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Smooth functions, orthogonal polynomials and rapidly decreasing sequences

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1 Introduction

Let $w : [a, b] \to \mathbb{R}$ be a real-valued L^1-function on a bounded interval with the following property:

$$\begin{gathered} \exists\, \delta > 0,\, \exists\, C > 0 \text{ and } \exists\, \nu > 0 \text{ such that for all } x_t \in [a, b] : \\ w(x) \geq C\, |x - x_t|^{\nu} \\ \text{for almost all } x \in [a, b] \text{ with } |x - x_t| < \delta. \end{gathered} \tag{1.1}$$

An example of a function w with this property is

$$w(x) \geq C$$

for all $x \in [a, b]$, where $C > 0$. Using w as a weight function, we define the weighted L^2 space $L^2([a, b], w(x)dx)$. Let the inner product on $L^2([a, b], w(x)dx)$ for all $f, g \in L^2([a, b], w(x)dx)$ be as follows

$$(f, g) = \int_a^b f(x)\overline{g(x)}w(x)dx.$$

We note that $1, x, x^2, \ldots$ are linearly independent elements of $L^2([a, b], w(x)dx)$. If we apply the Gram-Schmidt process, the process will give us a sequence of polynomials $\{p_n\}_{n=0}^{\infty}$ such that the degree of p_n is exactly n. The sequence of polynomials is an orthonormal system in $L^2([a, b], w(x)dx)$. On general grounds this is even an orthonormal basis which results in the fact that we can write every arbitrary $f \in L^2([a, b], w(x)dx)$ as

$$f = \sum_{n=0}^{\infty} (f, p_n)\, p_n.$$

Let us define the map $\mathcal{F} : L^2([a, b], w(x)dx) \to \ell^2$ that assigns to every element the sequence of coefficients with respect to the orthonormal basis $\{p_n\}_{n=0}^{\infty}$. Now the question rises what will happen with the image of a 'decent' function. An informal meta-principle tells us that smoothness will result in good convergence. This gives us the idea that we are able to say something about $\mathcal{F}(f)$ when f is an infinitely differentiable function. It turns out to be even more beautiful than one could hope for: infinitely differentiable functions are mapped bijectively via $\mathcal{F}$ to the rapidly decreasing sequences. A rapidly decreasing sequence $\{a_n\}_{n=0}^{\infty}$ has the property that it goes to 0 faster than every negative power of n. The main goal of this thesis is to give a proof of this statement.

This, however, will not be the first thesis or paper on this subject. In the late sixties Zerner stated a theorem [3] very similar to the theorem we will prove in this thesis. The main differences are the following. Firstly, his theorem covers a more restricted case, namely the case where the weight function w has the property $w(x) \geq C > 0$ for all $x \in [a, b]$. On the other hand, his theorem treated several dimensions instead of one, generalising the situation more than we will. He also gave a sketch of a proof of the theorem. Shortly after his French publication in Comptes Rendus, Pavec gave a full proof [2] of the theorem given the sketch by Zerner and taking $w(x) \equiv 1$ for all $x \in [a, b]$. These two underlying papers have been used to construct the proof given here and we will reflect on their results and use at the end of this thesis.

2 Formulation of the main theorem

We will have a closer and more formal look at the statement we want to prove. Let $a, b \in \mathbb{R}$ and $a < b$ throughout this whole thesis. We start by defining the following polynomial space.

Definition 2.1. The space $\mathcal{P}_k([a,b])$ is the space of real-valued polynomials in one variable on the interval $[a,b]$ with $a, b \in \mathbb{R}$ and with degree exactly $k \in \mathbb{N}$.

The polynomials in this thesis are all real-valued, because certain theorems are only applicable to real-valued polynomials. The 'decent' functions we mentioned in the introduction make up the following space.

Definition 2.2. Define the space $C^\infty([a,b]) \subset L^2([a,b], w(x)dx)$ as the space of infinitely differentiable real-valued functions on the interval $[a,b]$.

We will define the rapidly decreasing sequences formally now.

Definition 2.3. A sequence $\{a_n\}_{n=0}^\infty$ is called a rapidly decreasing sequence if for all $k \in \mathbb{N}$:

$$\lim_{n \to \infty} n^k a_n = 0.$$

The space consisting of these rapidly decreasing sequences is key in our proof.

Definition 2.4. Define the space (s) as the space of rapidly decreasing sequences $\{a_n\}_{n=0}^\infty$.

Now we are ready to state the theorem that will be proven in this thesis.

Theorem 2.5. *The restriction of $\mathcal{F}$ to the subset $C^\infty([a,b])$ is a bijection between the vector spaces $C^\infty([a,b])$ and (s).*

First, we will prove that $\mathcal{F}(C^\infty([a,b])) \subset (s)$. Then we will turn to orthogonal polynomials and give a bound for

$$\|p_k^{(j)}\|_\infty$$

in terms of $\|p_k\|_2$ for all $p_k \in \mathcal{P}_k([a,b])$ and $j = 0, 1, 2, \ldots$. Afterwards we will use that bound to show that $(s) \subset \mathcal{F}(C^\infty([a,b]))$. At that point we only have to conclude that $\mathcal{F}$ is indeed a bijection between vector spaces.

This result gives rise to the statement that says that there is an isomorphism between $C^\infty([a,b])$ and (s) as topological vector spaces. We will not prove that statement.

3 Smooth functions

In this section we will prove that $\mathcal{F}$ maps $C^\infty([a,b])$ injectively into (s).

Let $f \in C^\infty([a,b])$ be arbitrary. We define the following distances:

$$c_k = d(f, \mathcal{P}_k([a,b]))_\infty = \inf_{p_k \in \mathcal{P}_k([a,b])} \|f - p_k\|_\infty$$
$$\alpha_k = d(f, \mathcal{P}_k([a,b]))_2 = \inf_{p_k \in \mathcal{P}_k([a,b])} \|f - p_k\|_2.$$

We now use a result from [1] that will, after some elaboration, help us to bound the sequence $\{\alpha_k\}_{k=0}^\infty$. Let $g \in C^\infty([-1,1])$. First define

$$\Delta_n(x) = \begin{cases} 1 & \text{if } n = 0, \\ \max\left(\frac{\sqrt{1-x^2}}{n}, \frac{1}{n^2}\right) & \text{if } n = 1, 2, \dots. \end{cases}$$

This function has the modulus of continuity $\omega(g,h)$ defined as follows:

$$\omega(g,h) = \max_{x,t,|t|\le h} |g(x+t) - g(x)|.$$

Now we are ready to state a consequence of the theorem, cf. [1, p. 66].

Theorem 3.1. *For every $g \in C^\infty([-1,1])$ and for every $q = 1, 2, \dots$ there is a sequence of polynomials $p_n(x)$ for which*

$$|g(x) - p_n(x)| \le M_q \Delta_n(x)^q \omega(g^{(q)}, \Delta_n(x)), \qquad (3.1)$$
$$-1 \le x \le 1, \qquad n = q, q+1, \dots;$$

the constant M_q depends only upon q.

We take $x \in [-1,1]$ arbitrarily and apply this theorem. We then have that for $n \in \mathbb{N}_{>0}$

$$\Delta_n(x) \le \frac{1}{n}$$

and for $n = 0$

$$\Delta_n(x) = 1$$

by definition of Δ_n. Clearly

$$\omega(g^{(q)}, \Delta_n(x)) = \max_{x,t,|t|\le \Delta_n(x)} \left|g^{(q)}(x+t) - g^{(q)}(x)\right|$$
$$\le 2 \cdot \|g^{(q)}\|_\infty.$$

When we substitute these results into (3.1), we get

$$|g(x) - p_n(x)| \le 2M_q \|g^{(q)}\|_\infty \frac{1}{n^q}.$$

Since this holds for all $x \in [-1,1]$, we have

$$\|g - p_n\|_{\infty,[-1,1]} \leq 2M_q \|g^{(q)}\|_\infty \frac{1}{n^q}, \qquad n \geq q. \tag{3.2}$$

We can scale our $f \in C^\infty([a,b])$, turning it into a function g in $C^\infty([-1,1])$. Then, turning to the sequence $\{c_k\}_{k=0}^\infty$, (3.2) implies

$$c_k = \inf_{p_k \in \mathcal{P}_k} \|f - p_k\|_\infty \leq C_q \|f^{(q)}\|_\infty \frac{1}{k^q}, \qquad k \geq q; \tag{3.3}$$

where C_q depends only on q.

Corollary 3.2. *From* (3.3) *we conclude that for every* $f \in C^\infty([a,b])$

$$\lim_{k\to\infty} k^l c_k = 0 \qquad \textit{for all } l \in \mathbb{N}$$

and thus $\{c_k\}_{k=0}^\infty \in (s)$.

The most important lemma of this section will now be stated and proved.

Lemma 3.3. *Let* $\mathcal{F} : L^2([a,b], w(x)dx) \to \ell^2$ *be the map assigning to every element* f *the sequence of coefficients with respect to the orthonormal basis of* $L^2([a,b], w(x)dx)$. *Then* $\mathcal{F}(f) \in (s)$ *for all* $f \in C^\infty([a,b])$.

Proof. We will denote $\mathcal{F}(f)$ by $\{u_k\}_{k=0}^\infty$. We will have a closer look at the sequence $\{\alpha_k\}_{k=0}^\infty$:

$$\begin{aligned}
\alpha_k &= \inf_{p_k \in \mathcal{P}_k} \|f - p_k\|_2 \\
&= \inf_{p_k \in \mathcal{P}_k} \left(\int_a^b |f(x) - p_k(x)|^2 w(x)dx \right)^{\frac{1}{2}} \\
&\leq \inf_{p_k \in \mathcal{P}_k} \left(\int_a^b \|f - p_k\|_\infty^2 w(x)dx \right)^{\frac{1}{2}} \\
&= \inf_{p_k \in \mathcal{P}_k} \|f - p_k\|_\infty \left(\int_a^b w(x)dx \right)^{\frac{1}{2}}
\end{aligned}$$

and therefore we have that for all $k \in \mathbb{N}$:

$$\alpha_k \leq \left(\int_a^b w(x)dx \right)^{\frac{1}{2}} \cdot c_k.$$

Combining this with corollary 3.2 yields that $\{\alpha_k\}_{k=0}^\infty \in (s)$.

Since $\mathcal{P}_k = \operatorname{span}(p_0, p_1, \ldots, p_k)$, we have

$$\alpha_k^2 = \sum_{r \geq k+1} |u_r|^2$$

and so

$$|u_{k+1}| \leq |\alpha_k| \, .$$

We have showed already that $\{\alpha_k\}_{k=0}^{\infty} \in (s)$. Therefore we may now conclude that $\{u_k\}_{k=0}^{\infty} \in (s)$. Since we defined $\{u_k\}_{k=0}^{\infty}$ as being the image of an arbitrary f in $C^{\infty}([a,b])$ under $\mathcal{F}$, this gives the sought for result.

$\square$

4 Orthogonal polynomials

We will now have a look at polynomials of degree $k \in \mathbb{N}$ and bound the supremum norm of their j-th derivative by their 2-norm. We will use the following, cf. [1, p. 40].

Theorem 4.1 (Markov Inequality)**.** *If $q_k \in \mathcal{P}_k([-1,1])$, then*

$$\|q_k'(x)\|_\infty \leq k^2 \|q_k\|_\infty, \qquad -1 \leq x \leq 1.$$

Let $p_k \in \mathcal{P}_k([a,b])$. If we want the theorem to be applicable to p_k, we have to transform the interval used in this theorem. Therefore we define the polynomial $\overline{p}_k \in \mathcal{P}_k([-1,1])$ as follows:

$$\overline{p}_k(x) = p_k\left(\frac{b-a}{2}x + \frac{a+b}{2}\right).$$

Then

$$\overline{p}_k'(x) = p_k'\left(\frac{b-a}{2}x + \frac{a+b}{2}\right) \cdot \frac{b-a}{2}$$

and therefore we get for all $x \in [a,b]$

$$\begin{aligned} \left|p_k'(x)\right| &= \left|\overline{p}_k'\left(\frac{2}{b-a}x - \frac{a+b}{b-a}\right) \cdot \frac{2}{b-a}\right| \\ &\leq k^2 \frac{2}{b-a} \|p_k\|_\infty. \end{aligned}$$

Since this holds for all $x \in [a,b]$, we get the following corollary.

Corollary 4.2. *If $p_k \in \mathcal{P}_k([a,b])$, then*

$$\|p_k'\|_\infty \leq k^2 \frac{2}{b-a} \|p_k\|_\infty.$$

We will now prove the following key statement.

Theorem 4.3. *Let $\delta > 0$, $C > 0$, $\nu > 0$ so that property* (1.1) *of the weight function w given in the introduction hold and let $j = 0, 1, 2, \ldots$. For all polynomials $p_k \in \mathcal{P}_k([a,b])$ with $k \in \mathbb{N}$ such that*

$$\frac{b-a}{2k^2} \leq \delta,$$

we have

$$\|p_k^{(j)}\|_\infty \leq \left(\frac{k!}{(k-j)!}\right)^2 \left(\frac{2}{b-a}\right)^j k^{\nu+1} \sqrt{\frac{2^\nu(\nu+1)(\nu+2)(\nu+3)}{C(b-a)^{\nu+1}}} \|p_k\|_2.$$

Proof. Let $p_k \in \mathcal{P}_k$ be arbitrary. Furthermore, let $x_t \in [a,b]$ such that

$$|p_k(x_t)| = \|p_k\|_\infty$$

and we may assume without loss of generality that $p_k(x_t) \geq 0$. This implies that

$$p_k(x_t) = \|p_k\|_\infty.$$

Take $\delta > 0$, $C > 0$, $\nu > 0$ so that property (1.1) of the weight function w given in the introduction hold. Then for all $x \in [a,b]$ with $p_k(x_t) - \|p_k'\|_\infty |x_t - x| \geq 0$

$$\begin{aligned} p_k(x) &\geq p_k(x_t) - \|p_k'\|_\infty \, |x_t - x| \\ &= \|p_k\|_\infty \left(1 - \frac{\|p_k'\|_\infty}{\|p_k\|_\infty} \, |x_t - x|\right) \\ &\geq \|p_k\|_\infty \left(1 - \frac{\|p_k'\|_\infty}{\|p_k\|_\infty} \, |x_t - x|\right). \end{aligned}$$

From corollary 4.2 we know that

$$\frac{\|p_k'\|_\infty}{\|p_k\|_\infty} \leq \frac{2k^2}{b-a}$$

and thus

$$p_k(x) \geq \|p_k\|_\infty \left(1 - \frac{2k^2}{b-a} \, |x_t - x|\right)$$

for all $x \in [a,b]$ with $|x_t - x| \leq \frac{b-a}{2k^2}$.

We will look at the 2-norm of p_k. Note that at least one of the two intervals

$$\left[x_t, x_t + \frac{b-a}{2k^2}\right] \qquad \text{and} \qquad \left[x_t - \frac{b-a}{2k^2}, x_t\right]$$

is contained in the interval $[a,b]$. Since both intervals will give the same result, we use the first interval mentioned for the following computation. Let $k \in \mathbb{N}_{>0}$ be big enough for $\frac{b-a}{2k^2} \leq \delta$ to hold.

$$\begin{aligned} \|p_k\|_2^2 &= \int_a^b |p_k(x)|^2 \, w(x) dx \\ &\geq \int_{x_t}^{x_t + \frac{b-a}{2k^2}} |p_k(x)|^2 \, w(x) dx \\ &\geq \int_{x_t}^{x_t + \frac{b-a}{2k^2}} \|p_k\|_\infty^2 \left(1 - \frac{2k^2}{b-a} (x - x_t)\right)^2 w(x) dx \\ &\geq \int_{x_t}^{x_t + \frac{b-a}{2k^2}} \|p_k\|_\infty^2 \left(1 - \frac{2k^2}{b-a} (x - x_t)\right)^2 C \, (x - x_t)^\nu \, dx \\ &= C \|p_k\|_\infty^2 \frac{(b-a)^{\nu+1}}{2^\nu k^{2(\nu+1)} (\nu+1)(\nu+2)(\nu+3)}. \end{aligned}$$

From this we know now

$$\|p_k\|_\infty \leq k^{\nu+1}\sqrt{\frac{2^\nu(\nu+1)(\nu+2)(\nu+3)}{C\,(b-a)^{\nu+1}}}\cdot\|p_k\|_2.$$

This result can be combined with corollary 4.2. That will give us

$$\|p_k'\|_\infty \leq k^2\frac{2}{b-a}k^{\nu+1}\sqrt{\frac{2^\nu(\nu+1)(\nu+2)(\nu+3)}{C\,(b-a)^{\nu+1}}}\cdot\|p_k\|_2.$$

When we apply corollary 4.2 repeatedly, say j times, we obtain the desired result for the j-th derivative of p_k, namely

$$\|p_k^{(j)}\|_\infty \leq \left(\frac{k!}{(k-j)!}\right)^2\left(\frac{2}{b-a}\right)^j k^{\nu+1}\sqrt{\frac{2^\nu(\nu+1)(\nu+2)(\nu+3)}{C(b-a)^{\nu+1}}}\|p_k\|_2.$$

□

The theorem we have just proven gives rise to a remarkable corollary, that we state below.

Corollary 4.4. *If $\{p_k\}_{k=0}^\infty$ is defined to be an orthonormal basis for $L^2([a,b],w(x)dx)$, then for every $j=0,1,2,\ldots$ there is a constant C_j such that*

$$\|p_k^{(j)}\|_\infty \leq C_j k^{2j+\nu+1}, \qquad k=0,1,2,\ldots.$$

5 Rapidly decreasing sequences

The results we found in section 4 will help us to prove here that $(s) \subset \mathcal{F}(C^\infty([a,b]))$.

Lemma 5.1. *Let $\mathcal{F} : L^2([a,b], w(x)dx) \to \ell^2$ be the map assigning to every element f the sequence of coefficients with respect to the orthonormal basis of $L^2([a,b], w(x)dx)$. Then $(s) \subset \mathcal{F}(C^\infty([a,b]))$.*

Proof. Take an arbitrary sequence $\{\alpha_n\}_{n=0}^\infty \in (s)$. In appendix A we prove that, for $l = 0, 1, 2, \ldots$, $C^l([a,b])$ is a Banach space with the norm defined by

$$\|f\|_l = \sum_{i=0}^{l} \|f^{(i)}\|_\infty.$$

Recall that $\{p_n\}_{n=0}^\infty$ was defined as being an orthonormal system of polynomials of degree $n \in \mathbb{N}$ that formed a basis for $L^2([a,b], w(x)dx)$. Furthermore, for all $n \in \mathbb{N}$, we know from corollary 4.4

$$\begin{aligned}
\|p_n\|_l &= \sum_{j=0}^{l} \|p_n^{(j)}\|_\infty \\
&\leq \sum_{j=0}^{l} C_j n^{2j+\nu+1} \\
&\leq \sum_{j=0}^{l} C_j n^{2j+\nu+1} \\
&\leq \tilde{C}_l n^{2l+\nu+1}
\end{aligned}$$

for some $\tilde{C}_l \geq 0$. We have

$$\sum_{n=0}^{\infty} |\alpha_n| \, \|p_n\|_l \leq \sum_{n=0}^{\infty} |\alpha_n| \, \tilde{C}_l n^{2l+\nu+1}.$$

Since $\{\alpha_n\}_{n=0}^\infty \in (s)$, we know that for all $m \in \mathbb{N}$ holds that $\lim_{n\to\infty} \alpha_n n^m = 0$ for all $m \in \mathbb{N}$. From a certain $n_0 \in \mathbb{N}$ on, the product $|\alpha_n| \, \tilde{C}_l n^{2l+\nu+1}$ will be less or equal to $\frac{1}{n^2}$ and we know that

$$\sum_{n=0}^{\infty} \frac{1}{n^2} < \infty.$$

We can now conclude that

$$\sum_{n=0}^{\infty} |\alpha_n| \, \|p_n\|_l < \infty.$$

Hence $\sum_{n=0}^{\infty} \alpha_n p_n$ is absolutely convergent in $C^l([a,b])$ for all $l \in \mathbb{N}$, which implies that $\sum_{n=0}^{\infty} \alpha_n p_n$ is convergent in $C^l([a,b])$. Now set

$$s_m := \sum_{n=0}^{m} \alpha_n p_n.$$

For $l = 0, 1, 2, \ldots$ let

$$g_l = \lim_{m \to \infty} s_m \qquad \text{in } C^l([a,b]).$$

Since the inclusion $C^{l+1}([a,b]) \subset C^l([a,b])$ is continuous for $l = 0, 1, 2, \ldots$, we see that

$$g_0 = g_1 = g_2 = \ldots = f$$

for some $f \in C^{\infty}([a,b])$. Certainly

$$s_m \to f \qquad \text{in } C([a,b])$$

implies that

$$s_m \to f \qquad \text{in } L^2([a,b], w(x)dx),$$

so that

$$\mathcal{F}(s_m) \to \mathcal{F}(f) \qquad \text{in } \ell^2.$$

Since $\mathcal{F}(s_m) = (\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots)$, we see that $\mathcal{F}(f) = (\alpha_0, \alpha_1, \alpha_2, \ldots)$, as required. Therefore $(s) \subset \mathcal{F}(C^{\infty}([a,b]))$. $\square$

6 Proof of the main theorem

We will prove the main theorem using different lemmas that have been proven in previous chapters. Firstly, define $[a, b]$ as a finite closed interval with $a, b \in \mathbb{R}$. Let w be a real-valued weight function in $L^1([a, b], dx)$ for $L^2([a, b], w(x)dx)$ with the following property: $\exists\, \delta > 0$, $\exists\, C > 0$ and $\exists\, \nu > 0$ such that for all $x_t \in [a, b]$

$$w(x) \geq C\,|x - x_t|^{\nu}$$

for almost all $x \in [a, b]$ with $|x - x_t| < \delta$. The sequence of polynomials $\{p_n\}_{n=0}^{\infty}$ is the corresponding orthonormal basis of $L^2([a, b], w(x)dx)$. The map

$$\mathcal{F} : L^2\left([a, b], w(x)dx\right) \to \ell^2 \tag{6.1}$$

assigns to every element the sequence of coefficients with respect to that orthonormal basis.

Lemma 3.3 and lemma 5.1, both of which we have given a proof, have told us that $\mathcal{F}$ is injective and surjective respectively. They therefore give the proof of the following theorem.

Theorem 6.1. *The restriction of the map $\mathcal{F}$ to the subset $C^{\infty}([a, b])$ assigning to every element f the sequence of coefficients with respect to the orthonormal basis of $L^2([a, b], w(x)dx)$, is a bijection between the vector spaces $C^{\infty}([a, b])$ and (s).*

This result gives rise to the statement that says that there is an isomorphism between $C^{\infty}([a, b])$ and (s) as topological vector spaces.

7 Concluding remarks

The proof that we have seen in this thesis has been constructed with the help of the proof by Pavec [2]. That paper covers a case which is more restrictive in the sense that it is assumed that $w(x) \geq C > 0$. At the same time it is more general because it treats bounded domains (with sufficiently smooth boundary) in an arbitrary dimension. The proof given in [2] that $\mathcal{F}(f) \in (s)$ is a bit brief, consisting basically of a reference to [1]. However, we have not been able to find a multi-variable result as needed in [2], and the proof of the underlying result in [1] (i.e., of our theorem 3.1) does not seem to generalize easily to several variables. The proof of the other key step, namely that $(s) \subset \mathcal{F}(C^\infty([a,b]))$, as given in [2], is in fact given for $w \equiv 1$ but this is easily adapted for $w(x) \geq C > 0$. Our conclusion at this moment is that the result as claimed in [2] and [3] may well be true but that the argumentation seems not to be entirely complete yet. For one dimension it is certainly sound, and as we have shown, it can be generalized to more general weight functions than those bounded away from zero.

8 References

[1] G.G. Lorentz, *Approximation of Functions*, Holt, Rinehart and Winston, New York, 1966.

[2] M.R. Pavec, *Isomorphisme entre* $\mathcal{D}(\overline{\Omega})$ *et* (s)*, d'après une note de M. Zerner*, Publications des Séminaires de Mathématiques (Fasc. 1: Séminaires d'Analyse fonctionnelle), Rennes **6** (1969).

[3] M. Zerner, *Développement en séries de polynômes orthonormaux des fonctions indéfiniment différentiables*, C. R. Acad. Sc. Paris Sér. A-B **268** (1969), A218-A220.

A Banach space $C^l([a,b])$

In this appendix we will prove that the space $C^l([a,b])$, $l = 0,1,2,\ldots$, with a certain norm is a Banach space.

Lemma A.1. *The space $C^l([a,b])$ with the norm defined by*

$$\|f\|_l = \sum_{i=0}^{l} \|f^{(i)}\|_\infty$$

is a Banach space.

Proof. Take an arbitrary Cauchy sequence $\{f_n\}_{n=0}^{\infty}$ in $C^l([a,b])$, which is equivalent with

$$\{f_n\}_{n=0}^{\infty}, \{f_n'\}_{n=0}^{\infty}, \ldots, \left\{f_n^{(l)}\right\}_{n=0}^{\infty}$$

all being Cauchy sequences in $C([a,b])$ with the supremum norm. We will prove the lemma using induction on $l \in \mathbb{N}$.

Suppose $l = 1$. We already know that $\{f_n\}_{n=0}^{\infty}$ and $\{f_n'\}_{n=0}^{\infty}$ are both Cauchy in $C([a,b])$ with the norm $\|.\|_\infty$. The space $C([a,b])$ with the norm $\|.\|_\infty$ is a Banach space. That means that there are $g, h \in C([a,b])$ such that

$$\lim_{n\to\infty} f_n = g \qquad \text{in } C([a,b]) \text{ with } \|.\|_\infty$$
$$\lim_{n\to\infty} f_n' = h \qquad \text{in } C([a,b]) \text{ with } \|.\|_\infty.$$

Let $x \in [a,b]$ be arbitrary. From the fundamental theorem of calculus we know that

$$f_n(x) := \int_a^x f_n'(t)dt$$

is a differentiable function, so that for $n \to \infty$, since $f_n' \to h$ uniformly,

$$g(x) := \int_a^x h(t)dt.$$

This implies that g is a differentiable function and $g' = h$, so that

$$\lim_{n\to\infty} f_n' = g' \qquad \text{in } C([a,b]) \text{ with } \|.\|_\infty.$$

Therefore $g \in C^1([a,b])$. Now we may conclude that the Cauchy sequence $\{f_n\}_{n=0}^{\infty}$ in $C^1([a,b])$ converges to an element of $C^1([a,b])$ and thus that $C^1([a,b])$ is a Banach space with the norm $\|.\|_1$.

Let us assume that $C^l([a,b])$ is a Banach space for all $l \leq L$. Now take $l = L+1$ and let $\{f_n\}_{n=0}^{\infty}$ be a C-sequence in $C^{(L+1)}([a,b])$. Then there exist $g \in C^L([a,b])$ and $h \in C([a,b])$ such that

$$\begin{aligned}
\lim_{n\to\infty} f_n &= g && \text{in } C([a,b]) \text{ with } \|.\|_\infty \\
\lim_{n\to\infty} f_n' &= g' && \text{in } C([a,b]) \text{ with } \|.\|_\infty \\
&\vdots \\
\lim_{n\to\infty} f_n^{(L)} &= g^{(L)} && \text{in } C([a,b]) \text{ with } \|.\|_\infty \\
\lim_{n\to\infty} f_n^{(L+1)} &= h && \text{in } C([a,b]) \text{ with } \|.\|_\infty.
\end{aligned}$$

Again, we will use the fundamental theorem of calculus. We know that

$$f_n^{(L)}(x) := \int_a^x f_n^{(L+1)}(t)dt$$

is a differentiable function, so that for $n \to \infty$, since $f_n^{(L+1)} \to h$ uniformly,

$$g^{(L)}(x) := \int_a^x h(t)dt.$$

This results in $g^{(L)}$ being differentiable and therefore $g^{(L+1)} = h$. Then

$$\lim_{n\to\infty} f_n^{(L+1)} = g^{(L+1)} \qquad \text{in } C([a,b]) \text{ with } \|.\|_\infty.$$

We now know that $g^{(L)} \in C^1([a,b])$ and thus that the Cauchy sequence $\{f_n\}_{n=0}^{\infty}$ in $C^l([a,b])$ converges to an element of $C^l([a,b])$. We conclude that $C^l([a,b])$ is a Banach space with the norm $\|.\|_l$. □