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Operators between $C(K)$-spaces

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Contents

1 Introduction

In this thesis we characterize two types of operators between $C(K)$-spaces: lattice and algebra homomorphisms. This characterization is nice, because something that seems to be rather big and incomprehensible, suddenly becomes simple and comprehensible.
With $C(K)$ is meant the space of all continuous functions on a compact Hausdorff space K. One can give these spaces a special structure, namely the structure of a lattice or an algebra. In this thesis we consider the morphisms between $C(K)$-spaces that respect these structures.
$C(K)$-spaces are not simply lattices and algebras. Because of the completeness with respect to the norm $||f|| = \sup\{|f(x)| : x \in K\}$, these spaces are also Banach lattices and Banach algebras. We treat this in the second section. We also give some more examples of Banach lattices and Banach algebras.
To prove the characterization theorems, we need some useful results, like The Riesz Representation Theorem. We discuss these results in Section 3.
Before we formulate and prove the characterization theorems in Section 5, in Section 4 we give the proof of some lemmas that are essential in the proof.
Finally, we consider the lattice and algebra automorphisms. The structure of the groups of such automorphisms can be described completely, and this is done in the final section.
At the end of this introduction, I would like to thank my advisor M.F.E. de Jeu for his kind help and the friendly contact during my project.

2 Banach lattices and Banach algebras

In this section we will define Banach lattices and Banach algebras and their morphisms. We will also give some examples. To define Banach lattices, we need some preparation. First recall that a relation is an order relation if it is reflexive, antisymmetric and transitive.

Definition 2.1. *An ordered vector space is a vector space E over $\mathbb{R}$ equipped with an order relation $\geq$, such that*

1. *If $e \geq f$, then $e + g \geq f + g$ for all $g \in E$.*
2. *If $e \geq f$, then $\alpha e \geq \alpha f$ for all $\alpha \in \mathbb{R}$, with $\alpha \geq 0$.*

Definition 2.2. *A vector lattice is an ordered vector space E such that for every two elements $e, f \in E$ the supremum $e \vee f$ exists in E.*

Two vectors $e, f \in E$ have a supremum g in E if $g \geq e$ and $g \geq f$, and whenever h is an upper bound of $\{e, f\}$, then $h \geq g$. If the supremum of $e, f \in E$ exists, the infimum $e \wedge f$ also exists, indeed $e \wedge f = -(-e \vee -f)$. Before we go further, we still need a short definition, that corresponds to our intuition.

Definition 2.3. *The absolute value of a vector $e \in E$ is given by $|e| := e \vee -e$.*

Now we can define a lattice norm.

Definition 2.4. *A norm $|| \cdot ||$ on a vector lattice E is called a lattice norm whenever $|e| \leq |f|$ in E implies $||e|| \leq ||f||$.*

A vector lattice equipped with a lattice norm is called a normed vector lattice. Finally we come to our definition of a Banach lattice.

Definition 2.5. *A Banach lattice is a vector lattice E with a lattice norm such that E is complete.*

Let us consider some examples.

Example 2.6. Let X be a nonempty set. Then $B(X)$, the set of all real-valued bounded functions defined on X, is a vector lattice under the ordering $f \geq g$ whenever $f(x) \geq g(x)$ for all $x \in X$. We have $(f \vee g)(x) = \max\{f(x), g(x)\}$ for all $x \in X$. Under the norm $||f||_\infty = \sup\{|f(x)| : x \in X\}$, $B(X)$ is a Banach lattice.

Example 2.7. Let ℓ^1 denote the vector space of all real sequences $x = (x_1, x_2, ...)$ such that $\sum_{i=1}^{\infty} |x_i| < \infty$. With the pointwise algebraic operations and coördinatewise ordering (i.e., $x \geq y$ if $x_1 \geq y_1, x_2 \geq y_2, ...$ for each $x, y \in \ell^1$) ℓ^1 is a vector lattice. We have $x \vee y = (\max\{x_1, y_1\}, \max\{x_2, y_2\}, ...)$. Under the norm $||x|| = \sum_{i=1}^{\infty} |x_i|$ the vector lattice ℓ^1 becomes a Banach lattice.

Example 2.8. Let X be a topological space.

- Denote by $C_c(X)$ the vector space of all continuous functions on X that have compact support. In other words, a function $f : X \to \mathbb{R}$ which is continuous, belongs to $C_c(X)$ if the set $\{x \in X : f(x) \neq 0\}$ has compact closure.

- Let $C_b(X)$ be the space of all continuous bounded functions on X. So, $C_b(X)$ contains all continuous functions $f : X \to \mathbb{R}$ such that $\sup\{|f(x)| : x \in X\} < \infty$.

- $C_0(X)$ is the set of all continuous functions $f : X \to \mathbb{R}$ such that for all $\epsilon > 0$ the set $\{x \in X : |f(x)| \geq \epsilon\}$ is compact.

Under the pointwise ordering and the supremum norm (as in Example 2.6) $C_c(X)$, $C_b(X)$ and $C_0(X)$ are normed vector lattices. Here we have again $(f \vee g)(x) = \max\{f(x), g(x)\}$ for all $x \in X$. Moreover, $C_b(X)$ and $C_0(X)$ are Banach lattices. If X is not compact, $C_c(X)$ is not in general a Banach lattice, because it need not be complete. If X is compact, we have $C_c(X) = C_0(X) = C_b(X)$, and these spaces are all Banach lattices.

Now we move on to Banach algebras. First, we need the definition of an algebra over a field.

Definition 2.9. *An algebra over a field $\mathbb{F}$ is a vector space A over $\mathbb{F}$ that has a multiplication defined on it such that for all $\alpha \in \mathbb{F}$ and $a, b, c \in A$*

1. *the associative property holds: $a(bc) = (ab)c$;*
2. *the distributive properties hold: $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$;*
3. *the following property holds: $\alpha(ab) = (\alpha a)b = a(\alpha b)$.*

Definition 2.10. *A Banach algebra is an algebra A over the field $\mathbb{R}$ or $\mathbb{C}$ with a norm $||\cdot||$ such that A is complete and such that $||ab|| \leq ||a|| ||b||$ for all $a, b \in A$.*

Some books also require that A contains a unit element, but we do not. We will consider some examples with and without unit element.

Example 2.11. Let $C(K)$ be the space of all complex- or real-valued continuous functions on a nonempty compact space K, equipped with the supremum norm. Define multiplication in the usual way: $(fg)(x) = f(x)g(x)$ for all $f, g \in C(K)$ and $x \in K$. This makes $C(K)$ into a Banach algebra, which is also commutive. The constant function 1 is the unit element.

Example 2.12. Let X be a Banach space. Then $B(X)$ is the algebra of all bounded linear operators on X, whereby the multiplication is composition. This is a Banach algebra, with respect to the operator norm $||A|| = \sup\{||Ax|| : x \in X, ||x|| < 1\}$. The identity operator is its unit element.

Example 2.13. $B(X)$ as in Example 2.6 with the usual multiplication is a Banach algebra. The constant function 1 is its unit element.

Example 2.14. The space ℓ^1, defined in Example 2.7, equipped with the pointwise multiplication is a Banach algebra. ℓ^1 does not contain a unit element.

Example 2.15. Just as in Example 2.8 we consider $C_c(X)$, $C_b(X)$ and $C_0(X)$. Besides $\mathbb{R}$, we can also take $\mathbb{C}$ as field. These spaces are algebras when the multiplication is defined pointwise. They are also commutative. Moreover $C_b(X)$ and $C_0(X)$ are Banach algebras. If X is not compact, $C_0(X)$ and $C_c(X)$ need not have a unit element. If X is compact, the three spaces are equal and $C_c(X)$ is also a Banach algebra. We are then in the case of Example 2.11.

For more examples of Banach algebras, see [4, Chapter VII.1]

Now we want to consider morphisms between these two types of spaces.

Definition 2.16. *Let $T : X \to Y$ be a linear map between two vector lattices X and Y. Then T is a lattice homomorphism if $T(x \vee y) = Tx \vee Ty$ for all $x, y \in X$.*

It is easy to see that T is positive. Let $x \in X$ with $x \geq 0$. Then $x \vee 0 = x$, hence

$$\begin{aligned} T(x) &= T(x \vee 0) \\ &= T(x) \vee T(0) \\ &= T(x) \vee 0 \\ &\geq 0. \end{aligned}$$

Just like the definition of a lattice homomorphism the definition of an algebraic homomorphism is intuitively clear:

Definition 2.17. *Let $T : X \to Y$ be a linear map between two algebras X and Y. Then T is a algebra homomorphism if $T(xy) = T(x)T(y)$ for all $x, y \in X$.*

We do not require that $T(\mathbf{1_X}) = \mathbf{1_Y}$, when X and Y contain unit elements denoted by $\mathbf{1_X}$ and $\mathbf{1_Y}$. Note that the kernel of T is an ideal.

We will give two examples.

Example 2.18. Let Ω be a compact Hausdorff space. The real space $C(\Omega)$ and $\mathbb{R}$ are algebras and lattices. Let $\omega \in \Omega$ be an arbitrary element. Define

$$\begin{aligned} \delta_\omega : C(\Omega) &\to \mathbb{R} \\ f &\mapsto f(\omega). \end{aligned}$$

Clearly, δ_ω is linear. Moreover, $\delta_\omega(f \vee g) = (f \vee g)(\omega) = \max\{f(\omega), g(\omega)\} = \delta_\omega(f) \vee \delta_\omega(g)$ for all $f, g \in C(\Omega)$. Hence δ_ω is a lattice homomorphism. Finally, $\delta_\omega(fg) = (fg)(\omega) = f(\omega)g(\omega) = \delta_\omega(f)\delta_\omega(g)$ for all $f, g \in C(\Omega)$, thus δ_ω is also an algebra homomorphism.

The above example plays an important role in the proof of our theorems.

Example 2.19. Let T be the map given by

$$\begin{aligned} T : C([0,1]) &\to C([0,1]) \\ f &\mapsto \left(\tilde{f} : x \mapsto xf(x)\right). \end{aligned}$$

Then T is a lattice homomorphism, since

$$\begin{aligned} T(f \vee g)(x) &= x \cdot (f \vee g)(x) \\ &= x \cdot \max\{f(x), g(x)\} \\ &= \max\{x \cdot f(x), x \cdot g(x)\} \\ &= (T(f) \vee T(g))(x) \end{aligned}$$

for all $f, g \in C([0,1])$ and $x \in [0,1]$. But if we take $f = g$ the constant function 1, and $x = \frac{1}{2}$, we find:

$$T(fg)(x) = x(fg)(x) = \frac{1}{2} \neq \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = xf(x) \cdot xg(x) = (T(f)T(g))(x).$$

Hence T is not an algebra homomorphism.

3 Useful results

In this section we will give some results that are useful for the next section. Before we come to the first theorem, Urysohn's lemma, we need the definition of a normal space.

Definition 3.1. *A topological space X is a normal space if, given any disjoint closed sets E and F, there are open neighbourhoods U of E and V of F that are also disjoint.*

Theorem 3.2 (Urysohn's lemma)**.** *[2, Theorem 10.5] For a topological space X the following statements are equivalent:*

1. *X is a normal space.*
2. *Every pair of disjoint closed sets can be separated by a continuous function. In other words: for every pair A, B of disjoint and closed subsets of X there exists a continuous function $f : X \to [0, 1]$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.*

Since a compact Hausdorff space is normal [5, Example 3.5.11], Urysohn's lemma is very useful to us.

Another result we will use is The Riesz Representation Theorem. First some definitions.

Definition 3.3. *Let Ω be a set. A collection A of subsets of Ω is called a σ-algebra (in Ω) if it has the following properties:*

1. *$\Omega \in A$;*
2. *If $U \in A$ then $U^c \in A$;*
3. *If $(U_n)_{n\in\mathbb{N}} \in A$ then $\bigcup_{n\in\mathbb{N}} U_n \in A$.*

For every collection E of subsets of Ω, there is a smallest σ-algebra $\sigma(E)$ which contains E. $\sigma(E)$ is called the σ-algebra generated by E and E is called a generator of $\sigma(E)$.

Definition 3.4. *Let A be a σ-algebra in a set Ω and $\mu : A \to [0, +\infty]$ a function. μ is called a measure on A if $\mu(\emptyset) = 0$ and if for every sequence $(U_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets from A*

$$\mu\left(\bigcup_{n\in\mathbb{N}} U_n\right) = \sum_{n=1}^{\infty} \mu(U_n).$$

Definition 3.5. *Let Ω be a topological space. Denote by O the collection of its open subsets. The Borel σ-algebra $B(\Omega)$ in Ω is the σ-algebra generated by O, in other words $B(\Omega) = \sigma(O)$.*

From now on let X be a locally compact Hausdorff space. A space X is locally compact if each element of X has a compact neighbourhood.

Definition 3.6. *A measure μ on the σ-algebra $B(X)$ is called:*

1. *a Borel measure on X if $\mu(K) < \infty$ for every compact $K \subset X$;*
2. *inner regular if for every $A \in B(X)$*

$$\mu(A) = \sup\{\mu(K) : K\text{compact} \subset A\};$$

3. *outer regular if for every $A \in B(X)$*

$$\mu(A) = \inf\{\mu(U) : A \subset U\text{open}\};$$

4. *regular if it is both inner regular and outer regular.*

Now we have enough luggage for the theorem.

Theorem 3.7 (The Riesz Representation Theorem)**.** *[2, Theorem 38.3] For every positive linear functional F on $C_c(X)$, there exists a unique regular Borel measure μ such that $F(f) = \int f d\mu$ holds for every $f \in C_c(X)$.*

The next theorem is about the support of a measure.

Theorem 3.8. *[2, Theorem 31.12] Let μ be a regular Borel measure on X. Then there exists a unique closed subset E of X with the following two properties:*

1. *$\mu(E^c) = 0$;*
2. *If V is an open set such that $E \cap V \neq \emptyset$, then $\mu(E \cap V) > 0$.*

The set E is called the support of μ and is denoted by $\text{supp}(\mu)$.

4 Preparatory lemmas

From now on let Ω and Q be compact Hausdorff spaces. We can see $C(\Omega)$ and $C(Q)$ as Banach lattices, as in Example 2.8. They are also Banach algebras according to Example 2.11. Before we will describe the algebra and latticehomomorphisms between $C(K)$-spaces in the characterization theorems, we need a lemma which characterizes the lattice homomorphisms from a $C(\Omega)$-space to the real numbers.

Lemma 4.1. *A non-zero linear functional $\phi : C(\Omega) \to \mathbb{R}$ is a lattice homomorphism if and only if there exists some $c > 0$ and some $\omega_0 \in \Omega$, such that $\phi = c\delta_{\omega_0}$, with δ_{ω_0} as in Example 2.18.*
In this case c and ω_0 are both uniquely determined.

Proof. Let $\phi = c\delta_{\omega_0}$ for some $c > 0$ and $\omega_0 \in \Omega$. Clearly, ϕ is linear. Let $f, g \in C(\Omega)$. Because $c > 0$, we have

$$\begin{aligned} \phi(f \vee g) &= c \cdot \delta_{\omega_0}(f \vee g) \\ &= c \cdot (f \vee g)(\omega_0) \\ &= c \cdot \max\{f(\omega_0), g(\omega_0)\} \\ &= \max\{c \cdot f(\omega_0), c \cdot g(\omega_0)\} \\ &= \phi(f) \vee \phi(g). \end{aligned}$$

Thus ϕ is a lattice homomorphism.
For the converse, assume that $\phi : C(\Omega) \to \mathbb{R}$ is a lattice homomorphism. By definition, ϕ is linear and positive. Hence by the Riesz Representation Theorem, there exists a unique regular Borel measure μ on Ω such that $\phi(f) = \int f d\mu$ for all $f \in C(\Omega)$.
Now we will prove that $\text{supp}(\mu)$ contains exactly one element. Assume that the support of μ contains two distinct points, say s and t. Because Ω is Hausdorff, we can find two open sets U and V, such that $s \in U$, $t \in V$ and $U \cap V = \emptyset$. Let $T = \Omega \backslash U$ and $S = \Omega \backslash V$. Then T and S are closed. The singletons $\{s\}$ and $\{t\}$ are also closed, because Ω is Hausdorff. According to Urysohn's lemma, there exists a continuous function $f : \Omega \to [0, 1]$ with the following properties: $f(s) = 1$ and $f(a) = 0$ for all $a \in T$. In the same way we can make a continuous function $g : \Omega \to [0, 1]$ such $g(t) = 1$ and $g(a) = 0$ for all $a \in S$. So we have two functions $f, g \in C(\Omega)$ satisfying $f(s) = g(t) = 1$ and $f \wedge g = 0$, indeed $\min\{f(x), g(x)\} = 0$ for all $x \in \Omega$. Moreover, $\int f d\mu > 0$. To see this, recall that $F := f^{-1}((\frac{1}{2}, \infty))$ is an open set, because of the continuity of f. The element s belongs to F, so $\text{supp}(\mu) \cap F \neq \emptyset$ and therefore $\mu(\text{supp}(\mu) \cap F) > 0$, according to Theorem 3.6. Hence we have

$$\int f d\mu \geq \int_{\text{supp}(\mu) \cap F} f d\mu \geq \frac{1}{2}\mu(\text{supp}(\mu) \cap F) > 0$$

In the same way, we find that $\int g d\mu > 0$. This implies

$$\begin{aligned} 0 &= \phi(f \wedge g) \\ &= \min\{\phi(f), \phi(g)\} \\ &= \min\{\int f d\mu, \int g d\mu\} \\ &> 0, \end{aligned}$$

which is a contradiction. Thus $\text{supp}(\mu)$ contains at most one element. Since $\text{supp}(\mu)$ can not be empty, because ϕ is not the zero function, $\text{supp}(\mu)$ contains only one element, say ω_0. Thus,

$$\phi(f) = \int f d\mu = \int_{\{\omega_0\}} f d\mu = f(\omega_0)\mu(\{\omega_0\}) = \mu(\{\omega_0\}) \cdot \delta_{\omega_0}(f).$$

So we find $c = \mu(\{\omega_0\})$. This implies $c > 0$, because μ is non-negative. Finally, ω_0 and c are unique, because μ is unique. □

We also need the characterization of the algebra homomorphisms from a $C(\Omega)$-space over $\mathbb{R}$ or $\mathbb{C}$ to respectively the real or complex numbers.

Lemma 4.2. *Let $C(\Omega)$ be the real or complex vectorspace of functions. A non-zero linear functional $\phi : C(\Omega) \to \mathbb{F}$ (with $\mathbb{F} = \mathbb{R}$ if $C(\Omega)$ is real and $\mathbb{F} = \mathbb{C}$ if $C(\Omega)$ is complex) is an algebra homomorphism if and only if there exists a point $\omega_0 \in \Omega$ such that $\phi = \delta_{\omega_0}$.*
In this case ω_0 is uniquely determined.

Proof. Let $\phi = \delta_{\omega_0}$ for an $\omega_0 \in \Omega$. As in the previous lemma, ϕ is clearly linear. Furthermore, $\phi(fg) = \phi(f)\phi(g)$ for all $f, g \in C(\Omega)$, as we saw in Example 2.18. So ϕ is an algebra homomorphism.
For the converse, assume that $\phi : C(\Omega) \to \mathbb{F}$ is a non-zero algebra homomorphism. Let $\mathbf{1}$ be the constant function one on Ω. Because of the multiplicativity we have $\phi(\mathbf{1}) = \phi(\mathbf{1}^2) = (\phi(\mathbf{1}))^2$. So $\phi(\mathbf{1}) = 0$ or $\phi(\mathbf{1}) = 1$. Assume $\phi(\mathbf{1}) = 0$. Then $\phi(f) = \phi(f \cdot \mathbf{1}) = \phi(f)\phi(\mathbf{1}) = 0$ for all $f \in C(\Omega)$, which is a contradiction with the fact that ϕ is non-zero. Hence $\phi(\mathbf{1}) = 1$.
Let $M = \{f \in C(\Omega) : \phi(f) = 0\}$ be the kernel of ϕ. Denote by N_ω the set $\{f \in C(\Omega) : f(\omega) = 0\}$. Our claim is now that there exists a point $\omega_0 \in \Omega$ such that $M \subset N_{\omega_0}$.
Let us assume that there is no such point. Then for each $\omega \in \Omega$ we can find a function f_ω in M which is not in N_ω, in other words $f_\omega(\omega) \neq 0$. Since f_ω is continuous, the set $V_\omega = f_\omega^{-1}(\mathbb{F}\backslash\{0\})$ is open. We have $\omega \in V_\omega$ and $f_\omega(t) \neq 0$ for all $t \in V_\omega$. Then $(V_\omega)_{\omega \in \Omega}$ is an open cover of Ω. By the compactness of Ω, there exists a finite open subcover $(V_{\omega_i})_{i=1,\ldots,n}$ of Ω.
Let us consider the function

$$x = \sum_{i=1}^{n} f_{\omega_i} \overline{f_{\omega_i}},$$

where $\overline{f}(t) = \overline{f(t)}$ for all $t \in \Omega$. Let $t \in \Omega$. There exists an j such that $t \in V_{\omega_j}$, so $f_{\omega_j}(t) \neq 0$. We have $f_{\omega_i}(t)\overline{f_{\omega_i}(t)} \geq 0$ for all i, hence $x(t) > 0$. Therefore, we have $y = \frac{1}{x} \in C(\Omega)$ and

$$\phi(x)\phi(y) = \phi(xy) = \phi(\mathbf{1}) = 1.$$

Hence $\phi(x) \neq 0$. On the other hand, as $\phi(f_\omega) = 0$ for each $\omega \in \Omega$, we find $\phi(x) = 0$, because $\ker(\phi)$ is an ideal. This is a contradiction. Thus, there exists some $\omega_0 \in \Omega$ such that $M \subset N_{\omega_0}$.
Now we shall show that $N_{\omega_0} \subset M$. To do this, let $f \in C(\Omega)$ satisfy $f(\omega_0) = 0$. Consider the function $u = \phi(f)\mathbf{1} - f$. Then $\phi(u) = \phi(f)\phi(\mathbf{1}) - \phi(f) = 0$, hence $u \in M$. So u is also in N_{ω_0}, thus $u(\omega_0) = 0$. This together with $u(\omega_0) = \phi(f) - f(\omega_0) = \phi(f)$, gives us $\phi(f) = 0$. Hence $f \in M$, and thus $M = \{f \in C(\Omega) : f(\omega_0) = 0\}$.
Finally, let $f \in C(\Omega)$. Then $f - \phi(f)\mathbf{1} \in M$ because

$$\phi(f - \phi(f)\mathbf{1}) = \phi(f) - \phi(f) \cdot 1 = 0.$$

So $f - \phi(f)\mathbf{1} \in N_{\omega_0}$, hence $f(\omega_0) - \phi(f) = (f - \phi(f)\mathbf{1})(\omega_0) = 0$. In other words $\phi(f) = f(\omega_0)$.
The uniqueness of ω_0 is a special case of the uniqueness of c and ω_0 in Lemma 4.1.
□

5 The characterization theorems

Now we are ready to characterize the lattice homomorphisms between real $C(\Omega)$-spaces.

Theorem 5.1. *A positive operator $T : C(\Omega) \to C(Q)$ is a lattice homomorphism if and only if there exist a non-negative function $w \in C(Q)$ and a mapping $\tau : Q \to \Omega$ which is continuous on the set $\{q \in Q : w(q) > 0\}$, such that for each $f \in C(\Omega)$ and each $q \in Q$ we have*

$$Tf(q) = w(q)f(\tau(q)).$$

Moreover, in this case, $w = T\mathbf{1}_\Omega$ and the mapping τ is uniquely determined on the set $\{q \in Q : w(q) > 0\}$.

Proof. Let $T(f) = w \cdot (f \circ \tau)$. T is well-defined, because $T(f)$ is continuous. Indeed, if f is the constant function zero, $T(f)$ is also the zero function. So assume that f is not the zero function and let $q \in Q$ with $w(q) > 0$. Then τ is continuous in q, and we know that w and f are continuous functions, hence $T(f) = w \cdot (f \circ \tau)$ is continuous in q. So let $q \in Q$ with $w(q) = 0$ and let $\epsilon > 0$. Since w is continuous, the set $U_\delta = w^{-1}((-\delta, \delta))$ is an open neighbourhood of q for all $\delta > 0$, such that $|w(u)| < \delta$ for all $u \in U_\delta$. Let $\delta = \frac{\epsilon}{||f||_\infty}$. Note that we can divide by $||f||_\infty$, since f is not the zero function. Let $u \in U_\delta$. Then

$$\begin{aligned} |T(f)(q) - T(f)(u)| &= |w(q)f(\tau(q)) - w(u)f(\tau(u))| \\ &= |w(u)f(\tau(u))| \\ &\leq |w(u)| \cdot ||f||_\infty \\ &< \delta \cdot ||f||_\infty \\ &= \epsilon, \end{aligned}$$

hence $T(f)$ is continuous in q.
Clearly, T is linear. Because $w \geq 0$, we have $T(f \vee g)(q) = (T(f) \vee T(g))(q)$, just as in Lemma 4.1. Hence T is a lattice homomorphism.
For the converse, assume that $T : C(\Omega) \to C(Q)$ is a lattice homomorphism.
The following step is important. Let $q \in Q$. We compose T with δ_q, so we get

$$\delta_q \circ T : C(\Omega) \to \mathbb{R}$$

with $(\delta_q \circ T)(f) = \delta_q(T(f)) = T(f)(q)$. Because T and δ_q are lattice homomorphisms and the composition of two lattice homomorphisms is again a lattice homomorphism, $\delta_q \circ T$ is a lattice homomorphism. Thus, by Lemma 4.1, there exist a unique constant $c = w(q) \geq 0$ and an $\omega_0 = \tau(q) \in \Omega$ (only unique when $c > 0$) such that

$$T(f)(q) = (\delta_q \circ T)(f) = c \cdot \delta_{\tau(q)}(f) = w(q)f(\tau(q))$$

for all $f \in C(\Omega)$. So for all $f \in C(\Omega)$ and $q \in Q$ we have $T(f)(q) = w(q)f(\tau(q))$. Let $f = \mathbf{1}_\Omega$. Then

$$T(f)(q) = w(q)f(\tau(q)) = w(q) \cdot 1 = w(q),$$

thus $w = T(\mathbf{1}_\Omega)$ and $w \in C(Q)$.
We have already said that τ is unique on the set $\{q \in Q : w(q) > 0\}$, according to Lemma 4.1. What remains is to prove that τ is continuous on $\{q \in Q : w(q) > 0\}$.
Let $q \in Q$ such that $w(q) > 0$ but assume that τ is not continuous in q. That means there exists an open neighbourhood V of $\tau(q)$ such that for all open neighbourhoods U of q we have $\tau[U] \not\subset V$. By Urysohn's lemma there is a continuous $f : \Omega \to \mathbb{R}$ with $f(\tau(q)) = 1$ and $f(\omega) = 0$ for all $\omega \in \Omega \backslash V$. The function $T(f) = w \cdot (f \circ \tau)$ is an element of $C(Q)$, hence continuous. Let us define another function:

$$\begin{aligned} g : \{q \in Q : w(q) > 0\} &\to \mathbb{R} \\ q &\mapsto \frac{w(q) \cdot f(\tau(q))}{w(q)} = (f \circ \tau)(q). \end{aligned}$$

Clearly, g is also continuous. So $g^{-1}\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right)$ is open in $\{q \in Q : w(q) > 0\}$, hence in X because $\{q \in Q : w(q) > 0\}$ is open in X. Because $f(\tau(q)) = 1$ and $w(q) > 0$, we have $q \in g^{-1}\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right)$. Moreover, because f is zero on $\Omega \backslash V$, it follows that $f^{-1}\left(\frac{1}{2}, \frac{3}{2}\right) \subset V$, and thus

$$\begin{aligned} g^{-1}\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right) &= (f \circ \tau)^{-1}\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right) \cap \{q \in Q : w(q) > 0\} \\ &= \tau^{-1}\left(f^{-1}\left(\frac{1}{2}, \frac{3}{2}\right)\right) \cap \{q \in Q : w(q) > 0\} \\ &\subset \tau^{-1}(V). \end{aligned}$$

But that means that the image of $g^{-1}\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right)$ under τ is contained in V. So we have found an open neighbourhood U of q such that $\tau[U] \subset V$. This is a contradiction, hence τ is continuous in q. □

In $C(K)$-spaces we have the identity $(f \vee g)(x) = \max\{f(x), g(x)\}$. Therefore, $f \vee g$ and $f \wedge g$ satisfy also the identities

$$f \vee g = \frac{1}{2}(f + g + |f - g|) \text{ and } f \wedge g = \frac{1}{2}(f + g - |f - g|),$$

which we will use in the proof of the next theorem.

Theorem 5.2. *Every algebra homomorphism between real $C(K)$-spaces is a lattice homomorphism. However, the converse is false.*

Proof. Let $T : C(\Omega) \to C(Q)$ be an algebra homomomorphism. To prove that T is a lattice homomorphism, it is enough to show that $|T(f)| = T(|f|)$ for all $f \in C(\Omega)$. Because then we have

$$\begin{aligned} T(f \vee g) &= T\left(\frac{1}{2}(f + g + |f - g|)\right) \\ &= \frac{1}{2}(Tf + Tg + T|f - g|) \\ &= \frac{1}{2}(Tf + Tg + |Tf - Tg|) \\ &= Tf \vee Tg \end{aligned}$$

for all $f, g \in C(\Omega)$.
We know that T is positive. Indeed, if $g \geq 0$, then

$$Tg = T((\sqrt{g})^2) = (T(\sqrt{g}))^2 \geq 0.$$

Now if $f \in C(\Omega)$, we have $T(|f|) \geq 0$, because $|f| \geq 0$. We also have

$$|T(f)|^2 = (T(f))^2 = T(f^2) = T(|f|^2) = (T(|f|))^2,$$

whereby we used that $v^2 = |v|^2$, for $v = T(f)$ and $v = f$ and that T is an algebra homomorphism.
So $|T(f)(x)|^2 = (T(|f|)(x))^2$ for all $x \in Q$. And since $|T(f)(x)| \geq 0$ and $T(|f|)(x) \geq 0$, we have $|T(f)(x)| = T(|f|)(x)$ for all $x \in Q$. So we find $|T(f)| = T(|f|)$.
For an example of a lattice homomorphism which is not an algebra homomorphism, see Example 2.19. □

In the next theorem we will characterize the algebra homomorpisms.

Theorem 5.3. *An operator $T : C(\Omega) \to C(Q)$, with $C(\Omega)$ and $C(Q)$ real or complex spaces, is an algebra homomorphism if and only if there exists a clopen subset V of Q and a mapping $\tau : Q \to \Omega$ which is continuous on V such that for each $f \in C(\Omega)$ and each $q \in Q$ we have*

$$Tf(q) = \chi_V(q) f(\tau(q)).$$

In that case $\chi_V = T\mathbf{1}_\Omega$ and T is uniquely determined on V.

Proof. Assume that $Tf = \chi_V \cdot (f \circ \tau)$. T is well-defined, because $T(f)$ is continuous, since T is a special case of Theorem 5.1. T is also linear. Moreover,

$$\begin{aligned} T(f \cdot g)(x) &= \chi_V(x)(f \cdot g)(\tau(x)) \\ &= \chi_V(x) f(\tau(x)) g(\tau(x)) \\ &= (\chi_V(x))^2 f(\tau(x)) g(\tau(x)) \\ &= T(f)(x) \cdot T(g)(x), \end{aligned}$$

for all $f, g \in C(\Omega)$ and $x \in Q$ hence T is an algebra homomorphism.
For the converse, we distinguish two cases:

1. Assume that $T : C(\Omega) \to C(Q)$ is an algebra homomorphism between real $C(K)$-spaces.
   According to Theorem 5.2, T is a lattice homomorphism. So, by Theorem 5.1, there is a map $\tau : Q \to \Omega$ and a $w \in C(Q)$ such that for all $f \in C(\Omega)$ and $q \in Q$ we have
   $$Tf(q) = w(q) f(\tau(q)).$$
   And we also know that $w = T(\mathbf{1_\Omega})$. When we use that T is an algebra homomorphism, we get
   $$w^2 = (T(\mathbf{1_\Omega}))^2 = T(\mathbf{1_\Omega}^2) = T(\mathbf{1_\Omega}) = w$$
   This implies that for all $q \in Q$ we have $w(q) = 0$ or $w(q) = 1$. So $w = \chi_V$ for a unique clopen subset $V \subset Q$. From Theorem 5.1 it follows that τ is unique and continuous on V.

2. Assume that $T : C(\Omega) \to C(Q)$ is an algebra homomorphism between complex $C(K)$-spaces.
   Let $q \in Q$. Just as in the proof of Theorem 5.1, we compose T with δ_q, so we get
   $$\delta_q \circ T : C(\Omega) \to \mathbb{C}$$
   with $(\delta_q \circ T)(f) = \delta_q(T(f)) = T(f)(q)$. Because T and δ_q are algebra homomorphisms and the composition of two algebra homomorphisms is again an algebra homomorphism, $\delta_q \circ T$ is a algebra homomorphism. Thus, by Lemma 4.2, if $\delta_q \circ T \neq 0$, there exists a unique $\omega_0 = \tau(q) \in \Omega$ such that
   $$T(f)(q) = (\delta_q \circ T)(f) = \delta_{\tau(q)}(f) = f(\tau(q)),$$
   for all $f \in C(\Omega)$. So for all $f \in C(\Omega)$ and all $q \in Q$: if $\delta_q \circ T \neq 0$, then $T(f)(q) = f(\tau(q))$.
   From $(T(\mathbf{1_\Omega}))^2 = T(\mathbf{1_\Omega}^2) = T(\mathbf{1_\Omega})$, it follows that $T(\mathbf{1_\Omega}) = \chi_V$ for a unique clopen subset $V \subset Q$.
   Now let $q \in V$. Then $(\delta_q \circ T)(\mathbf{1}_\Omega) = T(\mathbf{1}_\Omega)(q) = \chi_V(q) = 1$, so $\delta_q \circ T \neq 0$. Therefore, we have
   $$T(f)(q) = f(\tau(q)) = \chi_V(q) f(\tau(q))$$
   for all $f \in C(\Omega)$.
   If $q \notin V$, then $\chi_V(q) = 0$, so $T(f)(q) = T(\mathbf{1}_\Omega \cdot f)(q) = (T(\mathbf{1}_\Omega)T(f))(q) = \chi_V(q)T(f)(q) = 0$. Choose a $\tau(q) \in \Omega$ arbitrary, then certainly $T(f)(q) = \chi_V(q)f(\tau(q))$. So we have
   $$T(f)(q) = \chi_V(q)(f(\tau(q))$$
   for all $q \in Q$ and all $f \in C(\Omega)$.
   Finally, the continuity of τ on V is a special case of the continuity of the map τ on the set $\{q \in Q : w(q) > 0\}$ in Theorem 5.1.

□

6 Lattice and algebra automorphisms

In this section we will consider the automorphisms of $C(K)$-spaces. Therefore we introduce some notation. Denote by $\mathrm{LatAut}(C(\Omega))$ the group of lattice automorphisms $T : C(\Omega) \to C(\Omega)$. Further, let $\mathrm{AlgAut}(C(\Omega))$ be the group of algebra automorphisms of $C(\Omega)$. Finally, $\mathrm{Homeom}(\Omega)$ is the group of homeomorphisms $\tau : \Omega \to \Omega$. Recall that a continuous map is a homeomorphism if it is bijective and its inverse is also continuous. In these groups the operation is composition.

The following theorem will characterize the lattice automorphisms. For this we need the group $G = \{f \in C(\Omega) : f > 0\}$. This is a group under multiplication. First we will consider the map

$$\begin{array}{rcl} \alpha : \mathrm{Homeom}(\Omega) & \to & \mathrm{Aut}(G) \\ \tau & \mapsto & \alpha_\tau : g \mapsto g \circ \tau^{-1}. \end{array}$$

α is well-defined, since α_τ is a surjective and injective homomorphism. Moreover, α is a homomorphism, because

$$\alpha(\tau \circ \sigma)(g) = g \circ (\tau \circ \sigma)^{-1} = g \circ \sigma^{-1} \circ \tau^{-1} = \alpha(\tau)(g \circ \sigma^{-1}) = (\alpha(\tau) \circ \alpha(\sigma))(g),$$

for all $\tau, \sigma \in \mathrm{Homeom}(\Omega)$ and $g \in G$. This gives us the semi-direct product $G \rtimes_\alpha \mathrm{Homeom}(\Omega)$ with the following multiplication:

$$(g, \tau) * (h, \sigma) = (g \cdot \alpha_\tau(h), \tau \circ \sigma) = (g \cdot (h \circ \tau^{-1}), \tau \circ \sigma).$$

Theorem 6.1. *The map*

$$\begin{array}{rcl} \phi : G \rtimes_\alpha \mathrm{Homeom}(\Omega) & \to & \mathrm{LatAut}(C(\Omega)) \\ (g, \tau) & \mapsto & \big(T : f \mapsto g \cdot (f \circ \tau^{-1})\big) \end{array}$$

is a group isomorphism.

Proof. We will show that ϕ is a homomorphism. First, we have:

$$\begin{array}{rcl} \phi((g,\tau) * (h,\sigma))(f) & = & \phi((g \cdot (h \circ \tau^{-1}), \tau \circ \sigma))(f) \\ & = & (g \cdot (h \circ \tau^{-1})) \cdot (f \circ (\tau \circ \sigma)^{-1}) \\ & = & (g \cdot (h \circ \tau^{-1})) \cdot (f \circ \sigma^{-1} \circ \tau^{-1}). \end{array}$$

And second, we have:

$$\begin{array}{rcl} \phi((g,\tau)) \circ \phi((h,\sigma))(f) & = & \phi((g,\tau))(h \cdot (f \circ \sigma^{-1})) \\ & = & g \cdot ((h \cdot (f \circ \sigma^{-1})) \circ \tau^{-1}) \\ & = & (g \cdot (h \circ \tau^{-1})) \cdot (f \circ \sigma^{-1} \circ \tau^{-1}), \end{array}$$

for all $(g, \tau), (h, \sigma) \in G \rtimes_\alpha \mathrm{Homeom}(\Omega)$ and $f \in C(\Omega)$. So ϕ is a homomorphism. Now we will prove that ϕ is surjective. Let $S \in \mathrm{LatAut}(C(\Omega))$. According to Theorem 5.1 for all $f \in C(\Omega)$ we have $S(f) = w \cdot (f \circ \sigma)$ for $w \in C(\Omega)$ a non-negative function and $\sigma : \Omega \to \Omega$ a continuous map. This map S has an inverse, say T with $T(f) = v \cdot (f \circ \tau)$, such that $T \circ S = \mathrm{id}_{C(\Omega)} = S \circ T$. This means

$$\begin{aligned} g &= (S \circ T)(g) \\ &= S(v \cdot (g \circ \tau)) \\ &= w \cdot ((v \cdot (g \circ \tau)) \circ \sigma) \\ &= w \cdot ((v \circ \sigma) \cdot (g \circ \tau \circ \sigma)) \\ &= w \cdot (v \circ \sigma) \cdot (g \circ \tau \circ \sigma) \end{aligned}$$

for all $g \in C(\Omega)$. If we take $g = \mathbf{1}$, we find

$$1 = g(x) = w(x) \cdot v(\sigma(x)) \cdot 1 = w(x) \cdot v(\sigma(x)),$$

for all $x \in \Omega$, thus $w(x) = \frac{1}{v(\sigma(x))}$, and because w is non-negative, we have $w(x) > 0$ for all $x \in \Omega$.
Now we get $(S \circ T)(g) = g \circ \tau \circ \sigma$ for all $g \in C(\Omega)$.
We know from Theorem 5.1 that $\mathrm{id}_{C(\Omega)}$ is of the form $\mathrm{id}_{C(\Omega)}(f) = v \cdot (f \circ \tau)$ for $v = \mathrm{id}_{C(\Omega)}(\mathbf{1}) = \mathbf{1}$ and a map τ which is unique on the set $\{\omega \in \Omega : v(\omega) > 0\} = \Omega$. So τ need to be id_Ω. But $S \circ T = \mathrm{id}_{C(\Omega)}$, and $S \circ T$ corresponds with the continuous map $\tau \circ \sigma$, so $\tau \circ \sigma = \mathrm{id}_\Omega$. Likewise, $\sigma \circ \tau = \mathrm{id}_\Omega$. Hence τ and σ are bijections and have continuous inverses. So these maps are homeomorphisms.
Finally, we find $\phi((w, \tau)) = \phi((w, \sigma^{-1})) = S$. Hence ϕ is surjective.

For the injectivity we have to show that $\ker \phi = \{0\}$, since ϕ is a homomorphism. Assume that $\phi((w, \tau)) = \mathrm{id}_{C(\Omega)}$. So $\mathrm{id}_{C(\Omega)}(f) = w \cdot (f \circ \tau^{-1})$. We just saw that $w = \mathrm{id}_{C(\Omega)}(\mathbf{1}) = \mathbf{1}$ and τ^{-1} need to be id_Ω, so $\tau = \mathrm{id}_\Omega$. Hence $\ker \phi = \{0\}$. □

In the next theorem we will see that looking at the algebra automorphisms on $C(\Omega)$ is actually the same as looking at the homeomorphisms on Ω.

Theorem 6.2. *The map*

$$\begin{aligned} \phi : \mathrm{Homeom}(\Omega) &\rightarrow \mathrm{AlgAut}(C(\Omega)) \\ \tau &\mapsto \left(T : f \mapsto f \circ \tau^{-1}\right) \end{aligned}$$

is a group isomorphism.

Proof. First, ϕ is a group homomorphism, because ϕ has the same structure as α in the proof of Theorem 6.1.

Now we will prove that ϕ is surjective. Let $S \in \mathrm{AlgAut}(C(\Omega))$. According to Theorem 5.3 for all $f \in C(\Omega)$ we have $S(f) = \chi_W \cdot (f \circ \sigma)$ for W a clopen subset of Q and $\sigma : \Omega \to \Omega$ a continuous map. This map S has an inverse, say T with $T(f) = \chi_V \cdot (f \circ \tau)$, such that $T \circ S = \mathrm{id}_{C(\Omega)} = S \circ T$. This means

$$\begin{aligned} g &= (S \circ T)(g) \\ &= S(\chi_V \cdot (g \circ \tau)) \\ &= \chi_W \cdot ((\chi_V \cdot (g \circ \tau)) \circ \sigma) \\ &= \chi_W \cdot ((\chi_V \circ \sigma) \cdot (g \circ \tau \circ \sigma)) \\ &= \chi_W \cdot (\chi_V \circ \sigma) \cdot (g \circ \tau \circ \sigma) \end{aligned}$$

for all $g \in C(\Omega)$. If we take $g = \mathbf{1}$, we find

$$1 = g(x) = \chi_W(x) \cdot (\chi_V(\sigma(x)) \cdot 1) = \chi_W(x) \cdot \chi_V(\sigma(x)),$$

for all $x \in \Omega$, thus $\chi_W = \mathbf{1}$. So $S(f) = f \circ \sigma$ for all $f \in C(\Omega)$.
Hence for each algebra automomorphism M there is a continuous map μ such that $M(f) = f \circ \mu$. Moreover, this map is unique, by Theorem 5.3.
For $\mathrm{id}_{C(\Omega)}$ this continuous map has to be id_Ω. But $S \circ T = \mathrm{id}_{C(\Omega)}$, and $S \circ T$ corresponds with $\tau \circ \sigma$, so $\tau \circ \sigma = \mathrm{id}_\Omega$. Likewise, $\sigma \circ \tau = \mathrm{id}_\Omega$. Hence τ and σ are bijections and have continuous inverses. So these maps are homeomorphisms. Finally, we find $\phi(\tau) = \phi(\sigma^{-1}) = S$, thus ϕ is surjective.

By the injectivity of ϕ is meant that for each algebra automorphism M there is a *unique* continuous map μ such that $M(f) = f \circ \mu$. This uniqueness is a special case of the uniqueness of τ and V in Theorem 5.3.

□

References

[1] Y.A. Abramovich, C.D. Aliprantis, *An Invitation to Operator Theory.* American Mathematical Society, Providence, 2002.

[2] C.D. Aliprantis, O. Burkinshaw, *Principles of Real Analysis.* Third Edition, Academic Press, San Diego, 1998.

[3] H. Bauer, *Measure and Integration Theory.* Walter de Gruyter, Berlin, 2001.

[4] J.B. Conway, *A Course in Functional Analysis.* Second Edition, Springer-Verlag, New York, 1990.

[5] V. Runde, *A Taste of Topology.* Springer Science+Business Media, New York, 2008.