David Nicolaas Leendert Kok

Lattice algebra representations of $L^1(G)$ on translation invariant Banach function spaces

Master's thesis, defended on October 28, 2016

Thesis advisor: Dr. M.F.E. de Jeu

Mathematisch Instituut, Universiteit Leiden

Abstract

If G is a locally compact group, then $L^1(G)$ acts by convolution on $L^p(G)$. This action yields a homomorphism of the Banach algebra $L^1(G)$ into the bounded operators on $L^p(G)$. For finite p it is known, with the help of rather complicated tools, that this action gives a lattice homomorphism of $L^1(G)$ into the regular operators on $L^p(G)$. In this master's thesis we generalize this result by letting $L^1(G)$ act on so-called translation invariant Banach function spaces on G, or, more precisely, on the largest subspaces of such spaces where this action can be meaningfully defined. We show, using methods that are considerably simpler than those used in the previous proof, that under mild assumptions this action is also a lattice homomorphism.

Acknowledgements

I wish to express my very great appreciation to my supervisor Marcel de Jeu for his guidance and help throughout my research project. Many thanks for the inspiring and fun conversations.

Furthermore I deeply thank Ben de Pagter and Werner Ricker for allowing me to use their currently unpublished work on compact groups. I would not have found several key insights without the help of this material.

Lastly I would like to thank my family for all their support and my student association for providing distractions throughout the year.

Contents

1 Introduction

The field of representation theory studies groups and algebras by assigning to each element a linear map, thereby attempting to reduce questions of abstract algebra to simpler questions of linear algebra. It is well-known that the convolution action of $L^1(G)$ on $L^p(G)$ defines such a representation of $\mathbb{R}$-algebras. It has previously been shown [4, 3] that this natural representation is a lattice homomorphism, but the proof of this statement uses advanced techniques from measure theory. In this thesis we will give a more elementary proof of this known statement, and develop more general tools to study the representation of $L^1(G)$ on translation invariant Banach function spaces.
The central idea of our proof consists of several different steps. We will identify the closure of $C_c(G)$ in an abstract translation invariant Banach function space. Having identified this we can make use of compatibilities between convolutions on $C_c(G)$, the inclusion map from $C_c(G)$ into such a translation invariant Banach function space and convolutions on the Banach function space. We then split the last part of the proof of the main result into three steps: we prove a result on lattice homomorphisms inside a well-chosen commutator subspace, based on the method used by [3] to show our main result for the action of $L^1(G)$ on $L^p(G)$. Secondly we then find a different description of this commutator subspace to show that under lenient conditions it is a sublattice, and lastly we finish the proof by using a density argument to extend all our results from $C_c(G)$ to $L^1(G)$.

This thesis is split into 8 sections, numbered 2-9. In Section 2 we introduce the definitions used in the study of partially ordered vector spaces, along with several well-known results that we will rely on later in the thesis. We also introduce the notion of a representation, explain what is meant by a lattice algebra representation and introduce translation invariant Banach function spaces and their strongly continuous part. In Section 3 we show that this strongly continuous part is a Banach lattice. Furthermore, motivated by the results of de Pagter and Ricker, we introduce the spaces $L_c^\infty(G)$ and $L_{loc}^1(G)$ and show that, under mild assumptions, these bound any translation invariant Banach function space from below and above, respectively. In Section 4 we introduce the convolution maps on $C_c(G)$ and show that we can extend these to the strongly continuous part of a Banach function space. In Section 5 we introduce an approximate unit for the convolution operators considered in the previous section and use this approximate unit to identify the closure of $C_c(G)$ in our Banach function space. In Section 6 we introduce a set of sufficient conditions for a particular map to be a lattice homomorphism, with emphasis on keeping the conditions as general as possible. In Section 7 we apply the theorem from Section 6 to our extended convolution operators and conclude that, under mild assumptions, the left representation of $C_c(G)$ on the strongly continuous part of a translation invariant Banach function space is a lattice algebra homomorphism. We furthermore show that under additional mild assumptions the left representation of $L^1(G)$ on this strongly continuous part is also a lattice algebra homomorphism. We also give several explicit examples of translation invariant Banach function spaces. Section 8 presents a sketch of an alternative proof of the main result of this thesis in special cases. Section 9 summarizes the main results from the previous sections.

2 Preliminaries

From Section 3 onward some familiarity with the theory of ordered vector spaces is required. This section, split into three subsections, introduces the definitions and states a few preliminary results on partially ordered vector spaces, representations and Banach function spaces that will be used throughout the rest of the thesis. The first subsection of this section introduces results on partially ordered vector spaces along with some key theorems, with references to their proofs. The next subsection introduces the notions of group and $\mathbb{R}$-algebra representations, strong continuity and the definition of a lattice algebra representation. In the third subsection we introduce Banach function spaces (B.f.s.'s), make an explicit assumption about the translation action of the group on a translation invariant B.f.s. and introduce the strongly continuous part of a B.f.s.

Throughout this thesis we will be working over the base field of the real numbers, as opposed to the complex numbers. In particular we will assume that any functionals of interest are real-valued and any scalars are real numbers.

2.1 Partially ordered vector spaces

Definition 2.1. A real vector space V is said to be a *partially ordered vector space* or *ordered vector space* if it is equipped with a partial ordering $\geq$ that satisfies:

(1) If $x \geq y$ then $x + z \geq y + z$ for all $z \in V$.

(2) If $x \geq y$ then $\alpha x \geq \alpha y$ for all $\alpha \in \mathbb{R}_{\geq 0}$.

Definition 2.2. An element x in an ordered vector space V is called *positive* if $x \geq 0$. We denote with $V^+ = \{x \in V : x \geq 0\}$ the *positive cone* of V.

Definition 2.3. Let V, W be ordered vector spaces and $T : V \to W$ a linear map. Then T is said to be *positive* (notation: $T \geq 0$) if $Tx \geq 0$ for all $x \geq 0$, i.e. $T(V^+) \subseteq W^+$.

Remark 2.4. For two ordered vector spaces V, W the set of all $\mathbb{R}$-linear maps from V to W, denoted with $L(V, W)$, can be equipped with the partial ordering where for $S, T \in L(V, W)$ we say $S \geq T$ if and only if $S - T$ is positive. Under this partial ordering $L(V, W)$ forms an ordered vector space.

Definition 2.5. A *Riesz space* (or *vector lattice*) is an ordered vector space where for every two elements $x, y \in V$ the supremum and infimum of $\{x, y\}$, denoted by $x \vee y$ and $x \wedge y$ respectively, exist in V.

Definition 2.6. Let V be a partially ordered vector space and $x \in V$. If the supremum of $\{x, -x\}$ exists in V this supremum is called the *absolute value* of x and denoted with $|x|$.

We remark that in a Riesz space every element has an absolute value.

Definition 2.7. Let V be a Riesz space. V is called *Archimedean* if $\inf \left\{ \frac{1}{n}x : n \in \mathbb{N} \right\} = 0$ for all $x \in V^+$.

We will from here on restrict ourselves to Archimedean Riesz spaces.

Definition 2.8. Two elements x, y in a Riesz space V are said to be *disjoint* if $|x| \wedge |y| = 0$. A sequence $(x_n)_{n \in \mathbb{N}}$ in V is said to be *disjoint* if it is pairwise disjoint, i.e. if $|x_n| \wedge |x_m| = 0$ whenever $n \neq m$.

Definition 2.9. Let V be a Riesz space and $W \subseteq V$ a linear subspace. Then W is called a *sublattice* of V if it has the property that if $x, y \in W$ then also $x \wedge y, x \vee y \in W$.

Definition 2.10. Let V be a Riesz space and $W \subseteq V$ a linear subspace. If W has the property that for every $y \in W$ and $x \in V$ with $|x| \leq |y|$ we have $x \in W$ then W is called an *ideal* in V.

We remark that every ideal is also a sublattice, since if $W \subseteq V$ is an ideal and $x, y \in W$ then $|x - y| \in W$ since $|(|x - y|)| \leq |x - y|$; the identities $x \vee y = \frac{1}{2}(x + y + |x - y|), x \wedge y = \frac{1}{2}(x + y - |x - y|)$ ([2, Theorem 1.7.(2)])then complete the proof.

Definition 2.11. Let V, W be Riesz spaces and $T : V \to W$ be linear. Then T is called a *lattice homomorphism* if $T(|x|) = |T(x)|$ for all $x \in V$.

Definition 2.12. Let V be a Riesz space and $x, y \in V$ with $x \leq y$. Then we define the *order interval* $[x, y]$ as $[x, y] = \{z \in V : x \leq z \leq y\}$. We say that a subset $A \subseteq V$ is *bounded above* (resp. *bounded below*) if there exists a $x \in V$ such that $a \leq x$ (resp $a \geq x$) for all $a \in A$. Lastly we say that A is *order bounded* if it is bounded both above and below, i.e. if it is contained in an order interval.

Definition 2.13. Let V, W be Riesz spaces and $T : V \to W$ a linear map. Then T is said to be *order bounded* if it maps order bounded subsets of V to order bounded subsets of W. We denote the set of all order bounded linear maps from V to W with $L_b(V, W)$. For convenience we write $L_b(V)$ for $L_b(V, V)$.

Definition 2.14. Let V, W be Riesz spaces and $T : V \to W$ a linear map. T is said to be *regular* if it can be written as the difference of two positive linear maps or, equivalently, if there exists a positive linear map $S : V \to W$ with $T \leq S$. We denote the set of all regular linear maps from V to W with $L_r(V, W)$. For convenience we write $L_r(V)$ for $L_r(V, V)$.

Definition 2.15. A Riesz space V is said to be *Dedekind complete* if every non-empty subset of V that is bounded above has a supremum.

Theorem 2.16 (Riesz-Kantorovich)**.** *Let V, W be Riesz spaces with W Dedekind complete. Then $L_b(V, W)$ is a Dedekind complete Riesz space, and we have $L_b(V, W) = L_r(V, W)$.*

Proof. A proof of the fact that $L_b(V, W)$ is a Dedekind complete Riesz space can be found in [2, Theorem 1.18]. Now since any positive map is order bounded we see that $L_r(V, W) \subseteq L_b(V, W)$. Now let $T \in L_b(V, W)$ be arbitrary, and set $T^+ = T \vee 0$, $T^- = (-T) \vee 0$. Then $T^+, T^- \geq 0$ and $T = T^+ - T^-$ so $T \in L_r(V, W)$. $\square$

Next we will consider Riesz spaces equipped with a suitable norm.

Definition 2.17. Let V be a Riesz space. A *lattice norm* or *Riesz norm* on V is a norm on V that satisfies $\|x\| \leq \|y\|$ for all $x, y \in V$ with $|x| \leq |y|$. A Riesz space equipped with a lattice norm is called a *normed Riesz space*.

Definition 2.18. A normed Riesz space that is complete with respect to its lattice norm is called a *Banach lattice*.

The definition of the lattice norm above has an interesting consequence.

Theorem 2.19. *Every positive map from a Banach lattice to a normed Riesz space is continuous.*

Proof. A proof can be found in [2, Theorem 4.3]. □

Definition 2.20. Let V, W be Dedekind complete Banach lattices. We define the *regular norm* on $L_r(V, W)$ as the operator norm of the absolute value, i.e. $\|T\|_{\text{reg}} = \||T|\|_{\text{operator}}$ for $T \in L_r(V, W)$.

Proposition 2.21. *Let V, W be Banach lattices with W Dedekind complete. Then $L_r(V, W)$ equipped with the regular norm is a Dedekind complete Banach lattice.*

Proof. See [2, Thm. 4.74]. □

Now we consider Riesz spaces with an even stronger compatibility between the partial ordering and the norm.

Definition 2.22. Let V be a normed Riesz space. V is said to have *order continuous norm* if every decreasing net in V with infimum 0 decreases in norm to 0.

Some important results on Banach lattices with order continuous norm are:

Proposition 2.23. *Every Banach lattice with order continuous norm is Dedekind complete.*

Proof. A proof can be found in [2, Corollary 4.10]. □

Theorem 2.24. *A Banach lattice has order continuous norm if and only if every order bounded disjoint sequence is norm convergent to* 0*.*

Proof. A proof can be found in [2, Theorem 4.14]. □

Lastly we remark that if V is a Banach lattice with order continuous norm and $W \subseteq V$ is a Banach sublattice (i.e. sublattice that is also a Banach lattice) then W also has order continuous norm:

Lemma 2.25. *If V is a Banach lattice with order continuous norm and W is a Banach sublattice then W also has order continuous norm.*

Proof. If V has order continuous norm then every order bounded disjoint sequence in V is norm convergent to 0 (Theorem 2.24). But every order bounded disjoint sequence in W is also an order bounded disjoint sequence in V. So every such sequence converges to 0. Since W is a Banach lattice we conclude that it has order continuous norm. □

2.2 Representations

In this section we will introduce two different types of representations, namely representations of a group on a real vector space and representations of an $\mathbb{R}$-algebra on a real vector space.

Definition 2.26. Let G be a group with unit element e and let V be a real vector space. Denote with $GL_{\mathbb{R}}(V)$ the general linear group of V, i.e. the set of al invertible linear maps from V to V. A *representation* of G on V is a group homomorphism from G to $GL_{\mathbb{R}}(V)$, i.e. a map $\pi : G \to GL_{\mathbb{R}}(V)$ satisfying $\pi(e) = Id_V$ and $\pi(a)\pi(b) = \pi(ab)$ for all $a, b \in G$.

Definition 2.27. Let G be a topological group and V a real topological vector space. Let $\pi : G \to GL_{\mathbb{R}}(V)$ be a representation. Then π is said to be *strongly continuous* if for each $v \in V$ the map $g \mapsto \pi(g)v$ is continuous.

Definition 2.28. Let R be an $\mathbb{R}$-algebra and let V be a real vector space. A *representation* of R on V is an $\mathbb{R}$-algebra homomorphism $\pi : R \to \mathrm{End}_{\mathbb{R}}(V)$, i.e. a map $\pi : R \to \mathrm{End}_{\mathbb{R}}(V)$ which satisfies for all $a, b \in R$ and $\lambda \in \mathbb{R}$:

(1) $\pi(a) + \pi(b) = \pi(a + b)$;

(2) $\pi(\lambda a) = \lambda\pi(a)$;

(3) $\pi(ab) = \pi(a)\pi(b)$.

Even if R is unital we do not demand that π is unital (i.e. $\pi(\mathbf{1}_R) \neq Id_V$ is allowed).

Definition 2.29. An $\mathbb{R}$-algebra that is also a Riesz space is called a *Riesz algebra*.

Remark 2.30. It is sometimes customary to additionally demand that in a Riesz algebra A the product of two positive elements is positive, i.e. $ab \geq 0$ for all $a, b \geq 0$. Since we do not need this assumption we do not include this as part of the definition of Riesz algebra.

Definition 2.31. Let R be a Riesz algebra, V a real ordered vector space and let S be a Riesz subalgebra of $\mathrm{End}_{\mathbb{R}}(V)$. Assume $\pi : R \to S$ a representation (i.e. a representation of R on V with image contained in S). Then π is called a *lattice algebra representation* if it is a lattice homomorphism.

2.3 Banach function spaces

Throughout this thesis we will denote with G a (not necessarily abelian) locally compact Hausdorff group with unit element e. We equip G with a left translation invariant Haar measure μ, i.e. a left translation invariant Borel measure that is finite on compact sets, outer regular on all Borel sets and inner regular on all open sets [6, p. 212]. For a Borel measurable set $Y \subseteq G$ we denote with χ_Y the indicator function of Y. Furthermore we denote with $L^0(\mu)$ the set of all μ-measurable functions (up to almost-everywhere equality) from G to $\mathbb{R}$. Later in this thesis we will need that if two measurable functions agree (μ-a.e.) on each compact set, then they also agree μ-a.e. on G. A sufficient condition for this implication to hold is that G is σ-compact, but we will just assume that μ has the property above.

We also introduce the modular fuction Δ corresponding to μ, i.e. the unique function $\Delta : G \to \mathbb{R}$ satisfying $\Delta(x)\mu(S) = \mu(Sx)$ for all $x \in G$ and $S \subseteq G$ Borel measurable. Recall that Δ is multiplicative and continuous (and therefore also bounded on compact sets), and furthermore that we have the chain rules $\int_G f(xy)d\mu(x) = \Delta(y)^{-1} \int_G f(x)d\mu(x)$ and $\int_G f(y^{-1})\Delta(y)^{-1}d\mu(y) = \int_G f(y)d\mu(y)$ for all μ-integrable functions $f : G \to \mathbb{R}$. Important to remark is that some authors prefer to adopt the convention $\int_G f(xy)d\mu(x) = \Delta(y) \int_G f(x)d\mu(x)$ instead of our definition, which is equivalent to calling the multiplicative inverse of our function above the modular function. Some caution is therefore required when relying on results about the modular function from external sources.

We remark that the set $L^0(\mu)$ defined above equipped with the almost everywhere point-wise partial ordering, i.e. $f \geq g \iff f(x) \geq g(x)$ μ-a.e., is a Riesz space and its absolute value is the point-wise absolute value.

Definition 2.32. A *Banach function space* (or *B.f.s.*) on G is an ideal E in $L^0(\mu)$ that is also a Banach lattice, i.e. is equipped with a lattice norm $\|\cdot\|$ with respect to which E is a Banach space.

We now introduce the left and right translation operators:

Definition 2.33. For $y \in G$ we denote with $\lambda_y : L^0(\mu) \to L^0(\mu)$ the *left translation operator* given by $\lambda_y(g)(x) = g(y^{-1}x)$ for all $g \in L^0(\mu)$. If E a Banach function space on G such that the image of E under λ_y lies inside E for all $y \in G$ (i.e $\lambda_y(E) \subseteq E$ for all $y \in G$) then E is called *left translation invariant*.

Definition 2.34. For $y \in G$ we denote with $\rho_y : L^0(\mu) \to L^0(\mu)$ the *right translation operator* given by $\rho_y(g)(x) = g(xy^{-1})$ for all $g \in L^0(\mu)$. If E is a Banach function space on G such that the image of E under ρ_y lies inside E for all $y \in G$ (i.e $\rho_y(E) \subseteq E$ for all $y \in G$) then E is called *right translation invariant*.

A Banach function space that is both left translation invariant and right translation invariant is called *translation invariant*. Throughout this thesis we will denote with E a translation invariant Banach function space on G, and write $\|\cdot\|_E$ for its norm. We remark that for a translation invariant B.f.s. E the left translation map $\lambda : G \to GL_{\mathbb{R}}(E)$ and the pre-composition of the right translation map with the inverse on G, $\rho \circ i : G \to GL_{\mathbb{R}}(E)$, are group representations of G on E. However, we wish to have slightly stronger compatibility between the group action and our Banach function spaces. Therefore we will throughout this thesis work with the following assumption on the continuity of the translation operators:

Assumption 2.35. We assume that for each $y \in G$ the operators $\lambda_y, \rho_y : E \to E$ are continuous, and furthermore that the maps $y \mapsto \|\lambda_y\|_{B(E,E)}$ and $y \mapsto \|\rho_y\|_{B(E,E)}$ are bounded on compact sets.

Using the translation operators introduced above we can now introduce the notion of a *strongly continuous* element in E:

Definition 2.36. An element $f \in E$ for which $\lim_{y\to x} \|\lambda_y f - \lambda_x f\|_E = 0$ for all $x \in G$ is called *left strongly continuous*, if $\lim_{y\to x} \|\rho_y f - \rho_x f\|_E = 0$ for all $x \in G$ it is called *right strongly continuous* if it is both left strongly continuous and right strongly continuous it is called strongly continuous.

Definition 2.37. For a translation invariant Banach function space E we denote with E_s its *strongly continuous part*, i.e. the linear subspace of all strongly continuous elements in E.

Remark 2.38. Since under Assumption 2.35 the left translation operator λ_x is continuous for each $x \in G$, and therefore bounded, we can for each $f \in E$ write

$$\|\lambda_y f - \lambda_x f\|_E = \|\lambda_x(\lambda_{x^{-1}y} f - f)\|_E \leq \|\lambda_x\|_{B(E,E)} \|\lambda_{x^{-1}y} f - f\|_E .$$

Therefore under Assumption 2.35 we see that an element $f \in E$ is left strongly continuous if and only if $\lim_{y\to e} \|\lambda_y f - f\|_E = 0$. Analogously we remark that under Assumption 2.35 an element $f \in E$ is right strongly continuous if and only if $\lim_{y\to e} \|\rho_y f - f\|_E = 0$.

Remark 2.39. In some textbooks our notion of strongly continuous is called *homogeneous*, and E_s is denoted as the *homogeneous part* of a B.f.s. E.

Remark 2.40. Clearly E_s is the maximal subspace of E on which both group representations λ and $\rho \circ i$ are strongly continuous representations.

Remark 2.41. Let $f \in E$ be arbitrary, and let $Y \subseteq G$ be a measurable set. We observe that since $|f\chi_Y| \leq |f|$ and E is an ideal in $L^0(\mu)$ we also have $f\chi_Y \in E$.

We can use this to introduce the set of all strongly continuous elements of E that vanish at infinity, which will be of interest later:

Definition 2.42. For a B.f.s. E we introduce the set of strongly continuous elements of E that vanish at infinity:

$$E_{s,0} := \{f \in E_s : \forall\ \epsilon > 0\ \exists\ K \subseteq G \text{ compact s.t. } \left\|f\chi_{G\setminus K}\right\|_E < \epsilon\}.$$

For a continuous function $f : G \to \mathbb{R}$ we write $\operatorname{supp}(f) = \overline{\{x \in G : f(x) \neq 0\}}$ for its support, and denote with $C_c(G)$ the set of all continuous functions from G to $\mathbb{R}$ with compact support. For any measurable subset $H \subseteq G$ we denote with $L^1(H)$ and $L^\infty(H)$ the μ-integrable and essentially bounded elements of $L^0(\mu)$, respectively, that vanish almost everywhere outside H. Lastly for any two normed vector spaces X, Y we denote with $B(X, Y)$ the set of bounded operators from X to Y.

3 Properties of translation invariant Banach function spaces

We will first prove some preliminary results on the spaces introduced in the previous section. Recall that throughout this thesis we are working under Assumption 2.35, i.e. the assumption that for all $y \in G$ the maps $\lambda_y : E \to E$ and $\rho_y : E \to E$ are bounded operators, and that for compact $K \subseteq G$ the suprema $\sup_{y\in K} \|\lambda_y\|_{B(E,E)}$ and $\sup_{y\in K} \|\rho_y\|_{B(E,E)}$ are finite.

We will see that $E_{s,0}$ and E_s are closed in E (and are therefore also Banach spaces), and are sublattices of E. Furthermore we will show that $E_{s,0}$ is an ideal in E_s, and show that in general neither of these is an ideal in E or $L^0(\mu)$. Furthermore we will see that $E_{s,0}$ and E_s are translation invariant. Next we will find necessary and sufficient conditions for $C_c(G)$ to be a subset of E, and conclude that in this case we even have $C_c(G) \in E_{s,0}$. Lastly we will introduce the space $L_c^\infty(G)$ of compactly supported essentially bounded functions and the space $L_{loc}^1(G)$ of locally integrable functions, and show that under Assumption 2.35 we have $L_c^\infty(G) \subseteq E \subseteq L_{loc}^1(G)$ with continuous inclusions.

3.1 The spaces E_s and $E_{s,0}$ are Banach lattices

In this subsection we will prove that E_s is closed in E and that $E_{s,0}$ is closed in E_s, and conclude that both of these spaces are Banach spaces. Furthermore we will show that E_s is a sublattice of E and that $E_{s,0}$ is a sublattice of E_s, and therefore of E. As was remarked in the previous section we note that an element $f \in E$ is strongly continuous if and only if $\lim_{y\to e} \|\lambda_y f - f\|_E = 0 = \lim_{y\to e} \|\rho_y f - f\|_E$.

Lemma 3.1. *The strongly continuous part E_s of E is closed in E.*

Proof. Let $(f_n)_{n\in\mathbb{N}} \subseteq E_s$ be a sequence converging to $f \in E$, and let $\epsilon > 0$ be arbitrary. We will find a neighbourhood K of e in G such that $\|\lambda_y f - f\|_E < \epsilon$ for all $y \in K$, proving that f is left strongly continuous. Pick a neighbourhood M of e in G with compact closure, and a constant $Q \in \mathbb{R}_{>0}$ such that $\|\lambda_y\|_{B(E,E)} < Q$ for all $y \in M$. Pick an $n \in \mathbb{N}$ such that $\|f - f_n\|_E < \frac{\epsilon}{2(Q+1)}$ and a neighbourhood N of e in G such that $\|\lambda_y f_n - f_n\|_E < \frac{\epsilon}{2}$ for all $y \in N$ (which exists because $f_n \in E_s$). Set $K = M \cap N$. Then it follows from the triangle inequality that for all $y \in K$ we have

$$\begin{aligned}
\|\lambda_y f - f\|_E &\leq \|\lambda_y f - \lambda_y f_n\|_E + \|\lambda_y f_n - f_n\|_E + \|f_n - f\|_E \\
&< \|\lambda_y\|_{B(E,E)} \|f - f_n\|_E + \frac{\epsilon}{2} + \|f_n - f\|_E \\
&= (\|\lambda_y\|_{B(E,E)} + 1) \|f - f_n\|_E + \frac{\epsilon}{2} \\
&< (Q+1)\frac{\epsilon}{2(Q+1)} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

It follows that f is left strongly continuous. The proof that f is right strongly continuous is analogous, completing the proof of the lemma. $\square$

Lemma 3.2. *The strongly continuous part E_s of E is a sublattice of E.*

Proof. It suffices to show that $|f| \in E_s$ for all $f \in E_s$. Let $y \in G$ and $f \in E_s$ be arbitrary. The reverse triangle inequality now gives us that

$$\big|\lambda_y|f| - |f|\big| = \big||\lambda_y f| - |f|\big| \leq |\lambda_y f - f|,$$

where we used that $|\lambda_y f| = \lambda_y |f|$. It follows from the inequalities above that $\big\|\lambda_y|f| - |f|\big\|_E \leq \|\lambda_y f - f\|_E$ for all y, so if f is left strongly continuous then so is $|f|$. An analogous proof shows that $|f|$ is right strongly continuous, so $|f| \in E_s$ for $f \in E_s$. We conclude that E_s is a sublattice of E. $\square$

Lemma 3.3. *$E_{s,0}$ is closed in E_s, and is therefore closed in E.*

Proof. Let $(f_n)_{n\in\mathbb{N}} \subseteq E_{s,0}$ be a sequence converging to $f \in E_s$, and let $\epsilon > 0$ be arbitrary. Pick an $n \in \mathbb{N}$ such that $\|f - f_n\|_E < \frac{\epsilon}{2}$ and a compact $K \subseteq G$ such that $\|f_n \chi_{G\setminus K}\|_E < \frac{\epsilon}{2}$. We then find that:

$$\|f\chi_{G\setminus K}\|_E \leq \|(f - f_n)\chi_{G\setminus K}\|_E + \|f_n\chi_{G\setminus K}\|_E \leq \|f - f_n\|_E + \|f_n\chi_{G\setminus K}\|_E < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So $f \in E_{s,0}$, completing the proof. $\square$

Lemma 3.4. *$E_{s,0}$ is a sublattice of E_s, and is therefore a sublattice of E.*

Proof. Let $f \in E_{s,0}$ be arbitrary. We need to show that $|f| \in E_{s,0}$, i.e. $|f| \in E_s$ and $|f|$ vanishes at infinity. For each compact $K \subseteq G$ we have $\big\||f|\chi_{G\setminus K}\big\|_E = \|f\chi_{G\setminus K}\|_E$ since $\chi_{G\setminus K} \geq 0$ and the norm on E is a lattice norm, so $|f|$ vanishes at infinity. Since $f \in E_{s,0} \subseteq E_s$ which is a sublattice of E it follows that $|f| \in E_s$. We conclude that $|f| \in E_{s,0}$. $\square$

Lemma 3.5. *$E_{s,0}$ is an ideal in E_s.*

Proof. Let $f \in E_{s,0}$ and $g \in E_s$ be such that $0 \leq |g| \leq |f|$. Then for all compact $K \subseteq G$ we have $\|g\chi_{G\setminus K}\|_E = \big\||g|\chi_{G\setminus K}\big\|_E \leq \big\||f|\chi_{G\setminus K}\big\|_E = \|f\chi_{G\setminus K}\|_E$, so g vanishes at infinity, i.e. $g \in E_{s,0}$. We conclude that $E_{s,0}$ is an ideal in E_s. $\square$

However, in general E_s and $E_{s,0}$ are not ideals in E or $L^0(\mu)$, as can be seen from the following example:

Example 3.6. If $E = L^\infty(G)$, then the set of all strongly continuous elements of E is precisely the set of (essentially) bounded uniformly continuous functions on G. But given a non-zero uniformly continuous function $f \in L^\infty(G)$ we can generally find a non-continuous function that lies below $|f|$, for example by multiplying $|f|$ by a suitable indicator function (note that this is not true in the case that G is discrete, but for example for $G = \mathbb{R}^n$ the above holds). We conclude that the set of strongly continuous elements of E is not in general an ideal in E or $L^0(\mu)$. A similar procedure gives a counterexample for $E_{s,0}$.

Lemma 3.7. *The spaces E_s and $E_{s,0}$ are translation invariant.*

Proof. We will first show that the translate of a function vanishing at infinity also vanishes at infinity. Suppose $f \in E$ vanishes at infinity, and let $\epsilon > 0$ and $y \in G$ be arbitrary. Pick compact $K \subseteq G$ such that $\left\|f\chi_{G\setminus K}\right\|_E < \frac{\epsilon}{\|\lambda_y\|_{B(E,E)}}$. We then see that

$$\left\|(\lambda_y f)\chi_{G\setminus yK}\right\|_E = \left\|\lambda(f\chi_{G\setminus K})\right\|_E \leq \|\lambda_y\|_{B(E,E)} \left\|f\chi_{G\setminus K}\right\|_E < \epsilon.$$

As yK is compact and $\epsilon > 0$ was arbitrary we conclude that $\lambda_y f$ vanishes at infinity. A similar proof shows that $\rho_y f$ vanishes at infinity. To complete the proof of the lemma it now suffices to show that E_s is translation invariant.
Let $f \in E_s$ and $y \in G$ be arbitrary. To show that $\lambda_y f \in E_s$ we need to show that $\lim_{x\to e} \|\lambda_x(\lambda_y f) - \lambda_y f\|_E = 0 = \lim_{x\to e} \|\rho_x(\lambda_y f) - \lambda_y f)\|_E$. The first equality follows immediately from the fact that f is left strongly continuous . For the second equality we remark that λ_y and ρ_x commute, so we can write

$$\|\rho_x(\lambda_y f) - \lambda_y f)\|_E = \|\lambda_y(\rho_x f - f)\|_E \leq \|\lambda_y\|_{B(E,E)} \|\rho_x f - f\|_E \,,$$

and the desired statement about the limit follows from the fact that f is right strongly continuous. We conclude that $\lambda_y f \in E_s$. An analogous proof shows that $\rho_y f \in E_s$, from which it follows that E_s is translation invariant. We conclude that E_s and $E_{s,0}$ are translation invariant. □

In conclusion, we have shown in this subsection that under Assumption 2.35 the space E_s of all strongly continuous elements of E and the space $E_{s,0}$ of all strongly continuous elements of E that vanish at infinity are translation invariant Banach sublattices of E, but are not in general Banach function spaces.

3.2 The inclusion of $C_c(G)$ in E

In this subsection we will find sufficient and necessary conditions for $C_c(G)$ to be a subset of a translation invariant Banach function space E. We introduce the spaces $L_c^\infty(G)$ and $L_{loc}^1(G)$ and show that we have $L_c^\infty(G) \subseteq E \subseteq L_{loc}^1(G)$ with continuous inclusions.
As we will show immediately below such sufficient and necessary conditions are given by the inclusion of a single non-zero element of $C_c(G)$ being contained in E:

Lemma 3.8. *Let E be a translation invariant Banach function space. Then the following are equivalent:*

(i) $C_c(G) \subseteq E$;

(ii) $C_c(G) \cap E \neq \{0\}$.

Proof. (i) $\implies$ (ii) is trivial.
(ii) $\implies$ (i): Let $0 \neq f \in C_c(G) \cap E$ be arbitrary, and pick an $\epsilon > 0$ such that $O = \{x \in G : |f(x)| > \epsilon\}$ is non-empty. Note that $0 < \epsilon\chi_O \leq |f|$ so $\chi_O \in E$. Now let $g \in C_c(G)$ be arbitrary, and set $K = \text{supp}(g)$. Note that $K \subseteq G = \cup_{y \in G}(yO)$, so since K is compact we can find a finite subcover, say $K \subseteq \cup_{y \in I}(yO)$ with I finite. Since E is translation invariant and $\chi_O \in E$ we have $\chi_{yO} = \lambda_y(\chi_O) \in E$ for all $y \in I$. In particular we have

$$0 \leq \chi_K \leq \sum_{y \in I} \chi_{yO} \in E.$$

So $\chi_K \in E$. Since $0 \leq |g| \leq \|g\|_\infty \chi_K$ it follows that $g \in E$, as was to be shown. □

The proof of Lemma 3.8 above actually gives us a stronger statement: if either of the equivalent statements in the Lemma is true, then also for any compact $K \subseteq G$ we have $\chi_K \in E$ so $L^\infty(K) \subseteq E$ (since E is an ideal in $L^0(\mu)$). Letting $f \in L^\infty(K)$ be arbitrary and denoting with $j : L^\infty(K) \to E$ the inclusion map we remark that $\|j(f)\|_E = \|j(|f|)\|_E \leq \|j(\|f\|_\infty \chi_K)\|_E = \|f\|_\infty \|\chi_K\|_E$, so the inclusion of $L^\infty(K)$ in E is continuous. This motivates us to introduce the space of L^∞ functions with compact support:

Definition 3.9. We introduce the vector space of *essentially bounded functions with compact support* on G as

$$L_c^\infty(G) = \left. \bigoplus_{\substack{K \subseteq G \\ K \text{ compact}}} L^\infty(K) \middle/ \sim \right.$$

as the direct limit of $L^\infty(K)$ where $K \subseteq G$ ranges over the compact sets (and the equivalence relation is defined through the inclusion maps $L^\infty(K) \subseteq L^\infty(M)$ for $K \subseteq M$, i.e. we identify an element of $L^\infty(K)$ with its image in $L^\infty(M)$ for any two such K, M). Since $L_c^\infty(G)$ is a direct limit of topological vector spaces it is naturally equipped with the inductive topology.

Remark 3.10. We recall that a map $j : L_c^\infty(G) \to X$ is continuous if and only if its precomposition with the inclusion map $L^\infty(K) \to L_c^\infty(G)$ is continuous for all compact K.

From the statement above the definition it follows that $L_c^\infty(G) \subseteq E$ (unless the intersection with $C_c(G)$ is trivial). Furthermore it follows immediately from Remark 3.10 that the inclusion map $L_c^\infty(G) \to E$ is continuous, so we have the following statement:

Theorem 3.11. *Let E be a translation invariant B.f.s. such that $E \cap C_c(G) \neq \{0\}$. Then $L_c^\infty(G) \subseteq E$ with continuous inclusion.*

Remark 3.12. We note that since for any continuous function with compact support $f \in C_c(G)$, say with support $\text{supp}(f) = K$, we have $\|\lambda_y f - f\|_\infty \to 0$ as $y \to e$, and since (if $C_c(G) \cap E \neq \{0\}$) the inclusions $C_c(K) \subseteq L^\infty(K) \subseteq E$ are continuous it follows that $\|\lambda_y f - f\|_E \to 0$. Combined with an analogous statement for the right translation we conclude that if $C_c(G) \subseteq E$ then also $C_c(G) \subseteq E_s$, so clearly even $C_c(G) \subseteq E_{s,0}$.

From the results above we see that any translation invariant B.f.s. is bounded from below by the space $L_c^\infty(G)$. In the next part of this section we will show that the space $L^1_{loc}(G)$ of locally integrable functions is an upper bound for any translation invariant Banach function space. To show this we need to show that elements of E are locally integrable. As a first step we introduce the Köthe dual of a function space:

Definition 3.13. Let A be a normed Riesz space of μ-measurable functions from G to $\mathbb{R}$. We define the *Köthe dual* $A^\times$ of A as

$$A^\times = \{g \in L^0(\mu) : \|g\|_{A^\times} = \sup_{\substack{f \in A \\ \|f\|_A \le 1}} \int_G |fg| d\mu < \infty\}.$$

Lemma 3.14. *Let $K \subseteq G$ be compact and denote the set of elements of E vanishing (almost everywhere) outside K with $E_K = \{\chi_K f : f \in E\}$ (equipped with $\|\cdot\|_E$). Then under Assumption 2.35 we have $E_K \subseteq L^1(K)$ with continuous inclusion.*

Proof. We follow an analogous proof to the one given by [5, Prop. 4.3] for the compact abelian case. If K has measure zero both E_K and $L^1(K)$ are the zero space, and the claim is trivial. From here on we assume that $\mu(K) > 0$. It follows from [9, Theorem 112.1] that the carrier of $E_K^\times$ is K, so in particular $E_K^\times \neq \{0\}$. Pick $0 \le g \in E_K^\times$ with $\|g\|_{E_K^\times} = 1$, and let $0 \le f \in E_K$ be arbitrary. We will compute the double integral of $f(xy)g(y)$ in two different ways below, allowing the value $+\infty$ at this point. Using Tonelli's theorem (note that all functions are positive) we see that

$$\begin{aligned}\int_G \left(\int_G f(xy)g(y)d\mu(y) \right) d\mu(x) &= \int_G \left(\int_G f(xy)d\mu(x) \right) g(y)d\mu(y) \\ &= \int_G \left(\int_G \Delta(y^{-1})f(x)d\mu(x) \right) g(y)d\mu(y) \\ &= \left(\int_G f(x)d\mu(x) \right) \left(\int_G \Delta(y^{-1})g(y)d\mu(y) \right).\end{aligned}$$

However we remark that both g and f are almost-everywhere zero outside K, so the integrand $f(xy)g(y)$ only contributes to the integral for $y \in K, x \in KK^{-1}$. In particular we see that we can also write the inner integral as

$$\begin{aligned}\int_G f(xy)g(y)d\mu(y) &= \int_K f(xy)g(y)d\mu(y) \\ &= \int_K (\lambda_{x^{-1}}(f))(y)g(y)d\mu(y) \\ &= \int_K (\lambda_{x^{-1}}(f)\chi_K)(y)g(y)d\mu(y).\end{aligned}$$

But since E is translation invariant we have $\lambda_{x^{-1}}(f) \in E$, and since $0 \le f$ we have $0 \le \lambda_{x^{-1}}(f)\chi_K \le \lambda_{x^{-1}}(f)$ so $\lambda_{x^{-1}}(f)\chi_K \in E_K$. We furthermore remark that $\|\lambda_{x^{-1}}(f)\chi_K\|_E \le$

$\|\lambda_{x^{-1}}(f)\|_E \leq \|\lambda_{x^{-1}}\|_{B(E,E)} \|f\|_E$. Using this and the definition of the norm on the Köthe dual (recall that $\|g\|_{E_K^\times} = 1$) we see that

$$\int_G f(xy)g(y)d\mu(y) = \int_K f(xy)g(y)d\mu(y) \leq \|\lambda_{x^{-1}}\|_{B(E,E)} \|f\|_E.$$

Now since $\|\lambda_{x^{-1}}\|_{B(E,E)}$ is bounded on compact sets and our integrand has a non-zero contribution only if $x \in KK^{-1}$, i.e. $x^{-1} \in (KK^{-1})^{-1} = KK^{-1}$, we can find a constant M such that $\|\lambda_{x^{-1}}\|_{B(E,E)} < M$ for all $x^{-1} \in KK^{-1}$. Using this we find that

$$\begin{aligned}\left(\int_G f(x)d\mu(x)\right)\left(\int_G \Delta(y)^{-1}g(y)d\mu(y)\right) &= \int_G \left(\int_G f(xy)g(y)d\mu(y)\right) d\mu(x) \\ &\leq \int_{KK^{-1}} \|\lambda_{x^{-1}}\|_{B(E,E)} \|f\|_E \, d\mu(x) \\ &\leq M\mu(KK^{-1}) \|f\|_E < \infty.\end{aligned}$$

Since $g \gneq 0$ we have $\int_G \Delta(y^{-1})g(y)d\mu(y) > 0$, so $\int_G f(x)d\mu(x) < \infty$. Furthermore we can pick $f \in E_K$ such that $f \gneq 0$ so $\int_G f(x)d\mu(x) > 0$, so also $\alpha = \int_G \Delta(y^{-1})g(y)d\mu(y) < \infty$. The above inequalities applied to $|f|$ can now be rewritten to read

$$\|f\|_1 \leq \alpha^{-1}M\mu(KK^{-1}) \|f\|_E,$$

so $E_K \subseteq L^1(K)$ and this inclusion is continuous. □

Motivated by this result we introduce the space of locally integrable functions:

Definition 3.15. A measurable function $f : G \to \mathbb{R}$ is called *locally integrable* if $f\chi_K \in L^1(K)$ for all $K \subseteq G$ compact. Denote the set of all locally integrable functions with $L^1_{loc}(G)$.

We equip $L^1_{loc}(G)$ with the topology generated by the seminorms $\|\cdot\|_{1,K}$ given by $\|f\|_{1,K} = \int_K |f(y)|d\mu(y)$ where $K \subseteq G$ is a compact set. Equipped with this topology we remark that a map $j : X \to L^1_{loc}(G)$ is continuous if and only if its composition with each of these seminorms is. Since E is an ideal in $L^0(\mu)$ we see that for each $f \in E$ and $K \subseteq G$ we have $f\chi_K \in E_K \subseteq L^1(K)$, so $f \in L^1_{loc}(G)$. We now conclude that Lemma 3.14 implies that the map $f \to \|f\|_{1,K} = \|f\chi_K\|_1$ is continuous, from which we immediately find the following.

Theorem 3.16. *Let E be a translation invariant B.f.s. such that the translation operators λ_y, ρ_y are continuous for each $y \in G$, and assume furthermore that $\|\lambda_y\|_{B(E,E)}, \|\rho_y\|_{B(E,E)}$ are bounded on compact sets. Then $E \subseteq L^1_{loc}(G)$ with continuous inclusion.*

Remark 3.17. In the proof above we only used the left translation operator, so in fact we could only require that λ_y is continuous for all $y \in G$ and uniformly bounded on compact sets.

Remark 3.18. De Pagter & Ricker show that for compact groups G we have $L^\infty(G) \subseteq E \subseteq L^1(G)$ with continuous inclusions [5, Prop 4.8(i)]. The spaces $L_c^\infty(G)$ and $L_{loc}^1(G)$ fulfil the closest analogous role in the locally compact case.

Remark 3.19. For any $K \subseteq G$ compact we have $L^\infty(K) \subseteq E_K^\times$ (since $E_K \subseteq L^1(K)$ with continuous inclusion). In particular $E_K^\times$ separates the points of E_K, which agrees with [9, Theorem 112.1].

We will use the fact that $E_K^\times$ separates the points of E_K in Section 4.

In this thesis we wish to exploit the compatibility between the inclusion map $C_c(G) \to E$ and extended convolution operators that will be introduced in the next section we will from here on restrict ourselves to those B.f.s.'s that contain $C_c(G)$, i.e. from here on we will assume that $C_c(G) \cap E \neq \{0\}$.

4 Extending the convolution on $C_c(G)$ to E_s

In this section we will consider the convolution defined on $C_c(G) \times C_c(G)$ given by

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)d\mu(y).$$

We will extend this convolution to $C_c(G) \times E_s$ and $E_s \times C_c(G)$. Recall that we assume that $C_c(G) \subseteq E_s$, as mentioned at the end of the previous section. Denote the inclusion map with $j : C_c(G) \to E_s$. By 'extend this convolution' we mean that we will introduce maps $l : C_c(G) \times E_s \to E_s$ and $r : E_s \times C_c(G) \to E_s$ such that the following diagrams commute

$$\begin{array}{ccc} C_c(G) \times C_c(G) & \xrightarrow{(f,g) \mapsto f * g} & C_c(G) \\ {\scriptstyle Id \times j}\downarrow & & \downarrow{\scriptstyle j} \\ C_c(G) \times E_s & \xrightarrow{(f,g) \mapsto l_f(g)} & E_s \end{array} \qquad \begin{array}{ccc} C_c(G) \times C_c(G) & \xrightarrow{(f,g) \mapsto f * g} & C_c(G) \\ {\scriptstyle j \times Id}\downarrow & & \downarrow{\scriptstyle j} \\ E_s \times C_c(G & \xrightarrow{(f,g) \mapsto r_f(g)} & E_s \end{array}$$

To extend the convolution maps like this we will first prove that the convolution of two elements $f, g \in C_c(G)$ as given above is indeed a continuous function with compact support. We will then formally define the desired map l as a Bochner integral and show that it indeed satisfies the commutative relation given above, which can be rewritten as $l_f(j(g)) = j(f*g)$ for all $f, g \in C_c(G)$. After that we repeat this procedure for the right convolution operator.

Lemma 4.1. *For $f, g \in C_c(G)$ the function $f * g$ as defined pointwise above is continuous and has compact support.*

Proof. Let $f, g \in C_c(G)$ be arbitrary. It is clear that $f*g$ has compact support (its support is contained in $\text{supp}(f)\text{supp}(g)$). We will show that $f * g$ is continuous.
Fix $x, y \in G$. We then see that

$$\begin{aligned} |(f * g)(x) - (f * g)(y)| &= \left| \int_G f(z)g(z^{-1}x)d\mu(z) - \int_G f(z)g(z^{-1}y)d\mu(z) \right| \\ &= \left| \int_G f(z)(g(z^{-1}x) - g(z^{-1}y))d\mu(z) \right| \\ &\leq \int_G |f(z)||g(z^{-1}x) - g(z^{-1}y)|d\mu(z) \\ &\leq \|f\|_\infty \int_G |g(z^{-1}x) - g(z^{-1}y)|d\mu(z) \\ &= \|f\|_\infty \int_G \Delta(z)|g(z^{-1}x) - g(z^{-1}y)|d\mu(z^{-1}) \\ &\leq \|f\|_\infty M \left\|\rho_{x^{-1}}g - \rho_{y^{-1}}g\right\|_1, \end{aligned}$$

where $M \in \mathbb{R}_{>0}$ is a constant such that $|\Delta(z^{-1})| < M$ for all $z^{-1} \in \mathrm{supp}(g)x^{-1} \cup \mathrm{supp}(g)y^{-1}$. We remark that as $x \to y$ we can pick M independent of x (i.e. if we fix a neighbourhood of y with compact closure we can pick an M such that the above inequality is satisfied for all x in that neighbourhood) and since g is uniformly continuous it is right strongly continuous so $\|\rho_{x^{-1}}g - \rho_{y^{-1}}g\|_1$ vanishes as $x \to y$. We conclude that $f * g$ is continuous. □

4.1 Extending the left convolution operator

Let $f \in C_c(G)$ and $g \in E_s$ be arbitrary. Recall that the map $y \mapsto \lambda_y(g)$ is continuous as g is left strongly continuous. Therefore the map $F : G \to E_s$ given by $y \mapsto f(y)\lambda_y(g)$ is also continuous. Since f has compact support F also does, so its image is compact and therefore separable. Since F is continuous it is also weakly measurable, and we conclude that F is Bochner measurable. Now pick an $M > 0$ such that $\|\lambda_y\|_{B(E_s,E_s)} < M$ for all $y \in \mathrm{supp}(f)$. We then see that:

$$\begin{aligned}\int_G \|F(y)\|_E \, d\mu(y) &= \int_G |f(y)| \, \|\lambda_y(g)\|_E \, d\mu(y) \\ &\leq \int_G |f(y)| \, \|\lambda_y\|_{B(E_s,E_s)} \, \|g\|_E \, d\mu(y) \\ &\leq M \, \|f\|_1 \, \|g\|_E < \infty.\end{aligned}$$

So F is Bochner integrable. Write $l_f(g) = \int_G F d\mu$ for the Bochner integral of F, and note that $l_f(g)$ is linear in both f and g and is continuous in g (since the constant M in the estimate above depends on $\mathrm{supp}(f)$ and therefore on f it is not clear that this expression is also continuous in f).
We will now prove that $l_f(g)$ extends the convolution on $C_c(G)$. To show this we will first give a point-wise expression for the Bochner integral above, and show that on E_s these two expressions agree by using the earlier proven fact that the Köthe dual of E_K separates the points of E_K. We will then use this point-wise expression to show that l extends the left convolution on $C_c(G)$.

Theorem 4.2. *Let E be a translation invariant B.f.s. such that both the left and the right translation operators λ_y, ρ_y are continuous for all y with operator norm bounded on compact sets. Denote with $j : C_c(G) \to E_s$ the inclusion map and with $l : C_c(G) \to B(E_s, E_s)$ the map given by $f \mapsto (g \mapsto l_f(g) = \int_G f(y)\lambda_y(g)d\mu(y))$ for all $g \in E_s$. Then $l_f(g)$ is continuous for all $f \in C_c(G), g \in E_s$ and $l_f(j(g)) = j(f * g)$ for all $f, g \in C_c(G)$, i.e. l extends the convolution on $C_c(G)$.*

Proof. We first introduce a point-wise expression. For $f \in C_c(G), g \in E$ and $x \in G$ we define

$$(f * g)(x) := \int_G f(y)g(y^{-1}x)d\mu(y).$$

Since f has compact support and $g \in E \subseteq L^1_{loc}(G)$ this integral is well-defined. We claim that $f * g$ is a continuous map. The proof of this claim is similar to the proof of Lemma 4.1.

Write $K = \text{supp}(f)$. Let $y \in G$ be arbitrary, and let $L \subseteq G$ be a compact neighbourhood of y. We then see for all $x \in L$ that:

$$\begin{aligned}
|(f * g)(x) - (f * g)(y)| &= \left| \int_G f(z)g(z^{-1}x)d\mu(z) - \int_G f(z)g(z^{-1}y)d\mu(z) \right| \\
&= \left| \int_K f(xz)g(z^{-1})d\mu(xz) - \int_K f(yz)g(z^{-1})d\mu(yz) \right| \\
&= \left| \int_{x^{-1}K} f(xz)g(z^{-1})d\mu(z) - \int_{y^{-1}K} f(yz)g(z^{-1})d\mu(z) \right| \\
&= \left| \int_{x^{-1}K \cup y^{-1}K} (f(xz) - f(yz))g(z^{-1})d\mu(z) \right| \\
&\leq \int_{L^{-1}K} |(f(xz) - f(yz))||g(z^{-1})|d\mu(z) \\
&\leq \int_{L^{-1}K} \|\lambda_{x^{-1}}f - \lambda_{y^{-1}}f\|_\infty |g(z^{-1})|d\mu(z) \\
&\leq \|\lambda_{x^{-1}}f - \lambda_{y^{-1}}f\|_\infty \int_{L^{-1}K} \Delta(z^{-1})|g(z)|d\mu(z) \\
&\leq M \|g\|_{1,L^{-1}K} \|\lambda_{x^{-1}}f - \lambda_{y^{-1}}f\|_\infty .
\end{aligned}$$

Here M is a constant such that $|\Delta(z^{-1})| < M$ for all $z \in L^{-1}K$. We remark that $\|g\|_{1,L^{-1}K} < \infty$ since $g \in E \subseteq L^1_{loc}(G)$. But since f is continuous with compact support we have $\|\lambda_x f - \lambda_y f\|_\infty \to 0$ as $x \to y$, so the expression above vanishes as $L \downarrow \{y\}$. We conclude that $f * g$ is continuous.

We will now show that for all $f \in C_c(G), g \in E_s$ we have $f * g = l_f(g)$. Let positive $f \in C_c(G), g \in E_s$ be arbitrary, and let $K \subseteq G$ be compact. Let $0 \leq h \in E_K^\times$ be arbitrary. Since $f * g$ is continuous we have $(f * g)\chi_K \in L^\infty(K) \subseteq E_K$. From this we see that

$$\langle l_f(g)\chi_K, h\rangle = \left\langle \int_G f(y)\lambda_y(g)\chi_K d\mu(y), h \right\rangle.$$

Since bounded linear functionals commute with taking Bochner integrals we can move the pairing inside the integral to get

$$\langle l_f(g)\chi_K, h\rangle = \int_G \langle f(y)\lambda_y(g)\chi_K, h\rangle d\mu(y)$$

$$= \int_G \left(\int_K f(y)g(y^{-1}x)\chi_K(x)h(x)d\mu(x) \right) d\mu(y).$$

We remark that the indicator function of K equals the constant 1 in K, so we may disregard it in the expression above. Tonelli's theorem allows us to rewrite this as

$$\begin{aligned}
\langle l_f(g)\chi_K, h\rangle &= \int_K \left(\int_G f(y)g(y^{-1}x)d\mu(y) \right) h(x)d\mu(x) \\
&= \int_K (f * g)(x)h(x)d\mu(x) \\
&= \langle (f * g)\chi_K, h\rangle.
\end{aligned}$$

Since $E_K \subseteq L^1(K)$ the points of E_K are separated by $L^\infty(K) \subseteq E_K^\times$, so $l_f(g)\chi_K = (f*g)\chi_K$. So $f * g$ and $l_f(g)$ agree almost everywhere on each compact set, so $f * g = l_f(g)$ as measurable functions (by assumption on (G, μ)), so in particular as elements of E. But since both sides of this expression are bilinear in f, g we conclude that this equality holds for all $f, g \in C_c(G)$ (as opposed to only for positive functions). We conclude that our point-wise expression agrees with the Bochner integral. To show that l extends the convolution on $C_c(G)$ we now remark that the point-wise expression introduced above coincides with the point-wise expression for the convolution on $C_c(G)$, i.e. for $f, g \in C_c(G)$ we have $j(f * g) = f * j(g) = l_f(g)$, as was to be shown. □

Lastly we remark that since $C_c(G)$ is norm-dense in $L^1(G)$ we can with another assumption extend the left convolution to $L^1(G) \times E_s$:

Lemma 4.3. *Let E be a translation invariant B.f.s. such that both the left and the right translation operators λ_y, ρ_y are continuous for all $y \in G$, that the right translation maps are uniformly bounded on compact sets and that the left translation operators are uniformly bounded (i.e. the operator norm $\|\lambda_y\|_{B(E_s,E_s)}$ is bounded on G). Then there is a unique continuous extension of $l : C_c(G) \to B(E_s, E_s)$ to $L^1(G)$.*

Proof. In the construction of $l : C_c(G) \to B(E_s, E_s)$ we have seen that for $f \in C_c(G)$ we can choose $M \in \mathbb{R}$ such that $\|\lambda_y\|_{B(E_s,E_s)} < M$ for all $y \in \text{supp}(f)$, and for all $g \in E_s$ we then find the inequality

$$\|l_f(g)\| \leq M \|f\|_1 \|g\|_E .$$

We remark that since by assumption the left translation operators are uniformly bounded we can pick an M that bounds all the norms of the left translation operators from above, so in the inequality above we can pick M independent of f. Evaluating the expression above for fixed M and g we see that $l_f(g)$ is continuous in f. Since $C_c(G)$ lies dense in $L^1(G)$ it therefore has a unique extension to $L^1(G)$ (where we use that E_s is a Banach lattice so $B(E_s, E_s)$ is complete), as was to be shown. □

4.2 Extending the right convolution operator

We will now extend the right convolution operator from $C_c(G)$ on $C_c(G)$ to E_s as well, similarly to the extension of the left convolution operator in the previous section. For completeness the proofs are given in full below, but the approach is similar to the previous case. We do remark that for non-abelian groups the definition of the right convolution is slightly non-intuitive, but we will later in this section show that the definition we give is the correct one.

Let $f \in C_c(G)$ and $g \in E_s$ be arbitrary. As earlier we remark that since $g \in E_s$ the map from G to E_s given by $y \mapsto \rho_y(g)$ is continuous, so the map $F : G \to E_s$ given by $y \mapsto \Delta(y^{-1})f(y)\rho_y(g)$ also is. Since f has compact support, say K, the map F does as well, so F is Bochner measurable. Furthermore we can pick constants $M, L \in \mathbb{R}_{>0}$ such that $\|\rho_y\|_{B(E_s,E_s)} < M$, $\Delta(y^{-1}) < L$ for all $y \in K$, from which we find the estimate

$$\int_G \left\|\Delta(y^{-1})f(y)\rho_y(g)\right\|_E d\mu(y) \leq \int_G M|f(y^{-1})|L\,\|g\|_E\, d\mu(y) = ML\,\|f\|_1\,\|g\|_E < \infty.$$

So F is Bochner integrable. We denote its integral with $r_f(g) = \int_G F(y)d\mu(y)$, and remark that $r_f(g)$ is bilinear in f and g and continuous in g. We will now show that $r_f(g)$ extends the regular right convolution on $C_c(G)$.

Theorem 4.4. *Let E be a translation invariant B.f.s. such that both the left and the right translation operators λ_y, ρ_y are continuous for all y with operator norm bounded on compact sets. Denote with $j : C_c(G) \to E_s$ the inclusion map and with $r : C_c(G) \to B(E_s, E_s)$ the map given by $f \mapsto (g \mapsto r_f(g) = \int_G \Delta(y)f(y)\rho_y(g)d\mu(y))$. Then $r_f(g)$ is continuous for all $f \in C_c(G), g \in E_s$ and $r_f(j(g)) = j(g * f)$ for all $f, g \in C_c(G)$, i.e. r extends the right convolution on $C_c(G)$.*

Proof. The proof is analogous to the proof of Theorem 4.2. We first introduce a point-wise expression for the Bochner integral. For $f \in C_c(G), g \in E$ and $x \in G$ arbitrary define

$$(g * f)(x) := \int_G \Delta(y^{-1})f(y)g(xy^{-1})d\mu(y).$$

Since f has compact support, Δ is continuous so bounded on compact sets and $g \in E \subseteq L^1_{loc}(G)$ is locally integrable this integral exists. The proof that $g * f$ is continuous is completely analogous to the proof that $f * g$ is continuous in Theorem 4.2.
Now let positive $f \in C_c(G), g \in E_s$ be arbitrary. Pick $K \subseteq G$ compact, and let $0 \leq h \in E_K^\times$ be arbitrary. Again we remark that since $g * f$ is continuous we have $(g * f)\chi_K \in L^\infty(K) \subseteq E_K$. From this we see that:

$$\langle r_f(g)\chi_K, h\rangle = \left\langle \int_G \Delta(y^{-1})f(y)\rho_y(g)d\mu(y), h \right\rangle.$$

Since bounded linear functionals commute with taking Bochner integrals we can move the pairing inside the integral to get:

$$\begin{aligned}\langle r_f(g)\chi_K, h\rangle &= \int_G \langle \Delta(y^{-1})f(y)\rho_y(g)\chi_K, h\rangle d\mu(y)\\ &= \int_G \left(\int_K \Delta(y^{-1})f(y)g(xy^{-1})\chi_K(x)h(x)d\mu(x)\right) d\mu(y).\end{aligned}$$

Since the indicator function is constant on K we may omit it. Tonelli's theorem allows us to rewrite this as:

$$\begin{aligned}\langle r_f(g)\chi_K, h\rangle &= \int_K \left(\int_G \Delta(y^{-1})f(y)g(xy^{-1})d\mu(y)\right) h(x)d\mu(x)\\ &= \int_K (g * f)(x)h(x)d\mu(x)\\ &= \langle (g * f)\chi_K, h\rangle.\end{aligned}$$

Since $E_K \subseteq L^1(K)$ the points of E_K are separated by $L^\infty(K) \subseteq E_K^\times$. So $r_f(g)$ and $j(g * f)$ agree almost everywhere on all compact sets. As before we conclude that $g * f = r_f(g)$.
We will now in two steps show that r extends the right convolution on $C_c(G)$. First for $f, g \in C_c(G)$ we introduce the function $g \star f : G \to \mathbb{R}$ given by

$$g \star f(x) = \int_G \Delta(y^{-1})f(y)g(xy^{-1})d\mu(y).$$

From the point-wise expression above we immediately see that $j(g \star f) = j(g) * f = r_f(j(g))$, so to show that r extends the right convolution (i.e. $j(g * f) = r_f(j(g))$ for all $f, g \in C_c(G)$) it suffices to show that $g \star f = g * f$ (here denoting the regular convolution on $C_c(G)$), which we will prove below with two chain rules.
For each $x \in G$ we see that:

$$(g \star f)(x) = \int_G \Delta(y^{-1})f(y)g(xy^{-1})d\mu(y).$$

Changing coordinates $z = y^{-1}$ lets us rewrite the expression above as:

$$(g \star f)(x) = \int_G \Delta(z)f(z^{-1})g(xz)d\mu(z^{-1}).$$

The chain rule $d\mu(z^{-1}) = \Delta(z)^{-1}d\mu(z)$ now gives us that the above equals:

$$\begin{aligned}(g \star f)(x) &= \int_G \Delta(z^{-1})\Delta(z)f(z^{-1})g(xz)d\mu(z)\\ &= \int_G f(z^{-1})g(xz)d\mu(z).\end{aligned}$$

Changing coordinates $w = xz$ so $z = x^{-1}w$ now gives us:

$$(g \star f)(x) = \int_G f(w^{-1}x)g(w)d\mu(x^{-1}w).$$

Since μ is left translation invariant we known that $d\mu(x^{-1}w) = d\mu(w)$, so the above equals:

$$\begin{aligned}(g \star f)(x) &= \int_G g(w)f(w^{-1}x)d\mu(w) \\ &= (g * f)(x).\end{aligned}$$

We conclude that r extends the right convolution on $C_c(G)$. □

To conclude this section we remark that if we assume that the function $y \mapsto \Delta(y)^{-1} \|\rho_y\|_{B(E_s,E_s)}$ is uniformly bounded we can extend r to $L^1(G)$ using an argument analogous to that used in the previous section to extend l to $L^1(G)$. However we know that the function $y \mapsto \Delta(y)^{-1}$ is generally not bounded (in fact, its image is a subgroup of $\mathbb{R}^*_{>0}$ and therefore either the single point $\{1\}$ or unbounded) so it is not clear that this assumption is valid. Therefore we will not assume that we can extend the right action of $C_c(G)$ on E_s to $L^1(G)$.

5 Continuous approximations

In this section we will introduce an approximate unit $(e_K)_{K\in\mathcal{K}}$ for the extended convolution operators, by which we mean that l_{e_K} and r_{e_K} converge to the identity map on E_s in the strong operator topology as K increases over our index set. We will then use this convergence to show that $E_{s,0}$ is the closure of $C_c(G)$ in E.

Definition 5.1. Denote with $\mathcal{K}$ the set of all compact subsets of G that have a non-empty interior containing the unit element e. We equip $\mathcal{K}$ with the reverse inclusion ordering, for $K, M \in \mathcal{K}$ we have $K \succeq M \iff K \subseteq M$. For each such $K \subseteq \mathcal{K}$ let $e_K : G \to \mathbb{R}$ be a positive continuous function with $\mathrm{supp}(e_K) \subseteq K$ and $\int_G e_K(x)d\mu(x) = 1$. Such a collection of functions $(e_K)_{K\in\mathcal{K}}$ is called a *standard peak set.*

Proposition 5.2. *Let $(e_K)_{K\in\mathcal{K}}$ be a standard peak set. Then $(e_K)_{K\in\mathcal{K}}$ is an approximate unit for the map $l : C_c(G) \times E_s \to E_s$, i.e. for all $g \in E_s$ we have $\lim_K \|l_{e_K}(g) - g\|_E = 0$.*

Proof. Let $g \in E_s$ be arbitrary. We then see that

$$\begin{aligned}
\|l_{e_K}(g) - g\|_E &= \left\| \int_G e_K(y)\lambda_y(g)d\mu(y) - \int_G e_K(y)gd\mu(y) \right\|_E \\
&\leq \int_G |e_K(y)| \, \|\lambda_y g - g\|_E \, d\mu(y) \\
&\leq \int_G |e_K(y)| \left(\sup_{y\in K} \|\lambda_y(g) - g\|_E \right) d\mu(y) \\
&= \sup_{y\in K} \|\lambda_y(g) - g\|_E .
\end{aligned}$$

Here the first and last equality follow from the observations that $\int_G e_K(y)d\mu(y) = 1$ and $e_K \geq 0$. Since $g \in E_s$ we have $\|l_y(g) - g\|_E \to 0$ as $y \to e$, so this supremum vanishes as $K \downarrow \{e\}$, completing the proof. ☐

Proposition 5.3. *Let $(e_K)_{K\in\mathcal{K}}$ be a standard peak set. Then $(e_K)_{K\in\mathcal{K}}$ is an approximate unit for the map $r : E_s \times C_c(G) \to E_s$, i.e. for all $g \in E_s$ we have $\lim_K \|r_{e_K}(g) - g\|_E = 0$.*

Proof. Let $g \in E_s$ be arbitrary. We then see that

$$\begin{aligned}
\|r_{e_K} g - g\|_E &= \left\| \int_G \Delta(y)^{-1} e_K(y)\rho_y(g)d\mu(y) - \int_G e_K(y)gd\mu(y) \right\|_E \\
&\leq \left\| \int_G \Delta(y)^{-1} e_K(y)\rho_y(g)d\mu(y) - \int_G e_K(y)\rho_y(g)d\mu(y) \right\|_E \\
&+ \left\| \int_G e_K(y)\rho_y(g)d\mu(y) - \int_G e_K(y)gd\mu(y) \right\|_E \\
&= \left\| \int_G (\Delta(y)^{-1} - 1)e_K(y)\rho_y(g)d\mu(y) \right\|_E + \left\| \int_g e_K(y)(\rho_y(g) - g)d\mu(y) \right\|_E
\end{aligned}$$

$$\leq \left(\sup_{y\in K}\left|\Delta(y)^{-1}-1\right|\right)\left(\sup_{y\in K}\|\rho_y\|_{B(E,E)}\right)\|g\|_E + \sup_{y\in K}\|\rho_y(g)-g\|_E\,.$$

Where we used that $\|e_K\|_1 = 1$. Since $\sup_{y\in K}|\Delta(y)^{-1}-1| \to 0$ as $K \downarrow \{e\}$ (since Δ is continuous and $\Delta(e)=1$), $\sup_{y\in K}\|\rho_y\|_{B(E,E}$ is bounded as $K\downarrow\{e\}$ and $\|g\|_E$ is independent of K, we see that the first term in the sum vanishes as $K\downarrow\{e\}$. But since $g\in E_s$ the second term also vanishes in this limit, so we conclude that $(e_K)_{K\in\mathcal{K}}$ is an approximate unit for r. □

Next we will show that $E_{s,0}$ is the closure of $C_c(G)$ in E.

Theorem 5.4. *Let E be a translation invariant Banach function space with non-zero intersection with $C_c(G)$, and assume that for all $y\in G$ the left and right translation maps are continuous and $\|\lambda_y\|_{B(E,E)}, \|\rho_y\|_{B(E,E)}$ are bounded on compact sets. Then $\overline{C_c(G)}^E = E_{s,0}$ (the space of strongly continuous functions that vanish at infinity).*

Proof. Since $E_{s,0}$ is closed in E and we already know that $C_c(G)\subseteq E_{s,0}$ it suffices to show that $E_{s,0}\subseteq\overline{C_c(G)}^E$. We pick a standard peak set $(e_K)_{K\in\mathcal{K}}$ and will show that for any $f\in E_{s,0}$ and $K\in\mathcal{K}$ we can approximate $l_{e_K}(f)$ with functions from $C_c(G)$, and conclude that $l_{e_K}(f)\in\overline{C_c(G)}^E$. Since $l_{e_K}(f)\to f$ as $K\downarrow\{e\}$ we can then conclude that $f\in\overline{C_c(G)}^E$. Let $f\in E_{s,0}$ and $K\in\mathcal{K}$ be arbitrary. It follows from Theorem 4.2 that $l_{e_K}(f)$ is continuous, and since $E_{s,0}$ is translation invariant $l_{e_K}(f)$ also vanishes at infinity. Let $\epsilon>0$ be arbitrary and pick an $L\subseteq G$ compact such that $\left\|l_{e_K}(f)\chi_{G\setminus L}\right\|_E<\frac{\epsilon}{2}$, and pick any open set V such that $L\subseteq V$ and V has compact closure. Urysohn's lemma now gives us that there exists a function $\phi: G\to\mathbb{R}$ that is continuous, satisfies $0\leq\phi\leq 1$, is identically 1 on L and 0 outside V. Pick such a ϕ. Then $\phi l_{e_K}(f)\in C_c(G)$ since ϕ has compact support, but we also see that

$$\|\phi l_{e_K}(f)-l_{e_K}(f)\|_E = \left\|(\phi l_{e_K}(f)-l_{e_K}(f))\chi_{G\setminus L}\right\|_E \leq 2\left\|l_{e_K}(f)\chi_{G\setminus L}\right\|_E < \epsilon,$$

since $0\leq\phi\leq 1$ and ϕ is equal to 1 on L. Therefore $l_{e_K}(f)\in\overline{C_c(G)}^E$. Since $K\in\mathcal{K}$ was arbitrary and $l_{e_K}(f)\to f$ as $K\downarrow\{e\}$ we conclude that $f\in\overline{C_c(G)}^E$, as was to be shown. □

6 Lattice homomorphism theorem

In this section we will give in a slightly more abstract setting a set of sufficient conditions for a map to be a lattice homomorphism. We will later combine this result with the observations from the previous sections to show that the map $l : C_c(G) \to B(E_{s,0}, E_{s,0})$ and even its extension to $L^1(G)$ commute with taking absolute values, provided we restrict the image to a suitable subspace. We will furthermore prove that under some further assumptions on E these maps are lattice algebra representations of $C_c(G)$ and $L^1(G)$ respectively on $E_{s,0}$.

The theorem below, and the proof thereof, rely on two key ideas. The first idea is that in a Riesz algebra A we can consider the map sending an element $f \in A$ to the left or right multiplication maps $g \mapsto fg$, $g \mapsto gf$. If we can then find a larger Riesz space V such that A can be densely included in V it might be possible to extend this multiplication map to $A \times V$, and this new map inherits some of the properties of the multiplication on A. An important observation is that by using an approximate unit below we do not require that the inclusion of A in V is continuous, but merely that it has a specific compatibility with the approximate unit. In the particular case where $A = C_c(G)$ this compatibility comes down to a 'local continuity' of the inclusion. We will exploit the fact that through this extended multiplication map we can interpret any element $a \in A$ either as an element in V as $j(a)$, or as an operator acting on A as $\pi(a)$. By switching between these two we will prove interesting properties of π.

Note that we do not assume that the inclusion map $j : A \to V$ is continuous, because in the cases we are interested in this is not in general true. A simple example to illustrate the importance of not assuming that the inclusion map is continuous would be the inclusion of $C_c(G)$ (equipped with the $\|\cdot\|_1$-norm) into $L^p(G)$ for $1 < p \leq \infty$, which is not in general continuous.

The second key ingredient is motivated by [3]. Instead of attempting to prove that the extended multiplication map is a lattice homomorphism, we remark that the left multiplication commutes with the right multiplication on A. If we can find a similar extension of the right multiplication then clearly the image of the extended left multiplication commutes with the image of the extended right multiplication. We will only show that inside the commutant of the extended right multiplication our map commutes with taking absolute values. To complete the proof of the desired statement that our extension is a lattice homomorphism it is therefore sufficient to show that this commutant is a sublattice of our space of interest, which will need to be done for each case individually.

Theorem 6.1. *Let A be a Riesz algebra equipped with a topology such that $|\cdot| : A \to A$ is continuous. Let V be a Riesz space equipped with a vector space topology such that $|\cdot|$ is continuous and the positive cone V^+ of V is closed. Let $j : A \to V$ be a lattice homomorphism with dense image. We denote with $L_{rc}(V)$ the set of regular continuous* *maps $T : V \to V$. Suppose we have two positive linear maps $\pi, \pi' : A \to L_{rc}(V)$ such that for all $a, b \in A$ we have*

$$\begin{aligned} \pi(a)\pi'(b) &= \pi'(b)\pi(a); && (\pi, \pi' \text{ commute}) \\ j(ab) &= \pi(a)j(b); \\ j(ab) &= \pi'(b)j(a); \\ \pi(a), \pi'(a) & \textit{ are continuous.} \end{aligned}$$

Lastly assume A contains a positive net $(e_i)_{i\in\Lambda}$ such that for all $a \in A$ we have

$$\lim_{i\in\Lambda} \pi(e_i)j(a) = j(a) = \lim_{i\in\Lambda} \pi'(e_i)j(a).$$

Write $\pi'(A)' := \{T \in L_{rc}(V) : T\pi'(a) = \pi'(a)T \ \forall a \in A\}$ for the commutant of the image of π'. We note that $\pi(A) \subseteq \pi'(A)'$. Then, for all $a \in A$, $|\pi(a)|$ exists in $\pi'(A)'$ and, in fact, $|\pi(a)| = \pi(|a|)$. If $\pi'(A)'$ is a lattice then π is a lattice homomorphism.

Proof. Let $a \in A$ be arbitrary. We will show that $\pi(|a|) = |\pi(a)|$ in $\pi'(A)'$, i.e. $\pi(|a|) \geq \pm\pi(a)$ and for all $T \in \pi'(A)'$ with $T \geq \pm\pi(a)$ we have $T \geq \pi(|a|)$.
Since $\pi \geq 0$ we have $\pi(|a|) \geq \pi(\pm a) = \pm\pi(a)$. To show that the second requirement holds we pick $b \in A^+$ and T as above arbitrary, and write

$$\begin{aligned} T(j(b)) &= T\left(\lim_{i\in\Lambda} j(e_i b)\right) && \text{(By assumption)} \\ &= \lim_{i\in\Lambda} T(j(e_i b)) && \text{(Since } T \text{ is continuous)} \\ &= \lim_{i\in\Lambda} (T \circ \pi'(b))j(e_i) && \text{(Per the relation between } \pi' \text{ and } j) \\ &= \lim_{i\in\Lambda} (\pi'(b) \circ T)j(e_i). && \text{(Since } T \in \pi'(A)') \end{aligned}$$

Now we remark that $e_i \geq 0$ for all $i \in \Lambda$ (by assumption), so since j is a lattice homomorphism we also have $j(e_i) \geq 0$. Since $T \geq \pm\pi(a)$ it follows that $Tj(e_i) \geq \pm\pi(a)j(e_i) = \pm j(ae_i)$. So $Tj(e_i) \geq |j(ae_i)| = j(|ae_i|)$, where again we used that j is a lattice homomorphism. Now since we picked $b \in A^+$ and π' is positive we have $\pi'(b) \geq 0$. Combining these inequalities gives us that $(\pi'(b) \circ T)j(e_i) \geq \pi'(b)j(|ae_i|)$ for all i. Since the positive cone of V is closed it therefore follows that $T(j(b)) \geq \lim_{i\in\Lambda} \pi'(b)j(|ae_i|)$, where we note that this limit exists precisely because it equals $\pi(|a|)j(b)$ as shown by

$$\begin{aligned} T(j(b)) &\geq \lim_{i\in\Lambda} \pi'(b)j(|ae_i|) \\ &= \pi'(b) \lim_{i\in\Lambda} j(|ae_i|) && \text{(Per continuity of } \pi'(b)) \\ &= \pi'(b) \lim_{i\in\Lambda} |j(ae_i)| && \text{(Since } j \text{ is a lattice homomorphism)} \\ &= \pi'(b) \left|\lim_{i\in\Lambda} j(ae_i)\right| && \text{(Since the absolute value on } V \text{ is continuous)} \end{aligned}$$

$$
\begin{aligned}
&= \pi'(b)|j(a)| && \text{(By assumption)} \\
&= \pi'(b)j(|a|) && \text{(Since } j \text{ is a lattice homomorphism)} \\
&= j(|a|b) \\
&= \pi(|a|)j(b).
\end{aligned}
$$

Now since j is a lattice homomorphism with dense image in V the image of A^+ is dense in V^+ (if $j(a_\nu) \to v$ then $j(a_\nu^+) = j(a_\nu)^+ \to v^+$, so $V^+ = \overline{j(A^+)}$). Since both T and $\pi(|a|)$ are continuous maps and we have $T \geq \pi(|a|)$ on a dense subset of V^+ it holds on the whole positive cone, i.e. $T \geq \pi(|a|)$ as operators. We conclude that $\pi(|a|)$ is the absolute value of $\pi(a)$ in $\pi'(A)'$. If $\pi'(A)'$ is a lattice then we furthermore see that $\pi(|a|) = |\pi(a)|$, so π is a lattice homomorphism onto $\pi'(A)'$. □

Remark 6.2. If V is a Banach lattice it is by definition equipped with a vector space topology such that $|\cdot|$ is continuous, has a closed positive cone, and any regular operator on V is continuous by Theorem 2.19. Therefore $L_{rc}(V) = L_r(V)$, i.e. we only need to check if π is positive.

Remark 6.3. As was noted in the paragraph above the theorem the maps π, π' extend the left and right multiplication on A. Since we demand that $j(A)$ is dense in V and that $\pi(a), \pi'(a)$ are continuous operators for all $a \in A$ the compatibility relations $\pi'(b)j(a) = j(ab) = \pi(a)j(b)$ mean that if j is given then π, π' are, if they exist, necessarily unique.

Remark 6.4. The compatibility requirement $\pi(a)j(b) = j(ab)$ for all $a, b \in A$ implies that $(\pi(a)\pi(b))j(c) = \pi(a)(\pi(b)j(c)) = \pi(a)j(bc) = j(a(bc)) = j((ab)c) = \pi(ab)j(c)$ for all $a, b, c \in A$, and since $\pi(a), \pi(b), \pi(ab)$ are continuous and the image of j is dense in V this means that $\pi(a)\pi(b) = \pi(ab)$ for all $a, b \in A$. Similarly $\pi'(a)\pi'(b) = \pi'(ba)$ for all $a, b \in A$.

7 Applying the homomorphism theorem to $C_c(G)$

In this section we will combine Theorem 6.1 with the observations made in the sections before. In particular we will show that if $L_r(E_{s,0})$ is a Riesz space the map $l : C_c(G) \to L_r(E_{s,0})$ is a lattice homomorphism, and we will show that under the stronger assumptions that $E_{s,0}$ is Dedekind complete and the left translation maps are uniformly bounded that the extension $l : L^1(G) \to L_r(E_{s,0})$ is a lattice homomorphism. Furthermore we will show that $E_{s,0}$ is Dedekind complete if E has order continuous norm.
We will also prove that l is a representation and conclude that under the stronger assumption above the map $l : L^1(G) \to L_r(E_{s,0})$ is a lattice algebra representation. Lastly we will give some examples of translation invariant Banach function spaces E and explicitly state the results we have obtained for those particular cases.

7.1 Proving that l is a lattice homomorphism

To apply Theorem 6.1 to the action of $C_c(G)$ on E_s we will pick $A = C_c(G)$ equipped with the $\|\cdot\|_1$-norm and set the convolution as multiplication. We choose $V = E_{s,0}$, which is a Banach lattice. We remark that, equipping $C_c(G)$ with the 1-norm, the absolute value on $C_c(G)$ is continuous. Setting $j : C_c(G) \to E_{s,0}$ the inclusion map and $\pi = l, \pi' = r$ we remark that j has dense image in $E_{s,0}$ (Theorem 5.4), l, r are positive and the desired compatibilities between j and l, r hold (Theorem 4.2 and Theorem 4.4). We did not show that for $f, g \in C_c(G)$ we have $l_f r_g = r_g l_f$, but since firstly for any $h \in C_c(G)$ we have $l_f r_g j(h) = j(fhg) = r_g l_f j(h)$, and secondly the operators l_f, r_g are continuous and the image of j lies dense in $E_{s,0}$ it follows immediately that these operators commute. Furthermore we have shown in Propositions 5.2 and 5.3 that a standard peak set satisfies the compatibility formula $\lim_{i\in\Lambda} j(e_i a) = j(a) = \lim_{i\in\Lambda} j(a e_i)$, so all conditions of Theorem 6.1 are met.

We will now show that if $L_r(E_{s,0})$ is a Riesz space then the commutator of the image of the right action, $r(C_c(G))' \subseteq L_r(E_{s,0})$ is a sublattice. To show this we will first show that the commutant of the image of $C_c(G)$ under r coincides with the commutant of all right translation operators ρ_y on $E_{s,0}$, and then conclude that this commutant is a sublattice of $L_r(E_{s,0})$.

Theorem 7.1. *Let E be a translation invariant Banach function space such that the left and right translation operators λ_y, ρ_y are bounded operators for each $y \in G$, and assume that the operator norms $\|\lambda_y\|_{B(E,E)}, \|\rho_y\|_{B(E,E)}$ are bounded on compact sets $K \subseteq G$. Furthermore assume that $C_c(G) \cap E \neq \{0\}$. Then the commutant of the image of $C_c(G)$ under r in $L_r(E_{s,0})$ is the commutant of all right translation operators, i.e. it equals the set $\{T \in L_r(E_{s,0}) : T\rho_y = \rho_y T \ \forall\ y \in G\}$.*

Proof. We will prove both inclusions.
Let T be a regular operator on $E_{s,0}$ that commutes with r_f for all $f \in C_c(G)$, so for all

$g \in E_{s,0}$ we have $(Tr_f - r_f T)g = 0$. We pick $z \in G$ arbitrary and consider this expression for $f = \rho_z(e_K)$ with $(e_K)_{e\in K\subseteq G}$ our approximate unit from before. Using the chain rule we find with an explicit computation that

$$r_{\rho_z(e_K)}g = \int_G \Delta(y)^{-1}\rho_z(e_K)(y)\rho_y(g)d\mu(y).$$

Intruducing $w = z^{-1}y$, so $y = zw$ allows us to rewrite this as:

$$\begin{aligned} r_{\rho_z(e_K)}g &= \int_G \Delta(zw)^{-1}\rho_z(e_K)(zw)\rho_{zw}(g)d\mu(zw) \\ &= \int_G \Delta(w)^{-1}\Delta(z)^{-1}e_K(zwz^{-1})\rho_w(\rho_z(g))d\mu(w) \\ &= r_{\tilde{e}_K}\rho_z(g). \end{aligned}$$

Here we introduced $\tilde{e}_K : G \to \mathbb{R}$ by $\tilde{e}_K(x) = \Delta(z)^{-1}e_K(zxz^{-1})$ (for our fixed z above). We remark that $\int_G \tilde{e}_K(x)d\mu(x) = \int_G \Delta(z)^{-1}e_K(zxz^{-1})d\mu(x) = \int_G e_K(s)d\mu(s) = 1$ (where we introduced $s = zxz^{-1}$ so $x = z^{-1}sz$, and used the translation properties of μ) and $\mathrm{supp}(\tilde{e}_K) = z^{-1}\mathrm{supp}(e_K)z$, so it follows that $(\tilde{e}_K)_{e\in K\subseteq G}$ is a standard peak set and therefore an approximate unit for r. Substituting this and our choice of functions in our expression above gives

$$0 = Tr_{\rho_z(e_K)}g - r_{\rho_z(e_K)}Tg = (Tr_{\tilde{e}_K}\rho_z(g) - r_{\tilde{e}_K}\rho_z(Tg)).$$

But since $(\tilde{e}_K)_{e\in K\subseteq G}$ is an approximate unit for the convolution we know that $r_{\tilde{e}_K}g \to g$ as $K \downarrow \{e\}$. Since T and ρ_z are continuous maps we therefore find that the above converges to

$$0 = (Tr_{\tilde{e}_K}\rho_z(g) - r_{\tilde{e}_K}\rho_z(Tg)) \to T\rho_z(g) - \rho_z(Tg) = (T\rho_z - \rho_z T)g.$$

So T commutes with ρ_z. Since $z \in G$ was arbitrary we conclude that T commutes with all right translations.

Conversely assume that $T \in L_r(E_s)$ commutes with all right translations, and let $f \in C_c(G)$ be arbitrary. For $g \in E_{s,0}$ we define $F_g : G \to E_{s,0}$ given by $F_g(y) = \Delta(y)^{-1}f(y)\rho_y(g)$, and we note that since T commutes with all right translations we have $TF_g(y) = F_{Tg}(y)$ for all y and g. So $0 = TF_g(Y) - F_{Tg}(y)$ for all y, so in particular the Bochner integral over this expression equals 0, i.e.

$$0 = \int_G (TF_g(y) - F_{Tg}(y))d\mu(y) = T(r_f(g)) - r_f(Tg).$$

So T commutes with all right convolutions with elements of $C_c(G)$, completing the proof. □

Corollary 7.2. *If $L_r(E_{s,0})$ is a Riesz space then the commutant $r(C_c(G))'$ is a sublattice of $L_r(E_s)$.*

Proof. We have shown above that $\rho(C_c(G))' = \{T \in L_r(E_{s,0}) : T\rho_y = \rho_y T \ \forall \ y \in G\}$. But composition with a right translation $\circ\rho_y$ (for fixed $y \in G$) is an endomorphism of $L_r(E_{s,0})$ with inverse $\circ\rho_{y^{-1}}$, and therefore a lattice automorphism of $L_r(E_{s,0})$. Therefore the map from $L_r(E_{s,0})$ to itself given by $T \mapsto \rho_{y^{-1}} \circ T \circ \rho_y$ is also a lattice automorphism. We remark that the set of regular operators commuting with ρ_y are precisely the fixpoints of this automorphism, but the fixpoints of a lattice automorphism of a Riesz space form a sublattice. The commutant of interest is therefore the intersection over all $y \in G$ of these sublattices, hence also a sublattice. □

Next we remark that the extended left convolution is a representation:

Lemma 7.3. *The map $l : C_c(G) \to L_r(E_{s,0})$ and, if the left translation operators are uniformly bounded, its extension to $l : L^1(G) \to L_r(E_{s,0})$ is a representation (of $\mathbb{R}$-algebras).*

Proof. We will make use of the inclusion map $j : C_c(G) \to E_{s,0}$. Note that for $f \in C_c(G)$ and $g \in E_{s,0}$ the expression $l_f(g)$ is linear in both f and g. Therefore all that needs to be shown is that $l_f l_h = l_{f*h}$ for $f, h \in C_c(G)$.
Now let $f, h, w \in C_c(G)$ be arbitrary. From the associativity of the commutation operator on $C_c(G)$ we see that we have

$$l_f l_h(j(w)) = l_f(j(h * w)) = j(f * (h * w)) = j((f * h) * w) = l_{f*h}(j(w)).$$

Since $j(C_c(G))$ is dense in $E_{s,0}$ it follows that $l_f l_h = l_{f*h}$. We conclude that l is a representation.
If the left translation operators $l_y : E_s \to E_s$ are uniformly bounded so $l : L^1(G) \to L_r(E_{s,0})$ is well-defined we note that $l : L^1(G) \to L_r(E_{s,0})$ is continuous. Therefore the map from $L^1(G) \times L^1(G)$ to $L_r(E_{s,0})$ given by $(f, g) \mapsto l_{f*g} - l_f \circ l_g$ is also continuous, but it is also identically zero on the dense subspace $C_c(G) \times C_c(G)$ and is therefore the zero map, proving that if l can be extended to $L^1(G)$ continuously this extension is also an algebra representation. □

Remark 7.4. It follows from an explicit computation that the map $l : C_c(G) \to E_s$ (including the elements that do not vanish at infinity) is a representation, but since we cannot use the result from Theorem 6.1 in this case this is of less interest.

Combining these results now gives the following statement:

Theorem 7.5. *Let E be a translation invariant Banach function space such that the left and right translation operators λ_y, ρ_y are bounded operators for each $y \in G$, and assume that the operator norms $\|\lambda_y\|_{B(E,E)}, \|\rho_y\|_{B(E,E)}$ are bounded on compact sets $K \subseteq G$. Furthermore assume that $C_c(G) \cap E \neq \{0\}$; this implies that $C_c(G) \subseteq E$. Then there exists an extension of the left convolution on $C_c(G)$ to the space of all strongly continuous elements in E that vanish at infinity, $l : C_c(G) \to L_r(E_{s,0})$. If $L_r(E_{s,0})$ is a Riesz space, for example if $E_{s,0}$ is Dedekind complete (in particular if E has o.c. norm), this extension is a lattice algebra representation. If $E_{s,0}$ is Dedekind complete and $\{\|\lambda_y\|_{B(E_{s,0},E_{s,0})} : y \in G\}$ is bounded the map l extends to a lattice algebra representation from $L^1(G)$ to $L_r(E_{s,0})$.*

Proof. The first claim, that $l : C_c(G) \to L_r(E_{s,0})$ is a lattice homomorphism is a direct consequence of Theorem 6.1 applied to $C_c(G)$ and $E_{s,0}$, along with the observation made in Corollary 7.2. For the second claim we remark that if $E_{s,0}$ is Dedekind complete its set of regular operators $L_r(E_{s,0})$ forms a Riesz space. Assuming the left translations are uniformly bounded we see that the extended map $l : L^1(G) \to L_r(E_{s,0})$ is positive (by construction) and linear. Note that $E_{s,0}$ is a Banach lattice, so $L_r(E_{s,0})$ equipped with the regular norm is a Banach lattice (so in particular it is a normed Riesz space). Since $L^1(G)$ is a Banach lattice and the map $l : L^1(G) \to L_r(E)$ is positive it is continuous by Theorem 2.19. But the map from $L^1(G)$ to $L_r(E_{s,0})$ given by $f \mapsto l_{|f|} - |l_f|$ is well-defined since $L_r(E_{s,0})$ is a Riesz space, continuous since both absolute values and l are continuous maps, and it vanishes on the dense subset $C_c(G)$ of $L^1(G)$. Therefore it equals the 0 map, i.e. the extension of l to $L^1(G)$ is also a lattice homomorphism.
Lastly we note that if E has o.c. norm then $E_{s,0}$ also does by Lemma 2.25, and it follows from Proposition 2.23 that $E_{s,0}$ is Dedekind complete. □

7.2 Examples

In this section we will give some examples of translation invariant Banach function spaces that satisfy all of the assumptions in Theorem 7.5.

Example 7.6. As our first example we consider the spaces $L^p(G)$ where $1 \leq p < \infty$. For $f \in L^p(G)$ and $y \in G$ we compute:

$$\begin{aligned} \|\lambda_y f\|_p &= \left(\int_G |(\lambda_y f)(x)|^p d\mu(x)\right)^{\frac{1}{p}} \\ &= \left(\int_G |f(y^{-1}x)|^p d\mu(x)\right)^{\frac{1}{p}}. \end{aligned}$$

Changing coordinates to $z = y^{-1}x$, so $x = yz$ this integral reduces to:

$$\begin{aligned} \|\lambda_y f\|_p &= \left(\int_G |f(z)|^p d\mu(yz)\right)^{\frac{1}{p}} \\ &= \left(\int_G |f(z)|^p d\mu(z)\right)^{\frac{1}{p}} \\ &= \|f\|_p. \end{aligned}$$

Here we used that μ is left translation invariant. Therefore the norm of each left translation operator λ_y is equal to 1, so each of these operators is continuous and their norms are globally bounded. For the right translation we find that:

$$\|\rho_y f\|_p = \left(\int_G |(\rho_y f)(x)|^p d\mu(x)\right)^{\frac{1}{p}}$$

$$= \left(\int_G |f(xy^{-1})|^p d\mu(x) \right)^{\frac{1}{p}}.$$

Changing coordinates to $z = xy^{-1}$, so $x = zy$ this integral reduces to:

$$\begin{aligned} \|\rho_y f\|_p &= \left(\int_G |f(z)|^p d\mu(zy) \right)^{\frac{1}{p}} \\ &= \left(\int_G \Delta(y)^{-1} |f(z)|^p d\mu(z) \right)^{\frac{1}{p}} \\ &= \Delta(y)^{-\frac{1}{p}} \|f\|_p. \end{aligned}$$

So the right translation operator ρ_y has norm $\Delta(y)^{-\frac{1}{p}}$, so these operators are continuous and their norm is bounded as y runs over compact sets. As was remarked near the end of Section 4 we see that the map $y \mapsto \Delta(y) \|\rho_y\|_{B(L^p(G),L^p(G))} = \Delta(y)^{1-\frac{1}{p}}$ is generally not globally bounded (for $p > 1$), so unless $p = 1$ we cannot use the methods from the previous sections to extend the right convolution operator to a lattice representation.
Next we note that $C_c(G) \cap L^p(G) \neq \{0\}$, and in fact we know that $C_c(G)$ is dense in $L^p(G)$ for $1 \leq p < \infty$ [6, Prop. 7.9], i.e. $L^p(G) = L^p(G)_{s,0}$. Since $L^p(G)$ is also Dedekind complete it follows from Theorem 7.5 that $l : L^1(G) \to L_r(L^p(G))$ is a lattice algebra representation, i.e. a representation of a Banach space that is also a lattice homomorphism. This particular example is the result proven by Arendt [3] with a different method, as was mentioned in the introduction of this thesis.

Example 7.7. For the next example we consider the case $p = \infty$. We see that on this space both the left and the right translation operators λ_y, ρ_y have norm 1 for all $y \in G$. We remark that the closure of $C_c(G)$ in $L^\infty(G)$ is precisely the set of continuous functions vanishing at infinity $C_0(G) = \{f \in L^\infty(G) : f \text{ continuous and } \forall\ \epsilon > 0\ \exists\ K \subseteq G \text{ compact s.t. } \left\|f|_{G\setminus K}\right\|_\infty < \epsilon\}$. Since this space is not in general Dedekind complete it does not follow that the left convolution operator $\ell : L^1(G) \to L_r(C_0(G))$ is a lattice algebra representation. However $C_0(G)$ does satisfy all the requirements of Theorem 6.1, so it still follows from this theorem along with Theorem 7.1 that if $f \in C_c(G)$ is arbitrary and $T : C_0(G) \to C_0(G)$ is a regular operator that commutes with all right translation operators and furthermore satisfies $T \geq \pm l_f$ then $T \geq l_{|f|}$.

Example 7.8. In the third example we consider rearrangement invariant B.f.s.'s on unimodular groups, i.e. groups G with constant modular function $\Delta(x) = 1$ for all $x \in G$. For a measurable function $f \in L^0(\mu)$ we define its distribution function $d_{|f|} : [0, \infty) \to [0, \infty]$ given by:

$$d_{|f|}(s) = \mu(\{x \in G : |f(x)| > s\}).$$

A Banach function space E is called *rearrangement invariant* if for every $f \in E, g \in L^0(\mu)$ we have $d_{|g|} = d_{|f|} \implies g \in E$. We note that for $y \in G$ and $f \in E$ we have $d_{|\lambda_y f|} = d_{|f|}$ and $d_{|\rho_y f|} = \Delta(y) d_{|f|} = d_{|f|}$, so $\lambda_y f, \rho_y f \in E$ for $f \in E$, i.e. E is translation invariant.

Example 7.9. For the fourth example we consider a weight function on G.
Let G be a locally compact group. A function $w : G \to \mathbb{R}$ is called a *weight function* if it has the following properties:

(1) $w(x) \geq 1$ for all $x \in G$;

(2) $w(xy) \leq w(x)w(y)$ for all $x, y \in G$;

(3) w is measurable and bounded on compact sets.

For a given weight function w the μ-measurable functions $f \in L^0(\mu)$ with the property that $fw \in L^1(G)$ form a Banach algebra under the norm $\|\cdot\|_{1,w}$ defined by:

$$\|f\|_{1,w} = \int_G |f(x)|w(x)d\mu(x).$$

This Banach algebra is called a *Beurling algebra* on G and denoted with $L^1(G, w)$.

It is a known result [7, Lemma 1.3.5(ii)] that $C_c(G)$ is dense in $L^1(G, w)$, i.e. $L^1(G, w) = L^1(G, w)_{s,0}$. Furthermore for every $x \in G$ and $f \in L^1(G, w)$ we have $\lambda_x(f) \in L^1(G, w)$ and $\|\lambda_x f\|_{1,w} \leq w(x) \|f\|_{1,w}$ [7, Lemma 1.3.6(i)]. By [8, Theorem 3.7.5] we may without loss of generality assume that w is continuous, and under this assumption we see that $\|\lambda_x e_K\|_{1,w} \to w(x) \|e_K\|_{1,w}$ as $K \downarrow \{e\}$ (with $(e_K)_{K \in \mathcal{K}}$ our approximate unit from before). From this we find that $\|\lambda_x\|_{B(L^1(G,w),L^1(G,w))} = w(x)$, which is not in general bounded on G. Next we remark that $L^1(G, w)$ is Dedekind complete, so $L_r(L^1(G, W))$ is a Riesz space. We conclude that since the left translation operators are not uniformly bounded we cannot extend the left convolution from $C_c(G)$ on $L^1(G, w)$ to $L^1(G)$ using our approach, but Theorem 6.1 still gives us that the extended left convolution operator $l : C_c(G) \to L_r(L^1(G, w))$ is a lattice homomorphism.

8 Discussion for order continuous norm

In this section we discuss an alternative approach to prove Theorem 7.5 in special cases by making use of the results on integral operators shown in [1, Section 5.1]. Important is to note which assumptions we have to make on our translation invariant B.f.s. E in order for these results to be applicable.

Definition 8.1. Let $(\mathcal{S}, \Sigma_1, \mu)$ and $(\mathcal{T}, \Sigma_2, \nu)$ be two σ-finite measure spaces. Let X and Y be linear subspaces of $L^0(\mu)$ and $L^0(\nu)$ respectively. An operator $T : X \to Y$ is said to be an *integral operator* if there exists a $\mu \times \nu$-measurable function $T(\cdot, \cdot) : \mathcal{S} \times \mathcal{T} \to \mathbb{R}$ such that for each $f \in X$ the function $x \mapsto T(x, y)f(x)$ belongs to $L^1(\mu)$ for ν-almost all $y \in \mathcal{T}$, and the equality

$$(Tf)(y) = \int_{\mathcal{S}} T(x, y)f(x)d\mu(x)$$

holds for ν-almost all $y \in \mathcal{T}$.

We wish to show that for fixed $f \in C_c(G)$ (or, if well-defined, $f \in L^1(G)$) the map $l_f : E_s \to E_s$ is an integral operator (with $\mathcal{S} = G = \mathcal{T}$, and $X = E_s = Y$ in the definition above). We remark that in our case the notion of an integral operator is only well-defined if G is σ-compact, which we will assume for the rest of this section.

Proposition 8.2. *Let G be σ-finite and let $f \in C_c(G)$ be arbitrary. Then $l_f : E_s \to E_s$ is an integral operator.*

Proof. Let $g \in E_s$ and $y \in G$ be arbitrary. We first use the chain rule to rewrite:

$$\begin{aligned} l_f(g)(y) &= \int_G f(x)g(x^{-1}y)d\mu(x) \\ &= \int_G f(yx)g(x^{-1})d\mu(yx) \\ &= \int_G f(yx)g(x^{-1})d\mu(x). \end{aligned}$$

Introducing $z = x^{-1}$ allows is to rewrite this as:

$$\begin{aligned} l_f(g)(y) &= \int_G f(yz^{-1})g(z)d\mu(z^{-1}) \\ &= \int_G f(yz^{-1})g(z)\Delta(z)^{-1}d\mu(z). \end{aligned}$$

We therefore introduce the map $T : G \times G \to \mathbb{R}$ given by $T(x, y) = \Delta(x)^{-1}f(yx^{-1})$, and remark that T is $\mu \times \mu$-measurable. Furthermore we note that for fixed $g \in E_s$ and $y \in G$ the map $x \mapsto T(x, y)g(x)$ is μ-integrable since f has compact support, Δ is bounded on compact sets and $g \in E_s \subseteq L^1_{loc}(G)$ is locally integrable. Since we have shown immediately above that $l_f(g)(y) = \int_G T(x,y)g(x)d\mu(x)$ for (almost) all $y \in G$ we conclude that l_f is an integral operator. The proof for $f \in L^1(G)$, if this extension of the left convolution operator is well-defined, is fully analogous. □

We now wish to apply the Bukhvalov-Luxemburg-Schep-Zaanen theorem to our integral operators, from which it will follow that our left convolution map is a lattice homomorphism:

Theorem 8.3 (Bukhvalov-Luxemburg-Schep-Zaanen). *Let E, F be two Banach function spaces that are order dense in $L^0(\mu)$. Then the vector space $\mathcal{L}_\kappa(E,F)$ of all regular integral operators from E to F is a band in $\mathcal{L}_r(E,F)$. Moreover for each $S, T \in \mathcal{L}_\kappa(E,F)$ and for each $g \in E$ we have*

$$\big[(S \vee T)g\big](y) = \int_{\mathcal{S}} \max\{S(x,y), T(x,y)\}g(x)d\mu(x) \qquad \textit{and}$$

$$\big[(S \wedge T)g\big](y) = \int_{\mathcal{S}} \min\{S(x,y), T(x,y)\}g(x)d\mu(x)$$

for μ-almost all $y \in \mathcal{T}$.

Proof. See [1, Theorem 5.14]. □

Since in our case the measurable functions in the integrand are given by an expression of the form $\Delta(x)^{-1}f(yx^{-1})$ we see that taking the point-wise maximum (respectively minimum) coincides with taking the point-wise maximum (respectively minimum) of the functions of $C_c(G)$ under consideration. Applying the above theorem to l_f and l_{-f} for $f \in C_c(G)$ will immediately show that l is a lattice homomorphism if $E_{s,0}$ is an order dense ideal of $L^0(\mu)$.

We remark that if E has o.c. norm all of its elements vanish at infinity:

Lemma 8.4. *If E is a B.f.s. with order continuous norm then all elements of E vanish at infinity.*

Proof. Let $f \geq 0 \in E$ be given, and let $K \subseteq G$ be compact. We remark that $f\chi_K \in E$. Now consider the net $(f - f\chi_K)_{K \subseteq G \text{ compact}}$, ordered under inclusion. This positive net is decreasing with infimum 0 (since $(f\chi_K)_{K \subseteq G \text{ compact}}$ is increasing with supremum f), and since E has order continuous norm our first net therefore converges in norm to 0, i.e. $f\chi_K \to f$, i.e. f vanishes at infinity. Since any element of E can be written as the difference of two positive functions we conclude that every element of E vanishes at infinity. □

To apply Theorem 8.3 we need that $E_{s,0} = E_s$ is an order dense ideal of $L^0(\mu)$. But we have shown in Example 3.6 that in general E_s is not an ideal in $L^0(\mu)$. In the case that G is compact and abelian de Pagter and Ricker [5, Theorem 4.17] show that $E = E_s$ if and only if E has order continuous norm. We suspect that even for G locally compact this equivalence may hold, in which case if E has order continuous norm the space $E_s = E$ is indeed an ideal in $L^0(\mu)$.

Conjecture 8.5. *A translation invariant Banach function space E on a locally compact Hausdorff group G has order continuous norm if and only if E coincides with its strongly continuous part, i.e.* $E = E_s = E_{s,0}$.

To motivate this conjecture we observe that even in the non-compact non-abelian case the set $C_c(G)$ is dense in each of the Banach function spaces $L^p(G)$ for $1 \leq p < \infty$ and Beurling algebras $L^1(G, w)$, and we furthermore note that these have order continuous norm.

In order for Theorem 8.3 to apply we then only need to assume that E is order-dense in $L^0(\mu)$.

Theorem 8.6. *Let E be a translation invariant Banach function space with order continuous norm, that is furthermore order dense in $L^0(\mu)$. Assume that the left and right translation operators λ_y, ρ_y are bounded operators for each $y \in G$, and assume that the operator norms $\|\lambda_y\|_{B(E,E)}, \|\rho_y\|_{B(E,E)}$ are bounded on compact sets $K \subseteq G$. Furthermore assume that $C_c(G) \cap E \neq \{0\}$. Then there exists an extension of the left convolution on $C_c(G)$ to E, $l : C_c(G) \to L_r(E)$. Under the assumption that Conjecture 8.5 holds this extension is a lattice representation. Furthermore if $\{\|\lambda_y\|_{B(E,E)} : y \in G\}$ is bounded the map l extends to a lattice algebra representation from $L^1(G)$ to $L_r(E)$.*

Proof. This is a direct corollary of Theorem 8.3 as well as of Theorem 7.5. □

Interesting to note are the differences between Theorem 7.5 and Theorem 8.6. Since according to Conjecture 8.5 together with Lemma 8.4 the spaces E and $E_{s,0}$ coincide under the assumption that E has o.c. norm, the only differences are the assumption that E is order-dense in $L^0(\mu)$ and that Theorem 7.5 does not rely on a conjecture. We conclude that using known results of integral operators gives us a second method of obtaining the result of Theorem 7.5, but only under specific assumptions on our B.f.s. E.

9 Conclusion

We have proven that for a locally compact Hausdorff topological group G equipped with a left translation invariant Haar measure and a translation invariant Banach function space E on G the set of all strongly continuous elements in E forms a closed sublattice of E. Furthermore we have introduced the spaces $L_c^\infty(G)$ and $L_{loc}^1(G)$. We have shown that under the assumptions that $C_c(G) \cap E \neq \{0\}$, that the translation operators λ_y, ρ_y are continuous for each $y \in G$ and that their norms are bounded on compact sets that $L_c^\infty(G) \subseteq E \subseteq L_{loc}^1(G)$ with continuous inclusions. This extends the result for compact abelian groups. Under the same assumptions as above we have shown that we can extend the convolution operators on $C_c(G)$ to the set of strongly continuous elements of E. We have shown that we can find a collection of functions that forms an approximate unit for the convolution on $C_c(G)$ and for the extensions introduced above. We used this approximate unit to show that the space of all strongly continuous elements of E that vanish at infinity, $E_{s,0}$, is the closure of $C_c(G)$ in E. Motivated by the compatibilities between the inclusion map $C_c(G) \subseteq E_{s,0}$ and the convolution operators we have derived an abstract result giving sufficient conditions for when such an extension of the convolution map is a lattice homomorphism. In the main result of this thesis these results are combined to conclude that if $L_r(E_{s,0})$ is a Riesz space, for example if $E_{s,0}$ is Dedekind complete (in particular if E has order continuous norm), that this extended convolution map $l : C_c(G) \to L_r(E_{s,0})$ is a lattice homomorphism. Under the stronger assumptions that $E_{s,0}$ is Dedekind complete and the left translation operators are uniformly bounded this convolution extends to a lattice algebra representation $L^1(G) \to L_r(E_{s,0})$.

References

[1] Y. A. Abramovich and C. D. Aliprantis. *An invitation to operator theory*, volume 50 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[2] C. D. Aliprantis and O. Burkinshaw. *Positive operators*. Springer, Dordrecht, 2006. Reprint of the 1985 original.

[3] W. Arendt. On the o-spectrum of regular operators and the spectrum of measures. *Math. Z.*, 178(2):271–287, 1981.

[4] B. Brainerd and R. E. Edwards. Linear operators which commute with translations. I. Representation theorems. *J. Austral. Math. Soc.*, 6:289–327, 1966.

[5] B. de Pagter and W. Ricker. Optimal order bounded integration maps and convolution operators. In preparation.

[6] G. B. Folland. *Real analysis: Modern Techniques and Their Applications*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[7] E. Kaniuth. *A course in commutative Banach algebras*, volume 246 of *Graduate Texts in Mathematics*. Springer, New York, 2009.

[8] H. Reiter and J. D. Stegeman. *Classical harmonic analysis and locally compact groups*, volume 22 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, second edition, 2000.

[9] A. C. Zaanen. *Riesz spaces. II*, volume 30 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1983.