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Topological Dimensions and the Löwenheim-Skolem Theorem

Bachelor thesis

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Contents

Introduction

We give several definitions for dimensions, i.e., ways to assign an integer to a topological space, that aim to formalize the intuitive ideas we have of dimension in Euclidean spaces. These definitions are topological invariants: they assign the same integer to spaces that are homeomorphic. We prove two inequalities concerning the covering dimension (dim), the large inductive dimension (Ind), and the Dimensionsgrad (Dg) for compact Hausdorff spaces X, namely $\dim X \leqslant \operatorname{Ind} X$ and $\dim X \leqslant \operatorname{Dg} X$. The proofs in this thesis were given by Hart ([7], 2005), and differ from the original proofs given by Vedenissoff ([12], 1939) and Fedorchuk ([4], 2003) respectively, in that they make use of the model theoretic notion of elementarity and the Löwenheim-Skolem theorem to reduce the case of compact Hausdorff spaces to that of compact metrizable spaces.

Historical Notes

Mathematicians and philosophers have long contented themselves with an intuitive idea of dimension. For example, Aristotle described the dimension of a magnitude as the number of directions in which it extends ([10], p. 2). With the invention of the Cartesian coordinate system, another common way to describe the dimension of an object was the minimal number of coordinates one needs to identify a point of the object. This attitude towards dimension changed with the development of modern mathematics, the advances made in analysis in the 18th century and the birth of set theory in the 19th century, which enlarged the domain of what could be considered geometrical objects and what could be considered continuous functions. Two counterintuitive results that led to a need for more formalization are the following.

In 1877 Cantor constructed a bijective map from the unit n-cube $[0,1]^n$, where n is a positive integer, to the unit interval $[0,1]$. This showed that one can uniquely determine a point in an n-cube with only one coordinate! The second troubling result was Peano's space-filling curve in 1890. This map from the unit interval onto the unit square was not only surjective, but in contrast to Cantor's map, also continuous, though not injective.

What was at stake here was the topological distinction between $\mathbb{R}^n$ and $\mathbb{R}^m$, whenever $n \neq m$. Brouwer took this issue to heart and proved in ([1], 1913) that the various $\mathbb{R}^n$ differed on the basis of a topological property, namely his *Dimensionsgrad* definition of dimension. In the following years more definitions were formulated that proved the same result. Having secured the topological distinctness of the various $\mathbb{R}^n$, mathematicians turned to (i) determining what the largest class of spaces was in which these different definitions coincide and (ii) the question how they differed in classes of spaces in which they did not necessarily coincide. It is in this last category that the theorems of this thesis fall.

1 Definitions for Topological Dimensions

In this section X will be a normal topological space. A topological space X is called *normal* if for all closed and disjoint subsets F and G of X there exist open subsets U and V of X such that $F \subseteq U$, $G \subseteq V$ and $U \cap V = \emptyset$. Throughout the first three sections we will translate definitions in terms of open sets to terms of closed sets (via complements of open sets). This will prove useful for our proofs in the last section. In this vein normality can be defined as: X is normal if for all closed and disjoint subsets F and G of X there exist closed subsets A and B of X such that $F \cap A = \emptyset$, $G \cap B = \emptyset$ and $A \cup B = X$ (take $A = U^{\mathsf{c}}$ and $B = V^{\mathsf{c}}$).

Moreover, we will assume that X satisfies the separation axiom T_1, which says that for every pair of distinct points each has a neighborhood not containing the other. This is equivalent to saying that $\{x\}$ is a closed set in X for all $x \in X$. Lastly, we will use the result that a compact and Hausdorff space is normal.

The first definition we give is the large inductive dimension, which uses the intuition found in Poincaré's work ([11], 1912), that to partition an n-dimensional object, n being an integer $1 \leqslant n \leqslant 3$, in general an object of dimension at least $n-1$ is needed. Thus a line can be cut into two parts by taking away a point; a plane by taking away a line, but not by taking away only a point; and a cube by taking away a plane, but not by taking away only a line.

1.1 The Large Inductive Dimension (Ind)

Definition 1.1. The *large inductive dimension of* X, denoted by $\operatorname{Ind} X$, is an integer or "the infinite number" ∞, and is defined inductively as follows. For a nonnegative integer n we say:

1. $\operatorname{Ind} X = -1$ if and only if $X = \emptyset$.
2. $\operatorname{Ind} X \leqslant n$ if for all closed subsets $F \subseteq X$ and all open neighborhoods $V \subseteq X$ of F there is an open subset $U \subseteq X$ such that $F \subseteq U \subseteq V$ and $\operatorname{Ind} \partial U \leqslant n-1$.
3. $\operatorname{Ind} X = n$ if $\operatorname{Ind} X \leqslant n$ and $\operatorname{Ind} X \nleqslant n-1$.
4. $\operatorname{Ind} X = \infty$ if for all $k \in \mathbb{Z}_{\geqslant -1}$ we have $\operatorname{Ind} X \nleqslant k$.

This version of the large inductive dimension uses the boundary of some sufficiently 'nice' open neighborhood 'close around' F to capture the intuition of a space of a smaller dimension separating F from closed sets in $\overline{U}^{\mathsf{c}}$.

Remarks.

1. From normality follows the stronger condition that there exists a U as in (2) such that $F \subseteq \overline{U} \subseteq V$. To see this, note that F and V^{c} are closed and disjoint, so there are disjoint open neighborhoods U_F and $U_{V^{\mathsf{c}}}$ of F and V^{c} respectively. Since $\overline{U}_F$ is the smallest closed set containing U_F, it

follows from $U_F \subseteq X \setminus U_{V^c}$ that $\overline{U}_F \subseteq X \setminus U_{V^c} \subseteq V$. Now choose U as in (2) for the neigborhood U_F of F.

2. The large inductive dimension is a topological invariant. This shouldn't come as too much of a surprise, since definition 1.1 hinges on topological notions such as 'open', 'closed', 'neighborhood' and 'boundary'. The same is true for the other definitions for dimension we will discuss.

3. As the name of our first definition already suggests, there is also a small inductive dimension (ind). In this definition one demands less in step (2), namely that there be an open neighborhood U with the required boundary for every *point* in X, not for every closed set. Demanding this allows for more candidate boundaries and in some cases pushes down the value of the small inductive dimension. (For an example see the example given by Dowker in ([3], 6.2.20).)

Example. For the unit interval we have $\operatorname{Ind}[0,1] = 1$. The unit interval is normal, since it is compact and Hausdorff. Let $F \subseteq [0,1]$ be a closed subset, and U an open set such that $F \subseteq U$. We can write $U = \bigcup \mathcal{U}$ for some family of open intervals $\mathcal{U}$. Since F is a compact subspace and $\mathcal{U}$ is an open cover of F, there is a finite subcover $\mathcal{V} \subseteq \mathcal{U}$ such that $F \subseteq \bigcup \mathcal{V}$. Define $V = \bigcup \mathcal{V}$. To prove that $\operatorname{Ind}[0,1] \leqslant 1$ it suffices to prove that $\operatorname{Ind} \partial V \leqslant 0$.
Let $p \in \partial V$ and W be an open neigborhood of p. Number the elements in $\partial W \setminus \{p\}$ as $x_1, x_2, \ldots, x_k$ and let $r > 0$ be a real number such that it is unequal to $d(p, x_i)$ for $i = 1, 2, \ldots, k$, and such that $B(p, r)$, the open interval around p with radius r, is a subset of W. Then we get that $\partial B(p, r) = \emptyset$ in ∂V, so $\operatorname{Ind} \partial B(p, r) = -1$. We can do this for each point in a closed set $G \subseteq \partial V$ and take for all open $W' \subseteq \partial V$ with $G \subseteq W'$ the union of such open balls that are subsets of W' to be the set with empty boundary. Hence $\operatorname{Ind} \partial V = 0$ and thereby $\operatorname{Ind}[0,1] \leqslant 1$. To see that $\operatorname{Ind}[0,1] \nleqslant 0$, note that because $[0,1]$ is connected the only open subsets that have empty boundaries are $\emptyset$ and $[0,1]$.

We now turn to another version of the definition for the large inductive dimension. This one stands closer to the original intuition as formulated by Poincaré.

Definition 1.2. A *partition* between two closed and disjoint sets A and B is a (closed) subset $L \subseteq X$ such that there exist open $U, V \subseteq X$ with $A \subseteq U$, $B \subseteq V$, $U \cap V = \emptyset$ and $L^c = U \cup V$.

Remark. The existence of U and V as in the definition above is equivalent to the existence of closed subsets F and G of X such that $F \cap A = \emptyset$, $G \cap B = \emptyset$, $F \cup G = X$ and $F \cap G = L$. Here too, one can take $F = U^c$ and $G = V^c$.

Proposition 1.3. *Given (1), (3) and (4) of definition 1.1, we get that (2) of the same definition is equivalent to the following statement.*

2. $\operatorname{Ind} X \leqslant n$ *if for all* $A, B \subseteq X$ *closed and disjoint there is a partition* L *between* A *and* B *with* $\operatorname{Ind} L \leqslant n-1$.

Proof. We write $\operatorname{Ind}_1 X$ for the large inductive dimension according to definition 1.1 and $\operatorname{Ind}_2 X$ for the large inductive dimension according to proposition 1.3.

We prove by induction on n that $\operatorname{Ind}_1 X \leqslant n \iff \operatorname{Ind}_2 X \leqslant n$ for all $n \in \mathbb{Z}_{\geqslant -1}$. For $n = -1$ we have

$$\operatorname{Ind}_1 X = -1 \iff X = \emptyset \iff \operatorname{Ind}_2 X = -1.$$

Assume that $\operatorname{Ind}_1 X \leqslant n$ for a nonnegative integer n. Let A and B be closed and disjoint subsets of X. Then B^{c} is an open neighborhood of A, and by assumption there is an open neighborhood U of A such that $A \subset \overline{U} \subseteq B^{\mathsf{c}}$ and $\operatorname{Ind}_1 \partial U \leqslant n-1$. By the induction hypothesis we have $\operatorname{Ind}_2 \partial U \leqslant n-1$. We can now take ∂U as our partition, since ∂U is closed, $A \subseteq U$ and $B \subseteq \overline{U}^{\mathsf{c}}$, $(\partial U)^{\mathsf{c}} = U \cup \overline{U}^{\mathsf{c}}$ and $U \cap \overline{U}^{\mathsf{c}} = \emptyset$.
Conversely, suppose $\operatorname{Ind}_2 X \leqslant n$. Let $F \subseteq X$ be closed and $V \subseteq X$ be an open neighborhood of F. Note that V^{c} is closed and that F and V^{c} are disjoint, so by assumption there is a partition L between F and V^{c} with $\operatorname{Ind}_2 L \leqslant n-1$. This means there are open U_1 and U_2 such that $F \subseteq U_1$, $V^{\mathsf{c}} \subseteq U_2$, $U_1 \cap U_2 = \emptyset$ and $L^{\mathsf{c}} = U_1 \cup U_2$.
We first prove that $\partial U_1 \subseteq L$. For $x \in \partial U_1$ we have $x \notin L^{\mathsf{c}} = U_1 \cup U_2$, since x can be an element of neither U_1 nor U_2. For in the case that $x \in U_1$ we would be able to find an open neighborhood of x which does not meet U_1^{c}. Similarly, if $x \in U_2$, we would be able to find an open neighborhood of x which is a subset of U_2, and hence does not meet U_1. We conclude that $\partial U_1 \subseteq L$, and by the induction hypothesis and theorem 7.1.3. of [3], which says that for a closed subset M of a normal space X we have $\operatorname{Ind}_1 M \leqslant \operatorname{Ind}_1 X$, it follows that

$$\operatorname{Ind}_1 \partial U_1 \underset{7.1.3}{\leqslant} \operatorname{Ind}_1 L \underset{\text{IH}}{=} \operatorname{Ind}_2 L \leqslant n-1,$$

and hence $\operatorname{Ind}_1 X \leq n$. Hereby we have also proven the case that $\operatorname{Ind}_1 X = \operatorname{Ind}_2 X = \infty$, for by the previous we have $\operatorname{Ind}_1 X \nleqslant n \iff \operatorname{Ind}_2 X \nleqslant n$. □

1.2 Brouwer's Dimensionsgrad (Dg)

The Dimensionsgrad goes back to the same intuition as the definition for the large inductive dimension, but differs from it in that it makes use of a cut instead of a partition.

Definition 1.4. A closed subset $C \subseteq X$ is called a *cut* between two closed and disjoint sets A and B if for all continua K (compact and connected spaces) in X that meet both A and B, we have $C \cap K \neq \emptyset$.

Definition 1.5. The *Dimensionsgrad of* X, denoted by $\operatorname{Dg} X$, is an integer or "the infinite number" ∞, and is defined inductively as follows. For a nonnegative integer n we say:

1. $\operatorname{Dg} X = -1$ if and only if $X = \emptyset$.

2. $\operatorname{Dg} X \leqslant n$ if for all $A, B \subseteq X$ closed and disjoint there is a cut C between A and B with $\operatorname{Dg} C \leqslant n-1$.

3. $\operatorname{Dg} X = n$ if $\operatorname{Dg} X \leqslant n$ and $\operatorname{Dg} X \nleqslant n-1$.

4. $\operatorname{Dg} X = \infty$ if for all $k \in \mathbb{Z}_{\geqslant -1}$ we have $\operatorname{Dg} X \nleqslant k$.

Proposition 1.6. *Let A and B be disjoint closed subsets of X. Every partition between A and B is a cut between A and B.*

Proof. Let L be a partition between A and B. Then there exist open $U, V \subseteq X$ such that $A \subseteq U$, $B \subseteq V$, $U \cap V = \emptyset$ and $L^{\mathsf{c}} = U \cup V$. Suppose that K is a continuum that meets both A and B, but does not meet L. Then K must be a subset of $L^{\mathsf{c}} = U \cup V$. Since K is connected and U and V are non-empty disjoint open sets, it follows that $K \subseteq U$ or $K \subseteq V$. However, K cannot be a subset of U, since it meets $B \subseteq V$, and similarly, K cannot be a subset of V, because it meets $A \subseteq U$. □

However, not every cut between disjoint closed sets A and B is a partition between A and B. In looking for an example of a cut that is not a partition, it is convenient to look for a space that is not locally-connected, because of the theorem that states that for a connected, locally connected and locally compact space there is a continuum for every pair of points that contains both points. As we will prove below, this has the consequence that every cut in the space is a partition.

Lemma 1.7. *Let Y be a connected, locally connected and locally compact space, and let A and B be two disjoint closed sets in Y. Then every cut between A and B is a partition between A and B.*

Proof. Let C be a cut between A and B. Define U as the set of all $y \in Y$ for which there exists a continuum K such that $y \in K$, $K \cap A \neq \emptyset$ and $K \cap C = \emptyset$. We claim that U is open. For $y \in K$ we have that there is a neigborhood $O \subseteq Y \setminus C$ such that O is connected and its closure is compact and disjoint from C. Consequently, $K \cup \overline{O}$ is again a continuum disjoint from C, so $O \subseteq U$. Similarly we can define an open set V of points that are in a continuum that meets B, but not C. We have that $U \cap V = \emptyset$, since every continuum that meets both A and B must meet C. Lastly, define $W = Y \setminus (U \cup V \cup C)$. This set is open, since for $x \in W$ we have that $Y \setminus C$ is an open neighborhood, so by locally compactness and locally connectedness there is an open neigborhood O_x of x such that $\overline{O}_x$ is a continuum that is a subset of $Y \setminus C$. This continuum does not meet U or V, otherwise there would exist a continuum that connects x with A or B. We now have that $A \subseteq U$, $B \subseteq V \cup W$, $U \cap (V \cup W) = \emptyset$, and $C^{\mathsf{c}} = U \cup (V \cup W)$, so C is a partition between A and B. □

Example. Consider the so-called *topologist's sine curve*, but with x ranging over part of the negative x-axis as well, defined as

$$T := \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \; : \; x \in [-1, 0) \cup (0, 1] \right\} \cup \{ \; (0, y) \; : \; y \in [-1, 1] \; \},$$

and let the topology on T be the one induced by the Euclidean plane. The origin $C = \{(0,0)\}$ is a cut between the closed sets $\{(-1,0)\}$ and $\{(1,0)\}$, because every continuum between these two points must meet C. However, C is not a partition between these points, since $T \setminus C$ cannot be written as the union of two disjoint open sets each containing one of the points: a part of the y-axis is also a subset of $T \setminus C$.

Corollary 1.8. *For a normal space X we have* $\operatorname{Dg} X \leqslant \operatorname{Ind} X$.

Proof. We prove by induction on n that $\operatorname{Ind} X \leqslant n \implies \operatorname{Dg} X \leqslant n$ for all $n \in \mathbb{Z}_{\geqslant -1}$. The cases that $\operatorname{Ind} X = -1$ and $\operatorname{Ind} X = \infty$ are trivial. Suppose that $\operatorname{Ind} X \leqslant n$ holds for a nonnegative integer n. Let A and B be any two disjoint closed subsets of X. Then there is a partition, and hence a cut, L between A and B with $\operatorname{Ind} L \leqslant n - 1$. By the induction hypothesis we have $\operatorname{Dg} L \leqslant n - 1$, so the inequality $\operatorname{Dg} X \leqslant n$ is established. □

1.3 The Covering Dimension (dim)

The covering dimension comes from the following observation made by Lebesgue in 1911. To cover an open interval I of real numbers with arbitrarily small open intervals, one can do that in such a way that each point of I is in *at most* two sets of the cover, namely by positioning the covering intervals next to each other, allowing them to only overlap around their boundary points. Moreover, if the open covering intervals are sufficiently small, smaller than the interval to be covered, there is in fact a point of I in *at least* two sets of the cover.

Similarly, to cover an open square I^2 with arbitrarily small open squares, one can do that in such a way that each point of I^2 is in *at most* three sets of the cover, namely by positioning the covering squares as bricks in a wall. And if the covering squares are small enough, there will in fact be a point in I^2 in *at least* three sets of the cover. This is the idea behind Lebesgue's more general 'Pflastersatz', or 'tiling theorem', which roughly states that (i) for any $\epsilon > 0$ an n-dimensional cube in $\mathbb{R}^n$ has a finite ϵ-cover (a cover of n-cubes with diameter at most ϵ), such that every point of the cube will be in *at most* $n+1$ elements of the cover, and that (ii) there exists an $\epsilon_0 > 0$ such that for all finite ϵ_0-covers of the n-dimensional cube, there is a point in the cube that is in *at least* $n+1$ of the elements of the cover.

This theorem in turn gave rise to our last definition for dimension. Before we can give it, we will need some terminology.

Definition 1.9. The *order* of a cover of a space X is the largest number n, if it exists, such that there is a point of X contained in n elements of the given cover. If such a number does not exist, we say that the order is "the infinite number" ∞.

So every $n+1$ element subfamily of a cover of order n has an empty intersection.

Definition 1.10. A *refinement* of a cover $\{A_i\}_{i \in I}$ of X is a cover $\{B_j\}_{j \in J}$ of X such that for all $j \in J$ there exists an $i \in I$ such that $B_j \subseteq A_i$.

Definition 1.11. The *covering dimension of* X, denoted by $\dim X$, is an integer or "the infinite number" ∞, and is defined as follows. For $n \in \mathbb{Z}_{\geqslant -1}$ we say:

1. $\dim X \leqslant n$ if every finite open cover has a refinement of order at most $n+1$.

2. $\dim X = n$ if $\dim X \leqslant n$ and $\dim X \nleqslant n-1$.

3. $\dim X = \infty$ if for all $k \in \mathbb{Z}_{\geqslant -1}$ we have $\dim X \nleqslant k$.

From the definition we directly get that the equality $\dim X = -1$ holds if and only if X is the empty set. We now define a special type of refinement.

Definition 1.12. A *shrinking* of a cover $\{A_i\}_{i\in I}$ of X is a cover $\{B_i\}_{i\in I}$ of X such that for all $i \in I$ we have $B_i \subseteq A_i$.

It is easy to see that the order of the shrinking $\{B_i\}_{i\in I}$ is less or equal than the order of $\{A_i\}_{i\in I}$. There is a handy characterization of the covering dimension, which we will use in our proofs. Instead of having to check all finite open covers of X, we only need to check those with $n+2$ elements to determine if $\dim X \leqslant n$.

Theorem 1.13 (Hemmingsen's characterization)**.** *The inequality* $\dim X \leqslant n$ *holds if and only if every* $n+2$*-element open cover of* X *has a shrinking with an empty intersection.*

In terms of closed sets of X Hemmingsen's characterization says that $\dim X \leqslant n$ if and only if for all closed $X_1, X_2, \ldots, X_{n+2} \subseteq X$ such that $X_1 \cap X_2 \cap \cdots \cap X_{n+2} = \emptyset$ (so the complements of the X_i form a cover), there are closed $Y_1, Y_2, \ldots, Y_{n+2} \subseteq X$ such that $Y_1 \cap Y_2 \cap \cdots \cap Y_{n+2} = \emptyset$, $X_i \subseteq Y_i$ for all $1 \leqslant i \leqslant n+2$ (the complements of the Y_i form a shrinking), and $Y_1 \cup Y_2 \cup \cdots \cup Y_{n+2} = X$ (the complements of the Y_i have an empty intersection). A proof for theorem 1.13 can be found in ([3], 7.2.13). We conclude this section with a theorem that states in what spaces our dimension formulas coincide.

Theorem 1.14 (The Urysohn identity)**.** *For all separable metrizable spaces* X*, that is, for all normal spaces* X *with a countable base, we have the identities*

$$\dim X = \operatorname{Ind} X = \operatorname{ind} X.$$

2 Elementarity

2.1 A Little Model Theory

The Löwenheim-Skolem theorem is one of the early fundamental results of model theory. Model theory is the branch of mathematical logic that studies classes of mathematical structures (such as groups, fields, partially ordered sets and graphs) by associating some formal language to those structures. This allows for an exploitation of the connections between on the one hand properties of (sets of) sentences of a certain formal language, and on the other hand the mathematical structures satisfying these sentences.

In this section we will give the basic model theoretic definitions needed to formulate the Löwenheim-Skolem theorem, in particular those of 'language', 'structure' and 'elementary substructure'.

Loosely speaking, a language is recursively defined to be the set consisting of all sentences formed according to certain rules from symbols specific to the language, together with fixed logical symbols. A *signature* consists of

- symbols for constants;
- relation or predicate symbols;
- function or operation symbols.

More formally, a *signature* σ is a triple $(S_{\text{rel}}, S_{\text{func}}, \text{ar})$, where S_{rel} and S_{func} are disjoint sets not containing any logical connectives, called the relation and function symbols respectively, and where ar is a function $S_{\text{rel}} \cup S_{\text{func}} \to \mathbb{Z}_{\geqslant 0}$ specifying the arity of a symbol. The *arity* of a symbol is the number of terms (such as variables) the symbol can take as arguments. Here constants are treated as nullary function symbols. The signature will play the role of the specific symbols of a language.

In addition to the specific symbols, every language comprises:

- countably infinite many variables '$x_0, x_1, x_2, ...$';
- logical connectives '$\vee$', '$\wedge$', '$\rightarrow$', '$\neg$', and the equality sign '$=$' ;
- the universal '$\forall$' and existential '$\exists$' quantifier;
- the seperation symbols '( )', and occasionally '[ ]'.

Up to now we have listed different labels. Their roles in formulas will be specified by rules of formation, of which we will now give only a rudimentary exposition.

The set of *terms* is the smallest set that contains (i) all constant symbols, (ii) all variables, and (ii) $f(t_1, t_2, ..., t_n)$, if $t_1, t_2, ..., t_n$ are terms and f is an n-ary function symbol.

The set of *atomic formulas* is the smallest set that contains (i) $s = t$ for all terms s and t, and (ii) $R(t_1, t_2, ..., t_n)$, if $t_1, t_2, ..., t_n$ are terms and R is an n-ary relation symbol.

The set of *formulas* is the smallest set that (i) contains all atomic formulas, (ii) contains $\forall x \phi$ and $\exists x \phi$, if ϕ is a formula and x a variable, and (iii) is closed under the logical connectives (if ϕ and ψ are formulas, then so are $\phi \vee \psi$, $\phi \wedge \psi$, $\phi \rightarrow \psi$, and $\neg \phi$).

A *free variable* is a variable in a formula that does not lie within the scope of a quantifier. A formula without free variables is called a *sentence*.

The *language* of a signature is the set of all sentences formed from the symbols in its signature together with the other more general symbols listed above.

So far we have strings of symbols which are formed according to rules. In order for these strings to have meaning, more specifically truth values, we need to interpret them. This happens with the help of models or structures, which provide the mathematical objects for which the symbols in a given signature can stand.

Definition 2.1. A *structure* or *model* $\mathcal{M}$ for a language $\mathcal{L}$, also called $\mathcal{L}$-structure or $\mathcal{L}$-model, is a triple (M, σ, I) where

- M is a non-empty set which is called the *domain* or *universe* of $\mathcal{M}$. The variables of $\mathcal{L}$ are supposed to range over the elements of M;
- $\sigma = (S_{\text{rel}}, S_{\text{func}}, \text{ar})$ is the signature of $\mathcal{L}$;
- I is the interpretation function which assigns
  - to each constant symbol c an element of M: $I(c) = c^M \in M$;
  - to each n-ary relation symbol R a subset of M^n: $I(R) = R^M \subset M^n$;
  - to each n-ary function symbol f a function from M^n to M: $I(f) = f^M : M^n \to M$.

  So formally, though perhaps not very aesthetically pleasing, I is a function from $S_{\text{rel}} \cup S_{\text{func}}$ to $(\bigcup_{i \in \mathbb{Z}_{>0}} \mathcal{P}(A^i)) \cup \{f : M^i \to M | i \in \mathbb{Z}_{\geqslant 0}\}$.

Sometimes no notational distinction is made between a structure and its domain, so one might speak of the elements of a structure. Also, often the superscript of the symbols is left out, because the context makes clear whether the symbol is meant or some specific interpretation. Lastly, it is common to write a structure for a language with n symbols in its signature as an $n+1$-tuple of the domain followed by the n symbols.

A sentence ϕ *is true in* a model $\mathcal{M}$, or *holds in* $\mathcal{M}$, if and only if its interpretation $I(\phi)$ is true in $\mathcal{M}$. This too is defined recursively. To give an idea, we say that $I(R(t_1, t_2, \ldots, t_n))$ holds in $\mathcal{M}$ for an $n-$ary relation symbol R and terms $t_1, t_2, \ldots, t_n$ if and only if $(I(t_1), I(t_2), \ldots, I(t_n)) \in R^M$. For formulas ϕ and ψ, we say that $\phi \wedge \psi$ holds in $\mathcal{M}$ if and only if both ϕ and ψ hold in $\mathcal{M}$, and so forth. We write $\mathcal{M} \models \phi$ for ϕ holds in $\mathcal{M}$.

Example. A group $(G, e, ^{-1}, *)$ is a structure for the language for groups. It has the constant e, the unary operation $^{-1}$ and the binary operation $*$. This structure is called *algebraic*, because is has no relation. The sentence $(\forall x)(\exists y)(x * y = e)$ is true in G.

Example. A partially ordered set $(P, \leqslant)$ is a structure which only has one binary relation $\leqslant$. Structures with no operations are called *relational*.

Definition 2.2. Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{L}$-structures. A *homomorphism* h between $\mathcal{M}$ and $\mathcal{N}$ is a function $h : \mathcal{M} \to \mathcal{N}$ such that:

- $h(c^M) = c^N$ for all constants c in $\mathcal{L}$;
- $h(f^M(m_1, m_2, ..., m_n)) = f^N(h(m_1), h(m_2), ..., h(m_n))$ for all n-ary function symbols f in $\mathcal{L}$ and all elements $m_1, m_2, ..., m_n \in \mathcal{M}$;
- $(m_1, m_2, ..., m_n) \in R^M$ implies $(h(m_1), h(m_2), ..., h(m_n)) \in R^N$ for all n-ary relation symbols R in $\mathcal{L}$.

A homomorphism $h : \mathcal{M} \Rightarrow \mathcal{N}$ is called an *embedding* if h is injective and $(h(m_1), h(m_2), \ldots, h(m_n)) \in R^N$ implies $(m_1, m_2, \ldots, m_n) \in R^M$. An equivalent definition is that for *atomic* formulas ϕ with n variables we have that

$$\mathcal{M} \models \phi(m_1, m_2, \ldots, m_n) \iff \mathcal{N} \models \phi(h(m_1), h(m_2), \ldots, h(m_n)).$$

Definition 2.3. If the inclusion between two $\mathcal{L}$-models $\mathcal{M} \hookrightarrow \mathcal{N}$ is an embedding, $\mathcal{M}$ is said to be a *substructure* of $\mathcal{N}$, and $\mathcal{N}$ an *extension* of $\mathcal{M}$.
We then have that (i) the domain M of $\mathcal{M}$ is a subset of that of $\mathcal{N}$, (ii) $R^M = R^N \cap M^n$ for each n-ary relation symbol R of $\mathcal{L}$, and (iii) $f^M = f^N|M^n$ for each n-ary relation symbol f of $\mathcal{L}$.

Definition 2.4. If the inclusion between two $\mathcal{L}$-models $\mathcal{M} \subseteq \mathcal{N}$ is an embedding, and moreover we have that

$$\mathcal{M} \models \phi(m_1, m_2, \ldots, m_n) \iff \mathcal{N} \models \phi(m_1, m_2, \ldots, m_n)$$

holds for all $m_1, m_2, \ldots, m_n \in \mathcal{M}$ and *all* formulas $\phi \in \mathcal{L}$ with n free variables, then the embedding is called *elementary*, and $\mathcal{M}$ is called an *elementary substructure* of $\mathcal{N}$, and $\mathcal{N}$ an *elementary extension* of $\mathcal{M}$.

A bijective homomorphism is called an *isomorphism*. Isomorphisms are elementary embeddings.

Theorem 2.5 (Tarski-Vaught test)**.** *Let $\mathcal{M}$ be a substructure of an $\mathcal{L}$-structure $\mathcal{N}$. Then $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$ if and only if for each formula $\phi(x, m_1, m_2, \ldots, m_n)$ in $\mathcal{L}$ with free variable x and elements m_i from $\mathcal{M}$ such that $\mathcal{N} \models (\exists x)\phi(x, m_1, m_2, \ldots, m_n)$, there is an $m \in \mathcal{M}$ such that $\mathcal{N} \models \phi(m, m_1, m_2, \ldots, m_n)$.*

Example. The field of rational numbers $(\mathbb{Q}, 0, 1, +, \cdot)$ is a substructure of the field of real numbers $(\mathbb{R}, 0, 1, +, \cdot)$, since it has the same signature, its domain $\mathbb{Q}$ is a subset of $\mathbb{R}$, and it is closed under its operations $+$ and $\cdot$. However, according to the Tarski-Vaught test it is not an elementary substructure: the sentence $(\exists x)(x \cdot x = 2)$ is true in $\mathbb{R}$, has only constants that are in $\mathbb{Q}$ also, but there is no element in $\mathbb{Q}$ that will satisfy the formula $x \cdot x = 2$.

2.2 The Löwenheim-Skolem Theorem

The Löwenheim-Skolem theorem says something about the cardinality of structures, by which we mean the cardinality of their domains. If we have an infinite structure, then for every infinite cardinal number we can find either an elementary substructure of that cardinality (downward Löwenheim-Skolem) or an elementary extension of that cardinality (upward Löwenheim-Skolem). In our proofs we will only make use of the downward version.

Theorem 2.6 (Downward Löwenheim-Skolem)**.** *Let $\mathcal{M}$ be a structure for some language $\mathcal{L}$ and let Y be a subset of $\mathcal{M}$. There is an elementary substructure $\mathcal{N}$ of $\mathcal{M}$ such that $|\mathcal{N}| \leqslant |Y| + |\mathcal{L}|$ and $Y \subseteq \mathcal{N}$.*

If $\mathcal{L}$ has a countable signature, its cardinality will be countable. Additionally, if $\mathcal{M}$ is infinite, we can take any countably infinite subset Y to ensure that

$\mathcal{M}$ has a countable elementary substructure. It follows that if a theory, a set of sentences closed under logical consequences, is countable and is true in an infinite model, it will also be true in a countably infinite model. This last fact leads to *Skolem's paradox*: Zermelo-Fraenkel set theory is countable (its axioms and all their logical consequences are countable), and it has an infinite model, so by Löwenheim-Skolem it must also have a countable model. At the same time, in every model for ZF set theory the statement 'there is no injection from $\mathbb{R}$ to $\mathbb{N}$' must be true, i.e., 'there is a set of uncountable cardinality', so it must also be true in a countable model, in which there can only be countably many real numbers. The solution to the paradox is that there is no bijection between $\mathbb{N}$ and that set of real numbers *inside* the countable model, though there is one outside of it.

The proof of the downward Löwenheim-Skolem theorem makes use of the axiom of choice: with the help of choice functions one adds elements to Y so that formulas of the form $\exists x \phi(x)$ that hold in $\mathcal{M}$ will also hold in the set that is to be the elementary substructure. Because there are formulas $\exists x \phi(x)$ characterizing the constants of $\mathcal{M}$ and elements in the images of its functions, we automatically get that the set contains the constants and is closed under the functions, so the set becomes a substructure. By the Tarski-Vaught test an elementary substructure is constructed.

3 Lattices and Wallman Representations

We now consider a specific type of structure, namely that of lattices. We will use lattices to contstruct Wallman representations, which play a key role in our proofs.

3.1 The Language of Lattices

Definition 3.1. A *lattice* $(L, \leqslant)$ is a non-empty partially ordered set in which every two elements $x, y \in L$ have a supremum '$x \vee y$' (also called a *join*) and an infimum '$x \wedge y$' (also called a *meet*).

The join and meet are necessarily unique. Given two joins u and v of x and y, we have $u \leqslant v$ as well as $v \leqslant u$, since u and v are both upper bounds and both least upperbounds. In a similar way it can be shown that meets are unique.

Example. Let X be a non-empty set. The powerset $(\mathcal{P}(X), \subseteq)$ is a lattice, since for every $A, B \in \mathcal{P}(X)$ we have that $A \cup B$ is the join and $A \cap B$ is the meet.

A different, but equivalent way to define lattices is to define them as algebraic structures.

Definition 3.2. A *lattice* $(L, \vee, \wedge)$ is a non-empty set with two binary operations $\vee$ (the *join*) and $\wedge$ (the *meet*) satisfying the following axioms for elements $x, y, z \in L$:

1.	(i)	$x \vee y = y \vee x$	
	(ii)	$x \wedge y = y \wedge x$	(commutative laws)
2.	(i)	$x \vee (y \vee z) = (x \vee y) \vee z$	
	(ii)	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	(associative laws)
3.	(i)	$x = x \vee (x \wedge y)$	
	(ii)	$x = x \wedge (x \vee y)$	(absorption laws)
4.	(i)	$x \vee x = x$	
	(ii)	$x \wedge x = x$	(idempotent laws)

It is well known that we can easily go from looking at a lattice as an algebraic structure to looking at it as a relational structure, and vice versa, with the help of our next proposition.

Proposition 3.3. *If L is a lattice according to one of the definitions, we can construct on the same set a lattice according to the other definition by the following precepts:*

(i) If L is a lattice according to definition 3.1, then define the operations $\vee$ and $\wedge$ by $x \vee y = \sup\{x, y\}$, and $x \wedge y = \inf\{x, y\}$.

(ii) If L is a lattice according to definition 3.2, then define the relation $\leqslant$ on L by $x \leqslant y$ if and only if $x = x \wedge y$.

A *bounded* lattice is a lattice that has a least element $\mathbf{0}$ (the lattice's *bottom*) and a greatest element $\mathbf{1}$ (the lattice's *top*), which satisfy $\mathbf{0} \leqslant x \leqslant \mathbf{1}$ for all x in L. Algebraically speaking, a bounded lattice L is a structure $(L, \vee, \wedge, \mathbf{0}, \mathbf{1})$ such that $(L, \vee, \wedge)$ is a lattice and $\mathbf{0}$ is the neutral element for the join operation $\vee$, and $\mathbf{1}$ is the neutral element for the meet operation $\wedge$:

$$x \vee \mathbf{0} = x \text{ and } x \wedge \mathbf{1} = x \text{ for all } x \in L.$$

A lattice L is called *distributive* if for all $x, y, z \in L$ we have $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. It turns out that distributivity of $\vee$ over $\wedge$ follows from distributivity of $\wedge$ over $\vee$, and vice versa.

Example. Coming back to our first example, let X be a non-empty set. The powerset $(\mathcal{P}(X), \cup, \cap, \emptyset, X)$ is bounded, since for every $A \in \mathcal{P}(X)$ we have that $A \cup \emptyset = A$ and $A \cap X = A$. Moreover, it is well known that in powersets $\cup$ is distributive over $\cap$.

A lattice L is *disjunctive* or *separative* if for all $x, y \in L$ we have if $x \nleqslant y$, then there is a $z \in L$ such that $z \neq \mathbf{0}$, $z \leqslant x$ and $y \wedge z = \mathbf{0}$.

A lattice L is *normal* if it is bounded and for all x and y in L with $x \wedge y = \mathbf{0}$ there are a and b in L such that $x \wedge a = \mathbf{0}$, $y \wedge b = \mathbf{0}$, and $a \vee b = \mathbf{1}$. Notice the similarity with the definition for normality for topological spaces.

3.2 Constructing Wallman Representations

Definition 3.4. Let L be a bounded and distributive lattice. $F \subseteq L$ is a *filter* if it satisfies (i) $\mathbf{0} \notin F$, (ii) if $x, y \in F$ then $x \wedge y \in F$, and (iii) if $x \in F$ and $x \leqslant y$, then we have $y \in F$. An *ultrafilter* is a maximal filter under inclusion among the filters on L.

Definition 3.5. Let L be a bounded and distributive lattice. The *Wallman representation* or *Wallman space* wL of L is the space with the set of all ultrafilters as its underlying set. A base for its closed sets is the family $\mathcal{B} = \{\overline{x} \mid x \in L\}$, where $\overline{x} = \{u \in wL \mid x \in u\}$.

Proposition 3.6. *The base $\mathcal{B}$ from definition 3.5 ordered by inclusion is a bounded and distributive lattice.*

Proof. Since $\mathcal{B} \subseteq \mathcal{P}(wL)$, $\mathcal{B}$ is a subset of a distributive bounded lattice. To show it is a substructure of a lattice, a sublattice, we have to show that $\emptyset, X \in \mathcal{B}$ and that $\mathcal{B}$ is closed under $\cup$ and $\cap$. By definition no filter has the element $\mathbf{0}$, so $\overline{\mathbf{0}} = \{u \in wL \mid \mathbf{0} \in u\} = \emptyset \in \mathcal{B}$. Similarly, all filters contain $\mathbf{1}$ by 3.4.(iii), so $\overline{\mathbf{1}} = wL \in \mathcal{B}$. Let $x, y \in L$. Since $x \wedge y, x \vee y \in L$ and $\overline{x \wedge y}, \overline{x \vee y} \in \mathcal{B}$, it suffices to show that $\overline{x} \cap \overline{y} = \overline{x \wedge y}$ and $\overline{x} \cup \overline{y} = \overline{x \vee y}$.
Suppose $u \in \overline{x} \cap \overline{y}$. This means that $x, y \in u$ and by 3.4.(ii) $x \wedge y \in u$, or equivalently, $u \in \overline{x \wedge y}$. If $u \in \overline{x \wedge y}$, we have $x \wedge y \in u$, and since $x \wedge y \leqslant x, y$, by 3.4.(iii) $x, y \in u$ and $u \in \overline{x} \cap \overline{y}$.
Now suppose $u \in \overline{x} \cup \overline{y}$. If $u \in \overline{x}$, then $x \in u$ and given that $x \leqslant x \vee y$, we get by 3.4.(iii) that $u \in \overline{x \vee y}$. Conversely, suppose $u \in \overline{x \vee y}$ and that $x \notin u$. Then there exists a $z \in u$ such that $x \wedge z = \mathbf{0}$. If that were not the case, x and all $w \in L$ with $x \leqslant w$ could be added to u to make an ultrafilter $u' \supsetneq u$, which is a contradiction with u being a maximal filter. Since we have $z, x \vee y \in u$, it follows from 3.4.(i) and (ii) that $(x \vee y) \wedge z \neq \mathbf{0}$. Because of the distributivity of L, we have that $(x \wedge z) \vee (y \wedge z) = \mathbf{0} \vee (y \wedge z) \neq \mathbf{0}$, so $y \wedge z \neq \mathbf{0}$. Lastly, from $y \wedge z \leqslant y$ and 3.4.(ii) follows that $y \in u$. Had we assumed that $y \notin u$, we would have deduced in the same way that $x \in u$. Hence $u \in \overline{x} \cup \overline{y}$. □

Proposition 3.7. *Let L be a bounded and distributive lattice and $\mathcal{B}$ the base of its Wallman representation wL we defined. Then $\mathcal{B}$ is a lattice-homomorphic image of L. Moreover, if L is separative, then L is isomorphic to $\mathcal{B}$.*

Proof. Define the function $f : L \to \mathcal{P}(wL)$ by $x \mapsto \overline{x}$. Clearly $\mathcal{B}$ is the image of L under f. From the proof of proposition 3.6 we know that $f(\mathbf{0}) = \emptyset$, $f(\mathbf{1}) = wL$, $f(x \wedge y) = f(x) \cap f(y)$, and $f(x \vee y) = f(x) \cup f(y)$ for $x, y \in L$, so f is a homomorphism. To show that L is isomorphic to $\mathcal{B}$ when L is separative it suffices to show that f is injective. Assume that $x \neq y$. Without loss of generality we can assume that $x \nleqslant y$. Because L is separative there exists a $z \in L$ such that $z \leqslant x$ and $y \wedge z = \mathbf{0}$. Take $u \in \overline{z}$. Then we have that $z \in u$, and consequently $x \in u$, but $y \notin u$. Hence we get $u \in \overline{x} \setminus \overline{y}$, which shows that $\overline{x} \neq \overline{y}$. □

Proposition 3.8. *The Wallman space wL of a lattice L is compact and satisfies T_1.*

Proof. Let $u \in wL$. If $\mathcal{B}$ is the base for the closed sets of wL as given in definition 3.5, $\{u\}$ is closed if $\{u\} = \bigcap \mathcal{C}$ for some subset $\mathcal{C} \subseteq \mathcal{B}$. We will take $\mathcal{C} = \{\overline{x} \mid x \in u\}$ to prove the last equality. For all $\overline{x} \in \mathcal{C}$ we have $x \in u$ and thereby $u \in \overline{x}$, so $\{u\} \subseteq \bigcap \mathcal{C}$. Suppose $v \in \bigcap \mathcal{C}$ and $u \neq v$. Then for all $x \in u$ we have $v \in \overline{x}$, so $x \in v$, which proves that $u \subseteq v$. However, u is a maximal filter, so $u = v$, giving us a contradiction.
We prove compactness by using the contrapositive of the finite intersection property. Let $\{F_i \mid i \in I\}$ be a family of closed sets in wL such that $\bigcap_{j \in J} F_j \neq \emptyset$ for all finite $J \subseteq I$. We will prove that $\bigcap_{i \in I} F_i \neq \emptyset$. For every $i \in I$ we have $F_i = \bigcap \mathcal{F}_i$ for some subset $\mathcal{F}_i \subseteq \mathcal{B}$. Define $\mathcal{F} := \bigcup_{i \in I} \mathcal{F}_i$ and let $\overline{x}_1, \ldots, \overline{x}_k \in \mathcal{F}$. Pick $i_1, i_2, \ldots, i_k \in I$ such that $F_{i_j} \subseteq \overline{x}_j$ for $j = 1, 2, \ldots, k$. Then we have that $\bigcap_{j=1}^{k} F_{i_j} \subseteq \bigcap_{j=1}^{k} \overline{x}_j$, and since we have by assumption that $\bigcap_{j=1}^{k} F_{i_j} \neq \emptyset$, it follows that $\overline{x}_1 \cap \overline{x}_2 \cap \cdots \cap \overline{x}_k \neq \emptyset$, and therefore $x_1 \wedge x_2 \wedge \cdots \wedge x_k \neq \mathbf{0}$. This means that finite intersections of elements from $u = \{x \in L \mid \overline{x} \in \mathcal{F}\}$ will be non-empty. Since $\mathbf{0} \notin u$ and $x \wedge y \neq \mathbf{0}$ for all $x, y \in u$, there is an ultrafilter v such that $u \subseteq v$. We have now found an element of $\bigcap_{i \in I} F_i$:

$$(\forall x \in u)(x \in v) \iff (\forall x \in u)(v \in \overline{x}) \iff$$
$$(\forall \overline{x} \in \mathcal{F})(v \in \overline{x}) \iff v \in \bigcap \mathcal{F} = \bigcap_{i \in I} F_i.$$

□

Proposition 3.9. *The Wallman space wL is Hausdorff if and only if the lattice L it is representing is normal.*

Proof. Suppose L is normal and let $u, v \in wL$ such that $u \neq v$. Without loss of generality we may assume that $u \not\subseteq v$. Then there is an element $a \in u \setminus v$. Since v is an ultrafilter, there exists a $b \in v$ such that $a \wedge b = \mathbf{0}$, otherwise a could be added to v along with other elements to make an ultrafilter $v' \supsetneq v$. Because of normality there are $c, d \in L$ such that $a \wedge c = \mathbf{0}$, $b \wedge d = \mathbf{0}$, and $c \vee d = \mathbf{1}$. This gives us $\overline{a} \cap \overline{c} = \emptyset$, $\overline{b} \cap \overline{d} = \emptyset$, and $\overline{c} \cup \overline{d} = wL$ in terms of the basic closed sets of wL we defined. Since $\overline{c}^{\mathsf{c}}$ and $\overline{d}^{\mathsf{c}}$ are open and disjoint, it suffices to show that $u \in \overline{c}^{\mathsf{c}}$ and $v \in \overline{d}^{\mathsf{c}}$. Since $a \in u$, we have $u \in \overline{a}$, and similarly we get $v \in \overline{b}$. Given that $a \wedge c = \mathbf{0}$, it follows that $c \notin u$, so $u \notin \overline{c}$. In the same way it follows that $v \notin \overline{d}$. This gives us that $u \in \overline{c}^{\mathsf{c}}$ and $v \in \overline{d}^{\mathsf{c}}$.
Suppose that wL is Hausdorff and let $a, b \in L$ be such that $a \wedge b = \mathbf{0}$. It follows that $\overline{a} \cap \overline{b} = \emptyset$, where $\overline{a}$ and $\overline{b}$ are elements of the base $\mathcal{B}$ homomorphic to L. Because wL is also compact, wL is normal, so there are closed $F, G \subseteq wL$ such that $F \cap \overline{a} = \emptyset$, $G \cap \overline{b} = \emptyset$, and $F \cup G = wL$. Moreover, there are $\mathcal{C}, \mathcal{D} \subseteq \mathcal{B}$ such that $F = \bigcap \mathcal{C}$ and $G = \bigcap \mathcal{D}$, and hence $\overline{a} \cap \bigcap \mathcal{C} = \emptyset$ and $\overline{b} \cap \bigcap \mathcal{D} = \emptyset$. By the finite intersection property there are $\overline{c}_1, \overline{c}_2, \ldots, \overline{c}_k \in \mathcal{C}$ and $\overline{d}_1, \overline{d}_2, \ldots, \overline{d}_l \in \mathcal{D}$ for some $k, l \in \mathbb{N}$ such that $\overline{a} \cap \overline{c}_1 \cap \overline{c}_2 \cap \cdots \cap \overline{c}_k = \emptyset$ and $\overline{b} \cap \overline{d}_1 \cap \overline{d}_2 \cap \cdots \cap \overline{d}_l = \emptyset$. As a consequence we get that $a \wedge (c_1 \wedge c_2 \wedge \cdots \wedge c_k) = \mathbf{0}$ and $b \wedge (d_1 \wedge d_2 \wedge \cdots \wedge d_l) = \mathbf{0}$. Lastly, because have that $wL = F \cup G \subseteq \overline{c_1 \wedge c_2 \wedge \cdots \wedge c_k} \cup \overline{d_1 \wedge d_2 \wedge \cdots \wedge d_l}$, we get that $(c_1 \wedge c_2 \wedge \cdots \wedge c_k) \vee (d_1 \wedge d_2 \wedge \cdots \wedge d_l) = \mathbf{1}$. □

4 The Proofs

Let X be a compact and Hausdorff space. We are now ready for the proofs that the inequalities $\dim X \leqslant \operatorname{Ind} X$ and $\dim X \leqslant \operatorname{Dg} X$ hold. The proof of the first inequality will be expounded more extensively than the second, because some of the arguments we give are more or less the same in both proofs. We will make use of formulas in the language of lattices. To avoid ambiguity, we write $\sqcap$ and $\sqcup$ for the meet and join, and $\wedge$ and $\vee$ for the logical conjunction ('and') and disjunction ('or').

4.1 $\dim X \leqslant \operatorname{Ind} X$

Step 1. First, define $Cl(X) := \{F \subseteq X \mid F \text{ is closed in } X\}$. Notice that $(Cl(X), \cup, \cap, \emptyset, X)$ is a bounded and distributive lattice, since it has a bottom $\emptyset$ and top X, it is a subset of the distributive lattice $\mathcal{P}(X)$, and finite unions and intersections of closed sets are again closed. Let $\mathcal{A} \subseteq Cl(X)$ be any countable subfamily. The Löwenheim-Skolem theorem gives us an elementary sublattice L of $Cl(X)$ such that $|L| \leqslant \aleph_0$ and $\mathcal{A} \subseteq L$.

Step 2. Given that L is a sublattice, it inherits the distributivity and constants of $Cl(X)$, so it is bounded and distributive. This means we can associate a compact T_1-space to it, its Wallman representation wL. To prove that L is separative, it suffices to show that $Cl(X)$ is separative, because of the elementarity of L, and the fact that separativity can be expressed in a formula in the language of lattices without reference to elements outside L:

$$(\forall x)(\forall y)(\exists z)\big[(x \nleqslant y) \to \big((z \neq \mathbf{0}) \wedge (z \leqslant x) \wedge (z \wedge y = \mathbf{0})\big)\big].$$

Let $F, G \in Cl(X)$ be sets such that $F \nsubseteq G$. Then there is a $x \in F \setminus G$. Since X satisfies T_1, $\{x\}$ is a closed set, and we have that $\{x\} \subseteq F$ and $\{x\} \cap G = \emptyset$. Having established the separativity of L, we get from proposition 3.7 that L is isomorphic to the base $\mathcal{B}$ for the closed sets of wL as defined in definition 3.5. Therefore, wL has a countable base.

Step 3. To apply the Urysohn identity, we need to show that wL is normal. Since X is compact and Hausdorff, it is normal. Clearly, as a consequence the lattice $Cl(X)$ is also, since the variables in the lattice-theoretic formula

$$(\forall x)(\forall y)(\exists a)(\exists b)\big[(x \sqcap y = \mathbf{0}) \to \big((x \sqcap a = \mathbf{0}) \wedge (y \sqcap b = \mathbf{0}) \wedge (a \sqcup b = \mathbf{1})\big)\big]$$

range over the closed sets of X. The elementarity of L gives us that it, too, is normal. By proposition 3.9 wL is Hausdorff, and since it is also compact, wL is normal. Applying the Urysohn identity we get the identity $\dim wL = \operatorname{Ind} wL$.

Step 4. Notice that Hemmingsen's characterization for $\dim X \leqslant n$ can be formulated in the following way for the lattice $Cl(X)$:

$$\begin{aligned}&(\forall x_1)(\forall x_2)\dots(\forall x_{n+2})(\exists y_1)(\exists y_2)\dots(\exists y_{n+2})\\&\big[(x_1 \sqcap x_2 \sqcap \dots \sqcap x_{n+2} = \mathbf{0}) \to \big((x_1 \leqslant y_1) \wedge (x_2 \leqslant y_2) \wedge \dots \wedge (x_{n+2} \leqslant y_{n+2})\\&\qquad \wedge (y_1 \sqcap y_2 \sqcap \dots \sqcap y_{n+2} = \mathbf{0}) \wedge (y_1 \sqcup y_2 \sqcup \dots \sqcup y_{n+2} = \mathbf{1})\big)\big].\end{aligned}$$

Let us call this formula δ_n. Because of elementarity, we get that $Cl(X) \models \delta_n$ if and only if $L \models \delta_n$. We now want to prove that $L \models \delta_n$ holds if and only if $Cl(wL) \models \delta_n$ holds. First, assume that $L \models \delta_n$. We use a 'swelling and shrinking' argument. Let $G_1, G_2, \ldots, G_{n+2} \in Cl(wL)$ be such that $G_1 \cap G_2 \cap \cdots \cap G_{n+2} = \emptyset$. The space wL is normal. It can be readily shown by induction that there are open sets $U_1, U_2, \ldots, U_{n+2} \subset wL$ such that $G_i \subset U_i$ for $i = 1, 2, \ldots, n+2$, and $U_1 \cap U_2 \cap \cdots \cap U_{n+2} = \emptyset$.
Note that $G_i = \bigcap\{\overline{x} \mid x \in L_i\}$, for some subset $L_i \subseteq L$. Furthermore, we have $G_i \cap U_i^{\mathsf{c}} = (\bigcap\{\overline{x} \mid x \in L_i\}) \cap U_i^{\mathsf{c}} = \emptyset$. Because wL is compact, we can use the finite intersection property to get the existence of $x_1, x_2, \ldots, x_k \in L_i$ for some $k \in \mathbb{N}$ such that $\overline{x}_1 \cap \overline{x}_2 \cap \cdots \cap \overline{x}_k \cap U_i^{\mathsf{c}} = \emptyset$. The homomorphism between L and $\mathcal{B}$ gives us that $\overline{x_1 \cap x_2 \cap \cdots \cap x_k} \cap U_i^{\mathsf{c}} = \emptyset$. Define $y_i = x_1 \cap x_2 \cap \cdots \cap x_k$. We have 'swollen' G_i to $\overline{y}_i$, since we now have that $G_i \subseteq \overline{y}_i \subseteq U_i$.
This gives us $y_1, y_2, \ldots, y_{n+2} \in L$ such that $\overline{y_1 \cap y_2 \cap \cdots \cap y_{n+2}} = \emptyset$, and hence $y_1 \cap y_2 \cap \cdots \cap y_{n+2} = \emptyset$. We can now use the fact that δ_n holds in L to find our required shrinking. There are $z_1, z_2, \ldots, z_{n+2} \in L$ such that

(i) $y_i \subseteq z_i$, so $G_i \subseteq \overline{y}_i \subseteq \overline{z}_i$ for all $i = 1, 2, \ldots, n+2$;

(ii) $z_1 \cap z_2 \cap \cdots \cap z_{n+2} = \emptyset$, so $\overline{z}_1 \cap \overline{z}_2 \cap \cdots \cap \overline{z}_{n+2} = \emptyset$;

(iii) and $z_1 \cup z_2 \cup \cdots \cup z_{n+2} = X$, so $\overline{z}_1 \cup \overline{z}_2 \cup \cdots \cup \overline{z}_{n+2} = wL$.

In fact, from our proof we get that if δ_n is true in *any* lattice base for the closed sets of wL, that is, a base for the closed sets that is closed under finite unions and intersections, it will be true in wL.

Conversely, suppose that $Cl(wL) \models \delta_n$. We can use a similar 'swelling and shrinking' argument. Let $x_1, x_2, \ldots, x_{n+2} \in L$ be such that $x_1 \cap x_2 \cap \cdots \cap x_{n+2} = \emptyset$. Then $\overline{x}_1 \cap \overline{x}_2 \cap \cdots \cap \overline{x}_{n+2} = \emptyset$, so by hypothesis there are $F_1, F_2, \ldots, F_{n+2} \in Cl(wL)$ such that $\overline{x}_i \subseteq F_i$ for all $i = 1, 2, \ldots, n+2$, $F_1 \cap F_2 \cap \cdots \cap F_{n+2} = \emptyset$, and $F_1 \cup F_2 \cup \cdots \cup F_{n+2} = wL$. Since wL is normal, there are open $U_i \subseteq wL$ such that $\overline{x}_1 \subseteq F_i \subseteq U_i$ for all $i = 1, 2, \ldots, n+2$ and $U_1 \cap U_2 \cap \cdots \cap U_{n+2} = \emptyset$. For all $i = 1, 2, \ldots, n+2$ we have $F_i = \bigcap \mathcal{F}_i$ for some $\mathcal{F}_i \subseteq \mathcal{B}$, so we get $(\bigcap \mathcal{F}_i) \cap U_i^{\mathsf{c}} = F_i \cap U_i^{\mathsf{c}} = \emptyset$. The finite intersection property again provides us with $\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_k \in \mathcal{F}_i$ for some number k such that $\overline{y}_1 \cap \overline{y}_2 \cap \cdots \cap \overline{y}_k \cap U_i^{\mathsf{c}} = \emptyset$. Define $z_i := y_1 \cap y_2 \cap \cdots \cap y_k$. This gives us our shrinking, for

(i) $\overline{x}_i \subseteq F_i \subseteq \overline{z}_i \subseteq U_i$, so $x_i \subseteq z_i$ for all $i = 1, 2, \ldots, n+2$;

(ii) $\overline{z}_1 \cap \overline{z}_2 \cap \cdots \cap \overline{z}_{n+2} = \emptyset$, since $U_1 \cap U_2 \cap \cdots \cap U_{n+2} = \emptyset$, so $z_1 \cap z_2 \cap \cdots \cap z_{n+2} = \emptyset$;

(iii) and $wL = F_1 \cup F_2 \cup \cdots \cup F_{n+2} \subseteq \overline{z}_1 \cup \overline{z}_2 \cup \cdots \cup \overline{z}_{n+2} = wL$, so $z_1 \cup z_2 \cup \cdots \cup z_{n+2} = X$.

Hence we get that $L \models \delta_n$.

To summarize, we now have the following:

$$\begin{aligned} \dim X \leq n &\iff Cl(X) \models \delta_n \iff L \models \delta_n \\ &\iff Cl(wL) \models \delta_n \iff \dim wL \leq n, \end{aligned}$$

so we may conclude that $\dim X = \dim wL$.

Step 5. Just like there is a lattice-theoretic formula that characterizes the covering dimension, there is a recursive formula $I_n(x)$ that stands for $\operatorname{Ind} x \leqslant n$:

$$(\forall a)(\forall b)(\exists l)\big[(a \leqslant x) \wedge (b \leqslant x) \wedge (a \sqcap b = \mathbf{0}) \rightarrow (\operatorname{part}(l, a, b, x) \wedge I_{n-1}(l))\big],$$

where $\operatorname{part}(l, a, b, x)$ states that l is a partition between a and b in the space x:

$$(\exists f)(\exists g)\big[(f \sqcap a = \mathbf{0}) \wedge (g \sqcap b = \mathbf{0}) \wedge (f \sqcup g = x) \wedge (f \sqcap g = \mathbf{l})\big].$$

We begin our recursive formula with $I_{-1}(x)$, short for $x = \mathbf{0}$.

Stating that $\operatorname{Ind} X \leqslant n$ is equivalent to saying $Cl(X) \models I_n(\mathbf{1})$, which by elementarity is equivalent to $L \models I_n(\mathbf{1})$.

To prove that $L \models I_n(\mathbf{1}) \implies Cl(wL) \models I_n(\mathbf{1})$, we prove the more general case that if any lattice base for a compact and Hausdorff space Y satisfies $I_n(\mathbf{1})$, then $Cl(Y)$ does also. Suppose $I_n(\mathbf{1})$ is true in some lattice base $\mathcal{C}$ for the closed sets of Y. We will prove by induction on n that $Cl(Y) \models I_n(\mathbf{1})$ for all $n \in \mathbb{Z}_{\geqslant -1}$. For $n = -1$ we get that $\mathbf{1} = \mathbf{0}$ holds in $\mathcal{C}$, so $\mathcal{C} = \{\emptyset\}$ must hold, and consequently $wL = \emptyset$, so $\operatorname{Ind} wL \leqslant -1$. Now assume that $n > -1$. Let $F, G \subseteq wL$ be closed and disjoint sets. There are $\mathcal{C}_F, \mathcal{C}_G \subseteq \mathcal{C}$ such that $F = \bigcap \mathcal{C}_F$ and $G = \bigcap \mathcal{C}_G$, and hence $F \cap (\bigcap \mathcal{C}_G) = \emptyset$ and $(\bigcap \mathcal{C}_F) \cap G = \emptyset$. Using the finite intersection property in Y we get that $F \cap G_1 \cap G_2 \cap \cdots \cap G_k = \emptyset$ and $F_1 \cap F_2 \cap \cdots \cap F_l \cap G = \emptyset$ for numbers k and l and $F_i, G_j \in \mathcal{C}$. Define $C_F := F_1 \cap F_2 \cap \cdots \cap F_l$ and $C_G := G_1 \cap G_2 \cap \cdots \cap G_k$. Then $C_F, C_G \in \mathcal{C}$, so by assumption there is a partition $P \in \mathcal{C}$ between C_F and C_G such that $I_{n-1}(P)$ holds in $\mathcal{C}$.
Now observe that $\mathcal{C}_P = \{R \in \mathcal{C} \mid R \subseteq P\} = \{R \cap P \mid R \in \mathcal{C}\}$ is a lattice base for P, since it is closed under finite unions and intersections, and all closed sets in P are of the form $R \cap P$, where R is closed in Y. Its bottom is $\emptyset$ and its top is P, so $\mathcal{C}_P \models I_{n-1}(\mathbf{1})$. By the induction hypothesis we get that $Cl(P) \models I_{n-1}(\mathbf{1})$, so $\operatorname{Ind} P \leqslant n-1$, which in turn gives us by definition that $\operatorname{Ind} Y \leqslant n$. Applying this more general result to wL, and using the fact that L is isomorphic to $\mathcal{B}$, we get $L \models I_n(\mathbf{1}) \implies Cl(wL) \models I_n(\mathbf{1})$, as required.

Summarizing what we have, we get:

$$\begin{aligned} \operatorname{Ind} X \leq n \iff Cl(X) \models I_n(\mathbf{1}) &\iff L \models I_n(\mathbf{1}) \\ &\implies Cl(wL) \models I_n(\mathbf{1}) \iff \operatorname{Ind} wL \leq n. \end{aligned}$$

The impication $Cl(Y) \models I_n(\mathbf{1}) \implies \mathcal{C} \models I_n(\mathbf{1})$ does not hold. As an example we take the unit interval $[0,1]$ and the lattice base for the closed sets $\mathcal{Q}$ generated by the subbase $\{[0,q] \mid q \in [0,1] \cap \mathbb{Q}\} \cup \{[p,1] \mid p \in [0,1] \setminus \mathbb{Q}\}$. Every closed set can be generated by $\mathcal{Q}$, because every rational endpoint can be reached by a sequence of irrational points, and vice versa. However, $\mathcal{Q}$ does not satisfy $I_1(\mathbf{1})$, and in fact doesn't satisfy $I_n(\mathbf{1})$ for any n, since all elements in $\mathcal{Q}$ are intervals, since it is closed under finite intersections only. We will never be able to find a partition $Q \in \mathcal{Q}$ such that $\operatorname{Ind} Q \leqslant 0$. Therefore, we can only conclude that $\operatorname{Ind} wL \leqslant \operatorname{Ind} X$.

Conclusion. Putting our pieces together, we conclude that

$$\dim X \underset{(4)}{=} \dim wL \underset{(3)}{=} \operatorname{Ind} wL \underset{(5)}{\leq} \operatorname{Ind} X.$$

4.2 $\dim X \leqslant \operatorname{Dg} X$

We can repeat step 1 and 2 of the last proof to get a countable elementary sublattice L of $Cl(X)$, and its Wallman representation wL with a countable base $\mathcal{B}$. We can also need the result from step 4 that $\dim X = \dim wL$. We will give a recursive formula $\Delta_n(X)$ to denote $\operatorname{Dg} X \leqslant n$. First, let $\operatorname{conn}(a)$ abbreviate the formula wich says that a is connected:

$$(\forall x)(\forall y)\big[\big((x \sqcap y = \mathbf{0}) \wedge (x \sqcup y = a)\big) \to \big((x = \mathbf{0}) \vee (x = a)\big)\big].$$

Moreover, we will use $\operatorname{cut}(c, x, y, a)$ to denote that c is a cut between x and y in the space a:

$$(\forall v)\big[\big((v \leqslant a) \wedge \operatorname{conn}(v) \wedge (v \sqcap x \neq \mathbf{0}) \wedge (v \sqcap y \neq \mathbf{0})\big) \to (v \sqcap c \neq \mathbf{0})\big].$$

The formula $\Delta_n(a)$ can now be given as:

$$(\forall x)(\forall y)(\exists c)$$
$$\big[\big((x \leqslant a) \wedge (y \leqslant a) \wedge (x \sqcap y = \mathbf{0})\big) \to \big(\operatorname{cut}(c, x, y, a) \wedge \Delta_{n-1}(c)\big)\big].$$

Here too, $\Delta_{-1}(a)$ stands for $a = \mathbf{0}$. Elementarity gives us that $Cl(X) \models \Delta_n(\mathbf{1}) \iff L \models \Delta_n(\mathbf{1})$. Also, recall that because L and $\mathcal{B}$ are isomorphic, we have $\mathcal{B} \models \Delta_n(\mathbf{1}) \iff L \models \Delta_n(\mathbf{1})$. However, the same example which showed that not every lattice base for a space Y satisfies $I_n(\mathbf{1})$ if $\operatorname{Ind} Y \leqslant n$, can be applied to the dimensionsgrad, so we can not conclude that $Cl(wL) \models \Delta_n(\mathbf{1}) \implies L \models \Delta_n(\mathbf{1})$. It is also possible that a lattice base for the closed sets satisfies $\Delta_0(\mathbf{1})$ while $\operatorname{Dg} Y > 0$. We can however prove for our elementary lattice L that $L \models \Delta_n(\mathbf{1}) \implies Cl(wL) \models \Delta_n(\mathbf{1})$.
Let $F, G \in Cl(wL)$ be closed and disjoint. We are in search of a cut in $Cl(wL)$ that satisfies Δ_{n-1} in $Cl(wL)$. With our swelling argument we can find disjoint basic closed sets $\overline{a}, \overline{b} \in \mathcal{B}$ such that $F \subseteq \overline{a}$ and $G \subseteq \overline{b}$. Since $a \cap b = \emptyset$, there is by assumption a cut $c \in L$ such that $L \models \Delta_{n-1}(c)$. That $\operatorname{Dg} \overline{c} \leqslant n-1$ follows from an induction argument similar to the one in step 5 of our last proof. We prove that $\overline{c}$ is a cut between $\overline{a}$ and $\overline{b}$, and hence between F and G. Let $K \subseteq wL$ be a closed set that meets both $\overline{a}$ and $\overline{b}$, but not $\overline{c}$. We will prove that K is not connected and thereby not a continuum, so that by contraposition $\overline{c}$ will be a cut. Again, with the finite intersection property we can get a basic set $\overline{d} \in \mathcal{B}$ such that $K \subseteq \overline{d}$ and $\overline{c} \cap \overline{d} = \emptyset$. Since c is a cut between a and b in X, no connected component of d meets both a and b, otherwise that component would be a continuum connecting a and b, so that $c \cap d \neq \emptyset$. So there are closed sets $f, g \in L$ such that $d = f \cup g$, $f \cap g = \emptyset$ and $a \cap d \subseteq f$ and $b \cap d \subseteq g$ ([3], 6.1.2). Going back to wL we now see that $\emptyset \neq K \cap \overline{a} \subseteq \overline{d} \cap \overline{a} \subseteq \overline{f}$, and $\emptyset \neq K \cap \overline{b} \subseteq \overline{d} \cap \overline{b} \subseteq \overline{g}$. But then K is not connected, since $\overline{f} \cap \overline{g} = \emptyset$. So $\overline{c}$ is indeed a cut between $\overline{a}$ and $\overline{b}$, and we thereby have $Cl(wL) \models \Delta_n(\mathbf{1})$. Since

$$\begin{aligned} \operatorname{Dg} X \leq n &\iff Cl(X) \models \Delta_n(\mathbf{1}) \iff L \models \Delta_n(\mathbf{1}) \\ &\implies Cl(wL) \models \Delta_n(\mathbf{1}) \iff \operatorname{Dg} wL \leq n, \end{aligned}$$

we get as a result that $\operatorname{Dg} wL \leqslant \operatorname{Dg} X$.

The theorem from [5] says that $\dim wL = \operatorname{Dg} wL$. We conclude that

$$\dim X \underset{(4)}{=} \dim wL = \operatorname{Dg} wL \leqslant \operatorname{Dg} X.$$

References

[1] L.E.J. Brouwer, *Über den Natürlichen Dimensionsbegriff,* Journal für die reine und angewandte Mathematik **142** (1913), 146-152.

[2] B.A. Davey and H.A. Priestley, *Introduction to Lattices and Order. Second Edition,* Cambridge University Press, Cambridge, 2002.

[3] R. Engelking, *General Topology. Revised and completed edition,* Sigma Series in Pure Mathematics, no. 6, Heldermann Verlag, Berlin, 1989. MR 91c:54001

[4] V.V. Fedorchuk, *On the Brouwer dimension for compact spaces,* Mathematical Notes **73** (2003), 271-279, Russian original: Matematicheskie Zametki **73** (2003), 295-304.

[5] V.V. Fedorchuk, M. Levin, and E.V. Shchepin, *On the Brouwer definition of dimension,* Uspekhi Matematischeskikh Nauk **54** (1999), no. 2(326), 193-194, translation in Russian Mathematical Surveys **54** (1999), no. 2, 432-433. MR 2000g:54068

[6] T. Gowers (Ed.), *The Princeton Companion to Mathematics,* Princeton University Press, Princeton, 2008.

[7] K.P. Hart, *Elementarity and Dimensions,* Mathematical Notes **78** (2005), 264-269.

[8] W. Hodges, *A Shorter Model Theory,* Cambridge University Press, Cambridge, 1997. MR 98i:03041

[9] W. Hurewicz and H. Wallman, *Dimension Theory,* Princeton University Press, Princeton, 1941.

[10] I.M. James (Ed.), *History of Topology,* Elsevier Science, Amsterdam, 1999.

[11] H. Poincaré, *Pourquoi l'espace a trois dimensions,* Revue de Métaphysique et de Morale **20** (1912), 483-504.

[12] N. Vedenissoff, *Remarks on the dimension of topological spaces,* Uchenye Zapiski Moskov. Gos. Univ. Mathematika **30** (1939), 131-140.