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Non-linear structured population models: an approach with semigroups on measures and Euler's method

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Abstract

In this thesis we study a measure-valued structured population model. We present a functional analytic framework in which we think the type of equations in this model are studied best and we formulate a technique to use the corresponding linear model to get solutions for the non-linear model.

A key in creating a convenient framework is embedding the space of Borel measures in a Banach space that is a subspace of the dual of the bounded Lipschitz functions. We give an existence result for positive mild solutions with values in a Banach space, based on a contraction argument, which yields positive measure-valued solutions to the (semi-) linear model.

To get approximations for the non-linear model, we freeze the coefficients in the equation on an equidistant grid in time and use the solutions of the linear model. These approximations are similar to those obtained by applying the Forward Euler Scheme for ordinary differential equations. We prove that the approximations form a Cauchy sequence that converges and we find a rate of convergence. We present a generalization of this technique that can be applied to a problem formulated in terms of a parametrised non-linear semigroup on a Banach space, where the parameter is determined by a feedback function.

Contents

1 Introduction

Measure valued evolution equations have become a study of interest the past few years. They find applications in population dynamics and crowd-dynamics, but also in stochastic differential equations. We will focus on the study of time evolution of physiologically structured populations.

There is need for a better functional analytic framework to study measure valued evolution equations [6, 19]. This thesis is an attempt to present such a framework and to argue that the framework we present is natural to study these equations. The main goal was to understand the work of Piotr Gwiazda *et al.* in [10] from a functional analytic point of view. Whereas [10] focusses on the dependence of the solution to the model ingredients, we will focus on creating a convenient framework and notation to make the theory of non-linear measure-valued models more readily understandable. The tools we provide can be used to obtain the same results, but can also be applied in different situations.

An interesting idea in [10] is the method of solving the non-linear equation in their model. The coefficients in the non-linear equation are frozen on a equidistant grid on a time-interval and then the solutions of the linear equation can be used on each grid-mesh. By letting the grid size vanish they find a weak solution. This procedure is similar to Euler's method for solving ordinary differential equations. They use the framework of 'mutational equations', which in our view makes this method less transparent than necessary. We present a theory, based on these ideas, that fits nicely in our framework for measure-valued models, that avoids the use of the mutational equations and that can also be used to solve similar non-linear models in function spaces.

We found that Hrvoje Šikić had published some interesting work in [18] that turned out to be useful in developing this functional analytic framework. We used an existence theorem for positive mild solutions and a result that showed the equivalence of two different variation of constants formulas without having to do calculations with generators of semigroups. These theorems were written down in a very general setting, and we reformulated these theorems in our setting. Although avoiding the use of generators was not needed in this thesis, it was interesting to study in its own right and these results can be useful when studying perturbations of semigroups that do not have a generator, or where the generator is difficult to compute, for example with Markov semigroups, which need not be strongly continuous. These results are discussed and presented in Section 2.

Section 3 explains how we can apply this theory to get results for the (linear) measure valued equations from [10]. In Section 4 we deal with the non-linear equations, by applying a method similar to Euler's method for ordinary differential equations.

1.1 Measure-valued models

Equations describing a structured population are usually formulated in terms of densities on the state space of an individual. Integrating this density over a set in the individual's state space yields the expected number of individuals in the population with state in that set on a particular time. In this case, the models are formulated in L^1 spaces. In some cases it is however more natural to formulate such models in terms of measures instead of densities. We will address some of the arguments for using measure-valued models here.

The approach with measures can have some technical advantages. In some cases, it is not clear what function space would be natural to work in when working with density functions. For example, the equations may not be regular enough to ensure that an initial condition that is in L^1 would stay in L^1. We will argue that for measures, there is a natural space to work with.

From a more philosophical perspective, one could also argue that individuals with are modelled best with Dirac measures on the state space. It would be interesting to study when the density models approximate the models with Dirac measures well. A model with Dirac measures is in fact a particle description of the system, which is often used in simulations, especially in crowd-dynamics.

In [4] the study of models with measures is motivated by the fact that in the selection-mutation equations that they study, some solutions tend to stationary states that are measures. These steady states are for example a Dirac mass at the evolutionary stable strategy value.

Another argument would be that a framework with measures could be useful when studying stochastic equations.

An important advantage of the measure-valued approach, mentioned in [10], is the ability to deal with a difficulty of the classical approach: the L^1 norm does not behave well with empirical data. When comparing the model with discrete data from experiments, the L^1 norm can give inconsistent information.

Suppose that we have a real population that has a distribution over some state variable (age, length, etc.) in $\mathbb{R}^+$ that is absolutely continuous with respect to the Lebesque measure. Data from experiments typically consists of the number of individuals that have a state in some interval in the state space for individuals, for different intervals. So this data only approximates the integrals of the density over these intervals, not the density itself. It would be natural that if the intervals are smaller (and thus the experiment is more accurate) then the densities that would fit are close in norm. This is however not the case with the L^1 norm.

For example, suppose that we have the empirical data $\{a_n\}_{n=1}^{\infty}$, where a_n is the number of individuals that have a state in the interval $[nh, (n+1)h)$. A density f that would fit would satisfy $a_n = \int_{[nh,(n+1)h)} f(x)\,\mathrm{d}x$ for all $n \in \mathbb{N}$. However two densities that have the same integral over some interval do not have to be close in L^1 norm: for example the L^1 distance between two peaks that do not overlap is the sum of the L^1 norms of these peaks, even if they are close. To be precise, consider the set

$$A = \left\{ \mu \in \mathcal{M}^+(\mathbb{R}^+) \,:\, a_n = \int_{[nh,(n+1)h)} \mathrm{d}\mu, \quad \text{for all } n \in \mathbb{N} \right\}.$$

Denote with $A \cap L^1$ the set of densities of the measures that are absolutely continuous w.r.t. the Lebesque measure, then $A \cap L^1$ has diameter $2\sum_{n=1}^{\infty} a_n$ with respect to L^1 norm. This does not depend on h, so more accurate experimental data would give no more information on which density to use. In the bounded Lipschitz norm (defined below), the diameter would be $h\sum_{n=1}^{\infty} a_n$. Hence, the L^1 norm may not be the most natural norm to use in equations describing a process when comparing with empirical data.

1.2 From densities to measures: an example

The structured population model that we introduce in this section will be the leading example in this thesis, as it is also studied in [10]. All results were first derived for this specific model and then were generalized as much as possible. In this way it is possible to compare with [10] and check if results are consistent with the existing theory.

The classical version of this structured population model is derived and studied in [19]. A solution $u(\cdot, t) \in L^1(\mathbb{R}^+)$ is found that satisfies

$$\begin{cases} \partial_t u(x,t) + \partial_x\big(F_2\big(u(\cdot,t),x,t\big)u(x,t)\big) = F_3\big(u(\cdot,t),x,t\big)u(x,t), \\ F_2\big(u(\cdot,t),0,t\big)u(0,t) = \int_{\mathbb{R}^+} F_1\big(u(\cdot,t),x,t\big)u(x,t)\,\mathrm{d}x \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

Here $\mathbb{R}^+$ is the state space for individuals and $u(x,t)$ is the density for the number of individuals that have a state $x \in \mathbb{R}^+$ at time $t \in [0,T]$. At zero, mass is inserted or removed and F_1 describes how this depends on the current density and time. When $F_1 \geq 0$, then F_1 can be interpreted as a birth law. F_2 describes a velocity field on the state space which results in a flow of mass on $\mathbb{R}^+$; it tells how the state of an individual changes (e.g. growth or ageing). F_3 is the rate of change of the population mass changes on all states and can be interpreted as death or growth.

To obtain a measure valued version of this model, one could start with substituting $u(\cdot,t)$ with measure valued solutions $\mu_t \in \mathcal{M}(\mathbb{R}^+)$. A first step would be to give meaning to the term $\partial_x(F_2(\mu_t,t)\mu_t)$, for example to interpret this in the sense of distributions. This would suggest to look for weak solutions. We take another approach from the perspective of semigroup theory, explained in Section 3.1 and Section 4.1. We stress that in either approach, the expression $\partial_x(F_2(\mu_t,t)\mu_t)$ is a formal expression and one should be careful how to interpret this.

In [10], the measure valued version of the second line in (1.1) is formulated as

$$F_2(\mu_t,t)(0)\mu_t(0) = \int_{\mathbb{R}^+} F_1(\mu_t,t)(x)\,\mathrm{d}\mu_t(x) \tag{1.2}$$

Yet this expression is erroneous. In general the evaluation of a measure on a set is not continuous. So if one interprets $\mu_t(0)$ as an evaluation on the set $\{0\}$, then (1.2) equates the continuous function on the right hand side to a function on the left that is not always continuous. Moreover, if the measure μ_t is absolutely continuous with the Lebesque measure on $\mathbb{R}^+$ such that it has density $u(\cdot,t)$, then $\mu_t(\{0\}) = 0$ while the expression on the right hand side is non-zero in a non-trivial case. Hence, (1.2) can only be viewed as *representing* the type of boundary condition that is envisioned: adding new mass at state 0 and start with velocity $F_2(u(\cdot,t),0,t)$.

A more natural approach to add mass in zero is to add a Dirac delta measure, which results in the following formal expression of the non-linear model we will investigate in this thesis,

$$\begin{cases} \partial_t\mu_t + \partial_x\big(F_2(\mu_t,t)\mu_t\big) = F_3(\mu_t,t)\mu_t + \big(\int_{\mathbb{R}^+} F_1(\mu_t,t)\,\mathrm{d}\mu_t\big)\,\delta_0 \\ \mu_0 = \nu_0 \in \mathcal{M}^+(\mathbb{R}^+). \end{cases} \tag{1.3}$$

That is, we search for a solution $\mu_t \in \mathcal{M}(\mathbb{R}^+)$ that satisfies (1.3) for $t \in [0,T]$ and $F_1, F_2, F_3 \colon \mathcal{M}(\mathbb{R}^+) \times [0,T] \to \mathrm{BL}(\mathbb{R}^+)$. Note that we take ν_0 to be a positive measure, because it counts the individuals of a certain state in $\mathbb{R}^+$. Furthermore, we could replace the state space $\mathbb{R}^+$ with some other space with a differentiable structure, but for now we stick to $\mathbb{R}^+$ as to compare with [10].

In [4,10] the interpretation of the formal expression (1.3) is done by defining a weak solution. A drawback of this approach is that it is not immediate where this expression comes from and what a weak solution looks like. Furthermore, the approach in [10] seems to involve a lot of tedious computations and they do not establish uniqueness of weak solutions, if it holds at all.

In this thesis, a different approach is investigated: we will study mild solutions. A key in this approach is choosing a suitable Banach space wherein the space of measures $\mathcal{M}(\mathbb{R}^+)$ can be embedded. This results in what we find an elegant and readable theory, where we are able to benefit from powerful tools from functional analysis and theory of linear evolution semigroups in e.g. [9,13].

Before we will study the full non-linear problem, we will turn to the linear version of (1.3). In Section 2 theory is developed for general Banach spaces and applied to this linear version in Section 3.2 to find global mild solutions. In Section 3.3, it is shown that the weak solution that is found in [10] equals the mild solution that is found in this thesis. Besides being more natural and readable, mild solutions of the linear model are unique.

In [10], weak solutions for the non-linear problem in (1.3) are found by using the framework of mutational equations, where existence follows from a compactness argument. We have found a constructive proof that yields a unique solution and a convergence rate for the approximations. In the non-linear case, it is not clear how a mild solution should be defined. We propose a definition, and we prove that our mild solution equals the weak solution in [10] at least for $F_1 \equiv F_3 \equiv 0$ in Section 4.1.

1.3 Notation

Here we briefly introduce and discuss some notation and conventions that are used throughout this thesis.

Let (S,d) be a separable complete metric space (a Polish space). We shall write $\mathrm{BL}(S)$ to denote the vector space of bounded Lipschitz functions from S to $\mathbb{R}$. For $f \in \mathrm{BL}(S)$ we define

$$\|f\|_{\mathrm{BL}} = \|f\|_\infty + |f|_{\mathrm{Lip}}.$$

Here $|f|_{\mathrm{Lip}}$ denotes the Lipschitz constant of f,

$$|f|_{\mathrm{Lip}} = \sup\left\{ \frac{|f(x)-f(y)|}{\mathrm{d}(x,y)} \,:\, x,y \in S, x \neq y \right\},$$

we will also use the shorter notation $|f|_{\mathrm{L}}$ for this. Note that the Sobolev space $W^{1,\infty}(\mathbb{R}^d)$ is isometrically isomorphic with $\mathrm{BL}(\mathbb{R}^d)$, where $\mathbb{R}^d$ is equipped with the usual Euclidean norm. For the dual norm on $\mathrm{BL}(S)^*$ we will write $\|\cdot\|^*_{\mathrm{BL}}$.

With $\mathcal{M}(S)$ we denote the space of signed finite Borel measures on S, and with $\mathcal{M}^+(S)$ we denote the positive cone. With $\mathcal{C}(X,Y)$ we denote the space of continuous functions from X to Y, with X,Y topological spaces.

With a *bounded map* we mean a map that is bounded on bounded sets. We call a map $f\colon S \to \mathbb{R}$ *uniformly bounded* if it has a uniform bound $M > 0$ such that $f(x) \leq M$ for all $s \in S$. Of course, when S is bounded these definitions coincide. There are two cases where this terminology may lead to confusion, so we explain these cases here. With a bounded Lipschitz function $f \in \mathrm{BL}(S)$ we mean a Lipschitz function that is uniformly bounded. With the space $\mathcal{C}_b^1(S)$ we mean the space of continuously differentiable real-valued functions on S that are uniformly bounded and have a derivative that is uniformly bounded.

In Section 2 we mainly deal with semigroups of linear operators, but we also encounter non-linear semigroups of operators. The semigroups we that denote with Roman letters are linear; for the non-linear semigroups we will use the letter Φ or ϕ.

1.4 Embedding of measures in a Banach space

We want to investigate mild solutions in the space $\mathcal{M}(S)$, where S is a Polish space. As mentioned earlier, it is convenient to work with a Banach space to apply results from [9, 13] and Section 2. In [4, Remark 2.6] it is stated that one cannot work in the dual space $[W^{1,\infty}(\mathbb{R}^d)]^*$, but it turns out that this is almost the space that will do if $S = \mathbb{R}^d$. In this section we will investigate the embedding of measures into $\mathrm{BL}(S)^*$, using the results of [12].

A measure $\mu \in \mathcal{M}(S)$ defines a linear functional on $\mathrm{BL}(S)$: $I_\mu(f) = \int_S f \, \mathrm{d}\mu$. The linear map $\mu \mapsto I_\mu : \mathcal{M}(S) \to \mathrm{BL}(S)^*$ is injective [8, Lemma 6], so we can embed $\mathcal{M}(S)$ into $\mathrm{BL}(S)^*$. If we view $\mathcal{M}(S)$ as a subspace of $\mathrm{BL}(S)^*$ than norm-convergence corresponds to narrow convergence. Furthermore, we can use the bounded Lipschitz norm $\|\cdot\|_{\mathrm{BL}}^*$ on measures; this corresponds to the flat metric used in [10]. Note that this norm is natural when studying transport equations, in contrast to the total variation norm. That is, for the total variation norm, denoted in this thesis by $\|\cdot\|_{\mathrm{TV}}$, it holds that $\|\delta_x - \delta_y\|_{\mathrm{TV}} = 2$ for $x, y \in S$, even if $d(x, y)$ is small.

Let

$$D := \operatorname{span}\{\delta_x : x \in S\} = \left\{ \sum_{k=1}^{n} \alpha_k \delta_{x_k} : n \in \mathbb{N}, \alpha_k \in \mathbb{R}, x_k \in S \right\}.$$

We define $\mathcal{S}_{\mathrm{BL}}$ to be the closure of D in $\mathrm{BL}(S)^*$ with respect to $\|\cdot\|_{\mathrm{BL}}^*$. When we use the notation $\mathcal{S}_{\mathrm{BL}}$ we will always mean that it is a normed with $\|\cdot\|_{\mathrm{BL}}^*$. Sometimes we will write $\mathcal{S}_{\mathrm{BL}}(S)$ to emphasis the use of the state space S. By [12, Corollary 3.10], $\mathcal{M}(S)$ is a $\|\cdot\|_{\mathrm{BL}}^*$-dense subspace of $\mathcal{S}_{\mathrm{BL}}$.

A remarkable property of the space $\mathcal{S}_{\mathrm{BL}}$ is that its dual $\mathcal{S}_{\mathrm{BL}}^*$ is isometrically isomorphic to $\mathrm{BL}(S)$ [12, Theorem 3.7], and the way a $\varphi \in \mathrm{BL}(S)$ works on a measure $\mu \in \mathcal{M}(S) \subset \mathrm{BL}(S)^*$ is natural:

$$\langle \mu, \varphi \rangle = I_\mu(\varphi) = \int_S \varphi \, \mathrm{d}\mu. \tag{1.4}$$

We can define an ordering on $\mathcal{S}_{\mathrm{BL}}$ by defining

$$D^+ = \left\{ \sum_{i=1}^{n} \alpha_i \delta_{x_i} : n \in \mathbb{N}, \alpha_i \in \mathbb{R}^+, x_i \in S \right\}, \tag{1.5}$$

and then define $\mathcal{S}_{\mathrm{BL}}^+$ to be the closure of D^+ with respect to $\|\cdot\|_{\mathrm{BL}}^*$. Now it holds that $\mathcal{M}^+(S) = \mathcal{S}_{\mathrm{BL}}^+$ because S is complete [12, Theorem 3.9].

Hence $\mathcal{S}_{\mathrm{BL}}$ is a convenient Banach space to work with. First, it is endowed with the bounded Lipschitz norm, which is a natural norm in this context. And second, if we want to ensure that mild solutions in $\mathcal{S}_{\mathrm{BL}}$ are measure-valued, we only have to require that they are positive, a requirement we had to make anyway in the population model.

2 The Perturbed Abstract Cauchy Problem

Let $\mathcal{X}$ be a Banach space and let $(\hat{T}_t)_{t\geq 0}$ be a strongly continuous semigroup (a C_0 semigroup) of bounded linear operators on $\mathcal{X}$ with generator $(A, \mathcal{D}(A))$. Let $F\colon \mathcal{X} \to \mathcal{X}$ be globally Lipschitz. In this section we consider the Perturbed Abstract Cauchy Problem,

$$\begin{cases} \partial_t u(t) &= Au(t) + F\big(u(t)\big) \\ u(0) &= x_0 \in \mathcal{X} \end{cases}. \tag{2.1}$$

This is the abstract formulation of the system in (1.3) if F_1, F_2 and F_3 do not depend on time and F_2 is linear. In Section 3 we will set $\mathcal{X} = \mathcal{S}_{\mathrm{BL}}$ and use the general theory in this section to obtain solutions for the measure-valued population model.

As explained in the introduction, we will investigate mild solutions.

Definition 2.1. Let $T > 0$. A *mild solution* of (2.1) on $[0, T]$ is a function $u \in \mathcal{C}([0, T], \mathcal{X})$ that satisfies

$$u(t) = \hat{T}_t x_0 + \int_0^t \hat{T}_{t-s} F\big(u(s)\big)\,\mathrm{d}s. \tag{2.2}$$

A global mild solution is a mild solution defined on $\mathbb{R}^+$.

The formula in (2.2) is called the *variation of constants formula*, or VOC. We take the integral in (2.2) to be a Bochner integral, as we are working in an abstract Banach space. A short overview of the theory of Bochner integration can be found in Appendix A. In Section 2.2 we will prove that the integral in (2.2) is well-defined.

In Section 2.1 we will prove an existence theorem for the system in (2.1). In Section 2.3 we will take for $\mathcal{X}$ an ordered vector space and investigate when solutions are positive.

2.1 Solutions to the Cauchy Problem

This section is concerned with proving the following theorem.

Theorem 2.2. *Under the assumptions that F is globally Lipschitz and $(\hat{T}_t)_{t\geq 0}$ is strongly continuous, there exists a unique mild solution $u(t)$ to* (2.1). *This solution has a Lipschitz dependence on the initial solution x_0 and exists globally for all time $t \geq 0$.*

This theorem is a special case of [13, Theorem 6.1.2]. We will give a more detailed proof for our case here. Different ingredients for the proof are formulated in separate lemmas. This is done to make the proof more readable and because these lemmas will be used to prove two variations on this theorem in Section 2.3 and Section C.

We often use the following property, which holds for all C_0 semigroups [13, theorem 1.2.2]:

$$\|\hat{T}_t\|_{\mathcal{L}(\mathcal{X})} \leq Me^{\omega t} \qquad \text{for some } M \geq 1 \text{ and } \omega \in \mathbb{R}. \tag{2.3}$$

for all $t \geq 0$.

First, let us prove that any mild solution of (2.1) will be unique.

Proposition 2.3 (Uniqueness). *Under the assumption that F is globally Lipschitz and $(\hat{T}_t)_{t\geq 0}$ is strongly continuous, every two mild solutions $u\colon I \to \mathcal{X}$ and $v\colon J \to \mathcal{X}$ of* (2.1) *satisfy $u(t) = v(t)$ for all $t \in I \cap J$.*

Proof. Let $u\colon I \to \mathcal{X}$ and $v\colon J \to \mathcal{X}$ be two functions that satisfy (2.2), where $I, J \subset \mathbb{R}^+$. Let $t \in I \cap J$. By Theorem A.7 and (2.3) it holds that,

$$\begin{aligned}\|u(t) - v(t)\| &= \left\| \int_0^t \hat{T}_{t-s} \left[F\big(u(s)\big) - F\big(v(s)\big)\right] \mathrm{d}s \right\| \\ &\leq \int_0^t Me^{\omega(t-s)} \left\|F\big(u(s)\big) - F\big(v(s)\big)\right\| \mathrm{d}s\end{aligned}$$

for some $\omega \in \mathbb{R}$ and $M \geq 1$. Use that F is Lipschitz with $|F|_{\mathrm{L}} \leq L$ to get

$$\|u(t) - v(t)\| \leq ML \int_0^t e^{\omega(t-s)} \|u(s) - v(s)\| \,\mathrm{d}s \tag{2.4}$$

Apply Gronwall's Inequality in Lemma B.1 with $r(t) = \|u(t) - v(t)\|$ and $a(t) = 0$. Then it follows that

$$\|u(t) - v(t)\| \leq 0,$$

so $u(t) = v(t)$. □

Local existence of solutions is obtained by using Banach's Fixed Point Theorem. Let $T > 0$ and define the (non-linear) operator Q on $\mathcal{C}([0,T], \mathcal{X})$ by

$$Q(u)(t) = \hat{T}_t x_0 + \int_0^t \hat{T}_{t-s} F\big(u(s)\big) \,\mathrm{d}s. \tag{2.5}$$

Note that a fixed point of Q will be a mild solution by definition. The fact that $Q(u)$ is continuous is not immediate, it depends on the fact that $\hat{T}_t$ is strongly continuous.

Lemma 2.4. *Under the assumptions of Theorem 2.2, the operator Q defined in* (2.5) *is a well-defined operator from $\mathcal{C}([0,T], \mathcal{X})$ to $\mathcal{C}([0,T], \mathcal{X})$.*

Proof. Let $u \in \mathcal{C}([0,T], \mathcal{X})$. We only have to prove that $Q(u)$ is continuous. Let $\varepsilon > 0$. Take $t, s \in [0,T]$ with $t > s$ and $|t - s| < \delta$, where $\delta > 0$ is to be determined. Compute

$$\begin{aligned}\big\|Qu(t) - Qu(s)\big\| \leq{}& \|\hat{T}_t x_0 - \hat{T}_s x_0\| \\ &+ \left\| \int_0^t \hat{T}_{t-r} F\big(u(r)\big) \,\mathrm{d}r - \int_0^s \hat{T}_{s-r} F\big(u(r)\big) \,\mathrm{d}r \right\|.\end{aligned} \tag{2.6}$$

Because $(\hat{T}_t)_{t\geq 0}$ is strongly continuous, the map $t \mapsto \hat{T}_t x$ is continuous for every $x \in \mathcal{X}$. So we can take $\delta_0 > 0$ such that $\|\hat{T}_t x_0 - \hat{T}_s x_0\| < \frac{1}{3}\varepsilon$, if $|t - s| < \delta_0$.

Rewrite the remaining part of (2.6) as

$$\begin{aligned}&\left\| \int_0^t \hat{T}_{t-r} F\big(u(r)\big) \,\mathrm{d}r - \int_0^s \hat{T}_{s-r} F\big(u(r)\big) \,\mathrm{d}r \right\| \\ &\leq \left\| \int_0^{t-s} \hat{T}_{t-r} F\big(u(r)\big) \,\mathrm{d}r \right\| + \left\| \int_0^s \hat{T}_{s-r} \left[F\big(u(r{-}s{+}t)\big) - F\big(u(r)\big)\right] \mathrm{d}r \right\|.\end{aligned} \tag{2.7}$$

Using Theorem A.7 and (2.3), the norm of first integral can be estimated as

$$\left\| \int_0^{t-s} \hat{T}_{t-r} F\big(u(r)\big)\,\mathrm{d}r \right\| \le \int_0^{t-s} M e^{\omega(t-r)} \left\|F\big(u(r)\big)\right\| \,\mathrm{d}r,$$

for some $M \ge 1$ and $\omega \in \mathbb{R}$. Because u is continuous, $B = \{u(r) : r \in [0,T]\}$ is a bounded set. Since F is Lipschitz, $F[B]$ is bounded in norm, say with $C > 0$. So we can proceed by writing

$$\begin{aligned} \left\| \int_0^{t-s} \hat{T}_{t-r} F\big(u(r)\big)\,\mathrm{d}r \right\| &\le CM \int_0^{t-s} e^{\omega(t-r)}\,\mathrm{d}r \\ &\le CM \max(1, e^{\omega T})(t-s). \end{aligned} \tag{2.8}$$

Let $L > 0$ be the Lipschitz constant of F. The last integral of (2.7) can be estimated as

$$\begin{aligned} &\left\| \int_0^{s} \hat{T}_{s-r} \left[F\big(u(r{-}t{+}s)\big) - F\big(u(r)\big) \right] \,\mathrm{d}r \right\| \\ &\qquad\qquad \le ML \int_0^{s} e^{\omega(s-r)} \left\| u\big(r-(t{-}s)\big) - u(r) \right\| \,\mathrm{d}r. \end{aligned} \tag{2.9}$$

Because u is continuous, we can take $\delta_1 > 0$ such that $\left\|u\big(r-(t{-}s)\big) - u(r)\right\| < \frac{1}{3}\varepsilon_1$ if $|t-s| < \delta_1$. Here we choose $\varepsilon_1 = (MLT\max(1,e^{\omega T}))^{-1}\varepsilon$.

Now we can see that we have to choose $\delta > 0$ such that

$$\delta < \min\left(\delta_0,\, \delta_1,\, \frac{\varepsilon}{3CM\max(1,e^{\omega T})}\right).$$

Going back to (2.6) and filling in the estimates in (2.8) and (2.9) gives us

$$\|Qu(t) - Qu(s)\| < \tfrac{1}{3}\varepsilon + CM\max(1,e^{\omega T})\,\delta + MLT\max(1,e^{\omega T})\tfrac{1}{3}\varepsilon_1 \;<\; \varepsilon.$$

So $Q(u)$ is continuous. □

In fact, Q maps the space $\mathcal{Z}$ of bounded measurable maps $u\colon [0,T] \to \mathcal{X}$ to itself. To prove this statement, the requirement of strong continuity of $(\hat{T}_t)_{t\ge 0}$ can be weakened. Accordingly the approach in the proof of the next lemma and of Theorem 2.2 can also be used to get a fixed point of Q in $\mathcal{Z}$. We then obtain a mild solution that is only bounded and measurable. This is formulated and proved in Section C.

Similarly, Q maps the space $\mathrm{BL}([0,T],\mathcal{X})$ of bounded Lipschitz functions to itself if one requires $\hat{T}_t x$ to be Lipschitz in time for all $x \in \mathcal{X}$. This can readily be seen from the proof of Lemma 2.4. Then we obtain a mild solution that is Lipschitz in time.

Now let's return to the proof of Theorem 2.2. The goal is to apply the Banach Fixed Point Theorem to Q. That is, we want Q to be a contraction on a Banach space. The space $\mathcal{C}([0,T],\mathcal{X})$ is indeed a Banach space if we endow it with the norm

$$\|u\|_\infty = \sup_{t\in[0,T]} \|u(t)\|.$$

The proof of the completeness of the space $\mathcal{C}([0,T],\mathcal{X})$ is exactly the same as for the space of real-valued bounded continuous functions $\mathcal{C}_b([0,T])$. See for example [5, page 65].

Now Q is almost a contraction on the Banach space $\mathcal{C}([0,T],\mathcal{X})$.

Lemma 2.5. *Under the assumption that F is Lipschitz continuous and $(\hat{T}_t)_{t\geq 0}$ is strongly continuous, the operator Q is a contraction on $\mathcal{C}([0,T'],\mathcal{X})$ for some $T' \leq T$.*

Proof. Let $u, v \in \mathcal{C}([0,T],\mathcal{X})$. Let $L > 0$ be the Lipschitz constant of F. Using Bochner's Theorem, the Lipschitz continuity of F and the bound in (2.3) we can write

$$\begin{aligned}\|Q(u)(t) - Q(v)(t)\| &= \left\| \int_0^t \hat{T}_{t-s} \left[F\big(u(s)\big) - F\big(v(s)\big) \right] \mathrm{d}s \right\| \\ &\leq LM \int_0^t e^{\omega(t-s)} \, \|u(s) - v(s)\| \, \mathrm{d}s \\ &\leq LM \max\big(1, e^{\omega t}\big) \; t \, \|u - v\|_\infty. \end{aligned} \tag{2.10}$$

There exists a $T' > 0$ such that $LM \max(1, e^{\omega t})\, t < 1$ for all $t \in [0, T']$. Then Q is a contraction on $\mathcal{C}([0,T'],\mathcal{X})$. □

Now we are in a position to prove Theorem 2.2.

Proof of Theorem 2.2. Let $T' > 0$ as in Lemma 2.5 and define the operator $Q\colon \mathcal{C}([0,T'],\mathcal{X}) \to \mathcal{C}([0,T'],\mathcal{X})$ as before by

$$Q(u)(t) = \hat{T}_t x_0 + \int_0^t \hat{T}_{t-s} F\big(u(s)\big) \, \mathrm{d}s.$$

Now Q is well-defined by Lemma 2.4 and from Lemma 2.5 it follows that Q is a contraction on the Banach space $(\mathcal{C}([0,T'],\mathcal{X}), \|\cdot\|_\infty)$.

By Banach's Fixed Point Theorem, there exists a unique fixed point of Q. By definition, this fixed point is a (local) mild solution of (2.1) with initial condition x_0.

Now we will prove that $u(t)$ is defined for all $t \geq 0$ (a more constructive argument will be given in the proof of Theorem 2.12). Let $\mathcal{U}$ be the set of all local mild solutions with initial condition x_0. For $u \in \mathcal{U}$ we denote by I_u the domain of u. Note that if $u, \hat{u} \in \mathcal{U}$ are such that $I_{\hat{u}} \subset I_u$, then it follows from the uniqueness of mild solutions, Proposition 2.3, that $\hat{u}$ is the restriction of u to the domain $I_{\hat{u}}$. Let $I_{\max} = \bigcup_{u\in\mathcal{U}} I_u$. It is now possible to define $u(\cdot, x_0)\colon I_{\max} \to \mathcal{X}$ by

$$u(t; x_0) = u(t) \quad \text{with } u \in \mathcal{U} \text{ such that } t \in I_u. \tag{2.11}$$

Indeed, if $u, \hat{u} \in \mathcal{U}$ both are such that $t \in I_u$ resp. $t \in I_{\hat{u}}$, then Proposition 2.3 guarantees that $u(t) = \hat{u}(t)$ and thus the function $u(\cdot, x_0)$ is well-defined. Note that we at least have $[0, T'] \subset I_{\max}$.

Let $T_{\max} = \sup I_{\max}$. If we assume that $T_{\max} < \infty$, then we would be able to construct a mild solution $\hat{u}\colon [0,\ T_{\max} + \frac{1}{2}T'] \to \mathcal{X}$ by defining

$$\hat{u}(t) = \begin{cases} u(t, x_0) & \text{if } t \in [0, T_{\max}) \\ u\big(t - T_{\max},\ u(T_{\max} - \tfrac{1}{2}T',\ x_0)\big) & \text{if } t \in [T_{\max},\ T_{\max} + \tfrac{1}{2}T']. \end{cases}$$

So by definition of $I_{\max}$, we have $[0,\ T_{\max} + \frac{1}{2}T'] \subset I_{\max}$, which contradicts with $T_{\max} = \sup I_{\max}$. Hence it holds that $T_{\max} = \infty$ and thus $u(t, x_0)$ is a global solution for $t \in \mathbb{R}^+$.

It remains to show the Lipschitz dependence of $u(\cdot, x_0)$ on x_0. Let $x, y \in \mathcal{X}$ be two initial conditions. Then

$$\begin{aligned}\|u(t,x)-u(t,y)\| &\leq \|\hat{T}_t x - \hat{T}_t y\| + \left\| \int_0^t \hat{T}_{t-s} \left[F\big(u(s,x)\big) - F\big(u(s,y)\big)\right] \mathrm{d}s \right\| \\ &\leq Me^{\omega t}\|x-y\| + \int_0^t Me^{\omega(t-s)} \left\|F\big(u(s,x)\big) - F\big(u(s,y)\big)\right\| \mathrm{d}s,\end{aligned}$$

for some $M \geq 1$ and $\omega \in \mathbb{R}$. Use that F is Lipschitz with Lipschitz constant $|F|_{\mathrm{L}} \leq L$ to get

$$\begin{aligned}\underbrace{\|u(t,x)-u(t,y)\|}_{r(t)} \leq \underbrace{M\max(1, e^{\omega t})\|x-y\|}_{a(t)} \\ + \int_0^t \underbrace{Me^{\omega(t-s)}L}_{b(s)} \|u(s,x)-u(s,y)\| \,\mathrm{d}s.\end{aligned}$$

Now apply Gronwall's lemma with the indicated variables. Equation (B.2) gives

$$r(t) \leq a(t) \left[1 + ML\int_0^t e^{\omega(t-s)}\,\mathrm{d}s \cdot \exp\left(ML\int_0^t e^{\omega(t-s)}\,\mathrm{d}s\right)\right]. \tag{2.12}$$

Denote the part between brackets with $1 + C(t)$. We now have

$$\|u(t,x)-u(t,y)\| \leq M\max(1, e^{\omega t})\|x-y\|\big(1+C(t)\big).$$

So $x \mapsto u(t,x)$ is Lipschitz continuous. □

2.2 The Variations of Constants Formula

Recall the variation of constants formula as introduced in the introduction of Section 2,

$$u(t) = \hat{T}_t x + \int_0^t \hat{T}_{t-s} F\big(u(s)\big)\,\mathrm{d}s.$$

Here $\mathcal{X}$ is a Banach space, $x \in \mathcal{X}$, $u \in \mathcal{C}([0,T], \mathcal{X})$, $F\colon \mathcal{X} \to \mathcal{X}$ is Lipschitz and $(\hat{T}_t)_{t\geq 0}$ is a C_0 semigroup of bounded linear operators on $\mathcal{X}$. If $u(\cdot, x)$ is the unique solution to this equation, then we can write $u(t) = V_t x$, where $(V_t)_{t\geq 0}$ is a strongly continuous semigroup on $\mathcal{X}$. Indeed, the semigroup property follows from the uniqueness and the strong continuity follows from the fact that $u(\cdot, x)$ is continuous for each $x \in \mathcal{X}$. So then we could also write

$$V_t x = \hat{T}_t x + \int_0^t \hat{T}_{t-s}\big[F\left(V_s x\right)\big]\,\mathrm{d}s.$$

Be aware that if F is not linear, then the operators V_t are not linear for all t.

Definition 2.6. The semigroup $(V_t)_{t\geq 0}$ constructed above will be called the semigroup of solutions associated to the mild solution u or to the model in (2.1).

Later, in Section 2.4, we will see that V_s and T_{t-s} can be interchanged in this expression if F is linear. This will become important when we will compare our results with [10] in Section 3.3.

Now we will turn our attention to the fact whether the VOC formula is well-defined. For this we will take a bit technical detour in Bochner measurability and integrability. The integral has to be well-defined and therefore it is natural to look closer at the concept of an integrable semigroup.

Definition 2.7. A semigroup $(T_t)_{t\geq 0}$ of operators on a Banach space $\mathcal{X}$ is an integrable semigroup if $t \mapsto T_t x$ is Bochner-measurable on $[0, \infty)$ for all $x \in \mathcal{X}$ and there exist $M \geq 1$ and $\omega \geq 0$ such that

$$\|T(t)\| \leq Me^{\omega t} \qquad \text{for all } t \geq 0. \tag{2.13}$$

Let $(T(t))_{t\geq 0}$ be an integrable semigroup and $x\colon \mathbb{R}^+ \to \mathcal{X}$ a bounded measurable map. The requirements in Definition 2.7 guarantee that $T_s[x(s)]$ is (Bochner) integrable on bounded intervals: by Lemma A.8 the function $s \mapsto T_s[x(s)]$ is measurable and if $I \subset \mathbb{R}^+$ is a bounded interval, then $\|x(t)\| \leq M_I$ for some $M_I > 0$ and it holds that

$$\int_I \|T_s[x(s)]\| \,\mathrm{d}s \leq \int_I De^{\omega t} M_I \,\mathrm{d}s < \infty, \tag{2.14}$$

so by Bochner's Theorem, $T_s[x(s)]$ is integrable.

We can now prove that the VOC formula is well-defined if the semigroup used is integrable. The map $u\colon \mathbb{R}^+ \to \mathcal{X}$ is continuous, so by Proposition A.9 it is measurable. If $F\colon \mathcal{X} \to \mathcal{X}$ is Lipschitz, then $s \mapsto F(u(s))$ is a bounded map and it is measurable by Lemma A.10. Set $x(s) = F(u(s))$ in the argument before and it follows that $T_s[F(u(s))]$ is integrable.

The following proposition guarantees that the mild solution (2.2) in Section 2.1 is well-defined.

Proposition 2.8. *A strongly continuous semigroup on a Banach space $\mathcal{X}$ is an integrable semigroup.*

Proof. Let $(\hat{T}_t)_{t\geq 0}$ be a strongly continuous semigroup on a Banach space $\mathcal{X}$. For every $z \in \mathcal{X}$ the map $t \mapsto \hat{T}_t z$ from $\mathbb{R}^+$ to $\mathcal{L}(\mathcal{X})$ is continuous, so by Proposition A.9 it is measurable. The bound in (2.13) holds for all strongly continuous semigroups [13, theorem 1.2.2]. □

In fact, an integrable semigroup is almost a C_0 semigroup. Theorem 10.2.3 in [11] states that if $t \mapsto \hat{T}_t x$ is measurable for all $x \in \mathcal{X}$, then $\hat{T}_t$ is strongly continuous for $t > 0$. So one could say that an integrable semigroup is strongly continuous but in 0.

Therefore it is not surprising that all theorems in Section 2.1 and Section 2.3 can be reformulated in terms of integrable semigroups without having to do major modifications to the proofs. An example can be found in Section C.

2.3 Positivity of solutions

Let $\mathcal{B}$ be an ordered Banach space over $\mathbb{F}$, and denote with $\mathcal{B}^+$ the cone of positive elements of $\mathcal{B}$. This section will be concerned with establishing the right conditions under which mild solutions of (2.1) will be positive.

Definition 2.9. A semigroup $(T(t))_{t\geq 0}$ of operators on an ordered Banach space $\mathcal{B}$ is a positive semigroup if $T(t)x \in \mathcal{B}^+$ for all $t \geq 0$ and $x \in \mathcal{B}^+$.

Let $(T_t)_{t\geq 0}$ be a positive C_0 semigroup of bounded linear operators on $\mathcal{B}$ with generator $(A, \mathcal{D}(A))$. Let $F\colon \mathcal{B}^+ \to \mathcal{B}$ be a Lipschitz map and consider

$$\begin{cases} \partial_t u(t) = Au(t) + F\big(u(t)\big) \\ u(0) = x \in \mathcal{B}^+. \end{cases} \tag{2.15}$$

A mild solution u of (2.15) is positive if $u(t) \in \mathcal{B}^+$ for all $t \geq 0$. In other words: a mild solution is positive if its corresponding semigroup is positive.

We will formulate a natural condition on F that ensures that there exists a unique mild solution that is positive. The approach we will use was published by Šikić in [18] for a very general setting. The framework that Šikić uses is so general that it is difficult to grasp the main idea of the approach. One of the goals of this section is to present the ideas of Šikić in our setting to show that these ideas are in fact quite powerful and useful. The connection between our framework and that of Šikić is explained later in this section.

To see the main idea in the approach of Šikić in [18], rewrite (2.15) as

$$\begin{cases} \partial_t u(t) = (A - B)u(t) + (F + B)u(t) \\ u(0) = x \in \mathcal{B}^+, \end{cases} \tag{2.16}$$

where B is an operator on $\mathcal{B}$. Note that for all $a \in \mathbb{F}$ the operator $(A - a\,\mathrm{I})$ is the generator the positive semigroup $e^{-at}T_t$. So if can we choose a such that $(F + a\,\mathrm{I})$ is a positive operator on $\mathcal{B}$ and take B such that $B(x) = ax$ then any classical solution of (2.16), and thus of (2.15), is positive.

So what about the mild solutions of (2.16)? Is a mild solution of (2.16) also a mild solution of (2.15) and can we then prove the positivity of this mild solution? Suppose that $(S_t)_{t\geq 0}$ is the semigroup with generator $A - B$. A mild solution of (2.16) is

$$u(t) = S(t)x + \int_0^t S(t-s)\left[(F+B)\big(u(s)\big)\right] \mathrm{d}s. \tag{2.17}$$

Now apply the same trick as before by setting $S_t = e^{-at}T_t$. Corollary 2.11 states that in this case u is a mild solution of (2.15). Positivity is proved in Theorem 2.12. The right hand side will turn out to be a positive function of $u(s)$, and with an induction argument the existence and positivity of u are established using this idea.

The point is in defining S_t. We could obtain S_t by using the Bounded Perturbation Theorem in [9, III 1.3], which states that $A - B$ indeed generates a strongly continuous semigroup if B is bounded and linear. Then Corollary 2.11 would indeed follow straight away. But then we heavily rely on the fact that T_t has a generator A.

Now Šikić uses a lemma that is independent of generators and therefore can be applied to cases where T_t is not strongly continuous. Here the idea is to let S_t be the semigroup of mild solutions for (2.1), but then with the bounded linear perturbation $-B$ instead of F. Then it is proved that (2.17) holds. See [18, Lemma 3.1] or our reformulation, Lemma 2.10 below.

But wait, a closer look on the definition of S_t preceding [18, Lemma 3.1] and in Lemma 2.10 reveals that a different version of the VOC-formula is used. This may seem as a concession to make the proof work, but it is in fact the main ingredient of a very nice and useful result: for bounded linear perturbations this different version of the VOC formula is equivalent to the normal one without using generators (see Section 2.4). Especially in this light the results needed to prove existence of a positive solution is only a special case of the lemma (formulated in Corollary 2.11).

Yet what makes this lemma elegant mostly is that it solves our problems on the level of mild solutions, without using generators and using only elementary or natural steps in the proof. Of course, here also lies its power, as it can be extended to situations where the semigroups do not have generators.

Returning once to the explanation of the main idea, using equation (2.16), Šikić says that this lemma shows that the mild solution u 'behaves nicely with respect to further linear perturbation'. Maybe a better way to put it is that u behaves nicely with respect to an *other* linear perturbation, and keep the formulation with generators in equation (2.16) in mind.

In [4, lemma 3.2], exactly the same approach is taken to prove that the solutions of a specific model is positive, and the same explanation with generators is given. They however prove the positivity of solutions first for functions in $L^1(\mathbb{R}^d)$, and then use a density argument to get positivity of their measure-valued solution. So their proof for positivity of solutions has to be done for every different model, although they skip the proof for other models because there 'analogous arguments are applicable'. Apparently this works for the models they present, but it does not give insight in what are precisely the requirements to ensure that the solution of a model will be positive. Our approach will be in a general ordered Banach space and we will formulate a general positivity requirement. The results can be applied to measures by setting $\mathcal{B} = \mathcal{S}_{\mathrm{BL}}$ and noting that the positive elements of $\mathcal{S}_{\mathrm{BL}}$ are precisely the positive measures. This method also works for measures on a Polish space S, where there is no natural candidate for a measure μ such that $L^1(S, \mu)$ is dense in $\mathcal{M}(S)$ with the $\|\cdot\|^*_{\mathrm{BL}}$-topology.

The following lemma is a reformulation of Lemma 3.1 in [18]. The operator F can also be taken measurable instead of continuous. Important to note is that equation (2.18) is not the same as the regular variation of constants formula. For the purpose of this section this does not give any problems, as can be seen in the proof of Corollary 2.11. In fact, it is the key to Corollary 2.13 in Section 2.2.

Lemma 2.10. *Let $\mathcal{X}$ be a Banach space and $\mathcal{Y} \subset \mathcal{X}$ be a subset. Let $(T_t)_{t\geq 0}$ be a C_0 semigroup of bounded linear operators on $\mathcal{X}$. Let $(S_t)_{t\geq 0}$ be a C_0 semigroup such that for all $x \in \mathcal{X}$*

$$S_t x = T_t x + \int_0^t S_s \left[B\, T_{t-s} x\right] \mathrm{d}s, \tag{2.18}$$

where $B\colon \mathcal{X} \to \mathcal{X}$ is a bounded linear operator. Let $F\colon \mathcal{Y} \to \mathcal{X}$ be a bounded continuous map. If $u \in \mathcal{C}(\mathbb{R}^+, \mathcal{Y})$ is a continuous map with $u(0) = x$ that satisfies the regular variation of constants formula,

$$u(t) = T_t x + \int_0^t T_{t-s} \left[F\big(u(s)\big)\right] \mathrm{d}s, \tag{2.19}$$

with $x \in \mathcal{Y}$, then for every $x \in \mathcal{Y}$ and $t \geq 0$,

$$u(t) = S_t x + \int_0^t S_{t-s}\left[(F-B)\big(u(s)\big)\right]\,\mathrm{d}s.$$

Proof. Let $x \in \mathcal{Y}$. Consider the following integral, where the equality follows from substituting S with the expression in (2.18),

$$\begin{aligned}&\int_0^t S_s\left[F\big(u(t-s)\big)\right]\,\mathrm{d}s\\ &= \int_0^t T_s\left[F\big(u(t-s)\big)\right]\,\mathrm{d}s + \int_0^t\int_0^s S_r\left[B\big[T_{s-r}\left[F\big(u(t-s)\big)\right]\big]\right]\,\mathrm{d}r\,\mathrm{d}s.\end{aligned} \tag{2.20}$$

If we replace s with $t-s$ in the integral in (2.19), then we get

$$\int_0^t T_s\left[F\big(u(t-s)\big)\right]\,\mathrm{d}s = u(t) - T_t x, \tag{2.21}$$

which yields an expression for the first part of (2.20). The rest of this proof is concerned with rewriting the double integral in (2.20).

First we apply Theorem A.11, the Fubini Theorem for Bochner integrals. The map $s \mapsto u(t-s)$ is measurable by Proposition 2.8 and since F is continuous, we can apply Lemma A.10 to get that $s \mapsto F(u(t-s))$ is measurable. Clearly $(s,r) \mapsto s-r$ is measurable so the composition $(s,r) \mapsto T_{s-r}\left[F(u(t-s))\right]$ is measurable by applying Lemma A.8 two times. By assumption B is continuous and $(S_r)_{r\geq 0}$ is an integrable semigroup, so we can respectively apply Lemma A.10 and Lemma A.8 again to get that the integrand is measurable with respect to the product measure. So the double integral can be written as

$$\int_0^t\int_r^t S_r\left[B\big[T_{s-r}\left[F\big(u(t-s)\big)\right]\big]\right]\,\mathrm{d}s\,\mathrm{d}r. \tag{2.22}$$

Notice that $S_r \circ B$ is a bounded linear operator, so (2.22) can be rewritten as

$$\begin{aligned}&\int_0^t (S_r\circ B)\int_r^t T_{s-r}\left[F\big(u(t-s)\big)\right]\,\mathrm{d}s\,\mathrm{d}r\\ &= \int_0^t (S_r\circ B)\int_0^{t-r} T_w\left[F\big(u(t-r-w)\big)\right]\,\mathrm{d}w\,\mathrm{d}r && \text{change } (s-r)\to w\\ &= \int_0^t (S_r\circ B)\left[u(t-r) - T_{t-r}x\right]\,\mathrm{d}r && \text{by (2.21)}\\ &= \int_0^t (S_r\circ B)u(t-r)\,\mathrm{d}r - \int_0^t (S_r\circ B)T_{t-r}x\,\mathrm{d}r && \text{by linearity of } S_r\circ B\\ &= \int_0^t S_r\left[Bu(t-r)\right]\,\mathrm{d}r - (S_t x - T_t x) && \text{by (2.18).}\end{aligned}$$

Turning back to equation (2.20), we now have

$$\int_0^t S_s\left[F\big(u(t-s)\big)\right]\,\mathrm{d}s = u(t) - T_t x + \int_0^t S_r\left[Bu(t-r)\right]\,\mathrm{d}r - S_t x + T_t x, \tag{2.23}$$

which, since S_s is linear, finishes the proof. $\square$

We could also write $V_t x$ instead of $u(t)$ in equation (2.19), where $(V_t)_{t\geq 0}$ would be a family of (non-linear) operators. Then the statement in Lemma 2.10 would be more symmetric, like in [18]. But here we stick to the notation $u(t)$ to stress the non-linearity and to avoid the suggestion that V_t would be a semigroup of linear operators (which it is not).

Now set $Bx = ax$ in Lemma 2.10 for some $a > 0$ to prove that any mild solution of (2.15) is a mild solution of (2.16) and vice versa.

Corollary 2.11. *Let $\mathcal{X}$ be a Banach space and $\mathcal{Y} \subset \mathcal{X}$ be a subset. Let $a > 0$ and let $G\colon \mathcal{Y} \to \mathcal{X}$ be a continuous map. Let $(\hat{T}_t)_{t\geq 0}$ be a C_0 semigroup of bounded linear operators on $\mathcal{X}$ and let $u \in \mathcal{C}(\mathbb{R}^+, \mathcal{Y})$ be a continuous function with $u(0) = x$. Then u is a solution of*

$$u(t) = \hat{T}_t x + \int_0^t \hat{T}_{t-s}\left[G\big(u(s)\big)\right] \mathrm{d}s \tag{2.24}$$

for all $x \in \mathcal{Y}$, if and only if it satisfies for all $x \in \mathcal{Y}$

$$u(t) = e^{-at}\hat{T}_t x + \int_0^t e^{-a(t-s)}\hat{T}_{t-s}\left[(G+a)\big(u(s)\big)\right] \mathrm{d}s. \tag{2.25}$$

Proof. The forward implication is a direct application of Lemma 2.10. Suppose that $u(t)$ satisfies (2.24), let $T_t = \hat{T}_t$ and define $S_t = e^{at}T_t$. The operator B given by $B(x) = ax$ is bounded and linear and we can write

$$\begin{aligned}\int_0^t S_s[B(T_{t-s}x)]\,\mathrm{d}s = \int_0^t ae^{as}T_t x\,\mathrm{d}s = \left(\int_0^t ae^{as}\,\mathrm{d}s\right) T_t x \\ = \left(e^{at}-1\right)T_t x = S_t x - T_t x,\end{aligned} \tag{2.26}$$

so equation (2.18) is satisfied. Now equation (2.25) follows from Lemma 2.10 by replacing a with $-a$ and setting $F = G$.

For the other implication suppose that $u(t)$ satisfies (2.25). Let $T_t = e^{-at}\hat{T}_t$ and define $S_t = e^{at}T_t = \hat{T}_t$. Again let the operator B now be given by $B(x) = ax$. Now equation (2.26) again holds, something that is seen best if one forgets about $\hat{T}_t$. So equation (2.18) is satisfied.

Define the operator F as $F(x) = (G+B)(x) = (G+a)(x)$. Now by assumption equation (2.19) is satisfied. So by Lemma 2.10 u now satisfies

$$u(t) = S_t x + \int_0^t S_{t-s}\left[(F-B)[u(s)]\right] \mathrm{d}s.$$

Substituting S_t by $\hat{T}_t$ and F by $G + B$ gives the desired result. □

The next theorem is a variation on two theorems of Šikić: [18, Theorem 3.1] and [18, Corollary 4.1]. However, Šikić requires that G should satisfy some boundedness condition and that $\mathcal{B}$ is a Banach lattice. In return the condition on G is (2.27) is weakened.

Furthermore, in [18] a generalized version of integrable semigroups is used instead of strongly continuous semigroups. Let us now explain the connection between this other definition and the definitions in this thesis. A positive integrable semigroup is the analogue of [18, Definition 1.3]. In [18] the notions of measurability and integration are more general than here and the integrability of $T_s[x(s)]$ is required a priori in the definition of a positive integrable semigroup. The proof that in our setting the integrability of $T_s[x(s)]$ follows from the requirements we make in Definition 2.7 can be found in [18, Example 1]. An alternative proof is given in Lemma A.8. In the example of Šikić, the results in [3, Lemma 6.4.6] and [11, Theorem 10.2.3] are used implicitly, whereas Lemma A.8 only uses Pettis' Measurability Theorem.

In Lemma 2.10 and Corollary 2.11 the strong continuity can be replaced by integrability without modifying the proof. In that case, Theorem 2.12 can also be proven for the integrable case.

Theorem 2.12. *Let $\mathcal{B}$ be an ordered Banach space such that the cone of positive elements $\mathcal{B}^+$ is closed and let $(T_t)_{t\geq 0}$ be a positive strongly continuous semigroup of bounded linear operators on $\mathcal{B}$. If $G\colon \mathcal{B}^+ \to \mathcal{B}$ is a Lipschitz map such that there exists an $a > 0$ for which*

$$G(x) + ax \in \mathcal{B}^+ \qquad \textit{whenever } x \in \mathcal{B}^+, \tag{2.27}$$

then there exists a unique mild solution $u(t) \in \mathcal{C}(\mathbb{R}^+, \mathcal{B})$,

$$u(t) = T_t x_0 + \int_0^t T_{t-s} G\big(u(s)\big)\,\mathrm{d}s, \tag{2.28}$$

such that $u(t) \in \mathcal{B}^+$ for all $t \geq 0$ and $x_0 \in \mathcal{B}^+$.

Proof. Let $K > 0$ and $a > 0$ such that the assumption in (2.27) holds. Let $T > 0$ and $x_0 \in \mathcal{B}^+$ be arbitrary. Define the (non-linear) operator Q on $\mathcal{C}([0,T],\mathcal{B})$ as

$$Q(u)(t) = e^{-at} T_t x_0 + \int_0^t e^{-a(t-s)} T_{t-s} \left[(G+a)u(t)\right]\,\mathrm{d}s. \tag{2.29}$$

By Lemma 2.4, Q is well-defined and by Lemma 2.5 there exists a $T' > 0$ such that Q is a contraction on $\mathcal{C}([0,T'],\mathcal{B})$ with respect to $\|\cdot\|_\infty$. Define $u_0 \equiv 0$ and $u_{k+1} = Su_k$ for $k \in \mathbb{N}$. By the proof of Banach's Fixed Point Theorem $u = \lim_{k\to\infty} u_k$ exists and is a fixed point of Q. By Corollary 2.11, u satisfies (2.28) for $t \in [0, T']$.

It remains to prove that $u(t)$ is positive and that it is defined for all $t \geq 0$. We will prove that $u_k(t) \in \mathcal{B}^+$ for all $t \geq 0$ and $k \in \mathbb{N}$ by induction over k. Clearly the claim holds for u_0. Suppose that the claim holds for $k \leq n$. For all $t \in [0, T']$ it holds that

$$u_{n+1}(t) = e^{-at} T_t x_0 + \int_0^t e^{-a(t-s)} T_{t-s} \left[(G+a)u_n(t)\right]\,\mathrm{d}s. \tag{2.30}$$

First note that $T_t x_0 \in \mathcal{B}^+$ because $(T_t)_{t\geq 0}$ is assumed to be positive. Since $e^{-at} \geq 0$ the first term of (2.30) is positive. By assumption $u_n(t) \in \mathcal{B}^+$, so from (2.27), the fact that T_t is positive and $e^{-a(t-s)} \geq 0$ it follows that the integrand in (2.30) is positive.

The fact that the integral in (2.30) is positive follows from a version of the mean value theorem for Bochner integrals: in [7, Corollary II 2.8] it is stated that for every Bochner-measurable function $f\colon [0,T] \to \mathcal{B}$ and $0 \leq t \leq T$ the integral $\frac{1}{t}\int_0^t f(s)\,\mathrm{d}s$ is contained in the closed convex hull of $f([0,t])$. If f is positive, so that $f([0,t]) \subset \mathcal{B}^+$, then the closed convex hull of $f([0,t])$ is contained $\mathcal{B}^+$ because $\mathcal{B}^+$ is closed and convex. So $\frac{1}{t}\int_0^t f(s)\,\mathrm{d}s \in \mathcal{B}^+$ and thus the integral $\int_0^t f(s)\,\mathrm{d}s$ is positive.

From the previous, it follows that the integral in (2.30) is positive. Hence $u_{n+1}(t) \in \mathcal{B}^+$ for all $t \in [0,T']$ and by induction the claim holds for all $k \in \mathbb{N}$. Since $\mathcal{B}^+$ was assumed to be closed we now have that $u(t) \in \mathcal{B}^+$ for all $t \in [0,T']$.

Because T' does not depend on the initial condition, we can extend $u(t)$ to a solution on $\mathbb{R}^+$: the full solution $u(t)$ equals on $[nT',(n+1)T']$ the positive solution to (2.28) with positive initial condition $x_0 = u(nT')$ for all $n \in \mathbb{N}$. Hence $u(t)$ is a positive global mild solution. □

2.4 The Variations of Constants Formula: the linear case

As already mentioned in the previous sections, the VOC formula can be written in a different way if the perturbation is linear. We will need this result in Section 3.3 when comparing our results with [10].

From [9, Corollary III 1.1.7] it already follows that the two VOC formulas are equivalent, but the advantage of the approach taken here is that it is avoids computations involving generators, and therefore the results can easily be generalized to semigroups which are not strongly continuous. We note that that this is not important for the application in Section 3.3, but it was a remarkable result from Lemma 2.10 that was worth investigating on its own right.

To prove Corollary 2.13, it would be natural to take $B = F$ in Lemma 2.10. If we prove that there exists an integrable semigroup S_t that satisfies the different variation of constants formula in (2.18), then Lemma 2.10 gives that $S_t x_0 = u(t)$ and we are done. The proof of the existence of S_t is straightforward however long, whereas the proof below is much shorter. Therefore, this 'natural' approach is deferred to the appendix, in Section C. The proof below only shows that the integrals in the normal and the new VOC-formulas are the same by smartly rewriting formulas and then applying Lemma 2.10. It uses the same reasoning as in [18, Section 4].

Corollary 2.13. *Let $\mathcal{X}$ be a Banach space and $G\colon \mathcal{X} \to \mathcal{X}$ a bounded linear operator. Let $(P_t)_{t\geq 0}$ and $(U_t)_{t\geq 0}$ be C_0 semigroups. Then U_t satisfies*

$$U_t x = P_t x + \int_0^t P_{t-s}\big[G(U_s x)\big]\,\mathrm{d}s \tag{2.31}$$

for all $x \in \mathcal{X}$ if and only if U_t also satisfies for all $x \in \mathcal{X}$

$$U_t x = P_t x + \int_0^t U_s\big[G(P_{t-s} x)\big]\,\mathrm{d}s \tag{2.32}$$

Proof. The backward implication immediately follows from Lemma 2.10 by setting $B = G$.

So suppose that U_t is the (unique) solution of (2.31). Rewrite (2.31), using the linearity the operators involved, to get

$$P_t x = U_t x + \int_0^t P_{t-s}[-G(U_s x)]\,\mathrm{d}s. \tag{2.33}$$

Set $S_t = P_t$ and $T_t = U_t$ and $B = -G$ in Lemma 2.10 and note that equation (2.18) is satisfied by doing a change of variables in (2.33).

Apply Theorem 2.2 with $\hat{T}_t = U_t$ and $F = -G$ to get a mild solution $u(t)$. Now $u(t)$ satisfies (2.19) with $F = -G$. That is,

$$u(t) = U_t x + \int_0^t U_{t-s}[-G(u(s))]\,\mathrm{d}s \tag{2.34}$$

Since $F - B = -G - (-G) = 0$, Lemma 2.10 states that $u(t) = P_t x$. So we can equate (2.33) and (2.34) and subsequently substitute $u(s)$ with $P_s x$ to get

$$\int_0^t P_{t-s}[G(U_s x)]\,\mathrm{d}s = \int_0^t U_{t-s}[G(P_s x)]\,\mathrm{d}s.$$

After substituting s with $t - s$ on the right hand side this yields that equations (2.31) and (2.32) are the same. □

3 The linear population model

Before we turn to the non-linear model in (1.3) we will study a linear version:

$$\begin{cases} \partial_t \mu_t + \partial_x(b\mu_t) = c\mu_t + \langle a, \mu_t \rangle \delta_0 \\ \mu_0 = \nu_0 \in \mathcal{M}^+(\mathbb{R}^+). \end{cases} \tag{3.1}$$

Here $a, b, c \colon \mathbb{R}^+ \to \mathbb{R}$ are bounded Lipschitz functions. We will find solutions to (3.1) as follows. We look at the semi-linear model

$$\begin{cases} \partial_t \mu_t + \partial_x(b\mu_t) = F(\mu_t) \\ \mu_0 = \nu_0 \in \mathcal{M}^+(\mathbb{R}^+), \end{cases} \tag{3.2}$$

where $F \colon \mathcal{S}_{\mathrm{BL}}(\mathbb{R}^+) \to \mathcal{S}_{\mathrm{BL}}(\mathbb{R}^+)$ is a Lipschitz map. First we will study this equation for $F \equiv 0$ in Section 3.1. We will define a semigroup $(P_t)_{t\geq 0}$ on $\mathcal{S}_{\mathrm{BL}}(\mathbb{R}^+)$ that is induced by the flow on the state space that corresponds to this transport equation. Then we apply the perturbation results from Section 2 to obtain a mild solution, and we take this as the definition of a mild solution of (3.2).

Definition 3.1. A mild solution of (3.2) is a function $\mu \in \mathcal{C}(\mathbb{R}^+, \mathcal{S}_{\mathrm{BL}}(\mathbb{R}^+))$ that satisfies

$$\mu_t = P_t \nu_0 + \int_0^t P_{t-s} F(\mu_s) \, \mathrm{d}s,$$

where $(P_t)_{t\geq 0}$ is the semigroup corresponding to the model in (3.3).

If in addition we require that F satisfies the positivity requirement in Theorem 2.12, then these mild solutions have range in $\mathcal{S}^+_{\mathrm{BL}}(\mathbb{R}^+)$, which equals $\mathcal{M}^+(\mathbb{R}^+)$. Thus, we find positive measure-valued mild solutions of (3.2). Then we set $F(\mu_t) = c\mu_t + \langle a, \mu_t \rangle \delta_0$ and we will find conditions on a, b and c such that F satisfies the conditions mentioned in Section 3.2. Hence, we find a positive mild solution to (3.1).

In (3.1) we use the state space $\mathbb{R}^+$ as to give meaning to birth in the point zero (the term with δ_0), to give meaning to the x-derivative and to compare with [10]. When considering only a flow on the state space then we can also find a semigroup induced by this flow and apply the perturbation results, so we could as well have chosen a general Polish space S and try to find solutions in $\mathcal{M}^+(S)$. In this case it is however not clear how to formulate the model as in (3.2).

In Section 3.3 we will compare these results to the results on the linear model from [10]. It will turn out that the mild solutions we find are the same as the solutions that are found in [10].

3.1 Construction of a semigroup on measures induced by a flow on the state space

The goal of this section is to define a semigroup on $\mathcal{S}_{\mathrm{BL}}(\mathbb{R}^+)$ that corresponds to the linear transport model

$$\begin{cases} \partial_t \mu_t + \partial_x(b\mu_t) = 0 \\ \mu_0 = \nu_0 \end{cases} \tag{3.3}$$

where $\nu_0 \in \mathcal{M}(\mathbb{R}^+)$ is given initial data and b is a function on $\mathbb{R}^+$. We will formulate appropriate conditions on b to ensure that the semigroup we are searching will be strongly continuous and positive.

If we view μ_t as a density, the system (3.3) is just the classical transport equation, where mass is transported in a way that is determined by b. The idea that densities change due to transportation of mass caused by an underlying flow on the state space $\mathbb{R}^+$, is used here to get a semigroup on measures. This will be done by implying sufficient conditions on b such that the underlying flow will be a Lipschitz semigroup and then use the concepts treated in [12] to define a strongly continuous semigroup on a well-suited space of measures that is induced by the flow.

Consider the ordinary differential equation for this underlying flow,

$$\begin{cases} \partial_t x(t) = b\big(x(t)\big) \\ x(0) = x_0, \end{cases} \tag{3.4}$$

where $b\colon \mathbb{R}^+ \to \mathbb{R}$ is a bounded Lipschitz continuous function with Lipschitz constant $|b|_{\mathrm{Lip}}$, such that $\mathbb{R}^+$ is positively invariant under b and where $x_0 \in \mathbb{R}^+$. Now we can apply Theorem 2.12 with $T_t = I$ for all $t \geq 0$, $\mathcal{B} = \mathbb{R}$ and $F = b$. It reduces to the well-known existence result for ordinary differential equations. Thus we get a unique (mild) solution $x(t; x_0)$ such that $x(t; x_0) \in \mathbb{R}^+$ for all $t \geq 0$ and $x_0 \in \mathbb{R}^+$. Note that from (2.27) it follows that we must require that $b(0) \geq 0$.

Because the solutions are unique and exist globally in time, we can associate a dynamical system ϕ_t to (3.4) by means of

$$\phi_t(x_0) = x(t, x_0).$$

The function $x \mapsto \phi_t(x)$ is Lipschitz continuous (by for example Theorem 2.2 or Lemma 3.3 below). So ϕ_t is a Lipschitz map on $\mathbb{R}^+$ and thus $(\phi_t)_{t\geq 0}$ is a Lipschitz semigroup according to [12, Def. 5.2].

Now the construction of a semigroup as explained in [12] can be applied here. Define $S_\phi(t)$ by

$$S_\phi(t)f := f \circ \phi_t \tag{3.5}$$

for $f \in \mathrm{BL}(\mathbb{R}^+)$ and $t \geq 0$. S_ϕ is a semigroup of bounded linear operators on $\mathrm{BL}(\mathbb{R}^+)$ and thus the dual operators S_ϕ^* form a semigroup of bounded linear operators on $\mathrm{BL}(\mathbb{R}^+)^*$. We now can define a semigroup $(P_t)_{t\geq 0}$ of bounded linear operators on $\mathcal{S}_{\mathrm{BL}}$ by restricting S_ϕ^* to $\mathcal{S}_{\mathrm{BL}}$.

In fact, $(P_t)_{t\geq 0}$ is a strongly continuous semigroup on $\mathcal{S}_{\mathrm{BL}}$ because the conditions of [12, Theorem 5.5] are satisfied: $(\phi_t)_{t\geq 0}$ is strongly continuous and by Lemma 3.3 *(i)* below,

$$\limsup_{t\downarrow 0} |\phi_t|_{\mathrm{Lip}} \leq \lim_{t\to 0} \left(1 + |b|_{\mathrm{L}} t e^{|b|_{\mathrm{L}} t}\right) = 1 < \infty. \tag{3.6}$$

By [12, Corollary 5.7], P_t leaves $\mathcal{M}^+(\mathbb{R}^+)$ invariant, so P_t is a strongly continuous semigroup on $\mathcal{M}^+(\mathbb{R}^+)$.

Definition 3.2. The semigroup $(P_t)_{t\geq 0}$ constructed above is called the semigroup induced by (the flow) ϕ_t, or the semigroup corresponding to the model in (3.3).

An important property of P_t is that it satisfies the following identity:

$$\langle P_t\mu, f\rangle = \langle \mu, f\circ\phi_t\rangle\,, \tag{3.7}$$

for all $f\in \mathrm{BL}(S)\cong \mathcal{S}_{\mathrm{BL}}^*$ and $\mu\in\mathcal{S}_{\mathrm{BL}}$. This identity can be obtained by using the dual semigroup $S_\phi(t)$ of P_t and equation (3.5). Alternatively, one can view $P_t\mu$ as the pushforward of μ under ϕ_t:

$$P_t\mu = \mu\circ\phi_t^{-1} = \phi_t\#\mu \tag{3.8}$$

Note that from a more general perspective we could also have started with a dynamical system ϕ_t on some Polish space S satisfying some conditions, instead of using the flow in (3.4) on $\mathbb{R}^+$. However in this case it is difficult to give meaning to the term $\partial_x(b\mu_t)$ in (3.3).

The following lemma shows some estimates of ϕ_t and P_t. We will use this later in Section 4.

Lemma 3.3. *Let $b\in\mathrm{BL}(\mathbb{R}^+)$ and let $(\phi_t)_{t\geq 0}$ be the semigroup of solutions of the associated flow in (3.4). Let $(P_t)_{t\geq0}$ be the induced semigroup on $\mathcal{S}_{\mathrm{BL}}(\mathbb{R}^+)$. Then the following assertions hold.*

(i) $|\phi_t(x)-\phi_t(y)|\leq \left(1+t|b|_{\mathrm{L}}e^{|b|_{\mathrm{L}}t}\right)|x-y|$ *for all* $x,y\in\mathbb{R}^+$ *and* $t\geq 0$.

(ii) $\|P_t\mu-P_t\nu\|_{\mathrm{BL}}^*\leq \left(1+t|b|_{\mathrm{L}}e^{|b|_{\mathrm{L}}t}\right)\|\mu-\nu\|_{\mathrm{BL}}^*$ *for all* $\mu,\nu\in\mathcal{S}_{\mathrm{BL}}(\mathbb{R}^+)$, $t\geq 0$.

(iii) $\|P_t\mu-P_s\mu\|_{\mathrm{BL}}^*\leq \|\mu\|_{\mathrm{TV}}\|b\|_\infty|t-s|$ *for all* $\mu\in\mathcal{S}_{\mathrm{BL}}(\mathbb{R}^+)$ *and* $t,s\geq 0$.

Proof. *(i)* This follows directly from the proof of Theorem 2.2 on page 13 by setting $M=1$ and $\omega=0$ in equation (2.12), but it is also straightforward to prove directly. Using the variation of constants formula for $\phi_t(x)$ and $\phi_t(y)$ we can write

$$\begin{aligned}|\phi_t(x)-\phi_t(y)| &\leq |x-y| + \int_0^t \left|b\big(\phi_r(x)\big)-b\big(\phi_r(y)\big)\right|\,\mathrm{d}r\\ &\leq |x-y| + |b|_{\mathrm{L}}\int_0^t|\phi_r(x)-\phi_r(y)|\,\mathrm{d}r.\end{aligned}$$

An application of Gronwall's Lemma gives the desired result.

(ii) For all $f\in\mathrm{BL}(\mathbb{R}^+)$ it holds that

$$|\left\langle P_t\mu-P_t\nu,\, f\right\rangle| = |\langle\mu-\nu,\, f\circ\phi_t\rangle|\leq \|\mu-\nu\|_{\mathrm{BL}}^*\|f\circ\phi_t\|_{\mathrm{BL}}.$$

We have $|f\circ\phi_t|_{\mathrm{L}}\leq|f|_{\mathrm{L}}|\phi_t|_{\mathrm{L}}$ and $\|f\circ\phi_t\|_\infty\leq\|f\|_\infty$, so

$$\|f\circ\phi_t\|_{\mathrm{BL}}\leq\max(1,|\phi_t|_{\mathrm{L}})\|f\|_{\mathrm{BL}}.$$

From *(i)* we know that $|\phi_t|_{\mathrm{L}}\leq 1+te^{|b|_{\mathrm{L}}t}$, so

$$|\left\langle P_t\mu-P_t\nu,\, f\right\rangle|\leq\left(1+te^{|b|_{\mathrm{L}}t}\right)\|f\|_{\mathrm{BL}}\|\mu-\nu\|_{\mathrm{BL}}^*,$$

which yields *(ii)* by definition of $\|\cdot\|_{\mathrm{BL}}^*$.

(iii) Using the variation of constants formula for ϕ_t and ϕ_s we can write

$$|\phi_t(x)-\phi_s(x)|\leq\int_s^t\left|b\big(\phi_r(x)\big)\right|\,\mathrm{d}r\leq\|b\|_\infty|t-s|.$$

so, using only the definition of P_t and $\langle \cdot, \cdot \rangle$, we see that for all $f \in \mathrm{BL}(\mathbb{R}^+)$ it holds that

$$
\begin{aligned}
|\langle (P_t - P_s)\mu, f \rangle| &\leq \int_{\mathbb{R}^+} |f|_{\mathrm{L}} |\phi_t(x) - \phi_s(x)| \, \mathrm{d}|\mu|(x) \\
&\leq \|\mu\|_{\mathrm{TV}} |f|_{\mathrm{L}} \|b\|_\infty |t - s|,
\end{aligned}
$$

which proves *(iii)*.

□

3.2 Perturbation of the constructed semigroup

Let $(A, \mathcal{D}(A))$ be the generator of the strongly continuous semigroup $(P_t)_{t \geq 0}$ found in Section 3.1. To get a mild solution of (3.1), we want to apply Theorem 2.12 to find a mild solution of

$$
\begin{cases}
\partial_t \mu_t = -A\mu_t + c\mu_t + \langle a, \mu_t \rangle \delta_0 \\
\mu_0 = \nu_0 \in \mathcal{M}^+(\mathbb{R}^+)
\end{cases}
\tag{3.9}
$$

Therefore, we use the Banach space $\mathcal{S}_{\mathrm{BL}}(\mathbb{R}^+)$ with positive cone $\mathcal{M}^+(\mathbb{R}^+)$ and $G = c\mu_t + \langle a, \mu_t \rangle \delta_0$. Recall from Section 1.4 that the positive cone of $\mathcal{S}_{\mathrm{BL}}(\mathbb{R}^+)$ indeed is equal to $\mathcal{M}^+(\mathbb{R}^+)$, so the positive mild solution found is measure-valued. If we apply Theorem 2.2 then we get a mild solution with range in $\mathcal{S}_{\mathrm{BL}}(\mathbb{R}^+)$, so solutions would not necessarily be measures-valued. Furthermore, since this is a population model only positive measures make sense.

In this section we will check which conditions we have to put on a and c such that we can apply Theorem 2.12. With the results of this section, we can prove the following theorem.

Theorem 3.4. *Let $a, b, c \colon \mathbb{R}^+ \to \mathbb{R}$ be bounded Lipschitz functions such that $b(0) \geq 0$. If a is a non-negative function then the model in* (3.1) *has a unique positive (measure-valued) mild solution.*

Proof. Let $(P_t)_{t \geq 0}$ be the semigroup corresponding to the model in (3.3). Note that this is possible because we required $b(0) \geq 0$. Since P_t leaves $\mathcal{M}^+(\mathbb{R}^+)$ invariant, it is a positive semigroup on $\mathcal{S}_{\mathrm{BL}}(\mathbb{R}^+)$.

Define $G = c\mu_t + \langle a, \mu_t \rangle \delta_0$. By the results in this section, G is Lipschitz continuous. The requirement $a \geq 0$ ensures that G satisfies the positivity requirement (2.27) in Theorem 2.12. Now all conditions of Theorem 2.12 are met, so there exists a unique positive mild solution of (3.1). That is, there exists a function $\mu \colon \mathbb{R}^+ \to \mathcal{S}_{\mathrm{BL}}(\mathbb{R}^+)$ such that $\mu_t \in \mathcal{M}^+(\mathbb{R}^+)$ for all $t \geq 0$. □

3.2.1 Conditions on the death operator

Let (S, d) be a metric space. Let $c \colon S \to \mathbb{R}$ be a (uniformly) bounded, real-valued Borel-measurable function on S and define the operator

$$
\begin{aligned}
F \colon \mathcal{M}(S) &\longrightarrow \mathcal{M}(S) \\
\mu &\longmapsto c(\cdot)\mu,
\end{aligned}
\tag{3.10}
$$

where $c(\cdot)\mu$ denotes the measure that has density c with respect to μ. For F to be well-defined we need conditions on c such that $c(\cdot)\mu$ is a finite Borel measure.

By definition of $c(\cdot)\mu$, it is finite only if $c \in L^1(\mu)$ [2, Def. 17.1]. So we have to require that $c \in L^1(\gamma)$ for all finite measures $\gamma \in \mathcal{M}(S)$. For this to hold, it suffices to require that c is uniformly bounded.

Lemma 3.5. *Let $c\colon S \to \mathbb{R}$ be a uniformly bounded function. The function F defined in* (3.10) *is Lipschitz continuous with respect to $\|\cdot\|_{\mathrm{BL}}^*$ if and only if c Lipschitz. In that case, $|c|_{\mathrm{L}} \leq |F|_{\mathrm{L}} \leq \|c\|_{\mathrm{BL}}$.*

Proof. Suppose that c is a bounded Lipschitz function. Let $\mu, \nu \in \mathcal{M}(S)$. By definition it holds that

$$\begin{aligned}\|F(\mu) - F(\nu)\|_{\mathrm{BL}}^* &= \|c(\cdot)\mu - c(\cdot)\nu\|_{\mathrm{BL}}^* \\ &= \sup\{|\langle f\,,\, c(\cdot)\mu - c(\cdot)\nu\rangle| : f \in \mathrm{BL}(S), \|f\|_{\mathrm{BL}} \leq 1\}\end{aligned}$$

By [2, Theorem 17.3] it holds that $\langle f, c(\cdot)\gamma\rangle = \langle cf, \gamma\rangle$ for all $f \in \mathrm{BL}(S)$ and $\gamma \in \mathcal{M}(S)$, so

$$\|F(\mu) - F(\nu)\|_{\mathrm{BL}}^* = \sup\{|\langle fc\,,\, \mu - \nu\rangle| : f \in \mathrm{BL}(S), \|f\|_{\mathrm{BL}} \leq 1\} \tag{3.11}$$

Next observe that $\mathrm{BL}(S)$ is a Banach algebra, so we have $\|fc\|_{\mathrm{BL}} \leq \|f\|_{\mathrm{BL}}\|c\|_{\mathrm{BL}}$. Now we can make the estimate

$$|\langle fc\,,\, \mu - \nu\rangle| \leq \|\mu - \nu\|_{\mathrm{BL}}^* \|fc\|_{\mathrm{BL}} \leq \|\mu - \nu\|_{\mathrm{BL}}^* \|f\|_{\mathrm{BL}} \|c\|_{\mathrm{BL}}$$

Hence

$$\|F(\mu) - F(\nu)\|_{\mathrm{BL}}^* \leq \|\mu - \nu\|_{\mathrm{BL}}^* \|c\|_{\mathrm{BL}}.$$

So F is Lipschitz continuous with Lipschitz constant $\|c\|_{\mathrm{BL}}$.

Now suppose that F is Lipschitz continuous with Lipschitz constant L. That is, for all $\mu, \nu \in \mathcal{M}(S)$ it holds that

$$\|F(\mu) - F(\nu)\|_{\mathrm{BL}}^* \leq L\|\mu - \nu\|_{\mathrm{BL}}.$$

Let $x, y \in S$ arbitrary. It holds that

$$\begin{aligned}|c(x) - c(y)| &= |\langle c, \delta_x - \delta_y\rangle| \\ &\leq \sup\{|\langle cf, \delta_x - \delta_y\rangle : f \in \mathrm{BL}(S), \|f\|_{BL} \leq 1\} \\ &= \|F(\delta_x) - F(\delta_y)\|_{\mathrm{BL}}^* \qquad \text{by (3.11)} \\ &\leq L\|\delta_x - \delta_y\|_{\mathrm{BL}}^*.\end{aligned}$$

By [12, lemma 3.5] this is well-defined because δ_x, δ_y are in $\mathrm{BL}(S)^*$ and furthermore

$$\|\delta_x - \delta_y\|_{\mathrm{BL}}^* \leq \min\bigl(2, d(x, y)\bigr) \leq d(x, y).$$

So it follows that

$$|c(x) - c(y)| \leq Ld(x, y).$$

Hence c is Lipschitz continuous. Since we required c to be bounded beforehand, we can conclude that $c \in \mathrm{BL}(S)$. ☐

It is straightforward to see that F defined by 3.10 satisfies the positivity requirement (2.27) in Theorem 2.12. Simply note that $c(x) + \|c\|_\infty \in \mathbb{R}^+$ for all $x \in S$ to get that for each $\mu \in \mathcal{M}^+(S)$

$$F(\mu) + \|c\|_\infty \mu = (c + \|c\|_\infty)\mu \in \mathcal{M}^+(S). \tag{3.12}$$

3.2.2 Lipschitz conditions for the birth operator

Recall from Section 1.4 that $\mathrm{BL}(\mathbb{R}^+) \cong \mathcal{S}_{\mathrm{BL}}(\mathbb{R}^+)^*$. So if we let $a \in \mathrm{BL}(\mathbb{R}^+)$, then we can define

$$\begin{aligned} F\colon \mathcal{M}(\mathbb{R}^+) &\longrightarrow \mathcal{M}(\mathbb{R}^+) \\ \mu &\longmapsto \langle a, \mu\rangle \delta_0. \end{aligned} \tag{3.13}$$

Here we work with the state space $\mathbb{R}^+$ to give meaning to birth in the point 0 in the state space.

First note that F is Lipschitz with respect to $\|\cdot\|_{\mathrm{BL}}^*$. Let $\mu, \nu \in \mathcal{M}(\mathbb{R}^+)$. Since $\|\delta_0\|_{\mathrm{BL}}^* = 1$ we have

$$\|F(\mu) - F(\nu)\|_{\mathrm{BL}}^* = |\langle a,\, \mu - \nu\rangle| \leq \|a\|_{\mathrm{BL}} \|\mu - \nu\|_{\mathrm{BL}}^*, \tag{3.14}$$

So F is Lipschitz with Lipschitz constant less than or equal to $\|a\|_{\mathrm{BL}}$.

The positivity requirement (2.27) in Theorem 2.12 states that there has to be a $d > 0$ such that for all $\mu \in \mathcal{M}^+(\mathbb{R}^+)$ it holds that

$$F(\mu) + d\mu = \langle a, \mu\rangle \delta_0 + d\mu \in \mathcal{M}^+(\mathbb{R}^+). \tag{3.15}$$

Here one can see that we have to require that $\langle a, \mu\rangle \geq 0$ for all $\mu \in \mathcal{M}^+(\mathbb{R}^+)$, in other words, that a is a positive functional. Hence the requirement in (2.27) is satisfied if we take for a a non-negative function in $\mathrm{BL}(\mathbb{R}^+)$.

3.3 Comparison with other approaches

In this section we will compare our results with [10]. To be precise, we will prove that the mild solution of (3.1) we found in Theorem 3.4 are the same as the solutions that are found in [10]. The solutions that are found in [10] will be called *solutions obtained via the dual problem.* Indeed, in [10] a dual problem is posed, solutions are found for this dual problem and they turn out to define the solutions of the original problem. It is however unclear at first sight how this dual problem relates to the original problem and what the solutions of the dual problem mean. It will turn out that the solutions of the dual problem are related to the dual semigroup of solutions, by reversing the time.

Indeed the expressions used in both approaches are closely related. However it took some time to understand the connection thoroughly find the and right proofs to show this.

To improve readability, we will first set $c \equiv 0$ as to see the model with only the birth operator and then set $a \equiv 0$ to study the model with only the death operator. Proposition 3.9 and Proposition 3.11 are the key in comparing the solutions for the problem with only birth and the problem with only death respectively. In Proposition 3.9, the definition of a mild solution is written down and then it follows that the definition of a *solution obtained via the dual problem* holds. In Proposition 3.11, it is done the other way around: it is proved that a *solution obtained via the dual problem* satisfies the variation of constants formula.

Again we will stick to the notation that is introduced in Section 3.1. So throughout this section $(\phi_t)_{t\geq 0}$ is the Lipschitz semigroup of solutions of (3.4) and P_t is the semigroup induced by $(\phi_t)_{t\geq 0}$ as defined in Section 3.1.

3.3.1 The linear population model with birth

Consider the linear structured population model with birth

$$\begin{cases} \partial_t \mu_t + \partial_x (b\mu_t) = \langle a, \mu_t \rangle \delta_0 \\ \mu_0 = \nu_0, \end{cases} \tag{3.16}$$

where $a, b\colon \mathbb{R}^+ \to \mathbb{R}$ are bounded Lipschitz functions with $b(0) > 0$ and $\nu_0 \in \mathcal{M}(\mathbb{R}^+)$ is given initial data. By definition $\mu\colon [0,T] \to \mathcal{S}_{\mathrm{BL}}(\mathbb{R}^+), t \mapsto \mu_t$ is called a mild solution to (3.16) if it is continuous and satisfies

$$\mu_t = P_t \nu_0 + \int_0^t P_{t-s} \left[\langle a, \mu_s \rangle \delta_0\right] \mathrm{d}s. \tag{3.17}$$

By Theorem 3.4, there exists a unique mild solution because $\mu_t \mapsto \langle a, \mu_t \rangle \delta_0$ is Lipschitz continuous on $\mathcal{S}_{\mathrm{BL}}(\mathbb{R}^+)$, with respect to $\|\cdot\|_{\mathrm{BL}}^*$.

In [10, Definition 3.1] the concept of a weak solution of (3.16) is defined. First let us write down the definition of a weak solution in our notation.

Definition 3.6. $\mu\colon [0,T] \to \mathcal{S}_{\mathrm{BL}}(\mathbb{R}^+), t \mapsto \mu_t$ is called a *weak solution* to (3.16) if it is continuous and for all $\varphi \in \mathcal{C}_b^1(\mathbb{R}^+ \times [0,T])$,

$$\langle \varphi(\cdot, T), \mu_T \rangle - \langle \varphi(\cdot, 0), \nu_0 \rangle = \int_0^T \left\langle \tfrac{\partial \varphi}{\partial t}(\cdot, t) + \tfrac{\partial \varphi}{\partial x}(\cdot, t) b(\cdot) + \varphi(0,t) a(\cdot)\,,\, \mu_t \right\rangle \mathrm{d}t$$

In [10], a weak solution is found, but since that weak solution is not necessarily unique, it is not necessarily the same as our mild solution. However it is not difficult to prove that the mild solution in (3.17) is a weak solution.

Proposition 3.7. *A mild solution of* (3.16) *is a weak solution of* (3.16).

Indeed the following proof of Proposition 3.7 is not a deep proof, it is just a bit long and consists of elementary steps and long expressions. It will turn out later that Proposition 3.7 is also a result of the stronger statements in Proposition 3.9. Still, this proof gives an idea of how the weak solution should be interpreted from the perspective of mild solutions and shows how one can use the flexible notation of our approach.

Proof. Let μ be a mild solution of (3.16). We have to check if the expression in Definition 3.6 holds. So let us calculate

$$\langle \varphi(\cdot, T), \mu_T \rangle - \langle \varphi(\cdot, 0), \nu_0 \rangle - \int_0^T \left\langle \tfrac{\partial \varphi}{\partial t}(\cdot, t) + \tfrac{\partial \varphi}{\partial x}(\cdot, t) b(\cdot)\,,\, \mu_t \right\rangle \mathrm{d}t \\ - \int_0^T \left\langle \varphi(0,t) a(\cdot)\,,\, \mu_t \right\rangle \mathrm{d}t \tag{3.18}$$

and hope that the result will be 0. Substitute the expression for the mild solution (3.17) into the terms of the expression above. The first term becomes

$$\langle \varphi(\cdot, T), \mu_T \rangle = \langle \varphi(\cdot, T), P_T \nu_0 \rangle + \int_0^T \left\langle \varphi(\cdot, T), P_{T-s}\left[\langle a, \mu_s \rangle \delta_0\right] \right\rangle \mathrm{d}s. \tag{3.19a}$$

Take the first integral in (3.18) and calculate

$$\int_0^T \left\langle \tfrac{\partial\varphi}{\partial t}(\cdot,t) + \tfrac{\partial\varphi}{\partial x}(\cdot,t)b(\cdot)\,,\, \mu_t \right\rangle \mathrm{d}t = \int_0^T \left\langle \tfrac{\partial\varphi}{\partial t}(\cdot,t) + \tfrac{\partial\varphi}{\partial x}(\cdot,t)b(\cdot)\,,\, P_t\nu_0 \right\rangle \mathrm{d}t$$
$$+ \int_0^T \int_0^t \left\langle \tfrac{\partial\varphi}{\partial t}(\cdot,t) + \tfrac{\partial\varphi}{\partial x}(\cdot,t)b(\cdot)\,,\, P_{t-s}\left[\langle a,\mu_s\rangle\delta_0\right] \right\rangle \mathrm{d}s\,\mathrm{d}t. \quad (3.19\text{b})$$

Now note that by the Fundamental Theorem of Calculus and Fubini's theorem [2, Thm. 23.7],

$$\left\langle \varphi\big(\phi_T(\cdot),T\big)\,,\, \nu_0\right\rangle - \left\langle \varphi\big(\phi_0(\cdot),0\big)\,,\, \nu_0\right\rangle = \int_0^T \left\langle \partial_t\left[\varphi\big(\phi_t(\cdot),t\big)\right](r)\,,\, \nu_0\right\rangle \mathrm{d}r.$$

Hence, using the chain rule, the fact that ϕ_t is a solution to (3.4) and the definition of P_t, it follows that

$$\langle \varphi(\cdot,T), P_T\nu_0\rangle - \langle \varphi(\cdot,0),\nu_0\rangle - \int_0^T \left\langle \tfrac{\partial\varphi}{\partial t}(\cdot,t) + \tfrac{\partial\varphi}{\partial x}(\cdot,t)b(\cdot)\,,\, P_t\nu_0 \right\rangle \mathrm{d}t = 0.$$

So after substituting (3.19a) and (3.19b) in (3.18) these terms cancel. Note that this already proves that $P_t\nu_0$ is a weak solution for the problem if $a \equiv 0$. We are left with

$$\int_0^T \left\langle \varphi(\cdot,T), P_{T-s}\left[\langle a,\mu_s\rangle\delta_0\right]\right\rangle \mathrm{d}s - \int_0^T \langle \varphi(0,t)a(\cdot)\,,\, \mu_t\rangle\ \mathrm{d}t$$
$$- \int_0^T \int_0^t \left\langle \tfrac{\partial\varphi}{\partial t}(\cdot,t) + \tfrac{\partial\varphi}{\partial x}(\cdot,t)b(\cdot)\,,\, P_{t-s}\left[\langle a,\mu_s\rangle\delta_0\right] \right\rangle \mathrm{d}s\,\mathrm{d}t,$$

which can be rewritten as

$$\int_0^T \langle a,\mu_s\rangle \left\langle \varphi\big(\phi_{T-s}(\cdot),T\big),\delta_0\right\rangle - \langle a,\mu_s\rangle\,\langle \varphi(\cdot,s),\delta_0\rangle\ \mathrm{d}s$$
$$- \int_0^T \int_0^t \langle a,\mu_s\rangle \left\langle \tfrac{\partial\varphi}{\partial t}\big(\phi_{t-s}(\cdot),t\big) + \tfrac{\partial\varphi}{\partial x}\big(\phi_{t-s}(\cdot),t\big)b\big(\phi_{t-s}(\cdot)\big)\,,\, \delta_0\right\rangle \mathrm{d}s\,\mathrm{d}t. \quad (3.20)$$

We see again something that hints for the use of the Fundamental Theorem of Calculus. Indeed, just like before,

$$\partial_t\varphi\big(\phi_{t-s}(\cdot),t\big) = \tfrac{\partial\varphi}{\partial t}\big(\phi_{t-s}(\cdot),t\big) + \tfrac{\partial\varphi}{\partial x}\big(\phi_{t-s}(\cdot),t\big)b\big(\phi_{t-s}(\cdot)\big).$$

Substitute this in (3.20) and change the order of integration of the double integral using Fubini's Theorem to get

$$\int_0^T \langle a,\mu_s\rangle \left\langle \varphi\big(\phi_{T-s}(\cdot),T\big),\delta_0\right\rangle - \langle a,\mu_s\rangle\,\langle \varphi(\cdot,s),\delta_0\rangle\ \mathrm{d}s$$
$$- \int_0^T \int_s^T \langle a,\mu_s\rangle \left\langle \partial_t\varphi\big(\phi_{t-s}(\cdot),t\big)\,,\, \delta_0\right\rangle \mathrm{d}t\,\mathrm{d}s. \quad (3.21)$$

Now rearrange the terms to get

$$\int_0^T \langle a,\mu_s\rangle \left\langle \varphi\big(\phi_{T-s}(\cdot),T\big) - \varphi(\cdot,s) - \int_s^T \partial_t\varphi\big(\phi_{t-s}(\cdot),t\big)\,\mathrm{d}t\,,\, \delta_0\right\rangle \mathrm{d}s. \quad (3.22)$$

By the Fundamental Theorem of Calculus it follows that

$$\varphi\big(\phi_{T-s}(\cdot),T\big)-\varphi\big(\phi_0(\cdot),s\big)=\int_s^T \partial_t\varphi\big(\phi_{t-s}(\cdot),t\big)\,\mathrm{d}t,$$

so the expression in (3.22) is 0. Hence the expression in (3.18) is 0, as was required.

□

In Definition 3.8 we will write down the solution that is found in [10, Lemma 3.5], but with $c \equiv 0$, and then we will prove that this solution is the same as the mild solution we found.

Definition 3.8. Define the function $\varphi_{t,\psi}\colon C^1(\mathbb{R}^+\times[0,t])$ by

$$\varphi_{t,\psi}(x,\tau)=\psi\big(\phi_{t-\tau}(x)\big)+\int_\tau^t a\big(\phi_{s-\tau}(x)\big)\varphi_{t,\psi}(0,s)\,\mathrm{d}s \tag{3.23}$$

with $\psi\in\mathrm{BL}(\mathbb{R}^+)$ and $t\in[0,T]$. This is the *solution of the dual problem* in [10]. We will call $\mu\colon[0,T]\to\mathcal{M}(\mathbb{R}^+)$ a *solution obtained via the dual problem* if for all $\psi\in\mathrm{BL}(\mathbb{R}^+)$ it satisfies

$$\langle\psi,\mu_t\rangle=\langle\varphi_{t,\psi}(\cdot,0),\nu_0\rangle. \tag{3.24}$$

The definition of $\varphi_{t,\psi}$ seems a bit arbitrary here as it results from the theory in [10, Lemma 3.5], but the Proposition 3.9 below tells us how to interpret these results in the framework we have developed.

Proposition 3.9. *Let $(V_t)_{t\geq 0}$ be the semigroup associated to the mild solution μ_t of the linear model with birth in* (3.16)*. Let U_t be the dual semigroup of V_t. The solution of the dual problem, defined in* (3.23)*, satisfies*

$$\varphi_{t,\psi}(\cdot,s)=U_{t-s}\psi(\cdot) \tag{3.25}$$

for all $\varphi\in\mathrm{BL}(\mathbb{R}^+)$. It follows that any solution obtained via the dual problem is a mild solution of (3.16) *(in the sense of Definition 3.1) and vice versa.*

Proof. First let's prove the last statement. Let μ_t be a mild solution of (3.16) and let $\bar{\mu}_t$ be a *solution obtained via the dual problem.* If (3.25) holds, then for each $\psi\in\mathrm{BL}(\mathbb{R}^+)$ we can write

$$\langle\psi,\bar{\mu}_t\rangle=\langle\varphi_{t,\psi}(\cdot,0),\,\nu_0\rangle=\langle U_t\psi(\cdot),\,\nu_0\rangle=\langle\psi,\,V_t\nu_0\rangle=\langle\psi,\mu_t\rangle.$$

It remains to prove equation (3.25). Let $\psi\in\mathrm{BL}(\mathbb{R}^+)$ and $\nu_0\in\mathcal{M}(\mathbb{R}^+)$. We will calculate

$$\langle U_{t-\tau}\psi,\nu_0\rangle=\langle\psi,V_{t-\tau}\nu_0\rangle.$$

Now we invoke Corollary 2.13: replace $V_{t-\tau}$ with the variant of the variation of constants formula in (2.32) to get

$$\langle U_{t-\tau}\psi,\nu_0\rangle=\langle\psi,P_{t-\tau}\nu_0\rangle+\int_0^{t-\tau}\big\langle\psi,V_s\left[\langle a,P_{t-\tau-s}\nu_0\rangle\delta_0\right]\big\rangle\,\mathrm{d}s.$$

By definition of U_t we have $\big\langle \psi, V_s\left[\langle a, P_{t-\tau-s}\nu_0\rangle\delta_0\right]\big\rangle = \langle a, P_{t-\tau-s}\nu_0\rangle\langle U_s\psi, \delta_0\rangle$. So, using the definition of P_t, it follows that

$$\langle U_{t-\tau}\psi, \nu_0\rangle = \big\langle \psi\big(\phi_{t-\tau}(\cdot)\big), \nu_0\big\rangle + \int_0^{t-\tau} \big\langle a\big(\phi_{t-\tau-s}(\cdot)\big), \nu_0\big\rangle\, U_s\psi(0)\,\mathrm{d}s.$$

When changing variables in the integral, from s to $t-s$, we see the definition of $\varphi_{t,\psi}(\cdot,\tau)$ appearing:

$$\langle U_{t-\tau}\psi, \nu_0\rangle = \left\langle \psi\big(\phi_{t-\tau}(\cdot)\big) + \int_\tau^t a\big(\phi_{s-\tau}(\cdot)\big)U_{t-s}\psi(0)\,\mathrm{d}s\,,\, \nu_0\right\rangle.$$

So $\langle U_{t-\tau}\psi, \nu_0\rangle = \langle \varphi_{t,\psi}(\cdot,\tau), \nu_0\rangle$ for all $\nu_0 \in \mathcal{M}(\mathbb{R}^+)$. In particular, if we set $\nu_0 = \delta_x$ then we have $U_{t-s}\psi(x) = \varphi_{t,\psi}(x,s)$ for all $x \in \mathbb{R}^+$ and hence equation (3.25) is satisfied. □

A *solution obtained via the dual problem* is a weak solution in the sense of Definition 3.6, according to [10, Lemma 3.6]. So Proposition 3.9 actually implies that a mild solution is a weak solution, so Proposition 3.7 also follows from Proposition 3.9 and [10, Lemma 3.6].

3.3.2 The linear population model with death

Consider the linear structured population model with death,

$$\begin{cases} \partial_t\mu_t + \partial_x(b\mu_t) = c\mu_t \\ \mu_0 = \nu_0, \end{cases} \tag{3.26}$$

where $b, c\colon \mathbb{R}^+ \to \mathbb{R}$ are bounded Lipschitz functions with $b(0) > 0$ and $\nu_0 \in \mathcal{M}(\mathbb{R}^+)$ is given initial data. We will prove that the mild solution is the same as a solution that is found in [10]. The solution that is found in [10] is defined in Definition 3.10. Recall that a mild solution $\mu\colon [0,T] \to \mathcal{M}(\mathbb{R}^+), t \mapsto \mu_t$ to (3.26) satisfies

$$\mu_t = P_t\nu_0 + \int_0^t P_{t-s}\left[c(\cdot)\mu_s\right]\,\mathrm{d}s. \tag{3.27}$$

As before we use use the following notation that is in line with [10, Lemma 3.5] and where now we have set $a \equiv 0$.

Definition 3.10. Define the function $\varphi_{t,\psi}\colon C^1(\mathbb{R}^+ \times [0,t])$ as

$$\varphi_{t,\psi}(x,\tau) = \psi\big(\phi_{t-\tau}(x)\big)e^{\int_\tau^t c(\phi_{r-\tau}(x))\,\mathrm{d}r}, \tag{3.28}$$

with $\psi \in \mathrm{BL}(\mathbb{R}^+)$ and $t \in [0,T]$. This is the *solution of the dual problem* in [10]. We will call $\mu\colon [0,T] \to \mathcal{M}(\mathbb{R}^+)$ a *solution obtained via the dual problem* if for all $\psi \in \mathrm{BL}(\mathbb{R}^+)$ it satisfies

$$\langle \psi, \mu_t\rangle = \langle \varphi_{t,\psi}(\cdot,0), \nu_0\rangle. \tag{3.29}$$

Proposition 3.11. *For the linear model with death, in* (3.26)*, a solution obtained via the dual problem is a mild solution.*

Proof. Let $\psi \in \mathrm{BL}(\mathbb{R}^+)$ be arbitrary. Let $\mu\colon t \mapsto \mu_t$ be a *solution obtained via the dual problem.* We will compute

$$\left\langle \psi\,,\, P_t\nu_0 + \int_0^t P_{t-s}\left[c(\cdot)\mu_s\right] \mathrm{d}s \right\rangle \tag{3.30}$$

and hope we will arrive at $\langle \psi, \mu_t\rangle$, since then equation (3.27) is satisfied and μ will be a mild solution. First note that (3.30) is equal to

$$\langle \psi, P_t\nu_0\rangle + \int_0^t \left\langle \psi\big(\phi_{t-s}(\cdot)\big)c(\cdot)\,,\, \mu_s\right\rangle \mathrm{d}s. \tag{3.31a}$$

Now use the definition of μ_s. In (3.29) replace ψ with $A(\cdot) = \psi\big(\phi_{t-s}(\cdot)\big)c(\cdot)$ to get an expression for the integrant in (3.31a). So we have to know $\varphi_{s,A}(\cdot,0)$, which we obtain using (3.28):

$$\varphi_{s,A}(\cdot,0) = \psi\big(\phi_{t-s}\circ\phi_s(\cdot)\big)c\big(\phi_s(\cdot)\big)e^{\int_0^s c(\phi_r(\cdot))\,\mathrm{d}r}.$$

Hence (3.31a) is equal to

$$\langle \psi, P_t\nu_0\rangle + \int_0^t \left\langle \psi\big(\phi_t(\cdot)\big)c\big(\phi_s(\cdot)\big)e^{\int_0^s c(\phi_r(\cdot))\,\mathrm{d}r}\,,\, \nu_0\right\rangle \mathrm{d}s \tag{3.31b}$$

Move the integral inside, as the rest does not depend on s,

$$\langle \psi, P_t\nu_0\rangle + \left\langle \psi\big(\phi_t(\cdot)\big)\int_0^t c\big(\phi_s(\cdot)\big)e^{\int_0^s c(\phi_r(\cdot))\,\mathrm{d}r}\,\mathrm{d}s\,,\, \nu_0\right\rangle. \tag{3.31c}$$

Now note that

$$\tfrac{\partial}{\partial s}e^{\int_0^s c(\phi_r(\cdot))\,\mathrm{d}r} = c\big(\phi_s(\cdot)\big)\,e^{\int_0^s c(\phi_r(\cdot))\,\mathrm{d}r},$$

so by the Fundamental Theorem of Calculus, (3.31c) is equal to

$$\langle \psi, P_t\nu_0\rangle + \left\langle \psi\big(\phi_t(\cdot)\big)\left[e^{\int_0^t c(\phi_r(\cdot))\,\mathrm{d}r} - 1\right]\,,\, \nu_0\right\rangle, \tag{3.31d}$$

which in turn is equal to

$$\langle \psi, P_t\nu_0\rangle + \left\langle \psi\big(\phi_t(\cdot)\big)e^{\int_0^t c(\phi_r(\cdot))\,\mathrm{d}r}\,,\, \nu_0\right\rangle - \left\langle \psi\big(\phi_t(\cdot)\big)\,,\, \nu_0\right\rangle. \tag{3.31e}$$

Here $\langle \psi, P_t\nu_0\rangle$ and the last term cancel and we are left with equation (3.29):

$$\left\langle \psi\big(\phi_t(\cdot)\big)e^{\int_0^t c(\phi_r(\cdot))\,\mathrm{d}r}\,,\, \nu_0\right\rangle = \left\langle \phi_{t,\psi}(\cdot,0)\,,\, \nu_0\right\rangle = \langle \psi, \mu_t\rangle \tag{3.31f}$$

Summarizing, we have calculated that for every $\psi \in \mathrm{BL}(\mathbb{R}^+)$ it holds that

$$\left\langle \psi\,,\, P_t\nu_0 + \int_0^t P_{t-s}\left[c(\cdot)\mu_s\right] \mathrm{d}s \right\rangle = \langle \psi, \mu_t\rangle. \tag{3.32}$$

So μ, a *solution obtained via the dual problem*, satisfies (3.27) and hence is a mild solution. ☐

Again we write down an equation like in (3.23) to understand what this function $\varphi_{t,\psi}$ means in the case $a \equiv 0$. A remarkable result is that we now have an explicit expression for the dual semigroup of the semigroup of solutions for the problem with death.

Corollary 3.12. *Let T_t be the semigroup of solutions associated to the mild solution of the linear model with death in* (3.26)*. Let Z_t be the dual semigroup of T_t. The solution of the dual problem, defined in* (3.28)*, satisfies*

$$\varphi_{t,\psi}(\cdot, s) = Z_{t-s}\psi(\cdot) \tag{3.33}$$

for all $\varphi \in \mathrm{BL}(\mathbb{R}^+)$.

Proof. Essential to this proof is the observation that

$$\varphi_{t-s,\phi}(x, 0) = \varphi_{t,\phi}(x, s). \tag{3.34}$$

Indeed, by setting $\tau = 0$ and replacing t with $t - s$ in the definition of φ in equation (3.28) we get

$$\varphi_{t-s,\phi}(x, 0) = \psi\big(\phi_{t-s}(x)\big)e^{\int_0^{t-s} c(\phi_r(x))\,\mathrm{d}r}.$$

When changing variables in the integral, from r to $r - s$, we arrive at the definition of $\varphi_{t,\phi}(x, s)$.

Let μ_t be a *solution obtained via the dual problem* as in Definition 3.10, with initial condition $\mu_0 = \delta_x$ for $x \in \mathbb{R}^+$ arbitrary. By Proposition 3.11, μ_t is a mild solution, so it holds that

$$Z_{t-s}\psi(x) = \langle \psi,\, T_{t-s}\delta_0\rangle = \langle \psi,\, \mu_{t-s}\rangle.$$

By Definition 3.10 and equation (3.34) we get

$$\langle \psi,\, \mu_{t-s}\rangle = \langle \varphi_{t-s,\psi}(\cdot, 0),\, \delta_x\rangle = \langle \varphi_{t,\psi}(\cdot, s),\, \delta_x\rangle$$

for all $x \in \mathbb{R}^+$, which finishes the proof.

□

3.3.3 The linear model with birth and death

Finally consider the linear model as studied in [10],

$$\begin{cases} \partial_t\mu_t + \partial_x(b\mu_t) = c\mu_t + \langle a, \mu_t\rangle\delta_0 \\ \mu_0 = \nu_0, \end{cases} \tag{3.35}$$

where $a, b, c\colon \mathbb{R}^+ \to \mathbb{R}$ are bounded Lipschitz functions with $b(0) > 0$ and $\nu_0 \in \mathcal{M}(\mathbb{R}^+)$ is given initial data. We are now in a position to formulate and prove a theorem that explains the results for the problem in (3.35) in [10] in the framework we developed in Section 3.1 and Section 3.2. The proof of this theorem is analogue to the proof of Proposition 3.9 and uses Corollary 3.12.

In [10, Lemma 3.5] it is proved that the solution to the 'dual problem' is given by the function $\varphi_{t,\psi} \in \mathcal{C}^1(\mathbb{R}^+ \times [0, t])$ that satisfies

$$\begin{aligned} \varphi_{t,\psi}(x, \tau) = {} & \psi\big(\phi_{t-\tau}(x)\big)e^{\int_\tau^t c(\phi_{r-\tau}(x))\,\mathrm{d}r} \\ & + \int_\tau^t a\big(\phi_{s-\tau}(x)\big)\varphi_{t,\psi}(0, s)e^{\int_\tau^t c(\phi_{r-\tau}(x))\,\mathrm{d}r}\,\mathrm{d}s \end{aligned} \tag{3.36}$$

with $\psi \in \mathrm{BL}(\mathbb{R}^+)$ and $t \in [0, T]$. This is the solution of the dual problem. In [10, Lemma 3.6] it is proved that the function $\mu\colon [0, T] \to \mathcal{M}(\mathbb{R}^+)$ that satisfies

$$\langle \psi, \mu_t\rangle = \langle \varphi_{t,\psi}(\cdot, 0), \nu_0\rangle \tag{3.37}$$

is a weak solution to the problem in (3.35).

Theorem 3.13. *A solution as defined in [10, Lemma 3.6] is a mild solution as defined in Definition 3.1. Moreover, the solution of the dual problem $\varphi_{t,\phi}$ in [10, Lemma 3.5] can be written as*

$$\varphi_{t,\psi}(\cdot, s) = U_{t-s}\psi(\cdot), \tag{3.38}$$

where U_t is the dual semigroup of the semigroup corresponding to the mild solution of (3.35)*.*

Proof. Let S_t be the semigroup corresponding to the mild solution of the linear problem in (3.35) and let U_t its dual semigroup. We can interpret this solution as a result of the perturbation of the mild solution of the linear model with death, in (3.26). So if T_t is the semigroup corresponding to solution of the linear model with death, then, using the variant VOC-formula of Corollary 2.13 in (2.32),

$$S_t\nu_0 = T_t\nu_0 + \int_0^t S_s\left[\langle a, T_{t-s}\nu_0\rangle\delta_0\right]\,\mathrm{d}s. \tag{3.39}$$

Let $\psi \in \mathrm{BL}(\mathbb{R}^+)$ and $\nu_0 \in \mathcal{M}(\mathbb{R}^+)$. As in the proof of Proposition 3.9, we will calculate

$$\langle U_{t-\tau}\psi, \nu_0\rangle = \langle \psi, S_{t-\tau}\nu_0\rangle.$$

Replace $S_{t-\tau}$ with the formula in (3.39) just found to get

$$\langle U_{t-\tau}\psi, \nu_0\rangle = \langle \psi, T_{t-\tau}\nu_0\rangle + \int_0^{t-\tau} \big\langle \psi, S_s\left[\langle a, T_{t-\tau-s}\nu_0\rangle\delta_0\right]\big\rangle\,\mathrm{d}s.$$

Do a change of variables in the integral, from s to $t-s$, to get

$$\langle U_{t-\tau}\psi, \nu_0\rangle = \langle \psi, T_{t-\tau}\nu_0\rangle + \int_\tau^{t} \big\langle \psi, S_{t-s}\left[\langle a, T_{s-\tau}\nu_0\rangle\delta_0\right]\big\rangle\,\mathrm{d}s. \tag{3.40}$$

By definition of U_t we have $\big\langle \psi, S_{t-s}\left[\langle a, T_{s-\tau}\nu_0\rangle\delta_0\right]\big\rangle = \langle a, T_{s-\tau}\nu_0\rangle\langle U_{t-s}\psi, \delta_0\rangle$. Corollary 3.12 implies

$$\langle a,\, T_{s-\tau}\nu_0\rangle = \langle Z_{s-\tau}a,\, \nu_0\rangle = \Big\langle a\big(\phi_{s-\tau}(\cdot)\big)e^{\int_\tau^s c(\phi_{r-\tau}(\cdot))\,\mathrm{d}r},\, \nu_0\Big\rangle.$$

So we can rewrite equation (3.40) to

$$\begin{aligned}\langle U_{t-\tau}\psi, \nu_0\rangle &= \big\langle a\big(\phi_{t-\tau}(\cdot)\big)e^{\int_\tau^t c(\phi_{r-\tau}(\cdot))\,\mathrm{d}r},\, \nu_0\big\rangle \\ &\quad + \int_\tau^t \big\langle a\big(\phi_{s-\tau}(\cdot)\big)e^{\int_\tau^s c(\phi_{r-\tau}(\cdot))\,\mathrm{d}r},\, \nu_0\big\rangle U_{t-s}\psi(0)\,\mathrm{d}s.\end{aligned}$$

If we set $\nu_0 = \delta_x$ for some $x \in \mathbb{R}^+$ then we see the definition of the solution of the dual problem from [10, Lemma 3.5], stated in equation (3.37), appearing:

$$\begin{aligned}U_{t-\tau}\psi(x) &= a\big(\phi_{t-\tau}(x)\big)e^{\int_\tau^t c(\phi_{r-\tau}(x))\,\mathrm{d}r} \\ &\quad + \int_\tau^t a\big(\phi_{s-\tau}(x)\big)e^{\int_\tau^s c(\phi_{r-\tau}(x))\,\mathrm{d}r}U_{t-s}\psi(0)\,\mathrm{d}s.\end{aligned}$$

Hence equation (3.38) is satisfied.

The solution μ from [10, Lemma 3.6] is given by equation (3.37). The proof that this solution is exactly the same as in Proposition 3.9: for all $\psi \in \mathrm{BL}(\mathbb{R}^+)$ it holds that

$$\langle \psi, \mu_t \rangle = \langle \varphi_{t,\psi}(\cdot, 0),\, \nu_0 \rangle = \langle U_t \psi(\cdot),\, \nu_0 \rangle = \langle \psi,\, S_t \nu_0 \rangle,$$

so $\mu_t = S_t \nu_0$ for all $t \geq 0$. $\square$

4 Non-linear models

Now we turn our attention to the non-linear problem from the introduction,

$$\begin{cases} \partial_t \mu_t + \partial_x \big(F_2(\mu_t, t) \mu_t \big) = F_3(\mu_t, t) \mu_t + \big(\int_{\mathbb{R}^+} F_1(\mu_t, t) \, \mathrm{d}\mu_t \big) \, \delta_0 \\ \mu_0 = \nu_0 \in \mathcal{M}^+(\mathbb{R}^+) \end{cases} \tag{4.1}$$

with $F_1, F_2, F_3 \colon \mathcal{M}(\mathbb{R}^+) \times [0, T] \to \mathrm{BL}(\mathbb{R}^+)$. Later we will formulate necessary conditions on F_1, F_2.

First problem is that it is not clear how to define a mild solution. We will come to that later. We followed the path that was set out by Gwiazda *et al.* in [10]. To find a weak solution, their approach was to successively freeze the coefficients on an equidistant grid of $[0, T]$. The resulting approximations turn out to converge if the grid size vanishes. This procedure is analogous to the Euler method for solving ordinary differential equations.

The approach taken is however very complicated and more general than needed, employing the theory of so-called 'mutational equations'. In [10], a 'transition' would simply correspond to the semigroup of solutions. The use of the framework of mutational equations makes it difficult to see what is the line of reasoning, where the important steps are taken and which requirements are actually made. Our goal was to present a self-contained theory such that one can avoid these mutational equations by applying the procedure analogue to Euler's method directly to the model.

This resulted in a general abstract theorem, presented in Section 4.2. Where in [10] a weak solution is found by using the theorem of Arzela-Ascoli, we have found a more constructive proof for the existence of a unique (mild) solution that also yields a rate of convergence of the approximations.

A point of discussion could be that we take as a definition of a mild solution the limit of the approximations. However it is possible to prove that a mild solution of (4.1) is weak solutions, and we will do so for a special case. Furthermore, one could take a more philosophical perspective by arguing that the approximations really model the process that was studied, and the limit is indeed what we are looking for.

The main theorem presented in Section 4.2 is not related to a specific model as in (4.1), but can be applied to any (non-linear) semigroup on any Banach space. It would be interesting to check if this theorem can be useful when using function spaces instead of spaces of measures. Therefore it is convenient to first introduce the concepts and ideas using a concrete model. This is done in Section 4.1. The result is that some proofs in Section 4.1 and 4.2 are very alike.

Still the reader should be warned that there are a lot of long computations in the sections below. Especially Lemma 4.4, which is the key lemma of the approach may look difficult to read, but it is worth reading for its powerful result relies on nothing but smartly chosen elementary calculations. To relieve the effort reading a lot of symbols, we will often write $\|\cdot\|$ instead of $\|\cdot\|^*_{\mathrm{BL}}$.

In Section 4.3 we will apply the main theorem stated in Section 4.2 to obtain solutions for the model in (4.1) and for an other general model.

4.1 Transport with a density-dependent velocity field

Consider the non-linear model

$$\begin{cases} \partial_t \mu_t + \partial_x \big(F(\mu_t, t)\mu_t\big) = 0 \\ \mu_0 = \nu_0 \in \mathcal{M}^+(\mathbb{R}^+), \end{cases} \tag{4.2}$$

with $F\colon \mathcal{M}(\mathbb{R}^+) \times [0,T] \to \{b \in \mathrm{BL}(\mathbb{R}^+) : b(0) \geq 0\}$. In Section 3.1 we found that we have to require that $\operatorname{ran} F \subset \{b \in \mathrm{BL}(\mathbb{R}^+) : b(0) \geq 0\}$. The space $\Lambda = \operatorname{ran} F$ will be referred to as the parameter space. The reason for this will become more clear in Section 4.2, where the results of this section will be generalized.

Let $(\phi_t^b)_{t\geq 0}$ be the semigroup of solutions of the associated flow in (3.4) with parameter $b \in \mathrm{BL}^+(\mathbb{R}^+)$. Let $(P_t^b)_{t\geq 0}$ be the induced semigroup on $\mathcal{S}_{\mathrm{BL}}(\mathbb{R}^+)$ (see Section 3.1). For each $n \in \mathbb{N}$, set

$$h_n = \frac{T}{2^n}, \qquad t_n^j = jh_n \quad \text{for } j = 0, \dots, 2^{n-1}$$

and define the sequence $(x_n)_n$ in $\mathcal{M}^+(\mathbb{R}^+)$ as

$$\begin{aligned} x_n(0) = \nu_0, \qquad x_0(\cdot) = \nu_0, \\ x_n(t) = P^{b_n^j}_{t-t_n^j} x_n(t_n^j) \quad \text{for } t \in \left(t_n^j, t_n^{j+1}\right], \text{ with } b_n^j = F\left(t_n^j, x_n(t_n^j)\right) \end{aligned} \tag{4.3}$$

Note that since $\Lambda \subset \{b \in \mathrm{BL}(\mathbb{R}^+) : b(0) \geq 0\}$, we have that $b_n^j(0) \geq 0$ for all $n \in \mathbb{N}$ and valid j, so $P^{b_n^j}$ is well-defined (see Section 3.1). Furthermore, because P_t^b is a positive semigroup and $\nu_0 \in \mathcal{M}^+(\mathbb{R}^+)$, we have $x_n(t) \in \mathcal{M}^+(\mathbb{R}^+)$ for all $n \in \mathbb{N}$ and $t \in [0,T]$.

Intuitively, if we would have something that we could call a solution of (4.2), then this sequence (x_n) would approximate this solution. This is the intuition will turn out to make sense and motivates the following definition.

Definition 4.1. A limit of a subsequence of (x_n) in the space $\mathcal{C}([0,T], \mathcal{X})$ will be called a mild solution to (4.2).

The goal of this subsection is to find such a convergent subsequence of (x_n) in the space $\mathcal{C}([0,T], \mathcal{X})$. In [10], a variation of the Arzela-Ascoli Theorem is used to prove this. In our setting, the theorem from [1, Lemma 2.1] would seem suitable. However, it is possible to take a completely different approach. That is, we show by tedious estimates that the sequence of functions (x_n) defined in (4.3) is a Cauchy sequence in $\mathcal{C}([0,T], \mathcal{M}^+(\mathbb{R}^+)$. It follows that (x_n) is convergent and thus we get a unique solution, as well as a rate of convergence for the approximations.

In an attempt to find a mild solution of (4.2), we tried to estimate how close $x_n(t)$ and $x_{n-1}(t)$ are for all $n \in \mathbb{N}$ and $t \in [0,T]$. This result is formulated in what will be the key lemma of this section, Lemma 4.4. This approach is very technical but turns out to be flexible and it is elegant in the sense that the proofs mainly use elementary and natural ideas. Let's start with an outline that will make clear why we will need all the technical lemmas that will follow, in order to prove Lemma 4.4.

Fix $\tau \in [0,T]$. By definition of x_n, we can write

$$\left\|x_{n-1}(\tau) - x_n(\tau)\right\|_{\mathrm{BL}}^* = \left\|P_t^{b_{n-1}^j} x_{n-1}(t_{n-1}^j) - P_t^{b_n^{2j}} x_n(t_n^{2j})\right\|_{\mathrm{BL}}^*,$$

where j is such that $\tau \in (t_n^{2j}, t_n^{2j+1}]$ and $t = \tau - t_n^{2j}$. As you see, the notation becomes cumbersome already. Let us write $\|\cdot\|$ instead of $\|\cdot\|_{\mathrm{BL}}^*$ from now on and put

$$\tau_0 = t_n^{2j} = t_{n-1}^j \qquad \text{and} \qquad \lambda_1 = b_n^{2j}, \qquad \lambda_3 = b_{n-1}^j.$$

This is consistent with the notation we will introduce later in this section that will make the proofs more readable. Key to the approach in this section is applying the triangle inequality such that we get

$$\big\|x_{n-1}(\tau) - x_n(\tau)\big\| \leq \big\|P_t^{\lambda_3} x_{n-1}(\tau_0) - P_t^{\lambda_3} x_n(\tau_0)\big\| - \big\|P_t^{\lambda_3} x_n(\tau_0) - P_t^{\lambda_1} x_n(\tau_0)\big\|.$$

First we will derive estimates that help us to estimate expressions like above. For example, we can use Lemma 3.3 *(ii)* to estimate the first term. For the second term, where only the parameter λ of the semigroup P^λ is changed, we need the following lemma.

Lemma 4.2. *Let $(\phi_t^b)_{t\geq 0}$ be the semigroup of solutions of the associated flow in* (3.4) *with parameter $b \in \mathrm{BL}(\mathbb{R}^+)$. Let $(P_t^b)_{t\geq 0}$ be the induced semigroup on $\mathcal{S}_{\mathrm{BL}}(\mathbb{R}^+)$. Then for all $t \geq 0$, $\mu \in \mathcal{S}_{\mathrm{BL}}(\mathbb{R}^+)$ and $b, b' \in \mathrm{BL}(\mathbb{R}^+)$ it holds that*

$$\|P_t^b \mu - P_t^{b'} \mu\|_{\mathrm{BL}}^* \;\leq\; \|b - b'\|_\infty \|\mu\|_{\mathrm{TV}}\, t\left(1 + Lte^{Lt}\right), \tag{4.4}$$

where $L = \min\big(|b|_{\mathrm{L}},\, |b'|_{\mathrm{L}}\big)$.

Proof. Let $f \in \mathrm{BL}(\mathbb{R}^+)$ arbitrary. It holds that

$$\left|\big\langle P_t^b\mu - P_t^{b'}\mu,\, f\big\rangle\right| = \left|\big\langle \mu,\, f \circ \phi_t^b - f \circ \phi_t^{b'}\big\rangle\right| \leq \|\mu\|_{\mathrm{TV}} |f|_{\mathrm{L}} \|\phi_t^b - \phi_t^{b'}\|_\infty. \tag{4.5}$$

Using the variation of constants formula for ϕ_t^b and $\phi_t^{b'}$ we can calculate that for all $x \in \mathbb{R}^+$

$$\big|\phi_t^b(x) - \phi_t^{b'}(x)\big| \leq \int_0^t \big|b\big(\phi_r^b(x)\big) - b'\big(\phi_r^{b'}(x)\big)\big|\,\mathrm{d}r$$

Adding and subtracting $b(\phi_r^{b'}(x))$ in the integrand and using the triangle inequality results in

$$\big|\phi_t^b(x) - \phi_t^{b'}(x)\big| \;\leq\; |b|_{\mathrm{L}} \int_0^t \big|\phi_r^b(x) - \phi_r^{b'}(x)\big|\,\mathrm{d}r \;+\; \|b - b'\|_\infty\, t$$

Applying Gronwall's lemma gives the inequality

$$\big|\phi_t^b(x) - \phi_t^{b'}(x)\big| \leq \|b - b'\|_\infty t\left(1 + |b|_{\mathrm{L}} t e^{|b|_{\mathrm{L}} t}\right)$$

for all $x \in \mathbb{R}^+$. Applying this to equation (4.5) gives

$$\|P_t^b\mu - P_t^{b'}\mu\|_{\mathrm{BL}}^* \;\leq\; \|b - b'\|_\infty \|\mu\|_{\mathrm{TV}} t\left(1 + |b|_{\mathrm{L}} t e^{|b|_{\mathrm{L}} t}\right)$$

Equation (4.4) follows from the fact that we can interchange b and b' in the last equation. $\square$

Note that the estimate (4.4) in Lemma 4.2 depends on the total variation norm of μ. Therefore, we need that there exists an $R > 0$ such that $\|x_n(t)\|_{\mathrm{TV}} < R$ for all $t \in [0,T]$ and $n \in \mathbb{N}$. This will be the content of Lemma 4.3.

Also note that in the estimates in Lemma 3.3 and Lemma 4.2 depend on $\|b\|_\infty$ and $\|b - b'\|_\infty$, where $b, b' \in \Lambda$. To keep these terms under control, we need two conditions on F. Hence we make the following assumptions for F:

(F1) $\sup_{t\in[0,T]} \sup_{\mu\in\mathcal{M}(\mathbb{R}^+)} \|F(t,\mu)\|_{\mathrm{BL}} = \sup_{\lambda\in\Lambda} \|\lambda\|_{\mathrm{BL}} < \infty$,

(F2) for any $R > 0$ there exist constants $L_R > 0$ and $\omega_R > 0$ such that

$$\|F(\mu, s) - F(\nu, t)\|_\infty \leq L_R \|\mu - \nu\|^*_{\mathrm{BL}} + \omega_R |t - s|$$

for all $\mu, \nu \in \mathcal{M}^+(\mathbb{R}^+)$ with $\|\mu\|^*_{\mathrm{BL}}, \|\nu\|^*_{\mathrm{BL}} \leq R$.

Lemma 4.3 and Lemma 4.4 rely on these assumptions.

Lemma 4.3. *Under assumption (F1) and $\nu_0 \in \mathcal{M}^+(\mathbb{R}^+)$, the set*

$$\mathcal{A} = \{x_n(t) : n \in \mathbb{N}, t \in [0,T]\}$$

is bounded with respect to the norms $\|\cdot\|^*_{\mathrm{BL}}$ and $\|\cdot\|_{\mathrm{TV}}$.

Proof. As mentioned before, $(P^b_t)_{t\geq 0}$ is a positive semigroup by construction, so we have $\mathcal{A} \subset \mathcal{M}^+(\mathbb{R}^+)$. The total variation-norm coindices with the bounded Lipschitz norm on $\mathcal{M}^+(\mathbb{R}^+)$ [12, Lemma 3.1], so the two norms $\|\cdot\|^*_{\mathrm{BL}}$ and $\|\cdot\|_{\mathrm{TV}}$ coincide on $\mathcal{A}$ and we will just write $\|\cdot\|$.

By Lemma 3.3 *(ii)* and assumption (F1) there exists a $K > 0$ such that

$$\|P^b_t \mu\| \leq (1 + tK)\|\mu\|$$

for all $t \in [0,T]$, $b \in \Lambda$ and $\mu \in \mathcal{A}$. Note that this argument depends on the fact that P^b_t is linear.

Let $n \in \mathbb{N}$ be arbitrary. By definition of x_n for each $t \in (0, h_n]$ and $j \in \{0, 1, \ldots, 2^n - 1\}$ we can write

$$x_n(t^j_n + t) = P^{b^j_n}_t x_n(t^j_n),$$

so it follows that

$$\sup_{t\in(0,h_n]} \|x_n(t^j_n + t)\| \leq (1 + h_n K)\|x_{n-1}(t^j_n)\|.$$

Applying this equation inductively over j we obtain that for all $t \in [0,T]$

$$\|x_n(t)\| \leq \left(1 + TK2^{-n}\right)^{2^n} \|\nu_0\|^*_{\mathrm{BL}}.$$

It holds that $\lim_{n\to\infty}(1 + TK2^{-n})^{2^n} = e^{TK}$, which is the limit definition for the exponential function, first given by Euler. It also follows from the proof of Lemma 4.5 later in this section.

Now we have a bound for $\|x_n(t)\|$ that does not depend on n nor on t. Therefore $\mathcal{A}$ is bounded for both norms. $\square$

Let us simplify the two estimates from Lemma 3.3 and 4.2 that we will use. Assumption (F1) implies that there exists a $K > 0$ such that

$$|\lambda|_{\mathrm{L}} e^{|\lambda|_{\mathrm{L}} t} \le K$$

for all $\lambda \in \Lambda$ and $t \in [0, T]$. This is the same K as used in the proof of Lemma 4.3. Define $\mathcal{A}$ as in Lemma 4.3. From Lemma 4.3 it follows that there exists a $R > 0$ such that $\|\mu\|_{\mathrm{TV}} \le R$ for all $\mu \in \mathcal{A}$. Set $C = (1 + TK)R$. Now the estimates from Lemma 3.3 and Lemma 4.2 that we will use can be written as

$$\big\|P_t^{\lambda}\mu - P_t^{\lambda}\nu\big\| \le (1 + tK)\|\mu - \nu\| \tag{4.6a}$$

$$\big\|P_t^{\lambda}\mu - P_t^{\lambda'}\mu\big\| \le tC\|\lambda - \lambda'\|_{\infty}. \tag{4.6b}$$

for $\mu, \nu \in \mathcal{A}$ and $\lambda \in \Lambda$. Note that K and C do not depend on λ or t.

Now we are finally able to prove an estimate like we wanted. It turns out to be a rather strong estimate. The next lemma is the key lemma of this section, where the ideas proposed at the beginning of this section will be worked out in detail. In fact, the proof can be used to prove a more general existence theorem than in this section, this will be done in Section 4.2.

Lemma 4.4. *Let $F\colon \mathbb{R}^+ \times \mathcal{M}^+(\mathbb{R}^+) \to \mathcal{M}^+(\mathbb{R}^+)$ be such that it satisfies (F1) and (F2). Let $x_n\colon [0, T] \to \mathcal{M}^+(\mathbb{R}^+)$ be the map as defined in* (4.3)*. Then there exists a constants $M > 0$ such that for all $n \in \mathbb{N}$*

$$\sup_{\tau \in [0,T]} \big\|x_{n-1}(\tau) - x_n(\tau)\big\|_{\mathrm{BL}}^{*} \le 2^{-n}M. \tag{4.7}$$

Proof. Fix $n \in \mathbb{N}$. First we will derive (4.7) for τ in a fixed time interval. That is, first we fix $j \in \{0, 1, \dots, 2^{n-1}\}$. To improve readability we use the following notation. See Figure 1 for a sketch of the situation and to see how this notation is used. Set

$$\tau_0 = t_{n-1}^{j} = t_n^{2j} \qquad \tau_1 = t_n^{2j+1} \qquad \tau_2 = t_{n-1}^{j+1} = t_n^{2j+2}$$

and write $\lambda_3 = F\big(x_{n-1}(\tau_0), \tau_0\big)$ so that $x_{n-1}(\tau_0 + t) = P_t^{\lambda_3} x_{n-1}(\tau_0)$ for $t \in [0, h_n]$. Furthermore, set $\lambda_1 = F\big(x_n(\tau_0), \tau_0\big)$ and $\lambda_2 = F\big(x_n(\tau_1), \tau_1\big)$.

Let $t \in [0, h_n]$ be arbitrary. First we will estimate

$$\big\|x_{n-1}(\tau_0 + t) - x_n(\tau_0 + t)\big\| = \big\|P_t^{\lambda_3} x_{n-1}(\tau_0) - P_t^{\lambda_1} x_n(\tau_0)\big\| \tag{4.8}$$

by using the triangle inequality and applying (4.6a) and (4.6b):

$$\begin{aligned}
&\big\|x_{n-1}(\tau_0 + t) - x_n(\tau_0 + t)\big\| \\
&\qquad \le \big\|P_t^{\lambda_3} x_{n-1}(\tau_0) - P_t^{\lambda_3} x_n(\tau_0)\big\| + \big\|P_t^{\lambda_3} x_n(\tau_0) - P_t^{\lambda_1} x_n(\tau_0)\big\| \\
&\qquad \le (1 + tK)\big\|x_{n-1}(\tau_0) - x_n(\tau_0)\big\| + tC\|\lambda_3 - \lambda_1\|_{\infty}.
\end{aligned}$$

From the definition of λ_1 and λ_3 and the requirement (F2) we obtain

$$\begin{aligned}
\|\lambda_3 - \lambda_1\|_{\infty} &= \big\|F\big(x_{n-1}(\tau_0), \tau_0\big) - F\big(x_n(\tau_0), \tau_0\big)\big\| \\
&\le L_R\big\|x_{n-1}(\tau_0) - x_n(\tau_0)\big\|.
\end{aligned}$$

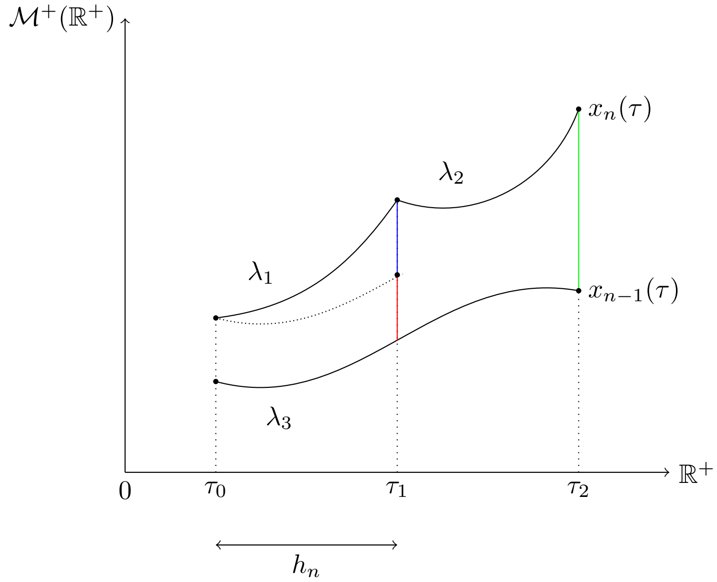

Figure 1: Sketch of the idea and notation in the proof of Lemma 4.4. The red and the blue line visualize the idea of applying the triangle inequality to equation (4.8) for $t = h_n$, the green line corresponds to the estimate in (4.10).

Thus for all $t \in [0, h_n]$,

$$\big\|x_{n-1}(\tau_0+t) - x_n(\tau_0+t)\big\| \leq \big(1 + h_n(K + CL_R)\big)\big\|x_{n-1}(\tau_0) - x_n(\tau_0)\big\|. \quad (4.9)$$

Next, consider the same expression as in (4.8), starting at time $\tau_1 = \tau_0 + h_n$ instead of τ_0. Again let $t \in [0, h_n]$ be arbitrary and write

$$\big\|x_{n-1}(\tau_1 + t) - x_n(\tau_1 + t)\big\| = \big\|P_t^{\lambda_3} x_{n-1}(\tau_1) - P_t^{\lambda_2} x_n(\tau_1)\big\|.$$

Working through the same steps as before,

$$\begin{aligned}
&\big\|x_{n-1}(\tau_1 + t) - x_n(\tau_1 + t)\big\| \\
&\qquad \leq \big\|P_t^{\lambda_3} x_{n-1}(\tau_1) - P_t^{\lambda_3} x_n(\tau_1)\big\| + \big\|P_t^{\lambda_3} x_n(\tau_1) - P_t^{\lambda_2} x_n(\tau_1)\big\| \\
&\qquad \leq (1 + tK)\big\|x_{n-1}(\tau_1) - x_n(\tau_1)\big\| \ + \ tC\|\lambda_3 - \lambda_2\|_\infty.
\end{aligned}$$

So for all $t \in [0, h_n]$ it holds that

$$\begin{aligned}
&\big\|x_{n-1}(\tau_1 + t) - x_n(\tau_1 + t)\big\| \\
&\qquad \leq (1 + h_n K)\big\|x_{n-1}(\tau_1) - x_n(\tau_1)\big\| \ + \ h_n C\|\lambda_3 - \lambda_2\|_\infty. \quad (4.10)
\end{aligned}$$

As before, by the definitions of λ_3 and λ_2 and the requirements on F we have

$$\begin{aligned}
\|\lambda_3 - \lambda_2\|_\infty &= \big\|F\big(x_{n-1}(\tau_0), \tau_0\big) - F\big(x_n(\tau_1), \tau_1\big)\big\| \\
&\leq \omega_R h_n + L_R\big\|x_{n-1}(\tau_0) - x_n(\tau_1)\big\|.
\end{aligned}$$

By using the triangle inequality and Lemma 3.3 *(iii)* we can rewrite this to

$$\begin{aligned}\|\lambda_3-\lambda_2\|_\infty &\le \omega_R h_n + L_R\big\|x_{n-1}(\tau_0)-x_n(\tau_0)\big\| + L_R\big\|x_n(\tau_0)-x_n(\tau_1)\big\|\\ &\le \omega_R h_n + L_R\big\|x_{n-1}(\tau_0)-x_n(\tau_0)\big\| + L_R R|\lambda_1|_{\mathrm{L}} h_n.\end{aligned}$$

Substitute this into equation (4.10) to get

$$\begin{aligned}&\big\|x_{n-1}(\tau_1+t)-x_n(\tau_1+t)\big\|\\ &\quad\le (1+h_nK)\big\|x_{n-1}(\tau_1)-x_n(\tau_1)\big\| + h_n^2\hat{C} + h_n C L_R\big\|x_{n-1}(\tau_0)-x_n(\tau_0)\big\|,\end{aligned}$$

where $\hat{C} = C(\omega_R + L_R R\sup_{\lambda\in\Lambda}|\lambda|_{\mathrm{L}})$. Note that it is trivial that we can replace $\|x_{n-1}(\tau_1)-x_n(\tau_1)\|$ by $\sup_{\tau\in[\tau_0,\tau_1]}\big\|x_{n-1}(\tau)-x_n(\tau)\big\|$. Then it follows that

$$\sup_{\tau\in[\tau_0,\tau_2]}\big\|x_{n-1}(\tau)-x_n(\tau)\big\| \le \big(1+h_n(K+CL)\big)\sup_{\tau\in[\tau_0,\tau_1]}\big\|x_{n-1}(\tau)-x_n(\tau)\big\| + h_n^2\hat{C}.$$

Use equation (4.9) to get

$$\sup_{\tau\in[\tau_0,\tau_2]}\big\|x_{n-1}(\tau)-x_n(\tau)\big\| \;\le\; (1+h_n\hat{K})\big\|x_{n-1}(\tau_0)-x_n(\tau_0)\big\| + h_n^2\hat{C}, \tag{4.11}$$

where $\hat{K}>0$ is such that $(1+h_n(K+CL))^2 \le 1+h_n\hat{K}$ for all $n\in\mathbb{N}$. Here we used that $h_n\le 1$ for all $n\in\mathbb{N}$.

Summarizing, we have found an estimate for $\|x_{n-1}(\tau)-x_n(\tau)\|$ for the case $\tau\in[\tau_0,\tau_1]$ in (4.9), for the case $\tau\in[\tau_1,\tau_2]$, and then finally for $\tau\in[\tau_0,\tau_2]$ in equation (4.11). Eventually we want an estimate for $\tau\in[0,T]$.

At this point, remember that we had set $\tau_0 = t_{n-1}^j$. If $j>0$, then we can set $\tau_{-2} = t_{n-1}^{j-1} = t_n^{2j-2}$ and replace $\|x_{n-1}(\tau_0)-x_n(\tau_0)\|$ in (4.11) with $\sup_{\tau\in[\tau_{-2},\tau_0]}\|x_{n-1}(\tau)-x_n(\tau)\|$, turning (4.11) into a kind of recurrence relation. Setting $j=0$ in (4.11) gives us an initial condition,

$$\sup_{\tau\in[0,t_{n-1}^1]}\big\|x_{n-1}(\tau)-x_n(\tau)\big\| \;\le\; h_n^2\hat{C}.$$

Replace h_n with $2^{-n}T$. By induction it follows that

$$\sup_{\tau\in[0,T]}\big\|x_{n-1}(\tau)-x_n(\tau)\big\| \;\le\; 2^{-2n}T^2\hat{C}\sum_{i=0}^{2^n-1}\big[1+2^{-n}\hat{K}T\big]^i. \tag{4.12}$$

Now we claim that

$$\big[1+2^{-n}\hat{K}\big]^i \le e^{\hat{K}T}$$

for all $i\le 2^n$. The proof is deferred to Lemma 4.5 below. By this claim we can replace the sum in (4.12) by $2^n e^{\hat{K}T}$. We finally arrive at the desired estimate,

$$\sup_{\tau\in[0,T]}\big\|x_{n-1}(\tau)-x_n(\tau)\big\| \;\le\; 2^{-n}T^2\hat{C}e^{\hat{K}T}.$$

□

It remains to prove a lemma that is actually the last technical part of the proof of the key lemma, Lemma 4.4 above. It turned out that this kind of calculation was needed more often than only in the proof above.

Lemma 4.5. *Let $m \in \mathbb{N}$ and let $0 < t_1 < \cdots < t_{m-1} < T$ be a partition of the interval $[0,T]$. Let X be a normed vector space and let $x\colon [0,T] \to X$ be a function such that there exist $A, B > 0$ for which x satisfies the following recurrence relation:*

$$\sup_{t \in (t_j, t_{j+1}]} \|x(t)\| \le \big(1 + m^{-1}A\big) \sup_{t \in (t_{j-1}, t_j]} \|x(t)\| + B, \tag{4.13}$$

for all $j \in \{1, \ldots, m-1\}$ and $\sup_{t \in [0,t_1]} \|x(t)\| \le C$. Then

$$\sup_{t \in [0,T]} \|x(t)\| \le m\, e^A \max(B, C).$$

Proof. By solving the recurrence relation in (4.13), or applying induction over j, we obtain the following estimate:

$$\sup_{t \in (t_{j-1}, t_j]} \|x(t)\| \le \hat{A}^{j-1} C + B \sum_{i=0}^{j-2} \hat{A}^i,$$

where $\hat{A} = (1 + m^{-1}A)$. It follows that

$$\sup_{t \in [0,T]} \|x(t)\| \le \max(B, C) \sum_{i=0}^{m-1} \big[1 + m^{-1}A\big]^i.$$

The claim now follows from the fact that $\big[1 + m^{-1}A\big]^i \le e^A$ for all $0 \le i \le m$. To prove this, use the Binomium of Newton to write

$$\big[1 + m^{-1}A\big]^i = \sum_{k=1}^{i} \binom{i}{k} \big[m^{-1}A\big]^k. \tag{4.14a}$$

Now use the definition of the Binomium coefficient to get

$$\binom{i}{k} = \frac{1}{k!} \prod_{l=0}^{k-1} (i - l) \le \frac{1}{k!} \prod_{l=0}^{k-1} (m - l) \le \frac{1}{k!} m^k, \tag{4.14b}$$

so it follows that

$$\big[1 + m^{-1}A\big]^i \le \sum_{k=1}^{i} \frac{1}{k!} A^k \le e^A, \tag{4.14c}$$

as was required. □

Now we are in a position to formulate and prove an existence theorem for the mild solutions of the model in (4.2). With Lemma 4.4 in the pocket, the proof is straightforward.

Theorem 4.6. *Let $F\colon \mathcal{M}(\mathbb{R}^+) \times [0,T] \to \mathrm{BL}^+(\mathbb{R}^+)$ be such that*

(F1) $\sup_{t \in [0,T]} \sup_{\mu \in \mathcal{M}(\mathbb{R}^+)} \|F(t, \mu)\|_{\mathrm{BL}} = \sup_{\lambda \in \Lambda} \|\lambda\|_{\mathrm{BL}} < \infty$,

(F2) for any $R > 0$ there exist constants $L_R > 0$ and $\omega_R > 0$ such that

$$\|F(\mu, s) - F(\nu, t)\|_\infty \le L_R \|\mu - \nu\|_{\mathrm{BL}}^* + \omega_R |t - s|$$

for all $\mu, \nu \in \mathcal{M}^+(\mathbb{R}^+)$ with $\|\mu\|_{\mathrm{BL}}^*, \|\nu\|_{\mathrm{BL}}^* \le R$.

Then there exists a unique solution of (4.2) *in the sense of Definition 4.1.*

Proof. Denote with $\|\cdot\|_\infty$ the supremum norm on $\mathcal{C}([0,T],\mathcal{S}_{\mathrm{BL}})$. By Lemma 4.4, the series

$$\sum_{n=1}^{\infty} \|x_n - x_{n-1}\|_\infty$$

is convergent. Since $\mathcal{C}([0,T],\mathcal{S}_{\mathrm{BL}})$ is a Banach space with the norm $\|\cdot\|_\infty$, the limit

$$\lim_{n\to\infty} x_n = x_0 + \sum_{n=1}^{\infty}(x_n - x_{n-1})$$

exists [16, Theorem 2.30]. $\square$

The estimates made in the proof of Lemma 4.4 also show that the mild solution that was found has a Lipschitz dependence on the initial values. Indeed, the proof of this dependence is a special case of the situation in the proof of Lemma 4.4. Therefore, we formulate the Lipschitz dependence as a Corollary.

Corollary 4.7. *Assume that (F1) and (F2) hold. The mild solution of* (4.2) *from Theorem 4.6 depends on its initial condition in a locally Lipschitz way. That is, for $R > 0$ and $\mu_0, \nu_0 \in \mathcal{M}^+(\mathbb{R}^+)$ with $\|\mu_0\|, \|\nu_0\| \le R$ there exists a constant L_R such that*

$$\sup_{t\in[0,T]} \|x(t) - y(t)\|^*_{\mathrm{BL}} \le L_R \|\mu_0 - \nu_0\|^*_{\mathrm{BL}},$$

where $x(t)$ and $y(t)$ are two such mild solutions of (4.2) *with initial values $x(0) = \mu_0$ and $y(0) = \nu_0$,*

Proof. Let $\hat{R} > 0$ and $\mu_0, \nu_0 \in \mathcal{M}^+(\mathbb{R}^+)$ with $\|\mu_0\|, \|\nu_0\| \le \hat{R}$. By Theorem 4.6 there exists approximating sequences (x_n) and (y_n) as defined in (4.3), with $x_n(0) = \mu_0$ and $y_n(0) = \nu_0$. Define

$$\mathcal{B} = \{x_n(t) : n \in \mathbb{N}, t \in [0,T]\} \cup \{y_n(t) : n \in \mathbb{N}, t \in [0,T]\}.$$

Lemma 4.3 ensures that there exists an $R > 0$ such that $\|\mu\| \le R$ for all $\mu \in \mathcal{B}$. So the estimates (4.6a) and (4.6b) again hold, for $\mu, \nu \in \mathcal{B}$.

Similarly to what is done in Lemma 4.4, fix $j \in \{0, 1, \dots, 2^n - 1\}$, set

$$\tau_0 = t_n^j \qquad \tau_1 = t_n^{j+1}.$$

and write $\lambda_3 = F\big(\tau_0,\, y_n(\tau_0)\big)$ so that $y_n(\tau) = P^{\lambda_3}_{\tau-\tau_0} y_n(\tau_0)$ for $\tau \in (\tau_0, \tau_2]$. Consequently, set $\lambda_1 = F\big(\tau_0,\, x_n(\tau_0)\big)$. See Figure 2 for a sketch of the situation.

We want to estimate

$$\big\|y_n(\tau_0 + t) - x_n(\tau_0 + t)\big\| = \big\|P_t^{\lambda_3} y_n(\tau_0) - P_t^{\lambda_1} x_n(\tau_0)\big\|.$$

Note that this is exactly the same equation as (4.8), but with x_{n-1} replaced by y_n. Indeed, we are in the same situation as in Lemma 4.4, but with different notation (compare Figure 1 and Figure 2). So doing the same calculations, we arrive at (4.9), with again x_{n-1} replaced by y_n:

$$\big\|y_n(\tau_0{+}t) - x_n(\tau_0{+}t)\big\| \le \big(1 + h_n(K + CL_R)\big)\big\|y_n(\tau_0) - x_n(\tau_0)\big\|.$$

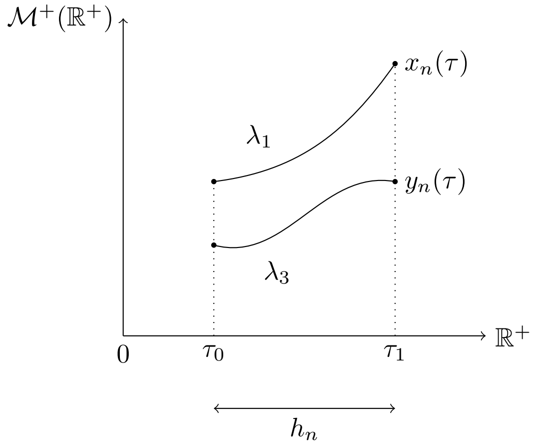

Figure 2: Sketch of the idea and notation in the proof of Lemma 4.7.

In contrast to Lemma 4.4 the situation on time h_n later is the same because we compare x_n and y_n on the same n-level. So we can skip these difficulties and see immediately (or also by induction, if you like) that

$$\sup_{\tau\in[0,T]} \|y_n(t) - x_n(t)\| \leq \left[1 + h_n \hat{K}\right]^{2^n} \|y_n(0) - x_n(0)\|,$$

where $\hat{K} = K + CL_R$. From equation (4.14c) it follows that

$$\|y_n(t) - x_n(t)\| \leq e^{\hat{K}T} \|\mu_0 - \nu_0\|,$$

for all $t \in [0, T]$.

□

Now we come back to the question how the definition of a mild solution as in Definition 4.1 relates to other solution concepts, in particular that of a weak solution, as used in [10]. The mild solution exists and is a result of a natural construction. But the name suggests that a solution as defined in Definition 4.1 should be a weak solution. In this particular case, this is true.

Definition 4.8. A weak solution of the model in (4.2) is a continuous function $\mu \colon [0, T] \to \mathcal{S}_{\mathrm{BL}}(\mathbb{R}^+), t \mapsto \mu_t$ such that for all $\varphi \in \mathcal{C}_b^1(\mathbb{R}^+ \times [0, T])$

$$\langle \varphi(\cdot, T), \mu_T \rangle - \langle \varphi(\cdot, 0), \nu_0 \rangle = \int_0^T \left\langle \tfrac{\partial \varphi}{\partial t}(\cdot, t) + \tfrac{\partial \varphi}{\partial x}(\cdot, t) F(\mu_t, t)\, ,\, \mu_t \right\rangle \mathrm{d}t$$

Proposition 4.9. *Assume (F1) holds. A mild solution of* (4.2) *as defined in Definition 4.1 is a weak solution in the sense of Definition 4.8.*

Proof. Let $x \colon [0, T] \to \mathcal{M}^+(\mathbb{R}^+)$ be a mild solution to (4.2) with $x(0) = \nu_0$. Let (x_n) be a sequence as defined in (4.3).

Note that for each $j \in \{0, 1, \ldots, 2^n - 1\}$, the function $x_n^j \colon [t_n^j, t_n^{j+1}] \to \mathcal{M}^+(\mathbb{R}^+)$ is a mild solution to the linear problem in (3.16) with $a \equiv 0$ and

$b = b_n^j = F(x_n(t_n^j), t_n^j)$, so by Proposition 3.7 the function x_n^j is a weak solution to the linear problem (see Definition 3.6):

$$\big\langle \varphi(\cdot, t_n^{j+1}),\, x_n(t_n^{j+1}) \big\rangle - \big\langle \varphi(\cdot, t_n^j),\, x_n(t_n^j) \big\rangle = \int_{t_n^j}^{t_n^{j+1}} \Big\langle \tfrac{\partial \varphi}{\partial t}(\cdot, t) + \tfrac{\partial \varphi}{\partial x}(\cdot, t) b_n^j(\cdot)\,,\, x_n(t) \Big\rangle \,\mathrm{d}t.$$

Taking the sum over all j results in a telescoping sum on the left hand side and we get

$$\big\langle \varphi(\cdot, T),\, x_n(T) \big\rangle - \big\langle \varphi(\cdot, 0),\, x_n(0) \big\rangle = \sum_{j=0}^{2^n-1} \int_{t_n^j}^{t_n^{j+1}} \Big\langle \tfrac{\partial \varphi}{\partial t}(\cdot, t) + \tfrac{\partial \varphi}{\partial x}(\cdot, t) b_n^j(\cdot)\,,\, x_n(t) \Big\rangle \,\mathrm{d}t.$$

Denote by $b_n \colon [0, T] \to \mathrm{BL}(\mathbb{R}^+)$ the simple function defined by $b_n(t) = b_n^j$ if $t \in (t_n^j, t_n^{j+1}]$. Then we can write

$$\big\langle \varphi(\cdot, T),\, x_n(T) \big\rangle - \big\langle \varphi(\cdot, 0),\, x_n(0) \big\rangle = \int_0^T \Big\langle \tfrac{\partial \varphi}{\partial t}(\cdot, t) + \tfrac{\partial \varphi}{\partial x}(\cdot, t) b_n(t)(\cdot)\,,\, x_n(t) \Big\rangle \,\mathrm{d}t.$$

Now take the limit of $n \to \infty$ and apply the Lebesque Dominated Convergence Theorem (e.g. [3, Theorem 2.8.1]). By Lemma 4.3, $\|x_n(t)\|_{\mathrm{TV}} \le R$ for all $t \in [0, T]$ and $n \in \mathbb{N}$, so

$$\Big| \Big\langle \tfrac{\partial \varphi}{\partial t}(\cdot, t) + \tfrac{\partial \varphi}{\partial x}(\cdot, t) b_n(t)(\cdot),\, x_n(t) \Big\rangle \Big| \le \Big(\big\| \tfrac{\partial \varphi}{\partial t}(\cdot, t) \big\|_\infty + \big\| \tfrac{\partial \varphi}{\partial x}(\cdot, t) \big\|_\infty \|b_n(t)\|_\infty \Big) R.$$

The partial derivatives are uniformly bounded because $\phi \in \mathcal{C}_b^1$, and assumption *(F1)* implies that there exists an $M > 0$ such that $b_n(t) \le M$ for all $n \in \mathbb{N}$ and $t \in [0, T]$. Now we can move the limit inside the integral and we get

$$\big\langle \varphi(\cdot, T),\, x(T) \big\rangle - \big\langle \varphi(\cdot, 0),\, x(0) \big\rangle = \int_0^T \Big\langle \tfrac{\partial \varphi}{\partial t}(\cdot, t) + \tfrac{\partial \varphi}{\partial x}(\cdot, t) F\big(x(t), t\big)(\cdot)\,,\, x(t) \Big\rangle \,\mathrm{d}t,$$

because $\lim_{n\to\infty} x_n(t) = x(t)$ and $\lim_{n\to\infty} b_n(t) = F(x(t), t)$ for all $t \in [0, T]$. $\square$

4.2 Parametrised semigroups with feedback functions

In Section 4.1 we used a specific non-linear model as a leading example. By freezing the coefficients in (4.2) we got a model for which we already had solutions: this was the linear model from Section 3.1. That is, for a coefficient $b \in \mathrm{BL}(\mathbb{R}^+)$ we used the semigroup of solutions $(P_t^b)_{t\geq 0}$. We can view $(P_t^b)_{t\geq 0}$ as a family of semigroups, parametrised by b, where we take take b to be in some parameter space, say Λ. If we now forget about the underlying model, we could as well have started with a family of semigroups and a parameter space.

First we will formulate a definition for the sequence x_n as in (4.3) from this more general perspective. In Section 4.1 we proved that this sequence converges for that specific example. In this section we will investigate which properties of the model were used, as to find sufficient assumptions in the more general case for this sequence x_n to converge. Of course, we do not restrict ourself to measure valued problems, but we use a general Banach space $(\mathcal{X}, \|\cdot\|)$ throughout this section.

The sequence (x_n) defined below is similar to the sequence of approximations to a system of ordinary differential equations obtained by applying the Forward Euler scheme. Therefore, we will will call this sequence x_n an *Euler sequence.*

Definition 4.10. Let (Λ, d) be a metric space: the *parameter space.* Let $(\Phi_t^\lambda)_{t\geq 0}$ be a family of (possibly non-linear) semigroups on $\mathcal{X}$, parametrised by $\lambda \in \Lambda$. Let $T > 0$ and let $F\colon \mathcal{X} \times [0,T] \to \Lambda$ be a function, the *feedback function.* For each $n \in \mathbb{N}$, set

$$h_n = \frac{T}{2^n}, \qquad t_n^j = jh_n \quad \text{for } j = 0, \ldots, 2^n.$$

A sequence $(x_n)_n$ in $\mathcal{X}$ will be called an *Euler sequence associated to the semigroup $(\Phi_t^\lambda)_{t\geq 0}$ and feedback function F (and initial condition ν_0)*, or shorthand an (Φ^λ, F)-Euler sequence, if

$$x_n(0) = \nu_0, \qquad x_0(\cdot) = \nu_0,$$
$$x_n(t) = \Phi_{t-t_n^j}^{\lambda_n^j}\left(x_n(t_n^j)\right) \quad \text{for } t \in \left(t_n^j, t_n^{j+1}\right], \text{ with } \lambda_n^j = F\left(x_n(t_n^j), t_n^j\right)$$

From a modeller's perspective, each x_n is a switched system where the switching is controlled by feedback that could depend on the time, the parameter used at that time and the trajectory that x_n has covered in the past. Here the feedback is modelled by F and it only depends on point of the trajectory of x_n at the last grid point. It does not depend on the parameter used somewhere else, as F does not have Λ in its domain. The feedback is held constant between the grid points t_n^j and changed at the grid points.

In Section 4.1 we investigated the convergence of the Euler sequence associated to the semigroup $(P_t^b)_{t\geq 0}$ on the Banach space $\mathcal{S}_{\mathrm{BL}}(\mathbb{R}^+)$ and feedback function F from (4.2), with parameter space $\Lambda = \operatorname{ran} F \subset \mathrm{BL}(\mathbb{R}^+)$. Theorem 4.6 tells us when this Euler sequence converges. In Theorem 4.11 we will formulate the conditions under which a general (Φ_t^λ, F)-Euler sequence x_n will converge.

The proof Theorem 4.6 mainly relies on only five assumptions: the estimates (4.6a) and (4.6b), the estimate in Lemma 3.3 *(iii)*, assumption (F2) and the boundedness of the set $\{x_n(t) : n \in \mathbb{N}, t \in [0,T]\}$.

The estimates in (4.6a), (4.6b) and Lemma 3.3 *(iii)* follow from the properties of P_t^b and we will take them as conditions on Φ_t^λ here. In Theorem

4.11, this is formulated in (C1), (C2) and (C3) respectively. The assumption (F2) is also taken as condition in the theorem. As for the boundedness of $\{x_n(t) : n \in \mathbb{N}, t \in [0,T]\}$, we need some bound on $\|\Phi_t^\lambda\|$ that is independent of λ. In Section 4.1 it just follows from (4.6a) that $\|P_t^\lambda \mu\|_{\mathrm{BL}}^* \leq (1+tK)\|\mu\|_{\mathrm{BL}}^*$ for all $\lambda \in \Lambda$, $t \in [0,T]$ and $\mu \in \mathcal{M}^+(\mathbb{R}^+)$. However, if we would require that $\|\Phi_t^\lambda(x)\| \leq (1+tK)\|x\|$ then this would imply that $x = 0$ is a fixed point of Φ_t^λ. This strong requirement is unnecessary: instead we use a milder condition, stated in (C4).

If we make the requirements on Φ_t^λ and F as mentioned, then the proof is mainly the same as the approach in Section 4.1, but with different notation. First, boundedness of $\{x_n(t) : n \in \mathbb{N}, t \in [0,T]\}$ is proved analogue to the proof of Lemma 4.3. Here however we need the same approximation that was used at the end of Lemma 4.4 and is stated in Lemma 4.5. Secondly, observe that Lemma 4.4 still holds. For reasons of clarity, the proof of Lemma 4.4 is just copied here with the right notation and references to the assumptions of this section. Then, when we have got an estimate like in Lemma 4.4, we can finish the proof like in the proof of Theorem 4.6.

We will apply the theorem to a non-linear semigroup on the Banach space $\mathcal{S}_{\mathrm{BL}}$, but in the end we would like to study measure-valued Euler sequences. Therefore, we will study the convergence of the Euler sequence in a closed subset of $\mathcal{X}$.

Theorem 4.11. *Let $\mathcal{Y} \subset \mathcal{X}$ be a closed subset of $\mathcal{X}$. Let $(\Phi_t^\lambda)_{t\geq 0}$ be a family of (possibly non-linear) semigroups on $\mathcal{X}$, parametrised by $\lambda \in \Lambda$, for some parameter space (Λ, d), and such that it leaves $\mathcal{Y}$ invariant. Impose the following conditions on Φ_t^λ:*

(C1) Every Φ_t^λ is locally lipschitz for $\|\cdot\|$ and for every $R > 0$ there exists an $K_R > 0$ such that for all $\lambda \in \Lambda$, $t \in [0,T]$ and $x, y \in \mathrm{ball}_R(\mathcal{X}) \cap \mathcal{Y}$

$$\|\Phi_t^\lambda(x) - \Phi_t^\lambda(y)\| \leq (1 + tK_R)\|x - y\|.$$

(C2) For every $R > 0$ there exists an $C_R > 0$ such that for all $\lambda, \lambda' \in \Lambda$, $t \in [0,T]$, $R > 0$ and $x \in \mathrm{ball}_R(\mathcal{X}) \cap \mathcal{Y}$

$$\|\Phi_t^\lambda(x) - \Phi_t^{\lambda'}(x)\| \leq tC_R\, \mathrm{d}(\lambda, \lambda').$$

(C3) The semigroups (Φ_t^λ) are locally Lipschitz in time and for every $R > 0$ there exists an $N_R > 0$ such that for all $\lambda \in \Lambda$, $t, s \in [0,T]$ and $x \in \mathrm{ball}_R(\mathcal{X}) \cap \mathcal{Y}$

$$\|\Phi_t^\lambda(x) - \Phi_s^\lambda(x)\| \leq N_R|t - s|.$$

(C4) There exists an $M > 0$ such that for all $\lambda \in \Lambda$, $t \in [0,T]$ and $x \in \mathcal{Y}$,

$$\|\Phi_t^\lambda(x)\| \leq \|x\| + tM\|x\| + tM.$$

Let $F \colon \mathcal{Y} \times [0,T] \to \Lambda$ be a locally Lipschitz feedback function. That is, for every $R > 0$ there exist constants $L_R, \omega_R > 0$ such that

$$\mathrm{d}\big(F(x,s), F(y,t)\big) \leq L_R\|x - y\| + \omega_R|t - s| \tag{4.15}$$

for any $x, y \in \mathrm{ball}_R(\mathcal{X}) \cap \mathcal{Y}$ and $t, s \in [0,T]$.

Then the Euler sequence (x_n), associated to the semigroup (Φ_t^λ) and feedback function F, converges in $\mathcal{C}([0,T],\mathcal{X})$ for the supremum norm. If the initial condition of (x_n) is in $\mathcal{Y}$, then the limit has range in $\mathcal{Y}$.

Proof. Let (x_n) be the (Φ^λ, F)-Euler sequence with initial condition $\nu_0 \in \mathcal{Y}$.

First we will show that the set $\{x_n(t) : n \in \mathbb{N}, t \in [0,T]\}$ is bounded in $\mathcal{Y}$. It follows from assumption (C4) that for all $n, j \in \mathbb{N}$ with $j \le 2^n - 1$,

$$\sup_{t\in(0,h_n]} \|x_n(t_n^j + t)\| \;\le\; h_n M + (1 + h_n M)\|x_{n-1}(t_n^j)\|.$$

By inserting $h_n = T2^{-n}$ and using Lemma 4.5 we get that for every $t \in [0,T]$,

$$\|x_n(t)\| \le TMe^{TM}\|\nu_0\|.$$

So there exists an $R > 0$ such that $\|x_n(t)\| < R$ for all $n \in \mathbb{N}$ and $t \in [0,T]$.

As we will apply the estimates in (C1), (C2), (C3) and (4.15) only to x and y of the form $x_n(t)$, we drop the subscript R in the constants K_R, C_R, N_R, ω_R and L_R from now on.

Now we proceed exactly as in the proof of Lemma 4.4 and Theorem 4.6. Again, the main effort in this proof is to show that there exists an $Z > 0$ such that for all $n \in \mathbb{N}$ the estimate

$$\sup_{\tau\in[0,T)} \big\|x_{n-1}(\tau) - x_n(\tau)\big\| \;\le\; 2^{-n} Z$$

holds. Fix $n \in \mathbb{N}$ and $j \in \{0, 1, \ldots, 2^n - 1\}$. To improve readability, we use the following notation. See Figure 1 for a sketch of the situation and to see how this notation is used. Set

$$\tau_0 = t_{n-1}^j = t_n^{2j} \qquad \tau_1 = t_n^{2j+1} \qquad \tau_2 = t_{n-1}^{j+1} = t_n^{2j+2},$$

and write $\lambda_3 = F\big(x_{n-1}(\tau_0),\, \tau_0\big)$ so that

$$x_{n-1}(\tau_0 + t) = \Phi_t^{\lambda_3}\big(x_{n-1}(\tau_0)\big) \quad \text{ for } t \in (0, h_n].$$

Consequently, set $\lambda_1 = F\big(x_n(\tau_0),\, \tau_0\big)$ and $\lambda_2 = F\big(x_n(\tau_1),\, \tau_1\big)$.

Let $t \in [0, h_n]$ be arbitrary. First we will estimate

$$\big\|x_{n-1}(\tau_0 + t) - x_n(\tau_0 + t)\big\| = \big\|\Phi_t^{\lambda_3}\big(x_{n-1}(\tau_0)\big) - \Phi_t^{\lambda_1}\big(x_n(\tau_0)\big)\big\| \tag{4.16}$$

by using the triangle inequality and applying (C1) and (C2):

$$\begin{aligned}
&\big\|x_{n-1}(\tau_0 + t) - x_n(\tau_0 + t)\big\| \\
&\qquad \le \big\|\Phi_t^{\lambda_3}\big(x_{n-1}(\tau_0)\big) - \Phi_t^{\lambda_3}\big(x_n(\tau_0)\big)\big\| + \big\|\Phi_t^{\lambda_3}\big(x_n(\tau_0)\big) - \Phi_t^{\lambda_1}\big(x_n(\tau_0)\big)\big\| \\
&\qquad \le (1 + tK)\big\|x_{n-1}(\tau_0) - x_n(\tau_0)\big\| + tC\,\mathrm{d}(\lambda_3, \lambda_1).
\end{aligned}$$

From the definition of λ_1 and λ_3 and the requirement on F we obtain

$$\begin{aligned}
\mathrm{d}(\lambda_3, \lambda_1) &= \mathrm{d}\left(F\big(x_{n-1}(\tau_0),\, \tau_0\big),\, F\big(x_n(\tau_0),\, \tau_0\big)\right) \\
&\le L\big\|x_{n-1}(\tau_0) - x_n(\tau_0)\big\|.
\end{aligned}$$

Thus for all $t \in [0, h_n]$,

$$\big\|x_{n-1}(\tau_0 + t) - x_n(\tau_0 + t)\big\| \le \big(1 + h_n(K + CL)\big)\big\|x_{n-1}(\tau_0) - x_n(\tau_0)\big\|. \tag{4.17}$$

Next, consider the same expression as in (4.16), starting on time $\tau_1 = \tau_0 + h_n$ instead of τ_0. Again let $t \in [0, h_n]$ be arbitrary and write

$$\big\|x_{n-1}(\tau_1 + t) - x_n(\tau_1 + t)\big\| = \big\|\Phi_t^{\lambda_3}\big(x_{n-1}(\tau_1)\big) - \Phi_t^{\lambda_2}\big(x_n(\tau_1)\big)\big\|.$$

Working through the same steps as before,

$$\begin{aligned}
&\big\|x_{n-1}(\tau_1 + t) - x_n(\tau_1 + t)\big\| \\
&\qquad \le \big\|\Phi_t^{\lambda_3}\big(x_{n-1}(\tau_1)\big) - \Phi_t^{\lambda_3}\big(x_n(\tau_1)\big)\big\| + \big\|\Phi_t^{\lambda_3}\big(x_n(\tau_1)\big) - \Phi_t^{\lambda_2}\big(x_n(\tau_1)\big)\big\| \\
&\qquad \le (1 + tK)\big\|x_{n-1}(\tau_1) - x_n(\tau_1)\big\| \; + \; tC\,\mathrm{d}(\lambda_3, \lambda_2).
\end{aligned}$$

As before, by the definitions of λ_3 and λ_2 and the requirement on F we have

$$\begin{aligned}
\mathrm{d}(\lambda_3, \lambda_2) &= \mathrm{d}\,\big(F\big(x_{n-1}(\tau_0), \tau_0\big), F\big(x_n(\tau_1), \tau_1\big)\big) \\
&\le \omega h_n + L\big\|x_{n-1}(\tau_0) - x_n(\tau_1)\big\|.
\end{aligned}$$

By using the triangle inequality and assumption (C3) we can rewrite this into

$$\begin{aligned}
\mathrm{d}(\lambda_3, \lambda_2) &\le \omega h_n + L\big\|x_{n-1}(\tau_0) - x_n(\tau_0)\big\| + L\big\|x_n(\tau_0) - x_n(\tau_1)\big\| \\
&\le \omega h_n + L\big\|x_{n-1}(\tau_0) - x_n(\tau_0)\big\| + LNh_n.
\end{aligned}$$

So for all $t \in [0, h_n]$ it holds that

$$\begin{aligned}
&\big\|x_{n-1}(\tau_1 + t) - x_n(\tau_1 + t)\big\| \\
&\qquad \le (1 + h_n K)\big\|x_{n-1}(\tau_1) - x_n(\tau_1)\big\| + h_n^2 \hat{C} + h_n CL\big\|x_{n-1}(\tau_0) - x_n(\tau_0)\big\|,
\end{aligned}$$

where we write $\hat{C} = C(\omega + LN)$. Note that we can replace $\|x_{n-1}(\tau_1) - x_n(\tau_1)\|$ by $\sup_{\tau\in[\tau_0,\tau_1]}\big\|x_{n-1}(\tau) - x_n(\tau)\big\|$ in this estimate. Then it follows that

$$\sup_{\tau\in[\tau_0,\tau_2]}\big\|x_{n-1}(\tau) - x_n(\tau)\big\| \le \big(1 + h_n(K + CL)\big) \sup_{\tau\in[\tau_0,\tau_1]}\big\|x_{n-1}(\tau) - x_n(\tau)\big\| + h_n^2 \hat{C}.$$

Use equation (4.17) to get

$$\sup_{\tau\in[\tau_0,\tau_2]}\big\|x_{n-1}(\tau) - x_n(\tau)\big\| \;\le\; (1 + h_n\hat{K})\big\|x_{n-1}(\tau_0) - x_n(\tau_0)\big\| + h_n^2\hat{C}, \tag{4.18}$$

where $\hat{K} > 0$ is such that $(1 + h_n(K + CL))^2 \le 1 + h_n\hat{K}$ for all $n \in \mathbb{N}$.

Summarizing, we have found an estimate for $\|x_{n-1}(\tau) - x_n(\tau)\|$ for the case $\tau \in [\tau_0, \tau_1]$ in (4.17), for the case $\tau \in [\tau_1, \tau_2]$, and then finally for $\tau \in [\tau_0, \tau_2]$ in equation (4.18). Eventually we want an estimate for $\tau \in [0, T]$.

At this point, remember that we had set $\tau_0 = t_{n-1}^j$. If $j > 0$, then we can set $\tau_{-2} = t_{n-1}^{j-1} = t_n^{2j-2}$ and replace $\|x_{n-1}(\tau_0) - x_n(\tau_0)\|$ in (4.18) with $\sup_{\tau\in[\tau_{-2},\tau_0]}\|x_{n-1}(\tau) - x_n(\tau)\|$, turning (4.18) into the kind of recurrence relation in Lemma 4.5. If $j = 0$ then it follows from (4.18) that

$$\sup_{\tau\in[0,t_{n-1}^1]}\big\|x_{n-1}(\tau) - x_n(\tau)\big\| \;\le\; h_n^2\hat{C}.$$

Apply Lemma 4.5 with $m = 2^n$, $A = T\hat{K}$ and $B = (2^{-n}T)^2\hat{C}$. We finally arrive at the desired estimate,

$$\sup_{\tau\in[0,T]}\big\|x_{n-1}(\tau) - x_n(\tau)\big\| \;\le\; 2^{-n}T^2\hat{C}\,e^{T\hat{K}}. \tag{4.19}$$

This estimate almost immediately gives the desired convergence of (x_n). Denote with $\|\cdot\|_\infty$ the supremum norm on $\mathcal{C}([0,T],\mathcal{X})$. By (4.19), the series

$$\sum_{n=1}^{\infty} \|x_n - x_{n-1}\|_\infty$$

is convergent. Since $\mathcal{C}([0,T],\mathcal{X})$ is a Banach space with the norm $\|\cdot\|_\infty$, the limit

$$\lim_{n\to\infty} x_n = x_0 + \sum_{n=1}^{\infty}(x_n - x_{n-1})$$

exists [16, Theorem 2.30]. Hence (x_n) converges uniformely.

Since Φ_t^λ leaves $\mathcal{Y}$ invariant and $x_n(0) \in \mathcal{Y}$ for all $n \in \mathbb{N}$, we have that $x_n(t) \in \mathcal{Y}$ for all $n \in \mathbb{N}$ and $t \in [0,T]$. Because $\mathcal{Y}$ is closed, each pointwise limit $\lim_{n\to\infty} x_n(t) \in \mathcal{Y}$, so the limit of (x_n) has range in $\mathcal{Y}$. □

4.3 Non-linear perturbed models

In this section we will apply Theorem 4.11 to two measure-valued models. The first example shows how Theorem 4.11 can be applied to a non-linear semigroup. We treat a model that is similar to the one studied in [4]. The second example shows how we can apply the theorem when we use a more complicated parameter space and solves the problem from Gwiazda *et al.* in [10].

4.3.1 An application of the theorem

Consider the non-linear model

$$\begin{cases} \partial_t \mu_t + \nabla\big(F(\mu_t, t)\mu_t\big) = G(\mu_t) \\ \mu_0 = \nu_0 \in \mathcal{M}^+(S). \end{cases} \tag{4.20}$$

with $S \subset \mathbb{R}^d$ a subset, a map $G\colon \mathcal{M}^+(S) \to \mathcal{M}(S)$ and a map

$$F\colon \mathcal{M}(S) \times [0,T] \longrightarrow \left\{b \in \mathrm{BL}(S, \mathbb{R}^d) \,:\, b \cdot \vec{n}\,(x) \leq 0 \text{ for } x \in \partial S\right\},$$

where $\vec{n}$ is the outward pointing normal field of the boundary ∂S of S. This model differs from the one studied in [4]: here, F depends on μ_t and G does not depend on t. Section 4.3.2 shows how to apply our results such that a dependence on t can be included.

Let $(P_t^b)_{t\geq 0}$ be the semigroup induced by the flow on S given by

$$\begin{cases} \partial_t x(t) = b\big(x(t)\big) \\ x(0) = x_0 \in S, \end{cases}$$

where $b\colon S \to \mathbb{R}^d$ is a velocity field that is Lipschitz continuous (w.r.t. the Euclidean metric) and satisfies $b \cdot \vec{n}\,(x) \leq 0$ for $x \in \partial S$ to ensure that the flow leaves S invariant. See also Definition 3.2 in Section 3.1. Note that if we take $S = \mathbb{R}^+$ we arrive at the condition $b(0) \geq 0$, which is the same as in the other examples in this thesis.

We take as the parameter space $\Lambda = \operatorname{ran} F$ with the metric induced by $\|\cdot\|_\infty$. We assume that G satisfies the positivity requirement in Theorem 2.12. So we assume that there exists an $a > 0$ for which

$$G(\mu) + a\mu \in \mathcal{M}^+(S) \qquad \text{whenever } \mu \in \mathcal{M}^+(S).$$

Let $(\Phi_t^b)_{t\geq 0}$ be the (non-linear) semigroup of solutions to the semi-linear model

$$\begin{cases} \partial_t \mu_t + \nabla(b\mu_t) = G(\mu_t) \\ \mu_0 = \nu_0 \in \mathcal{M}^+(S), \end{cases} \tag{4.21}$$

again with $b \in \mathrm{BL}(S, \mathbb{R}^d)$ such that $b \cdot \vec{n}\,(x) \leq 0$ for $x \in \partial S$. In Definition 3.1 it is explained what we mean by a solution to (4.21). By Theorem 2.12, such a semigroup of solutions exists.

Definition 4.12. Let (x_n) be a (Φ_t^b, F)-Euler sequence. A limit of a subsequence of (x_n) in the space $\mathcal{C}([0,T], \mathcal{M}^+(S))$ will be called a mild solution to (4.20).

The following two lemmas show that the requirements (C1)-(C4) in Theorem 4.11 are satisfied. The calculations are long, but the idea is the same for every estimate: use the variation of constants formula for Φ_t^b and apply Gronwall's lemma.

Lemma 4.13. *Assume that* $\sup_{b\in\Lambda}\|b\|_{\mathrm{BL}} < \infty$. *Then for all* $b \in \Lambda$, $t \in [0,T]$ *and* $\mu \in \mathcal{M}^+(S)$

$$\|\Phi_t^b(\mu)\| \leq \|\mu\| + tM\|\mu\| + tM.$$

Proof. Let $b \in \Lambda$, $t \in [0,T]$ and $\mu \in \mathcal{M}^+(S)$ be arbitrary. Using the variation of constants formula and Theorem A.7 we get

$$\|\Phi_t^b(\mu)\| \leq \|P_t^b(\mu)\| + \int_0^t \left\|P_{t-r}^b G\big(\Phi_r^b(\mu)\big)\right\| \,\mathrm{d}r.$$

By Lemma 3.3 *(ii)* and the assumption $\sup_{b\in\Lambda}\|b\|_{\mathrm{BL}} < \infty$ there exists a $\hat{K} > 0$ such that for all $\mu \in \mathcal{M}^+(S)$ and $t \in [0,T]$:

$$\|P_t^b\mu\| \leq (1 + t\hat{K})\|\mu\|.$$

So it follows that

$$\|\Phi_t^b(\mu)\| \leq (1 + t\hat{K})\|\mu\| + (1 + t\hat{K})\int_0^t \left\|G\big(\Phi_r^b(\mu)\big)\right\| \,\mathrm{d}r.$$

Rewrite using the triangle inequality such that we get

$$\|\Phi_t^b(\mu)\| \leq (1 + t\hat{K})\|\mu\| + (1 + t\hat{K})\|G(0)\| + L\int_0^t \|\Phi_r^b(\mu)\| \,\mathrm{d}r,$$

where $L = (1 + T\hat{K})|G|_{\mathrm{L}}$. By Gronwall's lemma it holds that

$$\|\Phi_t^b(\mu)\| \leq (1 + \hat{K})(\|\mu\| + t\|G(0)\|)\left(1 + tLe^{tL}\right), \tag{4.22}$$

and with this we can find an M as desired. □

Lemma 4.14. *Assume that* $\sup_{b\in\Lambda}\|b\|_{\mathrm{BL}} < \infty$. *Then there exists* $K > 0$ *such that for all* $\mu, \nu \in \mathcal{M}^+(S)$, $t \in [0,T]$ *and* $b \in \Lambda$,

$$\left\|\Phi_t^b(\mu) - \Phi_t^b(\nu)\right\| \leq (1 + tK)\|\mu - \nu\|. \tag{4.23a}$$

If we let $R > 0$ *and* $\mu \in \mathrm{ball}_R(\mathcal{M}^+(S))$ *then there exist* $C_R, N_R > 0$ *such that for all* $b, b' \in \Lambda$ *and* $t, s \in [0,T]$,

$$\left\|\Phi_t^b(\mu) - \Phi_t^{b'}(\mu)\right\| \leq tC_R\|b - b'\|_\infty, \tag{4.23b}$$

$$\left\|\Phi_t^b(\mu) - \Phi_s^b(\mu)\right\| \leq N_R|t - s| \tag{4.23c}$$

Proof. Let $\mu, \nu \in \mathcal{M}^+(S)$ be arbitrary. Using the variation of constants formula we can write

$$\left\|\Phi_t^b(\mu) - \Phi_t^b(\nu)\right\| \leq \left\|P_t^b\mu - P_t^b\nu\right\| + \int_0^t \left\|P_{t-r}^b G\big(\Phi_r^b(\mu)\big) - P_{t-r}^b G\big(\Phi_r^b(\nu)\big)\right\| \,\mathrm{d}r.$$

As with equation (4.6a), Lemma 3.3 *(ii)* and $\sup_{b\in\Lambda}\|b\|_{\mathrm{BL}} < \infty$ imply that there exists a constant $\hat{K} > 0$, such that for all $b \in \Lambda$ it holds that

$$\left\|P_t^b\mu - P_t^b\nu\right\| \leq (1 + t\hat{K})\|\mu - \nu\|. \tag{4.24}$$

It follows that for all $b \in \Lambda$

$$\left\|\Phi_t^b(\mu) - \Phi_t^b(\nu)\right\| \leq (1 + t\hat{K})\|\mu - \nu\| + L\int_0^t \left\|\Phi_r^b(\mu) - \Phi_r^b(\nu)\right\| \,\mathrm{d}r,$$

where $L = (1 + T\hat{K})|G|_{\mathrm{L}}$. Using Gronwall's lemma we get

$$\left\|\Phi_t^b(\mu) - \Phi_t^b(\nu)\right\| \leq \|\mu - \nu\|(1 + t\hat{K})\left(1 + tLe^{tL}\right).$$

for all $b \in \Lambda$, so equation (4.23a) holds.

The estimate in (4.23b) is obtained in a similar way. Now let $R > 0$ and let $\mu \in \mathrm{ball}_R(\mathcal{M}^+(S))$. Using the variation of constants formula we can write

$$\left\|\Phi_t^b(\mu) - \Phi_t^{b'}(\mu)\right\| \leq \|P_t^b\mu - P_t^{b'}\mu\| + \int_0^t \left\|P_{t-r}^b G\big(\Phi_r^b(\mu)\big) - P_{t-r}^{b'} G\big(\Phi_r^{b'}(\mu)\big)\right\| \,\mathrm{d}r.$$

Similarly to what we did to get equation (4.6b), set $\hat{C} = (1+T\hat{K})$. Then Lemma 4.2 and $\sup_{b\in\Lambda}\|b\|_{\mathrm{BL}} < \infty$ imply that

$$\left\|P_t^b\nu - P_t^{b'}\nu\right\| \leq t\hat{C}\|\nu\|\|b - b'\|_\infty, \tag{4.25}$$

for all $b \in \Lambda$ and $\nu \in \mathcal{M}^+(S)$.

Now we have to do some more work to estimate the integrant: use equations (4.24) and (4.25) to get

$$\begin{aligned}&\left\|P_{t-r}^b G\big(\Phi_r^b(\mu)\big) - P_{t-r}^{b'} G\big(\Phi_r^{b'}(\mu)\big)\right\| \\ &\qquad\leq t\hat{C}\left\|G\big(\Phi_r^b(\mu)\big)\right\|\|b - b'\|_\infty + (1 + t\hat{K})|G|_{\mathrm{L}}\left\|\Phi_r^b(\mu) - \Phi_r^{b'}(\mu)\right\|.\end{aligned}$$

By Lemma 4.13 there exists an $M_R > 0$ such that $\|\Phi_r^b(\nu)\| \leq M_R$ for all $b \in \Lambda$, $r \in [0, T]$ and $\nu \in \mathrm{ball}_R(\mathcal{M}^+(S))$. Note that G is bounded since G is Lipschitz. So there exists a constant $\hat{M}_R > 0$ such that $\|G(\Phi_r^b(\mu))\| \leq \hat{M}_R$ for all $b \in \Lambda$ and $r \in [0, T]$, and $\hat{M}_R$ does not depend on μ because $\|\mu\| < R$. It follows that

$$\left\|\Phi_t^b(\mu) - \Phi_t^{b'}(\mu)\right\| \leq \|b - b'\|_\infty\big(t\hat{C}R + \tfrac{1}{2}t^2\hat{C}\hat{M}_R\big) + L\int_0^t \left\|\Phi_r^b(\mu) - \Phi_r^{b'}(\mu)\right\| \mathrm{d}r,$$

where $L = (1 + T\hat{K})|G|_{\mathrm{L}}$. Using Gronwall's lemma we arrive at the estimate

$$\left\|\Phi_t^b\mu - \Phi_t^{b'}\mu\right\| \leq t\|b - b'\|_\infty\big(\hat{C}R + \tfrac{1}{2}t\hat{C}\hat{M}_R\big)\left(1 + tLe^{tL}\right).$$

Equation (4.23b) follows by setting $C_R = (\hat{C}R + \frac{1}{2}T\hat{C}\hat{M}_R)(1 + TLe^{TL})$.

It remains to prove equation (4.23c). The proof uses the same estimates as made in Lemma 2.4. Again let $R > 0$, $\mu \in \mathrm{ball}_R(\mathcal{M}^+(S))$ and let $t, s \in [0, T]$. Assume that $t > s$. We can write

$$\begin{aligned}\left\|\Phi_t^b(\mu) - \Phi_s^b(\mu)\right\| \leq \|P_t^b\mu - P_s^b\mu\| + \int_0^t \left\|P_{t-r}^b\left[G\big(\Phi_{r-s+t}^b(x)\big) - G\big(\Phi_r^b(\mu)\big)\right]\right\| \,\mathrm{d}r \\ + \int_0^{t-s} \left\|P_{t-r}^b G\big(\Phi_r^b(\mu)\big)\right\| \,\mathrm{d}r.\end{aligned} \tag{4.26}$$

By Lemma 3.3 *(iii)* and $\sup_{b\in\Lambda}\|b\|_{\mathrm{BL}} < \infty$ there exists an $\hat{N}_R > 0$ such that

$$\|P_t^b\nu - P_s^b\nu\| \leq \hat{N}_R|t-s|$$

for all $b \in \Lambda$ and $\nu \in \mathrm{ball}_R(\mathcal{M}^+(S))$, which gives an estimate for the first term. From equation (4.24) it follows that $\|P_t^b\nu\| \leq (1+t\hat{K})\|\nu\| = \hat{C}\|\nu\|$, and as before, we have $\|G(\Phi_r^b(\mu))\| \leq \hat{M}_R$ for all $b \in \Lambda$ and $r \in [0,T]$. Thus

$$\int_0^{t-s} \left\|P_{t-r}^b G\big(\Phi_r^b(x)\big)\right\| \,\mathrm{d}r \leq \hat{C}\hat{M}_R|t-s|.$$

Using again equation (4.24) for the integral left, equation (4.26) now reads

$$\left\|\Phi_t^b(\mu) - \Phi_s^b(\mu)\right\| \leq |t-s|\big(\hat{N}_R + \hat{C}\hat{M}_R\big) + L\int_0^t \left\|\Phi_{r-s+t}^b(\mu) - \Phi_r^b(\mu)\right\| \mathrm{d}r,$$

where $L = (1+T\hat{K})|G|_{\mathrm{L}}$. Gronwall's lemma implies that

$$\left\|\Phi_t^b(\mu) - \Phi_s^b(\mu)\right\| \leq |t-s|\big(\hat{N}_R + \hat{C}\hat{M}_R\big)\left(1+tLe^{tL}\right).$$

By setting $N_R = (\hat{N}_R + \hat{C}\hat{M}_R)(1+TLe^{TL})$ we arrive at equation (4.23c).

□

By Theorem 4.11 and the preceding two lemmas the following existence theorem holds.

Theorem 4.15. *Let $S \subset \mathbb{R}^d$. Let $F\colon \mathcal{M}(S)\times[0,T] \to \mathrm{BL}(S,\mathbb{R}^d)$ be such that*

(F1) $\sup_{t\in[0,T]}\sup_{\mu\in\mathcal{M}(S)}\|F(t,\mu)\|_{\mathrm{BL}} = \sup_{\lambda\in\Lambda}\|\lambda\|_{\mathrm{BL}} < \infty$,

(F2) for any $R>0$ there exist constants $L_R>0$ and $\omega_R>0$ such that

$$\|F(\mu,s) - F(\nu,t)\|_\infty \leq L_R\|\mu-\nu\|_{\mathrm{BL}}^* + \omega_R|t-s|$$

for all $\mu,\nu \in \mathcal{M}^+(S)$ with $\|\mu\|_{\mathrm{BL}}^*, \|\nu\|_{\mathrm{BL}}^* \leq R$.

(F3) $\mathrm{ran}\,F \subset \left\{b \in \mathrm{BL}(S,\mathbb{R}^d) \,:\, b\cdot\vec{n}\,(x) \leq 0 \text{ for } x \in \partial S\right\}$.

Let $G\colon \mathcal{M}^+(S) \to \mathcal{M}^+(S)$ be such that there exists an $a>0$ for which

$$G(\mu) + a\mu \in \mathcal{M}^+(S) \qquad \text{whenever } \mu \in \mathcal{M}^+(S).$$

Then there exists a unique mild solution of (4.20) *in the sense of Definition 4.12.*

4.3.2 Solutions to the non-linear population model

At last we are able to prove an existence theorem for the example in (4.1). For convenience, we state the model here again:

$$\begin{cases}\partial_t\mu_t + \partial_x\big(F_2(\mu_t,t)\mu_t\big) = F_3(\mu_t,t)\mu_t + \langle F_1(\mu_t,t),\, \mu_t\rangle\,\delta_0 \\ \mu_0 = \nu_0 \in \mathcal{M}^+(\mathbb{R}^+),\end{cases}$$

where $F_1, F_2, F_3\colon \mathcal{M}(\mathbb{R}^+)\times[0,T] \to \mathrm{BL}(\mathbb{R}^+)$ are such that F_1 is positive and $\mathrm{ran}\,F_2 \subset \{b \in \mathrm{BL}(\mathbb{R}^+) : b(0) \geq 0\}$.

We will use the feedback function $F\colon \mathcal{M}(\mathbb{R}^+) \times [0,T] \to (\mathrm{BL}(\mathbb{R}^+))^3$ defined by $F = (F_1, F_2, F_3)$. The parameter space will be the three dimensional space $\Lambda = \operatorname{ran} F \subset \{(a,b,c) \in (\mathrm{BL}(\mathbb{R}^+))^3 : b(0) \geq 0,\ a \geq 0\}$.

We can define two norms on Λ as follows. Let $\lambda = (a,b,c) \in \Lambda$ and define

$$\begin{aligned} \|\lambda\|_\infty &= \|a\|_\infty + \|b\|_\infty + \|c\|_\infty \\ \|\lambda\|_{\mathrm{BL}} &= \|a\|_{\mathrm{BL}} + \|b\|_{\mathrm{BL}} + \|c\|_{\mathrm{BL}}. \end{aligned}$$

It is easy and natural to take the for metric on Λ the metric induced by $\|\cdot\|_{\mathrm{BL}}$. Note that the requirement *(F2')* in Theorem 4.17 is different than the requirement *(F2)* in the previous sections. With this notation, we can formulate almost exactly the same theorem as in Section 4.1, but now for the perturbed model.

Definition 4.16. Let F be defined as above and let $\Lambda = \operatorname{ran} F$. Let Φ_t^λ be the semigroup of solutions of the linear model in (3.1), where $\lambda = (a,b,c) \in \Lambda$. Let (x_n) be a (Φ_t^λ, F)-Euler sequence. A limit of a subsequence of (x_n) in the space $\mathcal{C}([0,T], \mathcal{M}^+(\mathbb{R}^+))$ will be called a mild solution to (4.1).

Theorem 4.17. *Let $F\colon \mathcal{M}(\mathbb{R}^+) \times [0,T] \to (\mathrm{BL}(\mathbb{R}^+))^3$ be such that*

(F1) $\sup_{t\in[0,T]} \sup_{\mu\in\mathcal{M}(\mathbb{R}^+)} \|F(t,\mu)\|_{\mathrm{BL}} = \sup_{\lambda\in\Lambda} \|\lambda\|_{\mathrm{BL}} < \infty,$

(F2') for any $R > 0$ there exist constants $L_R > 0$ and $\omega_R > 0$ such that

$$\|F(\mu,s) - F(\nu,t)\|_{\mathrm{BL}} \leq L_R \|\mu - \nu\|^*_{\mathrm{BL}} + \omega_R |t-s|$$

for all $\mu, \nu \in \mathcal{M}^+(\mathbb{R}^+)$ with $\|\mu\|^*_{\mathrm{BL}}, \|\nu\|^*_{\mathrm{BL}} \leq R$,

(F3) $F_1(\mu) \geq 0$ *for all* $\mu \geq 0$ *and* $\operatorname{ran} F_2 \subset \{b \in \mathrm{BL}(\mathbb{R}^+) : b(0) \geq 0\}$.

Then there exists a unique mild solution of (4.1).

Proof. Let Φ_t^λ be the semigroup of solutions of the linear model in (3.1), where $\lambda = (a,b,c) \in \Lambda$. By definition of Λ, it holds that $a \geq 0$ and $b(0) \geq 0$, so by Theorem 3.4 such a semigroup exists and is positive.

Let (x_n) be a (Φ_t^λ, F)-Euler sequence. We will check the conditions of Theorem 4.11 to prove that x_n converges.

Define $G^\lambda \mu = c\mu + \langle a, \mu\rangle\, \delta_0$, for $\lambda = (a,b,c) \in \Lambda$ and $\mu \in \mathcal{M}^+(\mathbb{R}^+)$. By Lemma 3.5 and Section 3.2.2 it holds that $|G^\lambda|_{\mathrm{L}} \leq \|a\|_{\mathrm{BL}} + \|c\|_{\mathrm{BL}}$. We required that $\sup_{\lambda\in\Lambda} \|\lambda\|_{\mathrm{BL}} < \infty$, so there exists a $D > 0$ such that $|G^\lambda|_{\mathrm{L}} \leq \|\lambda\|_{\mathrm{BL}} < D$ for all $\lambda \in \Lambda$. With this fact, we can use the calculations from the previous example in Section 4.3.1.

First let us compare Lemma 4.13 with our situation. When replacing b with λ and G with G^λ in the proof, all calculations still hold. Note that $\|G^\lambda(0)\| = 0$ because G^λ is linear and replace $|G^\lambda|_{\mathrm{L}}$ with D in the last step, and it follows from equation (4.22) that for all $\lambda \in \Lambda$, $t \in [0,T]$ and $\mu \in \mathcal{M}^+(\mathbb{R}^+)$

$$\|\Phi_t^\lambda(\mu)\| \leq (1 + t\hat{K}) \|\mu\| \left(1 + tLe^{tL}\right),$$

where $L = (1 + T\hat{K})D$ and $\hat{K}$ is as defined in the proof of Lemma 4.13. Hence condition (C4) is satisfied.

Now we compare with Lemma 4.14. In the same manner, it follows from the proof of equation (4.23a) that there exist $K > 0$ such that for all $\mu, \nu \in \mathcal{M}^+(\mathbb{R}^+)$, $t \in [0,T]$ and $\lambda \in \Lambda$

$$\left\|\Phi_t^\lambda(\mu) - \Phi_t^\lambda(\nu)\right\| \leq (1+tK)\|\mu-\nu\|,$$

so condition (C1) is satisfied.

The proof of equation (4.23b) needs to be adjusted a little bit more. Let $R > 0$ and let $\mu \in \mathrm{ball}_R(\mathcal{M}^+(\mathbb{R}^+))$. Using the variation of constants formula we can write

$$\left\|\Phi_t^\lambda(\mu) - \Phi_t^{\lambda'}(\mu)\right\| \leq \|P_t^\lambda\mu - P_t^{\lambda'}\mu\| + \int_0^t \left\|P_{t-r}^\lambda G^\lambda\left[\Phi_r^\lambda(\mu)\right] - P_{t-r}^{\lambda'} G^{\lambda'}\left[\Phi_r^{\lambda'}(\mu)\right]\right\| \,\mathrm{d}r.$$

Equations (4.25) and (4.24) still hold and like before, we have to do some work to estimate the integrant. Use equations (4.25) and (4.24) to get

$$\begin{aligned}&\left\|P_{t-r}^\lambda G^\lambda\left[\Phi_r^\lambda(\mu)\right] - P_{t-r}^{\lambda'} G^{\lambda'}\left[\Phi_r^{\lambda'}(\mu)\right]\right\| \\ &\qquad \leq t\hat{C}\left\|G^\lambda\left[\Phi_r^\lambda(\mu)\right]\right\| \|b-b'\|_\infty + (1+t\hat{K})\left\|G^\lambda[\Phi_r^\lambda(\mu)] - G^{\lambda'}[\Phi_r^{\lambda'}(\mu)]\right\|.\end{aligned}$$

Here $\hat{K}$ is as in Lemma 4.14 and consequently $\hat{C} = 1 + T\hat{K}$. Since G^λ is linear, we have $\|G^\lambda[\nu]\| \leq |G^\lambda|_{\mathrm{L}}\|\nu\| < D\|\nu\|$ for all $\nu \in \mathcal{M}^+(\mathbb{R}^+)$ and $\lambda \in \Lambda$. So we can rewrite the last equation to

$$\begin{aligned}&\left\|P_{t-r}^\lambda G^\lambda\left[\Phi_r^\lambda(\mu)\right] - P_{t-r}^{\lambda'} G^{\lambda'}\left[\Phi_r^{\lambda'}(\mu)\right]\right\| \\ &\qquad\qquad \leq t\hat{C}D\left\|\Phi_r^\lambda(\mu)\right\|\|b-b'\|_\infty + \hat{C}\left\|G^\lambda[\Phi_r^\lambda(\mu)] - G^{\lambda'}[\Phi_r^{\lambda'}(\mu)]\right\|.\end{aligned}$$

We go on with estimating

$$\begin{aligned}&\left\|G^\lambda[\Phi_r^\lambda(\mu)] - G^{\lambda'}[\Phi_r^{\lambda'}(\mu)]\right\| \\ &\qquad\qquad \leq \left\|G^\lambda[\Phi_r^\lambda(\mu)] - G^{\lambda'}[\Phi_r^\lambda(\mu)]\right\| + |G^{\lambda'}|_{\mathrm{L}}\left\|\Phi_r^\lambda(\mu) - \Phi_r^{\lambda'}(\mu)\right\|.\end{aligned}$$

By definition of G^λ it holds that $G^\lambda - G^{\lambda'} = G^{\lambda-\lambda'}$. It follows that

$$\left\|G^\lambda[\Phi_r^\lambda(\mu)] - G^{\lambda'}[\Phi_r^\lambda(\mu)]\right\| \leq \left(\|a-a'\|_{\mathrm{BL}} + \|c-c'\|_{\mathrm{BL}}\right)\|\Phi_r^\lambda(\mu)\|.$$

Putting the last three equations together, we get

$$\begin{aligned}&\left\|P_{t-r}^b G^\lambda[\Phi_r^\lambda(\mu)] - P_{t-r}^{b'} G^{\lambda'}[\Phi_r^{\lambda'}(\mu)]\right\| \\ &\qquad \leq \hat{C}\|\Phi_r^\lambda(\mu)\|\big(tD\|b-b'\|_\infty + \|a-a'\|_{\mathrm{BL}} + \|c-c'\|_{\mathrm{BL}}\big) \\ &\qquad\qquad + \hat{C}|G^{\lambda'}|_{\mathrm{L}}\left\|\Phi_r^\lambda(\mu) - \Phi_r^{\lambda'}(\mu)\right\|,\end{aligned}$$

which we can rewrite to

$$\begin{aligned}&\left\|P_{t-r}^b G^\lambda[\Phi_r^\lambda(\mu)] - P_{t-r}^{b'} G^{\lambda'}[\Phi_r^{\lambda'}(\mu)]\right\| \\ &\qquad\qquad \leq \hat{C}M_R TD\|\lambda-\lambda'\|_{\mathrm{BL}} + \hat{C}D\left\|\Phi_r^\lambda(\mu) - \Phi_r^{\lambda'}(\mu)\right\|.\end{aligned}$$

Now Gronwall's lemma ensures that there exists a $C_R > 0$ such that

$$\|\Phi_t^\lambda(\mu) - \Phi_t^{\lambda'}(\mu)\| \leq C_R\|\lambda-\lambda'\|_{\mathrm{BL}}.$$

for all $\lambda, \lambda' \in \Lambda$, so condition (C2) is satisfied.

The proof of equation (4.23c) can be used to show that condition (C3 holds, by applying the same modifications as before. The condition in equation (4.15) follows directly from assumption $(F2')$.

Hence, Theorem 4.11 can be applied and thus the (Φ_t^λ, F)-Euler sequence converges. By definition, this limit is the unique mild solution. $\square$

A Bochner integration

In this section we will introduce the definitions and theorems we use concerning Bochner measurability and integrability. Unfortunately, the technical definitions used in the theory of Bochner integrals differ among common books used on this topic, like [7, 11, 15], which can be quite confusing. Therefore, these definitions are stated again in Section A.1. Theorems that are not proved are from [11, Section 3.5, 3.7]. In Section A.2 some technical results are proved which are used in Section 2.2 and Section 2.3.

Throughout this thesis, measurability and integrability are with respect to the Lebesque measure. In this section however, we work with a general σ-finite measure space.

A.1 General notions

Let (Ω, Σ, μ) be a σ-finite measure space. Let $\mathcal{X}$ be a Banach space. A function $x\colon \Omega \to \mathcal{X}$ is called *simple* if there exist $x_1, \ldots, x_n \in \mathcal{X}$ and $E_1, \ldots, E_n \in \Sigma$ such that $x(\omega) = x_i$ if $\omega \in E_i$ and $x(\omega) = 0$ otherwise. We will write $x = \sum_{i=1}^{n} x_i \chi_{E_i}$, where χ_{E_i} is the characteristic function of E_i.

Definition A.1. A function $x\colon \Omega \to \mathcal{X}$ is *measurable* if there exists a sequence of simple functions (x_n) that converges almost everywhere to x.

In most works, this form of measurability is called strong measurability, Bochner measurability or μ-measurability. This definition coincides with the regular definition for measurable numerical functions, see for example [2, Theorem 11.6] or [3, Lemma 2.18]. Yet important to realize is that in the numerical case the definition of measurability essentially only depends on the σ-algebra, and therefore is not connected with the measure, whereas in the vector-valued case measurability really depends on the measure.

As with numerical measurable functions, the vector-valued measurable functions respect the basic operations in the following proposition. The proof is almost immediate from the definitions.

Proposition A.2. *Let $x\colon \Omega \to \mathcal{X}$ and $y\colon \Omega \to \mathcal{X}$ be measurable functions. Let $f\colon \Omega \to \mathbb{R}$ be a numerically valued measurable function. Then the sum $x + y$ and the product fx are measurable.*

Proof. Let (x_n) and (y_n) be sequences of simple functions converging a.e. to x and y respectively. It is immediate from the definition that the functions $z_n = x_n + y_n$ are simple and converge a.e. to $x + y$, so $x + y$ is measurable.

To prove the second statement, let (f_n) be a sequence of simple functions converging a.e. to f and note that $f_n x_n$ are simple functions. It remains to prove that they converge a.e. to fx. Let A and B be sets of measure zero such that $f(\omega) = \lim_{n\to\infty} f_n(\omega)$ for all $\omega \in \Omega \setminus A$ and $x(\omega) = \lim_{n\to\infty} x_n(\omega)$ for all $\omega \in \Omega \setminus B$. Then for all $\omega \in \Omega \setminus (A \cup B)$ it holds that

$$\|f(\omega)x(\omega) - f_n(\omega)x_n(\omega)\| \le |f(\omega)|\|x(\omega) - x_n(\omega)\| + |f(\omega) - f_n(\omega)|\|x_n(\omega)\|.$$

By the convergence of f_n to f and x_n to x, this goes to zero as $n \to \infty$ pointwise almost everywhere. $\square$

A useful characterisation of measurability in terms of weak measurability is given by Pettis' Measurability Theorem. A function $x\colon \Omega \to \mathcal{X}$ is called *weakly measurable* if for all $\phi \in \mathcal{X}^*$ the numerical function $\phi \circ x$ is measurable. The concept that connects measurability to weak measurability is separability. A function is almost everywhere separably valued if there exists a set $A \in \Sigma$ of measure zero such that $f(\Omega \setminus A)$ is separable in $\mathcal{X}$.

Theorem A.3 (Pettis' Measurability Theorem, 3.5.2 in [11])**.** *A map $x\colon \Omega \to \mathcal{X}$ is measurable if and only if it is weakly measurable and almost everywhere separably valued.*

Let us look closer at Definition A.1 to show how differences in definitions can be confusing. In [11, Definition 3.4.5] Hille and Phillips define a function to be (strongly) measurable if it is the (a.e.) limit of countably valued functions. It is noted that it is 'easy to see' that this definition is the same as Definition A.1 if the measure space is finite. In the book of Diestel and Uhl [7] and in the original article of Pettis' Measurability Theorem [14], a finite measure space is used and the remark of Hille and Phillips in [11] is indeed a Corollary of Pettis' Measurability Theorem. Although this remark may suggest otherwise, these definitions for measurability coincide also for σ-finite spaces. To see this, compare (the proofs of) the two versions of Pettis' Measurability Theorem in [11, Theorem 3.5.2] and [15, Proposition 2.15][1].

Hence since our measure μ is σ-finite, a function is the limit of simple functions if and only if it is the limit of countably valued functions. And thus in our case this definition of measurability is compatible with the definition in [11].

The next proposition shows an easy application of Pettis' Measurability Theorem.

Proposition A.4. *The pointwise limit of a sequence of measurable functions is measurable.*

Proof. Let (x_n) be sequence of measurable functions from Ω to $\mathcal{X}$ with pointwise limit x. Each function x_n takes its values in a separable subspace of $\mathcal{X}$. The function x takes its values in the closed linear span of these subspaces, which is again separable.

For each $\phi \in \mathcal{X}^*$ the numerical function $\phi \circ x$ is measurable because it is the pointwise limit of the measurable numerical functions $\phi \circ x_n$ [3, Theorem 2.15]. So x is weakly measurable and by Pettis' Measurability Theorem, x is measurable. $\square$

Simple functions can be integrated in an obvious way. Let $x\colon \Omega \to \mathcal{X}$ be a simple function and write $x = \sum_{i=1}^n x_i \chi_{E_i}$ as before. Then for $A \subset \mathcal{X}$ it is natural to define $\int_A x \,\mathrm{d}\mu = \sum_{i=1}^n \mu(E_i) x_i$.

Definition A.5. A function $x\colon \Omega \to \mathcal{X}$ is (Bochner) integrable if there exists a sequence of simple functions x_n such that

$$\lim_{n\to\infty} \int_\Omega \|x_n - x\| \,\mathrm{d}\mu = 0.$$

[1]Be aware of a small leap in the argument in the proof of [15, Proposition 2.15]. Ryan claims that the inverse image of every weakly open set is measurable, which is not true. What is really needed for the proof to work is that pre-images of closed balls are measurable, which indeed is true.

Then by definition $\int_A x \,\mathrm{d}\mu$ is defined for each $A \subset \mathcal{X}$ as

$$\int_A x \,\mathrm{d}\mu = \lim_{n\to\infty} \int_A x_n \,\mathrm{d}\mu$$

Bochner's theorem gives a simple check determining for when a function is Bochner integrable.

Theorem A.6 (Bochner's Theorem, 3.7.4 in [11])**.** *A function $x\colon \Omega \to \mathcal{X}$ is Bochner integrable (with respect to μ) if and only if x is measurable and*

$$\int_\Omega \|x(\omega)\| \,\mathrm{d}\mu < \infty.$$

In this thesis, any $\mathcal{X}$-valued function which is Bochner integrable will be referred to as being integrable. A straightforward application of Bochner's theorem leads to an estimate for the norm of the integral which we often use.

Theorem A.7 (Theorem 3.7.6 in [11])**.** *If $x\colon \Omega \to \mathcal{X}$ is an integrable function, then*

$$\left\| \int_\Omega x(\omega) \,\mathrm{d}\mu \right\| \leq \int_\Omega \|x(\omega)\| \,\mathrm{d}\mu$$

A.2 Results needed for this thesis

In the variation of constants formula, we integrate over $T_\omega[x(\omega)]$ for some integrable semigroup $(T_\omega)_{\omega\in\Omega}$ and a measurable function $x\colon \Omega \to \mathcal{X}$. In Section 2.3, Bochner's Theorem is used to show the integrability of $T_\omega[x(\omega)]$. The following lemma states the requirements to meet one of the two requirements for this theorem: the measurability of $T_\omega[x(\omega)]$.

Lemma A.8. *Let $(T_\omega)_{\omega\in\Omega}$ be a family of bounded linear operators on $\mathcal{X}$ such that $\omega \mapsto T_\omega z$ is measurable for every $z \in \mathcal{X}$. Let $x\colon \Omega \to \mathcal{X}$ be a measurable function. Then $\omega \mapsto T_\omega[x(\omega)]$ is measurable.*

Proof. Let $(x_n)_n$ be a sequence of simple functions that converges almost everywhere to x. Write $x_n(\omega) = \sum_{i=1}^{N_n} \alpha_{n,i}\chi_{A_{n,i}}(\omega)$, where $N_n \in \mathbb{N}$, the coefficients $\alpha_{n,i}$ are in $\mathcal{X}$ and the sets $A_{n,i}$ are measurable. Now note that

$$T_\omega[x_n(\omega)] = \sum_{i=1}^{N_n} T_\omega[\alpha_{n,i}]\chi_{A_{n,i}}(\omega).$$

By assumption the map $\omega \mapsto T_\omega[\alpha_{n,i}]$ is measurable for each coefficient $\alpha_{n,i}$. By Proposition A.2, the map $\omega \mapsto T_\omega[x_n(\omega)]$ is measurable for each $n \in \mathbb{N}$. So $(T_\omega[x_n(\omega)])_n$ is a sequence of measurable functions, that for ω fixed almost always converges to $T_\omega[x(\omega)]$ by the continuity of T_ω. By Proposition A.4, this limit is measurable. $\square$

For proving that a strongly continuous semigroup $(T_\omega)_{\omega\in\Omega}$ is an integrable semigroup, we need that the continuous functions $\omega \mapsto T_\omega$ are measurable.

Proposition A.9. *If Ω is separable and the function $x\colon \Omega \to \mathcal{X}$ is continuous, then x is measurable.*

Proof. For each $\phi \in \mathcal{X}^*$, the numerical function $\phi \circ x$ is continuous and thus Borel measurable. The continuous image of a separable space is separable, so $x(\Omega)$ is separable. By Pettis' Measurability Theorem, x is measurable. $\square$

To prove that the variation of constants formula is well-defined, we need the following lemma.

Lemma A.10. *Let F be a continuous operator on $\mathcal{X}$ and let $x\colon \Omega \to X$ be a measurable function. Then $F \circ x\colon \Omega \to \mathcal{X}$ is measurable.*

Proof. Let (x_n) be simple functions converging a.e. to x. The functions $F \circ x_n$ are simple functions and by the continuity of F they converge to $F \circ x$. So by definition $F \circ x$ is measurable. $\square$

Essential to Lemma 2.10 is Fubini's Theorem for Bochner integrals.

Theorem A.11 (Theorem 3.7.13 in [11])**.** *Let $(A, \mathscr{A}, \mu)$ and $(B, \mathscr{B}, \nu)$ be σ-finite measure spaces. If the function $f\colon A \times B \to \mathcal{X}$ is Bochner integrable with respect to $\mu \otimes \nu$, then the functions $g(a) = \int_B f(a,b)\,\mathrm{d}\nu(b)$ and $h(b) = \int_A f(a,b)\,\mathrm{d}\mu(a)$ are defined almost everywhere in A resp. B and it holds that*

$$\int_{A\times B} f(a,b)\,\mathrm{d}(\mu\otimes\nu) = \int_A g(a)\,\mathrm{d}\mu = \int_B h(b)\,\mathrm{d}\nu \tag{A.1}$$

Here $\mathscr{A} \otimes \mathscr{B}$ is the product σ-algebra, which is generated by the rectangles $E \times F$, where $E \in \mathscr{A}$ and $F \in \mathscr{B}$. The product measure is denoted by $\mu \otimes \nu$. See for example [2, §23].

Theorem A.11 tells us that we can switch the order of integration of a double integral if the integrand is (Bochner) integrable with respect to the product measure. Lemma A.8 and A.10 are needed in Lemma 2.10 to prove this integrability.

B Gronwall's Lemma

We use the following version of Gronwall's Lemma, which can be found for example in [17].

Lemma B.1 (Gronwall's Lemma)**.** *Let r, K and a be functions from $\mathbb{R}^+$ to $\mathbb{R}^+$ that are integrable over any interval $[\alpha, \beta] \subset \mathbb{R}^+$ and let $b\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a bounded continuous map. Suppose that*

$$r(t) \le a(t) + K(t)\int_0^t b(s)r(s)\,\mathrm{d}s$$

for almost all $t \ge 0$ (with respect to the Lebesgue measure). Then for almost all $t \ge 0$

$$r(t) \le a(t) + K(t)\int_0^t a(s)b(s)\,\mathrm{d}s \cdot \exp\left(\int_0^t K(s)b(s)\,\mathrm{d}s\right). \tag{B.1}$$

In particular, if K and a are non-decreasing, then

$$r(t) \le a(t) \cdot \left[1 + K(t)\int_0^t b(s)\,\mathrm{d}s \cdot \exp\left(K(t)\int_0^t b(s)\,\mathrm{d}s\right)\right]. \tag{B.2}$$

C Some variations on the theorems derived

While developing the theory in Section 2, a small variation of Theorem 2.2 was found, as well as an alternative proof of Corollary 2.13. They are presented in this section, but they are not used in this thesis.

We start with the variation on Theorem 2.2. Here strong continuity is replaced by integrability and we obtain a mild solution that is bounded and measurable.

Proposition C.1. *Let $\mathcal{X}$ be a Banach space. Let $F\colon \mathcal{X} \to \mathcal{X}$ be globally Lipschitz and $(\hat{T}_t)_{t\geq 0}$ an integrable semigroup on $\mathcal{X}$. Then there exists a unique bounded measurable map $u\colon \mathbb{R}^+ \to \mathcal{X}$ that satisfies (2.2). That is,*

$$u(t) = \hat{T}_t x_0 + \int_0^t \hat{T}_{t-s} F\big(u(s)\big)\,\mathrm{d}s$$

for all $x_0 \in \mathcal{X}$.

Proof. The proof is analogue to the proof of Theorem 2.2. Let $T > 0$ and denote by $\mathcal{Z}([0,T],\mathcal{X})$ the space of bounded measurable maps from $[0,T]$ to $\mathcal{X}$. Define the operator Q on $\mathcal{Z}([0,T],\mathcal{X})$ as in (2.5), that is, for $u \in \mathcal{Z}([0,T],\mathcal{X})$ we have

$$Q(u)(t) = \hat{T}_t x_0 + \int_0^t \hat{T}_{t-s} F\big(u(s)\big)\,\mathrm{d}s. \tag{C.1}$$

It is not immediate that $Q(u)$ is bounded and measurable.

The set $B = \{u(s) : s \in [0,T]\}$ is bounded because u is bounded and thus $F[B]$ is bounded because F is Lipschitz. So there exists a $C > 0$ such that $\|F(u(s))\| \leq C$ for all $s \in [0,T]$. Since $(\hat{T}_t)_{t\geq 0}$ is integrable there exists an $M > 0$ such that $\|T_t\| \leq M$ for all $t \in [0,T]$. So boundedness of $Q(u)$ follows from

$$\|Q(u)(t)\| \leq \|T_t\|\|x_0\| + \int_0^t \|T_t\|\,\big\|F\big(u(s)\big)\big\|\,\mathrm{d}s \leq M\|x_0\| + tCM. \tag{C.2}$$

For obtaining measurability note that the integral in (C.1) is measurable with respect to t. Indeed, by Definition A.5 it is a limit of integrals of simple functions and it is straightforward to see that these are measurable. Furthermore, $\hat{T}_t x_0$ is measurable by definition and the sum of two measurable functions is measurable by Proposition A.2, so $Q(u)$ is measurable.

Lemma 2.5 still holds if we replace $\mathcal{C}$ by $\mathcal{Z}$ in the lemma and its proof. So Q is a contraction on $(\mathcal{Z}([0,T'],\mathcal{X}), \|\cdot\|_\infty)$ for some $T' > 0$.

By Banachs Fixed Point Theorem, there exists a unique fixed point u of Q in $\mathcal{Z}([0,T'],\mathcal{X})$. As desired, u satisfies (2.2).

Proposition 2.3 still holds if we use an integrable semigroup instead of a C_0 semigroup, so every $u \in \mathcal{Z}([0,T'],\mathcal{X})$ that satisfies (2.2) for some $x_0 \in \mathcal{X}$ and $T' > 0$ is unique. Using the same reasoning as in the proof of Theorem 2.2 on page 13, we can find a solution u that is defined on $[0,\infty)$. The prove that $u(t,x_0)$ is defined for all $t \in R^+$ also is exactly the same as in the proof of Theorem 2.2. □

With this proposition, we are able to give an alternative proof of Corollary 2.13. For this approach we need however that the Banach space is reflexive.

Theorem C.2. *Let $\mathcal{X}$ be a reflexive Banach space and $F\colon \mathcal{X} \to \mathcal{X}$ a bounded linear operator. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup. Let $(V_t)_{t\geq 0}$ be a C_0-semigroup that satisfies*

$$V_t x = T_t x + \int_0^t T_{t-s} F(V_s x)\,\mathrm{d}s \tag{C.3}$$

for all $x \in \mathcal{X}$. Then V_t also satisfies for all $x \in \mathcal{X}$

$$V_t x = T_t x + \int_0^t V_s F\big(T_{t-s} x\big)\,\mathrm{d}s \tag{C.4}$$

Proof. Take $B = F$ in Lemma 2.10. If we prove that that there exists an integrable semigroup S_t satisfies the different variation of constants formula in (2.18), then Lemma 2.10 yields $S_t = V_t$ and we are done.

Let $(T_t^*)_{t\geq 0}$ be the dual semigroup of $(T_t)_{t\geq 0}$, and let F^* be the dual operator of F. We are interested in mild solutions of (2.1) with $\hat{T}_t = T_t$ and F^* instead of F, as will become clear later in this proof. Generally $(T_t^*)_{t\geq 0}$ is not strongly continuous, but it is possible to prove that it is integrable. So here we need Proposition C.1. Let us check the requirements.

Because $(T_t)_{t\geq 0}$ is integrable we have that the function $t \mapsto T_t x$ is measurable for all $x \in \mathcal{X}$, so

$$\langle T_t^* \psi, x\rangle = \langle \psi, T_t x\rangle \tag{C.5}$$

is measurable for all $\psi \in \mathcal{X}^*$ and $x \in \mathcal{X}$. Using Pettis' Measureability Theorem and the reflexivity of $\mathcal{X}$, we see that $t \mapsto T_t^* \psi$ is measurable for all $\psi \in \mathcal{X}^*$. Since $\|T_t\| = \|T_t^*\|$ the bound in (2.13) holds for T_t^*, so $(T_t^*)_{t\geq 0}$ is integrable.

Since F is bounded and linear, F^* is bounded and linear and $\|F^*\| = \|F\|$. In particular, F^* is Lipschitz continuous.

The conditions of Proposition C.1 are satisfied, so there exists an integrable semigroup $(U_t)_{t\geq 0}$ on $\mathcal{X}^*$ that satisfies

$$U_t \psi = T_t^* x + \int_0^t T_{t-s}^* \left[F^*\big(U_s \psi\big)\right]\,\mathrm{d}s$$

for all $\psi \in \mathcal{X}^*$. So for all $\psi \in \mathcal{X}^*$ and $x \in \mathcal{X}$ it holds that

$$\begin{aligned} \langle U_t \psi,\, x\rangle &= \langle T_t^* \psi,\, x\rangle + \int_0^t \left\langle T_{t-s}^* \left[F^*\big(U_s\psi\big)\right],\, x\right\rangle\,\mathrm{d}s \\ &= \langle \psi,\, T_t x\rangle + \int_0^t \left\langle U_s\psi,\ F\big(T_{t-s}x\big)\right\rangle\,\mathrm{d}s. \end{aligned} \tag{C.6}$$

Let S_t be the restriction of U_t^* to $\mathcal{X} \cong \mathcal{X}^{**}$. From (C.6) it follows that for all $\psi \in \mathcal{X}^*$ and $x \in \mathcal{X}$ it holds that

$$\langle \psi,\, S_t x\rangle = \left\langle \psi,\ T_t x + \int_0^t F\big(T_{t-s}x\big)\,\mathrm{d}s \right\rangle,$$

so S_t satisfies (2.18).

Since $F = B$ is bounded and linear and V_t satisfies (2.19) by assumption, the conditions of Lemma 2.10 are satisfied. It follows that $V_t = S_t$, so V_t satisfies the equation in (C.4). $\square$

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