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Pink's conjecture on semiabelian varieties

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1 Introduction

A central object of study in algebraic geometry is the abelian variety, which has the structures of both an abelian group and an algebraic variety. Its one-dimensional examples are precisely elliptic curves, so one may regard abelian varieties as the generalisation of the concept of elliptic curves to higher dimensions.

Because of the double structure of abelian varieties, it is natural to ask questions that combine these two structures. One of these questions is the following statement, posed independently by Manin and Mumford and proven by Raynaud [35], [36]:

Theorem 1.1 (Raynaud)**.** *Let A be an abelian variety over $\mathbf{C}$ and let A_{tor} denote its torsion subgroup. Let $Z \subset A$ be an irreducible closed algebraic subvariety such that $Z \cap A_{\mathrm{tor}}$ is Zariski dense in Z. Then Z is a translate of an abelian subvariety of A.*

This conjecture has now been proven (in various ways, see chapter 8), as well as its generalisation to a wider class of commutative group varieties called semiabelian varieties. Still, its generalisation to families of semiabelian varieties, which is a group scheme X/S over a variety[1] such that every fibre is a semiabelian variety, leads to the following conjecture:

Conjecture 1.2. *Let $X \longrightarrow S$ be a family of semiabelian varieties, and let $X_0 = \bigcup_{s\in S} X_{s,\mathrm{tor}}$ be the union of the torsion subgroups of the fibres of $X \longrightarrow S$. Let $Z \subset X$ be an irreducible closed algebraic subvariety such that $Z \cap X_0$ is Zariski dense in Z. Then Z is contained in a proper closed subgroup scheme[2] of X/S.*

This is stated, in a more general way, in [33]. In this article, Pink claims to prove this conjecture from another conjecture regarding the connected mixed Shimura subvarieties (usually named *special subvarieties*) of connected mixed Shimura varieties, stated in chapter 8. The definition of these is quite abstract and complicated, but its importance lies in the fact that one of the main examples is that of a universal abelian variety, which has all abelian varieties with some extra given data as subvarieties in a 'natural' way. Therefore, theorems about connected mixed Shimura varieties may give us information about families of abelian and semiabelian varieties.

However, conjecture 1.2 is not true, as a counterexample was found by Bertrand [5]. Nevertheless it is not a counterexample to Pink's general conjecture; the reason for this is that there is a mistake in Pink's proof of conjecture 1.2 from his more general conjecture. In this thesis, I explain the theory on abelian varieties and mixed Shimura varieties necessary to formulate Pink's general conjecture. Furthermore, I explain Bertrand's counterexample, and I classify for which abelian varieties these counterexamples may occur. In order to do so, I classify the special subvarieties of universal abelian varieties.

[1] By a *variety* over a field k I mean, in this thesis, a seperated, geometrically integral k-scheme of finite type.

[2] I.e. a closed subvariety X' of X such that X'_s is a closed subgroup of X_s for every $s \in \operatorname{im} X'$.

2 Definitions

Before I can make any meaningful statements about Shimura varieties, I have to define them first. The definition of these is a rather complicated affair that requires the theory of Hodge structures, that also arise in the cohomology of Kähler manifolds, and linear algebraic groups. Before I come to Shimura varieties, I will first define, and give some essential properties of, Hodge structures and linear algebraic groups.

2.1 Hodge structures

Let V be a finite-dimensional vector space over $\mathbf{R}$. By $\mathbf{GL}(V)$ I denote the covariant functor $\mathbf{Alg_R} \longrightarrow \mathbf{Grp}$ given by $\mathbf{GL}(V)(B) = \mathrm{Aut}_{B-\mathbf{Mod}}(B \otimes_{\mathbf{R}} V)$. This is represented by an $\mathbf{R}$-scheme, which I also abusively denote as $\mathbf{GL}(V)$.

Let $\mathbf{S}$ denote $\mathrm{Res}_{\mathbf{C}/\mathbf{R}}\, \mathbf{G}_{m,\mathbf{C}}$, the Weil restriction from $\mathbf{C}$ to $\mathbf{R}$ of the multiplicative group over $\mathbf{C}$. In terms of the functor of points, for every $\mathbf{R}$-algebra B the set $\mathbf{S}(B)$ is equal to $(B \otimes_{\mathbf{R}} \mathbf{C})^\times$. Then $\mathbf{S} = \mathrm{Spec}\,\mathbf{R}[a, b, (a^2+b^2)^{-1}]$, so $\mathbf{S}$ is an affine group scheme over $\mathbf{R}$. One has that $\mathbf{S}(\mathbf{R}) = \mathbf{C}^\times$ and for any $\mathbf{C}$-algebra B, one has $\mathbf{S}(B) \cong B^\times \times B^\times$ via the isomorphism

$$\begin{array}{ccc} (B \otimes_{\mathbf{R}} \mathbf{C})^\times & \longrightarrow & B^\times \times B^\times \\ a \otimes z & \mapsto & (az, a\bar{z}). \end{array}$$

The induced map $\mathbf{C}^\times = \mathbf{S}(\mathbf{R}) \longrightarrow \mathbf{S}(\mathbf{C}) \cong \mathbf{C}^\times \times \mathbf{C}^\times$ is given by $z \mapsto (z, \bar{z})$. Furthermore, there is a natural injective morphism $\mathbf{G}_{m,\mathbf{R}} \longrightarrow \mathbf{S}$ coming from the inclusion map $\mathbf{R} \longrightarrow \mathbf{C}$.

Definition 2.1. Let V be a finite-dimensional $\mathbf{R}$-vector space. A *(mixed) Hodge structure* on V is a morphism $h : \mathbf{S} \longrightarrow \mathbf{GL}(V)$ of group schemes over $\mathbf{R}$[3].

Another way to regard these Hodge structures is given in the following proposition.

Theorem 2.2. *Let V be a finite-dimensional $\mathbf{R}$-vector space. Let H be the set of Hodge structures on V, and let $\tilde{H}$ be the set of decompositions of $V_{\mathbf{C}}$ into $\mathbf{C}$-subspaces $V^{p,q}$ indexed by $\mathbf{Z}^2$ such that $\overline{V^{p,q}} = V^{q,p}$. Then for any $(V^{p,q})_{p,q} \in \tilde{H}$, the map $\mathbf{C}^\times \longrightarrow \mathrm{GL}(V_{\mathbf{C}})$, through which z acts as $z^{-p}\bar{z}^{-q}$ on $V^{p,q}$, comes from a Hodge structure on V. This gives a bijection $H \cong \tilde{H}$.*

Proof. Take the action of $\mathbf{C}^\times$ on $V_{\mathbf{C}}$ as above. For any $v \in V^{p,q}$, one has that $\overline{z \cdot v} = \overline{z^{-p}\bar{z}^{-q}v} = z^{-q}\bar{z}^{-p}\bar{v} = z \cdot \bar{v}$, since $\bar{v} \in V^{q,p}$. This shows that the action commutes with complex conjugation, so it comes from an action $\mathbf{C}^\times \longrightarrow \mathrm{GL}(V)$. For a basis of eigenvectors of $V_{\mathbf{C}}$ for this action, the action of $a + bi$ is given by a diagonal matrix whose diagonal entries are of the form $(a+bi)^{-p}(a-bi)^{-q}$, which are algebraic in a, b and $(a^2+b^2)^{-1}$. This is still true if one changes to a basis of $V_{\mathbf{C}}$ which is also a basis of V. This shows that this map actually comes from a map $\mathbf{S} = \mathrm{Spec}\,\mathbf{R}[a, b, (a^2+b^2)^{-1}] \longrightarrow \mathbf{GL}(V)$.

[3] This notion is more split than the usual definition of a mixed Hodge structure as described, for example, in [12], but this does not matter for our purposes.

Conversely, consider a Hodge structure $h : \mathbf{S} \longrightarrow \mathbf{GL}(V)$. The identification $\mathbf{S}(\mathbf{C}) = \mathbf{C}^\times \times \mathbf{C}^\times$ comes from an isomorphism of algebraic groups $\mathbf{S}_\mathbf{C} \cong \mathbf{G}^2_{m,\mathbf{C}}$; hence h gives us a morphism $\mathbf{G}^2_{m,\mathbf{C}} \longrightarrow \mathbf{GL}(V_\mathbf{C})$ of algebraic groups over $\mathbf{C}$. This corresponds to a bigrading of $V_\mathbf{C}$, i.e. a decomposition $V_\mathbf{C} = \bigoplus_{p,q} V^{p,q}$, such that (a, b) acts on $V^{p,q}$ as $a^{-p}b^{-q}$.

Now let B be a $\mathbf{C}$-algebra, and let $a \in B$, $z \in \mathbf{C}$ be such that $a \otimes z \in \mathbf{S}(B)$. Suppose (a, z) corresponds to $(x, y) \in B^\times \times B^\times$, i.e. $(az, a\bar{z}) = (x, y)$; then $(\bar{a}, z)$ corresponds to $(\bar{a}z, \bar{a}\bar{z}) = (\bar{y}, \bar{x})$. This shows that complex conjugation on $\mathbf{S}_\mathbf{C}$ corresponds to complex conjugation and a coordinate swap on $\mathbf{G}^2_{m,\mathbf{C}}$. The map $\mathbf{G}^2_{m,\mathbf{C}} \longrightarrow \mathbf{GL}(V_\mathbf{C})$ must be invariant under complex conjugation. This means that for any $v \in V^{p,q}$ and any $(x, y) \in B^\times \times B^\times$ for any B, one has that

$$\begin{aligned} (\bar{y}, \bar{x}) \cdot \bar{v} &= \overline{(x, y) \cdot v} \\ &= \overline{x^{-p}y^{-q}v} \\ &= \bar{x}^{-p}\bar{y}^{-q}\bar{v}, \end{aligned}$$

which shows that $\bar{v} \in V^{q,p}$; in other words, the decomposition $(V^{p,q})_{p,q}$ is an element of $\tilde{H}$. The composition $\mathbf{S}(\mathbf{R}) \longrightarrow \mathbf{S}(\mathbf{C}) \xrightarrow{h} \mathbf{GL}(V_\mathbf{C})$ lets $z \in \mathbf{C}^\times$ act as $z^{-p}\bar{z}^{-q}$ on $V^{p,q}$, as was to be shown. □

Example 2.3. Let V be a complex vector space. If we regard V as a real vector space, we have a map $\mathbf{C} \longrightarrow \mathrm{End}_\mathbf{R}(V)$. For every $\mathbf{R}$-algebra B, this induces a map $B \otimes \mathbf{C} \longrightarrow \mathrm{End}_B(B \otimes V)$, and this in turn induces a map $(B \otimes \mathbf{C})^\times \longrightarrow \mathrm{Aut}_B(B \otimes V)$. Hence we get a morphism of real linear algebraic groups $\mathbf{S} \longrightarrow \mathbf{GL}(V)$, so this defines a Hodge structure on V. Its decomposition into $V^{p,q}$ is obtained as follows: let I be the $\mathbf{R}$-linear automorphism of V corresponding to complex multiplication by i. Its minimum polynomial is $X^2 + 1 \in \mathbf{R}[X]$. This splits into $(X + i)(X - i)$ in $\mathbf{C}[X]$. As this has distinct roots, I is diagonalisable; then $V^{-1,0} \subset V_\mathbf{C}$ is the eigenspace for the eigenvalue i, and $V^{0,-1}$ is the eigenspace of eigenvalue $-i$. Conversely, if V has a Hodge structure h so that $V_\mathbf{C} = V^{0,-1} \otimes V^{-1,0}$, then, for any $z_1, z_2 \in \mathbf{C}^\times$ with $z_1 + z_2 \neq 0$, the endomorphism $h(z_1) + h(z_2)$ acts the same as $h(z_1 + z_2)$ on both $V^{0,-1}$ and $V^{-1,0}$, so h extends to a ring homomorphism $\mathbf{C} \longrightarrow \mathrm{End}_\mathbf{R}(V)$, which gives V the structure of a complex vector space.

For a Hodge structure $h : \mathbf{S} \longrightarrow \mathbf{GL}(V)$, I write $W_n(V_\mathbf{C}) = \bigoplus_{p+q \leq n} V^{p,q}$; this is called the *weight filtration* on $V_\mathbf{C}$. The *Hodge type* of V is the set of (p, q) such that $V^{p,q}$ is nonzero. If for some n it holds that $W_k(V) = 0$ for all $k < n$ and $W_k(V) = V$ for all $k \geq n$, then V is said to be of *pure weight* n.

A rational Hodge structure is a rational vector space V with a Hodge structure on $V_\mathbf{R}$ such that the weight filtration is defined over $\mathbf{Q}$. One writes $\mathbf{Q}(n)$ for the one-dimensional rational vector space $(2\pi i)^n\mathbf{Q} \subset \mathbf{C}$ with the Hodge structure given by $V_\mathbf{C} = V^{(-n,-n)}$; similarly we define integral Hodge structures and $\mathbf{Z}(n)$. A *polarisation* of a pure Hodge structure V of weight n is a morphism of rational Hodge structures

$$\psi : V \otimes V \longrightarrow \mathbf{Q}(-n)$$

such that the induced map

$$\begin{array}{rcl}\psi_C : V_{\mathbf{R}} \times V_{\mathbf{R}} & \longrightarrow & \mathbf{R} \\ (x, y) & \mapsto & \psi(x, h(i)y)\end{array}$$

is symmetric and positive definite. As $\psi_{\mathbf{R}}$ is a morphism of Hodge structures one sees that for every $x, y \in V_{\mathbf{R}}$ one has

$$\begin{array}{rcl}\psi(x, y) & = & \psi(h(i)x, h(i)y) \\ & = & \psi(y, h(i)^2 x) \\ & = & \psi(y, (-1)^{-n} x),\end{array}$$

which means that ψ is symmetric if n is even and antisymmetric if n is odd.

2.2 Linear algebraic groups

In this section I will review some theory on linear algebraic groups. A more thorough treatment, along with definitions and proofs of the various statements, can be found in [7].

Let k be a field of characteristic zero. A *linear algebraic group* over k is an affine group scheme over k of finite type. As in 'ordinary' group theory, a linear algebraic group P is called *solvable* if there exists a chain $0 = P_0 \subset P_1 \subset \ldots \subset P_n = P$ of normal algebraic subgroups such that every P_{i+1}/P_i is commutative.

An important example of a linear algebraic group is the multiplicative group $\mathbf{G}_{m,k}$ over k. In general, a linear algebraic group P over k is called a *torus* if there exists a finite separable field extension $k \subset l$ such that $P_l \cong \mathbf{G}_{m,l}^n$ for some integer n; P is then said to be *split* over l. An example is of the Deligne torus $\mathbf{S}$ over $\mathbf{R}$, which is split over $\mathbf{C}$ but not over $\mathbf{R}$.

Let P be a linear algebraic group over k, let B be a k-algebra, and let $g \in P(B)$. Then g can be regarded as an endomorphism of the B-module $B \otimes_k \mathcal{O}_P(P)$. One can prove that $\mathcal{O}_P(P)$ has a finite-dimensional P-stable k-linear subspace V that generates $\mathcal{O}_P(P)$ as a k-algebra, such that the induced map $P \longrightarrow \mathbf{GL}(V)$ is injective. Any $g \in P(B)$ is called *unipotent* if $g - \mathrm{id}$ is nilpotent as an endomorphism of $B \otimes_k V$; this does not depend on the choice of V. P is called unipotent if for every B, and every $g \in P(B)$, the element g is unipotent. Every linear algebraic group has a maximal normal unipotent subgroup, called the *unipotent radical*. A linear algebraic group is called *reductive* if it is connected and its unipotent radical is trivial.

For a linear algebraic group P, its *adjoint* group P^{ad} is the linear algebraic group $P^{\mathrm{ad}} = P/Z(P)$. Its *derived* group P^{der} is the linear algebraic group $[P, P]$.

The tangent space of P at the origin is denoted $\mathrm{Lie}\, P$; this has the structure of a Lie algebra. For any k-algebra B, any $g \in P(B)$ acts by conjugation on $P(B)$. This fixes the identity, so by transport of structure one obtains an action of $P(A)$ on $\mathrm{Lie}\, P(B)$. This induces a morphism $P \longrightarrow \mathbf{GL}(\mathrm{Lie}\, P)$, called the *adjoint action* of P. This map factors through the adjoint group of P.

Now suppose $k = \mathbf{R}$, and let τ be an involution of P, i.e. an endomorphism of P such that $\tau^2 = \mathrm{id}$. τ is called a *Cartan involution* if the set

$$P^\tau(\mathbf{R}) = \{g \in P(\mathbf{C}) : \tau(\bar{g}) = g\}$$

is compact in the analytic topology.

Now suppose $k = \mathbf{Q}$. A *congruence subgroup* of $P(\mathbf{Q})$ is a subgroup of the form $P(\mathbf{Q}) \cap K$, where K is an open compact subgroup of $P(\mathbf{A}_f)$[4]. If P is a subgroup of $\mathrm{GL}_{n,\mathbf{Q}}$ that is defined over $\mathbf{Z}$, then any subgroup of $P(\mathbf{Z})$ containing the kernel of the map $P(\mathbf{Z}) \longrightarrow P(\mathbf{Z}/N\mathbf{Z})$ for some $N \in \mathbf{Z}_{>1}$ is a congruence subgroup.

Now I have set up the required terminology to define Shimura varieties.

2.3 Shimura varieties

In this section, I define the notion of connected mixed Shimura varieties as defined in [32]. These are the connected components of usual mixed Shimura varieties, which are defined in [31]; I omit this generalisation here in order to avoid the language of adèles. In order to proceed, I first need the notion of a connected mixed Shimura datum. First note that if P is a linear algebraic group over $\mathbf{Q}$, that $P(\mathbf{C})$ acts on $P_\mathbf{C}$ by conjugation, so it also acts on $\mathrm{Hom}(\mathbf{S}_\mathbf{C}, P_\mathbf{C})$. Furthermore, we say that a linear algebraic group G is an *almost direct product* of two linear algebraic groups A and B if G has normal subgroups A' and B' such that $A' \cong A$ and $B' \cong B$, such that $A' \cdot B' = G$, A' and B' commute and $A' \cap B'$ is finite.

Definition 2.4. A *connected mixed Shimura datum* is a pair (P, X^+) consisting of a linear algebraic group P over $\mathbf{Q}$ and a subset $X^+ \subset \mathrm{Hom}(\mathbf{S}_\mathbf{C}, P_\mathbf{C})$ with the following properties:

- There exists an algebraic subgroup U_P of the unipotent radical W_P of P, such that U_P is normal in P, and X^+ is a connected component of an orbit under the action of $P(\mathbf{R}) \cdot U_P(\mathbf{C}) \subset P(\mathbf{C})$, where $\mathrm{Hom}(\mathbf{S}_\mathbf{C}, P_\mathbf{C})$ is given the analytic topology;
- The following conditions hold for an $x \in X^+$ (or, equivalently, for all $x \in X^+$):
  1. The composite homomorphism $\mathbf{S}_\mathbf{C} \xrightarrow{x} P_\mathbf{C} \longrightarrow (P/U_P)_\mathbf{C}$ is defined over $\mathbf{R}$;
  2. The adjoint representation of $\mathbf{S}_\mathbf{C}$ on $\mathrm{Lie}\, P_\mathbf{C}$ induces a rational Hodge structure whose type is a subset of

  $$\{(1,-1),(0,0),(-1,1),(0,-1),(-1,0),(-1,-1)\};$$

  3. The weight filtration of $\mathrm{Lie}\, P$ coming from the Hodge structure above is given by

  $$W_n(\mathrm{Lie}\, P) = \begin{cases} 0, & \text{if } n < -2; \\ \mathrm{Lie}\, U_P, & \text{if } n = -2; \\ \mathrm{Lie}\, W_P, & \text{if } n = -1; \\ \mathrm{Lie}\, P, & \text{if } n \geq 0; \end{cases}$$

[4] Here $\mathbf{A}_f$ denotes the finite adèles over $\mathbf{Q}$.

4. The conjugation τ by $x(i)$ induces a Cartan involution on $(P/W_P)^{\mathrm{ad}}_{\mathbf{R}}$;
5. P/P^{der} is an almost direct product of a $\mathbf{Q}$-split torus with a torus T of compact type defined over $\mathbf{Q}$, i.e. $T(\mathbf{R})$ is compact when given the analytic topology;
6. P possesses no proper normal subgroup P' defined over $\mathbf{Q}$ such that x factors through $P'_{\mathbf{C}} \subset P_{\mathbf{C}}$.

Because of condition 3 and the connectedness of P, which follows from condition 6, the subgroup U_P is uniquely determined. If $W_P = 0$, I call (P, X^+) a *connected pure Shimura datum* or simply a *connected Shimura datum*. The notation X^+ comes from the convention to denote by X an orbit of $\mathrm{Hom}(\mathbf{S}_{\mathbf{C}}, P_{\mathbf{C}})$ under the action of $P(\mathbf{R}) \cdot U_P(\mathbf{C})$. The long list of conditions is there to ensure the truth of the following proposition.

Proposition 2.5. *Let (P, X^+) be a connected mixed Shimura datum.*

1. *X^+ has a unique structure of a complex manifold such that for every representation ρ of $P_{\mathbf{C}}$ on a complex vector space the Hodge filtration determined by $\rho \circ x$ varies holomorphically with $x \in X^+$.*
2. *Define $P(\mathbf{R})^+ \subset P(\mathbf{R})$ as the stabiliser of X^+. Then every sufficiently small congruence subgroup $G \subset P(\mathbf{R})^+$ works freely on X^+, so that $X^+ \longrightarrow G \setminus X^+$ is an unramified covering of complex manifolds.*
3. *$G \setminus X^+$ possesses a natural structure of a quasiprojective algebraic variety over $\mathbf{C}$.*

Proof. See [32, Facts 2.3]. □

I now have enough to define the notion of a connected mixed Shimura variety. Once again, let (P, X^+) be a connected mixed Shimura datum, and fix a model $P_{\mathbf{Z}}$ of P over $\mathbf{Z}$.

Definition 2.6. Let P, X^+, G be as above, then $G \setminus X^+$ is the *connected mixed Shimura variety* associated with (P, X^+, G). If (P, X^+) is a connected pure Shimura datum, then $G \setminus X^+$ is called a *(pure) Shimura variety.*

Our next aim is to define morphisms between connected mixed Shimura varieties. For this, I first need to define morphisms between connected mixed Shimura data.

Definition 2.7. A *morphism of connected mixed Shimura data* $(P, X^+) \longrightarrow (P', X'^+)$ is a morphism of linear algebraic groups $\varphi : P \longrightarrow P'$ such that the map $\mathrm{Hom}(\mathbf{S}_{\mathbf{C}}, P_{\mathbf{C}}) \longrightarrow \mathrm{Hom}(\mathbf{S}_{\mathbf{C}}, P'_{\mathbf{C}}) : x \mapsto \varphi \circ x$ maps X^+ into X'^+.

Definition 2.8. Let S and S' be connected mixed Shimura varieties associated with (P, X^+, G) and (P', X'^+, G') respectively. A *morphism of connected mixed Shimura varieties* is a map $S \longrightarrow S'$ induced from a morphism of connected mixed Shimura data $\varphi : (P, X^+) \longrightarrow (P', X'^+)$ such that $\varphi(G) \subset G'$.

Proposition 2.9. *Let $\varphi : S \longrightarrow S'$ be a morphism of connected mixed Shimura varieties. Then φ is holomorphic and algebraic with respect to the structure in 2.5.3, and its image is clsoed in S'.*

Proof. See [32, Facts 2.6]. □

Now I want to define a certain kind of subvarieties of connected mixed Shimura varieties, rather generically named special subvarieties. Much of this thesis will be dedicated to classifying these for certain connected mixed Shimura varieties.

Definition 2.10. A subvariety $Z \subset G \setminus X^+$ is called *special* if it is the image of a morphism of connected mixed Shimura varieties.

The following proposition greatly aids in classifying special subvarieties of connected mixed Shimura varieties.

Proposition 2.11. *Let $Z \subset G \backslash X^+$ be a special subvariety. Then there is a morphism of connected mixed Shimura varieties $\varphi : (P', X'^+, G') \longrightarrow (P, X^+, G)$ such that the induced map $P' \longrightarrow P$ is an immersion of linear algebraic groups, hence a closed immersion.*

Proof. See [32, Proposition 4.3]. □

In order to greatly simplify the statements of propositions later, the notion of a Hecke correspondence is needed.

Definition 2.12. Let (P, X^+) be a connected mixed Shimura datum, and let G, G' and G'' be congruence subgroups of P, with corresponding connected mixed Shimura varieties S, S' and S''. A *Hecke correspondence*[5] is a pair of morphisms of connected mixed Shimura varieties $(\varphi : S'' \longrightarrow S, \varphi' : S'' \longrightarrow S')$ for which there exists an automorphism α of P, inducing an automorphism of X^+, such that $\alpha^{-1}(G) \cap G' = G''$, and φ and φ' are the maps $S' \xleftarrow{\varphi'} G'' \setminus X^+ \xrightarrow{\varphi} S$ induced by α and the identity, respectively.

Given P, X^+, G and G' as above, one says that a closed subvariety $Z \subset S$ is said to be equal to another closed subvariety $Z' \subset S'$ *up to Hecke correspondence* if there is an α as above such that Z is a connected component of $\varphi(\varphi'^{-1}(Z))$. Despite the terminology this is not in general an equivalence relation. The significance of this definition is shown by the following proposition.

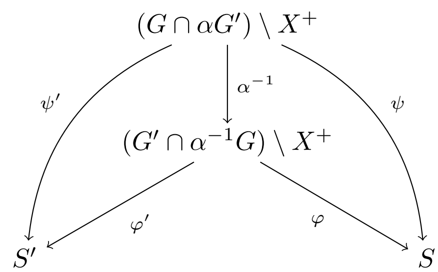
Proposition 2.13. *Let P, X^+, G, G' be as above. Suppose that $Z \subset G \setminus X^+$ and $Z' \subset G' \setminus X^+$ be irreducible closed algebraic subsets such that Z is equal to Z' up to Hecke correspondence. Then:*

1. *Z' is equal to Z up to Hecke correspondence;*
2. *if Z' is a special subvariety of S', then Z is a special subvariety of S.*

Proof.

[5] Actually, this is a generalisation of what is referred to as a generalised Hecke correspondence by [32]. In the terminology used there, a generalised Hecke correspondence must satisfy $G = G'$, whereas an ordinary Hecke correspondence requires α to be a conjugation by an element of $P(\mathbf{Q})$.

1. Suppose that Z is equal to Z' up to Hecke correspondence via an automorphism α. Consider the following commutative diagram:



Here φ' and ψ are induced by quotient maps on the level of congruence subgroups, φ is induced by α and ψ' is induced by α^{-1}. Since Z is irreducible and φ is finite, one sees that every irreducible component of $\varphi^{-1}(Z)$ maps surjectively to Z. Furthermore, there is such a component ζ such that $\varphi'(\zeta) = Z'$. Because φ' is finite, this means that ζ is an irreducible component of $\varphi' - 1(Z')$. Then $\alpha^{-1}\zeta$ is an irreducible component of $(G \cap \alpha G') \setminus X^+$ such that $\psi(\alpha^{-1}\zeta) = Z$ and $\psi'(\alpha^{-1}\zeta) = Z'$. As ψ' is a finite map, one has that $\alpha^{-1}\zeta$ is an irreducible component of $\psi'^{-1}(Z')$, so $Z = \psi(\alpha^{-1}\zeta)$ is an irreducible component of $\psi(\psi'^{-1}(Z'))$.

2. Let (Q, Y^+) be a connected mixed Shimura datum, H a congruence subgroup of Q inducing a connected mixed Shimura variety $T = H \setminus Y^+$, and $\psi : (Q, Y^+) \longrightarrow (P, X^+)$ a morphism of connected mixed Shimura data inducing a morphism of Shimura varieties $T \longrightarrow S'$ such that Z' is the image of T. Furthermore, let $\zeta \subset (\alpha^{-1} G \cap G') \setminus X^+$ be an irreducible component of $\varphi'^{-1}(Z')$ that maps to Z. Again, the image $\varphi'(\zeta)$ must be an irreducible component of Z', and $\varphi' : \zeta \longrightarrow Z$ is surjective.

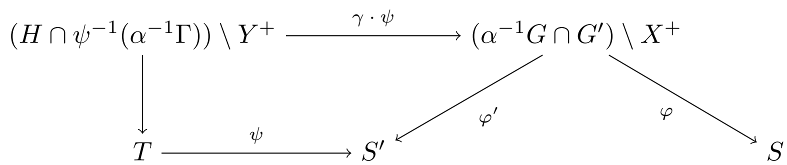
The morphism ψ induces a morphism of connected mixed Shimura varieties

$$(H \cap \psi^{-1}(\alpha^{-1} G)) \setminus Y^+ \longrightarrow (\alpha^{-1} G \cap G') \setminus X^+.$$

Now the following diagram commutes:

$$\begin{array}{ccc} (H \cap \psi^{-1}(\alpha^{-1} G)) \setminus Y^+ & \xrightarrow{\psi} & (\alpha^{-1} G \cap G') \setminus X^+ \\ \downarrow & & \downarrow \varphi' \\ T & \xrightarrow{\psi} & S' \end{array}$$

This means that the image K of the top arrow is also an irreducible subset of $\varphi'^{-1}(Z')$ that maps surjectively to Z'; again, the finiteness of φ' implies that K is an irreducible component of $\varphi'^{-1}(Z')$. The right action of G' on X^+ induces a transitive right action of G' on the irreducible components of $\varphi'^{-1}(Z')$, so there is a $\gamma \in G'$ such that $\gamma K = \zeta$. Since the map φ' trivialises the G'-action, the following diagram is again commutative:



Furthermore, the image of the composite map $\varphi \circ (\gamma \cdot \psi)$ is Z, so this establishes Z as a special subvariety of S.

$\square$

2.4 The Siegel upper half space

This section is devoted to a special example of Shimura varieties that will be of great importance for the rest of this thesis. Let $g > 0$ be an integer, and let J be the matrix $\begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$. One defines the algebraic group of symplectic similitudes $P_0 = \mathrm{GSp}_{2g,\mathbf{Q}}$ as follows, for any $\mathbf{Q}$-algebra B:

$$P_0(B) = \left\{(A \in \mathrm{GL}_{2g}(B) \mid \exists \lambda \in B^\times : A^T J A = \lambda J\right\}.$$

For any $a, b \in B$ one has $(a + bJ)(a - bJ) = a^2 + b^2$, so $a + bJ$ is invertible in $\mathrm{GL}_{2g}(B)$ if $a^2 + b^2$ is invertible in B. If this is the case, then

$$(a + bJ)^T J(a + bJ) = (a - bJ)J(a + bJ) = (a^2 + b^2)J,$$

which shows that $a + bJ \in P_0(B)$. As $J^2 = -1$, the map

$$\begin{array}{rcl} h_0 : \mathbf{S} & \longrightarrow & P_{0,\mathbf{R}} \\ a + bi & \mapsto & a + bJ \end{array}$$

is an injective morphism of linear algebraic groups over $\mathbf{R}$. As a differential manifold, the space $P_0(\mathbf{R})$ has two connected components; the identity component $P_0(\mathbf{R})^+$ consists of the elements with positive multiplier character λ. Now consider $X_0^+ = P_0(\mathbf{R})^+ \cdot h_0 \subset \mathrm{Hom}(\mathbf{S}, P_{0,\mathbf{R}})$. We show that (P_0, X_0^+) is a connected pure Shimura datum, by checking the different conditions of 2.4 voor h_0:

1. By definition the map $h_{0,\mathbf{C}} : \mathbf{S_C} \longrightarrow P_{0,\mathbf{C}}$ is defined over $\mathbf{R}$.

2. It is known that $\mathrm{Lie}\,\mathrm{GL}_{2g,\mathbf{Q}}$ is equal to $\mathrm{M}_{2g,\mathbf{Q}}$, with the adjoint action of $\mathrm{GL}_{2g,\mathbf{Q}}$ given by conjugation. As such, $\mathrm{Lie}\,P_0$ is a subspace of $\mathrm{M}_{2g,\mathbf{Q}}$. Since h_0 gives a Hodge structure of type

$\{(0,-1),(-1,0)\}$ on $\mathbf{R}^{2g}$, this conjugation gives a Lie structure whose type is a subset of $\{(1,-1),(0,0),(-1,1)\}$. As the scalars in $\mathrm{GL}_{2g,\mathbf{Q}}$ are contained in P_0, the scalars in $\mathrm{M}_{2g,\mathbf{Q}}$ are contained in $\mathrm{Lie}\, P_0$. These commute with any other matrix, in particular with elements $h_0(z)$, which shows that $(\mathrm{Lie}\, P_0(\mathbf{C}))^{0,0}$ is nonzero. Now consider $A = \begin{pmatrix} 1_g & i_g \\ i_g & -1_g \end{pmatrix} \in P_0(\mathbf{C})$; one has that $A^2 = 0$, so that $\exp(A) = 1 + A = \begin{pmatrix} 2_g & i_g \\ i_g & 0_g \end{pmatrix}$, which is an element of $P_0(\mathbf{C})$; this shows that $A \in \mathrm{Lie}\, P_0(\mathbf{C})$. On the other hand, for $z = a + bi \in \mathbf{C}^\times$ one has

$$\begin{aligned} h_0(z)Ah_0(z)^{-1} &= \frac{1}{a^2+b^2}\begin{pmatrix} a_g & b_g \\ -b_g & a_g \end{pmatrix}\begin{pmatrix} 1_g & i_g \\ i_g & -1_g \end{pmatrix}\begin{pmatrix} a_g & -b_g \\ b_g & a_g \end{pmatrix} \\ &= \frac{1}{a^2+b^2}\begin{pmatrix} (a^2-b^2+2abi)_g & (-2ab+(a^2-b^2)i)_g \\ (-2ab+(a^2-b^2)i)_g & (-a^2+b^2-2abi)_g \end{pmatrix} \\ &= \frac{z}{\bar{z}}A, \end{aligned}$$

which shows that $\mathrm{Lie}\, P_0(\mathbf{C})^{(-1,1)}$ (hence also $\mathrm{Lie}\, P_0(\mathbf{C})^{(1,-1)}$) must be nonzero, as was to be shown.

3. From the above we find that $U = W = 0$, so (P_0, X_0^+) is indeed a pure Shimura variety.

4. It is known that the center of $\mathrm{GSp}_{2g,\mathbf{Q}}$ consists of just the scalars, which we may identify with $\mathbf{G}_{m,\mathbf{Q}}$; its quotient is denoted $\mathrm{PGSp}_{2g,\mathbf{Q}}$. Now for any $A \in \mathrm{GSp}_{2g,\mathbf{Q}}$, let $d(A) \in \mathbf{G}_{m,\mathbf{Q}}$ be such that $A^T JA = d(A)J$. Then the kernel of the morphism $d : \mathrm{GSp}_{2g,\mathbf{Q}} \longrightarrow \mathbf{G}_{m,\mathbf{Q}} : A \mapsto d(A)$ is denoted $\mathrm{Sp}_{2g,\mathbf{Q}}$, the *symplectic group* over $\mathbf{Q}$ of dimension $2g$. This is a normal subgroup of $\mathrm{GSp}_{2g,\mathbf{Q}}$, and the quotient is $\mathbf{G}_{m,\mathbf{Q}}$. The intersection $\mathrm{Sp}_{2g,\mathbf{Q}} \cap \mathbf{G}_{m,\mathbf{Q}}$ consists of the scalars with square 1, which we may identify with $\mu_{2,\mathbf{Q}}$. One defines $\mathrm{Sp}_{2g,\mathbf{Q}} = \mathrm{PSp}_{2g,\mathbf{Q}} / \mu_{2g,\mathbf{Q}}$.

   Over $B = \mathbf{C}$ we see that the natural injective map $\mathrm{PSp}_{2g}(\mathbf{C}) \longrightarrow \mathrm{PGSp}_{2g}(\mathbf{C})$ is also surjective, because for every $A \in \mathrm{GSp}_{2g}(\mathbf{C})$ the matrices A and $\frac{1}{\sqrt{d(A)}}A \in \mathrm{Sp}_{2g}(\mathbf{C})$ have the same image in $\mathrm{PGSp}_{2g}(\mathbf{C})$; hence $\mathrm{PSp}_{2g}(\mathbf{C}) = \mathrm{PGSp}_{2g}(\mathbf{C})$. By definition $\tau = \mathrm{inn}(h_0(i)) = \mathrm{inn}(J)$ gives a Cartan involution if and only if $P_0^{\mathrm{ad},\tau}(\mathbf{R})$, the $\mathbf{R}$-points of the twist of $P_0^{\mathrm{ad}} = \mathrm{PGSp}_{2g,\mathbf{Q}}$ over $\mathbf{C}$ by τ, is compact. The involution τ comes from the involution $\tilde{\tau}$ on $\mathrm{Sp}_{2g,\mathbf{R}}$ given by $\tilde{\tau}(G) = JGJ^{-1}$. In this particular instance, we see, from the fact that $G = JG^{T,-1}J^{-1}$ for $G \in \mathrm{Sp}_{2g}(\mathbf{C})$ and that $J^2 = -1$, that

$$\begin{aligned} \tilde{\tau}(G) &= JGJ^{-1} \\ &= J(JG^{T,-1}J^{-1})J^{-1} \\ &= G^{T,-1}. \end{aligned}$$

   This gives us, for $G \in \mathrm{Sp}_{2g}(\mathbf{C})$ that $\tilde{\tau}(\bar{G}) = G$ if and only if $G\bar{G}^T = \mathrm{id}_{2g}$. In other words, $\mathrm{Sp}_{2g}^{\tilde{\tau}}(\mathbf{R})$ is a closed subgroup of the compact subgroup $\mathrm{U}_{2g}(\mathbf{C}) \subset \mathrm{GL}_{2g}(\mathbf{C})$ of unitary matrices. Furthermore, the preimage K of $\mathrm{PSp}_{2g}^{\tau}(\mathbf{R})$ in $\mathrm{Sp}_{2g}(\mathbf{C})$ consists of all G such that $\tilde{\tau}(\bar{G}) = \zeta G$ for some $\zeta \in \mu_2(\mathbf{C}) = \{\pm 1\}$; hence $K / \mathrm{Sp}_{2g}^{\tilde{\tau}}(\mathbf{R})$ has at most 2 elements, which means that K is compact as well. This implies that $\mathrm{PSp}_{2g}^{\tau}(\mathbf{R})$, as the image of a compact set, is compact as well.

5. For any $\mathbf{Q}$-algebra B, any commutator of $\mathrm{GSp}_{2g,\mathbf{Q}}(B)$ lies in $\mathrm{Sp}_{2g,\mathbf{Q}}(B)$ as it acts trivially on J, and by scaling we may assume that it is of the form $[A, A']$, for some $A, A' \in \mathrm{Sp}_{2g,\mathbf{Q}}(B)$. However, $\mathrm{Sp}_{2g,\mathbf{Q}}$ is a perfect group, so $[\mathrm{Sp}_{2g,\mathbf{Q}}, \mathrm{Sp}_{2g,\mathbf{Q}}] = \mathrm{Sp}_{2g,\mathbf{Q}}$; hence $\mathrm{GSp}_{2g,\mathbf{Q}} / \mathrm{GSp}_{2g,\mathbf{Q}}^{\mathrm{der}} = \mathrm{GSp}_{2g,\mathbf{Q}} / \mathrm{Sp}_{2g,\mathbf{Q}} = \mathbf{G}_{m,\mathbf{Q}}$, which is a split torus over $\mathbf{Q}$.

6. Suppose P' is a normal subgroup of P_0 containing $h_0(\mathbf{S})$. Then P' properly contains the scalars, so the quotient map $P_0 \longrightarrow P/P_0$ factors through $\mathrm{PGSp}_{2g,\mathbf{Q}}$ and its image there is nontrivial. However, this is a simple linear algebraic group, so this means it must be all of it; hence $P' = P$.

Now define the *Siegel upper half plane* $\mathcal{H}_g$ as the set of $g \times g$ symmetric complex matrices with positive definite imaginary part. It comes with a transitive action of $P_0(\mathbf{R})^+$ as follows: For $A \in P_0(\mathbf{R})^+$, write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathrm{M}_g(\mathbf{R})$; then for $\tau \in \mathcal{H}_g$, one has $A \cdot \tau = (a\tau + b)(c\tau + d)^{-1}$. As is shown by the following proposition, this action comes naturally from the action of $P(\mathbf{R})^+$ on X_0^+.

Proposition 2.14. *There is a unique isomorphism φ of complex manifolds $X_0^+ \longrightarrow \mathcal{H}_g$ that is equivariant under the $P_0(\mathbf{R})^+$-action such that $\varphi(h_0) = i$.*

Proof. Let $h \in X_0^+$, and write $h = \mathrm{Inn}(A) \circ h_0$ for some $A \in P_0(\mathbf{R})^+$. Let V_h be the Hodge structure on $\mathbf{R}^{2g}$ induced by h. Then $V_{\mathbf{C},h}^{-1,0} = AV_{\mathbf{C},h_0}^{-1,0}$. As $V_{\mathbf{C},h_0}^{-1,0} = \{\binom{iy}{y} : y \in \mathbf{C}^g\}$, we see that

$$V_{\mathbf{C},h}^{0,-1} = \left\{ \begin{pmatrix} (ai+b)y \\ (ci+d)y \end{pmatrix} : y \in \mathbf{C}^g \right\} = \left\{ \begin{pmatrix} (ai+b)(ci+d)^{-1}y \\ y \end{pmatrix} : y \in \mathbf{C}^g \right\}.$$

This shows that there is an injective map $\varphi : X_0^+ \longrightarrow \mathrm{GL}_g(\mathbf{C})$, sending $h \in X^+$ to the unique element $A \in \mathrm{GL}_g(\mathbf{C})$ such that $V_{\mathbf{C},h}^{-1,0} = \{\binom{Ay}{y} : y \in \mathbf{C}^g\}$. As above, if $h = \mathrm{Inn}(A) \circ h_0$, then $\varphi(h) = (ai + b)(ci + d)^{-1} = A \cdot i$. Now, for any $A_1 \in P_0(\mathbf{R})^+$ and any $h = A_2 \cdot h_0 \in X_0^+$ one has

$$\begin{aligned} \varphi(A_1 \cdot h) &= \varphi(A_1 A_2 \cdot h_0) \\ &= A_1 A_2 \cdot i \\ &= A_1(\varphi(h)), \end{aligned}$$

so the action is equivariant under the action of $P_0(\mathbf{R})$. As $\varphi(h_0) \in \mathcal{H}_g$, the group $P_0(\mathbf{R})^+$ works transitively on both $\mathcal{H}_g$ and X_0^+ the map φ is a bijection. It is also an isomorphism of smooth real manifolds. As the complex structure on X_0^+ is unique, φ must also be an isomorphism of complex manifolds. □

For a sufficiently small congruence subgroup $G_0 \subset P_0(\mathbf{Z})^+ = \mathrm{GL}_2(\mathbf{Z}) \cap P_0(\mathbf{R})^+$ we now have a connected Shimura variety $S_0 = G_0 \setminus \mathcal{H}_g$. A point $\tau \in \mathcal{H}_g$ induces an isomorphism of $\mathbf{R}$-vector spaces $t_\tau : \mathbf{C}^g \longrightarrow \mathbf{R}^{2g}$, defined by $t_\tau(\tau e_i) = f_i$, $t_\tau(e_i) = f_{i+g}$, where $e_1, \ldots, e_g$ is the standard basis for $\mathbf{C}^g$, and $f_1, \dots, f_{2g}$ is the standard basis for $\mathbf{R}^{2g}$. The morphism $h_\tau \in X_0^+$ corresponding to τ is on $\mathbf{R}$-points the composite map

$$\mathbf{C}^\times \longrightarrow \mathrm{Aut}_{\mathbf{R}}(\mathbf{C}^g) \xrightarrow{t_{\tau *}} \mathrm{Aut}_{\mathbf{R}}(\mathbf{R}^{2g}) = \mathrm{GL}_{2g}(\mathbf{R}).$$

For suitable choices of G_0, the connected Shimura variety S_0 becomes a Siegel modular variety, parametrising principally polarised abelian varieties with some level structure, as we will see in chapter 7.

3 Abelian and semiabelian varieties

Before continuing with more examples of Shimura varieties, I first introduce the notion of two specific kinds of group varieties, namely abelian and semiabelian varieties, starting with the former.

3.1 Abelian varieties

For a thorough treatment of abelian varieties in general, including proofs of the statements in this section, I refer to [17]. We begin, straightforwardly enough, with a definition.

Definition 3.1. Let k be a field. An *abelian variety* over k is a complete group variety.

The terminology comes from the fact that any such a group variety is automatically a commutative group variety. However, not every commutative group variety is an abelian variety; a counterexample is the group variety $\mathbf{G}_{a,k}$. An elliptic curve over k is a one-dimensional abelian variety over k. In fact, one can prove that every one-dimensional abelian variety is an elliptic curve.

The notion of an abelian variety can also be extended to a more scheme-theoretic definition.

Definition 3.2. Let S be a scheme. An *abelian scheme* X/S is a proper smooth group scheme $\pi : X \longrightarrow S$, whose fibres are geometrically connected.

An equivalent definition of an abelian scheme is then a smooth group scheme X/S whose fibres are abelian varieties. If S is itself a variety over a field k, we call X/S a *family of abelian varieties.* An abelian scheme of relative dimension 1 is the same as an elliptic curve over a scheme as defined in [21].

Now let X be an abelian variety, and let T be another variety over k. A *rigidified line bundle* on $X \times T$ is a pair $(\mathcal{L}, \alpha)$ of a line bundle, i.e. an invertible $\mathcal{O}_{X\times T}$-bundle, $\mathcal{L}$ on $X \times T$ and an isomorphism $\alpha : \mathcal{O}_T \xrightarrow{\sim} 0_T^*\mathcal{L}$ of line bundles on T. A morphism between two rigidified line bundles $(\mathcal{L}_1, \alpha_1)$ and $(\mathcal{L}_2, \alpha_2)$ on $X \times T$ is a morphism of line bundles $\varphi : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ such that $(0^*\varphi) \circ \alpha_1 = \alpha_2$. We can now define the following functor:

$$\begin{aligned} P_X : \mathbf{Sch}_{/k} &\longrightarrow \mathbf{Set} \\ T &\mapsto \{\text{isomorphism classes of rigidified line bundles on } X \times T\} \end{aligned}$$

One can show that P_X is representable by a k-scheme Y. As the tensor product of two rigidified line bundles is again rigidified, Y obtains the structure of a group variety. One can prove that its identity component $X^\vee$ is again an abelian variety, called the *dual abelian variety* of X. The line bundles on X coming from $X^\vee(k)$ are said to be algebraically equivalent to 0; in the case that X is a curve, this coincides with the usual definition of degree. The identity on $X^\vee$ induces a rigidified line bundle $(\mathcal{P}, \nu)$ on $X \times X^\vee$, called the Poincaré bundle, which has the following universal property: if T is a scheme over k and $(\mathcal{L}, \alpha)$ is a rigidified line bundle on $X \times T$ then there is a unique morphism $g : T \longrightarrow X^\vee$ such that $(\mathrm{id}_X \times g)^*\mathcal{P}$ and $\mathcal{L}$ are naturally isomorphic as line bundles over $X \times X^\vee$, and such that $(\mathrm{id}_X \times g)^*\nu = \alpha$. Because $\mathcal{P}$ is rigidified, we have an isomorphism $\mathcal{O}_{X^\vee} \cong \mathcal{P}|_{0\times X^\vee}$ of line bundles on $X^\vee$. On the other hand, the map $0 : \operatorname{Spec} k \longrightarrow X^\vee$ corresponds to the identity element of $P_X(\operatorname{Spec} k)$, so we find that $\mathcal{O}_X \cong \mathcal{P}|_{X\times 0}$ as line bundles on X. Furthermore, one can prove that $(X^\vee)^\vee \cong X$ and $P_X \cong P_{X^\vee}$ as line bundles over $X \times X^\vee = (X^\vee)^\vee \times X^\vee$. Also, a morphism of abelian varieties $f : X \longrightarrow Y$ induces a pullback morphism $f^\vee : Y^\vee \longrightarrow X^\vee$ called the *dual morphism* of f.

In the case of an abelian scheme X/S, the notion of a dual abelian scheme $X^\vee/S$ can be defined as well, but the proof of representability is more complicated; it can be found in [14, I].

Now let X be an abelian variety over a field k, and let $x \in X(k)$. For any k-scheme T, there is a map

$$\begin{array}{rcl} t_x(T) : X(T) & \longrightarrow & X(T) \\ y & \mapsto & y + x, \end{array}$$

and this is functorial in T, so it defines a morphism of varieties $t_x : X \longrightarrow X$. Now let $\mathcal{L}$ be a line bundle on X. Then $\mathcal{L}$ induces the following morphism of abelian varieties:

$$\begin{array}{rcl} \varphi_{\mathcal{L}} : X & \longrightarrow & X^\vee \\ x & \mapsto & t_x^*\mathcal{L} \otimes \mathcal{L}^{-1} \end{array}$$

If $\mathcal{L}$ is ample, then $\varphi_{\mathcal{L}}$ is an isogeny. An isogeny $X \longrightarrow X^\vee$ coming from an ample line bundle is called a *polarisation*; a polarisation that is an isomorphism is called a *principal polarisation*.

Now let X be an elliptic curve, and for any point $x \in X$, let $\mathcal{D}_x$ be the line bundle corresponding to the divisor $[x]$. Then one can take $\mathcal{L}$ to be the line bundle $\mathcal{D}_0$ corresponding to the divisor $[0]$; this gives an isomorphism $X \longrightarrow X^\vee$ that is defined on $\bar{k}$-points by sending x to $\mathcal{D}_{-x} \otimes \mathcal{D}_0^{-1} \cong \mathcal{D}_0 \otimes \mathcal{D}_x^{-1} \in \operatorname{Pic}^0(X)$; elliptic curves hence are canonically isomorphic to their dual. For general abelian varieties, however, this is not the case. Still, it can be proven that every abelian variety is projective, and an embedding of an abelian variety into a projective space defines an ample line bundle, hence a polarisation, so every abelian variety is isogenous to its dual.

3.2 Complex abelian varieties

We call a complex abelian variety X *simple* if X has precisely two abelian subvarieties, namely 0 and X itself. Furthermore, we define the category $\mathbf{Q} \otimes \mathbf{AbVar}_{\mathbf{C}}$ of *complex varieties up to isogeny* to have as objects complex abelian varieties, denoted $\mathbf{Q} \otimes X$, while the sets of morphisms between two abelian varieties X and X' is defined to be $\mathbf{Q} \otimes_{\mathbf{Z}} \operatorname{Hom}_{\mathbf{AbVar}_{\mathbf{C}}}(X, X')$, also denoted $\operatorname{Hom}(\mathbf{Q} \otimes X, \mathbf{Q} \otimes X')$. The terminology comes from the fact that for every isogeny $f : X \longrightarrow X'$ there is an isogeny $g : X' \longrightarrow X$ such that $gf \in \mathbf{Z}_{>0} \subset \operatorname{End}(X)$; this means that every isogeny is an isomorphism in $\mathbf{Q} \otimes \mathbf{AbVar}_{\mathbf{C}}$. The advantage of using this category lies in the following theorem.

Theorem 3.3 (Poincaré's Complete Reducibility Theorem)**.** *The category $\mathbf{Q} \otimes \mathbf{AbVar}_{\mathbf{C}}$ is semisimple, i.e. every complex abelian variety X is isogenous to a product of simple varieties that are uniquely determined up to permutation and isogeny.*

Proof. See [6, 5.3.7] □

Now let X be a complex abelian variety. The associated complex analytic manifold X^{an} can be regarded as a real analytic manifold, which makes it a Lie group. Since A is a commutative complete group variety, X^{an} is a compact commutative connected Lie group. By the classification of Lie groups[6], this means that X^{an} is a complex torus; let us write $X^{\mathrm{an}} = V/\Lambda$, where V is a complex vector space and Λ a lattice in V. Then $\Lambda = \pi_1(X) = \mathrm{H}_1(X, \mathbf{Z})$. The complex structure on $V = \mathbf{R} \otimes \Lambda$ gives $\mathrm{H}_1(X, \mathbf{Z})$ a $\mathbf{Z}$-Hodge structure of type $\{(0,-1),(-1,0)\}$. Now let X' be another complex abelian variety, and write $X'^{\mathrm{an}} = V'/\Lambda'$. A morphism $X \longrightarrow X'$ of complex abelian varieties corresponds to a $\mathbf{C}$-linear map $\varphi : V \longrightarrow V'$ such that $\varphi(\Lambda) \subset \Lambda'$. In other words, φ is a map $\Lambda \longrightarrow \Lambda'$ preserving the complex structure on $\mathbf{R} \otimes \Lambda$; but this is exactly the same as a morphism of $\mathbf{Z}$-Hodge structures. We thus have a fully faithful functor

$$\begin{array}{ccc} \mathbf{AbVar_C} & \longrightarrow & \mathbf{Z-Hodge} \\ A & \mapsto & \mathrm{H}_1(A, \mathbf{Z}). \end{array}$$

An isogeny from A to itself leaves the $\mathbf{Q}$-Hodge structure invariant, so we get a commutative diagram of functors

$$\begin{array}{ccc} \mathbf{AbVar_C} & \longrightarrow & \mathbf{Z-Hodge} \\ \downarrow & & \downarrow \\ \mathbf{Q} \otimes \mathbf{AbVar_C} & \longrightarrow & \mathbf{Q-Hodge} \end{array}$$

However, not every such a V/Λ gives a complex manifold that comes from an abelian variety. Before giving a necessary and sufficient condition for this, we first have to explain some theory.

A line bundle on a complex analytic torus $T = V/\Lambda$ is represented by an element of $\mathrm{H}^1(T, \mathcal{O}_T^\times)$. The exact sequence of sheaves on T

$$0 \longrightarrow \mathbf{Z}(1)_T \longrightarrow \mathcal{O}_T \xrightarrow{\exp} \mathcal{O}_T^\times \longrightarrow 0$$

induces a group homomorphism $\mathrm{H}^1(T, \mathcal{O}_T^\times) \longrightarrow \mathrm{H}^2(T, \mathbf{Z}(1))$. The image of a line bundle $\mathcal{L}$ is called the *first Chern class* of $\mathcal{L}$, written $c_1(\mathcal{L})$. One can prove (see [6, 2.1.2]) that $\mathrm{H}^2(T, \mathbf{Z}(1))$ is canonically isomorphic to $\mathrm{Alt}^2(\Lambda, \mathbf{Z}(1))$, the set of $\mathbf{Z}$-bilinear maps $\varphi : \Lambda \times \Lambda \longrightarrow \mathbf{Z}(1)$ satisfying $\varphi(x, y) = -\varphi(y, x)$. Under this identification the image of a line bundle $\mathcal{L}$ will be an alternating form $c_1(\mathcal{L}) : \Lambda \otimes \Lambda \longrightarrow \mathbf{Z}(1)$ that is a morphism of Hodge structures. The following theorem shows the importance of these definitions.

Theorem 3.4 (Lefschetz)**.** *Let $T = V/\Lambda$ be a complex analytic torus, and let $\mathcal{L}$ be a line bundle on T. Then the space of holomorphic sections of $\mathcal{L}^n$ defines a closed embedding as a closed complex submanifold of T into a projective space for each $n \geq 3$ if and only if $c_1(\mathcal{L})$ is a polarisation of the $\mathbf{Z}$-Hodge structure Λ.*

[6] See [3, 6.1].

Proof. See [8, 1.18]. □

This theorem has the following important corollary, which completely classifies which complex tori come from algebraic varieties.

Corollary 3.5. *Let $T = V/\Lambda$ be a complex torus. Then T is the analytic space corresponding to a complex algebraic variety if and only if Λ is polarisable. Therefore the functor $\mathbf{AbVar}_{\mathbf{C}} \longrightarrow$ $\{$polarisable $\mathbf{Z}$-Hodge structures of type $\{(0,-1),(-1,0)\}\}$ is an equivalence of categories.*

Proof. See [8, 1.20]. □

As is shown in [6, 2.4], for a complex abelian variety X with $X^{\mathrm{an}} = V/\Lambda$, the complex analytic variety corresponding to $X^\vee$ is equal to $\overline{\Omega}/\hat{\Lambda}$, where

$$\overline{\Omega} = \{f : V \longrightarrow \mathbf{C} \text{ antilinear}\}$$

and

$$\hat{\Lambda} = \{f \in \overline{\Omega} : \operatorname{Im} f(\Lambda) \subset \mathbf{Z}\}.$$

Its Hodge structure is stated in the following proposition.

Proposition 3.6. *Let $\Lambda^* = \operatorname{Hom}(\Lambda, \mathbf{Z}(1))$. The map $\hat{\Lambda} \longrightarrow \Lambda^*$ sending a functional f to $2\pi i \operatorname{Im} f$ is an isomorphism of $\mathbf{Z}$-Hodge structures.*

Proof. An inverse map can be given by $h \mapsto (z \mapsto \frac{1}{2\pi i}(ih(z) - h(iz)))$, so the map is a bijection. Now $\hat{\Lambda}$ and Λ^* are both $\mathbf{Z}$-Hodge structures of type $\{(0,-1),(-1,0)\}$, so it suffices to show that the complex structure on $\overline{\Omega} = \mathbf{R} \otimes \hat{\Lambda}$ is preserved by the bijection. For every $\alpha \in \mathbf{C}$, every $f \in \overline{\Omega}$ and every $z \in V$ one has $\operatorname{Im}(\alpha f(\alpha z)) = |\alpha|^2 \operatorname{Im} f(z)$. On the other hand, for every $\alpha \in \mathbf{C}$, $h \in V^* = \mathbf{R} \otimes \Lambda^*$ and $z \in V$ one has $(\alpha \cdot h)(\alpha z) = |\alpha|^2 z$, which shows that the two real vector spaces have the same complex structure. □

Under this identification, the polarisations of the complex analytic abelian variety V/Λ correspond to the polarisations of the $\mathbf{Z}$-Hodge structure Λ, as is shown in [6, 2.5.5].

Consider the universal covering $V = \mathbf{R} \otimes \mathrm{H}_1(A)$ with its complex structure, and let $\mathcal{O}_V^\times$ be the sheaf of nowhere zero holomorphic functions on V. The free group $\mathrm{H}_1(A)$ acts on $\mathrm{H}^0(V, \mathcal{O}_V^\times)$ by translations, which allows us to define the group of 1-cocycles $\mathrm{Z}^1(\mathrm{H}_1(A), \mathrm{H}^0(V, \mathcal{O}_V^\times))$. Such a cocycle is a function $f : \mathrm{H}_1(A) \times V \longrightarrow \mathbf{C}$, and it defines a line bundle by quotienting out $V \times \mathbf{C}$ by the action of $\mathrm{H}_1(A)$ given by $v_0 \cdot (v,t) = (v + v_0, f(v_0, v)t)$. As is shown in [6, B], this actually defines a group homomorphism

$$\mathrm{Z}^1(\mathrm{H}_1(A), \mathrm{H}^0(V, \mathcal{O}_V^\times)) \longrightarrow \operatorname{Pic} A.$$

It is surjective, and its kernel is $\mathrm{B}^1(\mathrm{H}_1(A), \mathrm{H}^0(V, \mathcal{O}_V^\times))$, so that we might consider it as an isomorphism

$$\mathrm{H}^1(\mathrm{H}_1(A), \mathrm{H}^0(V, \mathcal{O}_V^\times)) \xrightarrow{\sim} \mathrm{H}_1(A, \mathcal{O}_A^\times) = \operatorname{Pic} A.$$

Any polarisation $\psi : \Lambda \otimes \Lambda \longrightarrow \mathbf{Z}(1)$ induces an isomorphism $\tilde{\psi} : V \longrightarrow V^*$. If φ is an endomorphism of X in $\mathbf{Q} \otimes \mathbf{AbVar}_{\mathbf{C}}$, then let $R_\psi(\varphi)$ be $\tilde{\psi}^{-1} \circ \varphi^\vee \circ \tilde{\psi} \in \operatorname{End}(\mathbf{Q} \otimes X)$. The map R_ψ is an involution of $\operatorname{End}(\mathbf{Q} \otimes X)$ called the *Rosati involution* induced by ψ. Now let φ^T be the transpose of φ with respect to ψ. For every $v \in V$, one has

$$\begin{aligned} R_\psi(\varphi)(v) &= (\tilde{\psi}^{-1} \circ \varphi^\vee \circ \tilde{\psi})(v) \\ &= (\tilde{\psi}^{-1} \circ \varphi^\vee)(w \mapsto \psi(v, w)) \\ &= \tilde{\psi}^{-1}(w \mapsto \psi(v, \varphi(w))) \\ &= \tilde{\psi}^{-1}(w \mapsto \psi(\varphi^T(v), w)) \\ &= \varphi^T(v), \end{aligned}$$

so $R_\psi(\varphi)$ is the transpose of φ with respect to ψ.

3.3 Semiabelian varieties

We begin, straightforwardly enough, with a definition.

Definition 3.7. Let k be an algebraically closed field. A *semiabelian variety* over k is a commutative group variety G over k which fits in a short exact sequence of linear algebraic groups

$$0 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 0,$$

where T is a torus and A is an abelian variety.

Analogously, we can define a semiabelian scheme, and a family of semiabelian varieties.

Example 3.8. Let X be an abelian variety, and let $\mathcal{L} \in X^\vee$ be a degree zero line bundle on X. For any k-scheme T, let $\mathcal{G}(\mathcal{L})(T)$ be the set of pairs (x, φ), where $x \in X(T)$ and $\varphi : \mathcal{L}_T \longrightarrow t_x^*\mathcal{L}_T$ is an isomorphism. The set $\mathcal{G}(\mathcal{L})(T)$ carries an abelian group structure by the operation $(x_1, \varphi_1) \cdot (x_2, \varphi_2) = (x_1 + x_2, t_{x_2}^*\varphi_1 \circ \varphi_2)$, and this is functorial in T, so this defines a group functor $\mathcal{G}(\mathcal{L}) : \mathbf{Sch}_{/k} \longrightarrow \mathbf{Grp}$; see [17, VIII] for more details. One can show that $\mathcal{G}(\mathcal{L})$ is representable by $\mathbf{L}_{\mathcal{L}}^\times$, the geometric line bundle corresponding to $\mathcal{L}$ with its zero section removed. The isomorphism is defined as follows: The rigidification $r_T : \mathbf{G}_{m,T} \longrightarrow \mathbf{L}_{\mathcal{L},0,T}$ defines a point $P = r(1) \in \mathbf{L}_{\mathcal{L},0}(T)$ for every k-scheme T; then any $(x, \varphi) \in \mathcal{G}(\mathcal{L})(T)$ induces an isomorphism $\tilde{\varphi} : \mathbf{L}_{\mathcal{L},T} \xrightarrow{\sim} \mathbf{L}_{\mathcal{L},T}$ that makes the following diagram commute:

$$\begin{array}{ccc} \mathbf{L}_{\mathcal{L},T} & \xrightarrow{\tilde{\varphi}} & \mathbf{L}_{\mathcal{L},T} \\ \downarrow & & \downarrow \\ X & \xrightarrow{t_x} & X \end{array}$$

Then (x, φ) corresponds to the point $\tilde{\varphi}(T) \in \mathbf{L}_{\mathcal{L}}^\times(T)$. Now $\mathbf{L}_{\mathcal{L}}^\times$ fits into a short exact sequence of commutative group varieties

$$1 \longrightarrow \mathbf{G}_{m,k} \xrightarrow{r_k} \mathbf{L}_{\mathcal{L}}^\times \xrightarrow{\pi} X \longrightarrow 1,$$

where π is the projection morphism. This shows that $\mathbf{L}_{\mathcal{L}}^{\times}$ is a semiabelian variety. One can even prove that every extension of X by $\mathbf{G}_{m,k}$ is of this form.

Example 3.9. Let X be an abelian variety over a field k, and let $\mathcal{P}$ be the Poincaré bundle over $X \times X^{\vee}$. Let $q \in X^{\vee}$ be a point, and let $\mathcal{L}$ be the corresponding line bundle on X. Then there is a canonical isomorphism of line bundles $\pi_{1*}\mathcal{P}_{X\times\{q\}} = \mathcal{L}$. This isomorphism preserves the rigidification, so this isomorphism induces an isomorphism $\mathbf{L}_{\mathcal{P},q}^{\times} \cong \mathbf{L}_{\mathcal{L}}^{\times}$ of commutative group scheme extensions of X. This shows that $\mathbf{L}_{\mathcal{P},q}^{\times}$ is a semiabelian variety, and we can regard $\mathbf{L}_{\mathcal{P}}^{\times}$ as a semiabelian scheme over the basis $X^{\vee}$. On the other hand, we can also regard $\mathbf{L}_{\mathcal{P}}^{\times}$ as a semiabelian scheme over X. Now, if we denote the group law on $\mathbf{L}_{\mathcal{P},X\times\{q\}}^{\times}$ by $+^{q}$ for every point $q \in X^{\vee}$, and the group law on $\mathbf{L}_{\mathcal{P},\{p\}\times X^{\vee}}^{\times}$ by $+_p$ for every $p \in X$, then the two semiabelian scheme structures on $\mathbf{L}_{\mathcal{P}}^{\times}$ are compatible in the sense that for every $p_1, p_2 \in X$ and $q_1, q_2 \in X^{\vee}$ and for every quadruple $(g_{ij})_{i,j\in\{1,2\}}$ with $g_{ij} \in \mathbf{L}_{\mathcal{P},(p_i,q_j)}^{\times}$ one has

$$(g_{11} +_{p_1} g_{12}) +^{q_1+q_2} (g_{21} +_{p_2} g_{22}) = (g_{11} +^{q_1} g_{21}) +_{p_1+p_2} (g_{12} +^{q_2} g_{22}).$$

For this reason $\mathbf{L}_{\mathcal{P}}^{\times}$ is called a *biextension* of X and $X^{\vee}$; see [19, VII] for a more thorough treatment of this notion.

4 Classification of connected mixed Shimura varieties over Siegel modular varieties

Fix $(P_0, X_0^+) = (\mathrm{GSp}_{2g,\mathbf{Q}}, \mathcal{H}_g)$ as in section 2.4. In this section, I classify all Shimura varieties S (associated with a triple (P, X^+, G)) for which $P/W_P \cong P_0$, such that the morphism of linear algebraic groups $P \longrightarrow P_0$ induces a morphism of Shimura data $(P, X^+) \longrightarrow (P_0, X_0^+)$; these are called Shimura varieties over a Siegel modular variety. If U_P is the subgroup defined by point 3 of definition 2.4, then the morphism $P \longrightarrow P_0$ factors through the quotient map $\pi : P \longrightarrow P' = P/U_P$. Write $X'^+ = \pi(X^+)$, then (P', X'^+) is again a Shimura datum, and the map $(P, X^+) \longrightarrow (P_0, X_0^+)$ factors through (P', X'^+). For this reason, I first classify all such (P, X^+) such that $U_P = 0$.

4.1 Unipotent extensions of $\mathrm{GSp}_{2g,\mathbf{Q}}$ of weight -1

Our next objective is to classify all mixed Shimura data (P_1, X_1^+) such that $U_{P_1} = 0$ (in other words, $\mathrm{Lie}\, W_{P_1}$ is of pure Hodge weight -1) and $P_1/W_{P_1} \cong \mathrm{GSp}_{2g,\mathbf{Q}}$, with a morphism $(P_1, X_1^+) \longrightarrow (P_0, X_0^+)$ by quotienting out by W_{P_1}, such that X_1^+ maps surjectively to X_0^+. For such a (P_1, X_1^+), we have the following exact sequence:

$$1 \longrightarrow W_{P_1} \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 1$$

Since P_1 is an algebraic group over $\mathbf{Q}$, and $\mathbf{Q}$ has characteristic 0, it admits a Levi decomposition, i.e. there is a subgroup $M \subset P_1$ such that $P_1 \cong M \ltimes W_{P_1}$; see [23] for more details. By quotienting out W_{P_1}, we see that $M \cong P_0$, so we may write $P_1 = P_0 \ltimes W_{P_1}$. Now let $x : \mathbf{S}_{\mathbf{C}} \longrightarrow P_{\mathbf{C}}$ be an element of X_0^+. By point 1 of definition 2.4 we know that x is defined over $\mathbf{R}$. Now let $x_1 \in X^+$ be the composite map $\mathbf{S} \xrightarrow{x} P_0 \longrightarrow P_1$; then x_1 gives a Hodge structure of type $\{(0, -1), (-1, 0)\}$ via the adjoint representation on $\mathrm{Lie}\, W_{P_1}$. For any $\mathbf{R}$-algebra B, the action of $\lambda \in B^\times$ on $\mathrm{Lie}\, W_{P_1}(B)$ coming from the map $\mathbf{G}_{m,\mathbf{R}} \longrightarrow \mathbf{S}$, is multiplication by λ. The following lemma then shows that W_{P_1} is abelian.

Lemma 4.1. *Let V be a connected algebraic group over $\mathbf{R}$, and suppose that there exists a positive integer k and an action of $\mathbf{G}_{m,\mathbf{R}}$ on V through which, for any $\mathbf{R}$-algebra B, any element $\lambda \in B^\times$ acts as λ^k on $\mathrm{Lie}\, V(B)$. Then V is abelian.*

Proof. The given group acts through Lie algebra automorphisms. This means that for every $v, w \in \mathrm{Lie}\, V(B)$ and $\lambda \in B^\times$ the following holds:

$$\begin{aligned} \lambda^k[v, w] &= \varphi(\lambda)[v, w] \\ &= [\varphi(\lambda)v, \varphi(\lambda)w] \\ &= [\lambda^k v, \lambda^k w] \\ &= \lambda^{2k}[v, w]. \end{aligned}$$

This shows that $[v, w] = 0$, so $\mathrm{Lie}\, V$ is abelian; since V is connected, it must be abelian as well. $\square$

This shows us that $W_{P_1,\mathbf{R}}$ is abelian, so W_{P_1} itself must be abelian as well. Since W_{P_1} is unipotent and $\mathbf{Q}$ has characteristic 0, we know that there exists an n such that $W_{P_1} \cong \mathbf{G}_{a,\mathbf{Q}}^n$. Let $\mathrm{GL}(W_{P_1})$ be the linear algebraic group for which the B-points, for any $\mathbf{Q}$-algebra B, is the set

$$\mathrm{GL}(W_{P_1})(B) = \mathrm{Aut}_B(W_{P_1}(B)).$$

Then the semidirect product $P_1 = P_0 \ltimes W_{P_1}$ comes from a map $P_0 \longrightarrow \mathrm{GL}(W_{P_1})$.

Lemma 4.2. *Let V be a finite-dimensional representation of $\mathrm{GSp}_{2g,\mathbf{Q}}$ such that for some $h \in X_0^+$ the induced Hodge structure is of type $\{(-1,0),(0,-1)\}$. Then V is a direct product of copies of the standard representation $\mathbf{G}_{a,\mathbf{Q}}^{2g}$.*

Proof. Let us write $L_0 \cong \mathbf{G}_{a,\mathbf{Q}}^{2g}$ for the standard representation of P_0, and $L(w) \cong \mathbf{G}_{a,\mathbf{Q}}$ for the representation of $\mathbf{G}_{m,\mathbf{Q}}$ with the action of z given by multiplication by z^w. Finally, let L' be the representation $\mathbf{G}_{a,\mathbf{Q}}$ of P_0, with the action given by multiplication by the multiplier character. First we determine all irreducible representations of P_0. Since $P_0(\mathbf{C})$ is simply connected, we simply need to find all irreducible representations of $\mathrm{Lie}\, P_0 = \mathfrak{gsp}_{2g}$. As a Lie algebra, this is equal to $\mathfrak{sp}_{2g} \times \mathbf{G}_{a,\mathbf{Q}}$, where the factor $\mathbf{G}_{a,\mathbf{Q}}$ comes from the scalar matrices. By [15, 9.17] an irreducible representation of $\mathfrak{gsp}_{2g}$ is the tensor product of an irreducible representation of $\mathfrak{sp}_{2g}$ with a one-dimensional representation of $\mathbf{G}_{a,\mathbf{Q}}$. By [15, 17.5] the irreducible representations of $\mathfrak{sp}_{2g}$ are as follows: for nonnegative integers $a_1, a_2, \ldots, a_g$, let $V_{a_1,\ldots,a_g}$ be the subrepresentation of

$$\bigotimes_{k=1}^{g} \mathrm{Sym}^{a_k}(\bigwedge^k L_0)$$

generated as a Lie algebra module by $e_1^{a_1} \otimes (e_1 \wedge e_2)^{a_2} \otimes \ldots \otimes (e_1 \wedge \ldots \wedge e_g)^{a_g}$. Then $V_{a_1,\ldots,a_g}$ is irreducible, and every irreducible representation is of this form.

Now let us look at $\mathbf{C}$-points. The map

$$\begin{array}{rcl} \rho : \mathrm{Sp}_{2g}(\mathbf{C}) \times \mathbf{C} & \longrightarrow & \mathrm{GSp}_{2g}(\mathbf{C}) \\ (A, z) & \mapsto & e^z A \end{array}$$

is surjective, and its kernel is generated by $(e^{\frac{\pi i}{g}}, -\frac{\pi i}{g})$. Hence any irreducible representation of $\mathrm{GSp}_{2g}(\mathbf{C})$ is of the form $V_{a_1,\ldots,a_g} \otimes L(w)$ for some $a_1, \ldots, a_g$ and w, such that the action of $(e^{\frac{\pi i}{g}}, -\frac{\pi i}{g}) \in \mathrm{Sp}_{2g}(\mathbf{C}) \times \mathbf{C}$ is trivial. A straightforward calculation shows that $(e^{\frac{\pi i}{g}}, -\frac{\pi i}{g})$ acts by scalar multiplication by $\exp(\frac{\pi i}{g}(\sum_j j a_j - w))$; hence $w = \sum_j j a_j + 2kg$ for some $k \in \mathbf{Z}$. But then $V_{a_1,\ldots,a_g} \otimes L(w) = V_{a_1,\ldots,a_g}(\mathbf{C}) \otimes L'(\mathbf{C})^{\otimes k}$ as representations of $\mathrm{GSp}_{2g}(\mathbf{C})$; this can be seen because the actions of both $\mathrm{Sp}_{2g}(\mathbf{C})$ and the diagonal matrices are the same. In particular, this means that every irreducible representation of $\mathrm{GSp}_{2g,\mathbf{Q}}$ is of the form $V_{a_1,\ldots,a_g} \otimes L'^{\otimes k}$, and every such representation indeed is irreducible.

Under the map h_0 from section 2.4 the real vector space $L(\mathbf{R})$ gets a Hodge structure of type $\{(0,-1),(-1,0)\}$; a basis for $L(\mathbf{C})^{-1,0}$ is $\{e_j + ie_{g+j} : 1 \leq j \leq g\}$. Since the matrix

$\begin{pmatrix} 1_g & 0_g \\ i_g & 1_g \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbf{C})$ maps e_j to $v_j = e_j + ie_{g+j}$, we see that in the induced Hodge structure on $V_{a_1,\dots,a_g}(\mathbf{R})$, the vector $v_1^{a_1} \otimes \dots \otimes (v_1 \wedge \dots \wedge v_g)^{a_g}$ is an element of $V_{a_1,\dots,a_g}^{(-\sum_j ja_j,0)}$; hence $V_{a_1,\dots,a_g}(\mathbf{R}) \otimes L'(\mathbf{R})^{\otimes k}$ has a Hodge weight $(-2k - \sum_j ja_j, -2k)$. However, the difference between the two terms must be 1, so $\sum_j ja_j = 1$; hence $a_1 = 1$ and $a_j = 0$ for all $j > 1$. This also implies that $k = 0$, so $V_{a_1,\dots,a_g} \otimes L'^{\otimes k} = L_0$, as was to be shown. Since every representation is a direct sum of irreducible representations, this shows that any representations is a direct sum of copies of L_0. □

From this lemma it follows that W_{P_1} is the direct sum of a number of copies of the standard representation. Let us now write $P_1 = P_0 \ltimes W_{P_1}$. Let h_1 be the composite map

$$\mathbf{S} \xrightarrow{h_0} P_{0,\mathbf{R}} \longrightarrow P_{1,\mathbf{R}},$$

and let X_1 be the orbit of h_1 in $\mathrm{Hom}(\mathbf{S}, P_{1,\mathbf{R}})$ under the action of $P_1(\mathbf{R})$. Let X_1^+ be the connected component of X_1 containing h_1; then (P_1, X_1^+) is a connected mixed Shimura datum, as we can check for $h_1 \in X_1^+$:

1. Again by definition we know that $h_{1,\mathbf{C}} : \mathbf{S}_\mathbf{C} \longrightarrow P_{1,\mathbf{C}}$ is defined over $\mathbf{R}$.
2. Here $\mathrm{Lie}\, P_1(\mathbf{R}) = \mathfrak{gsp}_{2g}(\mathbf{R}) \times \mathbf{R}^{2gn}$ as vector spaces, and the action of $A \in h_0(\mathbf{C}^\times)$ is given by conjugation on $\mathfrak{gsp}_{2g}(\mathbf{R})$ and left multiplication on the copies $\mathbf{R}^{2gn}$, which gives us the desired Hodge structure type, i.e. $\mathfrak{gsp}_{2g}(\mathbf{R})$ has weights $\{(-1,1),(0,0)(1,-1)\}$ and $\mathbf{R}^{2n}$ has weights $\{(-1,0),(0,-1)\}$.
3. We see that $\mathrm{Lie}\, W_{P_1} = \mathbf{G}_{a,\mathbf{Q}}^{2gn} = W_{-1}(\mathrm{Lie}\, P)$, as was to be shown.
4. Since $P_1/W_{P_1} = P_0$, this was already shown in section 2.4.
5. Since $P_1^{\mathrm{der}} = \mathrm{Sp}_{2g,\mathbf{Q}} \ltimes \mathbf{G}_{a,\mathbf{Q}}^{2gn}$, we get that $P_1/P_1^{\mathrm{der}} = P_0/P_0^{\mathrm{der}} = \mathbf{G}_{m,\mathbf{Q}}$.
6. Using the composite map $h_0 : \mathbf{S} \longrightarrow P_{1,\mathbf{R}} \longrightarrow P_{0,\mathbf{R}}$, we see that any normal subgroup $P' \subset P_1$ for which $P'_\mathbf{R}$ contains $h_0(\mathbf{S})$ must map surjectively to P_0, so it is of the form $P' = P_0 \ltimes V$ for some linear subspace $V \subset W_{P_1}$ stable under the P_0-action. Let us prove that this inclusion is actual an equality. Writing $W_{P_1} = \mathbf{Q}^n \otimes_\mathbf{Q} \mathbf{G}_{a,\mathbf{Q}}^{2g}$ as P_0-modules, with the group acting on the right half of the product, one sees that $V = V' \otimes_{\mathbf{G}_{a,\mathbf{Q}}} \mathbf{G}_{a,\mathbf{Q}}^{2g}$ for some $\mathbf{Q}$-linear subspace $V' \subset \mathbf{Q}$. Now the image of P' in $G = P_0 \ltimes (W'/V' \otimes \mathbf{G}_{a,\mathbf{Q}}^{2g})$ is normal in G. This image is $P_0 \ltimes 0$, but this is normal only if $V' = W'$, which was to be proven.

As a set, $\mathcal{H}_g \times W_{P_1}(\mathbf{R}) \cong X_1^+$, with the map given by $(\tau, v) \mapsto \mathrm{Inn}(v) \circ h_\tau$. The action of $P_1(\mathbf{R})^+$ is then given by $(A, v)(\tau, v') = ({}^A\tau, Av' + v)$, where ${}^A\tau$ denotes the usual action of $\mathrm{GSp}_{2g}(\mathbf{R})^+$ on $\mathcal{H}_g$.

As a $\mathbf{Z}$-model for P_1 we take $\mathrm{GSp}_{2g} \ltimes \mathbf{G}_a^{2n}$; a congruence subgroup G_1 of $P_1(\mathbf{Z})$ in the classical sense of the word[7] is then a subgroup of $G \times \Lambda \subset \mathrm{Sp}_{2g}(\mathbf{Z}) \ltimes \mathbf{Z}^{2gn}$, where G is a congruence

[7] i.e. it contains the kernel of the map $\mathrm{Sp}_{2g}(\mathbf{Z}) \ltimes \mathbf{Z}^{2gn} \longrightarrow \mathrm{Sp}_{2g}(\mathbf{Z}/N\mathbf{Z}) \ltimes (\mathbf{Z}/N\mathbf{Z})^{2gn}$ for some $N \in \mathbf{Z}_{\geq 1}$.

subgroup of $\mathrm{Sp}_{2g}(\mathbf{Z})$, and Λ a maximal sublattice of $\mathbf{Z}^2$ fixed by G. For G sufficiently small, we get a connected mixed Shimura variety $S_1 = G_1 \setminus X_1^+$, that comes with a surjective morphism of connected mixed Shimura varieties $S_1 \longrightarrow G \setminus X_0^+$. The result of this section is the following theorem.

Theorem 4.3. *Let S be a connected mixed Shimura variety associated to (P, X^+, G), with unipotent radical $W_P \subset P$, such that $U_P = 0$ (in point 3 of definition 2.4), $P/W_P \cong \mathrm{GSp}_{2g,\mathbf{Q}}$, and the morphism $P \longrightarrow \mathrm{GL}_{2,\mathbf{Q}}$ induces a morphism of connected mixed Shimura data $(P, X^+) \longrightarrow (\mathrm{GSp}_{2g,\mathbf{Q}}, \mathcal{H}_g)$. Then there is an n such that $W_P \cong \mathbf{G}_{a,\mathbf{Q}}^{2gn}$, and $P = \mathrm{GSp}_{2g,\mathbf{Q}} \ltimes W_P$, $X^+ \cong \mathcal{H}_g \times \mathbf{R}^{2gn}$. Conversely, every such (P, X^+, G') defines a connected mixed Shimura variety S.*

4.2 General unipotent extensions of $\mathrm{GSp}_{2g,\mathbf{Q}}$

Again we take P_1, X_1^+ as above. Let us now look for extensions $f : P_2 \longrightarrow P_1$ by some U such that, for $W' = f^{-1}(W_{P_1})$, we have a connected mixed Shimura datum (P_2, X_2^+) with unipotent radical W' and such that the Hodge structure on $\mathrm{Lie}\, U$ induced by any $h \in X_2^+$ is of type $\{(-1,-1)\}$. This means that for any $\mathbf{R}$-algebra B, any $\lambda \in B^\times$ acts on $\mathrm{Lie}\, U(B)$ as λ^2, so by Lemma 4.1 $U \cong \mathbf{G}_{a,\mathbf{Q}}^k$ for some k. Furthermore, the Levi decomposition of P_2 is then $P_2 \cong P_0 \ltimes W$'. Since U is normal in P_2 there is an action of the subgroup P_0 of P_2 on U. for any $h \in X_0^+$, we see that any $h(z)$ acts as $z\bar{z} = d(h(z))$ on U. As the images of the different h generate P_0, the action of P_0 on U must be by multiplication through d.

The possible extensions are given in lemma 4.6. As in section 2.4, let $d : \mathrm{GSp}_{2g,\mathbf{Q}} \longrightarrow \mathbf{G}_{m,\mathbf{Q}}$ be the morphism of linear algebraic groups defined by $A^T J A = d(A) J$, where $J = \begin{pmatrix} 0 & \mathrm{id}_g \\ -\mathrm{id}_g & 0 \end{pmatrix}$, and $A \in \mathrm{GSp}_{2g}(B)$ for some $\mathbf{Q}$-algebra B.

Lemma 4.4. *Let V, H, I be rational vector spaces, regarded as algebraic groups over $\mathbf{Q}$. Let P be a linear algebraic subgroup of $\mathbf{GL}(V)$, and let $\rho : P \longrightarrow \mathbf{GL}(H)$ and $\sigma : P \longrightarrow \mathbf{GL}(I)$ be representations of P. Furthermore, let $\beta : H \times H \longrightarrow I$ be a bilinear map under which the P-action is invariant. Define the following map:*

$$\begin{array}{rcl} \tilde{\beta} : P \times H & \longrightarrow & \mathrm{H}^\vee \otimes I \\ (A, w) & \mapsto & (w' \mapsto \beta(w, Aw')) \end{array}$$

Then the morphism of varieties

$$\begin{array}{rcl} \varphi : P \times H \times I & \longrightarrow & \mathbf{GL}(V \oplus H \oplus I \oplus \mathbf{G}_{a,\mathbf{Q}}) \\ (A, w, u) & \mapsto & \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & \sigma(A) & \tilde{\beta}(A, w) & u \\ 0 & 0 & \rho(A) & w \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

is injective. Furthermore, let $\iota : I \longrightarrow P \times H \times I : u \mapsto (1, 0, u)$ *and* $\pi : P \times H \times I \longrightarrow P \ltimes H$ *denote the inclusion and projection map, respectively. Then the sequence*

$$1 \longrightarrow I \stackrel{\varphi\circ\iota}{\longrightarrow} \varphi(P \times H \times I) \stackrel{\pi\circ\varphi^{-1}}{\longrightarrow} P \ltimes H \longrightarrow 1$$

is a short exact sequence of linear algebraic groups over $\mathbf{Q}$*.*

I omit the proof, which is a straightforward verification. An alternative way to describe this group law on $P \times H \times I$ is by the formula

$$(A, w, u)(A', w', u') = (AA', Aw' + w, u' + u + \beta(w, Aw')).$$

Note that the matrices of the form $(1, w, u)$ for $w \in H$ and $u \in I$ form a normal subgroup of this group. Now, if we take $P = P_0$, $H = W_{P_1}$ and $I = U$, we get the following corollary.

Corollary 4.5. *Let* P_1 *and* W' *be as above, and let* $U = \mathbf{G}_{a,\mathbf{Q}}^k$ *for some integer* k*. Let* $\beta : W_{P_1} \times W_{P_1} \longrightarrow U$ *be a morphism of linear algebraic groups over* $\mathbf{Q}$ *such that for every* $\mathbf{Q}$*-algebra* B*, the induced map* $W_{P_1}(B) \times W_{P_1}(B) \longrightarrow U(B)$ *is* B*-bilinear, and for every* $A \in P_0(B)$ *and* $v, w \in W_{P_1}(B)$ *one has that* $\beta(Av, Aw) = d(A)\beta(Av, Aw)$*. Then for every* $\mathbf{Q}$*-algebra* B*, the set*

$$P_{2,\beta}(B) = P_0(B) \times W_{P_1}(B) \times U(B)$$

has a group structure given by

$$(A, w, u)(A', w', u') = (AA', Aw' + w, d(A)u' + u + \beta(w, Aw')).$$

This defines a linear algebraic group $P_{2,\beta}$*, which is an extension of* P_1 *by* U*. Furthermore, if* $X_{2,\beta}^+ = X_1^+ \times U(\mathbf{C})$*, then* $(P_{2,\beta}, X_{2,\beta}^+)$ *is a connected mixed Shimura datum via the identification*

$$\begin{aligned} X_0^+ \times W(\mathbf{R}) \times U(\mathbf{C}) &\longrightarrow \operatorname{Hom}(\mathbf{S}_{\mathbf{C}}, P_{2,\mathbf{C}}) \\ (h, w, u) &\mapsto \operatorname{Inn}(1, w, u) \circ h. \end{aligned}$$

For any $x \in X_{2,\beta}^+$*, the induced Hodge weight filtration on* $\operatorname{Lie} P_{2,\beta}$ *is given by*

$$W_n(\operatorname{Lie} P_{2,\beta}) = \begin{cases} 0, & \text{if } n < -2; \\ \operatorname{Lie} U, & \text{if } n = -2; \\ \operatorname{Lie} W', & \text{if } n = -1; \\ \operatorname{Lie} P_{2,\beta}, & \text{if } n \geq 0. \end{cases}$$

As the following lemma shows every 'suitable' extension of P_1 is of this form.

Lemma 4.6. *Let* P_1 *and* W' *be as above, and let* $U = \mathbf{G}_{a,\mathbf{Q}}^k$*. Let* P_2 *be an extension of* P_1 *by* U *such that* P_2 *is part of a connected mixed Shimura datum* (P_2, X_2^+) *such that, for some* $x \in X_2^+$*, the induced Hodge weight structure on* $\operatorname{Lie} P_2$ *is given by*

$$W_n(\operatorname{Lie} P_2) = \begin{cases} 0, & \text{if } n < -2; \\ \operatorname{Lie} U, & \text{if } n = -2; \\ \operatorname{Lie} W', & \text{if } n = -1; \\ \operatorname{Lie} P_2, & \text{if } n \geq 0. \end{cases}$$

Then $P_2 = P_{2,\beta}$ for a unique $\beta : W_{P_1} \times W_{P_1} \longrightarrow U$ such that for every $\mathbf{Q}$-algebra B, the induced map $W_{P_1}(B) \times W_{P_1}(B) \longrightarrow U(B)$ is B-bilinear, and for every $A \in P_0(B)$ and $v, w \in W_{P_1}(B)$ one has that $\beta(Av, Aw) = d(A)\beta(Av, Aw)$.

Proof. Let us write $W = W_{P_1}$. The short exact sequence of algebraic groups over $\mathbf{Q}$

$$1 \longrightarrow U \longrightarrow W' \longrightarrow W \longrightarrow 1$$

induces a short exact sequence of Lie algebras

$$0 \longrightarrow \operatorname{Lie} U \longrightarrow \operatorname{Lie} W' \longrightarrow \operatorname{Lie} W \longrightarrow 0.$$

Now take a morphism $x \in X_2^+$. As $x : \mathbf{S_C} \longrightarrow (P_2/U)_{\mathbf{C}}$ is defined over $\mathbf{R}$, so is the composite morphism

$$\mathbf{S_C} \longrightarrow (P_2/U)_{\mathbf{C}} \longrightarrow P_{0,\mathbf{C}} \longrightarrow P_{2,\mathbf{C}} \longrightarrow \mathrm{GL}(\operatorname{Lie} W')_{\mathbf{C}};$$

now consider the weight structure on $\operatorname{Lie} W'_{\mathbf{R}}$ given by the map $\varphi : \mathbf{S} \longrightarrow \mathrm{GL}(\operatorname{Lie} W'_{\mathbf{R}})$. The Lie bracket is an isomorphism of Hodge structures

$$\operatorname{Lie} W'_{\mathbf{R}} \otimes \operatorname{Lie} W'_{\mathbf{R}} \longrightarrow \operatorname{Lie} W'_{\mathbf{R}}.$$

By looking at weights, we find that $[\operatorname{Lie} W_{\mathbf{R}}, \operatorname{Lie} U_{\mathbf{R}}] = [\operatorname{Lie} U_{\mathbf{R}}, \operatorname{Lie} U_{\mathbf{R}}] = 0$ and $[\operatorname{Lie} W_{\mathbf{R}}, \operatorname{Lie} W_{\mathbf{R}}] \subset \operatorname{Lie} U_{\mathbf{R}}$. In terms of the algebraic groups themselves, this means that $U_{\mathbf{R}}$ is abelian, elements of $U_{\mathbf{R}}$ and $W_{\mathbf{R}}$ commute, and for two $w, w' \in W_{\mathbf{R}}$, one has that $(w, 0) \cdot (w', 0) = (w + w' + \beta_{\mathbf{R}}(w, w'))$ for some bilinear map $\beta_{\mathbf{R}} : W_{\mathbf{R}} \times W_{\mathbf{R}} \longrightarrow U_{\mathbf{R}}$. The group law on $W'_{\mathbf{R}}$ in general is then given by $(w, u)(w', u') = (w + w', u + u' + \beta_{\mathbf{R}}(w, w'))$. Since this must be defined over $\mathbf{Q}$, this implies that $\beta_{\mathbf{R}}$ comes from a bilinear form $\beta : W \times W \longrightarrow U$.

In the same manner we may write, as algebraic varieties, $P_2 \cong \mathrm{GSp}_{2g,\mathbf{Q}} \times W \times U$. We know that P_2 is isomorphic to the semidirect product $P_2 = \mathrm{GSp}_{2,\mathbf{Q}} \ltimes W'$ (this is again the Levi decomposition). The action of $\mathrm{GSp}_{2,\mathbf{Q}}$ is given by matrix multiplication on W. For any map $x : \mathbf{S} \longrightarrow P_{2,\mathbf{R}}$ and for any $\mathbf{R}$-algebra B, the action of $z = a + bi \in \mathbf{S}(B)$ on $U(B)$ is multiplication by $a^2 + b^2 = d(x(z))$. Since $\mathrm{GSp}_{2,\mathbf{R}}$ is is generated by the images of the various x, we see that the action of some $A \in \mathrm{GSp}_{2g}(B)$ on U is just given by multiplication by $d(B)$. Using this, and the group law we have on W', we can determine the group law of P_2:

$$\begin{aligned}
(A, w, u) \cdot (A', w', u') &= (1, w, u) \cdot (A, 0, 0) \cdot (1, w', u') \cdot (A', 0, 0) \\
&= (1, w, u) \cdot {}^{A}(1, w', u') \cdot (A, 0, 0) \cdot (A', 0, 0) \\
&= (1, w, u) \cdot (1, Aw', d(A)u') \cdot (AA', 0, 0) \\
&= (AA', Aw' + w, d(A)u' + u + \beta(w, Aw')).
\end{aligned}$$

It is easy to see that $(1, 0, 0)$ is a unit element and that

$$(A, w, u)^{-1} = \left(A^{-1}, -A^{-1}w, \frac{1}{d(A)}(\beta(w, w) - u) \right).$$

If this is indeed to be a group law, it has to be associative, so for every $\mathbf{Q}$-algebra B and for every $A \in P_0(B)$ and $v, w \in W(B)$ the following must hold:

$$\begin{aligned} (A, Aw + Av, \beta(Av, Aw)) &= (A, Av, 0)(1, w', 0) \\ &= ((A, 0, 0)(1, v, 0))(1, w', 0) \\ &= (A, 0, 0)((1, v, 0)(1, w', 0)) \\ &= (A, 0, 0)(1, v + w, 0) \\ &= (A, Aw + Av, d(A)\beta(v, w)), \end{aligned}$$

Which is easily seen to be true if and only if $\beta(Av, Aw) = d(A)\beta(v, w)$ for all A, v, w. As every choice of β gives a different group law on the variety $P_0 \times W \times U$, the actual β corresponding to P_2 is unique. □

Now take a P_2 satisfying the conditions of Lemma 4.6. By taking a connected component X_2^+ of the orbit of the map $h_2 : \mathbf{S_C} \longrightarrow P_{2,\mathbf{C}} : z \mapsto (h_0(z), 0, 0)$ under the action of $P_2(\mathbf{R}) \cdot U(\mathbf{C})$, one can show that (P_2, X_2^+) is a connected mixed Shimura datum, where X_2^+ can be identified with $\mathcal{H} \times \mathbf{R}^{2gn} \times \mathbf{C}^k$ under the action given by $(A, v, z)(\tau, v', z') = (A \cdot \tau, Av' + v, d(A)z' + \beta(v, Av') + z)$. The complex structure above a point $\tau \in \mathcal{H}_g$ is given by identifying $\mathbf{R}^{2g}$ with $\tau\mathbf{R}^g + \mathbf{R}^g$.

A congruence subgroup G_2 of P_2 is a subgroup of $P_2(\mathbf{Z})^+$, which we may write, as a set, as $G_2 = G \times \Lambda \times \Delta \subset \mathrm{Sp}_{2g}(\mathbf{Z}) \times \mathbf{Z}^{2gn} \times \mathbf{Z}^k$. Here G is a congruence subgroup of $\mathrm{Sp}_{2g}(\mathbf{Z})$, Λ is a lattice in $\mathbf{Z}^{2n}$ fixed by G, and Δ is a lattice in $\mathbf{Z}^k$ such that $\beta(\Lambda, \Lambda) \subset \Delta$. If G is sufficiently small this induces a connected mixed Shimura variety $S_2 = \Gamma_2 \setminus X_2^+$, which comes with a surjective morphism of connected mixed Shimura varieties $S_2 \longrightarrow (G \ltimes \Lambda) \setminus X_1^+$. Again, I state the result of this section in the following theorem. Note that it is a strengthening of a special case of [31, 2.16]

Theorem 4.7. *Let S be a connected mixed Shimura variety associated to (P, X^+, G), with unipotent radical $W_P \subset P$, such that $P/W_P \cong \mathrm{GSp}_{2g,\mathbf{Q}}$ and the morphism $P \longrightarrow \mathrm{GSp}_{2g,\mathbf{Q}}$ induces a morphism of connected mixed Shimura data $(P, X^+) \longrightarrow (\mathrm{GSp}_{2g,\mathbf{Q}}, \mathcal{H}_g)$. Then there exist integers n, k such that $U_P \cong \mathbf{G}_{a,\mathbf{Q}}^k$, $W_P/U_P \cong \mathbf{G}_{a,\mathbf{Q}}^{2gn}$, and $P/U_P = \mathrm{GSp}_{2g,\mathbf{Q}} \ltimes (W_P/U_P)$. For any $\mathbf{Q}$-algebra B, the group $P(B)$ is as a set equal to $\mathrm{GSp}_{2g}(B) \times B^{2gn} \times B^k$, and the group law is given by*

$$(A, w, u) \cdot (A', w', u') = (AA', Aw' + w, d(A)u' + u + \beta(w, Aw')),$$

for some bilinear map $\beta : \mathbf{G}_{a,\mathbf{Q}}^{2gn} \times \mathbf{G}_{a,\mathbf{Q}}^{2gn} \longrightarrow U_P$ satisfying $\beta(Aw, Aw') = d(A)\beta(w, w')$ for all A, w, w'. Furthermore, $X^+ \cong \mathcal{H}_g \times \mathbf{R}^{2gn} \times \mathbf{C}^k$, and G is a congruence subgroup of $P(\mathbf{Q})^+$. Conversely, every such (P, X^+, G) defines a connected mixed Shimura variety S.

5 Special subvarieties of extensions of Siegel modular varieties

In this chapter we classify the special subvarieties of the connected mixed Shimura varieties described in the last chapter. Hence we take $P_0 = \mathrm{GSp}_{2g,\mathbf{Q}}$ and $X_0^+ = \mathcal{H}_g$. Furthermore, let $G_0 \subset P_0(\mathbf{Z})$ be a congruence subgroup; then we consider the Shimura varieties $S_0 = G_0 \setminus X_0^+$.

For a subgroup Q of P_0, let Ξ_Q be the subset of morphisms $\mathbf{S}_\mathbf{C} \longrightarrow P_{0,\mathbf{C}}$ in X_0^+ factoring through $Q_\mathbf{C}$. By Lemma 2.11, a special subvariety arises from an injective morphism of connected mixed Shimura data $(Q, Y^+) \longrightarrow (P_0, X_0^+)$, so we may regard Q as a linear algebraic subgroup of P_0, and Y^+ as a connected component of Ξ_Q. However, not every subgroup of P_0 is part of a connected mixed Shimura datum (Q, Y^+) for some Y^+; let us call those that do *special*. Although I will not give an explicit classification of special subgroups of P_0, we will discuss specific examples of these connected mixed Shimura subdata in chapter 6. Suppose Q is a special subgroup of P_0; then $(Q, Y^+, G_0 \cap Q(\mathbf{Q}))$ defines a special subvariety Σ_{Q,Y^+} of S_0.

5.1 Special subvarieties of S_1

I keep the notation from section 4.1, so I consider the connected mixed Shimura datum (P_1, X_1^+), where P_1 is the linear algebraic group $P_0 \ltimes W$, where $W = W_{P_1}$ is a product of n distinct copies of the standard P_0-module $\mathbf{G}_{a,\mathbf{Q}}^{2g}$. As before, we can identify X_1^+ with $\mathcal{H}_g \times \mathbf{R}^{2gn}$. For G a congruence subgroup of P_0 and Λ a maximal sublattice of $\mathbf{Z}^{2gn}$ fixed by G, I consider the Shimura variety S_1 corresponding to $(P_1, X_1^+, G \ltimes \Lambda)$. The following proposition classifies the subgroups Q of P_1 for which there exists a connected subset $Y^+ \subset \mathrm{Hom}(\mathbf{S}_\mathbf{C}, Q) \cap X_1^+$ for which $(Q, Y^+, P(\mathbf{Z}) \cap G_1)$ defines a Shimura subvariety of S_1. In order to do this, we first need the following lemma from group cohomology.

Lemma 5.1. *Let k be a field of characteristic 0, B a subgroup of $\mathrm{GL}_n(k)$ containing the scalars, and let V' be a k-vector space that is a B-module, on which the scalars act by multiplication. Then $\mathrm{H}^1(B, V') = 0$.*

Proof. Let $s : B \longrightarrow V'$ be a cocycle. Then for all $A \in B$ one has that

$$2s(A) + s(2) = s(2A) = As(2) + s(A),$$

so $s(A) = (A-1)s(2)$, so s is a coboundary. □

Proposition 5.2. *Let $Q \subset P_1 = P_0 \ltimes W$ be a special algebraic subgroup. Then up to conjugation Q is of the form $Q' \ltimes W'$, where Q' is a special subgroup of P_0, and W' is a sub-Q'-module of W, and the section implicitly induced by the semidirect product is a restriction of the section $P_0 \longrightarrow P_1$.*

Proof. The image of the special subvariety Σ_Q in S_0 is again a special subvariety, so the image Q' of Q in P_0 is a special subgroup. Let Q^{red} be the reductive part of Q in the Levi decomposition. Then the map $Q \longrightarrow P_0$ factors through Q^{red}. On the other hand, since Q is a subgroup of P_1,

Q^{red} can be considered as a subgroup of P_0, so $Q' = Q^{\text{red}}$. Now let $W' \subset W$ be the kernel of the map $Q \longrightarrow P_0$; it is the unipotent radical of Q. Furthermore, by definition Q is a subgroup of $Q' \ltimes W$, where the semidirect product is induced from the semidirect product of $P_1 = P_0 \ltimes W$. We therefore have a morphism $Q \longrightarrow Q' \ltimes W/W'$ that respects the morphisms to Q'. The image of Q in $Q' \ltimes W/W'$ is isomorphic to Q', so it comes from a cocycle $Q' \longrightarrow W/W'$. By lemma 5.1 this is a coboundary, so the image of Q is up to conjugation equal to $Q' \ltimes 0$. Therefore Q is up to conjugation contained in $Q' \ltimes W'$, where the semidirect product is induced by the semidirect product $P_1 = P_0 \ltimes W$. Since the dimensions of the unipotent radical have to be the same, we find that Q must actually equal $Q' \ltimes W'$ up to conjugation, as was to be shown. □

From this classification of subgroups I can now derive a classification of the special subvarieties of S_1. In order to keep the classification simple, I list the subvarieties only up to Hecke correspondence, which is justified by proposition 2.13.

Proposition 5.3. *Let Z be a special subvariety of S_1. Then up to Hecke correspondence Z is of the form $\Sigma_{Q' \ltimes W', Y^+}$ for some special $Q' \subset P_0$ and some sub-Q'-module W' of W, and some connected component Y^+ of* $\mathrm{Hom}(\mathbf{S}_{\mathbf{C}}, Q' \ltimes W') \cap X_1^+$.

Proof. Suppose Z corresponds to a connected mixed Shimura datum (Q, Y^+). As we have seen above, we know that Q is a subgroup of P_0 that is a conjugate of a subgroup of the form $Q' \ltimes W'$. Without loss of generality, we may assume that this conjugations is by some element $w \in W'(\mathbf{Q})$. If $\pi : P_1 \longrightarrow P_0$ is the projection map, and $Y'^+ = \pi \circ Y^+$, one now has that $Y^+ = Y'^+ \times (W'(\mathbf{R}) + w) \subset \mathcal{H}_g \times W(\mathbf{R}) = X_1^+$.

Now let $G' \subset Q(\mathbf{Z}) \cap G_1$ be small enough, so that the inverse image of Z under the map $G' \setminus Y^+ \longrightarrow G_1 \setminus X_1^+ = S_1$ is of the form Σ_{Q,Y^+}; we may assume without loss of generality that $G' = \Gamma \ltimes \Lambda$, where Γ is a congruence subgroup of $Q'(\mathbf{Z})$, and Λ is a lattice in W'. Now let $m \in \mathbf{Z}_{>1}$ be such that $mw \in \Lambda$, and consider the Hecke correspondence

$$(\Gamma \ltimes \frac{1}{m}\Lambda) \setminus X^+ \xleftarrow{\varphi'} G' \setminus X^+ \xrightarrow{\varphi} (G \cap Q(\mathbf{Z})) \setminus X^+.$$

Then Z is an irreducible component of $\varphi(\varphi'^{-1}(\Sigma_{Q,Y^+}))$, as was to be shown. □

5.2 Special subvarieties of S_2

In this section, I classify the special subvarieties of S_2, keeping the notation from section 4.2. Thus we let $P_0, P_1, W, X_1^+, G, \Lambda$ be as above, and we let $U = \mathbf{G}_{a,\mathbf{Q}}^k$ for some integer k, and $\beta : W \times W \longrightarrow U$ a bilinear map so that $\beta(Av, Aw) = d(A)\beta(v, w)$ for all $A \in P_0$ and $v, w \in W$. We then let P_2 be the extension of P_1 by U as in corollary 4.5, and $X_2^+ = X_1^+ \times \mathbf{C}^k$. Choose some lattice $\Delta \subset \mathbf{Z}^k$ so that $\beta(\Lambda, \Lambda) \subset \Delta$; then S_2 is the connected mixed Shimura variety $G' \setminus X_2^+$, where G' is the congruence subgroup of P_0 corresponding to the set $G \times \Lambda \times \Delta$. Again, I start with classifying the special algebraic subgroups of P_2.

Lemma 5.4. *Let $Q \subset P_2$ be a special algebraic subgroup. Then up to conjugation Q is as a variety of the form $Q' \times W' \times U'$, where Q' is a special subgroup of P_0, W' is a sub-P'-module of W, and U' is a linear subspace of U such that the induced bilinear form $W' \times W' \longrightarrow U/U'$ is symmetric.*

Proof. Look at the kernel of $Q \longrightarrow P_1$; this will be a linear subspace U' of U. Now Q/U' is the image of a section $Q' \ltimes W' \longrightarrow P_2/U'$. We may assume that the image of Q' in P_1 equals $Q' \ltimes W'$ for some special subgroup Q' of P_0 and a sub-Q'-module W' of W, as any other special subgroup may be obtained by conjugating. Denote this section by $(A, v) \mapsto (A, v, z_{A,v})$; then one has $z_{AA',Av'+v} = d(A)z_{A',v'} + \beta(v, Av') + z_{A,v}$. In particular one has $z_{1,v} + z_{1,v'} + \beta(v', v) = z_{1,v+v} = z_{1,v} + z_{1,v'} + \beta(v, v')$, which shows that β must be symmetric on $W' \times W'$. If this is the case, then the map given by $z_{A,v} = \frac{1}{2}\beta(v, v)$ gives us a section. Any other section can be obtained from this one by conjugating. □

Again, the classification of special subgroups allows us to find the special subvarieties of S_2.

Theorem 5.5. *Let Z be a special subvariety of S_2. Then up to Hecke correspondence Z is of the form $\Sigma_{(P' \ltimes W') \ltimes U'), Y^+}$ for some P', W', U' such that P' is a special subgroup of P_0, W' is a sub-P'-module of W, and U' is a linear subspace of U such that the induced bilinear form $W' \times W' \longrightarrow U'$ is symmetric, and some connected component Y^+ of $X_2^+ \cap \mathrm{Hom}(\mathbf{S_C}, (P' \ltimes W') \ltimes U')$.*

Proof. The proof of this proposition is analogous to that of Proposition 5.3. □

6 Special subvarieties of modular curves and universal elliptic curves

This section is devoted to a specific example of the connected mixed Shimura varieties S_0, S_1 and S_2 as described above. First, let us take $g = 1$, and write $P_0 = \mathrm{GSp}_{2,\mathbf{Q}} = \mathrm{GL}_{2,\mathbf{Q}}$ and $X_0^+ = \mathcal{H}_1$. For a suitable choice of the congruence subgroup $G_0 \subset \mathrm{Sp}_2(\mathbf{Z}) = \mathrm{SL}_2(\mathbf{Z})$, the curve $S_0 = G_0 \setminus X_0^+$ is a moduli space for elliptic curves over $\mathbf{C}$ with some level structure; this means in this case that there is a 'natural' one-to-one correspondence between $\mathbf{C}$-points of $G_0 \setminus X_0^+$ and elliptic curves over $\mathbf{C}$ with a point of given order. This notion of naturality will be properly defined in the next chapter. A point $\tau \in X_0^+ \subset \mathbf{C}$ corresponds to the map $h_\tau : \mathbf{S} \longrightarrow \mathrm{GL}_2(\mathbf{R})$ obtained by taking the $\mathbf{R}$-basis $(\tau, 1)$ for $\mathbf{C}$. By a slight abuse of notation, I will still write τ for the image of $\tau \in X_0^+$ in S_0.

As $g = 1$, the variety S_0 is a one-dimensional quasiprojective complex variety, so a special subvariety of S_0 is either all of S_0 or a special point. The latter are classified by the following proposition.

Proposition 6.1. *Let $\tau \in \mathcal{H}_1 \subset \mathbf{C}$ be a point. Let E_τ be the elliptic curve $\mathbf{C}/(\mathbf{Z} \cdot \tau + \mathbf{Z})$. Then the image of τ in S_0 is special if and only if E_τ has complex multiplication.*

Proof. Suppose E_τ has complex multiplication; then $\mathbf{Q}(\tau)$ is an imaginary quadratic extension of $\mathbf{Q}$, so $\mathbf{Q}\tau + \mathbf{Q}$ is closed under multiplication by τ. By choosing $(\tau, 1)$ as a basis for $\mathbf{Q}(\tau)$, we get an injective morphism of algebraic groups $f : \mathrm{Res}_{\mathbf{Q}(\tau)/\mathbf{Q}} \mathbf{G}_{m,\mathbf{Q}(\tau)} \longrightarrow \mathrm{GL}_{2,\mathbf{Q}} = \mathrm{GSp}_{2,\mathbf{Q}}$; let T_τ be its image. The base-change of f to $\mathbf{R}$ is the morphism $h_\tau : \mathbf{S} \longrightarrow \mathrm{GSp}_{2,\mathbf{R}}$. It is easy to verify that $(T_\tau, \{\tau\})$ is a connected mixed Shimura subdatum, defining the special subvariety $\tau \in S_0$.

Conversely, if τ in S_0 is special, then the image $h_\tau(\mathbf{S})$ has to be contained in an algebraic subgroup of $P_{0,\mathbf{R}}$ defined over $\mathbf{Q}$. Let T_τ be the smallest of these algebraic subgroups defined over $\mathbf{Q}$. Then, as T_τ fixes τ, this subgroup must be commutative. However, $h_\tau(\mathbf{S})$ is a maximal abelian subgroup of $\mathrm{GL}_{2,\mathbf{R}}$, so $h_\tau(\mathbf{S}) = T_{\tau,\mathbf{R}}$. Now $T_\tau(\mathbf{Q})$ is a subgroup of $\mathbf{C}^\times$ that fixes the set $\mathbf{Q}\tau + \mathbf{Q}$. Now suppose that $T_\tau(\mathbf{Q})$ is contained in $\mathbf{R}^\times \subset \mathbf{C}^\times$; then $T_\tau(\mathbf{R})$ is contained in $\mathbf{R}^\times$. However, we know that $T_\tau(\mathbf{R}) = \mathbf{C}^\times$, which is a contradiction. Therefore there is a $z \in \mathbf{C} \setminus \mathbf{R}$ fixing $\mathbf{Q}\tau + \mathbf{Q}$. By multiplying z with a sufficiently large integer we may assume that z fixes $\mathbf{Z}\tau + \mathbf{Z}$; but then z is an endomorphism of E_τ that is not in $\mathbf{Z}$, which shows that E_τ has complex multiplication. □

6.1 Universal elliptic curves and their dual

Fix a suitably small congruence subgroup $G_0 \subset \mathrm{SL}_2(\mathbf{Z})$, and let P_0, X_0^+, S_0 be as in the preceding paragraph. Now I take $P_1 = P_0 \ltimes \mathbf{G}_{a,\mathbf{Q}}^2$. As was discussed in section 4.1, this gives rise to a connected mixed Shimura datum (P_1, X_1^+), where X_1^+ is the complex manifold $X_1^+ = \mathcal{H}_1 \times \mathbf{R}^2$, where the complex structure above a $\tau \in \mathcal{H}_1$ is determined by regarding $\mathbf{R}^2$ as $\mathbf{R}\tau + \mathbf{R} = \mathbf{C}$. Now we can take the quotient with the congruence subgroup $G_0 \ltimes \mathbf{Z}^2$ to obtain the connected mixed Shimura variety S_1, which comes with a projection morphism $S_1 \longrightarrow S_0$. Above every $\tau \in S_0$ one finds that $S_{1,\tau}$ is isomorphic to the elliptic curve $\mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z})$. Conversely, every complex elliptic curve is of this form; for this reason, we call S_1 the *universal elliptic curve* over S_0. In the next chapter we will formalise this notion. The following classification of special subvarieties of S_1 is a direct corollary of proposition 5.3.

Corollary 6.2. *Let Z be a special subvariety of S_1. Then up to Hecke correspondence Z is of one of the following forms:*

- *$Z = \{\tau\} \times 0$ for some special τ;*
- *$Z = E_\tau$ above a special τ;*
- *$Z = G_0 \setminus (\mathcal{H}_1 \times 0)$;*
- *$Z = S_1$.*

Note that this corollary implies that the special points of S_1 are exactly the torsion points of E_τ above a special $\tau \in S_0$.

Let $\mathbf{G}_{a,\mathbf{Q}}^{2\vee}$ be the linear algebraic group over $\mathbf{Q}$ defined for any $\mathbf{Q}$-algebra B as $\mathbf{G}_{a,\mathbf{Q}}^{2\vee}(B) = \mathrm{Hom}_B(B^2, B)$. We equip it with a left action of P_0, by having a $A \in P_0(B)$ act on some $\xi \in \mathbf{G}_{a,\mathbf{Q}}^{2\vee}(B)$ as $A \cdot \xi = (\det A)\xi \circ A^{-1}$. The induced Hodge structure on $\mathbf{G}_{a,\mathbf{Q}}^{2\vee}$ is of type $\{(0,-1),(-1,0)\}$, so by Lemma 4.2 it should be isomorphic to $\mathbf{G}_{a,\mathbf{Q}}^{2}$; indeed there exists the isomorphism of P_0-modules

$$\begin{array}{rcl} \Phi : \mathbf{G}_{a,\mathbf{Q}}^{2} & \longrightarrow & \mathbf{G}_{a,\mathbf{Q}}^{2\vee} \\ \begin{pmatrix} a \\ b \end{pmatrix} & \mapsto & \det \begin{pmatrix} a & - \\ b & - \end{pmatrix}. \end{array}$$

Now consider the connected mixed Shumura datum

$$(P_1', X_1') = (P_0 \ltimes (\mathbf{G}_{a,\mathbf{Q}}^{2} \times \mathbf{G}_{a,\mathbf{Q}}^{2\vee}), X_0^+ \times \mathbf{R}^2 \times \mathbf{R}^{2\vee}),$$

and let $G_1' = G_0 \ltimes (\mathbf{Z}^2 \times \mathbf{Z}^{2\vee})$; I denote by S_1' the connected mixed Shimura variety $G_1' \setminus X_1'$. Above every $\tau \in S_0$, the fibre of $S_1' \longrightarrow S_0$ is $E_\tau \times E_\tau^\vee$, where $E_\tau^\vee$ is the dual of the elliptic curve $E_\tau^\vee$, i.e. $E_\tau^\vee = \mathbf{R}^{2\vee}/\mathbf{Z}^{2\vee}$, with the complex structure given by

$$\hat{h}_\tau : \mathbf{C}^\times \longrightarrow \mathrm{GL}(\mathbf{R}^{2\vee}) : z \mapsto (\xi \mapsto |h_\tau(z)|\xi \circ h_\tau(z)^{-1}).$$

My next aim is to classify the special subvarieties of S_1'. For a special $\tau \in \mathcal{H}_1$ and $\lambda \in \mathbf{Q}(\tau)^\times$, we denote $V_{\tau,\lambda}$ for the sub-T_τ-module of $\mathbf{G}_{a,\mathbf{Q}}^{2} \times \mathbf{G}_{a,\mathbf{Q}}^{2\vee}$ given by $V_{\tau,\lambda}(B) = \{(x,y) \in B^2 \times B^{2\vee} : \Phi(x) = \lambda y\}$ for every $\mathbf{Q}$-algebra B. Again, the following classification follows directly from proposition 5.3. By picking a different set of 2 generators for $\mathbf{G}_{a,\mathbf{Q}}^{4}$ as a GL_2-module, we see that the special subvarieties $\{\tau\} \times E_\tau \times 0$, $\{\tau\} \times 0 \times E_\tau^\vee$ and $\{\tau\} \times V_{\tau,\lambda}(\mathbf{R})/(V_{\tau,\lambda}(\mathbf{R}) \cap \mathbf{Z}^2 \times \mathbf{Z}^{2\vee})$ for some $\lambda \in \mathbf{Q}(\tau)^\times$ are all equal up to Hecke correspondence, an analogous statement holds for the case that the special subvariety maps surjectively to S_0. We therefore get the following classification.

Corollary 6.3. *Let Z be a special subvariety of S_1'. Then up to Hecke correspondence Z is of one of the following forms:*

- $Z = \{\tau\} \times 0$, *where τ is special;*
- $Z = \{\tau\} \times E_\tau \times 0$, *where τ is special;*
- $Z = \{\tau\} \times E_\tau \times E_\tau^\vee$;
- $Z = G_0 \setminus \mathcal{H}_1 \times 0$;
- $Z = (G_0 \ltimes (\mathbf{Z}^2 \times 0)) \setminus (\mathcal{H}_1 \times \mathbf{R}^2 \times 0)$;
- $Z = S_1'$.

6.2 The Poincaré bundle

Now we follow the construction in section 4.2. We take P_2 to be the extension of P_1' by $U = \mathbf{G}_{a,\mathbf{Q}}$, given by the bilinear map

$$\begin{array}{rcl} \beta : (\mathbf{G}_{a,\mathbf{Q}}^2 \times \mathbf{G}_{a,\mathbf{Q}}^{2\vee}) \times (\mathbf{G}_{a,\mathbf{Q}}^2 \times \mathbf{G}_{a,\mathbf{Q}}^{2\vee}) & \longrightarrow & \mathbf{G}_{a,\mathbf{Q}} \\ ((v,\xi),(v',\xi')) & \mapsto & \xi(v'), \end{array}$$

where the action of a matrix $A \in P_0$ on U is given by multiplication. This gives us a connected mixed Shimura datum (P_2, X_2^+), where X_2^+ can be identified with $\mathcal{H} \times \mathbf{R}^2 \times \mathbf{R}^{2\vee} \times \mathbf{C}$. For any $x \in X_2^+$ the induced Hodge structure of $U(\mathbf{R})$ is of type $\{(-1,-1)\}$, so we may identify $U(\mathbf{Q})$ with $\mathbf{Q}(1)$ through multiplication with $2\pi i$.

We can divide out by a subgroup of the set-theoretic form $G_0 \times \mathbf{Z}^2 \times \mathbf{Z}^{2\vee} \times \mathbf{Z}$ to get a connected mixed Shimura variety S_2. Over a point $(\tau, v, \xi) \in S_1'$, we may identify the fibre $S_{2,(\tau,v,\xi)} = U(\mathbf{C})/U(\mathbf{Z}) = \mathbf{C}(1)/\mathbf{Z}(1)$ with $\mathbf{C}^\times$ via the exponential map. This connected mixed Shimura variety is then the geometric object corresponding to the Poincaré bundle over S_1' with the zero section removed, as can be shown by using the methods of [6, 2.5.1]. The two group laws on S_2 from its biextension structure as in example 3.9 are now as follows. For any $(\tau, v, \xi, z) \in \mathcal{H} \times \mathbf{R}^2 \times \mathbf{R}^{2\vee} \times \mathbf{C}$, let $\overline{(\tau, v, \xi, z)}$ be its image in S_2. The group laws are then given by

$$\overline{(\tau, v, \xi_1, z_1)} +_{\tau,v} \overline{(\tau, v, \xi_2, z_2)} = \overline{(\tau, v, \xi_1 + \xi_2, z_1 + z_2)}$$

and

$$\overline{(\tau, v_1, \xi, z_1)} +^{\tau,\xi} \overline{(\tau, v_2, \xi, z_2)} = \overline{(\tau, v_1 + v_2, \xi, z_1 + z_2)},$$

as one can check using the formulation of the group law in example 3.8.

There are several equivalent ways of describing P_2. First, we can use the isomorphism Φ to write $P_2(B) = \mathrm{GL}_2(B) \times B^2 \times B^2 \times B$, with multiplication given by $(A, v, w, z)(A', v', w', z') = (AA', Av' + v, Aw' + w, |A|z' + \langle Av', w\rangle + z)$, where $\langle \_, \_ \rangle : \mathbf{G}_{a,\mathbf{Q}}^2 \times \mathbf{G}_{a,\mathbf{Q}}^2 \longrightarrow \mathbf{G}_{a,\mathbf{Q}}$ is given by $\langle v, w\rangle = \Phi(w)(v)$. The Poincaré bundle is symmetric, which is reflected in the fact that there is an automorphism of P_2 given by $(A, v, w, z) \mapsto (A, w, v, z - \langle v, w\rangle)$, which is its own inverse.

Alternatively, one can identify (A, v, ξ, z) with the 4×4 matrix

$$\begin{pmatrix} \det A & \xi \circ A & z \\ 0 & A & v \\ 0 & 0 & 1 \end{pmatrix},$$

where $\xi \circ A$ is a row vector and v is a column vector. Let $f : S_2 \longrightarrow S_1'$ denote the morphism of Shimura varieties induced by the projection morphism

$$P_2 \longrightarrow \mathrm{GL}_{2,\mathbf{Q}} \ltimes (\mathbf{G}^2_{a,\mathbf{Q}} \times \mathbf{G}^{2\vee}_{a,\mathbf{Q}}).$$

The following results now follow straightforwardly from theorem 5.5.

Corollary 6.4. *Let Z be a special subvariety of S_2. Then up to Hecke correspondence Z is of one of the following forms, where groups are written in set-theoretic form, with the multiplication as in 4.5 understood:*

- $Z = f^{-1}(Z')$, *where Z' is a special subvariety of S_1';*
- $Z = \tau \times 0 \times 0 \times 0$, *for some special τ;*
- $Z = \{\tau\} \times E_\tau \times 0 \times 0$, *for some special τ;*
- $Z = (G_0 \times 0 \times 0 \times 0)) \setminus (\mathcal{H}_1 \times 0 \times 0 \times 0)$;
- $Z = (G_0 \times \mathbf{Z}^2 \times 0 \times 0) \setminus \mathcal{H}_1 \times \mathbf{R}^2 \times 0 \times 0$;
- $Z = \{\tau\} \times \overline{\{(v, \xi, z) : (v, \xi) \in V_{\tau,\lambda}(\mathbf{R}), 2z = \xi(v)\}} \subset \{\tau\} \times \mathbf{R}^2/\mathbf{Z}^2 \times \mathbf{R}^{2\vee}/\mathbf{Z}^{2\vee} \times \mathbf{C}/\mathbf{Z}$ *for some $\lambda \in \mathbf{Q}(\tau)^\times \cap i\mathbf{R}$ (regarded as a subset of $T_\tau(\mathbf{Q})$).*

Proof. Using theorem 5.5, the only nontrivial cases that need to be checked is for which V_λ and $V_{\tau,\lambda}$ the induced map of connected mixed Shimura varieties $f^{-1}(V_\lambda) \longrightarrow V_\lambda$ (or $f^{-1}(V_{\tau,\lambda}) \longrightarrow V_{\tau,\lambda}$) admits a section. First consider $V_{\tau,\lambda}$ as before for some $\lambda \in \mathbf{Q}(\tau)^\times$. Then the fact that β is symmetric implies that $\langle \lambda y, y'\rangle = \langle \lambda y', y\rangle$ for all $y, y' \subset B^2$, for any $\mathbf{Q}(\tau)$-algebra B. From the definition of the bilinear map one has that

$$\begin{aligned} \langle |\lambda| y, y'\rangle &= |\lambda| \langle y, y'\rangle \\ &= \langle \lambda y, \lambda y'\rangle \\ &= -\langle \lambda y', \lambda y\rangle \\ &= \langle -\lambda^2 y, y'\rangle. \end{aligned}$$

Since the bilinear form is degenerate, this implies that $|\lambda| = -\lambda^2$, so φ is symmetric on $V_{\tau,\lambda} \times V_{\tau,\lambda}$ if and only if λ is purely imaginary. For such a λ, the section $T_\tau \ltimes V_{\tau,\lambda} \longrightarrow T_\tau \times V_{\tau,\lambda} \times \mathbf{G}_{a,\mathbf{Q}}$ given by $(A, v) \mapsto (A, v, \frac{1}{2}\langle v, v\rangle)$ gives us the special subvariety Z'. If we now consider V_λ, the same calculation shows us that λ must be purely imaginary, but this is a contradiction with the fact that $\lambda \in \mathbf{Q}^\times$. $\square$

7 A moduli interpretation of the results

In this section, I interpret the connected mixed Shimura varieties discussed in the previous chapters within the theory of moduli spaces of principally polarised abelian varieties and their associated universal abelian varieties. I start with the pure Shimura variety S_0 as discussed in section 2.4.

7.1 Siegel modular varieties

Let S be a scheme, and let g be a positive integer. By $\mathbf{AbSch}_{S,g,1}$ I denote the category whose objects are pairs $(A/T, \varphi)$, where T is an S-scheme, A/T is an abelian scheme of relative dimension g, and $\varphi : A \longrightarrow A^\vee$ is a principal polarisation. Its morphisms are cartesian diagrams

$$\begin{array}{ccc} A' & \xrightarrow{\psi_A} & A \\ \downarrow & & \downarrow \\ T' & \xrightarrow{\psi} & T \end{array}$$

where ψ is a morphism of S-schemes such that $\psi_{A^\vee} \circ \varphi' = \psi_A \circ \varphi$, and $\psi^* 0_{A/T} = 0_{A'/T'} \in A'(T')$. Now to continue we need the following lemma.

Lemma 7.1. *Let X/S, Y/S be abelian schemes of relative dimension g, and let $\alpha : X \longrightarrow Y$ be an isogeny with dual isogeny $\alpha^\vee : Y^\vee \longrightarrow X^\vee$. Then there is a canonical perfect pairing*

$$\ker \alpha \times \ker \alpha^\vee \longrightarrow \mathbf{G}_{m,S}.$$

Proof. See [18, 2.1.5]. □

Now let S be a scheme over $\mathbf{C}$ and let A/S be a principally polarised abelian scheme, and let $N \geq 1$ be an integer. By $A[N]$ I denote the kernel of the multiplication map $N : A \longrightarrow A$. Through the principal polarisation, we may consider A as its own dual, and $N^\vee = N$ under this identification. The image of the perfect pairing $A[N] \times A[N] \longrightarrow \mathbf{G}_{m,S}$ lies in $\mu_{N,S}$. By fixing a N-th unit root $\zeta_N \in \mathbf{C}$ we can consider this as a pairing with values in the constant group scheme $(\mathbf{Z}/N\mathbf{Z})_S$.

The constant group scheme $(\mathbf{Z}/N\mathbf{Z})_S^{2g}$ comes with a natural symplectic pairing $(\mathbf{Z}/N\mathbf{Z})_S^{2g} \times (\mathbf{Z}/N\mathbf{Z})_S^{2g} \longrightarrow (\mathbf{Z}/N\mathbf{Z})_S$ induced by the matrix $\begin{pmatrix} 0_g & 1_g \\ -1_g & 0_g \end{pmatrix}$. Therefore we can define the following functor

$$\begin{array}{rcl} \mathcal{B}_{g,1,N} : \mathbf{AbSch}_{\mathbf{C},g,1} & \longrightarrow & \mathbf{Set} \\ (A/S, \varphi) & \mapsto & \mathrm{Isom}((\mathbf{Z}/N\mathbf{Z})_S^{2g}, A[N]), \end{array}$$

where the isomorphisms are to be understood as isomorphisms of symplectic modules over S. This in turn induces a contravariant functor

$$\begin{array}{rcl}\mathcal{A}_{g,1,N} : \mathbf{Sch}_{\mathbf{C}} & \longrightarrow & \mathbf{Set} \\ S & \mapsto & \{(A/S,\varphi,\alpha) : (A/S,\varphi) \in \mathbf{AbSch}_{\mathbf{C},g,1}, \alpha \in \mathcal{B}_{g,d,N}(A/S)\}/\sim\end{array}$$

where $\sim$ denotes 'up to isomorphism'; a morphism $(A/S,\varphi,\alpha) \longrightarrow (A'/S,\varphi',\alpha')$ is a morphism $\zeta : (A/S,\varphi) \longrightarrow (A'/S,\varphi')$ such that $\mathcal{B}_{g,1,N}(\zeta)(\alpha') = \alpha$. In other words, $\mathcal{A}_{g,1,N}$ sends a $\mathbf{C}$-scheme S to the set of isomorphy classes of abelian schemes A of relative dimension g over S, together with a principal polarisation $\varphi : A \xrightarrow{\sim} A^\vee$ and a given symplectic basis of the N-torsion. This functor connects with the rest of this thesis by the following proposition.

Proposition 7.2. *Let $N \geq 3$ and $g \geq 1$ be integers, and let $\Gamma(N) = \ker(\mathrm{Sp}_{2g}(\mathbf{Z}) \longrightarrow \mathrm{Sp}_{2g}(\mathbf{Z}/N\mathbf{Z}))$. Then the functor $\mathcal{A}_{g,1,N}$ is represented by the pure Shimura variety corresponding to $(\mathrm{GSp}_{2g}, \mathcal{H}_g, \Gamma(N))$.*

Proof. See [28, 7.3]. □

Hence, for $G_0 = \Gamma(N)$, the connected Shimura variety S_0 parametrises principally polarised abelian schemes with a given basis for the N-torsion. In particular, the set of $\mathbf{C}$-points $S_0(\mathbf{C})$ is the set of principally polarised abelian varieties over $\mathbf{C}$ with a given basis for the N-torsion with a certain symplectic structure. In fact, a point $\tau \in S_0$ corresponds to the abelian variety $\mathbf{R}^{2g}/\mathbf{Z}^{2g}$, with the complex structure induced by τ. The basis for the N-torsion is $(\frac{1}{N}e_1, \ldots, \frac{1}{N}e_{2g})$, where $e_1, \ldots, e_{2g}$ is the standard basis for $\mathbf{Z}^{2g}$.

7.2 The universal abelian variety

Again we take $S_0 = G_0 \setminus \mathcal{H}_g$, where $G_0 = \Gamma(N)$ for $N \geq 3$. We have seen that S_0 represents the functor $\mathcal{A}_{g,1,N} : \mathbf{Sch}_{\mathbf{C}} \longrightarrow \mathbf{Set}$. In particular, the identity map $S_0 \longrightarrow S_0$ corresponds to an abelian variety $\mathcal{E}/S_0$. Furthermore, if A/T is an element of $\mathbf{AbSch}_{\mathbf{C},g,1}$, then the morphism $\varphi : T \longrightarrow S_0$ to which it corresponds can be decomposed as $\mathrm{id}\circ\varphi$; this induces a cartesian diagram of abelian schemes $A/T \longrightarrow \mathcal{E}/S_0$, so we see that A is the pullback of $\mathcal{E}$ under the morphism φ. For this reason, $\mathcal{E}$ is called the *universal abelian variety*. It can be shown that $\mathcal{E} = S_1 = (G_0 \ltimes \mathbf{Z}^{2g})\backslash X_1^+$. Above a point $\tau \in S_0$, the equality comes from the fact that $S_{1,\tau} = \mathbf{C}^g/(\tau\cdot\mathbf{Z}^g + \mathbf{Z})$, which is indeed the complex abelian variety corresponding to τ, see section 2.4. If $g = 1$, we get the connected mixed Shimura variety S_1 from section 6.1.

For any $\mathbf{C}$-scheme S, we know that $S_0(S)$ equals the set of principally polarised abelian varieties over S with a given dimension g and with a given symplectic basis of the N-torsion, up to isomorphy. For a given morphism $f : S \longrightarrow S_0$, the abelian variety that it represents is $(S_1 \times_{S_0} S)/S$. Now let $\pi : S_1 \longrightarrow S_0$ be the structure morphism, and let $h : S \longrightarrow S_1$ be such that $\pi \circ h = f$. Then h induces an element of $(S_1 \times_{S_0} S)(S)$, the abelian variety induced by f. Therefore S_1 represents the functor

$$\begin{array}{rcl}\mathfrak{G}(N) : \mathbf{Sch}_{\mathbf{C}} & \longrightarrow & \mathbf{Set} \\ S & \mapsto & \left\{(A/S,\varphi;R) : \begin{array}{l} A/S \text{ p.p. ab. schemes of relative dimension } g, \\ \varphi : (\mathbf{Z}/N\mathbf{Z})^{2g}_S \xrightarrow{\sim} A(S)[N], R \in A(S)\end{array}\right\}/\sim .\end{array}$$

Now I wish to describe the special subvarieties of S_1 in the light of this moduli interpretation. For simplicity, I assume that $g = 1$, so that I may take the classification these special subvarieties from section 6. Above a special point τ, there are only two possible special subvarieties, namely a torsion point and all of E_τ; these are the elliptic curves with complex multiplication. The special subvarieties mapping surjectively to S_0 are more interesting.

Proposition 7.3. *Let Z be a special subvariety of $S_1 = (\Gamma(N) \ltimes \mathbf{Z}^2) \setminus X_1^+$, not equal to S_1, that maps surjectively to S_0. Then there exists a multiple N' of N and integers a, b such that Z represents the image of the natural transformation of functors*

$$\begin{aligned} \mathcal{A}_{1,1,N'} &\longrightarrow \mathfrak{G}(N) \\ \mathcal{A}_{1,1,N'}(S) &\longrightarrow \mathfrak{G}(N)(S) \\ (E/S, P, Q) &\mapsto (E/S, \frac{N'}{N}P, \frac{N'}{N}Q; aP + bQ). \end{aligned}$$

Proof. A connected component Z' of the inverse image of Z of S_1 in X_1^+ is of the form $\mathcal{H}_1 \times \{v\}$, for some $v \in \mathbf{Q}^2$. Now let N' be such that $N \mid N'$ and $N'v \in \mathbf{Z}^2$; then there exist $a, b \in \mathbf{Z}/N'\mathbf{Z}$ such that $v = a\frac{e_1}{N'} + b\frac{e_2}{N'}$ in $\mathbf{Q}^2/\mathbf{Z}^2$, where (e_1, e_2) is the standard basis of $\mathbf{Z}^2$. Now consider the quotient map $\pi : X_1^+ \longrightarrow (\Gamma(N') \ltimes \mathbf{Z}^2) \setminus X_1^+$; then $K = \pi(Z')$ equals $(\Gamma(N') \setminus \mathcal{H}_1) \times \{v\}$. This is the image of a morphism of connected mixed Shimura varieties

$$\begin{aligned} \Gamma(N') \setminus \mathcal{H}_1 &\longrightarrow (\Gamma(N') \ltimes \mathbf{Z}^2) \setminus (\mathcal{H}_1 \times \mathbf{R}^2) \\ \tau &\mapsto (\tau, v). \end{aligned}$$

Now, for any $\mathbf{C}$-scheme S, an element of $(\Gamma(N') \setminus \mathcal{H})(S)$ corresponds to an elliptic curve E/S with two given sections $P, Q \in E[N'](S)$ that generate $E[N']$. Over a $\mathbf{C}$-point x of S mapping to $\tau \in \Gamma(N') \setminus \mathcal{H}$, we see that $E_x \cong \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$, and $P = \frac{1}{N'}\tau$, $Q = \frac{1}{N'}$; therefore, $v = aP + bQ$ in E_x. The morphism above can therefore be represented in terms of contravariant functors on $\mathbf{AbSch_C}$ as follows:

$$\begin{aligned} \mathcal{A}_{1,1,N}(N') &\longrightarrow \mathfrak{G}(N') \\ \mathcal{A}_{1,1,N}(N')(S) &\longrightarrow \mathfrak{G}(N')(S) \\ (E/S, P, Q) &\mapsto (E/S, P, Q; aP + bQ). \end{aligned}$$

The inclusion $\Gamma(N') \subset \Gamma(N)$ induces a morphism $(\Gamma(N') \ltimes \mathbf{Z}^2) \setminus X_1^+ \longrightarrow S_1$. In terms of contravariant functors on $\mathbf{AbSch_C}$, this morphism is defined as follows:

$$\begin{aligned} \mathfrak{G}(N') &\longrightarrow \mathfrak{G}(N) \\ \mathfrak{G}(N')(S) &\longrightarrow \mathfrak{G}(N)(S) \\ (E/S, P, Q; R) &\mapsto (E/S, P, Q, \frac{N'}{N}P, \frac{N'}{N}Q; R). \end{aligned}$$

Under this morphism, the image of K is Z. In other words, we can interpret Z as the image of the natural transformation of functors

$$\begin{array}{rcl}\Gamma(N') & \longrightarrow & \mathfrak{G}(N)\\ \Gamma(N')(S) & \longrightarrow & \mathfrak{G}(N)(S)\\ (E/S,P,Q) & \mapsto & (E/S,\dfrac{N'}{N}P,\dfrac{N'}{N}Q;aP+bQ),\end{array}$$

as was to be shown. □

In the same way $S_1' = (\Gamma(N) \ltimes (\mathbf{Z}^2 \times \mathbf{Z}^{2\vee})) \setminus (\mathcal{H}_1 \times \mathbf{R}^2 \times \mathbf{R}^{2\vee})$ represents the contravariant functor

$$\begin{array}{rcl}\mathfrak{G}'(N): \mathbf{Sch_C} & \longrightarrow & \mathbf{Set}\\ S & \mapsto & \{(E/S,P,Q;R,R') : P,Q \text{ generate } E(S)[N], R,R' \in E(S)\}/\sim .\end{array}$$

Again, we wish to regard the special subvarieties of S_1' as the images of natural transformations of functors. In order to simplify notation, I use the identification $S_1' = G_0 \ltimes \mathbf{Z}^4 \setminus \mathcal{H}_1 \times \mathbf{R}^4$.

Let $Z \subset S_1'$ be a special subvariety such that, for the map $\pi : S_1' \longrightarrow S_0$, the image $\pi(Z)$ is a special point $\tau \in S_0$. If Z is zero-dimensional, we see that Z is a torsion point on $E_\tau \times E_\tau$, and if Z is two-dimensional, we see that $Z = E_\tau \times E_\tau$. Now suppose $G_0 = \Gamma(N)$ for some N, and that $Z \subset S_1'$ is a one-dimensional special subvariety mapping to some $\tau \in S_0$. Then the preimage Z' of Z in $\{\tau\} \times \mathbf{R}^4 \cong (\mathbf{R} \otimes_{\mathbf{Q}} \mathbf{Q}(\tau))^2$ is a translate of a one-dimensional $\mathbf{Q}(\tau)$-linear subspace V of $\mathbf{Q}(\tau)^2$ by an element of $\mathbf{Q}(\tau)$. Because V is defined over $\mathbf{Q}(\tau)$, the free abelian group $\Lambda = (\mathbf{Z} + \mathbf{Z}\tau)^2 \cap V \subset V$ is of maximal rank, so $V_{\mathbf{R}}/\Lambda$ is an abelian subvariety of $\mathbf{C}^2/(\mathbf{Z} + \mathbf{Z}\tau)^2$. Also, every one-dimensional subvariety of $\mathbf{C}^2/(\mathbf{Z}+\mathbf{Z}\tau)^2$ corresponds to such a V. Combining this with the zero-dimensional and two-dimensional cases, we see that the special subvarieties above some $\tau \in S_0$ are exactly the translates of abelian subvarieties by torsion points.

Again, the case that Z maps surjectively to S_0 allows for a nicer description. Here too the cases that $Z \longrightarrow S_0$ has zero-dimensional or two-dimensional fibres are not interesting.

Proposition 7.4. *Let Z be a special subvariety of $S_1' = (\Gamma(N) \ltimes \mathbf{Z}^4) \setminus (\mathcal{H}_1 \times \mathbf{R}^4)$ such that the projection morphism $Z \longrightarrow S_0$ is surjective and has one-dimensional fibres. Then there exists a multiple N' of N, integers a, b, c and d, and coprime integers p and q such that Z represents the image of the natural transformation of functors*

$$\begin{array}{rcl}\mathfrak{G}(N') & \longrightarrow & \mathfrak{G}'(N)\\ \mathfrak{G}(N')(S) & \longrightarrow & \mathfrak{G}'(N)(S)\\ (E/S,P,Q;R) & \mapsto & (E/S,\dfrac{N'}{N}P,\dfrac{N'}{N}Q;aP+bQ+pR,cP+dQ+qR).\end{array}$$

Proof. Let Z' be an irreducible component of the inverse image of Z in $\mathcal{H}_1 \times \mathbf{R}^4$; then $Z' = v_0 + V_{\mathbf{R}}$ for some $v_0 \in \mathbf{Q}^4$ and some sub-$\mathrm{GL}_2(\mathbf{Q})$-module V of $\mathbf{Q}^2 \times \mathbf{Q}^2$. We may write $V = \{(x,y) \in \mathbf{Q}^2 \times \mathbf{Q}^2 : qy = px\}$ for some coprime integers $p, q \in \mathbf{Z}$. Then the map $\mathbf{Q}^2 \longrightarrow V : v \mapsto (pv, qv)$ induces an isomorphism $\mathbf{Q}^2/\mathbf{Z}^2 \longrightarrow V/(V \cap \mathbf{Z}^4)$. As before, let N' be such that $N \mid N'$ and $N'v_0 = 0$. Then the image of Z' in $(\Gamma(N') \ltimes \mathbf{Z}^4) \setminus X_1'^+$ is of the form $(\Gamma(N') \setminus X_0^+) \times (V_\mathbf{R}/(V_\mathbf{R} \cap \mathbf{Z}^4) + v_0)$. This is the image of the morphism of connected mixed Shimura varieties

$$\begin{array}{rcl} (\Gamma(N') \ltimes \mathbf{Z}^2) \setminus X_1^+ & \longrightarrow & (\Gamma(N') \ltimes \mathbf{Z}^4) \setminus X_1'^+ \\ (\tau, v) & \mapsto & (\tau, (pv, qv) + v_0). \end{array}$$

Now let $v_0 = \frac{a}{N'}e_1 + \frac{b}{N'}e_2 + \frac{c}{N'}e_3 + \frac{d}{N'}e_4$ for some integers a, b, c, d; then this morphism corresponds to the natural transformation of functors

$$\begin{array}{rcl} \mathfrak{G}(N') & \longrightarrow & \mathfrak{G}'(N') \\ \mathfrak{G}(N')(S) & \longrightarrow & \mathfrak{G}'(N')(S) \\ (E/S, P, Q; R) & \mapsto & (E/S, P, Q; aP + bQ + pR, cP + dQ + qR). \end{array}$$

Furthermore, the morphism of connected mixed Shimura varieties $(\Gamma(N') \ltimes \mathbf{Z}^4) \setminus X_1'^+$ corresponds to the following natural transformation of functors:

$$\begin{array}{rcl} \mathfrak{G}'(N') & \longrightarrow & \mathfrak{G}'(N) \\ \mathfrak{G}'(N')(S) & \longrightarrow & \mathfrak{G}'(N)(S) \\ (E/S, P, Q; R, R') & \mapsto & (E/S, \frac{N'}{N}P, \frac{N'}{N}Q; R, R'). \end{array}$$

Composing these two functors gives us the desired result. □

7.3 The Poincaré bundle

In this section, we look at the connected mixed Shimura variety S_2 as defined, for elliptic curves, in section 6.2. The construction for general dimensions is as follows. Let V be the $\mathrm{GSp}_{2g,\mathbf{Q}}$-module $\mathbf{G}_{a,\mathbf{Q}}^{2g\vee}$, where for every $\mathbf{Q}$-algebra B, the action of a matrix $A \in \mathrm{GSp}_{2g}(B)$ on a functional $\xi \in V(B)$ is given by ${}^A\xi = d(A)\xi \circ A^{-1}$. Now any $h \in X_0^+$ gives V a Hodge structure of type $\{(0, -1), (-1, 0)\}$. By lemma 4.2, there is an isomorphism $\Phi : V \longrightarrow \mathbf{G}_{a,\mathbf{Q}}^{2g}$ of $\mathrm{GSp}_{2g,\mathbf{Q}}$-modules. In the case that $g = 1$, we have explicitly given this isomorphism in section 6.2. In general, we may define Φ by means of the symplectic form J used to define $\mathrm{GSp}_{2g,\mathbf{Q}}$. By a base change on either side we may assume that $\Phi(\mathbf{Z}^{2g}) = \mathbf{Z}^{2g\vee}$. Now, in the notation of corollary 4.7, we take $U = \mathbf{G}_{a,\mathbf{Q}}$, $W_{P_1} = \mathbf{G}_{a,\mathbf{Q}}^{2g} \times \mathbf{G}_{a,\mathbf{Q}}^{2g\vee}$, and the bilinear map

$$\begin{array}{rcl} \beta : W_{P_1} \times W_{P_1} & \longrightarrow & U \\ ((v, \xi), (v', \xi')) & \mapsto & \xi(v'). \end{array}$$

We then take G to be the congruence subgroup of P_2 corresponding to the set $\Gamma(N) \times \mathbf{Z}^2 \times \mathbf{Z}^{2\vee} \times \mathbf{Z}$, and we take $S_2 = G \setminus \mathcal{H}_g \times \mathbf{R}^4 \times \mathbf{C}$. Again, as in [6, 2.5.1], one can prove that S_2 as defined in this way is the geometric object corresponding to the Poincaré bundle over $S_1' = S_1 \times_{S_0} S_1^\vee$ with the zero section removed. Now denote by C_S the category of commutative group schemes over S.

One can prove that S_2 represents the following functor:

$$\begin{array}{rcl}\mathfrak{G}_2(N) : \mathbf{Sch}_{\mathbf{C}} & \longrightarrow & \mathbf{Set} \\ S & \mapsto & \left\{ (A/S, \varphi, G; R) : \begin{array}{l} A/S \text{ p.p. ab. schemes of relative dimension } g, \\ \varphi : (\mathbf{Z}/N\mathbf{Z})^{2g}_S \xrightarrow{\sim} A[N], \\ G \text{ an extension of } A \text{ by } \mathbf{G}_{m,S}, R \in G(S) \end{array} \right\} / \sim .\end{array}$$

On $\mathbf{C}$-points the correspondence works as follows. Let x be a point in $S_2(\mathbf{C})$, and let τ be its image in S_0, and $\mathcal{L}$ its image in $A^\vee_\tau = S^\vee_{1,\tau}$. Then $\mathcal{L}$ corresponds to an extension G of A_τ by $\mathbf{G}_m$, and this extension is $S_2 \times_{S'_1} \mathcal{L}$; therefore x corresponds to a point $R \in G(\mathbf{C})$. Furthermore, the isomorphism of $\mathrm{GSp}_{2g,\mathbf{Q}}$-modules $\Phi : \mathbf{G}^{2g}_{a,\mathbf{Q}} \longrightarrow V$ induces, when viewed as $\mathbf{S}$-modules via h_τ, an isomorphism of $\mathbf{Z}$-Hodge structures $\mathbf{Z}^{2g} \longrightarrow \mathbf{Z}^{2g\vee}(1)$; this yields an isomorphism $\varphi : A_\tau \longrightarrow A^\vee_\tau$. In this notation, $x \in S_2(\mathbf{C})$ corresponds to $(A_\tau, \varphi, G; R) \in \mathfrak{G}_2(N)$.

Now let us look at the special subvarieties of S_2 for the case $g = 1$. Following section 4.2, we can categorize them into three types:

- inverse images of special subvarieties of S'_1;
- sections of the Poincaré bundle above special subvarieties of S'_1 above which this bundle is trivial;
- $Z = \{\tau\} \times \overline{\{(v, \xi, z) : (v, \xi) \in V_{\tau,\lambda}(\mathbf{R}), 2z = \xi(v)\}} \subset \{\tau\} \times \mathbf{R}^2/\mathbf{Z}^2 \times \mathbf{R}^{2\vee}/\mathbf{Z}^{2\vee} \times \mathbf{R}/\mathbf{Z}$ for some $\lambda \in \mathbf{Q}(\tau)^\times \cap i\mathbf{R}$ (regarded as a subset of $T_\tau(\mathbf{Q})$), up to Hecke correspondence.

Of these, only the last one is interesting to describe. For a Z of the last kind, the map $Z \longrightarrow V' = \overline{(v, \xi) + V_{\tau,\lambda}(\mathbf{R})} \subset E_\tau \times E^\vee_\tau$ is either a double or a single covering, depending on whether the image of $V_{\tau,\lambda}(\mathbf{R}) \cap \mathbf{Z}^2 \times \mathbf{Z}^{2\vee}$ under the canonical map

$$\mathbf{Z}^2 \times \mathbf{Z}^{2\vee} \longrightarrow \mathbf{Z}$$

lies in $2\mathbf{Z} \subset \mathbf{Z}$ or not. If we write $\mathcal{P}$ for the Poincaré bundle on $E_\tau \times E^\vee_\tau$, and regard Z as a subset of $\mathbf{L}^\times_{\mathcal{P}}$, then the map $\mathbf{L}^\times_{\mathcal{P}} \longrightarrow \mathbf{L}^\times_{\mathcal{P}\otimes\mathcal{P}}$, defined by the morphism of line bundles $\mathcal{P} \longrightarrow \mathcal{P} \otimes \mathcal{P}$ defined by $z \mapsto z^2$ on local sections, maps Z into the image of the section $(v, \xi) \mapsto \overline{(v, \xi, \exp(\xi(v)))}$ of $\mathbf{L}^\times_{\mathcal{P}\otimes\mathcal{P}|_{V'}} \longrightarrow V'$. This shows that the line bundle $\mathcal{P} \otimes \mathcal{P}$ is trivial over V'; hence $\mathcal{P}|_{V'}$ is of order 2 or 1 in $\mathrm{Pic}^0(V')$. And vice versa, from section 4.2 it follows that every special subvariety V' of S'_1 above which $\mathcal{P}$ is of order one or two is of the form $\overline{(v, \xi) + V_{\tau,\lambda}(\mathbf{R})}$.

8 Pink's conjecture on semiabelian varieties

As Pink in [32], I start by listing the history of various conjectures.

8.1 Introduction

The following theorem, before being proven by Faltings [13] in 1983, was conjectured by Mordell [27].

Theorem 8.1 (Faltings). *For any geometrically irreducible smooth projective algebraic curve Z of genus ≥ 2 over a number field K, the set of rational points $Z(K)$ is finite.*

One can translate this into a question about abelian varieties by embedding Z into its Jacobian variety J (see [25]), so that $Z(K) = J(K) \cap Z$. Since $J(K)$ is finitely generated by the Mordell-Weil theorem, it is sufficient to prove that for every abelian variety A over a field of of characteristic zero, any finitely generated subgroup $\Lambda \subset A$ and any irreducible curve $Z \subset A$ of genus ≥ 2, the intersection $Z \cap \Lambda$ is finite. This is true, although the only known proof of this is from the above conjecture. Still, considering objects similar to $Z \cap \Lambda$ led to other conjectures, such as the following statement, first conjectured independently by Manin and Mumford before proven by Raynaud [35],[36]:

Theorem 8.2 (Raynaud). *Let A be an abelian variety over $\mathbf{C}$ and let A_{tor} denote its subgroup of all torsion points. Let $Z \subset A$ be an irreducible closed algebraic subvariety such that $Z \cap A_{\mathrm{tor}}$ is Zariski dense in Z. Then Z is a translate of an abelian subvariety of A.*

There are also other proofs, see [32] for details. The following theorem, due to McQuillan [24], implies the two above:

Theorem 8.3 (McQuillan). *Let A be a semiabelian variety over $\mathbf{C}$, let Λ_0 be a finitely generated subgroup of A, and let*

$$\Lambda = \{a \in A : \exists n \in \mathbf{Z}_{>0} : na \in \Lambda_0\}$$

be the division group of Λ_0. Let $Z \subset A$ be an irreducible closed algebraic subvariety such that $Z \cap \Lambda$ is Zariski dense in Z. Then Z is a translate of a semiabelian subvariety of A.

The analogous (and weaker) statement regarding abelian varieties is called the Mordell-Lang conjecture, and has been proven by the combined work of Faltings [13], Raynaud [34], Vojta [37] and Hindry [20].

On the other hand, there are Shimura varieties which act as universal polarised abelian varieties with a given dimension and level structure, as we have seen in chapter 7. Also, from Proposition 5.3, it follows that above special points of such a Siegel modular variety, the torsion points of these abelian varieties are exactly the special points of these varieties as Shimura varieties. This suggests the following analog of the previous conjecture, posed independently by André [2] and Oort [29]:

Conjecture 8.4 (André-Oort). *Let S be a pure Shimura variety over $\mathbf{C}$ and let $\Lambda \subset S$ denote the set of all its special points. Let $Z \subset S$ be an irreducible closed algebraic subvariety such that $Z \cap \Lambda$ is Zariski dense in Z. Then Z is a special subvariety of S.*

Although special cases of this conjecture, mostly under additional assumptions, have been proven by Moonen [26], André [1], Edixhoven [9], [10], Edixhoven-Yafaev [11],Yafaev [38], [39], Pila-Tsimerman [30] and Gao [16], this conjecture remains open. It should be noted, however, that it has recently been solved under the assumption of the Generalised Riemann Hypothesis by Klingler-Yafaev [22] for pure Shimura varieties.

The conjectures of Mordell-Lang and André-Oort were combined by Pink [32] into a conjecture about Shimura varieties.

Conjecture 8.5 (Pink). *Consider a mixed Shimura variety S over $\mathbf{C}$ and an irreducible closed subvariety Z, and let S_Z be the smallest special subvariety of S containing Z. Then the intersection of Z with the union of all special subvarieties of S of dimension $< \dim S_Z - \dim Z$ is not Zariski dense in Z.*

In [32], Pink showed how the conjectures of André-Oort and Mordell-Lang follow from this conjecture. He also claimed to prove the following conjecture concerning families of abelian varieties under the assumption of conjecture 8.5. In order to state the conjecture, we first need some notation. Suppose $B \longrightarrow X$ is a family of semiabelian varieties, and let $x \in X$ be a point. For any integer d, let $B_x^{[>d]}$ be the set of points of the semiabelian variety B_x contained in an algebraic subgroup of codimension $> d$. Furthermore, we set

$$B^{[>d]} = \bigcup_{x \in X} B_x^{[>d]}.$$

Conjecture 8.6 (Pink). *Consider an algebraic family of semiabelian varieties $B \longrightarrow X$ over $\mathbf{C}$ and an irreducible closed subvariety $Y \subset B$ of dimension d that is not contained in any proper closed subgroup scheme of $B \longrightarrow X$. Then $Y \cap B^{[>d]}$ is not Zariski dense in Y.*

A counterexample to this conjecture was found by Bertrand [5]. However, this counterexample does not disprove conjecture 8.5. There is a mistake in Pink's proof of the implication 8.5 ⇒ 8.6. In section 8.2, I will explain the counterexample found by Bertrand.

8.2 Bertrand's counterexample

In [5], Bertrand gives a counterexample to conjecture 8.6. This counterexample can be constructed as a special subvariety of the mixed Shimura variety S_2 as defined in section 6.2, in the following way. Let $\tau \in \mathcal{H}_1$ be a point, with corresponding elliptic curve $E = E_\tau$. As in example 3.8, the geometric line bundle $\mathbf{L}_{\mathcal{P}}^{\times}$ with the zero section removed corresponding to the Poincaré bundle $\mathcal{P}$ on $E \times E = E \times E^\vee$ can be regarded as a family of semiabelian varieties over $E^\vee$, as every point of $E^\vee$ corresponds to an extension of E by $\mathbf{G}_{m,\mathbf{C}}$.

Theorem 8.7. *Suppose $\tau \in \mathcal{H}$ is such that $\mathbf{Q}(\tau)$ is imaginary quadratic over $\mathbf{Q}$, and let $\lambda \in \mathbf{Q}(\tau)^\times$ be totally imaginary. Let $\mathcal{G}_\lambda$ be the image of the set*

$$V_\lambda = \{(x, y) \in \mathbf{C}^2 : x = \lambda y\}$$

in $(\mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z}))^2 = E \times E$, and suppose that the pairing $\langle , \rangle : (\mathbf{Z} + \tau\mathbf{Z})^2 \longrightarrow \mathbf{Z}(1)$ induced by the identification $E = E^\vee$ maps $V_\lambda \cap (\mathbf{Z} + \tau\mathbf{Z})^2$ to $2\mathbf{Z}(1) \subset \mathbf{Z}(1)$. Then there is a special subvariety T of $\mathbf{L}_{\mathcal{P}}^\times$ that projects one-to-one to $\mathcal{G}_\lambda$, such that T is not contained in any proper closed subgroup scheme of $\mathbf{L}_{\mathcal{P}}^\times$, but T contains infinitely many torsion points on fibres.

Proof. As we have seen in section 7.3, the map $V_\lambda \longrightarrow \mathbf{C} : (v, w) \mapsto \frac{1}{2}\langle v, w\rangle$ induces a map $s : \mathcal{G}_\lambda \longrightarrow \mathbf{C}/\mathbf{Z}$, which is a section of $\mathbf{L}_{\mathcal{P},\mathcal{G}_\lambda}^\times \longrightarrow \mathcal{G}_\lambda$.

Now $Z = s(\mathcal{G}_\lambda)$ is a one-dimensional subvariety of $\mathbf{L}_{\mathcal{P}}^\times$. Furthermore, it is not contained in any proper closed subgroup scheme. One way to see this is as follows. Suppose Z is contained in a closed subgroup scheme $H \subset \mathbf{L}_{\mathcal{P}}^\times$; as Z is connected, we may assume without loss of generality that H is connected. As Z maps surjectively to $E^\vee$, the same must hold for H. Now let H' be the image of H in $E \times E^\vee$. Suppose the fibre H' is a proper closed subgroup scheme of $(E \times E^\vee)/E^\vee$. As E is one-dimensional, its proper closed subgroups are finite; therefore $H' \longrightarrow E^\vee$ is a finite morphism. This means that above every $\xi \in E^\vee$, the fibre $H'_\xi \subset E$ consists of torsion points only. However, $Z_\xi \in E$ is a nontorsion point if ξ is nontorsion. This is a contradiction, so $H' = E \times E^\vee$.

Now consider the kernel K of the map $H \longrightarrow E \times E^\vee$, which is a closed subgroup scheme of $\mathbf{G}_{m,E^\vee}$. Again, as $\mathbf{G}_{m,E^\vee}$ is one-dimensional over $E^\vee$, K is either finite or all of $\mathbf{G}_m$. Let k be an integer such that $(k, \lambda k) \in (\mathbf{Z} + \tau\mathbf{Z})^2$, denote $m = \langle k, \lambda k\rangle$ in $\mathbf{Z}(1)$, and let p be a prime number not dividing $\frac{m}{2\pi i}$. Let $x \in E$ be the image of $\frac{k}{p}$ in $\mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z})$. The image of $(x, \lambda x)$ under the section s is the point $\overline{(x, \lambda x, \exp(\frac{1}{2}\langle x, \lambda x\rangle))} \in \mathbf{L}_{\mathcal{P},(x,\lambda x)}^\times$. If we regard $s(x, \lambda x)$ as an element of the semiabelian variety $\mathbf{L}_{\mathcal{P},\lambda x}^\times$, then $p \cdot s(x, \lambda x)$ is the point $\overline{(0, \lambda x, \exp(\frac{p}{2}\langle x, \lambda x\rangle))}$, which is annihilated by p, and is nontrivial, so $s(x, \lambda x)$ has order p^2 in the semiabelian variety $\mathbf{L}_{\mathcal{P},\lambda x}^\times$. This gets arbitrarily large as p increases, so it follows that $K = \mathbf{G}_{m,E^\vee}$, so $H = \mathcal{P}^\times$. However, this also shows that $s(\mathcal{G}_\lambda)$ contains infinitely many torsion points. □

According to Pink's conjecture 8.6, the intersection $\mathcal{G}_\lambda \cap \mathbf{L}_{\mathcal{P}}^{\times[>1]}$ is not Zariski dense in $\mathcal{G}_\lambda$. As $\mathbf{L}_{\mathcal{P}}^\times$ is of relative dimension 2 over $E^\vee$, $\mathcal{P}^{\times[>1]}$ is the set of torsion points in the fibres of $\mathbf{L}_{\mathcal{P}}^\times$, so conjecture 8.6 would predict that set of points of $\mathcal{G}_\lambda$ that are torsion in their fibre are not Zariski dense in $\mathcal{G}_\lambda$. However, we have seen before, the image of a torsion point ξ of $E^\vee$ is a torsion point of $\mathbf{L}_{\mathcal{P}}^\times|_{E\times\{\xi\}}$; this section thus gives a counterexample to conjecture 8.6.

On the other hand, it is not a counterexample to conjecture 8.5, because $\mathcal{G}_\lambda$ is a special subvariety of $\mathbf{L}_{\mathcal{P}}^\times$. This apparent contradiction stems from the fact that there is a mistake in Pink's proof of the implication 8.5 ⇒ 8.6. The mistake of the proof lies in the fact that Pink claims that any special subvariety of S_2 is a translate of a semiabelian subgroup scheme by a torsion point; in particular it is contained in a closed subgroup scheme. He uses this to translate a statement about mixed Shimura varieties into one about semiabelian schemes. However, this claim is false. In our example, the subvariety $\mathcal{G}_\lambda$ is special, but not contained in any closed subgroup scheme of $\mathbf{L}_{\mathcal{P}}^{\times} \longrightarrow E^{\vee}$. In the next subsections we will classify over which abelian varieties Bertrand's construction can be generalised.

8.3 Abelian subvarieties of $A \times A^{\vee}$ over which the Poincaré bundle is trivial

In this section, I classify the complex abelian varieties over which Bertrand's construction from section 8.2 works, so, for an abelian variety A, subvarieties $X \subset A \times A^{\vee}$ such that the Poincaré bundle restricted to X is trivial. In the terminology of [4], these are called *isotropic*. I start by classifying, for a simple abelian variety A, the set of abelian subvarieties of $A \times A^{\vee}$ over which the square of the Poincaré bundle is trivial. As the Poincaré bundle is birigidified, it is trivial over the abelian subvarieties 0, $A \times 0$ and $0 \times A^{\vee}$ of $A \times A^{\vee}$; the following proposition classifies the other cases. As in section 3.2, for V a (rational, integral) Hodge structure of type $\{(0,-1),(-1,0)\}$, I denote with V^* the Hodge structure $\mathrm{Hom}(V, \mathbf{R}(1))$ (or $\mathrm{Hom}(V, \mathbf{Q}(1))$, $\mathrm{Hom}(V, \mathbf{Z}(1))$, respectively).

Theorem 8.8. *Let $A = \Lambda_{\mathbf{R}}/\Lambda$ be a simple complex abelian variety with dual variety $A^{\vee}$ and let $\mathcal{P}$ be the Poincaré bundle on $A \times A^{\vee}$. Suppose that $\varphi \in \mathrm{Isom}(\Lambda_{\mathbf{Q}}\Lambda_{\mathbf{Q}}^*)$ is such that $\varphi(v)(w) = \varphi(w)(v)$ for all $v, w \in \mathbf{Q} \otimes \Lambda$. Let X be the image of the graph of $\varphi_{\mathbf{R}} : \Lambda_{\mathbf{R}} \longrightarrow \Lambda_{\mathbf{R}}^*$ in $A \times A^{\vee}$. Then X is an abelian subvariety of $A \times A^{\vee}$ such that $\mathcal{P}^2|_X$ is trivial. Conversely, every abelian subvariety X of $A \times A^{\vee}$ such that $\mathcal{P}^2|_X$ is trivial and $X \notin \{0, A \times 0, 0 \times A^{\vee}\}$ is of this form.*

Proof. First, suppose that X is an abelian subvariety of $A \times A^{\vee}$ such that $\mathcal{P}^2|_X$ is trivial, and $X \notin \{0, A \times 0, 0 \times A^{\vee}\}$. Now the first Chern class of the Poincaré bundle is an element of

$$\mathrm{H}^2(A \times A^{\vee}, \mathbf{Q}(1)) \quad = \quad \mathrm{Alt}^2(\Lambda_{\mathbf{Q}} \oplus \Lambda_{\mathbf{Q}}^*, \mathbf{Q}(1))$$

One can calculate (using, for instance, [6, 2.1.2]) that the class of the Poincaré bundle is the alternating bilinear form

$$\begin{aligned} F : (\Lambda_{\mathbf{Q}} \oplus \Lambda_{\mathbf{Q}}^*)^2 &\longrightarrow \mathbf{Q}(1) \\ ((v_1, \xi_1), (v_2, \xi_2)) &\mapsto \xi_1(v_2) - \xi_2(v_1). \end{aligned}$$

Now X corresponds to a sub-$\mathbf{Q}$-Hodge structure V of $\Lambda_{\mathbf{Q}} \oplus \Lambda_{\mathbf{Q}}^*$, with projection maps $\pi_1 : V \longrightarrow \Lambda_{\mathbf{Q}}$ and $\pi_2 : V \longrightarrow \Lambda_{\mathbf{Q}}^*$. Because A and $A^{\vee}$ are both simple, and the Poincaré bundle, or its square, is not trivial over $A \times A^{\vee}$, we get that V is the graph of the $\mathbf{Q}$-linear map $\varphi = \pi_2 \circ \pi_1^{-1} : \Lambda_{\mathbf{Q}} \longrightarrow \Lambda_{\mathbf{Q}}^*$. This map is induced by some element of $\varphi \in \mathrm{Hom}(\mathbf{Q} \otimes A, \mathbf{Q} \otimes A^{\vee})$ such that X is the graph of φ. We may write $V = \Gamma_\varphi(\Lambda_{\mathbf{Q}})$, with

$$\begin{aligned} \Gamma_\varphi : \Lambda_{\mathbf{Q}} &\longrightarrow \Lambda_{\mathbf{Q}} \oplus \Lambda_{\mathbf{Q}}^* \\ v &\mapsto (v, \varphi(v)). \end{aligned}$$

This induces the pullback map $\Gamma_\varphi^*: H^2(A \times A^\vee, \mathbf{Q}) \longrightarrow H^2(A, \mathbf{Q})$, which we can regard as a map $\mathrm{Alt}^2(\Lambda_\mathbf{Q} \oplus \Lambda_\mathbf{Q}^*)(1) \longrightarrow \mathrm{Alt}^2(\Lambda_\mathbf{Q})(1)$. The fact that $\mathcal{P}^2|_X$ is trivial implies that the image of $2F$ under this map is trivial, so the image of F must be as well. But $\Gamma_\varphi^* F$ is the bilinear map

$$\begin{array}{rcl} \Gamma_\varphi^* F : \Lambda_\mathbf{Q} \times \Lambda_\mathbf{Q} & \longrightarrow & \mathbf{Q}(1) \\ (v, w) & \mapsto & \varphi(v)(w) - \varphi(w)(v), \end{array}$$

so the fact that this map is trivial implies that $\varphi(v)(w) = \varphi(w)(v)$ for all $v, w \in \Lambda_\mathbf{Q}$.

Conversely, suppose that φ is of this form. Regarding $\Lambda_\mathbf{R} \oplus \Lambda_\mathbf{R}^*$ as $V \oplus \overline{\Omega}$ again as in section 3.2, we find that $\mathcal{P}$ is represented by the 1-cocycle

$$\begin{array}{rcl} a_\mathcal{P} : \Lambda \times \hat{\Lambda} \times V \times \overline{\Omega} & \longrightarrow & \mathbf{C}^\times \\ (v_0, \xi_0, v_1, \xi_1) & \mapsto & \exp(\pi(\xi_0(v_0) + \overline{\xi_1}(v_0) + \xi_0(v_1)) \end{array}$$

in $\mathrm{H}^1(\Lambda \oplus \Lambda^\vee, \mathrm{H}^0(V \oplus \bar{\Omega}, \mathcal{O}_{V \oplus \bar{\Omega}}^\times))$. In particular on $\Gamma_\varphi(V)$ this is of the form

$$a_{\mathcal{P}|_X}(v_0, \varphi(v_0), v_1, \varphi(v_1)) = \exp(\pi(\varphi(v_0)(v_0) + \overline{\varphi(v_1)(v_0)} + \varphi(v_0)(v_1))),$$

for every $v_0 \in \Lambda$ such that $\varphi(v_0) \in \hat{\Lambda}$. In this terminology, we know that $\operatorname{Im} \varphi(v_0)(v_1) = \operatorname{Im} \varphi(v_1)(v_0)$. This means that we can write this as

$$a_{\mathcal{P}|_X}(v_0, \varphi(v_0), v_1, \varphi(v_1)) = \exp(\pi \varphi(v_0)(v_0)) \cdot \exp(2\pi \operatorname{Re} \varphi(v_0)(v_1))).$$

Now $\exp(2\pi \operatorname{Re} \varphi(v_0)(v_1)))$ only takes real values. The only way this can be holomorphic in v_1, for a fixed $(v_0, \varphi(v_0))$, is if it is constant; call this constant value $f(v_0, \varphi(v_0))$; by substituting 0 for v_1 one sees that $f(v_0, \varphi(v_0)) = \exp(\pi \varphi(v_0)(v_0))$. Because $\operatorname{Im} \varphi(v_0)(v_0) \in \mathbf{Z}$, the complex number $f(v_0, \varphi(v_0))$ is actually real. Furthermore, as the action of $V' \cap (\Lambda \oplus \hat{\Lambda})$ on the set of constant functions on V' is trivial, one sees that

$$\begin{array}{rcl} f(v_0 + w_0, \varphi(v_0 + w_0)) & = & a_{\mathcal{P}|_X}(v_0 + w_0, \varphi(v_0 + w_0), 0, \varphi(0)) \\ & = & a_{\mathcal{P}|_X}(v_0, \varphi(v_0), 0, \varphi(0)) \cdot ((v_0, \varphi(v_0)) \cdot a_{\mathcal{P}|_X}(w_0, \varphi(w_0), 0, \varphi(0))) \\ & = & a_{\mathcal{P}|_X}(v_0, \varphi(v_0), 0, \varphi(0)) \cdot a_{\mathcal{P}|_X}(w_0, \varphi(w_0), 0, \varphi(0)) \\ & = & f(v_0, \varphi(v_0)) \cdot f(w_0, \varphi(w_0)), \end{array}$$

so $f : V' \cap (\Lambda \oplus \hat{\Lambda}) \longrightarrow \mathbf{R}^\times : (v, \varphi(v))$ is a group homomorphism. This means that we have

$$\begin{array}{rcl} \exp(4\pi \varphi(v)(v)) & = & f(2v, \varphi(2v)) \\ & = & f(v, \varphi(v))^2 \\ & = & \exp(2\pi \varphi(v)(v)), \end{array}$$

which shows that $f(v, \varphi(v))^2 = 1$ for all $(v, \varphi(v)) \in V' \cap (\Lambda \oplus \hat{\Lambda})$. But this means that $a_{\mathcal{P}|_X}^2 = a_{\mathcal{P}^2|_X}$ is trivial, so $\mathcal{P}^2|_X$ is trivial. □

Corollary 8.9. *Let A be a complex abelian variety with dual variety $A^\vee$, and let $\psi : \mathbf{Q} \otimes H_1(A) \times \mathbf{Q} \otimes H_1(A) \longrightarrow \mathbf{Q}(1)$ be a polarisation. Let $\varphi \in \mathrm{Aut}(\mathbf{Q} \otimes A)$, and let X be the graph of $\tilde{\psi} \circ \varphi$ in $A \times A^\vee$. Then $\mathcal{P}^2|_X$ is trivial if and only if $R_\psi(\varphi) = -\varphi$.*

Proof. From the previous theorem we see that $\mathcal{P}^2|_X$ is trivial if and only if $(\tilde{\psi} \circ \varphi)(v)(w) = (\tilde{\psi} \circ \varphi)(w)(v)$ for all $v, w \in H_1(A)$. This equation can also be written as $\psi(\varphi(v), w) = \psi(\varphi(w), v)$. As ψ is alternating, this is true if and only if the transpose of φ with respect to ψ is equal to $-\varphi$; but this transpose is equal to $R_\psi(\varphi)$. □

In general, to find endomorphisms φ of A such that $R_\psi(\varphi) = -\varphi$, the following classification of endomorphism algebras of abelian varieties up to isogeny is very useful.

Lemma 8.10. *Let A be a simple abelian variety, and let ψ be a polarisation of A. Let $D = \mathrm{End}(\mathbf{Q} \otimes A)$, $K = Z(D)$, $^\dagger = R_\psi$, and $K_0 = \{x \in K : x^\dagger = x\}$. Then $(D, ^\dagger)$ is of one of the following four types:*

1. *$D = K = K_0$ is a totally real number field, and $^\dagger = id_D$.*
2. *$K_0 = K$ is a totally real number field, and D is a quaternion algebra over K with $D \otimes_{K,\sigma} \mathbf{R} \cong \mathrm{M}_2(\mathbf{R})$ for every embedding $\sigma : K \longrightarrow \mathbf{R}$. There is an $a \in D$. such that $d^\dagger = a d^* a^{-1}$ for all $d \in D$, where * is the quaternion conjugation.*
3. *$K_0 = K$ is a totally real number field, and D is a quaternion algebra over K with $D \otimes_{K,\sigma} \mathbf{R} \cong \mathbf{H}$ for every embedding $\sigma : K \longrightarrow \mathbf{R}$. † is the normal quaternion conjugation.*
4. *K_0 is a totally real number field, K is a totally imaginary quadratic field extension of K_0, and D is a central simple algebra over K such that*
   - *(a) for every finite place v of K with $v = \bar{v}$ one has that $\mathrm{inv}_v(D) = 0$, where $^-$ denotes complex conjugation on K;*
   - *(b) For every place of K one has $\mathrm{inv}_v(D) + \mathrm{inv}_{\bar{v}}(D) = 0$ in $\mathbf{Q}/\mathbf{Z}$.*

   If m is the degree of D as a K-algebra, there is an isomorphism $D \otimes_{\mathbf{Q}} \mathbf{R} \longrightarrow \prod_{\sigma : K_0 \longrightarrow \mathbf{C}} \mathrm{M}_m(\mathbf{C})$ such that † on $D \otimes_{\mathbf{Q}} \mathbf{R}$ corresponds to the involution $(A_1, \ldots, A_t) \mapsto (\overline{A_1}^T, \ldots, \overline{A_t}^T)$.

Proof. See [17, XII]. □

As is remarked in [5], for non-simple A, there are generally many isotropic $X \subset A \times A^\vee$. In fact, the only case when they do not exist (except for the trivial ones $0 \times A$ and $A^\vee \times 0$) is when A is simple and there are restrictions on its endomorphism algebra, as is reflected in the proposition below.

Proposition 8.11. *Let A be an abelian variety. Then there exist isotropic abelian subvarieties X of $A \times A^\vee$, not subvarieties of $A \times 0$ or $0 \times A^\vee$, if and only if $\mathrm{End}(\mathbf{Q} \otimes A)$ is not a totally real number field.*

Proof. Suppose no such X exist; we may ignore isogenies and work in the category $\mathbf{Q}\otimes\mathbf{AbVar_C}$. If A is not simple, say via an isomorphism $A = B \times C$, then $B \times 0 \times 0 \times C^\vee \subset A \times A^\vee$ is an abelian subvariety over which $\mathcal{P}$ is trivial, which is a contradiction, so A is simple. Since the image of every endomorphism of A must either be 0 or all of A, one has that every element of $\mathrm{End}(\mathbf{Q}\times A)\setminus\{0\}$ is invertible, so it has the structure of a division ring. Now fix a polarisation ψ of A and its Rosati involution R_ψ. If there is an element of φ in $\mathrm{End}(\mathbf{Q}\times A)$ such that $R_\psi(\varphi) = -\varphi$, let then $N \in \mathbf{Z}_{>0}$ be such that $\chi(\xi) := N\varphi\circ\tilde{\psi}^{-1}(\xi) \in 2\Lambda$ for all $\xi \in \hat{\Lambda}$. If X is the graph of χ with corresponding subspace $V \subset W \oplus \overline{\Omega}$, then the element of $\mathrm{H}^1(V \cap (\Lambda\cap\hat{\Lambda}), \mathcal{O}_V^\times)$ corresponding to $\mathcal{P}|_X$ is of the form

$$a_{\mathcal{P}|_X}(\chi(\xi_0), \xi_0, \chi(\xi_1), \xi_1) = \exp(\pi\xi_0(\chi(\xi_0))).$$

As was shown in the proof of 8.8, this map factors through $\{\pm 1\}$; but the choice of N ensures that this map is trivial, so $\mathcal{P}|_X$ is trivial. We thus have a contradiction, so no such φ exists; lemma 8.10 now implies that $\mathrm{End}(\mathbf{Q}\times A)$ is a totally real number field.

Conversely, suppose A is an abelian variety such that $\mathrm{End}(\mathbf{Q}\times A)$ is a totally real number field. Fix a polarisation ψ of A, and write $A = \prod_i A_i^{n_i}$, where the A_i are pairwise nonisomorphic simple abelian varieties. Then if $D_i = \mathrm{End}(\mathbf{Q}\otimes A_i)$, one has that $\mathrm{End}(\mathbf{Q}\otimes A) = \prod_i \mathrm{M}_{n_i}(D_i)$. This is only a field if $i = 1$ and $n_1 = 1$, and then (A, R_ψ) is of type 1 in lemma 8.10. By corollary 8.9, the only subvarieties of $A \times A^\vee$ over which $\mathcal{P}$ is trivial are $0, A\times 0$ and $0 \times A^\vee$. □

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