|^\otimes_{\varphi^*} := \sup_{\|y\|_\varphi \le 1} |\int_\Omega x(t)y(t)d_\mu| < \infty\}$$

Definition 4.19. Let $\Omega = \mathbb{N}$, let μ be given by $\mu(\{n\}) = 1$, then the modular space X_ρ is called a sequence space.

The associate space is however not always isometrically isomorphic to the dual space, $X'_{\|\cdot\|_\varphi}$, of $X_{\|\cdot\|_\varphi}$, but only to a subset of it. With Lemma 2.10 it is easy to see that $X_{\|\cdot\|^\otimes_\varphi}$ is isometrically isomorphic to $X'_{\|\cdot\|_\varphi}$ if $X_{\|\cdot\|_\varphi}$ is a normed sequence space. It is even possible to prove that this is the case too, if $X_{\|\cdot\|_\varphi}$ is reflexive, but we will not prove that in this work.

Corollary 4.20. *Let $X_{\|\cdot\|_\varphi}$ an Orlicz sequence space. Then $(X_{\|\cdot\|_\varphi})'$ is isometrically isomorphic to $X_{\|\cdot\|^0_{\varphi^*}}$.*

Example 4.21. Now let us consider Example 4.9. Note that for every $u \ge 0$

$$\varphi_p(u) = \int_0^{|u|} pt^{p-1}dt.$$

So φ_p has the function $f : [0, \infty) \to [0, \infty), u \mapsto pu^{p-1}$ as representation.
Now the complementary function φ^*_p is given by it's representation g where g is given by $g(t) = \sup\{s : f(s) \le t\} = (\frac{t}{p})^{\frac{1}{p-1}}$, hence $\varphi^*_p(u) = \frac{p-1}{p} \cdot \frac{1}{p^{\frac{1}{p-1}}}|u|^{\frac{p}{p-1}} = \frac{p-1}{p^{\frac{p}{p-1}}}|u|^{\frac{p}{p-1}}$.
According to Corollary 4.20 we have:

$$\|x\|^1_{\varphi^*_p} = \|x\|^0_{\varphi^*_p} = \inf\{\frac{1}{t}(1 + \rho_{\varphi^*}(tx)) : t > 0\}$$
$$= \inf\{\frac{1}{t}(1 + \sum_{i\in\mathbb{N}} \frac{p-1}{p^{\frac{p}{p-1}}}(t|x_i|)^{\frac{p}{p-1}}) : t > 0\}$$
$$= \inf\{\frac{1}{t} + \frac{p-1}{p^{\frac{p}{p-1}}}t^{\frac{p}{p-1}-1}\sum_{i\in\mathbb{N}} |x_i|^{\frac{p}{p-1}} : t > 0\} = (*)$$

Let q be the conjugate number of p, then $p - 1 = \frac{p}{q}$. Consider:

$$h(t) = \frac{1}{t} + \frac{p-1}{p^{\frac{p}{p-1}}}t^{\frac{p}{p-1}-1}\sum_{i\in\mathbb{N}} |x_i|^{\frac{p}{p-1}} = \frac{1}{t} + \frac{p-1}{p^q}t^{q-1}\sum_{i\in\mathbb{N}} |x_i|^q.$$

We can find the infimum of the expression by looking at it's extreme values:

$$h'(t) = \frac{(p-1)(q-1)}{p^q}t^{q-2}(\sum_{i\in\mathbb{N}} |x_i|^q) - \frac{1}{t^2} = \frac{1}{p^q}t^{q-2}(\sum_{i\in\mathbb{N}} |x_i|^q) - \frac{1}{t^2} = 0 \Leftrightarrow t = p\left(\sum_{i\in\mathbb{N}} |x_i|^q\right)^{-\frac{1}{q}}.$$

Now that we have find the infimum, we only need to substitute it in $(*)$:

$$\begin{aligned}\|x\|^1_{\varphi^*_p} &= \frac{1}{p}(\sum_{i\in\mathbb{N}}|x_i|^q)^{\frac{1}{q}} + \frac{p-1}{p^q}\cdot p^{q-1}(\sum_{i\in\mathbb{N}}|x_i|^q)^{1-\frac{q-1}{q}} \\ &= \frac{1}{p}(\sum_{i\in\mathbb{N}}|x_i|^q)^{\frac{1}{q}} + \frac{p-1}{p}(\sum_{i\in\mathbb{N}}\frac{p-1}{p^{\frac{p-2}{p-1}}}|x_i|^q)^{1-\frac{1}{p}} \\ &= \frac{1}{p}(\sum_{i\in\mathbb{N}}|x_i|^q)^{\frac{1}{q}} + \frac{1}{q}(\sum_{i\in\mathbb{N}}\frac{p-1}{p^{\frac{p-2}{p-1}}}|x_i|^q)^{\frac{1}{q}} \\ &= (\sum_{i\in\mathbb{N}}\frac{p-1}{p^{\frac{p-2}{p-1}}}|x_i|^q)^{\frac{1}{q}} = \|x\|_q\end{aligned}$$

With Theorem 4.17 we can easily prove the Hölder inequality.

Corollary 4.22. *(Hölder)*
Let $X_{\|\cdot\|_\varphi}$ be an Orlicz space and let $x, y \in X$. Then

$$\int_\Omega x(t)y(t)d\mu \leq \|x\|_\varphi \|y\|^0_{\varphi^*}.$$

Proof. Let $k \in \mathbb{F}$ such that $\|\frac{x}{k}\|_\varphi = 1$ then $\|x\|_\varphi = k$. Then

$$\int_\Omega x(t)y(t)d\mu = k\int_\Omega \frac{x(t)}{k}y(t)d\mu \leq k\|y\|^1_{\varphi^*} = k\|y\|^0_{\varphi^*} = \|x\|_\varphi\|y\|^0_{\varphi^*}.$$

□

5 Combination of Orlicz norms

Though we have shown that all p-norms are Luxemburg-norms, it is not certain that combinations of p-norms are always Luxemburg-norms as well. In fact, it is a general belief that this is not the case, though this has not been proven. Therefore we wil introduce combinations of Luxemburg-norms and combinations of Ameniya-norms to capture the combinations of p-norms in our generalisation.
We will achieve this by splitting up Ω in measurable, pairwise disjoint subsets $(T_i)_{i\in I}$, and treat the index-set I for the subsets as a measurespace where $\{i\}$ has measure 1 for all $i \in I$. A $x : \Omega \to \mathbb{F}$, we then split up in $(x_{T_i})_i$ where we can give each x_{T_i} a different norm $\|\cdot\|_{\varphi_i}$.
Elements in the index sets then are as follows $x_T : I \to \mathbb{F}, i \mapsto \|x_{T_i}\|_{\varphi_i}$.

Definition 5.1. Let (Ω, Σ, μ) a σ-finite complete measure space and let $X = S(\Omega, \Sigma, \mu)$. Note that every measurable subset is a σ-finite complete measure space too.
Let (I, Σ_I, μ_I) a σ-finite complete measure space with for every $i \in I$ we have $\mu_I(i) = 1$ and $T = \{T_i\}_{i\in I}$ with $\emptyset \neq T_i \subset \Omega$, $T_i \cap T_j = \emptyset$, $\bigcup_{i\in I} T_i = \Omega$, $T_i \in \Sigma$ and let $Y = S(I, \Sigma_I, \mu_I)$.
Let $x \in X$ and $x_{T_i} := \{x(j)\}_{j\in T_i}$, let $x_T \in Y$ such that $x_T(i) = \|x_{T_i}\|_{\varphi_i}$ and $x^*_T \in Y$ such that $x^*_T(i) = \|x_{T_i}\|^0_{\varphi^*_i}$. Let $\{\varphi_i\}_{i\in I}$ and φ be elements of Φ.
A combination of modulars ρ_{φ,φ_I} is given by:

$$\rho_{\varphi,\varphi_I}(x) = \int_I \varphi(\|x_{T_i}\|_{\varphi_i})d\mu_I = \sum_{i\in I}\varphi(\|x_{T_i}\|_{\varphi_i}).$$

The complementary modular of a combination of Orlicz modulars is given by:

$$\rho_{(\varphi,\varphi_I)^*}(x) = \sum_{i\in I}\varphi^*(\|x_{T_i}\|^0_{\varphi^*_i}).$$

Definition 5.2. Let X, Y, T as in 5.1. Then $X_{\|\cdot\|_{\varphi,\varphi_I}} := \{x \in X : \|x\|_{\varphi,\varphi_I} < \infty\}$ and $X_{\|\cdot\|^0_{\varphi^*,\varphi_I^*}} := \{x \in X : \|x\|^0_{\varphi^*,\varphi_I^*} < \infty\}$ are normed Orlicz spaces, and we call $Y_{\|\cdot\|_{\varphi,\varphi_I}} = \{x_T \in Y : \|x_T\|_\varphi < \infty\}$ and $Y^0_{\varphi^*,\varphi_I^*} = \{x^*_T \in Y : \|x^*_T\|_{\varphi^*} < \infty\}$ their respective index spaces.

Now that we have described combinations of Luxemburg-norms, and their complementary norms, we naturally want that the Hölder inequality and the Young-inequality still hold.

Theorem 5.3. *(Combination Hölder)*
Let $X_{\|\cdot\|_{\varphi,\varphi_I}}$ be a normed Orlicz space and $Y_{\|\cdot\|_{\varphi,\varphi_I}}$ its index space, and let $x, y \in X$. Then

$$\int_\Omega x(t)y(t)d\mu \le \|x\|_{\varphi,\varphi_I}\|y\|^0_{(\varphi,\varphi_I)^*}$$

Proof. Consider:

$$\begin{aligned}
\int_\Omega x(t)y(t)d\mu &= \sum_{i\in I}\int_{T_i} x(t)y(t)d\mu \\
&\overset{\text{Hölder}}{\le} \sum_{i\in I}\|x_{T_i}\|_{\varphi_i}\|y_{T_i}\|^0_{\varphi_i^*} \\
&\overset{\text{Hölder}}{\le} \|x_T\|_\varphi\|y^*_T\|^0_{\varphi^*} \\
&= \|x\|_{\varphi,\varphi_I}\|y\|^0_{(\varphi,\varphi_I)^*}
\end{aligned}$$

□

Theorem 5.4. *(Combination Young)*
Let $X_{\|\cdot\|_{\varphi,\varphi_I}}$ be a normed Orlicz space and $Y_{\|\cdot\|_{\varphi,\varphi_I}}$ its index space, and let $x, y \in X$. Then

$$\int_\Omega x(t)y(t)d\mu \le \rho_{\varphi,\varphi_I}(x_T) + \rho_{(\varphi,\varphi_I)^*}(y_T).$$

Proof. Consider:

$$\begin{aligned}
\int_\Omega x(t)y(t)d\mu &= \sum_{i\in I}\int_{T_i} x(t)y(t)d\mu \\
&\overset{\text{Hölder}}{\le} \sum_{i\in I}\|x_{T_i}\|_{\varphi_i}\|y_{T_i}\|^0_{\varphi_i^*} \\
&\overset{\text{Young}}{\le} \sum_{i\in I}\varphi(\|x_{T_i}\|_{\varphi_i}) + \varphi^*(\|y_{T_i}\|^0_{\varphi_i^*}) \\
&= \rho_\varphi(x_T) + \rho_{\varphi^*}(y^*_T) \\
&= \rho_{\varphi,\varphi_I}(x_T) + \rho_{(\varphi,\varphi_I)^*}(y_T).
\end{aligned}$$

□

Theorem 5.5. *Let $X_{\|\cdot\|_{\varphi,\varphi_I}}$ be a normed Orlicz space and $Y_{\|\cdot\|_{\varphi,\varphi_I}}$ its index space. Then:*

$$\|x\|^0_{\varphi^*,\varphi_I^*} = \|x\|^\otimes_{\varphi^*,\varphi_I^*} = \sup_{\|y\|_{\varphi,\varphi_I}\le 1} |\int_\Omega x(t)y(t)d\mu|.$$

Proof. Let $c_T \in Y$ such that $\|c_{T_i}\|_{\varphi_i} = 1$ for all $i \in I$.
Then

$$\begin{aligned} \|y\|_{\varphi,\varphi_I} = \|y_T\|_\varphi &= \sum_{i \in I} \varphi(\|y_{T_i}\|_{\varphi_i}) \\ &= \sum_{i\in I} \varphi(\|y_{T_i}\|_{\varphi_i} \|c_{T_i}\|_{\varphi_i}) \\ &= \sum_{i \in I} \varphi(\| \|y_{T_i}\|_{\varphi_i} c_{T_i}\|_{\varphi_i}) \end{aligned}$$

Hence:

$$\begin{aligned} \|x\|^{\otimes}_{\varphi^*,\varphi_I^*} &= \sup_{\|y\|_{\varphi,\varphi_I} \le 1} \int_\Omega x(t)y(t)d\mu \\ &= \sup_{\|y\|_{\varphi,\varphi_I} \le 1} \sum_{i\in I} \sup_{d:T_i \to \mathbb{F}, \|d\|_{\varphi_i}=1} \int_{T_i} \|y_{T_i}\|_{\varphi_i} d(t)x(t)d\mu \\ &= \sup_{\|y\|_{\varphi,\varphi_I} \le 1} \sum_{i\in I} \|y_{T_i}\|_{\varphi_i} \sup_{d:T_i \to \mathbb{F}, \|d\|_{\varphi_i}=1} \int_{T_i} d(t)x(t)d\mu \\ &= \sup_{\|y\|_{\varphi,\varphi_I} = 1} \sum_{i\in I} \|y_{T_i}\|_{\varphi_i} \|x_{T_i}\|^{\otimes}_{\varphi_i^*} \end{aligned}$$

With Theorem 4.17 and Lemma 4.16 we now find that $\|x_{T_i}\|^{\otimes}_{\varphi_i^*} = \|x_{T_i}\|^0_{\varphi_i^*}$.
Hence we find:

$$\begin{aligned} \|x\|^{\otimes}_{\varphi^*,\varphi_I^*} &= \sup_{\|y\|_{\varphi,\varphi_I} \le 1} \sum_{i \in I} \|y_{T_i}\|_{\varphi_i} \|x_{T_i}\|^0_{\varphi_i^*} \\ &= \sup_{\|y_T\|_\varphi \le 1} \sum_{i\in I} \|y_{T_i}\|_{\varphi_i} \|x_{T_i}\|^0_{\varphi_i^*} \\ &= \|x_T^*\|^{\otimes}_{\varphi^*} \end{aligned}$$

By using Theorem 4.17 again we find $\|x_T^*\|^{\otimes}_{\varphi^*} = \|x_T^*\|^0_{\varphi^*}$.
Hence we find:

$$\|x\|^{\otimes}_{\varphi^*,\varphi_I^*} = \|x_T^*\|^{\otimes}_{\varphi^*} = \|x_T^*\|^0_{\varphi^*} = \|x\|^0_{\varphi^*,\varphi_I^*}$$

□

Corollary 5.6. *Let $X_{\|\cdot\|_{\varphi,\varphi_I}}$ an Orlicz sequence space. Then $(X_{\|\cdot\|_{\varphi,\varphi_I}})'$ is isometrically isomorphic to $X_{\|\cdot\|^0_{\varphi^*,\varphi_I^*}}$.*

6 References

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