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Some results on the existence of division algebras over $\mathbb{R}$

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Contents

0.1 Introduction

This thesis divides naturally into two chapters. In the first chapter, the concept of *division algebra* is defined as a (not necessarily associative) algebra in which left- and right-multiplication with a non-zero element is bijective. It is noted that the zero algebra, the Real numbers and the Complex numbers form division algebras of respective dimension 0, 1 and 2 over $\mathbb{R}$. In the rest of the chapter, it is proven that furthermore, the Hamilton numbers (otherwise known as the Quaternions) form a 4-dimensional division algebra over $\mathbb{R}$, and that the Cayley numbers (otherwise known as the Octonions) form an 8-dimensional division algebra over $\mathbb{R}$. The first chapter is based on [Baez 2001] and it assumes basic familiarity with linear algebra.

It is known that the five algebras mentioned above are in fact the only five finite-dimensional division algebras over $\mathbb{R}$. A proof of this is far beyond the scope of this thesis, but in the second chapter at least it is shown that there exist no division algebras over $\mathbb{R}$ of odd dimension greater than 1. To achieve this we prove that the existence of division algebras of dimension n over $\mathbb{R}$ implies the *parallelisability* of the $n-1$-sphere, a definition of which is provided at the beginning of that chapter. To prove that for even n the n-sphere is not parallelisable we make use in section 2.3 of the *Brouwer degree*. Before the Brouwer degree can even be defined however we have to establish reduced singular homology in section 2.2, which actually takes up the largest part of chapter 2. The general idea and proofs of many of the lemmata and propositions of Chapter 2 have been adapted from [Hatcher 2002]. The second chapter assumes basic familiarity with topology, category theory and homological algebra. For a good introduction to both category theory and homological algebra, see [Doray 2007].

Chapter 1

Division algebras over $\mathbb{R}$ of dimension 0,1,2,4 and 8

In what follows we will prove that there exist n-dimensional division algebras over $\mathbb{R}$ for $n = 0, 1, 2, 4, 8$, namely, respectively the zero algebra and the structures known as the real numbers ($\mathbb{R}$), the complex numbers ($\mathbb{C}$), the Hamilton numbers (or Quaternions) $\mathbb{H}$ and the Cayley numbers (or Octonions) $\mathbb{O}$. We will start by introducing some terminology.

1.1 Algebras

Definition 1.1. *An* algebra $(A, \cdot)$ over a field K, *is a vector space A over K equipped with a bilinear map* $\cdot$*:*

$$- \cdot - : A \times A \longrightarrow A,$$

and with a constant $e \in A$ such that for all $a \in A$, $e \cdot a = a \cdot e = a$.

Remark. The map $\cdot$ and the element e shall respectively be called *multiplication* and the *multiplicative unity* henceforth, and for any $a, b \in A$ we shall write ab instead of $a \cdot b$.

Remark. Let A be an algebra over K and suppose that e is the multiplicative unity in A. Then, due to the bilinearity of multiplication, for any $a \in A, \lambda \in K$ we have $(\lambda e)a = \lambda(ea) = \lambda a$. Hence we can identify λ and λe and in particular, we can identify $1 \in K$ and e. Moreover, for any $b \in A$, we will say that $b \in K$ if there exists a $\mu \in K$ such that $b = \mu e$.

Definition 1.2. *A* subalgebra *B of A is a linear subspace of A that contains 1 and which is closed under multiplication. For $a_1, a_2, \ldots, a_n \in A$, denote by $\langle a_1, a_2, \ldots, a_n \rangle$ the subalgebra of A* generated *by $a_1, a_2, \ldots, a_n$, that is, the minimal subalgebra of A that contains $a_1, a_2, \ldots, a_n$.*

1.2 Some possible properties of algebras

What distinguishes algebras from vector spaces is their multiplication, and it makes sense therefore to differentiate between algebras on the basis of the different properties satisfied by their respective multiplications. To begin with, we have the properties of commutativity, associativity, alternativity and power-associativity, the latter of which we won't need here, but nevertheless include for completeness sake:

Definition 1.3. *Let A be an algebra. The* commutator *is the bilinear map*

$$[-,-] : A \times A \longrightarrow A$$

given, for $a, b \in A$ by

$$[a, b] = ab - ba.$$

Definition 1.4. *An algebra A is said to be* commutative *if for all $a, b \in A$, $[a, b] = 0$.*

Definition 1.5. *Let A be an algebra. The* associator *is the trilinear map*

$$[-,-,-] : A \times A \times A \longrightarrow A$$

given, for $a, b, c \in A$ by

$$[a, b, c] = (ab)c - a(bc).$$

Definition 1.6. *An algebra A is said to be* associative *if for all $a, b, c \in A$, $[a, b, c] = 0$.*

Definition 1.7. *An algebra A is said to be* alternative *if for all $a, b \in A$,*

$$[a, b, b] = [b, a, b] = [b, b, a] = 0.$$

Definition 1.8. *An algebra A is said to be* power-associative *if for all $a \in A$,*

$$[a, a, a] = 0.$$

Remark. It is clear that for an algebra A to be associative under the definition above is equivalent to the equality for all $a, b, c \in A$ of $(ab)c$ to $a(bc)$. Likewise, alternativity is equivalent to the equalities

$$\begin{aligned} (aa)b &= a(ab), \\ (ab)a &= a(ba), \\ (ba)a &= b(aa). \end{aligned}$$

And power-associativity is equivalent to the equality of $(aa)a$ to $a(aa)$. In the context of associative, alternative or power-associative algebras we can and will therefore unambigously leave out all or some of the brackets.

An essential connection between associativity and alternativity is provided by Artin's lemma:

Lemma 1.1 (Artin's Lemma). *An algebra A is alternative if and only if every subalgebra generated by two of its elements is associative.*

Proof. Let A be an algebra. If for all $a, b \in A$, the subalgebra $\langle a, b \rangle$ is associative, then we specifically have $[a, a, b] = [a, b, a] = [b, a, a] = 0$, thus A is alternative. Proving the converse is a much more tedious task. It can be done using induction and a couple of identies involving the associator, but it will not be done here. A complete proof can be found on pages 27—30 of [Schafer 1966]. □

We are of course especially interested in the property of being a division algebra. The term *division algebra* has been used to denote several related structures. Our definition is the following:

Definition 1.9. *An algebra A is said to be a* division algebra *if for any $a \in A$, with $a \neq 0$, the left multiplication l_a and right multiplication r_a*

$$l_a, r_a : A \longrightarrow A$$

given by, respectively,

$$\begin{array}{ccc} z & \longmapsto & az, \\ z & \longmapsto & za \end{array}$$

are bijective.

Two other properties for which the term division algebra has been used are:

Definition 1.10. *An algebra A is said to* have no zero divisors *if for any $a, b \in A$ the following holds:*

$$(ab = 0) \Longrightarrow (a = 0 \vee b = 0).$$

Definition 1.11. *An algebra A is said to have* two-sided multiplicative inverses *if for every $a \in A$, $a \neq 0$, there exists an $a^{-1} \in A$ such that $aa^{-1} = a^{-1}a = 1$.*

These definitions are related, though in general not equivalent. Specifically, we have:

Lemma 1.2. *Let A be an algebra. Then*

$$A \textit{ is a division algebra} \Longrightarrow A \textit{ does not have zero divisors.}$$

Proof. Let $a \in A, a \neq 0$. Suppose A is a division algebra, then l_a is bijective and its kernel only contains 0. So, for any $b \in A$, if $l_a(b) = ab = 0$, we must have $b = 0$. □

If A is finite-dimensional, the converse is also true:

Lemma 1.3. *Let A be a finite-dimensional algebra. Then*

$$A \textit{ is a division algebra} \Longleftarrow A \textit{ does not have zero divisors.}$$

Proof. Let $a \in A, a \neq 0$. Suppose A does not have zero divisors, then the kernels of l_a and r_a only contain 0 and these maps are therefore injective. But being injective maps, their image must be of dimension no less than that of their domain, A, and therefore, A being finite-dimensional, can only be A itself. □

Remark. To see that this implication does not necessarily hold for infinite-dimensional algebras, consider the algebra $\mathbb{R}[X]$ over $\mathbb{R}$, which does not have zero divisors but where right or left multiplication by X is not bijective.

Other implications also only hold in special circumstances:

Lemma 1.4. *Suppose A is a commutative division algebra. Then A has two-sided multiplicative inverses.*

Proof. For every $a \in A$, with $a \neq 0$, the maps l_a and r_a are bijections, and thus a has a right inverse $l_a^{-1}(1)$ and a left inverse $r_a^{-1}(1)$. But since A is commutative, both inverses are two-sided inverses, thus A has two-sided multiplicative inverses. □

Proposition 1.5. *Suppose A is an associative algebra with two-sided multiplicative inverses. Then A has no zero divisors.*

Proof. Let $a, b \in A, a, b \neq 0$, and suppose $ab = 0$. Since a is associative, we have

$$0 = 0b^{-1} = (ab)b^{-1} = a(bb^{-1}) = a \neq 0,$$

contradiction. It follows that if $ab = 0$, then a or b equals 0, in other words, A has no zero divisors. □

Remark. Combining Lemma 1.3 and Proposition 1.5 gives us that finite-dimensional associative algebras with multiplicative inverses are division algebras.

The algebras $\mathbb{R}$ and $\mathbb{C}$ are indeed finite-dimensional and associative and have multiplicative inverses, and we will see that this is also true for $\mathbb{H}$. We will have to do a bit more work for $\mathbb{O}$ however, which does have multiplicative inverses but is not associative, only alternative. We will show that the same argument used for associative algebras can still be used for algebras like $\mathbb{O}$. But first we must define $\mathbb{H}$ and $\mathbb{O}$, show that they are respectively associative and alternative and prove that they have multiplicative inverses.

1.3 The division algebra $\mathbb{H}$

Remark. Let A be an n-dimensional algebra over a field K and $(v_0, v_1, \ldots, v_{n-1})$ a basis of A. Due to the bilinearity of its multiplication, we must have, for any $a = \sum_{0 \leq i \leq n-1} a_i v_i \in A$, and for any $b = \sum_{0 \leq j \leq n-1} b_j v_j \in A$, where $a_i, b_i \in K$, that

$$ab = \sum_{0 \leq i \leq n-1} a_i v_i \sum_{0 \leq j \leq n-1} b_j v_j = \sum_{0 \leq i,j \leq n-1} (a_i b_j)(v_i v_j). \qquad (*)$$

In other words, the multiplication of A is completely determined by the n^2 products of elements from a chosen basis of A. Conversely, if we have an n-dimensional vector space V over a field K and a multiplication $\cdot$ defined on a basis $(v_0, v_1, \ldots, v_{n-1})$ of V, then we can extend $\cdot$ to the whole of V using as a definition $(*)$. The multiplication $\cdot$ extended this way is a linear map: let

$a, b, c \in V, \lambda, \mu \in K$, we have

$$\begin{aligned}
(\lambda a)(\mu(b+c)) &= \left(\lambda \sum_{0 \le i \le n} a_i v_i\right)\left(\mu\left(\sum_{0 \le j \le n} b_j v_j + \sum_{0 \le j \le n} c_j v_j\right)\right) \\
&= \sum_{0 \le i \le n} \lambda a_i v_i \sum_{0 \le j \le n} \mu(a_j + b_j) v_j \\
&= \sum_{0 \le i,j \le n} \lambda\mu(a_i(b_j + c_j))(v_i v_j) \\
&= \lambda\mu \sum_{0 \le i,j \le n} (a_i b_j + a_i c_j)(v_i v_j) \\
&= \lambda\mu \left(\sum_{0 \le i,j \le n} (a_i b_j)(v_i v_j) + \sum_{0 \le i,j \le n} (a_i c_j)(v_i v_j)\right) \\
&= \lambda\mu(ab + ac)
\end{aligned}$$

Analogously, we find $(\lambda(a+b))(\mu c) = \lambda\mu(ac+bc)$. Thus if addionately we have a multiplicative identity, then V equipped with $\cdot$ forms an algebra.

Definition 1.12. *Let the* Hamilton numbers $\mathbb{H}$ *be the algebra* $(\mathbb{R}^4, \cdot)$ *defined by:*

$$\begin{aligned}
e_0 &= 1 \\
e_1^2 = e_2^2 = e_3^2 &= -1 \\
e_1 e_2 = -e_2 e_1 &= e_3 \\
e_2 e_3 = -e_3 e_2 &= e_1 \\
e_3 e_1 = -e_1 e_3 &= e_2.
\end{aligned}$$

Remark. The Hamilton numbers are otherwise known as the *Quaternions*.

Lemma 1.6. *The Hamilton numbers are associative.*

Proof. for any $a, b, c \in \mathbb{H}$, we have

$$\begin{aligned}
(ab)c &= \left(\sum_{0 \le i \le 3} a_i e_i \sum_{0 \le j \le 3} b_j e_j\right) \sum_{0 \le k \le 3} c_k e_k \\
&= \sum_{0 \le i,j \le 3} (a_i b_j)(e_i e_j) \sum_{0 \le k \le 3} c_k e_k \\
&= \sum_{0 \le i,j,k \le 3} (a_i b_j c_k)((e_i e_j) e_k).
\end{aligned}$$

The associativity of $\mathbb{H}$ thus depends on whether $(e_i e_j)e_k = e_i(e_j e_k)$. As this is indeed the case, we have

$$\begin{aligned}
(ab)c &= \sum_{0 \le i,j,k \le 3} a_i b_j c_k (e_i e_j) e_k \\
&= \sum_{0 \le i,j,k \le 3} a_i b_j c_k e_i (e_j e_k) \\
&= a(bc).
\end{aligned}$$

$\square$

Definition 1.13. *Define the* conjugation *on* $\mathbb{H}$ *to be the linear map*

$$-^* : \mathbb{H} \longrightarrow \mathbb{H}$$

given by, for any $a \in \mathbb{H}$,

$$a = a_0 + \sum_{1 \leq i \leq 3} a_i e_i \longmapsto a_0 - \sum_{1 \leq i \leq 3} a_i e_i = a^*.$$

Remark. For any $a \in \mathbb{H}$, a^* is called the *conjugate* of a.

Lemma 1.7. *The conjugation on* $\mathbb{H}$ *satisfies the following properties, for every* $a, b \in \mathbb{H}$*:*

1. $(a^*)^* = a$.
2. $(ab)^* = b^* a^*$.
3. $a + a^* \in \mathbb{R}$.
4. *If* $a \neq 0$*, then* $aa^* = a^* a \in \mathbb{R} \setminus \{0\}$

Proof. 1. This is clear from the definition of $-^*$.

2. It can easily be checked that this holds for any two elements e_i and e_j of the standard basis of $\mathbb{H}$. Due to the linearity of conjugation, it is matter of straightforward calculation that this is then also true for any two elements of $\mathbb{H}$.
3. From the definition of $-^*$ it directly follows that $a + a^* = 2a_0 \in \mathbb{R}$.
4. Straightforward calculation gives $aa^* = a^* a = \sum_{0 \leq i \leq 3} a_i a_i \in \mathbb{R}_{>0}$. □

Lemma 1.8. *The Hamilton numbers have two-sided multiplicative inverses.*

Proof. For every $a \in \mathbb{H} \setminus \{0\}$, we have $aa^* \in \mathbb{R} \setminus \{0\}$, and we can take

$$a^{-1} = \frac{1}{aa^*} a^*,$$

as

$$aa^{-1} = \frac{1}{aa^*} aa^* = 1$$

and

$$a^{-1} a = \frac{1}{aa^*} a^* a = \frac{1}{aa^*} aa^* = 1.$$

□

Corollary 1.9. *The Hamilton numbers form a 4-dimensional division algebra over* $\mathbb{R}$.

Proof. The Hamilton numbers are associative according to Lemma 1.6, they have multiplicative inverses according to Lemma 1.8 and they are finite-dimensional, thus they form a division algebra. □

We would like to directly define $\mathbb{O}$ as an 8-dimensional algebra over $\mathbb{R}$ by simply defining its multiplication on a basis, but this would make proving the alternativity of $\mathbb{O}$ very tedious. We shall therefore take a detour and define $\mathbb{O}$ using an instance of what is known as the *Cayley-Dickson construction*.

1.4 The division algebra $\mathbb{O}$

Definition 1.14. *The* Cayley numbers $\mathbb{O}$ *are the algebra* $(\mathbb{H} \times \mathbb{H}, \cdot)$, *where the bilinear multiplication* $\cdot$ *is given as follows, for* $a, b, c, d \in \mathbb{H}$:

$$(a, b)(c, d) = (ac - bd^*, a^*d + cb),$$

and where $(1, 0)$ *is the multiplicative unity.*

Remark. The Cayley numbers are otherwise known as the *Octonions*.

Lemma 1.10. *The Cayley numbers are alternative.*

Proof. This is a matter of straightforward and perhaps uninteresting calculation, but we won't skip over it as the alternativity of $\mathbb{O}$ forms a central argument for its constituting a division algebra.
Let $a, b, c, d \in \mathbb{H}$. Since the multiplication on $\mathbb{H}$ is bilinear and associative and because of the properties satisfied by conjugation on $\mathbb{H}$, we have

$$\begin{aligned}
((a,b)(a,b))\,(c,d) &= (aa - bb^*, a^*b + ab)\,(c,d) \\
&= ((aa - bb^*)c - d(a^*b + ab)^*, (aa - bb^*)^*d + c(a^*b + ab)) \\
&= (aac - bb^*c - (a^* + a)db^*, a^*a^*d - b^*bd + (a^* + a)cb) \\
&= (aac - adb^* - a^*db^* + cbb^*, a^*a^*d + a^*cb + acb - db^*b)
\end{aligned}$$

and

$$\begin{aligned}
(a,b)((a,b)(c,d)) &= (a,b)(ac - db^*, a^*d + cb) \\
&= (a(ac - db^*) - (a^*d + cb)b^*, a^*(a^*d + cb) + (ac - db^*)b) \\
&= (aac - adb^* - a^*db^* + cbb^*, a^*a^*d + a^*cb + acb - db^*b).
\end{aligned}$$

Furthermore it can be shown by similar computations that $((a,b)(c,d))(c,d) = (a,b)((c,d),(c,d))$ and $((a,b)(c,d))(a,b) = (a,b)((c,d)(a,b))$ also hold, and thus $\mathbb{O}$ is alternative. $\square$

Definition 1.15. *Define the* conjugation $-^*$ *on* $\mathbb{O}$

$$-^* : \mathbb{O} \longrightarrow \mathbb{O}$$

as follows, for any $a, b \in \mathbb{H}$,

$$(a, b) \longmapsto (a^*, -b).$$

Lemma 1.11. *The conjugation on* $\mathbb{O}$ *satisfies the same properties which were proven for conjugation on* $\mathbb{H}$ *in Lemma 1.7. For every* $a, b \in \mathbb{O}$:

1. $(a^*)^* = a$.

2. $(ab)^* = b^*a^*$.

3. $a + a^* \in \mathbb{R}$.

4. *If* $a \neq 0$, *then* $aa^* = a^*a \in \mathbb{R} \setminus \{0\}$

Proof. This is a matter of straightforward calculation. □

Corollary 1.12. *The Cayley numbers have two-sided multiplicative inverses.*

Proof. Since conjugation on $\mathbb{O}$ satisfies the same properties as conjugation on $\mathbb{H}$, this is true for analagous reasons. □

Definition 1.16. *Define the maps Re and Im:*

$$Re, Im : \mathbb{O} \longrightarrow \mathbb{O}$$

by, for any $a \in \mathbb{O}$, respectively

$$a \longmapsto \frac{a + a^*}{2}$$

$$a \longmapsto \frac{a - a^*}{2}.$$

Remark. Since for any $a \in \mathbb{O}$, we have $a + a^* \in \mathbb{R}$, we also have $Re(a) \in \mathbb{R}$.

Lemma 1.13. *For any $a \in \mathbb{O}$, we have $a, a^* \in \langle Im(a) \rangle$.*

Proof. Since $Re(a) \in K$,

$$a = \frac{a - a^*}{2} + \frac{a + a^*}{2} = Im(a) + Re(a) \in \langle Im(a) \rangle .$$

Likewise,

$$a^* = \frac{a - a^*}{2} - \frac{a + a^*}{2} = Im(a) - Re(a) \in \langle Im(a) \rangle .$$

□

Theorem 1.14. *The Cayley Numbers form an 8-dimensional division algebra over $\mathbb{R}$.*

Proof. Let $a, b \in \mathbb{O}$, with $a, b \neq 0$, and suppose $ab = 0$. The Cayley numbers have multiplicative inverses and $a \in \langle Im(a) \rangle$ and $b, b^* \in \langle Im(b) \rangle$, hence also $b^{-1} \in \langle Im(b) \rangle$, and thus $a, b, b^{-1} \in \langle Im(a), Im(b) \rangle$. But $\langle Im(a), Im(b) \rangle$ is a subalgebra of $\mathbb{O}$ generated by two elements, and, since $\mathbb{O}$ is alternative, according to Artin's Lemma (Lemma 1.1) therefore associative. We find

$$0 = 0b^{-1} = (ab)b^{-1} = a(bb^{-1}) = a \neq 0,$$

contradiction. It follows that if $ab = 0$, then a or b equals 0, and $\mathbb{O}$ has no zero divisors. Then according to Lemma 1.3 the Cayley numbers form a division algebra. □

Chapter 2

The non-existence of division algebras over $\mathbb{R}$ of odd dimension greater than 1

We have established that there exist division algebras of dimension 0, 1, 2, 4 and 8 over $\mathbb{R}$. In fact the five algebras we found are the only finite-dimensional division algebras over $\mathbb{R}$. Proving that there exist no division algebras over $\mathbb{R}$ outside these dimensions is however very hard, and in general outside the scope of this thesis. In this chapter we will restrict ourselves to a proof for odd dimension greater than 1. For the proof we make use of topology, namely the concept of *parallelisability* of the n-sphere.

2.1 Parallelisability of the n-sphere

Definition 2.1. *Let $n \geq -1$ and let $\|-\|$ be the euclidean norm on $\mathbb{R}^n$. The n-*sphere *is the topological subspace $S^n = \{a \in \mathbb{R}^{n+1} | \, \|a\| = 1\} \subset \mathbb{R}^{n+1}$.*

The cases $n = -1$ and $n = 0$ are specifically included in this definition; we have $S^{-1} = \emptyset$ and $S^0 = \{-1, 1\}$.

Definition 2.2. *For any $n, k \geq -1$, S^n is said to be* k-dimensionally combable*, or* combable in k dimensions*, if there exist $k+1$ continous maps*

$$\phi_0, \phi_1, \ldots, \phi_k : S^n \longrightarrow S^n,$$

with $\phi_0 = Id_{S^n}$ and such that for every $a \in S^n$, the images $\phi_0(a), \phi_1(a), \ldots, \phi_k(a)$ are linearly independent in $\mathbb{R}^{n+1}$. If S^n is 1-dimensionally combable, it is said to be just combable*. The n-sphere is said to be* parallelisable *if it is n-dimensionally combable.*

Remark. It is clear that for $k > n \geq 0$, the n-sphere is never k-dimensionally combable, and that for $n \geq 1$, parallelisability implies combability. The -1-sphere is combable in any number of dimensions, for every i we can just take ϕ_i to be the only map there is on S^{-1} and the conditions will automatically be met as S^{-1} does not contain any points.

The concept of parallelisability is relevant to the existence of division algebras by way of the following implication:

Proposition 2.1. *Suppose that for $n \geq 0$, there exists an n-dimensional division algebra A over $\mathbb{R}$. Then the $(n-1)$-sphere is parallelisable.*

Proof. Identify A with $\mathbb{R}^n$. Let $v_0, v_1, \ldots, v_{n-1} \in \mathbb{R}^n$ be n linearly independent vectors, with v_0 the multiplicative identity. Let

$$p : \mathbb{R}^n \setminus \{0\} \longrightarrow S^{n-1}$$

be the projection of elements onto the $(n-1)$-sphere via division by their euclidean norm. Now for $0 \leq i \leq n-1$ set $\phi_i = p \circ l_{v_i}$, where l_{v_i} is left multiplication by v_i. Because A has no zero divisors, this composition is well defined. Moreover, since for any i both l_{v_i} and p are continuous, so is ϕ_i, for every i. Note that $\phi_0(a) = a$. Also note that for every i we have $l_{v_i}(a) = r_a(v_i)$, where r_a is right multiplication by a. But since A is a division algebra, r_a is a bijection, and $r_a(v_0), r_a(v_i), \ldots, r_a(v_{n-1})$ are again linearly independent. (If this were not true, then since any element of A can be written as a linear combination of elements from this basis, the dimension of the image of r_a would be less than the dimension of A, and r_a could not be bijective.) Clearly p also preserves linear independence, and hence it follows that $\phi_0(a), \phi_1(a), \ldots, \phi_{n-1}(a)$ are linearly independent. $\square$

Corollary 2.2. *The -1-sphere, the 0-sphere, the 1-sphere, the 3-sphere and the 7-sphere are parallelisable.*

Proof. Given Proposition 2.1, this now directly follows from the existence of division algebras over $\mathbb{R}$ of dimension 0, 1, 2, 4 and 8 found in Section 1. $\square$

Thanks to Proposition 2.1, if we find that the n-sphere is not parallelisable, we also find that there exist no division algebras of dimension $n+1$ over $\mathbb{R}$. In the rest of this section, this will be shown for even n greater than 0. For this we employ the *Brouwer degree*. There are several different ways in which the Brouwer degree could be defined and its properties be proven; we make use of reduced singular homology. It may not be the fastest route, but the machinery developed along the way is useful for many other applications as well. The rest of the chapter is mostly an adaptation of [Hatcher 2002].

2.2 Reduced singular homology

Reduced singular homology is based on continous maps from *standard n-simplices* to a topological space X. *Reduced* singular homology differs from ordinary singular homology because unlike the latter the former also looks at dimension -1, which in turn affects results for dimension 0. In our case, this works out rather elegantly and it is why with the definition of S^n, we included the case $n = -1$. In time we will drop the qualifications *reduced* and *singular* and write just *homology* where reduced singular homology is to be understood.

2.2.1 Definition of the reduced singular homology functors

Definition 2.3. *Let $n \geq -1$, let X be a real vector space and let $v_0, v_1, \ldots, v_n \in X$. The n-*simplex *spanned by $v_0, v_1, \ldots, v_n$ is the set*

$$\langle v_0, v_1, \ldots, v_n \rangle = \left\{ \sum_{0 \leq i \leq n} t_i v_i \in X | t_0, t_1, \ldots, t_n \in \mathbb{R}_{\geq 0}, \sum_{0 \leq i \leq n} t_i = 1 \right\}.$$

Definition 2.4. *Let $n \geq -1$ and let $(e_0, e_1, \ldots, e_n)$ be the standard basis of $\mathbb{R}^{n+1}$. The* standard n-simplex Δ^n *is the n-simplex $\langle e_0, e_1, ..., e_n \rangle$.*

Remark. So $\Delta^{-1} = \emptyset$ and $\Delta^0 = \{1\}$, and in general, the standard n-simplex is the set

$$\left\{ (w_0, w_1, \ldots, w_n) \in \mathbb{R}^{n+1} | \sum_{0 \leq i \leq n} w_i = 1, w_0, w_1, \ldots, w_n \geq 0 \right\}.$$

Definition 2.5. *Let $n \geq k \geq -1$ and let $\langle v_0, v_1, \ldots, v_n \rangle$ be an n-simplex. A k-*face *of $\langle v_0, v_1, \ldots, v_n \rangle$ is a k-simplex spanned by $k+1$ elements of $\{v_0, v_1, \ldots, v_n\}$.*

Definition 2.6. *Let X be a topological space. For any $n \geq -1$, a* singular n-simplex σ in X *is a continuous map:*

$$\sigma : \Delta^n \longrightarrow X.$$

Definition 2.7. *For every $n \in \mathbb{Z}$, let*

$$Hom_{\mathfrak{Top}}(\Delta^n, -) : \mathfrak{Top} \longrightarrow \mathfrak{Set}$$

be the functor that for $X, Y \in \mathfrak{Top}$, and $f \in Mor(X, Y)$, sends X to the set of all singular n-simplices in X (for $n < -1$, this is the empty set) and where $Hom_{\mathfrak{Top}}(\Delta^n, f)$ is given by:

$$\sigma \longmapsto f \circ \sigma.$$

Definition 2.8. *Let*

$$F : \mathfrak{Set} \longrightarrow \mathfrak{Ab}$$

be the functor that assigns to a set $X \in \mathfrak{Set}$ the free abelian group generated by X. For every $n \in \mathbb{Z}$, define the functor C_n as the composition $F \circ Hom_{\mathfrak{Top}}(\Delta^n, -)$.

Remark. Because for every $X \in \mathfrak{Top}$ and every $n \in \mathbb{Z}$, the free group $C_n(X)$ is generated by the singular n-simplices in X, any abelian group homomorphism from $C_n(X)$ is determined by the images of the singular n-simplices. For this reason it will often be suficient to prove a lemma for singular n-simplices, after which the result extends naturally to all elements of $C_n(X)$.

Definition 2.9. *Let $s = \langle v_0, v_1, \ldots, v_n \rangle$ be an n-simplex, then for any $k \in \mathbb{N}, k \leq n+1$ and for any $Q = \{w_1, w_2, \ldots, w_k\} \subseteq \{v_1, v_2, \ldots, v_k\} = P$, define $s \setminus\!\!\setminus \langle w_1, w_2, \ldots, w_k \rangle$ to be the $n-k$-face spanned by $P \setminus Q$.*

Definition 2.10. *Let $X \in \mathfrak{Top}$, and for any $n \geq -1$, let $s = \langle v_0, v_1, \ldots, v_n \rangle \subseteq X$ be an n-simplex. Denote by*

$$\hookrightarrow_s : \Delta_n \longrightarrow X$$

the unique linear map that for every $0 \leq i \leq n$ sends e_i to v_i.

Definition 2.11. *For every $n \in \mathbb{Z}$,*

$$\partial_n : C_n \longrightarrow C_{n-1}$$

is the map that has for every $X \in \mathfrak{Top}$ component

$$\partial_n^X : C_n(X) \longrightarrow C_{n-1}(X)$$

given, for $\sigma \in Hom_{\mathfrak{Top}}(\Delta^n, X)$, by

$$\sigma \longmapsto \sum_{0 \leq i \leq n} (-1)^i (\sigma \circ \hookrightarrow_{\Delta_n \setminus\setminus e_i})$$

and extended linearly.

Lemma 2.3. *For every $n \in \mathbb{Z}$, the map ∂_n is a natural transformation.*

Proof. Let $X, Y \in \mathfrak{Top}$, $f \in Cont(X, Y)$. For all singular n-simplices $\sigma \in C_n(X)$, we have

$$\begin{aligned} \partial_n (Y) (C_n (f) (\sigma)) &= \partial_n (Y) (f \circ \sigma) \\ &= \sum_{0 \leq i \leq n} (-1)^i \left((f \circ \sigma) \circ \hookrightarrow_{\Delta_n \setminus\setminus \langle e_i \rangle} \right), \end{aligned}$$

and

$$\begin{aligned} C_{n-1} (f) \left(\partial_n^X (\sigma) \right) &= C_{n-1} (f) \left(\sum_{0 \leq i \leq n} (-1)^i \left(\sigma \circ \hookrightarrow_{\Delta_n \setminus\setminus \langle e_i \rangle} \right) \right) \\ &= \sum_{0 \leq i \leq n} (-1)^i \left(f \circ \left(\sigma \circ \hookrightarrow_{\Delta_n \setminus\setminus \langle e_i \rangle} \right) \right). \end{aligned}$$

Because the composition of maps is associative, these two expressions are identical. □

Remark. When there can be no confusion, in the future a component ∂_n^X of ∂_n, for certain $X \in \mathfrak{Top}$, shall simply be referred to as ∂_n. Similarly, in a measure that will hopefully improve legibility, the brackets around its argument shall be dropped, as will happen with certain other maps defined on the chain groups which we encounter later.

Lemma 2.4. *For every $X \in \mathfrak{Top}$, and every $n \in \mathbb{Z}$, we have $\partial_n \circ \partial_{n+1} = 0$.*

Proof. For every singular n-simplex $\sigma \in C_{n+1}(X)$, we have

$$\begin{aligned} \partial_n \partial_{n+1} \sigma &= \partial_n \sum_{0 \leq i \leq n+1} (-1)^i \left(\sigma \circ \hookrightarrow_{\Delta_{n+1} \setminus\setminus \langle e_i \rangle} \right) \\ &= \sum_{0 \leq i \leq n+1} \left((-1)^i \sum_{0 \leq j \leq n} (-1)^j \left(\left(\sigma \circ \hookrightarrow_{\Delta_{n+1} \setminus\setminus \langle e_i \rangle} \right) \circ \hookrightarrow_{\Delta_n \setminus\setminus \langle e_j \rangle} \right) \right) \\ &= \sum_{0 \leq j < i \leq n+1} (-1)^{i+j} \left(\sigma \circ \hookrightarrow_{\Delta_{n+1} \setminus\setminus \langle e_j, e_i \rangle} \right) \\ &+ \sum_{0 \leq i < j \leq n+1} (-1)^{i+j-1} \left(\sigma \circ \hookrightarrow_{\Delta_{n+1} \setminus\setminus \langle e_j, e_i \rangle} \right) \\ &= 0. \end{aligned}$$

□

Definition 2.12. *Let*

$$C_\bullet : \mathfrak{Top} \longrightarrow \mathfrak{Ch}_\bullet(\mathfrak{Ab})$$

be the functor that sends a topological space X to the chain complex composed for $n \in \mathbb{Z}$ of the chain groups $C_n(X)$ and the boundary maps ∂_n^X, and a continuous map $f \in Mor(X,Y)$ to the chain map that consists for each $n \in \mathbb{Z}$ of the group homomorphisms $C_n(f)$.

Remark. That $C_\bullet(f)$ is in fact a chain map follows from the fact that the ∂_ns are natural transformations, as was proven in Lemma 2.3.

Definition 2.13. *Let*

$$H_n : \mathfrak{Ch}_\bullet(\mathfrak{Ab}) \longrightarrow \mathfrak{Ab}$$

be the n^{th} homology functor. Define

$$H_n^\sigma : \mathfrak{Top} \longrightarrow \mathfrak{Ab}$$

to be the composition $H_n \circ C_\bullet$.

Remark. In the future, we will just write H_n instead of H_n^σ and since no other form of homology is adressed in this thesis, call it the n^{th} homology functor, trusting that this will not lead to any confusion. Likewise, we will call $H_n(X)$ the n^{th} homology group of X.

For very simple topological spaces, the homology groups can be calculated directly.

2.2.2 The homology groups of $\emptyset$ and of $\{x\}$

Proposition 2.5. *The homology group $H_{-1}(S^{-1})$ is isomorphic to $\mathbb{Z}$, and for every $m \neq -1$, the homology group $H_m(S^{-1})$ is trivial.*

Proof. We have $\Delta^{-1} = \emptyset$, so $Hom_{\mathfrak{Top}}(\Delta^{-1}, S^{-1})$ consists of only one element and $C_{-1}(S^{-1})$ is isomorphic to $\mathbb{Z}$. Since S^{-1} also equals $\emptyset$, and for $n \geq 0$, $\Delta^n \neq \emptyset$, there are no singular n-simplices in S^{-1} and hence for every $m \neq -1$, $Hom_{\mathfrak{Top}}(\Delta^m, S^{-1}) = \emptyset$, and $C_m(S^{-1})$ contains only the identity. It immediately follows that hence also $Ker(\partial_m^{S^{-1}})$ and thus also $H_m(S^{-1})$ are trivial. Furthermore, for every $n \in \mathbb{Z}$, and specifically for $n = 0$, the image $Im(\partial_n^{S^{-1}})$ is trivial, and $Ker(\partial_{-1}^{S^{-1}})$ equals $C_{-1}(S^{-1})$, so $H_{-1}(S^{-1})$ is also isomorphic to $\mathbb{Z}$. □

Proposition 2.6. *Let $\{x\} \in \mathfrak{Top}$ be a singleton, that is a topological space consisting of one point only. For every $n \in \mathbb{Z}$, the homology group $H_n(\{x\})$ is trivial.*

Proof. For every $n \geq -1$, the set $Hom(\Delta^n, \{x\})$ contains only one singular n-simplex, so $C_n(\{x\})$ is isomorphic to $\mathbb{Z}$. Furthermore, for even n, both $Ker(\partial_n^{\{x\}})$ and $Im(\partial_{n+1}^{\{x\}})$ are trivial, and for odd n, we have $Ker(\partial_n^{\{x\}}) = Im(\partial_{n+1}^{\{x\}}) = C_n(\{x\})$. □

Calculating the homology groups of other topological spaces is not such a straightforward matter. In order to reach our goals, the n^{th} homology group of the n-sphere and the Brouwer degree, we first have to develop some tools. The homology functors send maps between spaces to maps between the homology groups of these spaces. We start by proving that if two maps are homotopic, they are in fact sent to the same map by a homology functor, a consequence of which is that homotopy equivalent topological spaces have isomorphic homology groups.

2.2.3 How the homology functors factor through homotopy

Definition 2.14. *For $n, i \in \mathbb{N}$, with $0 \leq i \leq n$ define $p_i\Delta^n$ to be the $n+1$-simplex $\langle (e_0, 0), \ldots, (e_i, 0), (e_i, 1), \ldots, (e_n, 1) \rangle \subseteq \Delta^n \times I$.*

Definition 2.15. *Let $X, Y \in \mathfrak{Top}$, let $f, g \in Mor(X, Y)$ be homotopic, and let*

$$F : X \times I \longrightarrow Y$$

be a homotopy between f and g. For every $n \in \mathbb{Z}$, define the n^{th} prism operator

$$P_n : C_n(X) \longrightarrow C_{n+1}(Y)$$

in the following way on a regular n-simplex σ:

$$\sigma \longmapsto \sum_{0 \leq i \leq n} (-1)^i F \circ (\sigma \times I) \circ \hookrightarrow_{p_i\Delta^n}$$

and extend it linearly to all elements of $C_n(X)$.

Informally we can say that using a homotopy between two maps f and g, the prism operator sends a singular n-simplex σ to a combination of singular $n+1$-simplices which together form a 'prism' where the images of σ under f and g form the opposite bases. Corresponding to this view, we have a lemma that says that the boundary of this prism over σ is made up of its opposite bases, and its side, which is the prism over the boundary of σ.

Definition 2.16. *Let $A_\bullet$ and $B_\bullet$ be chain complices with boundary maps*

$$d_n : A_n \longrightarrow A_{n-1}$$

and

$$d_n : B_n \longrightarrow B_{n-1}$$

for every $n \in \mathbb{Z}$ and let f and g be chain maps between $A_\bullet$ and $B_\bullet$. A collection of maps $s = (s_n)$, with $n \in \mathbb{Z}$ and

$$s_n : X_n \longrightarrow Y_{n+1}$$

is said to be a chain homotopy *between $f_\bullet$ and $g_\bullet$ if for every n,*

$$d_{n+1} \circ s_n + s_{n-1} \circ d_n = g_n - f_n.$$

Lemma 2.7. *Let $X, Y \in \mathfrak{Top}$ and let $f, g \in Mor(X, Y)$ be homotopic. Then P is a chain homotopy between $C_\bullet(f)$ and $C_\bullet(g)$.*

Proof. A formal proof of this lemma, which is basically just a tedious rearranging of the many maps involved, can be found on pages 111—113 of [Hatcher 2002], starting towards the bottom of the page. $\square$

Proposition 2.8. *Let $X, Y \in \mathfrak{Top}$. If $f, g \in Cont(X, Y)$ are homotopic, then for all $n \in \mathbb{Z}$, $H_n(f) = H_n(g)$.*

Proof. Let $\alpha \in \ker(\partial_n^X)$, then $P_{n-1}\partial_n^X \alpha = 0$, and so because of Lemma 2.7,

$$(C_n(g) - C_n(f))(\alpha) = \partial_{n+1}^Y P_n \alpha,$$

thus

$$(C_n(g) - C_n(f))(\alpha) \in Im(\partial_{n+1}^Y).$$

And hence for every $[a] \in H_n(X)$,

$$(H_n(g))([\alpha]) - (H_n(f))([\alpha]) = (H_n(g) - H_n(f))([\alpha]) = 0,$$

so we have $H_n(g) = H_n(f)$. $\square$

Corollary 2.9. *Let $X, Y \in \mathfrak{Top}$, and let $f \in Cont(X, Y)$ be a homotopy equivalence. Then for every $n \in \mathbb{Z}$, the induced homomorphism $H_n(f)$ is an isomorphism between $H_n(X)$ and $H_n(Y)$.*

Proof. Because f is a homotopy equivalence, there exists a $g \in Cont(Y, X)$ such that $g \circ f \simeq Id_X$ and $f \circ g \simeq Id_Y$, and

$$H_n(g) \circ H_n(f) = H_n(g \circ f) = H_n(Id_X) = Id_{H_n(X)},$$
$$H_n(f) \circ H_n(g) = H_n(f \circ g) = H_n(Id_Y) = Id_{H_n(Y)},$$

so $H_n(f)$ is an isomorphism between $H_n(X)$ and $H_n(Y)$. $\square$

Corollary 2.10. *Let $A \in \mathfrak{Top}$ be contractible. Then all homology groups of A are trivial.*

Proof. If A is contractible, then A is homotopy equivalent to a singleton, and thus according to Proposition 2.8, its homology groups are isomorphic to those of a singleton. But according to Proposition 2.6, all homology groups of a singleton are trivial. $\square$

With the help of the preceding, we construct another tool, the *Meyer-Vietoris sequence*, which we can then use to calculate the homology groups of the n-sphere.

2.2.4 The Meyer-Vietoris sequence

The Meyer-Vietoris sequence is a long exact sequence that enables us to calculate the homology groups of a topological space X from the homology groups of two of its subspaces A and B, provided that X is the union of the interiors of A and B. It features the direct sums of the homology groups of A and B and the homology groups of their intersection and their union, the latter of which is X. The Meyer-Vietoris sequence is derived from a short exact sequence of chain complices that will be described next.

Definition 2.17. *Let $X \in \mathfrak{Top}$, let $\mathcal{U} \subseteq \mathcal{P}(X)$ and let $n \in \mathbb{Z}$. Denote by $C_\bullet^{\mathcal{U}}$ the subcomplex of $C_\bullet(X)$ made up of, for each $n \in \mathbb{Z}$, the chain subgroups $C_n^{\mathcal{U}}$ of $C_n(X)$ generated by the singular n-simplices whose image is contained in one of the sets of $\mathcal{U}$, and the boundary maps $d_n^{\mathcal{U}} = d_n^{C(X)}|_{C_n^{\mathcal{U}}}$.*

Definition 2.18. *Let $X \in \mathfrak{Top}$, let $A, B \subseteq X$ and let $n \in \mathbb{Z}$. The* direct sum *of* $C_\bullet(A)$ *and* $C_\bullet(B)$ *is the complex $(C(A) \oplus C(B))_\bullet \in \mathfrak{Ch}_\bullet(\mathfrak{Ab})$, made up of, for each $n \in \mathbb{Z}$, the direct sums $C_n(A) \oplus C_n(B)$ and the boundary maps $d^{C(A)\oplus C(B)} = (d^{C(A)}, d^{C(B)})$.*

The short exact sequence involves the following two chain maps:

Definition 2.19. *Let $X \in \mathfrak{Top}$ and let $A, B \subseteq X$ such that $int(A) \cup int(B) = X$. Define*

$$\phi : C_\bullet(A \cap B) \longrightarrow (C(A) \oplus C(B))_\bullet$$

by letting for each $n \in \mathbb{Z}$ the abelian group homomorphism ϕ_n be given by:

$$x \longmapsto (x, -x).$$

Definition 2.20. *Let $X \in \mathfrak{Top}$ and let $A, B \subseteq X$ such that $int(A) \cup int(B) = X$. Define*

$$\psi : (C(A) \oplus C(B))_\bullet \longrightarrow C_\bullet^{\{A,B\}}$$

by letting for each $n \in \mathbb{Z}$ the abelian group homomorphism ψ_n be given by:

$$(x, y) \longmapsto x + y.$$

Lemma 2.11. *Let $X \in \mathfrak{Top}$, let $A, B \subseteq X$ such that $int(A) \cup int(B) = X$ and let $n \in \mathbb{Z}$. The short sequence*

$$0 \longrightarrow C_\bullet(A \cap B) \xrightarrow{\phi} (C(A) \oplus C(B))_\bullet \xrightarrow{\psi} C_\bullet^{\{A,B\}} \longrightarrow 0$$

is exact.

Proof. It follows straight from their definitions that for each $n \in \mathbb{Z}$, ϕ_n is injective, ψ_n is surjective and that $Im(\phi_n) = Ker(\psi_n)$. ☐

The Meyer-Vietoris sequence involves the homology groups of this short exact sequence. If we apply the homology functors to ϕ and ψ, we get induced homomorphisms between respectively the homology groups of $C_\bullet(A \cap B)$ and $(C(A) \oplus C(B))_\bullet$ and the homology groups of $(C(A) \oplus C(B))_\bullet$ and $C_\bullet^{\{A,B\}}$. We still need homomorphisms between the homology groups of $C_\bullet^{\{A,B\}}$ and $C_\bullet(A \cap B)$. These exist, and are called *connecting* morphisms.

Remark. Let $A_\bullet, B_\bullet, C_\bullet \in \mathfrak{Ch}_\bullet(\mathfrak{Ab})$, and $\phi \in Mor(A_\bullet, B_\bullet)$, $\psi \in Mor(B_\bullet, C_\bullet)$ such that

$$0 \longrightarrow A_\bullet \xrightarrow{\phi} B_\bullet \xrightarrow{\psi} C_\bullet \longrightarrow 0$$

is a short exact sequence. This gives the following commutative diagram:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r, "d^A_{n+2}"] & A_{n+1} \arrow[r, "d^A_{n+1}"] \arrow[d, "\phi_{n+1}"] & A_n \arrow[r, "d^A_n"] \arrow[d, "\phi_n"] & A_{n-1} \arrow[r, "d^A_{n-1}"] \arrow[d, "\phi_{n-1}"] & \cdots \\
\cdots \arrow[r, "d^B_{n+2}"] & B_{n+1} \arrow[r, "d^B_{n+1}"] \arrow[d, "\psi_{n+1}"] & B_n \arrow[r, "d^B_n"] \arrow[d, "\psi_n"] & B_{n-1} \arrow[r, "d^B_{n-1}"] \arrow[d, "\psi_{n-1}"] & \cdots \\
\cdots \arrow[r, "d^C_{n+2}"] & C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d] & C_n \arrow[r, "d^C_n"] \arrow[d] & C_{n-1} \arrow[r, "d^C_{n-1}"] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Since for every $n \in \mathbb{Z}$, ψ_n is surjective and ϕ_n is injective, for every $c \in C_n$ there exists a $b \in B_n$ such that $\psi(b) = c$ and for every $b \in B_n$, if there exists an $a \in A_n$ such that $\phi(a) = b$, it is unique.

Lemma 2.12. *Using the same short exact sequence as in the previous remark, for each $n \in \mathbb{N}$, and for each $[c] \in H_n(C_\bullet)$, there exists a unique $[a] \in H_{n-1}(A_\bullet)$ such that for any inverse b under ψ_n of any representative c of $[c]$,*

$$d_n^B(b) = \phi_{n-1}(a).$$

Proof. This can be proven through diagram chasing, showing that wherever there are multiple options, the choice made does not affect the eventual outcome. For a detailed account of this, see the lower half of page 116 in [Hatcher 2002]. □

Definition 2.21. *For every $n \in \mathbb{Z}$, let $[c] \in H_n(C_\bullet)$ and $[a] \in H_{n-1}(A_\bullet)$ as found in Lemma 2.12. Define the* connecting morphism

$$\omega_n : H_n(C_\bullet) \longrightarrow H_{n-1}(A_\bullet)$$

by

$$[c] \longmapsto [a].$$

Before we can define the Meyer-Vietoris sequence, we need to make one adjustment. We want to replace the homology groups $H_n^{\{A,B\}}$ with the homology groups $H_n(X)$.

Definition 2.22. *Let $X \in \mathfrak{Top}$ and let $A, B \subseteq X$ such that $int(A) \cup int(B) = X$. The* inclusion

$$\iota : C_\bullet^{\{A,B\}} \longrightarrow C_\bullet(X)$$

is the chain map composed of the inclusions

$$\iota_n : C_n^{\{A,B\}} \longrightarrow C_n(X).$$

Lemma 2.13. *For every $n \in \mathbb{Z}$, the induced homomorphism $H_n(\iota)$ is an isomorphism.*

Proof. This lemma may seem innocent enough, but a proof would spread over many pages. What is needed is a chain map

$$\rho : C_n(X) \longrightarrow C_n^{\{A,B\}}$$

such that both $\rho \circ \iota$ and $\iota \circ \rho$ are homotopic to the identity. In that case, ι is a homotopy equivalence, and hence due to Corollary 2.9, $H_n(\iota)$ is an isomorphism. For constructing ρ the idea is that through repeated application of *barycentric subdivision*, an n-simplex can be subdivided into arbitrarily smaller n-simplices. The challenge is then to translate barycentric subdivision to singular n-simplices. For a comprehensive proof, see proposition 2.21 of [Hatcher 2002], pages 119—124. $\square$

We can now define the Meyer-Vietoris sequence.

Definition 2.23. *Let $X \in \mathfrak{Top}$, let $A, B \subseteq X$ such that $int(A) \cup int(B) = X$ and let $n \in \mathbb{Z}$. The* Meyer-Vietoris sequence *is the following sequence:*

$$\begin{tikzcd}
 & & \cdots \arrow[dll, "\omega_{n+1}\circ H_{n+1}(\iota)^{-1}"'] \\
H_n(A \cap B) \arrow[r, "H_n(\phi)"] & H_n(A) \oplus H_n(B) \arrow[r, "H_n(\iota\circ\psi)"] & H_n(X) \arrow[dll, "\omega_n\circ H_n(\iota)^{-1}"'] \\
H_{n-1}(A \cap B) \arrow[r, "H_{n-1}(\phi)"] & H_{n-1}(A) \oplus H_{n-1}(B) \arrow[r, "H_{n-1}(\iota\circ\psi)"] & H_{n-1}(X) \arrow[dll, "\omega_{n-1}\circ H_{n-1}(\iota)^{-1}"'] \\
\cdots & &
\end{tikzcd}$$

The reason why we are interested in the Meyer-Vietoris sequence is because it is exact. This follows from the following lemma.

Lemma 2.14. *Let $A_\bullet, B_\bullet, C_\bullet \in \mathfrak{Ch}_\bullet(\mathfrak{Ab})$, and suppose there exist chain morphisms $\phi \in Mor(A_\bullet, B_\bullet)$ and $\psi \in Mor(B_\bullet, C_\bullet)$ such that*

$$0 \longrightarrow A_\bullet \xrightarrow{\phi} B_\bullet \xrightarrow{\psi} C_\bullet \longrightarrow 0$$

is a short exact sequence. Then the following long sequence is exact:

$$\begin{tikzcd}
 & & \cdots \arrow[dll, "\omega_{n+1}"'] \\
H_n(A_\bullet) \arrow[r, "H_n(\phi)"] & H_n(B_\bullet) \arrow[r, "H_n(\psi)"] & H_n(C_\bullet) \arrow[dll, "\omega_n"'] \\
H_{n-1}(A_\bullet) \arrow[r, "H_{n-1}(\phi)"] & H_{n-1}(B_\bullet) \arrow[r, "H_{n-1}(\psi)"] & H_{n-1}(C_\bullet) \arrow[dll, "\omega_{n-1}"'] \\
\cdots & &
\end{tikzcd}$$

Proof. In order to prove this, one has to check that the images of the boundary operators are included in the kernels of the respective subsequent boundary operator and vice-versa. This is done in Theorem 2.16 of [Hatcher 2002] on page 117. ☐

Corollary 2.15. *The Meyer-Vietoris sequence is exact.*

Proof. This follows from Lemma 2.11, Lemma 2.13 and Lemma 2.14. ☐

Now we are ready to calculate the homology groups of the n-sphere.

2.2.5 The homology groups of the n-sphere

Proposition 2.16. *For every $n \geq -1$, $H_n(S^n)$ is isomorphic to $\mathbb{Z}$, and for every $m \neq n$, $H_m(S^n)$ is trivial.*

Proof. Let $A = \{(v_0, v_1, \ldots, v_n) \in S^n | v_0 < 1\}$ and $B = \{(v_0, v_1, \ldots, v_n) \in S^n | v_0 > -1\}$. Then A and B are both open, and thus their own interiors, and $A \cup B = S^n$. Furthermore, A and B deformation retract to a point, and thus for every $k \in \mathbb{Z}$, we have $H_n(A)$ and $H_n(B)$ are both trivial. The intersection $A \cap B$ deformation retracts to the set $\{(v_0, v_1, \ldots, v_n) \in S^n | v_0 = 0\}$, which is homeomorphic to S^{n-1}. Applying the Meyer-Vietoris sequence then yields that for all $k \in \mathbb{Z}$,

$$0 \xrightarrow{H_k(\iota \circ \psi)} H_k(S^n) \xrightarrow{\omega_k \circ H_k(\iota)^{-1}} H_{k-1}(S^{n-1}) \xrightarrow{H_{k-1}(\phi)} 0$$

is exact, which means that $H_k(S^n)$ and $H_{k-1}(S^{n-1})$ are isomorphic. Having found in Proposition 2.5 that $H_{-1}(S^{-1})$ is isomorphic to $\mathbb{Z}$, and that for all $m \neq -1$, $H_m(S^{-1})$ is trivial, straightforward induction on n then finishes the proof. ☐

Because the homoloy group $H_n(S^n)$ is isomorphic to $\mathbb{Z}$ as an abelian group, the ring $Hom_{\mathfrak{Ab}}(H_n(S^n), H_n(S^n))$ is canonically isomorphic to the ring $\mathbb{Z}$.

Lemma 2.17. *For $n \in \mathbb{N}$, let $\sigma_+^n, \sigma_-^n \in Hom_{\mathfrak{Top}}(\Delta^n, S^n)$ be singular n-simplices covering opposite halves of the n-sphere, such that $Im(\sigma_+^n) \cap Im(\sigma_-^n)$ is homeomorphic to S^{n-1} and $\partial_n^{S^n} \sigma_+^n = \partial_n^{S^n} \sigma_-^n$. Then $[\sigma_+^n - \sigma_-^n]$ is a generator of $H_n(S^n)$.*

Proof. This can be proven using induction and the Meyer-Vietoris sequence, or alternatively as in the second half of Example 2.23 on page 125 of [Hatcher 2002]. □

Now we are ready to define the Brouwer degree.

2.3 The Brouwer degree and the non-combability of the even-dimensional sphere

Definition 2.24. *For any $n \geq -1$, let χ_n be the canonical isomorphism between $Hom_{\mathfrak{Ab}}(H_n(S^n), H_n(S^n))$ and $\mathbb{Z}$ that sends $Id_{H_n(S^n)}$ to 1. The* Brouwer degree

$$deg_n : Mor(S^n, S^n) \longrightarrow \mathbb{Z}$$

is the composition $\chi_n \circ H_n$.

The Brouwer degree satisfies many properties, four of which we need.

Proposition 2.18. *Let $n \geq -1$. Let $f, g \in Mor(S^n, S^n)$. The Brouwer degree satisfies the following properties:*

1. *$deg_n(g \circ f) = deg_n(f) deg_n(g)$.*
2. *$deg_n(Id_{S^n}) = 1$.*
3. *If $f \sim g$, then $deg_n(f) = deg_n(g)$.*
4. *Let $0 \leq i \leq n$, then $refl_i$ is the map that sends a point $(v_0, v_1, \ldots, v_i, \ldots, v_n)$ to $(v_0, v_1, \ldots, -v_i, \ldots, v_n)$. We have that $deg_n(refl_i) = -1$.*

Proof.

1. This is a direct consequence of the fact that H_n is a functor and that χ_n is a ring isomorphism.
2. H_n is a functor, so sends Id_{S^n} to $Id_{H_n(S^n)}$, and ϕ was defined to send $Id_{H_n(S^n)}$ to 1.
3. Due to Proposition 2.8, $H_n(f) = H_n(g)$, and thus also $deg_n(f) = deg_n(g)$.
4. As $refl_i \circ refl_i = Id_{S^n}$, the value $deg_n(refl_i)$ must be either 1 or -1. Choose a $\sigma_+^n \in Hom_{\mathfrak{Top}}(Delta^n, S^n)$ that maps onto the part of S^n that consists of points with non-negative i^{th} coordinate and that satisfies the requirements of Lemma 2.17, and let $\sigma_-^n \in Hom_{\mathfrak{Top}}(Delta^n, S^n)$ be the mirror image of σ_+^n under $refl_i$. Since $H_n(refl_i)$ sends $[\sigma_+ - \sigma_-]$ to $[\sigma_- - \sigma_+] = -[\sigma_+ - \sigma_-] \neq [\sigma_+ - \sigma_-]$, we must have $deg_n(refl_i) = -1$. □

Corollary 2.19. *Let $n \geq -1$ and let $-Id_{S^n}$ be the antipodal map. We have that $deg_n(-Id_{S^n}) = (-1)^{n+1}$.*

Proof. The antipodal map is the composition of $n+1$ reflections, hence this follows from properties 1 and 4 of Proposition 2.18. $\square$

Lemma 2.20. *Let $n \geq -1$. Then for any continuous map*

$$\phi : S^n \longrightarrow S^n$$

such that for every $x \in S^n$, $\phi(x) \neq -x$, we have $\phi \sim Id$.

Remark. Recall that in the proof of Proposition 2.1 we defined

$$p : R^{n+1} \setminus \{0\} \longrightarrow S^n$$

to be the projection onto the n-sphere given by

$$x \longmapsto x/\|x\|.$$

Proof. We can use the homotopy

$$F : S^n \times I \longrightarrow S^n$$

given by

$$(x,t) \longmapsto p(tx + (1-t)\phi(x)).$$

$\square$

Lemma 2.21. *Let $n \geq -1$. Then for any continuous map*

$$\phi : S^n \longrightarrow S^n$$

such that for every $x \in S^n$, $\phi(x) \neq x$, we have $\phi \sim -Id$.

Proof. We can use the homotopy

$$F : S^n \times I \longrightarrow S^n$$

given by

$$(x,t) \longmapsto p(-tx + (1-t)\phi(x)).$$

$\square$

Proposition 2.22. *Let $n \in \mathbb{N}$ be even. Then the n-sphere is not combable.*

Proof. Suppose that the n-sphere is combable. Then there exists a map

$$\phi : S^n \longrightarrow S^n$$

such that for every $a \in S^n$, the vectors a and $\phi(a)$ are linearly independent, which is equivalent to $\phi(a) \neq a$ and $\phi(a) \neq -a$. According to Lemma 2.20 and Lemma 2.21 we then have that $Id \sim \phi \sim -Id$, and according to Proposition 2.18 this means that $1 = deg(Id) = deg(-Id) = (-1)^{n+1}$, which for n even is not true. $\square$

Theorem 2.23. *There exist no division algebras over $\mathbb{R}$ of odd dimension greater than 1.*

Proof. From Proposition 2.22 it follows that for n even, the n-sphere is not combable, and for $n > 0$, thus also not parallelisable. Proposition 2.1 then gives us that it cannot be the case that there exist $n+1$-dimensional division algebras over $\mathbb{R}$. $\square$

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