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Compact non-Hausdorff Manifolds

Bachelor Thesis

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Date Bachelor Exam: June 5th, 2014

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Contents

Chapter 1

Introduction

One of the simplest examples of a compact manifold (definitions 2.0.1 and 2.0.2) that is not Hausdorff (definition 2.0.3), is the space X obtained by glueing two circles together in all but one point. Since the circle is a compact Hausdorff manifold, it is easy to prove that X is indeed a compact manifold. It is also easy to show X is not Hausdorff. Many other examples of compact manifolds that are not Hausdorff can also be created by glueing together compact Hausdorff manifolds. This gives rise to the question: "Can every compact manifold be obtained by glueing together a finite number of compact Hausdoff manifolds?". In chapter 2 we will prove that the answer to this question is no, by constructing a compact manifold that can not be obtained this way.

In chapter 3 we will ask a second question: "Given a compact manifold Y, is there always a surjective étale map (definition 2.2.1) from a compact Hausdorff manifold to Y?". Even though there is a surjective étale map from a compact Hausdorff manifold to the space constructed in chapter 2, we will show that the answer to this second question is also no by constructing a different compact manifold.

Finally, in chapter 4 we will take a closer look at the compact manifolds constructed in chapters 2 and 3. We will compute the fundamental groups of each of these compact manifolds, which leads to the observation that for both of these spaces, there is a compact Hausdorff manifold with the same fundamental group. In fact, the compact manifold constructed in chapter 2 resembles a torus and also has the same fundamental group as a torus and the compact manifold constructed in chapter 3 resembles a sphere and has the same fundamental group as a sphere.

We have tried to find a way to describe all compact manifolds in terms of compact Hausdorff manifolds. However the answers to our questions in chapters 2 and 3 were no. The findings in chapter 4 give rise to the question: "Given a compact manifold M, are there always a cell complex C [2, chapter 0] and an open continuous and surjective map $f\colon M \to C$ such that f induces an isomorphism on all homotopy groups [2, section 4.1]?". We did not have time to investigate this question, so this is a possible point of further study.

Chapter 2

The First Question: "Can every compact manifold be obtained by glueing together a finite number of compact Hausdorff manifolds?"

We will start by rigorously defining what a compact manifold is.

Definition 2.0.1. A topological space (X, T_X) is called *compact* if for all collections $\{U_i\}_{i\in I}$ such that $U_i \in T_X$ for all $i \in I$ and $X = \bigcup_{i\in I} U_i$, there is a finite $J \subseteq I$ such that $X = \bigcup_{j\in J} U_j$.
Definition 2.0.2. A topological space (X, T_X) is called a *manifold* if for all $x \in X$ there is an open neighbourhood U of x in X, such that there is a $n \geq 0$, $V \subseteq \mathbb{R}^n$ open and a homeomorphism $g\colon U \to V$.

In this chapter we will ask ourselves: "Can every compact manifold be obtained by glueing together a finite number of compact Hausdorff manifolds?". To state this in mathematical terms, we use the following definitions.

Definition 2.0.3. Let (X, T_X) be a topological space. The space X is called *Hausdorff* if for all $x, y \in X$ such that $x \neq y$ there are $U, V \in T_X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.
Definition 2.0.4. Let (X, T_X) be a compact manifold. Then X is called *almost Hausdorff* if there is a finite set $\{Y_i\}_{i\in I}$ of compact Hausdorff manifolds and continuous open injective maps $\{\pi_i\colon Y_i \to X\}_{i\in I}$ such that $\bigcup_{i\in I} \pi_i(Y_i) = X$.

We can now ask the following question.

Question 2.0.5. *Are all compact manifolds almost Hausdorff?*

In this chapter we will prove that the answer to question 2.0.5 is no. We will construct a compact manifold (Z, T_Z) which is not almost Hausdorff. The construction of (Z, T_Z) is done in section 2.1, we then prove (Z, T_Z) is a compact manifold in section 2.2 and finally prove it is not almost Hausdorff in section 2.3. Since any compact Hausdorff manifold is also almost Hausdorff, it is clear that our space (Z, T_Z) will not be Hausdorff. This is one of the properties of (Z, T_Z) that we will use this to prove it is not almost Hausdorff.

2.1 Construction of (Z, T_Z)

The space (Z, T_Z) that will be constructed in construction 2.1.3 will imply a negative answer to question 2.0.5. Before we can construct our space (Z, T_Z) we need to introduce the following concepts, which will be used in the construction.

Definition 2.1.1. Let (X, T_X) be a topological space and $\sim$ an equivalence relation on X. Then the *quotient topology* $T_\sim$ on the *quotient space* $X/\sim$ is defined as

$$T_\sim = \{U \subseteq X/\sim \,:\, q^{-1}(U) \in T_X\} \tag{2.1}$$

where $q\colon X \to Y$ is given by $x \mapsto [x]$ and called the *quotient map*.
Definition 2.1.2. A *group action* ϕ of a group G on a set X is a map $\phi\colon G \times X \to X$ that satisfies the following conditions:

- $\phi(gh, x) = \phi(g, \phi(h, x))$ for all $g, h \in G$ and $x \in X$.
- $\phi(e, x) = x$ for all $x \in X$, where e is the identity element of G.

Now we can construct the space (Z, T_Z).

Construction 2.1.3. Define $X_1 = X_2 = \mathbb{R}^2$ with the Euclidean topologies T_1 and T_2 respectively. Define $U_1 = U_2 = \mathbb{R}^2 \setminus (\mathbb{R} \times \mathbb{Z})$. Define $X = X_1 \times \{1\} \cup X_2 \times \{2\}$ the disjoint union of X_1 and X_2 and $U = U_1 \times \{1\} \cup U_2 \times \{2\}$ the disjoint union of $U_1 \subseteq X_1$ and $U_2 \subseteq X_2$. We define T_X, the topology on X, as the natural topology on the disjoint union, i.e.

$$T_X = \{U \subseteq X \colon \phi_1^{-1}(U) \in T_1 \wedge \phi_2^{-1}(U) \in T_2\} \tag{2.2}$$

where $\phi_i\colon X_i \to X$ are the injections given by $(x, y) \mapsto (x, y, i)$ for $i \in \{1, 2\}$.

Define an equivalence relation $\sim_X$ such that $(x, y, i) \sim_X (u, v, j)$ if and only if $(x, y, i) = (u, v, j)$ or $(x, y) = (u, v)$ and $(x, y, i), (u, v, j) \in U$ for all $(x, y, i), (u, v, j) \in X$. We will leave it up to the reader to verify this is indeed an equivalence relation. We now have a topological space (X, T_X) and an equivalence relation $\sim_X$ on this topological space. Therefore we can define $Y = X/\sim_X$ as the quotient space with the quotient topology T_Y and $q_1\colon X \to Y$ the quotient map. This effectively defines Y as the xy-plane where all lines $y \in \mathbb{Z}$ are doubled as shown in figure 2.1a.

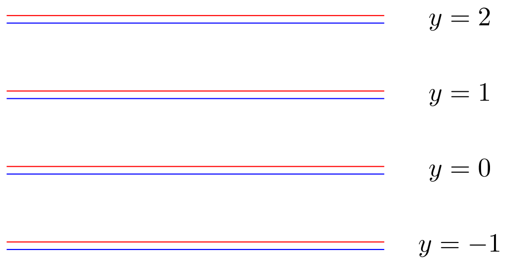
Define $G = \mathbb{Z}^2$ the additive group. Then we can define a group action of G on Y by:

$$\psi\colon G \times Y \to Y, ((g_1, g_2), [(x, y, z)]) \mapsto \begin{cases} [(x_1 + g_1, x_2 + g_2, z)], \ g_1 \in 2\mathbb{Z} \\ [(x_1 + g_1, x_2 + g_2, 3 - z)], \ g_1 \notin 2\mathbb{Z} \end{cases} \tag{2.3}$$

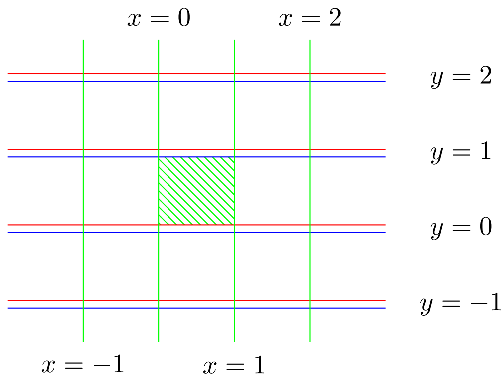
This action can be described as translating an element of Y by an element of G and "jumping to the other line" whenever the element (x, y) is on a line $y = n$ where $n \in \mathbb{Z}$ and the translation in the direction of the x-axis is odd.

The group action ψ defines an equivalence relation $\sim_Y$ on Y where two elements $x, y \in Y$ are equivalent if and only there is a $g \in G$ such that $\psi(g, x) = y$. We leave it up to the reader to verify that ψ is indeed an group action and $\sim_Y$ an equivalence relation. We now have a topological space (Y, T_Y) and an equivalence relation $\sim_Y$ and therefore we can define $Z = Y/\sim_Y$ as the quotient space with T_Z the quotient topology and $q_2\colon Y \to Z$ the quotient map. The resulting space Z has similar shape to that of a torus, as shown in figure 2.1d. The process of constructing Z from Y is illustrated in figure 2.1.

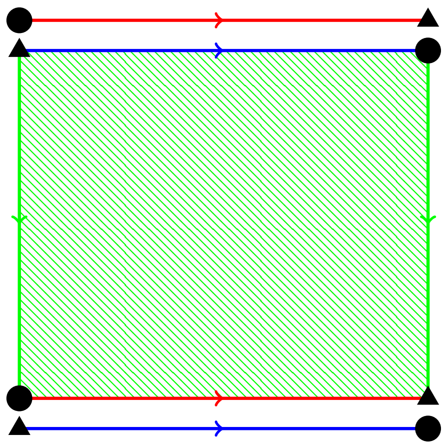
All notation introduced in construction 2.1.3 will be used further on in this chapter. To speed up some of the proofs that will follow later, we observe that the following is true.



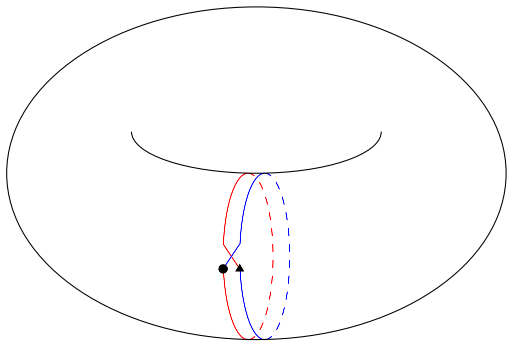
(a) The space (Y, T_Y) can be seen as $\mathbb{R}^2$ where all lines $y = n$ with $n \in \mathbb{Z}$ are doubled.



(b) Part of the space (Y, T_Y). The equivalence relation $\sim_Y$ will identify all squares, such as the hatched square in the centre, to each other.



(c) A close-up of the hatched square shown in figure 2.1b. All lines with the same color are identified to each other by $\sim_Y$ as indicated by the arrows. The relation $\sim_Y$ also identifies all points indicated with a circle to each other and all points indicated by a triangle to each other.



(d) The space (Z, T_Z), which closely resembles a torus. One circle around the body of the torus is doubled, this circle is indicated by the red and blue lines. The doubled circle contains a twist, as shown, which results in it being one connected circle instead of two seperate circles.

Figure 2.1 – Several stages of the construction of (Z, T_Z) out of (Y, T_Y) are shown.

Proposition 2.1.4. *$q_2 \circ q_1 \circ \phi_i$ is an open map for all $i \in \{1, 2\}$.*

In fact, it can be proven that ϕ_1, ϕ_2, q_1 and q_2 are all open maps. We leave it up to the reader to verify this proposition.

2.2 The space (Z, T_Z) is a Compact Manifold

We will continue by proving that the space (Z, T_Z) as constructed in construction 2.1.3 is a compact manifold. In order to do so we will use the following concept.

Definition 2.2.1. An *étale map* over a space (Z, T_Z) is a continuous map $f: W \to Z$, such that W is a topological space and for all $w \in W$ there is an open neighbourhood V of w such that f restricted to V is a homeomorphism from V to $f(V)$, where $f(V)$ is open in Z.

This concept of an étale map is also known as a local homeomorphism. We will prove (Z, T_Z)

is a compact manifold by defining a surjective étale map from a torus to (Z, T_Z). Since the torus is a compact manifold, it will then easily follow that (Z, T_Z) must be a compact manifold as well. First we construct the torus and define our map.

Construction 2.2.2. We will start by constructing a space (W, T_W), which will be a torus. We define the group $G = \mathbb{Z}^2$ as in construction 2.1.3 and define an action $\sigma\colon G \times \mathbb{R}^2 \to \mathbb{R}^2$ by $((g_1, g_2), (x_1, x_2)) \mapsto (x_1 + g_1, x_2 + g_2)$. We can define an equivalence relation $\sim_W$ on $\mathbb{R}^2$, where any $x, y \in \mathbb{R}^2$ are equivalent if and only if there is a $g \in G$ such that $\sigma(g, x) = y$. We leave it up to the reader to verify that σ is indeed a group action and $\sim_W$ an equivalence relation.

Define W as the quotient space $\mathbb{R}^2/\!\sim_W$ where T_W is the quotient topology and $q_3\colon \mathbb{R}^2 \to W$ is the quotient map. Define a map $f\colon W \to Z$ by $[(x_1, x_2)]_W \mapsto [(2x_1, 2x_2, 1)]_Z$.

We will leave it up to the reader to verify that f is well-defined, continuous and surjective. Before we prove that f is an étale map we will prove the following.

Proposition 2.2.3. *The map f is open.*

We will use the following definition in our proof.

Definition 2.2.4. Let $x \in \mathbb{R}^2$ and $\epsilon > 0$ then

$$B_\epsilon(x) = \{y \in X\colon d_E(x, y) < \epsilon\} \tag{2.4}$$

is *the open sphere of radius ϵ* where d_E is the Euclidean distance.

We continue by proving proposition 2.2.3.

Proof. We will use the same notation as used in construction 2.1.3 and construction 2.2.2. Therefore $X_1 = X_2 = \mathbb{R}^2$ and X is the disjoint union of X_1 and X_2 where ϕ_i is the inclusion map of X_i into X for $i \in \{1, 2\}$. Also $Y = X/\!\sim_X$ with q_1 the quotient map and $Z = Y/\!\sim_Y$ with q_2 the quotient map. Lastly $W = \mathbb{R}^2/\!\sim_W$ with q_3 the quotient map and $f\colon W \to Z$ is given by $[(x_1, x_2)]_W \mapsto [(2x_1, 2x_2, 1)]_Z$.

Let $A \subseteq W$ be open. We need to prove that $f(A)$ is open in Z. Define $A_1 = \phi_1^{-1}(q_1^{-1}(q_2^{-1}(f(A))))$. If $A_1 = \emptyset$ then A_1 is open in X_1. Assume $A_1 \neq \emptyset$ and let $x = (x_1, x_2) \in A_1 \subseteq X_1 = \mathbb{R}^2$. Then $[(x_1, x_2, 1)]_Z \in f(A)$. Therefore there must be a $y = [(y_1, y_2)]_W \in A$ such that $f(y) = [(x_1, x_2, 1)]_Z$. This shows that $[(2y_1, 2y_2, 1)]_Z = [(x_1, x_2, 1)]_Z$. Then $(x_1, x_2, 1) \sim_Y (2y_1, 2y_2, 1)$ or $(x_1, x_2, 1) \sim_X (2y_1, 2y_2, 1)$. In both cases there must be a $g = (g_1, g_2) \in G$ such that $(x_1, x_2) + (g_1, g_2) = (2y_1, 2y_2)$ where g_1 is even. Thus showing that $(g_1, g_2) = (2y_1, 2y_2) - (x_1, x_2)$ where g_1 is even.

We observe that $q_3^{-1}(A)$ is open in $\mathbb{R}^2$ and $(y_1, y_2) \in q_3^{-1}(A) \subseteq \mathbb{R}^2$ therefore there is an $\epsilon > 0$ such that $B_\epsilon(y_1, y_2) \subseteq q_3^{-1}(A)$. We will look at $B_\epsilon(x) \subseteq X_1 = \mathbb{R}^2$ and prove it must be a subset of A_1. Let $z = (z_1, z_2) \in B_\epsilon(x)$, then $d_E(z, x) < \epsilon$. Define $u = (u_1, u_2) = (y_1, y_2) + \frac{1}{2}(z - x)$. Then $d_E(u, (y_1, y_2)) = d_E\left(\frac{1}{2}(z - x), 0\right) = \frac{1}{2}\epsilon < \epsilon$ thus showing that $u \in B_\epsilon(y_1, y_2) \subseteq q_3^{-1}(A)$. Then $[u]_W \in A$ so $f([u]) \in f(A)$ and $f([u]) = [(2u_1, 2u_2, 1)]_Z = [(2y_1 + z_1 - x_1, 2y_2 + z_2 - x_2, 1)]_Z$

We now know $[(2y_1 + z_1 - x_1, 2y_2 + z_2 - x_2, 1)]_Z \in f(A)$ and $(g_1, g_2) = (2y_1, 2y_2) - (x_1, x_2)$ where g_2 is even. This shows that $[(z_1 + g_1, z_2 + g_2, 1)]_Z \in f(A)$ where g_1 is even. Since g_1 is even we know $[(z_1 + g_1, z_2 + g_2, 1)]_Z = [(z_1, z_2, 1)]_Z = \phi_1(q_1(q_2(z)))$ thus showing that $\phi_1(q_1(q_2(z))) \in f(A)$ therefore $z \in A_1$. We now conclude $B_\epsilon(x) \subseteq A_1$ thus showing that A_1 is open in X_1. Analogously we can prove that $A_2 = \phi_2^{-1}(q_1^{-1}(q_2^{-1}(f(A))))$ is open in X_2. This proves that $f(A)$ is open in Z. □

Using proposition 2.2.3, we can easily verify that f is indeed a surjective étale map over (Z, T_Z).

Proposition 2.2.5. *The map f is a surjective étale map over (Z, T_Z).*

Proof. Let $w \in W$. Write $w = [(w_1, w_2)]$ where $(w_1, w_2) \in \mathbb{R}$. Then we can define $B = B_\epsilon(w_1, w_2)$ where $\epsilon = \frac{1}{4}$. We know B is open in $\mathbb{R}^2$. We define $V = q_3(B)$, then clearly $w \in V$. Also observe that $q_3^{-1}(V) = \bigcup_{g \in G} B_\epsilon(x + g)$, which shows $q_3^{-1}(V)$ is a union of opens in $\mathbb{R}^2$ so it must be open in $\mathbb{R}^2$. We conclude V is an open neighbourhood of w in W.

We know f is continuous so the restriction $f|_V$ of f to V must be continuous. Also by definition $f|_V : V \to f(V)$ is surjective. Proposition 2.2.3 states that f is an open map, therefore $f|_V$ must also be an open map and $f(V)$ must be open in Z. The only condition left to prove is that $f|_V$ is an injective map.

Let $a, b \in V$ such that $f(a) = f(b)$. Since $V = q_3(B)$ we know that all elements $x \in V$ can be written as $x = [(x_1, x_2)]$ where $(x_1, x_2) \in B$. Write $a = [(a_1, a_2)]$ and $b = [(b_1, b_2)]$ where $(a_1, a_2), (b_1, b_2) \in B$. We then conclude that

$$d_E((a_1, a_2), (b_1, b_2)) \leq d_E((a_1, a_2), (x_1, x_2)) + d_E((b_1, b_2), (x_1, x_2)) < 2\epsilon = \frac{1}{2}.$$

Since $f(a) = f(b)$ we conclude that $[(2a_1, 2a_2, 1)] = [(2b_1, 2b_2, 1)]$ in Z therefore $(2a_1, 2a_2, 1) \sim_X (2b_1, 2b_2, 1)$ or $[(2a_1, 2a_2, 1)] \sim_Y [(2b_1, 2b_2, 1)]$ in Y. This shows that there must be a $g = (g_1, g_2) \in G$ such that $2a_1 + g_1 = 2b_1$ and $2a_2 + g_2 = 2b_2$. Since $d_E((a_1, a_2), (b_1, b_2)) < \frac{1}{2}$ we know $d_E((2a_1, 2a_2), (2b_1, 2b_2)) < 1$ thus showing that $g_1 = g_2 = 0$ therefore $a_1 = b_1$ and $a_2 = b_2$. We now conclude that $a = b$ thus showing that $f|_V$ is injective. □

Now that we know there is a surjective étale map $f : W \to Z$ from the torus to our space (Z, T_Z) we can easily prove that (Z, T_Z) is a compact manifold.

Proposition 2.2.6. *The space (Z, T_Z) is a compact manifold.*

Proof. We know the torus is compact and the continuous image of a compact space is compact [4, proposition 3.3.8]. Since our surjective étale map $f : W \to Z$ is a continuous and surjective map from a compact space to (Z, T_Z), we conclude that (Z, T_Z) is compact.

We have a surjective étale map $f : W \to Z$ and know that the torus W is a manifold. Let $z \in Z$ then there must be a $w \in W$ such that $f(w) = z$. For $w \in W$ there is an open neighbourhood U_1 of w such that $f|_{U_1}$ is a homeomorphism and $f(U_1)$ open in Z. There also is an open neighbourhood U_2 of w such that there is a $n \geq 0$ and open $V \subseteq \mathbb{R}^2$ and a homeomorphism $g : U_2 \to V$. We define U the intersection of U_1 and U_2. Then U is an open neighbourhood of w and $f|_U$ is a homeomorphism from U to $f(U) \subseteq Z$ open with $z \in f(U)$ and $g|_U$ is a homeomorpishm from U to $g(U) \subseteq V \subseteq \mathbb{R}^n$ open. This shows that $(g \circ (f|_V)^{-1})|_{f(U)}$ is a homeomorphism from an open neighbourhood $f(U)$ of z to an open $g(U) \in \mathbb{R}^n$, thus showing (Z, T_Z) is a manifold. □

2.3 A Negative Answer to our Question

Now that we know the space (Z, T_Z) is a compact manifold, we can prove that the existence of (Z, T_Z) implies a negative answer to question 2.0.5, by showing it is not almost Hausdorff. We will do so by deriving a contradiction. We will show that if question 2.0.5 is true, then there must be a Hausdorff open subset in Z that contains the entire doubled circle drawn in red and blue in figure 2.1d. To be able to derive our contradiction we first prove that this cannot be the case.

Proposition 2.3.1. *Define $C = q_2(q_1(X \setminus U))$. Let $V \subseteq Z$ be open such that $C \subseteq V$. Then there are $a, b \in V$ such that $A \cap B \neq \emptyset$ for all open $A, B \subseteq V$ such that $a \in A$ and $b \in B$.*

Proof. We choose $x = (0, 0, 1) \in X$ and $y = (0, 0, 2) \in X$. Then $x, y \notin U$ and therefore $[x]_Y \neq [y]_Y$ and $[x]_Z, [y]_Z \in C$. Also there is no $g \in G$ such that $\psi(g, x) = y$ therefore $[x]_Z \neq [y]_Z$. Let $a = q_2(q_1(x)) = [x]_Z$ and $b = q_2(q_1(y)) = [y]_Z$, then $a, b \in V$.

Let A and B be open neighbourhoods of a and b in V respectively. Then $A_1 = \phi_1^{-1}(q_1^{-1}(q_2^{-1}(A)))$ is open in $X_1 = \mathbb{R}^2$. Since $(q_2 \circ q_1 \circ \phi_1)(0, 0) = a \in A$ we can conclude that $(0, 0) \in A_1$. Since A_1 is an open in $\mathbb{R}^2$ there must be an $\epsilon_1 > 0$ such that $B_{\epsilon_1}(0, 0) \subseteq A_1$. Analogously we can define $B_2 = \phi_2^{-1}(q_1^{-1}(q_2^{-1}(B)))$ and choose an $\epsilon_2 > 0$ such that $B_{\epsilon_2}(0, 0) \subseteq B_2$.

Define $\epsilon = \min(\epsilon_1, \epsilon_2)$ then $B_\epsilon(0, 0) \subseteq B_{\epsilon_1}(0, 0) \subseteq A_1$ and $B_\epsilon(0, 0) \subseteq B_{\epsilon_2}(0, 0) \subseteq B_2$ and $\epsilon > 0$. Then $B_\epsilon(0, 0) \cap \mathbb{R}^2 \setminus (\mathbb{R} \times \mathbb{Z}) \neq \emptyset$. Let $z = (z_1, z_2) \in B_\epsilon(0, 0) \cap \mathbb{R}^2 \setminus (\mathbb{R} \times \mathbb{Z})$. Then $z \in B_\epsilon(0, 0)$ therefore $z \in A_1$ and $z \in B_2$ thus showing that $[(z_1, z_2, 1)]_Z \in A$ and $[(z_1, z_2, 2)]_Z \in B$. Also since $z \in \mathbb{R}^2 \setminus (\mathbb{R} \times \mathbb{Z})$ we know $(z_1, z_2, 1) \sim_X (z_1, z_2, 2)$ and therefore $[(z_1, z_2, 1)]_Z = [(z_1, z_2, 2)]_Z$. This shows that $A \cap B \neq \emptyset$. □

Finally we will prove that the existence of (Z, T_Z) implies a negative answer to question 2.0.5 by deriving a contradiction.

Proposition 2.3.2. *The space (Z, T_Z) is not almost Hausdorff.*

Proof. Assume (Z, T_Z) is almost Hausdorff, then there is a finite set $\{Y_i\}_{i \in I}$ of compact Hausdorff manifolds such that there are continuous open injective maps $\pi_i \colon Y_i \to Z$ for all $i \in I$ with $\bigcup_{i \in I} \pi_i(Y_i) = Z$. Define $C = q_2(q_1(X \setminus U))$.

We know $X \setminus U$ is closed in X, thus showing that $\phi_1^{-1}(X \setminus U)$ is closed in X_1 and then proposition 2.1.4 shows that $C = q_2(q_1(\phi_1(\phi_1^{-1}(X \setminus U))))$ is closed in Z. We know $\bigcup_{i \in I} \pi_i(Y_i) = Z$. Let $p \in C$ then there must be a $i \in I$ such that $p \in \pi_i(Y_i)$. We observe that for this $i \in I$, we know $\pi_i(Y_i) \cap C \neq \emptyset$.

Since π_i is an open map we know $\pi_i(Y_i)$ is open in Z. Therefore $\pi_i(Y_i) \cap C$ is open in $C \subseteq Z$.

We know C is closed in Z, so $\pi_i(Y) \cap C$ must be closed in $\pi_i(Y_i)$. Since π_i is continuous and Y_i is compact, $\pi_i(Y_i)$ must be compact. Then $\pi_i(Y_i) \cap C$ is a closed subset of a compact space, therefore it must be compact [4, proposition 3.3.6.(i)]. We leave it up to the reader to verify that C is homeomorphic to a circle and in particular a Hausdorff and path-connected space. Since $\pi_i(Y_i) \cap C$ is compact and a subset of C, which is Hausdorff, we conclude $\pi_i(Y_i) \cap C$ is closed in C [4, proposition 3.3.6.(ii)].

We know C is path-connected and $\pi_i(Y_i) \cap C$ is closed and open in C and non-empty. Then $\pi_i(Y_i) \cap C = C$, which shows that $C \subseteq \pi_i(Y_i)$. We now have an open subset $\pi_i(Y_i)$ of Z such that $C \subseteq \pi_i(Y_i)$. Since π_i is an open continuous injective map, we know π_i is a homeomorphism from Y_i to $\pi_i(Y_i)$. Then $\pi_i(Y_i)$ must be Hausdorff, because Y_i is Hausdorff. This is a contradiction with proposition 2.3.1. □

Chapter 3

The Second Question: "If (R, T_R) is a compact manifold, is there always a surjective étale map from a compact Hausdorff manifold to R?"

In chapter 2 we saw that the answer to question 2.0.5 is no. This means there are compact manifolds that cannot be constructed by glueing together a finite number of compact Hausdorff manifolds. However the space (Z, T_Z) we saw could be described in terms of a compact Hausdorff manifold, because there was a surjective étale map from a torus to Z. This gives rise to the following question.

Question 3.0.3. *If (R, T_R) is a compact manifold, is there always a surjective étale map from a compact Hausdorff manifold to R?*

The answer to this question is also no, which we will prove in this chapter. In section 3.1 we will construct a space (R, T_R) that will imply a negative answer to question 3.0.3. In section 3.2 we will relate our notion of a surjective étale map to the notion of a covering space, proving that these notions are the same under certain conditions. Finally, in section 3.3, we will use the theorem we prove in section 3.2, to prove that existence of the space (R, T_R) indeed implies a negative answer to question 3.0.3.

3.1 Construction of (R, T_R)

We start by constructing a compact manifold (R, T_R) that will imply a negative answer to question 3.0.3. The manifold we construct will be very similar to the space (Z, T_Z) as constructed in construction 2.1.3. The space we will construct can be seen as a sphere where the equator is a doubled circle, similar to the doubled circle in (Z, T_Z).

There are two ways to construct our space (R, T_R). We can construct it by either starting with a sphere and defining an equivalence relation or construct it from our space (Z, T_Z). We will construct (R, T_R) from a sphere. To be able to do so we first introduce a way to identify the

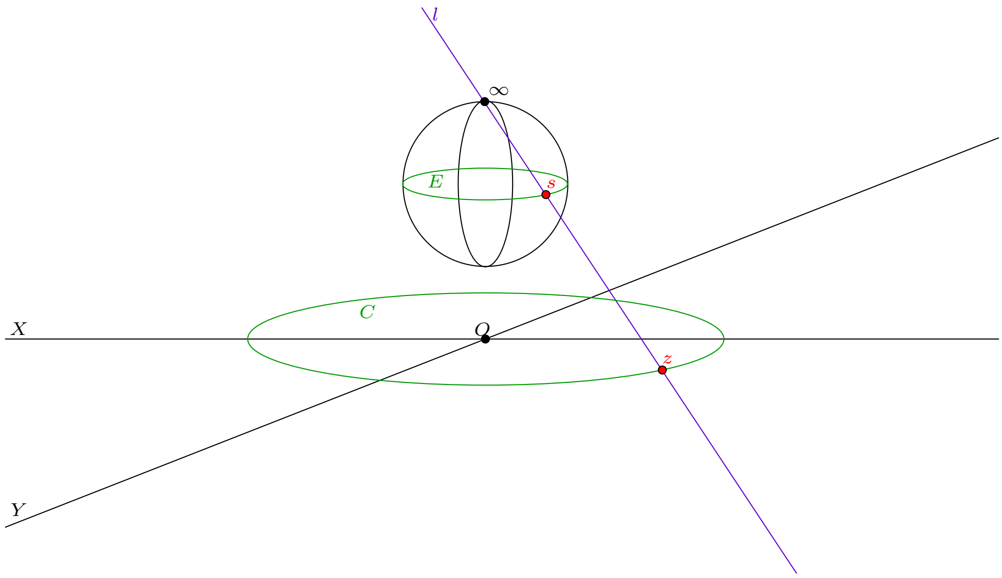


Figure 3.1 – The identification of S with $\mathbb{C} \cup \{\infty\}$.

sphere with $\mathbb{C} \cup \{\infty\}$.

Construction 3.1.1. Let $S = S^2$ be a sphere with T_S the Euclidean topology, where

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 + z^2 = 1\}. \tag{3.1}$$

We can identify S with $\mathbb{C} \cup \{\infty\}$. We do this by identifying the north pole $(0, 0, 1)$ of the sphere with ∞ and identifying $S \setminus \{(0, 0, 1)\}$ with $\mathbb{C}$ using a map $\pi \colon S \setminus \{(0, 0, 1)\} \to \mathbb{C}$ given by $(x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$. The map π is given by a well-defined rational function and therefore it is a homeomorphism from $S \setminus \{(0, 0, 1)\}$ to $\mathbb{C}$.

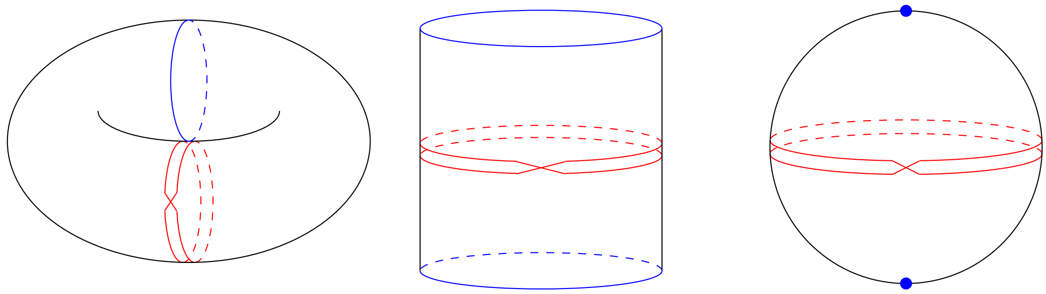
We can view the map π as follows. We place the sphere above an xy-plane representing $\mathbb{C}$. Any point $s \in S \setminus \{(0, 0, 1)\}$ is identified with a point in $\mathbb{C}$ by drawing a line through the point ∞ and s and identifying s with the intersection z of this line and the xy-plane. We choose to place the sphere such that the equator E of S^2 corresponds to the unit circle $C = \{z \in \mathbb{C} \colon |z| = 1\}$ in $\mathbb{C}$ and the south pole $(0, 0, -1)$ of the sphere is identified with $0 \in \mathbb{C}$ as shown in figure 3.1. We leave it up to the reader to verify that this corresponds to the homeomorphism π as given above.

We can now construct our space (R, T_R) by defining an equivalence relation on the sphere, using the identification with $\mathbb{C} \cup \{\infty\}$. Define an equivalence relation $\sim_S$ on $\mathbb{C} \cup \{\infty\}$ by $x \sim_S x$ for all $x \in \mathbb{C} \cup \{\infty\}$ and $z \sim_S -z$ for all $z \in \mathbb{C}$ such that $|z| \neq 1$. We leave it up to the reader to verify this is indeed an equivalence relation.

We have an equivalence relation on $\mathbb{C} \cup \{\infty\}$ and identify S with $\mathbb{C} \cup \{\infty\}$, so we can view $\sim_S$ as an equivalence relation on S. Define $R = S/\sim_S$ as the the quotient space with T_R the quotient topology and $q \colon S \to R$ the quotient map.

The equivalence relation $\sim_S$ can be described in the following way. Given a $z \in (-1, 0) \cup (0, 1)$ we can observe the circle $C = \{(u, v, w) \in S \subseteq \mathbb{R}^3 \colon w = z\}$ where $\sim_S$ identifies all points lying opposite of each other in C, to each other. Then all equivalence classes of elements of C under $\sim_S$ form another circle where any element has two inverse elements under the quotient map. Since this is done by $\sim_S$ for any $z \in (-1, 0) \cup (0, 1)$ and not for the equator, we obtain the sphere with a doubled equator as shown in figure 3.2c.

The notation introduced during the construction of (R, T_R) as given in construction 3.1.1 will be used throughout this chapter. A way to construct (R, T_R) starting with (Z, T_Z) is illustrated



(a) The space (Z, T_Z) can be cut along the blue line.

(b) After cutting the space (Z, T_Z) along the blue line as shown in figure 3.2a we can straighten it into the cylinder shown.

(c) We identify the blue circles in figure 3.2b to one point, which are marked by the blue dots and shape the cylinder into a sphere. The result is homeomorphic to the space (R, T_R).

Figure 3.2 – An alternative way to construct the space (R, T_R).

in figure 3.2. We will not give this construction in rigorous mathematical terms, nor will we prove that the result of the two constructions is the same space.

Before we continue we note the following.

Proposition 3.1.2. *The space (R, T_R) is a compact manifold.*

We know the sphere is a compact space and since q is a continuous map, this shows that (R, T_R) is compact. We leave it up to the reader to prove (R, T_R) is a manifold.

3.2 Étale maps and Covering Spaces

The definition of an étale map is given in definition 2.2.1. We define a covering space as follows.

Definition 3.2.1. A continuous map $f \colon Y \to X$ is a covering space if for all $x \in X$ there a set I and an open neighbourhood U of x such that $f^{-1}(U) = \bigcup_{i \in I} U_i$ where all U_i are open in Y, $U_i \cap U_j = \emptyset$ if $i \neq j$ and f restricted to U_i is a homeomorphism from U_i to U for all $i \in I$ and $I \neq \emptyset$.

The notions of étale maps and covering spaces have some similarities. Every covering space is also an étale map. However there are étale maps that are not covering spaces. For example, the map from the disjoint union of the open intervals $(0, 1)$ and $(-1, 1)$ to the open interval $(-1, 1)$ given by the identity is étale, however we can prove this is not a covering space by observing the point $0 \in (-1, 1)$.

In this section we will prove the following theorem.

Theorem 3.2.2. *Let $f \colon Y \to X$ be a surjective étale map where Y and X are compact Hausdorff manifolds. Then f is a covering space.*

We observe that the conditions that Y is compact and f continuous and surjective imply that X is compact as well. Not all conditions will be used in our proof of theorem 3.2.2, we only use that Y is a compact Hausdorff space and X a Hausdorff and locally path-connected space. However when we use theorem 3.2.2, to prove that the existence of our space (R, T_R) implies a negative answer to question 3.0.3, we will use it in the situation described in it.

We first prove the following lemma, which we will use when proving theorem 3.2.2.

Lemma 3.2.3. *Let $f\colon Y \to X$ be a surjective étale map such that X is Hausdorff and Y is compact. Then $f^{-1}(x)$ is finite for all $x \in X$.*

Proof. Let $x \in X$. We know X is Hausdorff therefore $\{x\}$ is closed in X. Since f is continuous, $f^{-1}(x)$ is then closed in Y and since Y is compact we conclude that $f^{-1}(x)$ is compact in Y.

We know f is an étale map, therefore for any $y \in f^{-1}(x)$ there is an open neighbourhood U_y of y such that f restricted to U_y is a homeomorphism from U_y to $f(U_y)$. Clearly $f^{-1}(x) \subseteq \bigcup_{y \in f^{-1}(x)} U_y$ thus showing there must be a finite $J \subseteq f^{-1}(x)$ such that $f^{-1}(x) \subseteq \bigcup_{y \in J} U_y$.

Let $z \in f^{-1}(x)$. Then $z \in \bigcup_{y \in J} U_y$. We know f restricted to U_y is a homeomorphism for all $y \in f^{-1}(x)$, therefore $z \notin U_y$ for all $y \in f^{-1}(x) \setminus \{z\}$. Thus we can conclude that $z \in J$ thus showing that $f^{-1}(x) \subseteq J$ and therefore $f^{-1}(x)$ is finite. □

Before we can prove theorem 3.2.2 we need to define the following notions.

Definition 3.2.4. Let (X, T_X) be a topological space S a subset of X. Then a point $x \in X$ is called a limit point of S if for all open neighbourhoods U of x there exists a $y \in U \cap S$ such that $y \neq x$.
Definition 3.2.5. Let (X, T_X) be a topological space and let $x, y \in X$. Then a *path from x to y in X* is a continuous map $\delta\colon [0,1] \to X$ such that $\delta(0) = x$ and $\delta(1) = y$.

We use these notions to formulate the following proposition.

Proposition 3.2.6. *Let $\delta\colon [0,1] \to X$ be a path. Let $C = \delta([0,1]) \in X$ and $S_1, S_2 \subseteq C$ such that $S_1 \neq \emptyset$, $S_2 \neq \emptyset$, $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = C$. Then there is a $s \in \{1,2\}$ such that there is a $t \in [0,1]$ such that $\delta(t) \in S_s$ and $\delta(t)$ a limit point of S_{3-s}.*

We leave it up to the reader to verify this proposition. We are now ready to prove theorem 3.2.2, so we begin our proof.

Proof of 3.2.2. Let $x \in X$. We will start by constructing the open neighbourhood U of x as mentioned in definition 3.2.1 of a covering space. Lemma 3.2.3 states that $f^{-1}(x)$ must be finite. Since Y is Hausdorff we can then define open neighbourhoods U_y of all $y \in f^{-1}(x)$ such that $U_y \cap U_z = \emptyset$ if $y \neq z$. We know f is an étale map and X is a manifold. Because $f^{-1}(x)$ is finite we can use these properties and take a finite intersection of open neighbourhoods of x, to construct a path-connected open neighbourhood U of x such that f restricted to $f^{-1}(U) \cap U_y$ is a homeomorphism to U for all $y \in f^{-1}(x)$.

Now that we have a neighbourhood U of x, we need to show that its pre-image consists of disjoint open subsets of Y which are mapped homeomorphically to U by f. Define $W = \bigcup_{y \in f^{-1}(x)} (f^{-1}(U) \cap U_y)$. We need to show that $f^{-1}(U) = W$ to prove that f is a covering space. We know $W \subseteq f^{-1}(U)$. We observe that for any $z \in U$ we know $\#f^{-1}(z) \geq \#f^{-1}(x)$ and if $\#f^{-1}(z) = \#f^{-1}(x)$ then $f^{-1}(z) \subseteq W$.

We will prove that $\#f^{-1}(z) = \#f^{-1}(x)$ for all $z \in U$, which shows that $f^{-1}(U) \subseteq W$. We will do so by assuming there are $z \in U$ such that $\#f^{-1}(z) > \#f^{-1}(x)$. Then there is a point $u \in U$ such that any open neighbourhood of u in U contains both points that have the same number of pre-images as x and points that have more pre-images as x. If u has the same number of pre-images as x, then we can derive a contradiction using that Y is compact, because the part of $f^{-1}(U)$ that is not contained in W must then have an "open border" at some point. If u has more pre-images than x, we can derive a contradiction using that Y is Hausdorff, because there then is an open neighbourhood of one of the pre-images of u which is not contained in W and is mapped homeomorphically to an open neighbourhood of u. However all points in this open neighbourhood of u must then have more pre-images than x, while we chose u such that any open neighbourhood of u contains points that have the same number of pre-images as x.

We first construct our point $u \in U$. Assume there is a $z \in U$ such that $\#f^{-1}(z) > \#f^{-1}(x)$. Since U is path-connected and $x, z \in U$, there must be a path $\delta\colon [0,1] \to U$ such that $\delta(0) = z$ and $\delta(1) = x$. Define $C = \delta([0,1])$ and define $S_1 = \{y \in C\colon \#f^{-1}(y) = \#f^{-1}(x)\}$ and $S_2 = \{y \in C\colon \#f^{-1}(y) > \#f^{-1}(x)\}$. Then $x \in S_1$, so S_1 is not empty. Also $z \in S_2$ therefore S_2 is not empty. Since for all $y \in U$ we know $\#f^{-1}(y) \geq \#f^{-1}(x)$, we conclude $S_1 \cup S_2 = C$. Also by definition $S_1 \cap S_2 = \emptyset$. Proposition 3.2.6 now states that there is a $s \in \{1,2\}$ such that there is a $t \in [0,1]$ such that $\delta(t) \in S_s$ and $\delta(t)$ a limit point of S_{3-s}. Define $u = \delta(t)$.

Assume $s = 1$, then $u \in S_1$ and u is a limit point of S_2. In this case u has the same number of pre-images as x, we will derive a contradiction using the compactness of Y.

We know $[0,1]$ is compact and the continuous image of a compact space is compact, therefore $C = \delta([0,1])$ is compact. Since X is Hausdorff and $C \subseteq X$ this shows that C must be closed in X. Since f is continuous we then know $f^{-1}(C)$ must be closed in Y. We observe that W is open in Y and therefore $W \cap f^{-1}(C)$ is open in $f^{-1}(C)$. Then $f^{-1}(C) \setminus \big(W \cap f^{-1}(C)\big) = f^{-1}(C) \setminus W$ must be closed in $f^{-1}(C)$, which in turn is closed in Y. Since Y is Hausdorff and $f^{-1}(C) \setminus W$ is closed in Y, we conclude $f^{-1}(C) \setminus W$ is compact.

We observe that $S_2 = f(f^{-1}(C) \setminus V)$ must then be compact. Since C is a subset of a Hausdorff space, C must be Hausdorff and then S_2 must be closed in C. Therefore S_2 contains all of its limit points in C [3]. Since $u \in C$ is a limit point of S_2 we know $u \in S_2$. But $u \in S_1$ and $S_1 \cap S_2 = \emptyset$ therefore this is a contradiction and the assumption that $s = 1$ must be incorrect.

We now conclude $s = 2$ therefore $u \in S_2$ and u a limit point of S_1. Then u has more pre-images than x, therefore there must be a $v \in f^{-1}(u)$ such that $v \notin W$. We will derive a contradiction using the fact that Y is Hausdorff.

Using the fact that f is an étale map and Y is Hausdorff and $f^{-1}(x)$ is finite, we can take a finite intersection to construct an open neighbourhood A of v and open neighbourhoods B_a of all $a \in f^{-1}(u) \cap W$ such that f restricted to A and f restricted to B_a are homeomorphisms to $f(A) = f(B_a)$, $f(A) = f(B_a)$ is open in X and $A \cap B_a = \emptyset$ for all $a \in f^{-1}(u) \cap W$.

We have an open neighbourhood $f(A)$ of u and u is a limit point of S_1 therefore there is a $w \in f(A)$ such that $w \in S_1$, which means that w has the same number of pre-images as x. However we know for all $w \in f(A)$ that w has more pre-images than x, because there are exactly $\#f^{-1}(x)$ elements in W that are mapped to w and then at least another element in A that is mapped to w by f. This is a contradiction thus showing that the assumption that there is a $z \in U$ such that $\#f^{-1}(z) > \#f^{-1}(x)$ was false.

We now conclude that $\#f^{-1}(z) = \#f^{-1}(x)$ for all $z \in U$ therefore $f^{-1}(U) \subseteq W$. Thus showing that $W = f^{-1}(U)$. □

3.3 A Negative Answer to our Question

In this section we will prove that there is no surjective étale map from a compact Hausdorff manifold to the compact manifold (R, T_R). We first observe there is a surjective étale map from (R, T_R) to the sphere.

Construction 3.3.1. We define an equivalence relation $\sim_T$ on $\mathbb{C} \cup \{\infty\}$ by $w \sim_T z$ if and only if $z^2 = w^2$ for all $z, w \in \mathbb{C}$ and $\infty \sim_T \infty$. We leave it up to the reader to verify this is indeed an equivalence relation. We can use the identification of $S = S^2$ with $\mathbb{C} \cup \{\infty\}$ as described in construction 3.1.1 to view $\sim_T$ as an equivalence relation on S. We note that the equivalence relation $\sim_S$ as described in construction 3.1.1 is a subset of $\sim_T$. Therefore $\sim_T$ induces a well-defined equivalence relation $\sim_R$ on R. We define $g\colon R \to R/\sim_R$ as the quotient map.

Proposition 3.3.2. *The space $R/\sim_R$ as constructed in construction 3.3.1 is homeomorphic to a sphere S^2 and g is a surjective étale map.*

We leave it up to the reader to prove this. We will prove the existence of (R, T_R) implies a negative answer to question 3.0.3 and we will use theorem 3.2.2 to do so. Before we can start our proof, we first observe the following.

Proposition 3.3.3. *If $p\colon Y \to S^2$ is a covering space, then $Y \cong S^2 \times I$ for a discrete space I and p restricted to $S^2 \times \{i\}$ is a homeomorphism to S^2 for all $i \in I$.*

This proposition can be proven using the theory of a universal covering space and the fact that the sphere is simply connected. The theory needed can be found in section 2.4 of Thamás Szamuely's book on Galois groups and fundamental groups [5]. We leave it up to the reader to prove propostion 3.3.3. We will now prove that the existence of (R, T_R) implies a negative answer to question 3.0.3.

Proposition 3.3.4. *There is no surjective étale map from a compact Hausdorff manifold to (R, T_R).*

Proof. Assume there is a surjective étale map f from a compact Hausdorff manifold (Y, T_Y) to (R, T_R). Proposition 3.3.2 states that there is a surjective étale map g from (R, T_R) to S^2 with the Euclidean topology. We observe that $p = g \circ f$ is then a surjective étale map from Y to S^2. We leave it up to the reader to verify this.

We can now apply theorem 3.2.2 to the map p, which shows that p is a covering space. Next we use proposition 3.3.3 to conclude that $Y \cong \bigcup_{i \in I} S^2 \times \{i\}$ and p restricted to $S^2 \times \{i\}$ is a homeomorphism to S^2 for all $i \in I$. We observe that Y is compact, therefore I must be finite. Also $p|_{S^2 \times \{i\}} = \big(g \circ f|_{S^2 \times \{i\}}\big)$ is a homeomorphism, thus showing that $f|_{S^2 \times \{i\}}$ is a continuous open injective map.

We now conclude that (R, T_R) is almost Hausdorff. However analogously to the proof of proposition 2.3.2, which states that (Z, T_Z) is not almost Hausdorff, we can prove that (R, T_R) is not almost Hausdorff. This is a contradiction therefore we conclude there is no surjective étale map from a compact Hausdorff manifold to (R, T_R). □

Chapter 4

The Fundamental Groups of (R, T_R) and (Z, T_Z)

In this section we will compute the fundamental groups of our space (Z, T_Z) as constructed in section 2.1 and our space (R, T_R) as constructed in section 3.1. We first define a group and a fundamental group.

Definition 4.0.5. A *group* is a category G containing only one object x, where all morphisms are isomorphisms, i.e. any morphism $f\colon x \to x$ has an inverse $f^{-1}\colon x \to x$ such that $f \circ f^{-1} = id_x = f^{-1} \circ f$ where id_x is the identity map and called the *identity element* of G. The composition $\circ$ is called the *group operation*.

Definition 4.0.6. Let (X, T) be a topological space and let $x \in X$ be a base point. Then the *fundamental group* $\pi_1(X, x)$ is the category consisting of the object x and where $\mathrm{Hom}(x, x)$ is the set of all equivalence classes of loops based at x in X up to path homotopy. This is a group with path composition as group operation.

In section 4.1, we will introduce a theorem known as Seifert - Van Kampen's theorem for fundamental groups. We will introduce Brown's version of this theorem, who proved that the same statement is true for fundamental groupoids, a concept we will also introduce in section 4.1. We will then use this theorem to compute the fundamental group of the line with a doubled origin as an example. In section 4.2 we will compute the fundamental group of the space (R, T_R) as constructed in construction 3.1.1. We will use both Seifert - Van Kampen's theorem and the example introduced in section 4.1 to do so. Finally, in section 4.3 we will give a sketch of how this theorem can be used to compute the fundamental group of the space (Z, T_Z) as constructed in construction 2.1.3.

4.1 Seifert - Van Kampen's Theorem for Fundamental Groupoids

We will first give all definitions we need. Then we will give a theorem that will allow us to compute the fundamental groups of spaces. For more background information and proofs of the proposition and theorem stated in this section, we will refer to Ronald Brown's book on topology and groupoids [1].

We start by defining a pushout diagram. This is a concept that can be defined within any category.

Definition 4.1.1. Let C_0, C_1, C_2 and C be objects of a category $\mathbf{C}$. Let i_1, i_2, u_1 and u_2 be morphisms in $\mathbf{C}$ such that a diagram

$$\begin{array}{ccc} C_0 & \xrightarrow{i_1} & C_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle u_1} \\ C_2 & \xrightarrow{u_2} & C \end{array} \tag{4.1}$$

can be formed. Diagram (4.1) is called a *pushout diagram* if the diagram commutes, i.e. $u_1 \circ i_1 = u_2 \circ i_2$, and for any commutative diagram

$$\begin{array}{ccc} C_0 & \xrightarrow{i_1} & C_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle v_1} \\ C_2 & \xrightarrow{v_2} & C' \end{array} \tag{4.2}$$

with C' an object in $\mathbf{C}$ and v_1 and v_2 morphisms in $\mathbf{C}$, there is a unique morphism $v\colon C \to C'$ in $\mathbf{C}$ such that $v \circ u_1 = v_1$ and $v \circ u_2 = v_2$. In this case we call C the *pushout* of i_1 and i_2.

The following statement follows from the definition of a pushout.

Proposition 4.1.2. *Let* (4.1) *be a pushout diagram. Then* (4.2) *is a pushout diagram if and only if there exists an isomorphism* $v\colon C \to C'$ *such that* $v\colon u_1 = v_1$ *and* $v\colon u_2 = v_2$.

This proposition follows easily from the definition of a pushout diagram. More information on pushouts and this proposition can be found in section 6.6 of Ronald Brown's book on topology and groupoids [1].

We now introduce the following concepts.

Definition 4.1.3. A *groupoid* is a category containing a set of objects, where all morphisms are isomorphisms, i.e. every morphism has an inverse.
Definition 4.1.4. Let (X, T) be a topological space and A a set. Than the *fundamental groupoid* πXA is the category containing $X \cap A$ as objects and for any $x, y \in X \cap A$ the set $\mathrm{Hom}(x, y)$ contains all equivalence classes of paths from x to y in X under path-homotopy. The composition of morphisms in πXA is given by path composition.
Definition 4.1.5. A set A is called *representative* of a topological space (X, T) if A contains at least one point from every path-connected component of X.

We observe that given the fundamental groupoid πXA of a space X for a set A, we can easily find the fundamental group $\pi_1(X, x)$ for an $x \in X \cap A$, because $\pi_1(X, x)$ is the subcategory of πXA containing only x as an object and $\mathrm{Hom}(x, x)$ as in πXA. We will introduce a theorem that will allow us to compute the fundamental groupoid of a space X for certain A, which then allows us to find the fundamental group.

Theorem 4.1.6. *Let* (X, T) *be a topological space. Let* $X_1, X_2 \subseteq X$ *be open and* $X_1 \cup X_2 = X$ *and* $X_0 = X_1 \cap X_2$. *Then the diagram*

$$\begin{array}{ccc} X_0 & \xrightarrow{i_1} & X_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle u_1} \\ X_2 & \xrightarrow{u_2} & X \end{array} \tag{4.3}$$

where i_1, i_2, u_1 and u_2 are the inclusions, is a pushout diagram in the category of topological spaces. Let A be a set which is representative of X_0, X_1 and X_2. Then diagram (4.3) *induces a diagram*

$$\begin{array}{ccc} \pi X_0 A & \xrightarrow{i_{1*}} & \pi X_1 A \\ {\scriptstyle i_{2*}}\downarrow & & \downarrow{\scriptstyle u_{1*}} \\ \pi X_2 A & \xrightarrow{u_{2*}} & \pi X A \end{array} \tag{4.4}$$

where i_{1*}, i_{2*}, u_{1*} and u_{2*} are the induced inclusions, and (4.4) *is a pushout diagram of groupoids.*

Proof. It is easy to check that (4.3) is a pushout diagram. For the proof that (4.4) is a pushout diagram, we will refer to 6.7.2 in Ronald Brown's book on topology and groupoids [1]. ☐

Remark 4.1.7. *When all conditions of theorem 4.1.6 are satisfied and $A = \{x\} \subseteq X_0$ then* (4.4) *is a pushout diagram of groups, where $\pi XA = \pi_1(X, x)$, $\pi X_0 A = \pi_1(X_0, x)$, $\pi X_1 A = \pi_1(X_1, x)$ and $\pi X_2 A = \pi_1(X_2, x)$.*

Theorem 4.1.6 is also known as Seifert - Van Kampen's theorem for fundamental groupoids, as mentioned in our introduction. Remark 4.1.7 corresponds to Seifert - Van Kampen's theorem for fundamental groups. Remark 4.1.7 follows from theorem 4.1.6 because a pushout of groupoids, where all groupoids have only one object, is also a pushout of groups.

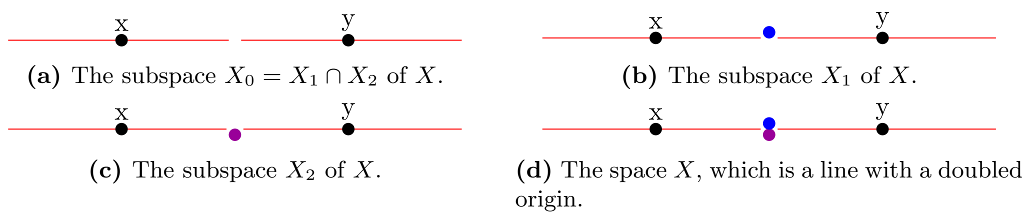
We will now continue by using theorem 4.1.6 to compute the fundamental group of the line with a doubled origin. We first construct this space.

Construction 4.1.8. Define $A = (\mathbb{R} \times \{0\}) \cup (\mathbb{R} \times \{1\})$ where $\mathbb{R}$ has the Euclidean topology and A the product topology. Define an equivalence relation $\sim_A$ on A by $(x, i) \sim_A (y, j)$ if and only if $(x, i) = (y, j)$ or $x = y \neq 0$. Define $X = A/\sim_A$ with the quotient topology and q the quotient map. Then X is the space to which we will refer as a line with a doubled origin. The space X is drawn in figure 4.1d

We can now prove the following about the fundamental group of the space (X, T_X) we just constructed.

Proposition 4.1.9. *Let (X, T_X) be the space constructed in construction 4.1.8. Let $x \in X$, then $\pi_1(X, x) \cong \mathbb{Z}$.*

Proof. We can choose $X_1 = q(\mathbb{R} \times \{0\})$ and $X_2 = q(\mathbb{R} \times \{1\})$. Then $X_1, X_2 \subseteq X$ and $q^{-1}(X_1) = A \setminus \{(0,1)\}$ and $q^{-1}(X_2) = A \setminus \{(0,0)\}$ therefore X_1 and X_2 are open in X. Furthermore $X_1 \cup X_2 = X$. Define $X_0 = X_1 \cap X_2 = X \setminus \{[(0,0)], [(0,1)]\}$. Choose $x = [(-1,0)] \in X$ and $y = [(1,0)]$ and $A = \{x, y\}$. The subspaces X_0, X_1 and X_2 of X are drawn in figure 4.1. The points x and y are also drawn.



(a) The subspace $X_0 = X_1 \cap X_2$ of X.

(b) The subspace X_1 of X.

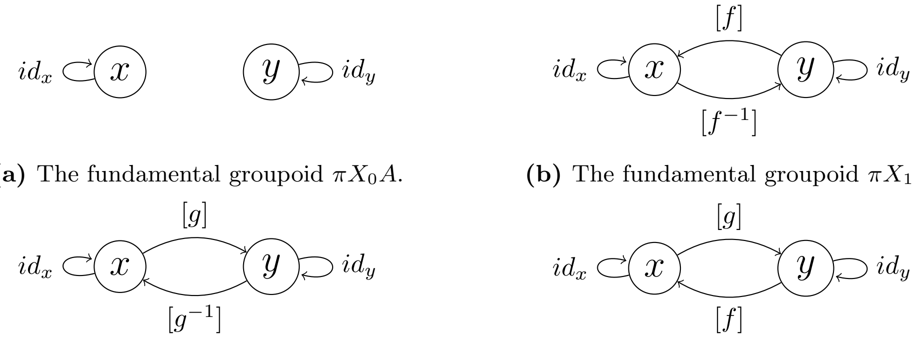
(c) The subspace X_2 of X.

(d) The space X, which is a line with a doubled origin.

Figure 4.1 – The line with a doubled origin and all its subspaces.

Let i_1 and i_2 be the inclusions of X_0 in X_1 and X_2 respectively and u_1 and u_2 be the inclusions of X_1 and X_2 in X respectively. Then we have a pushout diagram as shown in (4.3). Also A is representative for X_0, X_1 and X_2 therefore theorem 4.1.6 states that the fundamental groupoids form a pushout diagram as shown in (4.4).

Since X_1 and X_2 are both homeomorphic to $\mathbb{R}$, which is contractible, we can conclude their fundamental groupoids on A are as shown in figures 4.2b and 4.2c respectively, where $f\colon [0,1] \to X_1$ is given by $t \mapsto [(1-2t, 0)]$ and $g\colon [0,1] \to X_2$ is given by $t \mapsto [(-1+2t, 1)]$ and id_x and id_y are the identity elements in $\mathrm{Hom}(x,x)$ and $\mathrm{Hom}(y,y)$ respectively, given by the constant paths. When δ is a path we use δ^{-1} to notate the inverse path given by $t \mapsto \delta(1-t)$. Since X_0 is homeomorphic to $\mathbb{R}_{<0} \cup \mathbb{R}_{>0}$ and therefore consists of two path-connected components which are both contractible, we can conclude that $\pi X_0 A$ is as shown in figure 4.2a. Then i_{1*} and i_{2*} are the inclusions of $\pi X_0 A$ into $\pi X_1 A$ and $\pi X_2 A$ respectively.



(a) The fundamental groupoid $\pi X_0 A$.

(b) The fundamental groupoid $\pi X_1 A$.

(c) The fundamental groupoid $\pi X_2 A$.

(d) The generators of the fundamental groupoid $\pi X A$.

Figure 4.2 – The fundamental groupoids of X and its subspaces.

Let G be the groupoid containing two elements x' and y', generated by two maps $f'\colon y' \to x'$ and $g'\colon x' \to y'$. Then $\mathrm{Hom}(x', x') = \{(f'g')^n \colon n \in \mathbb{Z}\} \subseteq G$ where $(f'g')^n$ is given by composing the identity with $f'g'$ n times when $n \geq 0$ and composing the identity with $(f'g')^{-1}$ $|n|$ times when $n < 0$. This shows that $\mathrm{Hom}(x', x') \cong \mathbb{Z}$. The reader can easily check that G is a pushout of i_{1*} and i_{2*}. Proposition 4.1.2 now states that G must be isomorphic to $\pi X A$, thus showing that $\pi X A$ is generated by the elements shown in figure 4.2d.

We now conclude that $\mathbb{Z} \cong \mathrm{Hom}(x,x) \subseteq \pi X A$ and because X is path connected, we know $\pi_1(X,x) \cong \pi_1(X,y)$ for all $x, y \in X$ [4, corollary 5.1.24]. We can conclude $\pi_1(X,x) \cong \mathbb{Z}$ for all $x \in X$. □

4.2 The Fundamental Group of (R, T_R)

In this section we will prove that the space (R, T_R) is simply connected, i.e. its fundamental group $\pi_1(R, r)$ is trivial for all $r \in R$. We will do so using theorem 4.1.6 and remark 4.1.7. We first choose open subspaces X_1 and X_2 of our space (R, T_R) as shown in figure 4.3. We also define $X_0 = X_1 \cap X_2$.

We define $A = \{a, b\}$ where a is the north pole and b the south pole of the sphere, as shown in figure 4.3. Before we can compute the fundamental group of (R, T_R) we must first compute the fundamental groupoid $\pi X_0 A$ and use this to compute the fundamental group $\pi_1(X_0, a)$. We define the paths f, g, f' and g' as shown in figure 4.3.

Proposition 4.2.1. *The elements $[fg]$, $[f'g']$, $[f'g]$ and $[fg']$ generate the fundamental group $\pi_1(X_0, a)$.*

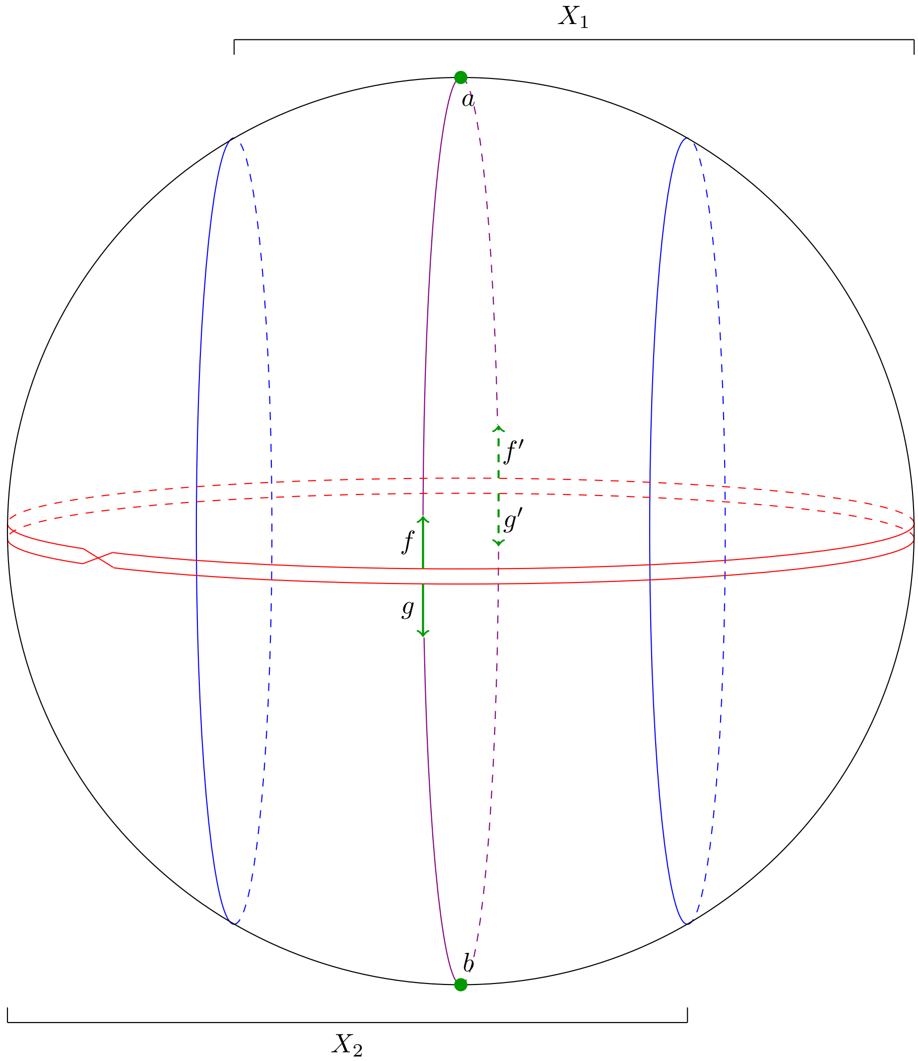
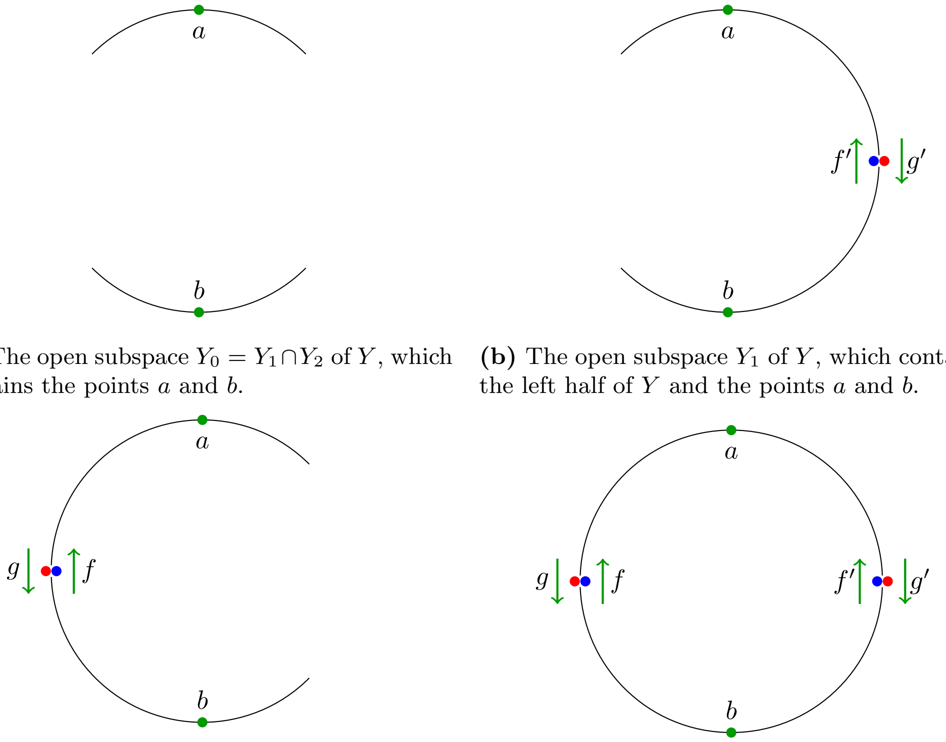


Figure 4.3 – The space (R, T_R). The red lines represent the doubled equator. The blue lines are the borders of the open subspace $X_0 = X_1 \cap X_2$, where X_1 is the open subspace containing the right part of the sphere bordered by the left blue line and X_2 is the open subspace containing the left part of the sphere bordered by the right blue line, as indicated in the figure. The point a is the north pole and the point b the south pole, as indicated by the green dots. The paths f and f' are both paths from b to a along the circle drawn in purple, where f' runs along the back of the sphere on the dashed part of the purple circle and f along the front of the sphere. Similarly g and g' are paths from a to b along the purple circle. The direction of the green arrows indicate the direction in which these paths run. The start of the arrows indicate across which of the two red lines the paths pass the equator. The purple circle is shown in figure 4.4d, where the points a and b and the paths f, g, f' and g' are also shown.

Remark 4.2.2. *While there are relations between these generators of $\pi_1(X_0, a)$, we will not use these when computing the fundamental group of (R, T_R). Note that $(fg)(f'g)^{-1}(f'g') = fg'$. In fact, it can be shown that $\pi_1(X_0, a)$ is the free group generated by $[fg]$, $[f'g']$ and $[f'g]$.*

We will now prove proposition 4.2.1.



(a) The open subspace $Y_0 = Y_1 \cap Y_2$ of Y, which contains the points a and b.

(b) The open subspace Y_1 of Y, which contains the left half of Y and the points a and b.

(c) The open subspace Y_2 of Y, which contains the left half of Y and the points a and b.

(d) The space Y, which is a circle with two doubled points.

Figure 4.4 – The space Y and its open subspaces Y_1, Y_2 and $Y_0 = Y_1 \cap Y_2$. The points a and b are contained in all these spaces. The paths f and f' are paths from b to a and the paths g and g' are paths from a to b. The path f and g run along the left side of the circle and cross the doubled point on the blue and red point respectively. The paths f' and g' run along the right side of the circle and cross the doubled point on the blue and red point respectively.

Proof of 4.2.1. We remark that the purple circle Y as shown in figure 4.3 is homotopy equivalent to X_0. Therefore $\pi_1(Y, a) \cong \pi_1(X_0, a)$ [4, corollary 5.1.26]. We can view Y as the circle $S^1 = \{z \in \mathbb{C} \colon |z| = 1\}$ where the points -1 and 1 are doubled and $a = i$ and $b = -i$ as drawn in figure 4.4d.

We can now define Y_1 as the union of $\left\{e^{it} \colon t \in \left(-\frac{3\pi}{4}, 0\right) \cup \left(0, \frac{3\pi}{4}\right)\right\}$ and the doubled points at 1. Similarly we define Y_2 as the union of $\left\{e^{it} \colon t \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{7\pi}{4}\right)\right\}$ and the doubled points at -1. Then Y_1 and Y_2 are open subsets of Y, they are shown in figures 4.4b and 4.4c respectively.

Let i_1 and i_2 be the inclusions of Y_0 into Y_1 and Y_2 respectively and u_1 and u_2 be the inclusions of Y_1 and Y_2 into Y respectively. Then the paths $g \colon [0,1] \to Y_2 \subseteq Y$ and $f \colon [0,1] \to Y_2 \subseteq Y$ are given by $t \mapsto e^{\pi i t + \frac{\pi}{2}}$ and $t \mapsto e^{-\pi i t - \frac{\pi}{2}}$ respectively for $t \neq \frac{1}{2}$ and $g\left(\frac{1}{2}\right)$ one of the two points at -1 and $f\left(\frac{1}{2}\right)$ the other point. Similarly we the paths f' and g' in Y_1 run along the other side of the circle as drawn in figure 4.4.

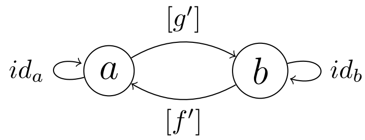
We now observe that Y_0 consists of two contractible subsets, one of which contains a and the other contains b, therefore $\pi Y_0 A$ is as shown in figure 4.5a. Also Y_1 and Y_2 can be seen as lines with a doubled origin. The proof of proposition 4.1.9 shows that their fundamental groupoids $\pi Y_1 A$ and $\pi Y_2 A$ must be generated by the elements shown in figures 4.5b and 4.5c respectively.

Theorem 4.1.6 states there is a pushout diagram

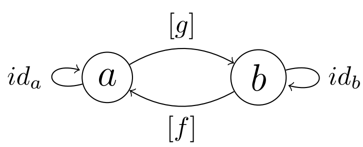
$$\begin{array}{ccc} \pi Y_0 A & \xrightarrow{i_{1*}} & \pi Y_1 A \\ {\scriptstyle i_{2*}}\downarrow & & \downarrow{\scriptstyle u_{1*}} \\ \pi Y_2 A & \xrightarrow{u_{2*}} & \pi Y A \end{array} \tag{4.5}$$

Since $\pi Y_1 A$ and $\pi Y_2 A$ are generated by the elements shown in figures 4.5b and 4.5c respectively, we conclude that $\pi Y A$ must be generated by the elements shown in figure 4.5d.

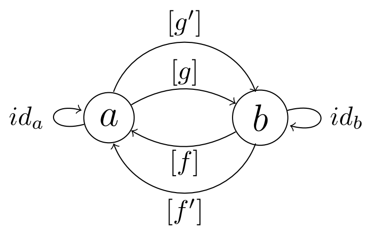
(a) The fundamental groupoid $\pi Y_0 A$.



(b) The generators of the fundamental groupoid $\pi Y_1 A$.



(c) The generators of the fundamental groupoid $\pi Y_2 A$.



(d) The generators of the fundamental groupoid $\pi Y A$.

Figure 4.5 – The fundamental groupoids of Y and its subspaces.

We now conclude that the elements $[fg]$, $[f'g']$, $[f'g]$ and $[fg']$ generate $\mathrm{Hom}(a, a) \subseteq \pi Y A$. $\square$

Now that we know more about $\pi_1(X_0, a)$ we can prove the following.

Proposition 4.2.3. *The fundamental group $\pi_1(R, x)$ is trivial for all $x \in R$.*

Proof. We define $A = \{a\}$ and $X_1, X_2 \subseteq R$ and $X_0 = X_1 \cap X_2$ as shown in figure 4.3. Let i_1 and i_2 be the inclusions of X_0 into X_1 and X_2 respectively and let u_1 and u_2 be the inclusions of X_1 and X_2 into R respectively. Remark 4.1.7 states there is a pushout diagram

$$\begin{array}{ccc} \pi_1(X_0, a) & \xrightarrow{i_{1*}} & \pi_1(X_1, a) \\ {\scriptstyle i_{2*}}\downarrow & & \downarrow{\scriptstyle u_{1*}} \\ \pi_1(X_2, a) & \xrightarrow{u_{2*}} & \pi_1(R, a) \end{array} \tag{4.6}$$

Proposition 4.2.1 and Remark 4.2.2 show that $[fg]$, $[f'g']$ and $[f'g]$ generate the fundamental group $\pi_1(X_0, a)$. We observe that X_1 and X_2 are homotopy equivalent to a line with a doubled origin, thus showing that $\pi_1(X_1, a)$ and $\pi_1(X_2, a)$ have one generator and are isomorphic to $\mathbb{Z}$. Both

$\pi_1(X_1, a)$ and $\pi_1(X_2, a)$ are generated by $[fg]$, we leave it up to the reader to verify this. Since diagram (4.6) is a pushout diagram, this shows that $\pi_1(R, a)$ is generated by $[fg]$.

We now look at the map i_{1*}. This map is induced by the inclusion i_1. The paths f' and f as drawn in figure 4.3 are path-homotopic in X_1, we leave it up to the reader to verify this. We conclude that $i_{1*}([f'g]) = [fg]$. Next we look at the map i_{2*}, which is induced by the inclusion i_2. The paths f' and g^{-1} are path-homotopic in X_2, from which we conclude that $i_{2*}([f'g]) = id_a$.

We know that u_{2*} must be a group homomorphism. Therefore $u_{2*}(id_a) = id_a$ thus showing that $(u_{2*} \circ i_{2*})([f'g]) = id_a$. Since diagram (4.6) must commute, we know $id_a = (u_{1*} \circ i_{1*})([f'g]) = u_{1*}([fg])$, thus showing that fg is path-homotopic to the constant path in a in R. Then $[fg] = id_a$ in $\pi_1(R, a)$. Since $\pi_1(R, a)$ is generated by $[fg]$ this shows that $\pi_1(R, a)$ is the trivial group. Since R is path connected we conclude that $\pi_1(R, x)$ is the trivial group for all $x \in R$. □

We now know that in the case of the sphere with the twisted doubled equator, the fundamental group is the same as for the sphere. In the case of the sphere with a regular doubled equator, we can use theorem 4.1.6 to show that its fundamental group is isomorphic to $\mathbb{Z}$.

4.3 The Fundamental Group of (Z, T_Z)

In this section we will show part of the computation of the fundamental group of (Z, T_Z) as constructed in construction 2.1.3. We will use theorem 4.1.6.

Proposition 4.3.1. *The fundamental group $\pi_1(Z, z)$ is the free abelian group on two generators for all $z \in Z$.*

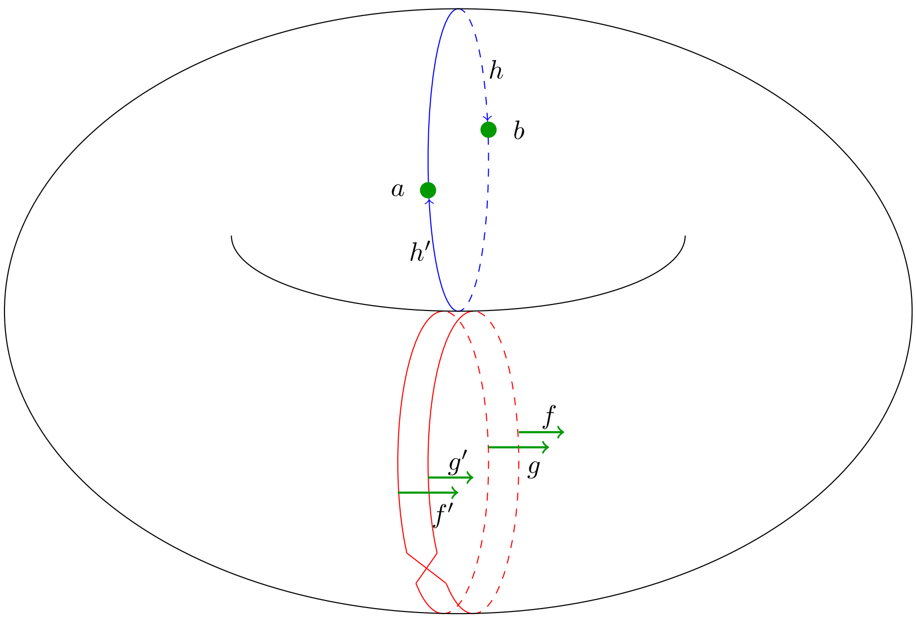
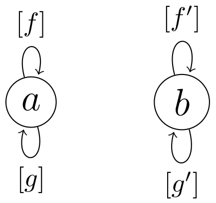
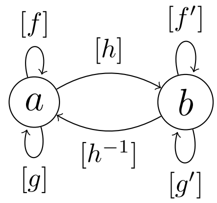


Figure 4.6 – The space (Z, T_Z). The points a and b are indicated by the green dots. The path h' from a to b and h' from b to a are drawn in blue. The start of the green arrows indicate where the paths f, g, f' and g' cross the doubled line, drawn in red. The direction of the arrows indicate in which direction they pass. The paths f and g are both loops based at a and the paths f' and g' are both paths based at b. The paths f and g form a circle around the inside of the torus, the paths f' and g' form a larger circle around the outside of the torus.

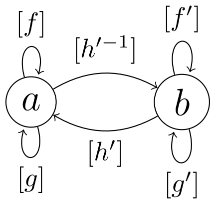
Proof. We choose X_1 to be an open subspace of X containing the upper half of the torus and X_2 the open subspace of X containing the lower half of the torus. We choose the points a and b as shown in figure 4.6 and define $A = \{a, b\}$. We choose these subspaces such that $X_0 = X_1 \cap X_2$ consists of two path-connected components, that are both homotopy equivalent to the circle with a doubled point, where a is contained in one of these path-connected components and b in the other. We also define the paths f, f', g, g', h and h' as shown in figure 4.6.



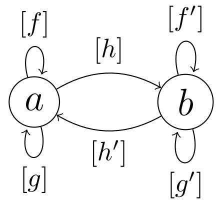
(a) The generators of the fundamental groupoid $\pi X_0 A$.



(b) The generators of the fundamental groupoid $\pi X_1 A$.



(c) The generators of the fundamental groupoid $\pi X_2 A$.



(d) The generators of the fundamental groupoid $\pi X A$.

Figure 4.7 – The fundamental groupoids of X and its subspaces.

Using theorem 4.1.6 we can show that the fundamental group of the circle with a doubled point, is the free group on two generators. This shows that the fundamental groupoid $\pi X_0 A$ is generated by the elements shown in figure 4.7a.

Since X_1 is also homotopy equivalent to a circle with a doubled point, we know that $\mathrm{Hom}(a, a)$ and $\mathrm{Hom}(b, b)$ in $\pi X_1 A$ are the same as in $\pi X_0 A$. This shows that the fundamental groupoid $\pi X_1 A$ is generated by the elements shown in figure 4.7b. Analogously we can show that $\pi X_2 A$ is generated by the elements shown in figure 4.7c. We can conclude that $\pi X A$ must be generated by the elements shown in figure 4.7d, using theorem 4.1.6.

We observe that hfh^{-1} is path homotopic to f' and hgh^{-1} is path homotopic to g' in X_1. Analogously, $h'^{-1}gh'$ is path homotopic to f' and $h'^{-1}fh'$ is path homotopic to g' in X_2. Using these relations we can prove that $f(h'h)$ is path homotopic to $(h'h)g$ in X. We can also show that $f(h'h)$ is path homotopic to $(h'h)f$ in X, thus showing that f is path homotopic to g in X.

We now conclude that $\mathrm{Hom}(a, a)$ is generated by $[h'h]$ and $[f]$ where these to elements commute. Using theorem 4.1.6 to show that $\pi X A$ is a pushout, we can show there are no other relations on the generators of $\mathrm{Hom}(a, a)$. Therefore $\mathrm{Hom}(a, a)$ is the free abelian group on two generators. Since Z is path-connected, we conclude that $\pi_1(Z, z)$ is the free abelian group on two generators for all $z \in Z$. □

Using theorem 4.1.6 we can also show that the fundamental group of a torus with a regular doubled circle, is generated by three elements f, f' and g, where $fg = gf$ and $f'g = gf'$ but $ff' \neq f'f$.

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