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Electrical networks and Markov chains

Classical results and beyond

Masterthesis

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Contents

Abstract

In this thesis we will discuss the connection between Markov chains and electrical networks. We will explain how weighted graphs are linked with Markov chains. We will then present classical results regarding the connection between reversible Markov chains and electrical networks. Based on recent work, we can also make a connection between general Markov chains and electrical networks, where we also show how the associated electrical network can be physically interpreted, which is based on a non-existing electrical component. We will then especially see which specific results still apply for these general Markov chains. This thesis concludes with an application based on this connection, called the Transfer current theorem. Proving this theorem relies upon the connection between random spanning trees with Markov chains and electrical networks.

Chapter 1

Introduction

Stochastic processes are central objects in probability theory and they are widely used in many applications in different fields, such as chemistry, economy, physics and mathematics. Markov chains are a specific type of stochastic processes, which has been formalized by Andrey Markov, where the process posses a property which is usually referred to as *memorylessness*. This means that the probability distribution of our process only depends on the current state of the process.
Electrical networks are used in daily life and hence, a lot of interest is devoted to these objects. Doyle and Snell were the first to mathematically formulate the connection between reversible Markov chains and electrical networks in 1984 [9]. Their work provides a way to solve problems from Markov chain theory by using electrical networks.
The first mathematical formulations regarding electrical networks theory could be backtracked to Kirchoff. He gave some mathematical formulation in his work from 1847 [13] regarding electrical networks.
One of the most famous results which uses the connection between reversible Markov chains and electrical networks is Pólya's theorem. Pólya proved that a random walk on an infinite 2-dimensional lattice has probability one of returning to the starting point, but that a a random walk on an infinite 3-dimensional lattice has probability bigger than zero of not returning to the starting point. This proof relies on the connection between reversible Markov chains and electrical networks [18].
The work mentioned above focuses on a specific class of Markov chains, namely where reversibility is guaranteed. The specific condition which guarantees reversibility. In recent years, there has been more research

done to the class of non-reversible Markov chains and to connect these with electrical networks. Mathematical formulation has been given by Balazs and Folly in 2014 [3], where they made non-reversibility possible by introducing a *voltage amplifier*, which is a new electrical component. They also give a physical interpretation of this component, where we specifically note that this is a theoretical extension of the classical electrical network.
Random spanning trees are objects from combinatorics which can be connected with Markov chains and electrical networks. Using the connection between random spanning trees, reversible Markov chains and electrical networks, Burton and Pemantle have proven the Transfer current theorem in 1993 [5]. The Transfer current theorem tells us that the probability distribution of seeing certain edges in a random spanning tree is a determinantal point process.
The Transfer current theorem has also been proven in terms of non-reversible Markov chains. Results on this can be found in [2] and [6].

The goal of the thesis is to give an overview of classical results regarding reversible Markov chains and electrical networks, to show the extension with results for non-reversible Markov chains, associated electrical networks and to show the connection between random rooted spanning trees and Markov chains.
By specifically looking at reversible Markov chains, electrical networks and random spanning trees, the Transfer current theorem will also be proven.

We will start by introducing two related problems in section 1.1 which lead to some intuition regarding the connection between Markov chains and electrical networks. The first problem will look at random walks and the second one will look at electrical networks. After that, the structure of the thesis will be stated in section 1.2.

1.1 Problems

This section is devoted to giving two problems which shows the relation between Markov chains and electrical networks.

A Markov chain problem

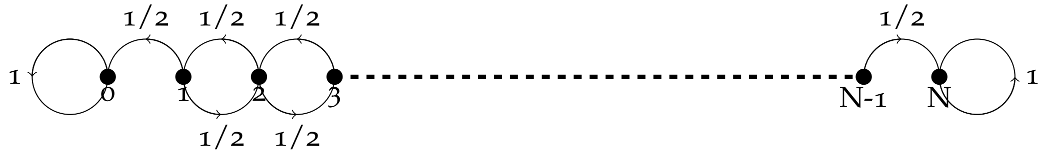

Figure 1.1: A Markov chain problem

Suppose that we have a casino game where we win 1 euro with probability $\frac{1}{2}$ or we lose 1 euro with probability $\frac{1}{2}$. We leave the casino when we have $N > 0$ euros or when we have lost all our money. Suppose that we enter the casino with $0 < u < N$ euros. Now define the integer-set $V := \{0, 1, ..., N\}$.
We can interpret this as a simple random walk $X = (X_n)_{n\geq 0}$ on V where at $u \in V \setminus \{0, N\}$ the walk has probability $\frac{1}{2}$ to jump to $u-1$ and probability $\frac{1}{2}$ to jump to $u+1$. Whenever we are at $\{0, N\}$, we stay there.
Now let $p(u)$ be the probability that we hit N before we hit 0, when we start from u. This can be written as

$$p(u) = \mathbf{P}_u(\tau_N < \tau_0), u \in V$$

where $\mathbf{P}_u$ stands for the probability law of X given $X_0 = u$ and where for all $u \in V$ we define the first hitting of time u by

$$\tau_u := \inf\{n \geq 0 : X_n = u\}.$$

We see directly that $p(N) = 1, p(0) = 0$.
Another equality we get is the following, for all $u \in V \setminus \{0, N\}$:

$$p(u) = \mathbf{P}_u(\tau_N < \tau_0) = \mathbf{P}_u(\tau_N < \tau_0, X_1 = u+1) + \mathbf{P}_u(\tau_N < \tau_0, X_1 = u-1)$$

$$= \mathbf{P}_u(X_1 = u+1)\mathbf{P}_u(\tau_N < \tau_0 | X_1 = u+1) + \mathbf{P}_u(X_1 = u-1)\mathbf{P}_u(\tau_N < \tau_0 | X_1 = u-1)$$

$$= \frac{\mathbf{P}_{u+1}(\tau_N < \tau_0) + \mathbf{P}_{u-1}(\tau_N < \tau_0)}{2} = \frac{p(u+1) + p(u-1)}{2}.$$

We just state that the solution of this problem is $p(u) = \frac{u}{N}$ for all $0 \leq u \leq N$. Showing this is left to the reader.
This problem is an specific form of the **Dirichlet problem**. This problem will be introduced in more generality in section 2.2.2.

An electrical network problem

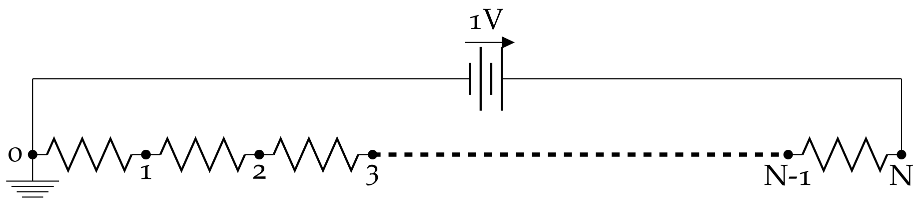

Figure 1.2: An electrical network problem

Consider an electrical network where we put resistors with equal resistance R in series between the points $\{0, 1, ..., N\}$ and where we have a unit voltage between the endpoints, such that the voltage is 0 at zero and 1 at N.
Let $\phi(u)$ be the **voltage** at $u \in \{0, 1, ..., N\}$. We have then that $\phi(N) = 1, \phi(0) = 0$. By **Kirchoff's node law**, we know that the **current flow** into u must be the same as the current flow out from u. By **Ohm's law**, we get that if u and v are connected by a resistance with R Ohm, then the current flow from u to v satisfies:

$$i(u, v) = \frac{\phi(u) - \phi(v)}{R}.$$

So that means for $u \in \{1, 2, ..., N-1\}$:

$$\frac{\phi(u-1) - \phi(u)}{R} + \frac{\phi(u+1) - \phi(u)}{R} = 0.$$

$$\phi(u) = \frac{\phi(u-1) + \phi(u+1)}{2}.$$

This problem has as solution $\phi(u) = \frac{u}{N}, 0 \leq u \leq N$.
We now refer back to the *Markov chain problem* and we notice that we have

the following connection:

$$\begin{cases} p(N) = 1 \text{ and } \phi(N) = 1 \\ p(0) = 0 \text{ and } \phi(0) = 0 \\ p(u) = \frac{p(u-1)+p(u+1)}{2} \text{ and } \phi(u) = \frac{\phi(u-1)+\phi(u+1)}{2}, 1 \leq u \leq N-1. \end{cases}$$

As we will see in section 2.2.2, this means that p and ϕ are **harmonic** on $\{1, 2, ..., N-1\}$ and we note that they admit the same boundary conditions. In this case, we see that $\phi(u) = p(u)$ for all $0 \leq u \leq N$.
These are both specific forms of the **Dirichlet problem** [8] and we will see that these problems are equivalent, hence the solutions are also equivalent for the same boundary conditions.

This specific connection between the *Markov chain problem* and the *Electrical network problem* gives rise to a connection between **Markov chains** and **electrical networks**. The connection between Markov chains and electrical networks is actually much more general and how to make this connection in more generality will be one of the main topics of the thesis.

1.2 Structure of the thesis

In this first chapter an introduction is given. Section 1.1 gives two examples which sketch the connection between Markov chains and electrical networks. This section states the structure of thesis.

The second chapter is devoted to graphs and Markov chains. The first section, section 2.1, is devoted to basic theory on graphs and the second section, section 2.2, introduces theory on Markov chains and in specific on the Dirichlet problem.

The third chapter considers classical theory regarding reversible Markov chains and electrical networks. This chapter focuses in specific on voltages and current flows, effective conductance and resistance, the Green function and it introduces the classical series, parallel and Delta-Star transformation laws. The rest of the chapter is devoted to the notion of energy, theory on the star and cycle spaces and it concludes by presenting Thomson's, Dirichlet's and Rayleigh's principles.

The fourth chapter focuses on the extension of the theory from the fourth chapter, by first looking at general Markov chains in section 4.1, and this will be used to obtain the electrical network associated to these general Markov chains. This chapter looks specifically at theory about voltages and current flows, it discusses effective conductance and resistance and it will give a non-physical interpretation of the electrical components which lead to the electrical networks associated to general Markov chains. This leads to results on the series, parallel and Delta-Star transformation laws for these particular electrical networks. The chapter concludes with sections on the notion of energy and it concludes by presenting Thomson's and Dirichlet's principles.

The fifth chapter focuses on random spanning trees and the connection with Markov chains. This chapter starts by introducing theory regarding spanning trees and rooted spanning trees. The subsequent section introduces Wilson algorithm and it gives results on the probability of picking a random (rooted) spanning tree corresponding with the probability that Wilson's algorithm samples a random (rooted) spanning tree. The chapter concludes with two sections, where the first one is devoted to a connection between current flows and random spanning trees. The final section present and proves the Transfer current theorem for the reversible setting, by using the theory on electrical networks from chapter 3.

Chapter 2

Finite graphs and associated Markov chains

The central objects under investigation are finite weighted graphs and the Markov chains associated to this. This chapter will be used to introduce notation and basic results for these objects.
Sections 2.1 and 2.2 will be devoted to graphs and Markov chains, respectively.

2.1 Graphs

In this section, we will state the framework regarding finite weighted graphs, which are the objects which we will look at throughout this thesis. More information on graphs can be found in [10].

Let $G = (V, E)$ be a finite graph, where V is the vertex-set, with **cardinality** $|V| < \infty$, and E is the edge-set. We will either have **undirected** or **directed** graphs.

Whenever we talk about undirected graphs, we assume that $E \subseteq V \times V$ and that E is a symmetric set. This means that for all $u, v \in V : \langle u, v \rangle \in E$ if and only if $\langle v, u \rangle \in E$. We call u and v then the **endpoints** of $\langle u, v \rangle$.

If we are talking about directed graphs, we will assume that $E \subseteq V \times V$,

Figure 2.1: Loop, parallel edges e_1, e_2, an edge with endpoints u and v and an directed edge e with tail e^- and head e^+

where E is not-necessarily symmetric. For every edge $e \in E$ we have endpoints of $e = \langle e^-, e^+ \rangle, e^-, e^+ \in V$, where e^- and e^+ will be referred to as the **tail** and **head** of the edge e. When we refer to $-e$, we mean the edge $-e := \langle e^+, e^- \rangle$. Note that for all $e \in E$ we also have $-e \in E$, whenever G is an undirected graph. If G is a directed graph, this is not true in general.

The notation $u \sim v$ means that there exists an edge $e = \langle u, v \rangle \in E$.
The **in**$-$**degree** (respectively, **out**$-$**degree**) of a vertex u is the number of edges where u is the head (respectively, tail) of an edge. Whenever we consider an undirected graph, the in-degree is equal to the out-degree of a vertex. We refer to this simply as the **degree** of the vertex.
We call a **path** a sequence of vertices u_i with $u_i \sim u_{i+1}$, where $u_i \in V, i \geq 1$. which are connected via edges. Whenever a path consists of all different vertices except that it has the same starting and ending point, we call this path a **cycle**. If there exists a path from all vertices to any other vertex, we call a graph **connected**. Whenever a graph is directed, this is also referred to as **irreducible** or **strongly connected** graph.
Suppose now that we have a graph $G = (V, E)$ and a graph $H = (W, F)$. Whenever $W \subseteq V$ and $F \subseteq E$, we call H a **subgraph** of G. If $W = V$, we call H a spanning subgraph.

Let us state how weights are defined on a graph.
A **weight** function on $G = (V, E)$ is a function $c : V \times V \to [0, \infty)$ such that $c(u, v) = 0$ if $\langle u, v \rangle \notin E$. If G is undirected we assume that for all

$u, v \in V$ with $\langle u, v \rangle \in E : c(u,v) = c(v,u)$. We refer then to (G,c) as a **weighted undirected graph**.
Whenever G is directed, we refer to (G,c) as a **weighted directed graph**.
We will assume in general that for all $v \in V$:

$$\sum_{u \in V} c(u,v) = \sum_{u \in V} c(v,u) \tag{2.1}$$

We refer to this condition as the **balance of the weights**.
If it is not explicitly stated, we **assume** that our graph is **connected and finite**.

2.2 Background on Markov chains

This section is devoted to basic results about discrete-time Markov chains on a finite state space. After we have recalled the definitions of a stationary and reversible measure, the link between Markov chains and weighted graphs will be made in section 2.2.1.
In section 2.2.2 the Dirichlet problem will be introduced and this will be the starting point of the connection between Markov chains and electrical networks.
More information about Markov chains can be found in [4, 17, 10, 1].

Let us start by recalling the definition of discrete-time Markov chains on a finite state space.

Definition 2.1. *A* ***discrete-time Markov chain*** *is a sequence $X = (X_n)_{n \in \mathbf{N}}$ of random variables on a probability space $(\Omega, \mathbf{P})$ taking their values in a finite state space V and satisfying the so-called* ***Markov property****:*

$$\mathbf{P}(X_{n+1} = x | X_0 = x_0, X_1 = x_1, X_2 = x_2, ..., X_n = x_n) = \mathbf{P}(X_{n+1} = x | X_n = x_n)$$

for every $x_0, x_1, ..., x_n, x \in V$.

The Markov property can be referred to as memorylessness.

We will assume that our Markov chain X is **time-homogeneous**, which means that

$$p_{uv} := \mathbf{P}(X_{i+1} = v | X_i = u) \text{ for all } i \geq 0.$$

We will refer to $P := [p_{uv}]_{u,v \in V}$ as the **transition matrix** of the Markov chain X.
Note that we have for all $u \in V : \sum_{v \in V} p_{uv} = 1$.

The transition matrix holds the information with regard to the probability distribution from each state.

Notation 2.1. $\mathbf{E}(\cdot)$ *refers to the* ***expectation*** *with respect to* $\mathbf{P}$.
For all $u \in V$*:*

$$\mathbf{P}_u(\cdot) := \mathbf{P}(\cdot | X_0 = u)$$

$$\mathbf{E}_u(\cdot) := \mathbf{E}(\cdot | X_0 = u).$$

We call a Markov chain **irreducible** if for all $u, v \in V$ there exists a $n \geq 0$ such that $p^n_{uv} := \mathbf{P}(X_n = v | X_0 = u) > 0$.
Irreducibility is an important property of a Markov chain which tells us that it is possible to get from any state to any other state.

An important object in Markov chain theory is the **stationary probability measure**. This is defined as follows:

Definition 2.2 (Stationary probability measure). *Given a probability measure* π *on* V*, that is a function* $\pi : V \to [0, 1]$ *satisfying* $\sum_{u \in V} \pi(u) = 1$*, we say that* π *is stationary for the Markov chain with transition matrix* P*, if for all* $v \in V$*:*

$$\pi(v) = \sum_{u \in V} \pi(u) p_{uv}.$$

We sometimes write equivalently $\pi = \pi P$*.*

We will prove that for every finite state irreducible Markov chain a unique stationary probability measure exists. Let us start by proving existence:

Lemma 2.1 (Existence of stationary probability measure). *Every finite state Markov chain has at least one stationary probability measure.*

Proof Let V be our finite state space.
Let f be constantly equal to 1. This means that $Pf = f$, which follows by calculation. Hence, 1 is an eigenvalue of P and f is the right-eigenvector corresponding to it. That means that there exists a left-eigenvector q,

which is not constantly equal to 0, such that $qP = q$. We note that q can hold negative values.
We have that whenever $q = qP$, then $|q| = |qP|$.
This is because of the following:

$$\text{For all } u \in V : |q(u)| = |\sum_{v \in V} q(v) p_{vu}| \leq \sum_{v \in V} |q(v) p_{vu}| \leq \sum_{v \in V} |q(v)| p_{vu}.$$

The last step used the non-negativity of P.
So we get that

$$\sum_{u \in V} |q(u)| \leq \sum_{u \in V} \sum_{v \in V} |q(v)| p_{vu} = \sum_{v \in V} |q(v)| \sum_{u \in V} p_{vu} = \sum_{v \in V} |q(v)|.$$

So we get equalities, which means that $|q| = |qP| = |q|P$.
Now set

$$\pi = \frac{|q|}{\sum |q|} = \frac{|q|}{\sum |q|} P = \pi P.$$

So that means that π is a stationary probability measure with respect to P. □

We will now show that for any finite state irreducible Markov chain an unique probability measure exists. We will first show that all its elements must be positive.

Lemma 2.2 (Positiveness of stationary probability measure). *For a finite state irredudible Markov chain every stationary probability measure is strictly positive.*

Proof Suppose that π is a stationary probability measure. We say that π is strictly positive if $\pi(u) > 0$ for all $u \in V$.
First note that there exist a $u \in V$ such that $\pi(u) > 0$. Fix $v \neq u$. By irreducibility, there exists a $n > 0$ such that $p^n(u, v) > 0$.
So that means that

$$\pi(v) = \sum_{w \in V} \pi(w) p^n(w, v) \geq \pi(u) p^n(u, v) > 0.$$

Hence, we see that for all $u \in V : \pi(u) > 0$. □

Let us now show that there exist a unique stationary probability measure for finite state irreducible Markov chains.

Lemma 2.3 (Uniqueness of stationary probability measure). *For a finite state irreducible Markov chain, there exists a unique stationary probability measure.*

Proof Consider two probability measures π_1, π_2 which are stationary with respect to P:

$$\begin{cases} \pi_1 = \pi_1 P \\ \pi_2 = \pi_2 P. \end{cases}$$

Define $\delta\pi := \pi_1 - \pi_2 = (\pi_1 - \pi_2)P = (\delta\pi)P$.
By the same argument as in the proof of the existence, we get then that $|\delta\pi| = |\delta\pi|P$.
So that means that $\delta\pi + |\delta\pi| = (\delta\pi + |\delta\pi|)P$.
So element-wise, we have that $\delta\pi + |\delta\pi|$ equals either 0 or $2\delta\pi$.
By the previous lemma, we know that $\delta\pi + |\delta\pi|$ is strictly positive and hence element-wise equals $2\delta\pi$.
So that means that $\delta\pi = |\delta\pi|$.
So that means that $\delta\pi(u) = \pi_1(u) - \pi_2(u) \geq 0$ for all $u \in V$. Now note that if $\delta\pi(u) > 0$ for some $u \in V$, we get that

$$1 = \sum_{u \in V} \pi_1(u) > \sum_{u \in V} \pi_2(u) = 1.$$

which is a contradiction. Hence, it follows that $\pi_1 = \pi_2$. So we have a unique stationary probability measure. □

Notation 2.2. *We have now constructed a transition matrix P and stationary probability measure π from our Markov chain X. We will refer to this as the pair (π, P).*

An important class of Markov chains are the so-called **reversible** Markov chains. Whenever (π, P) satisfies the following, we call our Markov chain reversible:

Definition 2.3 (Detailed balance condition). *We say that a transition matrix P and a probability measure π satisfy the detailed balance condition if*

$$\text{For all } u, v \in V : \pi(u)p_{uv} = \pi(v)p_{vu}.$$

We can directly note that whenever P and π satisfy the detailed balance condition, then π is stationary for P.

Let us introduce **hitting times**, which will play an important role in the rest of the thesis.

Definition 2.4 (Hitting times). *Consider a Markov chain $X = (X_n)_{n\in\mathbf{N}}$. Given a non-empty set $W \subseteq V$, we define the hitting time of the set W as follows:*

$$\tau_W := \inf\{i \geq 0 : X_i \in W\}, \tau_W^+ := \inf\{i > 0 : X_i \in W\}.$$

Notation 2.3. *Suppose that $W = \{u\}$ is a singleton. To simplify notation, let us refer to τ_u in place of $\tau_{\{u\}}$. We use the same notation for τ_u^+.*

2.2.1 Link between graphs and Markov chains

We will now explain how to construct a weighted graph from a Markov chain and vice versa.
Suppose that we have a Markov chain X with associated pair (π, P) on a finite state space V. Let G be the graph with vertex-set V and edge-set E, where E is defined by

$$E := \Big\{\langle u, v\rangle | \text{ for all } u, v \in V \text{ with } p_{uv} > 0\Big\}.$$

Note that this automatically means that $E \subseteq V \times V$.
Now assign the weights by

$$c(u, v) := \pi(u)p_{uv} \text{ for all } u, v \in V.$$

This means that $c(u, v) = 0$ if $u \not\sim v$.
Whenever c is a symmetric function, we interpret this as a undirected graph. This is the case when our Markov chain is reversible. Else, our graph is directed.
When our Markov chain is irreducible, this directly implies that our graph is connected. Let us make the remark that the obtained graph does not have any multiple edges.

Now start with a weighted graph (G, c) with $G = (V, E)$. We show how to construct a Markov chain associated to (G, c).

Definition 2.5. *Define* $C : V \to (0, \infty)$ *as follows:*

$$C(u) := \sum_{v \in V} c(u, v), \text{ for all } u \in V.$$

Set $p_{uv} = \frac{c(u,v)}{C(u)}$ for all $u, v \in V$, which gives that $\sum_{v \in V} p_{uv} = 1$ for all $u \in V$. Hence, $P = [p_{uv}]_{u,v \in V}$ is a transition matrix for a Markov chain X on V.

Moreover, by defining

$$\pi(u) := \frac{C(u)}{\sum_{u \in V} C(u)} \text{ for all } u \in V,$$

such π becomes a stationary probability measure. It follows now by the balance of weights (equation 2.1) that

$$\sum_{u \in V} C(u) p_{uv} = \sum_{u \in V} c(u, v) = \sum_{u \in V} c(v, u) = C(v).$$

If c is symmetric, (π, P) satisfies definition 2.3, hence our Markov chain is reversible. So we have a (one-to-one) correspondence between reversible Markov chains on a finite state space and undirected connected weighted finite graphs.

Similarly, we get a (one-to-one) correspondence between Markov chains on a finite state space and directed connected weighted finite graphs.

To match the assumption about graphs being connected and finite, we will only consider irreducible Markov chains.

2.2.2 Dirichlet problem and escape probabilities

This section is devoted to harmonic functions (with respect to Markov chains) and to the so-called Dirichlet problem. After defining the Dirichlet problem, we will prove that there exists a uniquely defined function which satisfies the Dirichlet problem.

We will start by defining the Dirichlet problem:

Definition 2.6 (Dirichlet problem). *Let* V, A, Z *be sets with* $A, Z \subset V$ *and* $A \cap Z = \emptyset$. *Consider a function* $f : V \to \mathbf{R}$ *and a transition matrix* P, *where:*

$$\begin{cases} f(u) = \sum_{v \in V} p_{uv} f(v) := (Pf)(u) \text{ for all } u \in V \setminus (A \cup Z) \\ f \restriction A = M \\ f \restriction Z = m \end{cases}$$

with $M > m$. This is called the Dirichlet problem.

The type of functions which satisfy the Dirichlet problem are **harmonic functions**. These functions are defined as follows:

Definition 2.7 (Harmonic function). *Let $U \subseteq V$. We call the function $f : V \to \mathbf{R}$ harmonic on U with respect to a given transition matrix P if*

$$f(u) = (Pf)(u), \text{ for all } u \in U.$$

Definition 2.8. *For $\emptyset \neq U \subseteq V$, define $\overline{U} = \{v \in V : v \sim u; \text{ for some } u \in U\} \cup U$.*

So $\overline{U}$ is the set of all vertices which are either in U or are neighbours of vertices in U.

To abbreviate notation, let us use the following notation for functions with respect to sets:

Notation 2.4. *Let $\upharpoonright$ note the restriction of a function to a set.*
Suppose that we have the sets U, V with $U \subseteq V$.
This means that when we have a function $f : V \to \mathbf{R}$, then $f \upharpoonright U : U \to \mathbf{R}$.

To prove that there exists an uniquely defined solution for the Dirichlet problem, we need to prove the maximum, uniqueness, superposition and existence principle. Let us start by stating and proving the maximum principle.

Theorem 2.1 (Maximum principle). *Let V, W both be sets with $W \subseteq V$, being a finite proper subset of V, and consider a function $f : V \to \mathbf{R}$, where f is harmonic on W with respect to P.*
If then $\max f \upharpoonright W = \max f$, then f is constant on $\overline{W}$.

Proof Let us first construct the set $K := \left\{v \in V : f(v) = \max f\right\}$.
Note that if we have a vertex $u \in (W \cap K)$, that means that f is harmonic at u with respect to P. Hence, it follows that the neighbors of u are also equal to $\max f$. So we have that $\overline{\{u\}} \subseteq K$.
We do know that $W \cap K \neq \emptyset$ (which is required by letting W be a proper subset). But that means that $f \upharpoonright W = \max f$. Hence, we know that $K \supseteq W$.
But that means again that $K \supseteq \overline{W}$.

So that gives that $f \restriction \overline{W} = \max f$, which means that f is constant on $\overline{W}$. □

We will now use the maximum principle to prove the uniqueness principle, which is the following:

Theorem 2.2 (Uniqueness principle). *Let V, W both be sets with $W \subseteq V$, being a finite proper subset of V.*
Now consider two functions $f, g : V \to \mathbf{R}$ where f, g are harmonic on W are harmonic with respect to P, and $f \restriction (V \setminus W) = g \restriction (V \setminus W)$.
Then $f = g$.

Proof Define $h := f - g$. First note that $h \restriction (V \setminus W) = 0$.
Suppose now that $h \nleq 0$ on W, hence h is positive at some point in W. That implies then that $\max h \restriction W = \max h$.
The maximum principle tells us then that $h \restriction W = \max h$. Hence, we know that $h \restriction \overline{W} = \max h > 0$. So we know then that $h \restriction (\overline{W} \setminus W) = \max h > 0$. Note that $\overline{W} \setminus W$ is non-empty, which is required by letting W be a proper subset.
Now note that $\overline{W} \setminus W \subseteq V \setminus W$, so that means that $h \restriction (\overline{W} \setminus W) = 0$. This leads to a contradiction and hence we know that $h = 0$ on V.
The same argument holds for $h \ngeq 0$ on W by symmetry.
That means that $f = g$. □

A direct consequence is that we are capable of proving the superposition principle, which is the following:

Proposition 2.1 (Superposition principle). *Let V, W both be sets with $W \subset V$.*
If f, f_1, f_2 are harmonic on W with respect to P and $\alpha_1, \alpha_2 \in \mathbf{R}$ with $f = \alpha_1 f_1 + \alpha_2 f_2$ on $V \setminus W$, then it follows that $f = \alpha_1 f_1 + \alpha_2 f_2$.

Proof This follows immediately by the uniqueness principle. (This is a specific form of the superposition principle). □

The last principle we need to prove is the existence principle.

Theorem 2.3 (Existence principle). *Let V, W both be sets and let $W \subset V$. Suppose now that $f_0 : V \setminus W \to \mathbf{R}$ is bounded, then $\exists f : V \to \mathbf{R}$ such that $f \restriction (V \setminus W) = f_0$ and f is harmonic on W with respect to P.*

Proof Let $X = (X_n)_{n \in \mathbf{N}}$ be the Markov chain on V with respect to P. For $u \in V$, define

$$f(u) := \mathbf{E}_u\Big(f_0(X_{\tau_{V \setminus W}})\Big).$$

Now note that $f(u) = f_0(u)$ for $u \in V \setminus W$.
For all $u \in W$:

$$f(u) = \mathbf{E}_u\Big(f_0(X_{\tau_{V \setminus W}})\Big) = \sum_{v \in V} \mathbf{P}_u\Big(X_1 = v\Big)\mathbf{E}_u\Big(f_0(X_{\tau_{V \setminus W}})|X_1 = v\Big).$$

We get then by the Markov property and the time-homogeneity that

$$f(u) = \sum_{v \in V} p_{uv} f(v) = (Pf)(u).$$

Hence this means that f is harmonic on W with respect to P. □

We now refer to definition 2.6. By the existence principle, we get that such a function exists and the uniqueness principle then tells us that this is a uniquely defined function. Hence, there exists a uniquely defined function which satisfies the Dirichlet problem.

We now refer back to the *Markov chain problem* and *Electrical network problem* from section 1.1, where $A = \{N\}, Z = \{0\}, M = 1, m = 0$ and where

$$\begin{cases} p \restriction A = \phi \restriction A = 1 \\ p \restriction Z = \phi \restriction Z = 0 \\ p, \phi \text{ are harmonic at } V \setminus (A \cup Z). \end{cases}$$

This is a specific form of the Dirichlet problem and hence, there exists a uniquely defined solution to this specific Dirichlet problem, which is $p(u) = \phi(u) = \frac{u}{N}$ for all $u \in V$.

This means that functions coming from Markov chains and electrical networks which both satisfy the same Dirichlet problem are equivalent. This gives rise to extending this in a more general framework in the rest of the thesis, where we will use the unique correspondence between Markov chains and electrical networks.

We now define the hitting probability function, which is a harmonic function which satisfies the Dirichlet problem. By using the superposition principle, we can directly see that any linear combination of this function can be used to solve any Dirichlet problem with respect to a given transition matrix P.

Definition 2.9 (Hitting probability function). *Let V, A, Z be sets with $A, Z \subset V$ and $A \cap Z = \emptyset$ and let a transition matrix P be given.*
Let $X = (X_n)_{n \in \mathbf{N}}$ be the Markov chain on V with respect to P.
Define $F : V \to [0,1]$ by $F(u) := \mathbf{P}_u(\tau_A < \tau_Z)$ for all $u \in V$.
This function can be interpreted as the probability that X starting at u hits A before it hits Z.

We have the following boundary conditions:
$F \upharpoonright A = 1, F \upharpoonright Z = 0$.
We get then that for all $u \in V \setminus (A \cup Z)$:

$$\begin{aligned} F(u) &= \sum_{v \in V} \mathbf{P}_u(X_1 = v)\mathbf{P}_u(\tau_A < \tau_z | X_1 = v) \\ &= \sum_{v \in V} p_{uv}\mathbf{P}_v(\tau_A < \tau_z) = \sum_{v \in V} p_{uv}F(v). \end{aligned}$$

We used here the Markov property and time-homogeneity. It directly follows that F is harmonic at $V \setminus (A \cup Z)$ with respect to P.

Chapter 3

Electrical networks and reversible Markov chains

This chapter will be devoted to classical results about electrical networks and reversible Markov chains. More information and background on the following chapter can be found in [15, 1].

This chapter starts with definitions to clarify the link between electrical networks and reversible Markov chains in section 3.1. Then the notions of voltages and current flows will be introduced, together with the two Kirchoff laws and Ohm's law. Section 3.1.1 gives results on the effective conductance and resistance and this section gives a probabilistic interpretation of the voltage. Section 3.1.2 will includes theory related to the Green function. Then the classical transformation laws, namely the series, parallel and Delta-Star laws will be presented in section 3.1.3. These laws can be used to reduce a graph and to obtain the effective resistance or equivalently the effective conductance.
Section 3.1.4 is devoted to the concept of energy. Section 3.1.5 introduces the star and cycle spaces. Finally, section 3.1.6 will present results on Thomson's, Dirichlet's and Rayleigh's principles.
This chapter only considers undirected graphs, as a consequence, the associated Markov chain is automatically reversible.

3.1 Electrical networks

This section states basic notation and definitions for electrical networks. We will furthermore define conductances, flows, potentials and the operators associated to these functions. We will then give the classical Kirchoff laws and Ohms's law, which gives rise to propositions which together defines voltages and current flows. After that, we state what we mean by flow conservation and conservation of energy. These general properties of the electrical networks will be used in the subsequent subsections.

We will start this section with defining an electrical network and how to physically interpret this.

Notation 3.1 (Electrical network). *Suppose that we have a weighted graph (G, c) with $G = (V, E)$ and where $c : E \to [0, \infty)$.*
Now suppose that $\emptyset \neq A, Z \subset V$ be given such that $A \cap Z = \emptyset$.
We refer to this as the ***electrical network*** *(G, c, A, Z).*

We can interpret an electrical network (G, c, A, Z) by taking the classical physical interpretation. The graph $G = (V, E)$ then refers to the underlying circuit, where the vertex-set V refers to the components of an electrical network and the edge-set E refers to how the components are connected. The function c is then interpreted as the conductance and in specific if two components, $u, v, u \neq v$ are connected, then $c(u, v)$ is the conductance between the components u and v. The set A is usually interpreted as the battery and Z as the ground.

Let (G, c, A, Z) be an electrical network with $G = (V, E)$. The weight $c(u, v)$ of an edge $\langle u, v \rangle$ is then interpreted as the **conductance** of a resistor connecting the endpoints u and v of the edge $\langle u, v \rangle$.

Define $r : V \times V \to (0, \infty]$ to be the reciprocal of c, i.e. $r(u, v) := \frac{1}{c(u,v)}$ for all $u, v \in V$. We refer to $r(u, v)$ as the **resistance** between u and v.

In this chapter, we will assume that our conductances are symmetric, hence the resistances are also symmetric.

Let us now refer to section 2.2.1, where we showed how to define from a weighted graph (G, c) a Markov chain X with associated pair (π, P). Now follow this approach and let the conductances be given by $c(u, v) = \pi(u)p_{uv}$, for all $u, v \in V$. Cause we assume that our conductances are symmetric, we see that our Markov chain is reversible and satisfies the detailed balance condition, definition 2.3.
Let us further assume that each edge occurs with both orientations.

Central objects in electrical networks are voltages and current flows. To define these objects, we first need some definitions.

Definition 3.1. *Define a **flow** to be an antisymmetric function $\theta : V \times V \to \mathbf{R}$, i.e. $\theta(u, v) = -\theta(v, u)$ for all $u, v \in V$.*
Define $\ell^2_-(E)$ to be the real Hilbert space of flows on $E \subset V \times V$ associated with the inner product

$$\langle \theta, \theta' \rangle := \frac{1}{2} \sum_{e \in E} \theta(e)\theta'(e) = \sum_{e \in E_{\frac{1}{2}}} \theta(e)\theta'(e)$$

with $E_{\frac{1}{2}} \subset E$ a set which contains exactly one of each pair $e, -e$. Formally:

$$\ell^2_-(E) = \{\theta : E \to \mathbf{R}, \theta(e) = -\theta(-e) \text{ for all } e \in E \text{ with } \langle \theta, \theta \rangle < \infty\}.$$

Definition 3.2. *Define a **potential** to be a function $f : V \to \mathbf{R}$.*
Define $\ell^2(V)$ to be the real Hilbert space of potentials f, g associated with the inner product

$$\langle f, g \rangle := \sum_{v \in V} f(v)g(v).$$

Formally:

$$\ell^2(V) = \{f : V \to \mathbf{R} \text{ with } \langle f, f \rangle < \infty\}.$$

Definition 3.3. *Define the **coboundary operator** $d : \ell^2(V) \to \ell^2_-(E)$ by*

$$df(e) := f(e^-) - f(e^+).$$

Note the following:

Remark 3.1. *Let f be a potential.*
Let $v_1 \sim v_2 \sim ... \sim v_n \sim v_{n+1} = v_1$ be a cycle of G. Let $e_i = \langle v_i, v_{i+1} \rangle, 1 \le i \le n$ be the same oriented cycle of G. Then:

$$0 = \sum_{i=1}^{n} [f(v_i) - f(v_{i+1})] = \sum_{i=1}^{n} df(e_i).$$

Definition 3.4. *Define the* ***boundary operator*** $d^* : \ell^2_-(E) \to \ell^2(V)$ *by*

$$d^*\theta(u) := \sum_{v:v\sim u} \theta(u,v) = \sum_{e:e^-=u} \theta(e).$$

We are now capable of defining a voltage.

Definition 3.5 (Voltage). *A* ***voltage*** ϕ ***with boundary*** $A \cup Z$ *is a potential which is harmonic at* $V \setminus (A \cup Z)$ *with respect to* P*.*

We are now ready to state the classical Kirchoff laws and Ohm's law. These classical laws could be backtracked to the work of Kirchoff [13].

We say that a flow θ satisfies **Kirchoff's node law** with boundary $A \cup Z$ if it's divergence-free at $V \setminus (A \cup Z)$, that is:

$$\text{For all } u \in V \setminus (A \cup Z) : 0 = d^*\theta(u).$$

We say that a flow θ satisfies **Kirchoff's cycle law** if for every oriented cycle $e_1, e_2, ..., e_n$ of G:

$$0 = \sum_{i=1}^{n} \theta(e_i) r(e_i).$$

We say that a flow θ and a potential f satisfy **Ohm's law** if

$$f(u) - f(v) = \theta(u,v)r(u,v), u \sim v, u, v \in V, \text{ equivalently } df(e) = \theta(e)r(e), e \in E.$$

By using these classical laws, we are ready to define a current flow.

Definition 3.6 (Current flow). *A* ***current flow*** *is a flow with boundary* $A \cup Z$ *that satisfies Ohm's law together with a voltage with boundary* $A \cup Z$*.*

Whenever we mention the electrical network (G, c, A, Z) and we refer to the voltage ϕ and current flow i, we mean that ϕ is a potential which is harmonic at $V \setminus (A \cup Z)$ and that i is a flow which is divergence-free at $V \setminus (A \cup Z)$. We also mean that ϕ and i satisfy Ohm's law with respect to the resistance r or equivalently conductance c.

Theorem 3.1. *Suppose that we have an electrical network* (G, c, A, Z) *with a voltage* ϕ *with boundary* $A \cup Z$ *and a flow* θ *which satisfies two of the three laws, then we can always deduce the other law.*

Proving this theorem is done by proving propositions 3.1, 3.2 and 3.3, which are introduced and proven below.

Proposition 3.1. *Consider the electrical network (G, c, A, Z) and the associated Markov chain X with the pair (π, P).*
Suppose that ϕ is a voltage with respect to P with boundary $A \cup Z$ and that our flow θ satisfies Kirchoff's node law with boundary $A \cup Z$ and Ohm's law with respect to ϕ, which means that θ is a current flow. Then θ satisfies Kirchoff's cycle law.

Proof We can directly see by Ohm's law that $\theta(e)r(e) = df(e)$ for all $e \in E$. Hence, for every oriented cycle $e_1, ..., e_n$ of G, we get that

$$\sum_{i=1}^{n} \theta(e_i)r(e_i) = \sum_{i=1}^{n} df(e_i) = 0.$$

So θ indeed satisfies Kirchoff's cycle law. □

Proposition 3.2. *Consider the electrical network (G, c, A, Z) and the associated Markov chain X with the pair (π, P).*
Suppose that ϕ is a voltage with respect to P with boundary $A \cup Z$ and that our flow θ satisfies Kirchoff's cycle law with boundary $A \cup Z$ and Ohm's law with respect to ϕ. Then θ satisfies Kirchoff's node law, which means that θ is a current flow.

Proof We obtain the following for our flow θ:

$$\text{For all } u \in V : d^*\theta(u) = \sum_{v:v\sim u} \theta(u,v) = \sum_{v:v\sim u} c(u,v)(\phi(u) - \phi(v))$$

$$= \phi(u) \sum_{v:v\sim u} c(u,v) - \sum_{v:v\sim u} c(u,v)\phi(v) = \phi(u)\pi(u) - \pi(u) \sum_{v:v\sim u} p_{uv}\phi(v)$$

$$= \pi(u)(\phi(u) - (P\phi)(u)).$$ □

Now note that ϕ is harmonic at $V \setminus (A \cup Z)$, i.e. $\phi(u) = (P\phi)(u)$ for all $u \in V \setminus (A \cup Z)$.
Hence, we see that $d^*\theta(u) = 0$ for all $u \in V \setminus (A \cup Z)$, so θ satisfies Kirchoff's node law with boundary $A \cup Z$.

Proposition 3.3. *Consider the electrical network (G, c, A, Z) and the associated Markov chain X with the pair (π, P).*
Suppose that our flow θ satisfies Kirchoff's node law with boundary $A \cup Z$ and Kirchoff's cycle law. Then θ is a current flow with respect to a voltage f, which means that θ satisfies Ohm's law with respect to f.

Proof First note that we have that for every oriented cycle $e_1, ..., e_n$ of G : $0 = \sum_{i=1}^{n} \theta(e_i) r(e_i) = 0$.
That means that there exists a potential f such that $\theta(e)r(e) = df(e)$, cause $0 = \sum_{i=1}^{n} df(e_i) = \sum_{i=1}^{n} \theta(e_i) r(e_i)$.
Note now that we obtain the following by Kirchoff's node law:

$$\text{For all } u \in V \setminus (A \cup Z) : 0 = d^*\theta(u) = \sum_{v:v\sim u} c(u,v)(f(u) - f(v))$$

$$= f(u) \sum_{v:v\sim u} c(u,v) - \sum_{v:v\sim u} c(u,v) f(v) = f(u)\pi(u) - \pi(u) \sum_{v:v\sim u} p_{uv} f(v)$$
$$= \pi(u)(f(u) - (Pf)(u)).$$

So that means that $f(u) = (Pf)(u)$ for all $u \in V \setminus (A \cup Z)$, hence f is a voltage with boundary $A \cup Z$.
So we see that $\theta(e)r(e) = df(u)$, which means that θ satisfies Ohm's law with respect to the voltage f. □

Hence, theorem 3.1 has been proven by the above three propositions. □

We now show that our operators from definitions 3.4 and 3.3 are adjoint with respect to flows and potentials in the following way:

Lemma 3.1 (Adjointness of operators).

$$\textit{For all } f \in \ell^2(V) \textit{ for all } \theta \in \ell^2_-(E) : \langle \theta, df \rangle = \langle d^*\theta, f \rangle.$$

Proof

$$\langle d^*\theta, f \rangle = \sum_{v\in V} d^*\theta(v) f(v) = \sum_{v\in V} f(v) \Big(\sum_{e:e^-=v} \theta(e) \Big)$$
$$= \sum_{v\in V} \sum_{e:e^-=v} f(v)\theta(e) = \sum_{e\in E} f(e^-)\theta(e).$$

We can also sum over $E_{\frac{1}{2}}$ and then take for each edge e in there the tail e^- of e and the head e^+ of $-e$. That gives that

$$\langle d^*\theta, f\rangle = \sum_{e\in E_{\frac{1}{2}}} [f(e^-)\theta(e) + f(e^+)\theta(-e)] = \sum_{e\in E_{\frac{1}{2}}} (f(e^-) - f(e^+))\theta(e)$$

$$= \sum_{e\in E_{\frac{1}{2}}} df(e)\theta(e) = \langle df, \theta\rangle = \langle \theta, df\rangle. \qquad \square$$

Remark 3.2. *Fixing the boundary conditions of a voltage gives a unique voltage by the uniqueness and existence principle.*

Let us first state some notation for a current flow.

Notation 3.2. *We call a current flow i a current flow from A to Z if*

$$\sum_{v:v\sim a} i(a,v) \geq 0 \text{ for all } a \in A, \sum_{v:v\sim z} i(z,v) \leq 0 \text{ for all } z \in Z$$

$$\text{or equivalently } d^*i(a) \geq 0 \text{ for all } a \in A, d^*i(z) \leq 0 \text{ for all } z \in Z.$$

We now define the so-called **strength** of a current flow:

Definition 3.7 (Strength). *For a current flow i, we define*

$$Strength(i) := \sum_{a\in A}\sum_{v:v\sim a} i(a,v) = \sum_{a\in A} d^*i(a).$$

Remark 3.3. *If $|Strength(i)| = 1$, we call i a* ***unit current flow****.*

By using the adjointness of operators from lemma 3.1 and the properties of a current flow, we can prove the following lemma's with regard to conservationof flow and energy, respectively.

Lemma 3.2 (Conservation of flow). *Let (G, c, A, Z) be an electrical network with $G = (V, E)$.*
If i is a current flow from A to Z, then

$$\sum_{a\in A} d^*i(a) = -\sum_{z\in Z} d^*i(z).$$

Proof

$$\sum_{a\in A} d^*i(a) + \sum_{z\in Z} d^*i(z) = \sum_{v\in V} d^*i(v) = \langle d^*i, 1\rangle = \langle i, d1\rangle = \langle i, 0\rangle = 0. \qquad \square$$

Lemma 3.3 (Conservation of energy). *Let (G, c, A, Z) be an electrical network with $G = (V, E)$.*
If i is a current flow that is divergence-free at $V \setminus (A \cup Z)$ from A to Z and f is a potential with $f \upharpoonright A = f_A, f \upharpoonright Z = f_Z$, with f_A, f_Z being constants, then

$$\langle i, df \rangle = \mathit{Strength}(i)(f_A - f_Z).$$

Proof

$$\langle i, df \rangle = \langle d^* i, f \rangle = \sum_{v \in V} d^* i(v) f(v) = \sum_{a \in A} d^* i(a) f(a) + \sum_{z \in Z} d^* i(z) f(z)$$

$$= f_A \cdot \text{Strength}(i) - f_Z \cdot \text{Strength}(i) = \text{Strength}(i)(f_A - f_Z). \qquad \square$$

We have now defined a framework for electrical networks with as central objects the current flows and voltages. The results in this section are results which directly follow by the definitions. We will use these definitions and results in the rest of this chapter.

3.1.1 Effective conductance and resistance

This section will be focussing on escape probabilities and effective conductance and resistance arising from escape probabilities.

Let (G, c, A, Z) be an electrical network with $G = (V, E)$, where $A = \{a\}$ and consider the associated Markov chain X with the pair (π, P).
Define the probability that a random walk starting at a will hit Z before it returns to a as

$$\mathbf{P}[a \to Z] := \mathbf{P}_a\Big(\tau_Z < \tau_a^+\Big).$$

Let i be the current flow and ϕ be the corresponding voltage with respect to P with boundary conditions $\phi \upharpoonright A = \phi_A$ and $\phi \upharpoonright Z = \phi_Z$, where ϕ_A, ϕ_Z are constants.
We refer to definition 2.9, where $F(u) = \mathbf{P}_u(\tau_A < \tau_Z)$ for all $u \in V$. Note that F is harmonic on $V \setminus (A \cup Z)$ with respect to P.
Also note that ϕ is a harmonic function at $V \setminus (A \cup Z)$ with respect to P. Hence, we can write by the superposition principle:

$$\mathbf{P}_u(\tau_A < \tau_Z) = \frac{\phi(u) - \phi_Z}{\phi_A - \phi_Z} \text{ for all } u \in V.$$

That gives then that:

$$\mathbf{P}[a \to Z] = \mathbf{P}_a\Big(\tau_Z < \tau_a^+\Big) = \sum_{v:v\sim a} p_{av}(1 - \mathbf{P}_v\Big(\tau_a < \tau_Z\Big))$$

$$= \sum_{v:v\sim a} \frac{c(a,v)}{\pi(a)}(1 - \frac{\phi(v) - \phi_Z}{\phi_A - \phi_Z}) = \frac{1}{\pi(a)}\frac{1}{\phi_A - \phi_Z}\sum_{v:v\sim a} c(a,v)(\phi_A - \phi(v))$$

$$= \frac{1}{\pi(a)}\frac{1}{\phi_A - \phi_Z}\sum_{v:v\sim a} i(a,v).$$

So we get that

$$\phi_A - \phi_Z = \frac{\sum_{v:v\sim a} i(a,v)}{\pi(a)\mathbf{P}[a \to Z]} = \frac{d^* i(a)}{\pi(a)\mathbf{P}[a \to Z]} = \frac{\text{Strength}(i)}{\pi(a)\mathbf{P}[a \to Z]}.$$

Let us first define effective conductance and resistance for elecrical networks:

Definition 3.8 (Effective conductance and resistance). *Let (G, c, A, Z) be an electrical network with $G = (V, E)$ and consider the associated Markov chain X with the pair (π, P).*
Define the effective conductance between the sets A and Z as follows:

$$\mathscr{C}(A \leftrightarrow Z) := \sum_{a\in A} \mathscr{C}(a \leftrightarrow Z) := \sum_{a\in A} \pi(a)\mathbf{P}[a \to Z].$$

Define the effective resistance between the sets A and Z as follows:

$$\mathscr{R}(A \leftrightarrow Z) := \mathscr{C}(A \leftrightarrow Z)^{-1}.$$

By using the relation between the strength of a current flow, the boundary conditions of a voltage and the escape probabilities, as abovely shown with the just defined effective conductance and resistance, we are capable of proving the following lemma. This lemma gives a direct way to calculate effective conductance and resistance.

Lemma 3.4. *Let (G, c, A, Z) be an electrical network with $G = (V, E)$ and consider the associated Markov chain X with the pair (π, P). Let i be a current flow and let ϕ be the corresponding voltage with respect to P with boundary conditions $\phi \upharpoonright A = \phi_A, \phi \upharpoonright Z = \phi_Z$, where ϕ_A, ϕ_Z are constants. Then*

$$\phi_A - \phi_Z = \frac{\mathit{Strength}(i)}{\mathscr{C}(A \leftrightarrow Z)}.$$

Proof Identify $\mathscr{C}(A \leftrightarrow Z)$ as if all the vertices in A were identified to a single vertex a.

$$\mathscr{C}(A \leftrightarrow Z) = \sum_{a\in A} \mathscr{C}(a \leftrightarrow Z) = \sum_{a\in A} \pi(a)\mathbf{P}[a \to Z]$$

$$= \frac{\sum_{a\in A}\sum_{u\sim a} i(a,u)}{\phi_A - \phi_Z} = \frac{\text{Strength}(i)}{\phi_A - \phi_Z}.$$

So that gives then that

$$\phi_A - \phi_Z = \frac{\text{Strength}(i)}{\mathscr{C}(A \leftrightarrow Z)} = \text{Strength}(i)\mathscr{R}(A \leftrightarrow Z).$$ □

Remark 3.4. *Consider an electrical network (G, c, A, Z), where $A = \{a\}$ and consider the associated Markov chain X with the pair (π, P).*
From definition 3.8, we see that $\mathbf{P}[a \to Z]^{-1} = \pi(a)\mathscr{C}(a \leftrightarrow Z)$.
Now consider the number of visits to a before hitting Z. This is then a geometric random variable with success probability $\mathbf{P}[a \to Z]^{-1}$.
Now let i be a unit current flow and let ϕ be the correspondige voltage with respect to P with boundary conditions $\phi \restriction Z = 0$ and where $\phi \restriction A = \phi(a)$.
We see then by the previous lemma, that

$$\mathbf{E}[\textit{Number of visits to a before hitting } Z] = \mathbf{P}[a \to Z]^{-1} = \phi(a)\pi(a).$$

3.1.2 Green function

An interesting object with regard to electrical networks and Markov chains, is the so-called Green function. This function is defined as follows:

Definition 3.9 (Green function). *Let (G, c, A, Z) be an electrical network with $G = (V, E)$, where $A = \{a\}$ and consider the associated Markov chain $X = (X_n)_{n\in\mathbf{N}}$ with the pair (π, P).*
Let $\mathcal{G}^{\tau_Z}(a, u)$ be the expected number of visits to u strictly before hitting Z by a random walk started at a, that is:

$$\mathcal{G}^{\tau_Z}(a,u) := \mathbf{E}_a(\mathcal{C}_u(\tau_z)), \textit{ where } \mathcal{C}_u(\tau_z) := \sum_{i=0}^{\tau_z - 1} \mathbb{1}_{\{X_i = u\}}.$$

The function $\mathcal{G}^{\tau_Z}(\cdot,\cdot)$ is called the Green function.

Note that the variable $\mathcal{C}_a(\tau_Z)$ is a geometric random variable if the random walk starts at a, with p being the success probability, where $p := \mathbf{P}_a(\tau_Z <$

$\tau_a^+) = \mathbf{P}[a \to Z]$.
Now recall that $\mathcal{G}^{\tau_Z}(a,a) = \mathbf{E}_a(\mathcal{C}_a(\tau_Z))$, hence

$$\mathcal{G}^{\tau_Z}(a,a) = \mathbf{P}[a \to Z]^{-1} = \pi(a)\mathscr{R}(a \leftrightarrow Z).$$

Because we are considering reversible Markov chains, we have the following relation for the Green function:

Lemma 3.5. *Let (G, c, A, Z) be an electrical network with $G = (V, E)$ and consider the associated Markov chain $X = (X_n)_{n\in\mathbf{N}}$ with the pair (π, P).*
We have then for all $u, v \in V$:

$$\pi(u)\mathcal{G}^{\tau_Z}(u,v) = \pi(v)\mathcal{G}^{\tau_Z}(v,u).$$

Proof Consider the associated Markov chain $X = (X_n)_{n\in\mathbf{N}}$ with the pair (π, P).
We have for all $u, v \in V$:

$$\mathcal{G}^{\tau_Z}(u,v) = \mathbf{E}_u(\mathcal{C}_v(\tau_Z)) = \mathbf{E}_u(\sum_{i=0}^{\tau_Z-1} \mathbb{1}_{\{X_i=v\}}).$$

Now use that we have the following:

$$\mathbf{P}_{x_0}\Big(X_1 = x_1, X_2 = x_2, .., X_{n-1} = x_{n-1}, X_n = x_n\Big) = p_{x_0x_1}p_{x_1x_2}...p_{x_{n-1}x_n} = \prod_{i=0}^{n-1} p_{x_ix_{i+1}}.$$

$$\mathbf{P}_{x_n}\Big(X_1 = x_{n-1}, X_2 = x_{n-2}, .., X_{n-1} = x_1, X_n = x_0\Big) = p_{x_nx_{n-1}}p_{x_{n-1}x_{n-2}}...p_{x_1x_0} = \prod_{i=0}^{n-1} p_{x_{n-i}x_{n-i-1}}.$$

By reversibility:

$$\prod_{i=0}^{n-1} p_{x_ix_{i+1}} = \prod_{i=0}^{n-1}\Big(p_{x_{i+1}x_i}\frac{\pi(x_{i+1})}{\pi(x_i)}\Big) = \frac{\pi(x_n)}{\pi(x_0)}\prod_{i=0}^{n-1} p_{x_{i+1}x_i} = \frac{\pi(x_n)}{\pi(x_0)}\prod_{i=0}^{n-1} p_{x_{n-i}x_{n-i-1}}.$$

So that means that

$$\pi(x_0)\prod_{i=0}^{n-1} p_{x_ix_{i+1}} = \pi(x_n)\prod_{i=0}^{n-1} p_{x_{n-i}x_{n-i-1}}.$$

For all $i \geq 1$:

$$\pi(u)\mathbf{P}_u(X_i = v, i < \tau_Z) = \sum_{x_1,...,x_{i-1}\in(V\setminus Z)} \pi(u)\mathbf{P}_u(X_1 = x_1, ..., X_{i-1} = x_{i-1}, X_i = v)$$

$$= \sum_{x_1,...,x_{i-1} \in (V \setminus Z)} \pi(v) \mathbf{P}_v(X_1 = x_{i-1}, ..., X_{i-1} = x_1, X_i = u) = \pi(v)\mathbf{P}_v(X_i = u, i < \tau_Z).$$

We see then that

$$\pi(u)\mathbf{E}_u\Big(\sum_{i=0}^{\tau_Z - 1} \mathbb{1}_{\{X_i = v\}}\Big) = \pi(u)\mathbf{E}_u\Big(\sum_{i=0}^{\infty} \mathbb{1}_{\{X_i = v\}} \mathbb{1}_{\{i < \tau_Z\}}\Big) = \pi(u)\sum_{i=0}^{\infty} \mathbf{P}_u(X_i = v, i < \tau_Z)$$

$$= \sum_{i=0}^{\infty} \pi(v)\mathbf{P}_v(X_i = u, i < \tau_Z) = \pi(v)\mathbf{E}_v\Big(\sum_{i=0}^{\infty} \mathbb{1}_{\{X_i = u\}} \mathbb{1}_{\{i < \tau_Z\}}\Big) = \pi(v)\mathbf{E}_v\Big(\sum_{i=0}^{\tau_Z - 1} \mathbb{1}_{\{X_i = u\}}\Big).$$

Hence, we get that for all $u, v \in V$:

$$\pi(u)\mathcal{G}^{\tau_Z}(u, v) = \pi(v)\mathcal{G}^{\tau_Z}(v, u).$$ □

This previous lemma has as consequence that we can interpret the Green function as a harmonic function. This directly gives a relation between the voltage and the Green function for certain boundary conditions. Correspondence follows from theory stated in chapter 2 with regard to the uniqueness and existence of a harmonic function.

Lemma 3.6 (Green function as voltage). *Let (G, c, A, Z) be an electrical network with $G = (V, E)$, where $A = \{a\}$ and consider the associated Markov chain $X = (X_n)_{n \in \mathbf{N}}$ with the pair (π, P) Now let i be a unit current flow and let ϕ be the correspondige voltage with respect to P with boundary conditions $\phi \restriction Z = 0$ and where $\phi \restriction A = \phi(a)$. Then*

$$\phi(u) = \frac{\mathcal{G}^{\tau_Z}(a, u)}{\pi(u)} \text{ for all } u \in V.$$

Proof

For all $u \in V \setminus \{A \cup Z\}$: $\mathcal{G}^{\tau_Z}(u, a) = \sum_{v: v \sim u} \mathbf{P}_u\Big(X_1 = v\Big)\mathcal{G}^{\tau_Z}(v, a) = \sum_{v: v \sim u} p_{uv}\mathcal{G}^{\tau_Z}(v, a).$

Hence $\mathcal{G}^{\tau_Z}(\cdot, \cdot)$ is harmonic at $V \setminus \{A \cup Z\}$ with boundary conditions $\mathcal{G}^{\tau_Z}(z, a) = 0$ and $\mathcal{G}^{\tau_Z}(a, a) = \pi(a)\phi(a)$.
Now observe that $\frac{\mathcal{G}^{\tau_Z}(u,a)}{\pi(a)} = \phi(u)$ for all $u \in V$ by the uniqueness principle.
We have then by lemma 3.5 : $\frac{\mathcal{G}^{\tau_Z}(a,u)}{\pi(u)} = \frac{\mathcal{G}^{\tau_Z}(u,a)}{\pi(a)}$ for all $u \in V$.
Hence: $\phi(u) = \frac{\mathcal{G}^{\tau_Z}(a,u)}{\pi(u)}$ for all $u \in V$. □

By using the Green function, we can also look at the number of transitions over edges. This object is defined as follows:

Definition 3.10. *Let (G, c, A, Z) be an electrical network with $G = (V, E)$ and consider the associated Markov chain $X = (X_n)_{n\in\mathbf{N}}$ with the pair (π, P). Define $S^{\tau_Z}(u, v)$ as the number of transitions from u to v, strictly before hitting Z, by*

$$S^{\tau_Z}(u,v) := \sum_{i=0}^{\tau_Z-1} \mathbb{1}_{\{X_i=u, X_{i+1}=v\}}, \text{ for all } u, v \in V.$$

The abovely defined function can be interpreted as a unit current flow by using the following proposition:

Proposition 3.4. *Let (G, c, A, Z) be an electrical network with $G = (V, E)$. where $A = \{a\}$ and consider the associated Markov chain $X = (X_n)_{n\in\mathbf{N}}$ with the pair (π, P). Now let i be a unit current flow and let ϕ be the correspondige voltage with respect to P with boundary conditions $\phi \restriction Z = 0$ and where $\phi \restriction A = \phi(a)$. Then for all $u, v \in V$:*

$$i(u,v) = \mathbf{E}_a(S^{\tau_Z}(u,v) - S^{\tau_Z}(v,u)).$$

Proof For all $u, v \in V$:

$$\mathbf{E}_a(S^{\tau_Z}(u,v)) = \mathbf{E}_a(\sum_{i=0}^{\tau_Z-1} \mathbb{1}_{\{X_i=u, X_{i+1}=v\}}) = \sum_{i=0}^{\tau_Z-1} \mathbf{P}(X_i = u, X_{i+1} = v)$$

$$= \sum_{i=0}^{\tau_Z-1} \mathbf{P}_a(X_i = u) p_{uv} = \mathbf{E}_a(\sum_{i=0}^{\tau_Z-1} \mathbb{1}_{\{X_i=u\}}) p_{uv} = \mathcal{G}^{\tau_Z}(a,u) p_{uv}.$$

Hence, we get for all $u, v \in V$:

$$\mathbf{E}_a(S^{\tau_Z}(u,v) - S^{\tau_Z}(v,u)) = \mathbf{E}_a(S^{\tau_Z}(u,v)) - E_a(S^{\tau_Z}(v,u)) = \mathcal{G}^{\tau_Z}(a,u)p_{uv} - \mathcal{G}^{\tau_Z}(a,v)p_{vu}.$$

By using lemma 3.6, we get that

$$\mathbf{E}_a(S^{\tau_Z}(u,v) - S^{\tau_Z}(v,u)) = \phi(u)\pi(u)p_{uv} - \phi(v)\pi(v)p_{vu} = i(u,v). \quad \square$$

We do see from this section that the Green function, the voltage and current flow are related.

3.1.3 Series, parallel and delta-star transformation law

Physicists have specific interest into reduction laws on electrical networks. This section shows the classical transformation laws for electrical networks. These transformation laws can be used for an underlying graph structure which allows multiple edges between two vertices. We will denote such a graph by $G = (V, E, -, +)$ where

$$- : E \to V, e \mapsto e^- \text{ and } + : E \to V, e \mapsto e^+$$

We will first give the series and parallel laws. These are transformations, which are valid under specific equations for the voltage and curent flow of the network for and the network after transformation. We then show the Delta-Star transformation law, which is a transformation following from applying the series and parallel law.
To conclude this section, we will give an example where we show that these laws can be used to reduce a graph to obtain the effective resistance. This is often the goal of applying transformation laws, to obtain effective conductance and resistance.

We start by giving a definition with regard to matching voltages and current flows on different electrical networks. This will be used to mathematically ensure that after transforming, the non-transformed part of a electrical network still has the same voltage and current flow.

Definition 3.11. *Consider two electrical networks $(G, c, A, Z), (G', c, A, Z)$ with $G = (V, E, -, +), G' = (V', E', -, +)$ with a voltage ϕ, ϕ',respectively, with boundary $A \cup Z$ and corresponding current flow i, i',respectively.*
The voltages ϕ and ϕ' match if $\phi(v) = \phi'(v)$ for all $v \in (V \cap V')$.
The current flows i and i' match if $i(e) = i'(e)$ for all $e \in (E \cap E')$.

The first transformation law which we will prove is the **series law**. This law is used to reduce electrical components which are in series.

Lemma 3.7 (Series law). *Consider an electrical network (G, c, A, Z) with $G = (V, E, -, +)$, where ϕ is the voltage with boundary $A \cup Z$ with corresponding current flow i. Suppose that $w \in V \setminus (A \cup Z)$ is a node of degree 2 with neighbors $u_1, u_2 \in V$.*

Now consider the electrical network (G', c, A, Z) with $G' = (V', E', -, +), V' = V \setminus \{w\}, E' = (E \cup \{\langle u_1, u_2 \rangle\}) \setminus \{\langle u_1, w\rangle \cup \langle w, u_2\rangle\}$, where ϕ' is the voltage with boundary $A \cup Z$ with corresponding current flow i'.
Suppose that the voltages ϕ and ϕ' match and that i and i' match.
Then $i(u_1, u_2) = i(u_1, w) = i(w, u_2)$ if and only if $r(u_1, u_2) = r(u_1, w) + r(w, u_2)$.

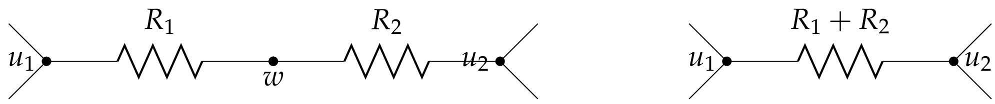

Figure 3.1: Series law with $R_1 = r(u_1, w), R_2 = r(w, u_2)$

Proof By Ohm's law, we have the following equalities:

$$(1) : \phi(u_1) - \phi(w) = i(u_1, w) r(u_1, w).$$

$$(2) : \phi(w) - \phi(u_2) = i(w, u_2) r(w, u_2).$$

$$(3) : \phi'(u_1) - \phi'(u_2) = i'(u_1, u_2) r(u_1, u_2).$$

By Kirchoff's node law, we get that

$$0 = d^* i(w) = i(u_1, w) - i(w, u_2) \text{ hence } i(u_1, w) = i(w, u_2).$$

We see then that combining (1) and (2) with the above observations gives:

$$(4) : \phi(u_1) - \phi(u_2) = i(u_1, w) r(u_1, w) - i(w, u_2) r(w, u_2) = i(u_1, w)(r(u_1, w) + r(w, u_2)).$$

Suppose that the voltages ϕ and ϕ' match and that i and i' match, where we refer to definition 3.11.
Now use (3) to observe that then

$$(5) : \phi(u_1) - \phi(u_2) = i'(u_1, u_2) r(u_1, u_2).$$

We can directly see then that if $i(u_1, w) = i'(u_1, u_2)$ gives that $r(u_1, u_2) = r(u_1, w) + r(w, u_2)$.
Similarly, if $r(u_1, u_2) = r(u_1, w) + r(w, u_2)$ it follows that $i(u_1, w) = i'(u_1, u_2)$.

$\square$

Remark 3.5. *Note that if $i(u_1, w) = i'(u_1, u_2) = i(w, u_2)$, this means that current flow going out of u_1 and into u_2 is unchanged. This is in fact, the transformation assumed by physicist to reduce the precise configuration with the series law for an electrical network.*

The second transformation law which we will prove is the **parallel law**. This law is used to reduce electrical components which are in parallel.

Lemma 3.8 (Parallel law). *Consider an electrical network (G, c, A, Z) with $G = (V, E, -, +)$, where ϕ is the voltage with boundary $A \cup Z$ with corresponding current flow i. Suppose that $e_1 = \langle u_1, u_2 \rangle = e_2$, where $u_1, u_2 \in V$. This means that e_1 and e_2 are in parallel.*
Now consider the electrical network (G', c, A, Z) with $G' = (V, E', -, +), E' = (E \cup \{e\}) \setminus (\{e_1\} \cup \{e_2\})$, with $e = \langle u_1, u_2 \rangle$, where ϕ' is the voltage with boundary $A \cup Z$ with corresponding current flow i'.
Suppose that the voltages ϕ and ϕ' match and that i and i' match.
Then $i(e) = i(e_1) + i(e_2)$ if and only if $c(e) = c(e_1) + c(e_2)$.

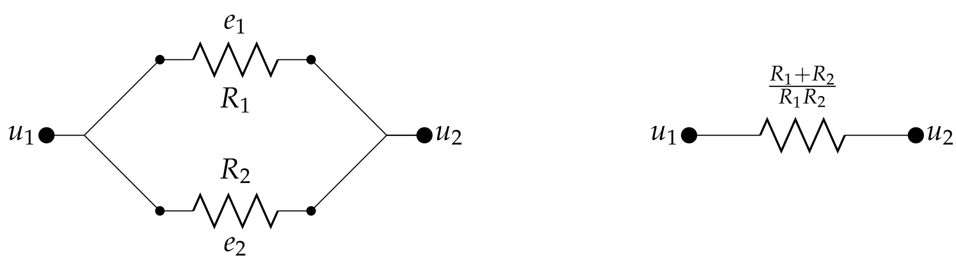

Figure 3.2: Parallel law with $R_1 = \frac{1}{c(e_1)}, R_2 = \frac{1}{c(e_2)}, R = \frac{1}{c(e)}$

Proof By Ohm's law, we have the following equalities:

$$(1) : i(e_1) = c(e_1)d\phi(e_1).$$

$$(2) : i(e_2) = c(e_2)d\phi(e_2).$$

$$(3) : i'(e) = c(e)d\phi'(e).$$

Suppose that the voltages ϕ and ϕ' match and that i and i' match, where we refer to definition 3.11.

That gives then that $d\phi(e_1) = d\phi(e_2) = d\phi'(e)$.
If then $i(e) = i(e_1) + i(e_2)$, it follows then by combining $(1),(2),(3)$ that $c(e) = c(e_1) + c(e_2)$.
If then $c(e) = c(e_1) + c(e_2)$, it follows then by combining $(1),(2),(3)$ that $\frac{i(e)}{d\phi'(e)} = \frac{i(e_1)}{d\phi'(e)} + \frac{i(e_2)}{d\phi'(e)}$, hence $i(e) = i(e_1) + i(e_2)$. □

Remark 3.6. *Note that if $i'(e) = i(e_1) + i(e_2)$, this means that current flow going out of u_1 and into u_2 is unchanged. This is in fact, the transformation assumed by physicist to reduce the precise configuration with the parallel law for an elecitrcal network*

We can now use the parallel law to reduce an electrical network (G, c, A, Z) with $G = (V, E, -, +)$ to an electrical network (G', c, A, Z) with $G = (V, E)$, such that the voltages and current flows of these two systems match and follow the relation defined in lemma 3.8.
We will now prove an third transformation law, the Delta-Star transformation law. Note that this transformation law only uses the series and parallel law. So we assume the change of the current flow for the series and parallel law exactly as these are given. Hence, we obtain a unique resistance for the changed network under these laws. That means that the delta and star configuration are equivalent with a unique one-to-one correspondence.

Lemma 3.9 (Delta-Star transformation). *Consider an electrical network (G, c, A, Z) with $G = (V, E, -, +)$, where ϕ is the voltage with boundary $A \cup Z$ with corresponding current flow i. Suppose that $u_1, u_2, u_3 \in V, w \in V \setminus (A \cup Z)$ with $u_1 \sim w, u_2 \sim w, u_3 \sim w$. We refer to this network as* ***star****.*
Now consider the electrical network (G', c, A, Z) with $G' = (V', E', -, +), V' = V \setminus \{w\}, E' = (E \cup \{\langle u_1, u_2\rangle \cup \langle u_2, u_3\rangle \cup \langle u_3, u_1\rangle\}) \setminus \{\langle u_1, w\rangle \cup \langle u_2, w\rangle \cup \langle u_3, w\rangle\}$, where ϕ' is the voltage with boundary $A \cup Z$ with corresponding current flow i'. We refer to this network as ***delta****.*
It follows then that

$$\textit{for all } i \in \{1, 2, 3\} : c(w, u_i)c(u_{i-1}, u_{i+1}) = \gamma$$

with indices being taken mod 3 *and then*

$$\gamma := \frac{\prod_i c(w,u_i)}{\sum_i c(w,u_i)} = \frac{\sum_i r(u_{i-1},u_{i+1})}{\prod_i r(u_{i-1},u_{i+1})}.$$

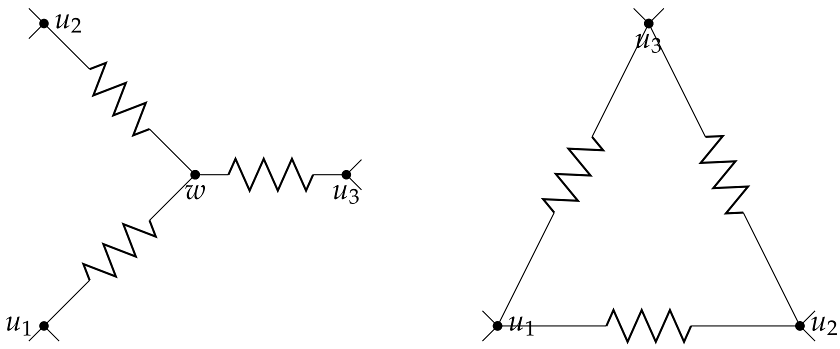

Figure 3.3: Delta-Star transformation

Proof Because we only use the series and parallel laws with the earlier specified assumption on the change of the current flow, we only have to talk about the changed resistances. We will still specificly state about which configuration we talk and which law we apply. Let us specify the star and delta configuration by T and D, respectively. To be specific, we will let $R(\cdot,\cdot)$ and $C(\cdot,\cdot)$ denote the effective resistance and conductance, respectively. Our indices will be taken mod 3.
We see then for T by applying the series law and using the symmetry of the resistances, that

$$(1): r(u_i,w) + r(u_j,w) =_T r(u_i,w) + r(w,u_j) = R(u_i,u_j), i \neq j.$$

Now apply for D the parallel law, combined with the series law. That gives that

$$C(u_i,u_j) = \frac{1}{r(u_i,u_j)} + \frac{1}{r(u_i,u_k)+r(u_k,u_j)}$$

$$= \frac{\sum_i r(u_{i-1},u_{i+1})}{r(u_i,u_j)(r(u_i,u_k)+r(u_k,u_j))}, i \neq j, i \neq k, j \neq k.$$

This means that

$$(2): R(u_i,u_j) = \frac{r(u_i,u_j)r(u_i,u_k) + r(u_i,u_j)r(u_k,u_j)}{\sum_i r(u_{i-1},u_{i+1})}.$$

Now use (1), which gives that

$$2\sum_i r(u_i, w) = \sum_i R(u_i, u_j) = 2\frac{\sum_i r(u_i, u_{i+1})r(u_i, u_{i+1})}{\sum_i r(u_{i-1}, u_{i+1})} := 2K.$$

$$r(u_i, w) = K - (r(u_{i-1}, w) + r(u_{i+1}, w)) = K - R(u_{i-1}, u_{i+1}).$$

Now apply (2), to obtain

$$K - R(u_{i-1}, u_{i+1}) = \frac{r(u_i, u_{i+1})r(u_i, u_{i-1})}{\sum_i r(u_{i-1}, u_{i+1})}.$$

So that gives then that

$$r(u_i, w)r(u_{i-1}, u_{i+1}) = \frac{\prod_i r(u_{i-1}, u_{i+1})}{\sum_i r(u_{i-1}, u_{i+1})} =: \frac{1}{\gamma}.$$

And that automatically gives that

$$c(u_i, w)c(u_{i-1}, u_{i+1}) = \gamma.$$

It follows then directly that

$$\frac{\prod_i c(w, u_i)}{\sum_i c(w, u_i)} = \frac{\gamma^3 \sum_i r(u_{i-1}, u_{i+1})}{\gamma(\sum_i r(u_{i-1}, u_{i+1}))} = \gamma^2 \frac{\prod_i r(u_{i-1}, u_{i+1})}{\sum_i r(u_{i-1}, u_{i+1})} = \gamma. \qquad \square$$

We will now give an **example** of how to use these laws to reduce a graph to a single edge, which then gives the effective resistance of our network. We state that the values are all resistance-values. We are looking for $\mathscr{R}(a \leftrightarrow z)$. To obtain this, we use the series, parallel and Delta-Star transformation laws.

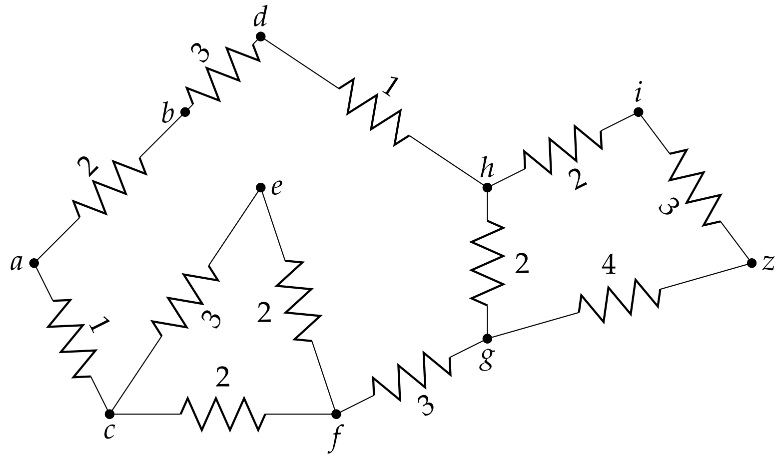

Figure 3.4: Circuit: Example 1.

We see by the series law that

$$r(a,h) = r(a,b) + r(b,d) + r(d,h) = 2 + 3 + 1 = 6.$$

$$r(c,f) = r(c,e) + r(e,f) = 3 + 2 = 5.$$

$$r(h,z) = r(h,i) + r(i,z) = 2 + 3 = 5.$$

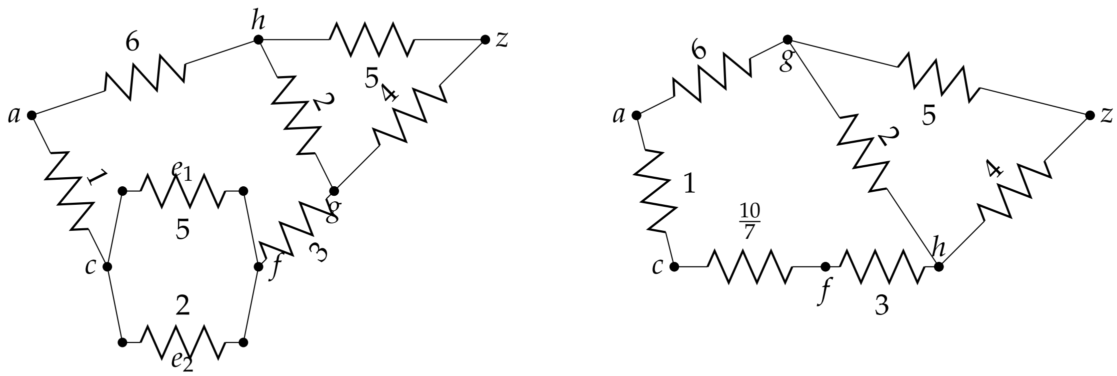

Figure 3.5: First picture: Reduced by using series law; Second picture: Reduced by using parallel law.

We use the parallel law in the first picture:

$$r(c,f) = \frac{1}{\frac{1}{r(e_1)} + \frac{1}{r(e_2)}} = \frac{1}{\frac{1}{5} + \frac{1}{2}} = \frac{10}{7}.$$

We use the series law in the second picture:

$$r(a,g) = r(a,c) + r(c,f) + r(f,g) = 1 + \frac{10}{7} + 3 = \frac{38}{7}.$$

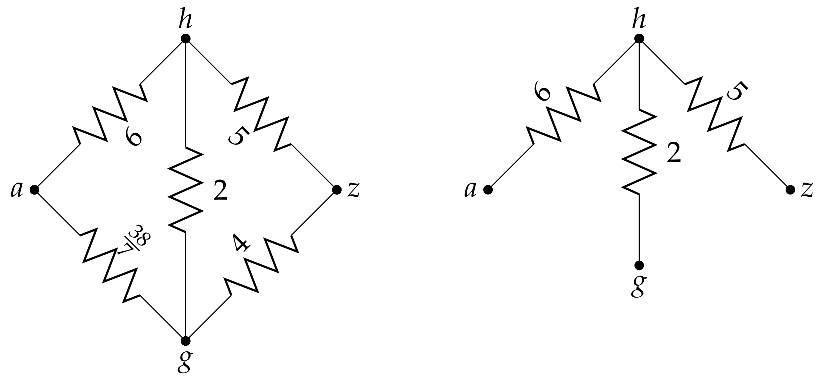

Figure 3.6: First picture: Reduced by using series law; Second picture: Configuration to use Delta-Star law.

Consider now the configuration as in the second picture. We will now use the Delta-Star transformation law. First note that we have

$$c(h,a) = \frac{1}{6}, c(h,z) = \frac{1}{5}, c(h,g) = \frac{1}{2}.$$

That means that

$$\prod_{v=a,g,z} c(h,v) = \frac{1}{60}, \sum_{v=a,g,z} c(h,v) = \frac{52}{60}, \gamma = \frac{\prod_{v=a,g,z} c(g,v)}{\sum_{v=a,g,z} c(h,v)} = \frac{1}{52}.$$

We see then directly by the Delta-Star transformation law, that

$$r(a,g) = \frac{\gamma}{c(h,z)} = \gamma r(h,z) = \frac{5}{52}.$$

$$r(a,z) = \frac{\gamma}{c(h,g)} = \gamma r(h,g) = \frac{2}{52}.$$

$$r(z,g) = \frac{\gamma}{c(h,a)} = \gamma r(h,a) = \frac{6}{52}.$$

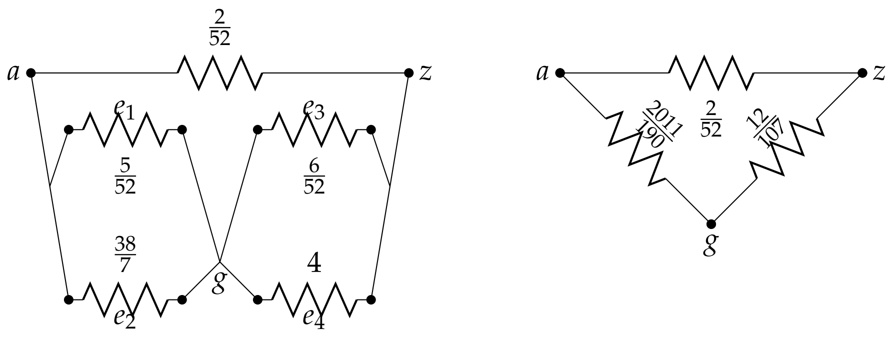

Figure 3.7: First picture: Reduced by using Delta-Star transformation law; Second picture: Reduced by using parallel law.

In the first picture, we use the parallel law:

$$r(a,g) = \frac{1}{\frac{1}{r(e_1)} + \frac{1}{r(e_2)}} = \frac{1}{\frac{52}{5} + \frac{7}{38}} = \frac{2011}{190}.$$

$$r(g,z) = \frac{1}{\frac{1}{r(e_3)} + \frac{1}{r(e_4)}} = \frac{1}{\frac{52}{6} + \frac{1}{4}} = \frac{12}{107}.$$

In the second picture, we use the series law:

$$r(a,z) = r(a,g) + r(g,z) = \frac{2011}{190} + \frac{12}{107} = \frac{217457}{20330}.$$

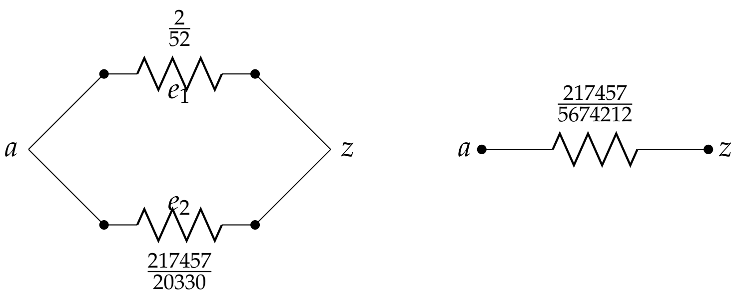

Figure 3.8: First picture: Reduced by using series law; Second picture: Reduced by using parallel law

In the first picture, we use the parallel law:

$$r(a,z) = \frac{1}{\frac{1}{r(e_1)} + \frac{1}{r(e_2)}} = \frac{1}{\frac{52}{2} + \frac{20330}{217457}} = \frac{217457}{5674212}.$$

3.1.4 Energy

An interesting quantity in electrical networks is the **energy**. This section will be devoted to defining energy and showing a relation between the effective resistance and energy of an electrical network.

Let (G, c, A, Z) be an electrical network with $G = (V, E)$.
We refer to definitions 3.1 and 3.2, where we defined $\ell^2_-(E), \ell^2(V)$. Now define the following $\langle \cdot, \cdot \rangle_h$, with $h : E \to \mathbf{R}_+$ with the norm $\| \cdot \|_h$:

Definition 3.12. *We define for $f, g \in \ell^2_-(E)$:*

$$\langle f, g \rangle_h := \langle fh, g \rangle = \langle f, gh \rangle.$$

$$\|f\|_h := \sqrt{\langle f, f \rangle_h}.$$

We define this inner product and norm similarly for $f, g \in \ell^2(V)$.

We are now capable of defining the energy of a flow. That is defined as follows:

Definition 3.13 (Energy of flow). *For $\theta \in \ell^2_-(E)$, define the energy as follows*

$$\mathscr{E}(\theta) := \|\theta\|_r^2$$

with r being the collection of resistances.

Now suppose that we have a network (G, c, A, Z) with a voltage ϕ and associated current flow i. We see then by Ohm's law that $i \cdot r = d\phi$. Hence, we see that

$$\mathscr{E}(i) = \|i\|_r^2 = \langle i, i \rangle_r = \langle i, ir \rangle = \langle i, d\phi \rangle.$$

Now suppose that the voltage ϕ has boundary conditions $\phi \restriction A = \phi_A$ and $\phi \restriction Z = \phi_Z$, where ϕ_A, ϕ_Z are constants. Then by lemma 3.3 we have that

$$\mathscr{E}(i) = \text{Strength}(i)(\phi_A - \phi_Z).$$

Now suppose that i is a unit current flow, then lemma 3.4 gives

$$\mathscr{E}(i) = \phi_A - \phi_Z = \mathscr{R}(A \leftrightarrow Z).$$

3.1.5 Star and cycle spaces

The star and cycle spaces are spaces which are defined in such a way that they can tell whether a flow satisfies Kirchoff's node and cycle law. These spaces will also be used in chapter 5 to proof the Transfer current theorem. Let (G, c, A, Z) be an electrical network with $G = (V, E)$.

We first define the following object, which is a unit current flow from e^- to e^+:

Definition 3.14. *Let $\chi^e : E \to \mathbf{R}$ with*

$$\chi^e = \mathbb{1}_{\{e\}} - \mathbb{1}_{\{-e\}}.$$

We obtain then directly the following:

Lemma 3.10.

$$\textit{For all } \theta \in \ell^2_-(E) : \langle \chi^e, \theta \rangle_r = \theta(e) r(e).$$

Proof First note that

$$\chi^e(f) = \begin{cases} 1 \text{ for } f = e \\ -1 \text{ for } f = -e \\ 0 \text{ otherwise.} \end{cases}$$

$$\text{For all } \theta \in \ell^2_-(E) : \langle \chi^e, \theta \rangle_r = \langle \chi^e, \theta \cdot r \rangle = \frac{1}{2} \sum_{f \in E} \chi^e(f) \cdot \theta(f) r(f)$$

$$= \frac{1}{2} \Big(\chi^e(e)\theta(e)r(e) + \chi^e(-e)\theta(-e)r(-e) \Big) = \frac{1}{2}(\theta(e)r(e) - \theta(-e)r(-e))$$

$$= \theta(e) r(e).$$ □

And we also get the following:

Lemma 3.11.

$$\textit{For all } v \in V : \langle \sum_{e: e^- = v} c(e) \chi^e, \theta \rangle_r = d^* \theta(v).$$

Proof First denote that $\langle \cdot, \cdot \rangle$ is a bi-linear operator, hence we get that

$$\text{For all } v \in V : \langle \sum_{e: e^- = v} c(e) \chi^e, \theta \rangle_r = \sum_{e: e^- = v} \langle c(e) \chi^e, \theta \cdot r \rangle = \sum_{e: e^- = v} \theta(e)$$

$$= d^* \theta(v).$$ □

It follows then immediately that we can write the two Kirchoff laws as follows:
A flow i satisfies *Kirchoff's node law* if and only if

$$\text{For all } v \in V \setminus (A \cup Z) : \langle \sum_{e:e^-=v} c(e)\chi^e, i\rangle_r = 0.$$

A flow i satisfies *Kirchoff's cycle law* if and only if

$$\text{For any oriented cycle } e_1, ..e_n \text{ in } G : \langle \sum_{k=1}^{n} \chi^{e_k}, i\rangle_r = 0.$$

We are now capable of defining the star and cycle space. These are defined as follows:

Definition 3.15 (Star space). *Let $\bigstar \subset \ell^2_-(E)$ be as follows:*

$$\bigstar := span\Big\{ \sum_{e:e^-=v} c(e)\chi^e; v \in V \Big\}.$$

Definition 3.16 (Cycle space). *Let $\Diamond \subset \ell^2_-(E)$ be as follows:*

$$\Diamond := span\Big\{ \sum_{k=1}^{n} \chi^{e_k}; \{e_1, ..., e_n\} \in C \Big\}$$

$$\textit{with } C = \{e_1, e_2, ..e_n \in E | e_1, e_2, ..., e_n \textit{ forms an oriented cycle } \}.$$

An important property of these two spaces is the following:

Lemma 3.12. *The star and cycle space are orthogonal to each other.*

Proof We will show that this is true by contradiction:
Suppose that we have a cycle $v_1 \sim v_2 \sim ... \sim v_n \sim v_{n+1}$ with $v_i \in V$ for all $1 \leq i \leq n$ and with $v_{n+1} = v_1$ and with edges $e_i = \langle v_i, v_{i+1}\rangle$ for all $1 \leq i \leq n$.
Now suppose that there does not exists any $1 \leq i \leq n$ such that $v = v_i$.
Then it directly follows that $\sum_{e:e^-=v} c(e)\chi^e$ and $\sum_{k=1}^{n} \chi^{e_k}$ are orthogonal, because the sum runs over disjoint sets of edges.
Suppose that there exists exactly one $1 \leq i \leq n$ such that $v = v_i = e^-_{i+1} = e^+_i$.
Then the edges $e_1, ..., e_{i-1}, e_{i+2}, ..., e_n$ are disjoint with v.
We get then that

$$\langle \sum_{k=1}^{n} \chi^{e_k}, \sum_{e:e^-=v} c(e)\chi^e\rangle_r = \langle \sum_{k=1}^{n} \chi^{e_k}, \sum_{e:e^-=v} \chi^e\rangle$$

$$= \frac{1}{2} \sum_{f \in E} \Big(\sum_{k=1}^{n} \chi^{e_k}(f) \sum_{e:e^- = v} \chi^{e}(f) \Big).$$

Now note that $v = e_{i+1}^- = e_i^+$, so we get that

$$\frac{1}{2} \sum_{f \in E} \Big(\sum_{k=1}^{n} \chi^{e_k}(f) \sum_{e:e^- = v} \chi^{e}(f) \Big) = \frac{1}{2} \Big(\chi^{e_i}(e_i) \sum_{e:e^- = v} \chi^{e}(e_i) + \chi^{e_{i+1}}(e_{i+1}) \sum_{e:e^- = v} \chi^{e}(e_{i+1}) \Big)$$

$$= \frac{1}{2}(-1+1) = 0.$$

We can immediately see that this also holds if there are more vertices in the cycle. □

This has as consequence the following:

Proposition 3.5.

$$\ell^2_-(E) = \bigstar \oplus \Diamond.$$

Proof We need to show that only the zero vector is orthogonal to both $\bigstar$ and $\Diamond$.
Suppose that $\theta \in \ell^2_-(E)$, which is orthogonal to both $\bigstar$ and $\Diamond$, we will then show that θ is indeed the zero vector.
First note that θ is orthogonal to both $\bigstar$ and $\Diamond$, which means that θ is a current flow for some voltage F with an empty boundary. But that means that $dF = 0$ everywhere. Now note that $\theta = cdF$, where c is the set of conductances. So that means that $\theta = 0$ everywhere. □

3.1.6 Thomson's, Dirichlet's and Rayleigh's principles

Interesting principles for electrical networks are Thomson's, Dirichlet's and Rayleigh's principles. These principles will be defined in this section and will also be proven.

Let (G, c, A, Z) be an electrical network with $G = (V, E)$.
Let us start with two lemma's which relate current flows with the star and cycle space from the previous section.

Lemma 3.13. *Let θ be a flow with the same divergence as i, i.e.$d^*\theta = d^*i$, where i is a current flow. Then*

$$\theta = i + (\theta - i)$$

is the orthogonal decomposition of θ relative to $\ell^2_-(E) = \bigstar \oplus \Diamond$.

Proof First note that $\theta \in \ell^2_-(E)$ and that i being a current flow, means that $i \perp \bigstar$, hence $i \in \Diamond$ (it satisfies Kirchoff's cycle law).
Now note that if $d^*\theta = d^*i$, then $d^*(\theta - i) = 0$, hence it is orthogonal to $\bigstar$ and thereby $\theta - i \in \Diamond$. This proves the orthogonal decomposition. □

Lemma 3.14. *Let i be a current flow, let $\theta \in \ell^2_-(E)$ with the same divergence-free components as i and let r be the collection of resistances.*
The orthogonal projection $P_\bigstar : \ell^2_-(E) \to \bigstar$ gives that

$$i = P_\bigstar \theta.$$

$$\|\theta\|_r^2 = \|i\|_r^2 + \|\theta - i\|_r^2.$$

Proof This immediately follows by the previous lemma. □

Let us now define the set of **unit flows** and **unit potentials**:

Definition 3.17. *Define the set of unit flows as follows:*

$$\mathcal{I}^1_{A,Z} := \Big\{\theta \in \ell^2_-(E) \text{ with } d^*\theta \restriction V \setminus (A \cup Z) = 0, d^*\theta \restriction A = -d^*\theta \restriction Z = 1\Big\}.$$

Define set of unit potentials as follows:

$$\mathcal{U}^1_{A,Z} := \Big\{F \in \ell^2(V) \text{ where } F \text{ is a voltage with } F \restriction A = 1, F \restriction Z = 0\Big\}.$$

We are now ready to prove Thomson's principle:

Theorem 3.2 (Thomson's principle). *Let (G, c, A, Z) be an electrical network with $G = (V, E)$.*
Let $i \in \mathcal{I}^1_{A,Z}$ be a current flow. Then

$$\mathscr{E}(i) = \min_{\theta \in \mathcal{I}^1_{A,Z}} \mathscr{E}(\theta)$$

with the minimum uniquely attained at $\theta = i$.

Proof Let $\delta(e) = \theta(e) - i(e)$ for all $e \in E$. We have then that δ is a flow from A to Z with $\delta \upharpoonright A = \delta \upharpoonright Z = 0$. We get that:

$$\mathscr{E}(\theta) = \langle \theta, \theta \rangle_r = \langle i+\delta, i+\delta \rangle_r = \langle i, i \rangle_r + 2\langle i, \delta \rangle_r + \langle \delta, \delta \rangle_r = \mathscr{E}(i) + 2\langle i, \delta \rangle_r + \mathscr{E}(\delta).$$

Let ϕ the voltage corresponding to i. We get then that

$$\langle i, \delta \rangle_r = \langle d\phi, \delta \rangle = \langle \phi, d^*\delta \rangle = \text{Strength}(\delta)(\phi_A - \phi_Z)$$

and Strength$(\delta) = 0$, hence $\mathscr{E}(\theta) = \mathscr{E}(i) + \mathscr{E}(\delta)$. Note that $\mathscr{E}(\delta) \geq 0$ with equality if and only if $\delta = 0$ everywhere, which means that $\theta = i$. □

Let us define the energy of a potential:

Definition 3.18 (Energy of potential). *For any potential F, define the energy as follows*

$$\mathscr{E}(F) := \|dF\|_c^2 = \sum_{e \in E_{\frac{1}{2}}} c(e)(dF(e))^2.$$

As before, we will prove two lemma's which relate potentials with the star and cycle space from the previous section.

Lemma 3.15. *Let (G, c, A, Z) be an electrical network with $G = (V, E)$.*
Let F be a potential with the same boundary conditions at A and Z as ϕ, where ϕ is a voltage with boundary $A \cup Z$. Then

$$dF = d\phi + d(F - \phi)$$

is the orthogonal decomposition of dF relative to $\ell^2_-(E) = \lozenge \oplus \bigstar$.

Proof First note that $F \in \ell^2(V)$, hence $dF \in \ell^2_-(E)$ and that ϕ is a voltage. That means that $d\phi \in \lozenge$, so $d\phi \perp \bigstar$ (it satisfies Kirchoff's node law). Now note that if $dF = d\phi$, then $d(F - \phi) = 0$, hence it is orthogonal to $\lozenge$ and thereby $d(F - \phi) \in \bigstar$. This proves the orthogonal decomposition. □

Lemma 3.16. *Let ϕ be a voltage, let $F \in \ell^2(V)$ with the same boundary conditions at A and Z as ϕ and let c be the collection of resistances.*
The orthogonal projection $P_\lozenge : \ell^2_-(E) \to \lozenge$ gives that

$$d\phi = P_\lozenge dF.$$

$$\|dF\|_c^2 = \|d\phi\|_c^2 + \|d(F - \phi)\|_c^2.$$

Proof This immediately follows by the previous lemma. □

We are now ready to prove Dirichlet's principle:

Theorem 3.3 (Dirichlet's principle). *Let (G, c, A, Z) be an electrical network with $G = (V, E)$.*
Let $\phi \in \mathcal{U}^1_{A,Z}$ satisfy Kirchoff laws. We get that:

$$\mathscr{E}(\phi) = \min_{F \in \mathcal{U}^1_{A,Z}} \mathscr{E}(F)$$

with the minimum uniquely attained at $F = \phi$.

Proof Let $\delta(u) = F(u) - \phi(u)$ for all $u \in V$. We have then that δ is a potential with $\delta \upharpoonright A = \delta \upharpoonright Z = 0$. We get that:

$$\mathscr{E}(F) = \langle dF, dF \rangle_c = \langle d(\phi+\delta), d(\phi+\delta) \rangle_c = \langle d\phi, d\phi \rangle_c + 2\langle d\phi, d\delta \rangle_c + \langle d\delta, d\delta \rangle_c$$

$$= \mathscr{E}(\phi) + 2\langle d\phi, d\delta \rangle_c + \mathscr{E}(\delta).$$

Let i be the current flow corresponding to ϕ. We get then that

$$\langle d\phi, d\delta \rangle_c = \langle i, d\delta \rangle = \langle d^* i, \delta \rangle = \text{Strength}(i)(\delta \upharpoonright A - \delta \upharpoonright Z)$$

and $\delta \upharpoonright A = \delta \upharpoonright Z = 0$, hence $\mathscr{E}(F) = \mathscr{E}(\phi) + \mathscr{E}(\delta)$. Note that $\mathscr{E}(\delta) \geq 0$ with equality if and only if $\delta = 0$ everywhere, which means that $F = \phi$. □

We can now interpret effective conductance and resistance also in terms of energy as follows:

Corollary 3.1. *Let (G, c, A, Z) be an electrical network with $G = (V, E)$. We have then that*

$$\max_{F \in \mathcal{U}^1_{A,Z}} \frac{1}{\mathscr{E}(F)} = \mathscr{R}(A \leftrightarrow Z) = \min_{\theta \in \mathcal{I}^1_{A,Z}} \mathscr{E}(\theta).$$

$$\min_{F \in \mathcal{U}^1_{A,Z}} \mathscr{E}(F) = \mathscr{C}(A \leftrightarrow Z) = \max_{\theta \in \mathcal{I}^1_{A,Z}} \frac{1}{\mathscr{E}(\theta)}.$$

Proof Now note that for a current flow i and voltage ϕ with boundary conditions $\phi \upharpoonright A = \phi_A, \phi \upharpoonright Z = \phi_Z$, where ϕ_A, ϕ_Z are constants, we get that

$$\mathscr{E}(i) = \langle i, i \rangle_r = \langle i, d\phi \rangle = \langle d^* i, \phi \rangle = \text{Strength}(i)(\phi_A - \phi_Z).$$

$$\mathscr{E}(\phi) = \langle d\phi, d\phi \rangle_c = \langle i, d\phi \rangle = \langle d^* i, \phi \rangle = \text{Strength}(i)(\phi_A - \phi_Z).$$

From lemma 3.4 we know that

$$\mathscr{C}(A \leftrightarrow Z) = \frac{\text{Strength}(i)}{\phi_A - \phi_Z} = \frac{1}{\mathscr{R}(A \leftrightarrow Z)}.$$

Hence, we get

$$\text{If } \phi \in \mathcal{U}^1_{A,Z} : \mathscr{E}(\phi) = \mathscr{C}(A \leftrightarrow Z).$$

$$\text{If } i \in \mathcal{I}^1_{A,Z} : \mathscr{E}(i) = \mathscr{R}(A \leftrightarrow Z).$$

By using Thomson's and Dirichlet's principles, we immediately get our corollary. □

Another interesting principle, is Rayleigh's monotonicity principle:

Theorem 3.4 (Rayleigh's monotonicity principle). *Let (G, c, A, Z) be an electrical network with $G = (V, E)$.*
If c, c' are two sets of conductances on G with $c \leq c'$, then $\mathscr{C}_c(A \leftrightarrow Z) \leq \mathscr{C}_{c'}(A \leftrightarrow Z)$, where $\mathscr{C}_c(A \leftrightarrow Z)$ defines the conductance between A and Z with respect to the electrical network (G, c, A, Z).

Proof Let $i_c, i_{c'}$ be two unit current flows for the electrical networks (G, c, A, Z) and (G, c', A, Z), respectively.
It follows then that $\mathscr{C}_c(A \leftrightarrow Z) = \frac{1}{\mathscr{E}_c(i_c)}$.
Notice that

$$\mathscr{E}_c(i_c) = \|i_c\|^2_{\frac{1}{c}} = \frac{1}{2}\sum_{e \in E} i_c(e)^2 \frac{1}{c(e)} \geq \frac{1}{2}\sum_{e \in E} i_c(e)^2 \frac{1}{c'(e)} = \|i_c\|^2_{\frac{1}{c'}} = \mathscr{E}_{c'}(i_c).$$

By Thomson's Principle, we get that $\mathscr{E}_{c'}(i_c) \geq \mathscr{E}_{c'}(i_{c'})$.
Hence:

$$\mathscr{E}_c(i_c) \geq \mathscr{E}_{c'}(i_c) \geq \mathscr{E}_{c'}(i_{c'}).$$

So that means that $\mathscr{C}_c(A \leftrightarrow Z) \leq \mathscr{C}_{c'}(A \leftrightarrow Z)$. □

As stated in the introduction in chapter 1, a famous result coming from the connection between electrical networks and reversible Markov chains is Pólya's theorem.

Theorem 3.5 (Pólya's theorem). *Simple random walk on a d-dimension lattice has probability one of returning to its starting point for $d = 1, 2$ and has probability bigger than zero of not returning to its starting point for $d > 2$.*

We refer for the original proof of this to [18] and to [9] for a proof which relies on Rayleigh's monotonicity principle and use of the transformation laws.

Chapter 4

Electrical networks and general Markov chains

This chapter will be devoted to linking electrical networks and general Markov chains and to give results similar to the results presented in chapter 3. More background information on this chapter can be found in the recent work from [3].

The approach in this chapter is to extend the classical theory from chapter 3 to the so-called general setting, where reversibility does not have to be guaranteed. We specifically note that the physical interpretation given in this chapter is a theoretical extension and that it is unknown so far if this is physically possible.

This chapter starts with results on general Markov chains in section 4.1. To make the connection between weighted graphs and general Markov chains, we need a different type of concept, which will be explained in section 4.1.2.
In section 4.2 we will state definitions, notation and results on electrical networks. Section 4.2.1 contains content regarding voltages and flows for electrical networks. Section 4.2.2 will be devoted to effective conductance and resistance. Section 4.2.3 holds the information of the physical interpretation of the electrical component for the electrical networks associated to general Markov chains.
As before, we still have transformation laws associated to our electrical

network. Results on this will be given in section 4.2.4. Section 4.2.5 will be devoted to the notion of energy. Section 4.2.6 will conclude this chapter about Thomson's and Dirichlet's principles.

4.1 General Markov chains

This first section will be devoted to results on Markov chains where reversibility is not necessarily guaranteed. Section 4.1.1 holds information about the time-reversed Markov chain and section 4.1.2 will clarify the correspondence between general Markov chains with weighted graphs.

Consider an irreducible Markov chain $X = (X_n)_{n\in\mathbf{N}}$ on a finite state space V in discrete time.
Consider the associated transition matrix P and stationary distribution π.
We assume that our transition matrix P satisfies the following condition:

$$\text{For all } u, v \in V : p_{uv} > 0 \text{ if and only if } p_{vu} > 0.$$

Let G be the graph with vertex-set V and edge-set E, where E is defined by

$$E := \Big\{ \langle u, v \rangle | \text{ for all } u, v \in V \text{ with } p_{uv} > 0 \Big\}.$$

Hence, we get that $E \subseteq V \times V$.
The associated directed graph $G = (V, E)$ is strongly connected and finite, does not have multiple edges and edges occur with both orientations.
Note that whenever (π, P) satisfy the detailed balance condition, see definition 2.3, the Markov chain becomes reversible.

4.1.1 Time-reversed Markov chains

This section shows results about the time-reversed Markov chain.
We start with defining the time-reversed Markov chain by starting from the Markov chain X with the pair (π, P) associated to X.

Proposition 4.1. *Consider a Markov chain $X = (X_n)_{n\in\mathbf{N}}$ with stationary distribution π and transition matrix P and a Markov chain $\hat{X} = (\hat{X}_n)_{n\in\mathbf{N}}$ with* *stationary distribution π and transition matrix $\hat{P}$. First note that π is strictly positive. Now define $\hat{P} := [\hat{p}_{uv}]_{u,v\in V}$ with*

$$\hat{p}_{uv} := p_{vu}\frac{\pi(v)}{\pi(u)} \text{ for all } u,v \in V.$$

We see then that whenever X is an irreducible Markov chain, that $\hat{X}$ is an irreducible Markov chain and whenever P is a transition matrix and π is stationary with respect to P, $\hat{P}$ is a transition matrix and π is stationary with respect to $\hat{P}$.

Proof Suppose that P is a transition matrix and that π is stationary with respect to P. We see then that $\hat{P}$ is a transition matrix:

$$\sum_{v\in V}\hat{p}_{uv} = \frac{1}{\pi(u)}\sum_{v\in V} p_{vu}\pi(v) = 1.$$

Also note that π is stationary with respect to $\hat{P}$:

$$\sum_{v\in V}\pi(v)\hat{p}_{vu} = \sum_{v\in V}\pi(u)p_{uv} = \pi(u).$$

Since X is irreducible, there exists for every pair $u,v \in V$ a path $u = u_0 \sim u_1 \sim ... \sim u_{n-1} \sim u_n = v$ with $p_{u_0u_1}...p_{u_{n-1}u_n} > 0$. This means that

$$\hat{p}_{u_nu_{n-1}}...\hat{p}_{u_1u_0} = p_{u_{n-1}u_n}\frac{\pi(u_{n-1})}{\pi(u_n)}...p_{u_0u_1}\frac{\pi(u_0)}{\pi(u_1)} = p_{u_0u_1}...p_{u_{n-1}u_n}\frac{\pi(u_0)}{\pi(u_n)} > 0.$$

Hence, we see that $\hat{X}$ is also irreducible. □

We assume that P is a transition matrix, that π is stationary with respect to P and that X is an irreducible Markov chain. We refer to $\hat{X}$ as the **time-reversed** Markov chain. This means for finite $i > 0$ that $(X_0, X_1, ..., X_i)$ and $(\hat{X}_i, \hat{X}_{i-1}, ..., \hat{X}_0)$ have the same probability law.

Let us introduce the following notation:

Notation 4.1. $\hat{\mathbf{P}}(\cdot), \hat{\mathbf{E}}(\cdot)$ *respectively refer to the* ***probability measure*** *and* ***expectation*** *of the space in which the time-reversed Markov chain $\hat{X}$ is defined.*
For all $u \in V$:

$$\hat{\mathbf{P}}_u(\cdot) := \hat{\mathbf{P}}(\cdot|\hat{X}_0 = u)$$

$$\hat{\mathbf{E}}_u(\cdot) := \hat{\mathbf{E}}(\cdot|\hat{X}_0 = u).$$

Let us now make the following remark on the correspondence between P and $\hat{P}$:

Remark 4.1. *Whenever $\hat{P} = P$ we see that $\pi(u)p_{uv} = \pi(v)p_{vu}$ for all $u, v \in V$. So that means that the associated Markov chain is reversible.*

4.1.2 Unique correspondence

This section shows the correspondence between weighted graphs and general Markov chains. This correspondence will be used throughout this chapter.
To do so, we need a weight pair corresponding to the weight function of the reversible setting.

Suppose that we have a directed graph $G = (V, E)$ with $E \subseteq V \times V$.

Definition 4.1 (Weight pair). *Define $D : V \times V \to [0, \infty)$ and $\gamma : V \times V \to (0, \infty)$ such that D is symmetric and γ is log anti-symmetric, i.e.:*

$$\text{For all } u, v \in V : D(u,v) = D(v,u) \text{ and } \gamma(u,v) = \frac{1}{\gamma(v,u)}.$$
$$\text{If } u \not\sim v : D(u,v) = 0 \text{ and } \gamma(u,v) = 1.$$

We make the following remark:

Remark 4.2. *We refer to the weights c from section 2.2.1 with*

$$\text{For all } u, v \in V : c(u,v) = D(u,v)\gamma(u,v).$$

We assume that c satisfies the balance of weights, see equation 2.1, for all $v \in V$:

$$\sum_{u \in V} c(u,v) = \sum_{u \in V} c(v,u).$$

So that means that for all $v \in V$:

$$\sum_{u \in V} D(u,v)\gamma(u,v) = \sum_{u \in V} D(v,u)\gamma(v,u) = \sum_{u \in V} D(u,v)\gamma(v,u).$$

We further assume that c is a normalized weight function, in the sense that

$$1 = \sum_{v \in V} \sum_{u \in V} c(u,v) = \sum_{v \in V} \sum_{u \in V} D(u,v)\gamma(u,v).$$

We have a correspondence between a strongly connected directed graph G and the irreducible Markov chain X. This correspondence will be shown in propositions 4.2 and 4.3.

Suppose that G is a strongly connected graph and suppose that our weight pair (D, γ) is known.
We have then a corresponding irreducible Markov chain X, with probability measure π and transition matrix P.

Proposition 4.2. *Take the following:*

$$\text{For all } u \in V : \pi(u) = C(u) = \sum_{v \in V} c(u,v) = \sum_{v \in V} D(u,v)\gamma(u,v)$$

$$\text{and } P = [p_{uv}]_{u,v \in V} \text{ with } p_{uv} = \frac{c(u,v)}{C(u)} \text{ for all } u, v \in V.$$

This specifies our associated Markov chain X with the pair (π, P) such that
(1) : P is a transition matrix of X.
(2) : π is a probability measure.
(3): π is stationary with respect to P.

Proof We will prove this in 3 parts.
(1): First note that our graph is strongly connected. Also note that for all $u \in V$:

$$\sum_{v \in V} p_{uv} = \frac{1}{C(u)} \sum_{v \in V} c(u,v) = 1.$$

Hence, P is a transition matrix.
(2): Note that:

$$\sum_{u \in V} \pi(u) = \sum_{u \in V} \sum_{v \in V} c(u,v) = 1.$$

Also note that $\pi(u) > 0$ for all $u \in V$, cause our graph is connected.
(3): Note that for all $u \in V$:

$$\pi(u) = \sum_{v \in V} \pi(u) p_{uv} = \sum_{v \in V} \pi(u) \frac{c(u,v)}{C(u)} = \frac{\pi(u)}{C(u)} \sum_{v \in V} c(u,v) = \sum_{v \in V} c(v,u)$$
$$= \sum_{v \in V} C(v) p_{vu} = \sum_{v \in V} \pi(v) p_{vu}. \qquad \square$$

Suppose now that we have an irreducible Markov chain X with probability measure π and transition matrix P.
We have then a unique corresponding weight pair (D, γ) for the graph G.

Remark 4.3. *First note that the weights c (section 2.2.1) satisfy*

$$c(u,v) = \pi(u)p_{uv} \text{ for all } u, v \in V.$$

This means that we can find D and γ coming from the weighted graph (G, c) by using the following proposition:

Proposition 4.3. *We immediately see that*

$$D(u,v) = \sqrt{c(u,v)c(v,u)} \text{ and } \gamma(u,v) = \sqrt{\frac{c(u,v)}{c(v,u)}} \text{ for all } u, v \in V.$$

Proof For all $u, v \in V$:

$$c(u,v) = D(u,v)\gamma(u,v).$$

$$D(u,v) = \frac{c(u,v)}{\gamma(u,v)} = \frac{c(v,u)}{\gamma(v,u)}.$$

$$\gamma(u,v) = \frac{c(u,v)}{D(u,v)} = \frac{D(u,v)}{c(v,u)}.$$

So that means that

$$D(u,v) = \sqrt{c(u,v)c(v,u)} \text{ and } \gamma(u,v) = \sqrt{\frac{c(u,v)}{c(v,u)}} \text{ for all } u, v \in V. \quad \square$$

We similarly define our weight pair for the time-reversed Markov chain $\hat{X}$ as follows:

Definition 4.2. *Define the weight pair for the time-reversed Markov chain $\hat{X}$ with $\hat{D} : V \times V \to [0, \infty)$ and $\hat{\gamma} : V \times V \to [0, \infty)$, by:*

$$\hat{D}(u,v) := \sqrt{\pi(u)\hat{p}_{uv}\pi(v)\hat{p}_{vu}} = \sqrt{\pi(v)p_{vu}\pi(u)p_{uv}} = D(u,v).$$

$$\hat{\gamma}(u,v) := \sqrt{\frac{\pi(u)\hat{p}_{uv}}{\pi(v)\hat{p}_{vu}}} = \sqrt{\frac{\pi(v)p_{vu}}{\pi(u)p_{uv}}} = \gamma(v,u).$$

Since $\hat{D} = D$, we refer to D instead of $\hat{D}$.
We refer to the weight pair $(D, \hat{\gamma})$ as $\hat{c}$.

Directly note that $\hat{c}(u,v) = D(u,v)\hat{\gamma}(u,v) = D(u,v)\gamma(v,u) = c(v,u)$. That also means that the balance of weights, see equation 2.1, can be written as

$$\sum_{u \in V} c(u,v) = \sum_{u \in V} \hat{c}(u,v).$$

Note that we have found a correspondence between the Markov chain X with associated pair (π, P) and the weighted direct graph (G, c) with $G = (V, E)$ and $c = (D, \gamma)$.
We similarly have a correspondence between the time-reversed Markov chain $\hat{X}$ with associated pair $(\pi, \hat{P})$ and the weighted directed graph $(G, \hat{c})$ with $G = (V, E)$ and $\hat{c} = (D, \hat{\gamma})$.

Remark 4.4. *If $c = \hat{c}$, then we see that $\gamma = \hat{\gamma}$, which means that $\gamma(u,v) = 1$ for all $u, v \in V$. As a direct consequence, we see that (G, c) and $(G, \hat{c})$ are the same graphs with the same weights.*

4.2 Electrical networks (beyond the reversible case)

This section will be devoted to results which connect the electrical networks with general Markov chains.

We will assume that we have a correspondence between the Markov chains X and $\hat{X}$ with associated pairs (π, P) and $(\pi, \hat{P})$ and weighted directed graphs (G, c) and $(G, \hat{c})$ with $G = (V, E)$ where $c = (D, \gamma)$ and $\hat{c} = (D, \hat{\gamma})$, respectively, as in the first section of this chapter.

Let us follow the approach from chapter 3 in the rest of this chapter. At first, we will consider electrical networks, but they will be associated to directed weighted graphs. Then conductance and resistance will be introduced in terms of weighted directed graphs.
We will then give definitions of flows and voltages and we will determine whether they satisfy Kirchoff's laws in section 4.2.1.
Then the effective conductance and resistance will be introduced and results will be given in section 4.2.2.
We will then give a physical interpretation of electrical networks corresponding to these directed weighted graphs, which will be called an elec-

trical component in section 4.2.3. To make the connection with the classical transformation laws, we will first show the physical interpretation for electrical networks and this will be used to show the series, parallel and Delta-Star transformation law in section 4.2.4.
We will conclude this chapter with section 4.2.5 about energy of voltage and flow and this then leads to the ending section, section 4.2.6, concerning Dirichlet and Thomson's principle.

Let us recall notation 3.1 and use the following notation for electrical networks associated to weighted directed graphs:

Notation 4.2. *Suppose that we have a weighted directed graph* (G, c) *with* $G = (V, E)$ *and with* $c = (D, \gamma)$*. Now suppose that* $\emptyset = A, Z \subset V$ *be given such that* $A \cup Z = \emptyset$*. We refer to this as the electrical network* (G, c, A, Z)*.*

We similarly define an electrical network $(G, \hat{c}, A, Z)$.
Whenever $c = \hat{c}$, we see that our corresponding Markov chain is reversible.

Let us define conductance and resistance:

Definition 4.3 (Conductance and resistance). *Define* $c^s : V \times V \to [0, \infty)$ *by* $c^s(u, v) := \frac{1}{2}(c(u, v) + \hat{c}(u, v))$ *for all* $u, v \in V$*. Define* $r^s : V \times V \to (0, \infty]$ *to be the reciprocal of* c^s*, i.e.* $r^s(u, v) := \frac{1}{c^s(u,v)}$ *for all* $u, v \in V$*. We directly note that* c^s *is symmetric, hence* r^s *is also symmetric.*

Suppose that we have the electrical networks (G, c, A, Z) and $(G, \hat{c}, A, Z)$. We refer to $G = (V, E)$ as the underlying circuit, with the vertex-set V referring to the underlying circuit and the edge-set E referring to how the components are connected.
We refer to $c^s(u, v), r^s(u, v)$ as the conductance, respectively resistance, between u and v. We usually interpret A as the battery and Z as the ground.

Note that the Markov chains X and $\hat{X}$ corresponding with (π, P) and $(\pi, \hat{P})$, respectively, and the associated directed weighted graphs (G, c) and $(G, \hat{c})$, respectively, tell us that each edge occurs with both orientations.

We refer to proposition 4.2 and to the balance of weights, see equation 2.1, where we saw that

$$\pi(u) = C(u) = \sum_{u \in V} c(u,v) = \sum_{u \in V} \hat{c}(u,v).$$

We directly see that $\pi(u) = \sum_{u \in V} c^s(u,v)$.

We are interested in the symmetrized Markov chain $X^s = (X^s)_{n \in \mathbf{N}}$, where its transition matrix is defined as follows:
Let us define $P^s := [p^s_{uv}]_{u,v \in V}$ with $p^s(u,v) = \frac{c^s(u,v)}{C(u)}$ for all $u, v \in V$.
Note that $c^s(u,v) = \frac{c(u,v)+\hat{c}(u,v)}{2} = \pi(u)\frac{p_{uv}+\hat{p}_{uv}}{2}$. So this means that $p^s(u,v) = \frac{p_{uv}+\hat{p}_{uv}}{2}$.

We see that π is a stationairy probability measure with respect to P and $\hat{P}$. Hence, π is also stationary with respect to P^s.
Note that π satisfies the the detailed balance condition, see definition 2.3, with respect to P^s:

$$\text{For all } u, v \in V : \pi(u)p^s(u,v) = \frac{\pi(u)p_{uv} + \pi(u)\hat{p}_{uv}}{2}$$

$$= \frac{\pi(v)\hat{p}_{vu} + \pi(v)p_{vu}}{2} = \pi(v)p^s(v,u).$$

This means that X^s with associated pair (π, P^s) is a reversible Markov chain and that the corresponding weighted directed graph (G, c^s) with $G = (V, E)$ can be interpreted as a weighted undirected graph, because c^s is symmetric.
This leads to an electrical network (G, c^s, A, Z), similarly defined as the electrical networks (G, c, A, Z) and $(G, \hat{c}, A, Z)$, where now its corresponding Markov chain is reversible.

4.2.1 Flows and voltages

Central objects in electrical networks are voltages and (current) flows. This section looks at these objects and whether the classical Kirchoff's laws are still satisfied.

Consider the electrical networks $(G, c, A, Z), (G, \hat{c}, A, Z), (G, c^s, A, Z)$ with $G = (V, E)$ and the corresponding Markov chains $X, \hat{X}, X^s$ with associated pairs $(\pi, P), (\pi, \hat{P}), (\pi, P^s)$, respectively.

Definition 4.4 (Voltage). *We refer to definition 3.5, and let $\phi, \hat{\phi}, \phi^s$ be the* ***voltage*** *with boundary $A \cup Z$ which is a potential which is harmonic at $V \setminus (A \cup Z)$ with respect to $P, \hat{P}, P^s$, respectively.*

Let us define the following set of flows:

Definition 4.5 (Flows). *Define the following* ***flows*** *associated to the voltages $\phi, \hat{\phi}, \phi^s$ as defined above, for all $u, v \in V$:*

$$i^{\phi}(u,v) := \phi(u)c(u,v) - \phi(v)c(v,u)$$

$$\hat{i}^{\phi}(u,v) := \phi(u)\hat{c}(u,v) - \phi(v)\hat{c}(v,u)$$

$$i^{\hat{\phi}}(u,v) := \hat{\phi}(u)c(u,v) - \hat{\phi}(v)c(v,u)$$

$$\hat{i}^{\hat{\phi}}(u,v) := \hat{\phi}(u)\hat{c}(u,v) - \hat{\phi}(v)\hat{c}(v,u)$$

$$i_s^{\phi}(u,v) := \phi(u)c^s(u,v) - \phi(v)c^s(v,u) = d\phi(u,v)c^s(u,v)$$

$$i_s^{\hat{\phi}}(u,v) := \hat{\phi}(u)c^s(u,v) - \hat{\phi}(v)c^s(v,u) = d\hat{\phi}(u,v)c^s(u,v).$$

Note that $\hat{i}^{\phi}, i^{\hat{\phi}}$ both satisfy **Kirchoff's node law** with boundary $A \cup Z$, cause they are both divergence-free at $V \setminus (A \cup Z)$, since for all $u \in V \setminus (A \cup Z)$:

$$d^*\hat{i}^{\phi}(u) = \sum_{v:v\sim u} \hat{i}^{\phi}(u,v) = \phi(u) \sum_{v:v\sim u} \hat{c}(u,v) - \sum_{v:v\sim u} \phi(v)\hat{c}(v,u)$$

$$= \phi(u) \sum_{v:v\sim u} \pi(u)\hat{p}_{uv} - \pi(u) \sum_{v:v\sim u} p_{uv}\phi(v) = 0.$$

We used here the correspondence between the weights c and (π, P) and that ϕ is harmonic at $V \setminus (A \cup Z)$ with respect to P.
We similarly see that $0 = d^* i^{\hat{\phi}}(u)$ for all $u \in V \setminus (A \cup Z)$.

Now note that that $i_s^{\phi}, i_s^{\hat{\phi}}$ both satisfy **Kirchoff's cycle law**, since for every oriented cycle $e_1, e_2, ..., e_n$ of G:

$$\sum_{i=1}^{n} i_s^{\phi}(e_i) r_s(e_i) = \sum_{i=1}^{n} d\phi(e_i) = 0.$$

This similarly holds for $i_s^{\hat{\phi}}$.

Note that $i_s^{\phi} = \frac{i^{\phi}+\hat{i}^{\phi}}{2}$ and similarly $i_s^{\hat{\phi}} = \frac{i^{\hat{\phi}}+\hat{i}^{\hat{\phi}}}{2}$.

Note that $\phi, \hat{\phi}, \phi^s \in \ell^2(V)$ and $i^{\phi}, \hat{i}^{\phi}, i^{\hat{\phi}}, \hat{i}^{\hat{\phi}}, i_s^{\phi}, i_s^{\hat{\phi}} \in \ell^2_-(E)$. By lemma 3.1, we get:

For all $f \in \ell^2(V)$ for all $\theta \in \ell^2_-(E) : \langle \theta, df \rangle = \langle d^*\theta, f \rangle$.

We now recall lemma 3.2 and see that $\hat{i}^{\phi}$ satisfies:

$$\sum_{a \in A} d^*\hat{i}^{\phi}(a) = -\sum_{z \in Z} d^*\hat{i}^{\phi}(z).$$

This proof is similar to the proof of lemma 3.2. This similarly holds for $i^{\hat{\phi}}$.

Recall definition 3.7 and let us similarly define

$$\text{Strength}(\hat{i}^{\phi}) := \sum_{a \in A} d^*\hat{i}^{\phi}(a).$$

$$\text{Strength}(i^{\hat{\phi}}) := \sum_{a \in A} d^*i^{\hat{\phi}}(a).$$

If $|\text{Strength}(\hat{i}^{\phi})| = 1$, we call $\hat{i}^{\phi}$ an **unit flow**. This is similarly for $i^{\hat{\phi}}$.

Recall lemma 3.3. Suppose that $\phi, \hat{\phi}$ are voltages with respect to $P, \hat{P}$ with boundary conditions $\phi \restriction A = \phi_A, \phi \restriction Z = \phi_Z, \hat{\phi} \restriction A = \hat{\phi}_A, \hat{\phi} \restriction Z = \hat{\phi}_Z$, with $\phi_A, \phi_Z, \hat{\phi}_A, \hat{\phi}_Z$ being constants. Let $\hat{i}^{\phi}, i^{\hat{\phi}}$ be the associated flows for ϕ and $\hat{\phi}$. We see then that:

$$\langle \hat{i}^{\phi}, d\phi \rangle = \text{Strength}(\hat{i}^{\phi})(\phi_A - \phi_Z).$$

$$\langle i^{\hat{\phi}}, d\hat{\phi} \rangle = \text{Strength}(i^{\hat{\phi}})(\hat{\phi}_A - \hat{\phi}_Z).$$

The proof is similar to the proof of lemma 3.3.

These results and definitions for voltages and flows will be used in the subsequent sections.

We now give an extra interpretation for flows, by using an assymetric conductance and a certain operator:

Definition 4.6 (Assymetric conductance). *Define* $c^a : V \times V \to (-\infty, \infty)$ *by* $c^a(u,v) := \frac{c(u,v)-c(v,u)}{2}$ *for all* $u, v \in V$. *Observe that* c^a *is an antisymmetric function.*

Definition 4.7. *Define* $h : \ell^2(V) \to \ell^2_-(E)$ *by* $hf(e) := f(e^-) + f(e^+)$. *Observe that* hf *is an symmetric function.*

We now write the first four flows from definition 4.5 as follows:

Corollary 4.1. *For all* $e \in E$:

$$i^\phi(e) = c^s(e)d\phi(e) + c^a(e)h\phi(e)$$

$$\hat{i}^\phi(e) = c^s(e)d\phi(e) - c^a(e)h\phi(e)$$

$$i^{\hat{\phi}}(e) = c^s(e)d\hat{\phi}(e) + c^a(e)h\hat{\phi}(e)$$

$$\hat{i}^{\hat{\phi}}(e) = c^s(e)d\hat{\phi}(e) - c^a(e)h\hat{\phi}(e)$$

Proof This immediately follows by rearranging the terms. □

Let us first define the hitting times associated to the Markov chains $\hat{X}$ and X^s with associated pairs $(\pi, \hat{P})$ and (π, P^s) similarly as in definition 2.4:

Definition 4.8 (Hitting times). *Given a non-empty set* $W \subseteq V$, *we define the hitting times of the set* W *with respect to* $\hat{X}$ *and* X^s *as follows:*

$$\hat{\tau}_W := \inf\{i \geq 0 : \hat{X}_i \in W\}, \hat{\tau}_W^+ := \inf\{i > 0 : \hat{X}_i \in W\}.$$

$$\tau^s_W := \inf\{i \geq 0 : X^s_i \in W\}, {\tau^s}^+_W := \inf\{i > 0 : X^s_i \in W\}.$$

We use similar notation for the hitting times of singletons as the notation introduced in notation 2.3.

4.2.2 Effective conductance and resistance

Specific interest in electrical networks is devoted to escape probabilities and in specific to effective conductance and resistance. Results on these objects will be given in this section.

Consider the electrical networks $(G, c, A, Z), (G, \hat{c}, A, Z)$ with $G = (V, E)$, with $A = \{a\}$. Consider the corresponding Markov chains X and $\hat{X}$ with the associated pairs $(\pi, P), (\pi, \hat{P})$, respectively.

Recall definition 2.9 where $F(u) = \mathbf{P}_u(\tau_A < \tau_Z)$ for all $u \in V$ and similarly define $\hat{F} : V \to [0,1]$ by $\hat{F}(u) := \hat{\mathbf{P}}_u(\hat{\tau}_A < \hat{\tau}_Z)$.
This is the probability that the random walk starting at u hits A before it hits Z with respect to $P, \hat{P}$, respectively.

Recall from section 3.1.1 that $\mathbf{P}[a \to Z] = \mathbf{P}_a(\tau_Z < \tau_a^+)$ and similarly define $\hat{\mathbf{P}}[a \to Z] := \hat{\mathbf{P}}_a(\hat{\tau}_Z < \hat{\tau}_a^+)$.

We have the following relation between voltages, flows and escape probabilities:

Lemma 4.1. *Suppose that $\phi, \hat{\phi}$ are voltages with respect to $P, \hat{P}$, respectively, with boundary conditions $\phi \upharpoonright A = \phi_A, \phi \upharpoonright Z = \phi_Z, \hat{\phi} \upharpoonright A = \hat{\phi}_A, \hat{\phi} \upharpoonright Z = \hat{\phi}_Z$, with $\phi_A, \phi_Z, \hat{\phi}_A, \hat{\phi}_Z$ being constants and let $\hat{i}^{\phi}, i^{\hat{\phi}}$ be defined as in definition 4.5. Then,*

$$\mathbf{P}[a \to Z] = \frac{d^* \hat{i}^{\phi}(a)}{\pi(a)(\phi_A - \phi_Z)}$$

$$\hat{\mathbf{P}}[a \to Z] = \frac{d^* i^{\hat{\phi}}(a)}{\pi(a)(\hat{\phi}_A - \hat{\phi}_Z)}.$$

Proof First note that $F(u) = \mathbf{P}_u(\tau_A < \tau_Z)$ is harmonic at $V \setminus (A \cup Z)$ with respect to P. Note that $F \upharpoonright A = 1, F \upharpoonright Z = 0$. Also note that ϕ is a harmonic function at $V \setminus (A \cup Z)$ with respect to P. Hence, we can write by the superposition principle:

$$\mathbf{P}_u(\tau_A < \tau_Z) = \frac{\phi(u) - \phi_Z}{\phi_A - \phi_Z} \text{ for all } u \in V.$$

That gives then that

$$\mathbf{P}[a \to Z] = \sum_{v:v\sim a} p_{av}(1 - \mathbf{P}_v(\tau_a < \tau_Z)) = \sum_{v:v\sim a} p_{av}(\frac{\phi_A - \phi(v)}{\phi_A - \phi_Z})$$

$$= \frac{1}{\pi(a)} \frac{1}{\phi_A - \phi_Z} \Big(\phi_A \sum_{v:v\sim a} \pi(a) p_{av} - \pi(a) \sum_{v:v\sim a} p_{av}\phi(v) \Big)$$

$$= \frac{1}{\pi(a)} \frac{1}{\phi_A - \phi_Z} \Big(\phi_A \pi(a) \sum_{v:v\sim a} \hat{p}_{av} - \pi(a) \sum_{v:v\sim a} \phi(v)\hat{p}_{va})$$

$$= \frac{1}{\pi(a)} \frac{1}{\phi_A - \phi_Z} \sum_{u:u\sim a} \phi(a)\pi(a)\hat{p}_{au} - \phi(u)\pi(u)\hat{p}_{ua}$$

$$= \frac{1}{\pi(a)} \frac{1}{\phi_A - \phi_Z} \sum_{u:u\sim a} \hat{i}^{\phi}(a,u) = \frac{d^*\hat{i}^{\phi}(a)}{\pi(a)(\phi_A - \phi_Z)}.$$

We find by symmetry that

$$\hat{\mathbf{P}}[a \to Z] = \frac{d^* i^{\hat{\phi}}(a)}{\pi(a)(\hat{\phi}_A - \hat{\phi}_Z)}. \qquad \square$$

Consider the electrical networks (G, c, A, Z) and $(G, \hat{c}, A, Z)$ with $G = (V, E)$ corresponding to the Markov chains X and $\hat{X}$ with associated pairs (π, P) and $(\pi, \hat{P})$, respectively.

Recall definition 3.8 where the effective conductance between the sets A and Z with respect to P is as follows:

$$\mathscr{C}(A \leftrightarrow Z) = \sum_{a\in A} \mathscr{C}(a \leftrightarrow Z) = \sum_{a\in A} \pi(a)\mathbf{P}[a \to Z]$$

Similarly define

$$\hat{\mathscr{C}}(A \leftrightarrow Z) := \sum_{a\in A} \hat{\mathscr{C}}(a \leftrightarrow Z) := \sum_{a\in A} \pi(a)\hat{\mathbf{P}}[a \to Z].$$

Also recall from definition 3.8 that the effective resistance between the sets A and Z with respect to P is as follows:

$$\mathscr{R}(A \leftrightarrow Z) = \mathscr{C}(A \leftrightarrow Z)^{-1}$$

Similarly define

$$\hat{\mathscr{R}}(A \leftrightarrow Z) := \hat{\mathscr{C}}(A \leftrightarrow Z)^{-1}.$$

Using the result from lemma 4.1, we are able to prove the following correspondence between effective conductance, the strength of a flow and boundary conditions of a voltage:

Lemma 4.2. *Let (G, c, A, Z) and $(G, \hat{c}, A, Z)$ be electrical networks with $G = (V, E)$. Let $\phi, \hat{\phi}$ be a voltage with boundary conditions $\phi \upharpoonright A = \phi_A, \phi \upharpoonright Z = \phi_Z, \hat{\phi} \upharpoonright A = \hat{\phi}_A, \hat{\phi} \upharpoonright Z = \hat{\phi}_Z$, with $\phi_A, \phi_Z, \hat{\phi}_A, \hat{\phi}_Z$ being constants and let $\hat{i}^{\phi}, i^{\hat{\phi}}$ be defined as in definition 4.5. Then*

$$\phi_A - \phi_Z = \frac{Strength(\hat{i}^{\phi})}{\mathscr{C}(A \leftrightarrow Z)}$$

$$\hat{\phi}_A - \hat{\phi}_Z = \frac{Strength(i^{\hat{\phi}})}{\hat{\mathscr{C}}(A \leftrightarrow Z)}.$$

Proof This proof is similar to the proof of lemma 3.4. $\square$

4.2.3 Electrical component

We show in this section what type of physical interpretation for the electrical component can be used.

Consider the electrical networks (G, c, A, Z) and $(G, \hat{c}, A, Z)$ with $G = (V, E)$ corresponding to the Markov chains X and $\hat{X}$ with associated pairs (π, P) and $(\pi, \hat{P})$, respectively.
Note that we can find back D and $\gamma, \hat{\gamma}$ by using proposition 4.3.

Definition 4.9 (Voltage amplifier). *Define $\lambda : V \times V \to (0, \infty)$ by $\lambda(u, v) := \gamma(u, v)^2$ for all $u, v \in V$. Note that λ is log anti-symmetric. We refer to $\lambda(u, v)$ as the voltage amplifier between u and v.*

Consider a voltage ϕ with respect to P with boundary conditions $\phi \restriction A = \phi_A, \phi \restriction Z = \phi_Z$, where ϕ_A, ϕ_Z are constants and we recall $\hat{i}^\phi$ from definition 4.5, that for all $u, v \in V$::

$$\hat{i}^\phi(u, v) = \hat{c}(u, v)\phi(u) - \hat{c}(v, u)\phi(v) = D(u, v)\big(\gamma(v, u)\phi(u) - \gamma(u, v)\phi(v)\big).$$

We now note that for all $u, v \in V$:

$$\frac{2c^s(u, v)}{1 + \lambda(v, u)} = D(u, v)\frac{\gamma(u, v) + \gamma(v, u)}{1 + \lambda(v, u)} = \frac{D(u, v)}{\gamma(v, u)}\frac{\gamma(u, v) + \gamma(v, u)}{\gamma(u, v) + \gamma(v, u)} = D(u, v)\gamma(u, v).$$

By rewriting, we obtain the following for all $u, v \in V$:

$$\phi(v) = (\phi(u) - \hat{i}^\phi(u, v)\frac{r^s(u, v)}{2})\lambda(v, u) - \hat{i}^\phi(u, v)\frac{r^s(u, v)}{2}. \tag{4.1}$$

Equivalently, for all $u, v \in V$:

$$\hat{i}^\phi(u, v) = \frac{2c^s(u, v)}{1 + \lambda(v, u)}(\lambda(v, u)\phi(u) - \phi(v)). \tag{4.2}$$

Let us denote the pair of a resistance r and voltage amplifier λ by (r, λ), we call this an **electrical component**.
We find then for every edge $\langle u, v\rangle$ an associated electrical component $\big(r^s(u, v), \lambda(v, u)\big)$.

We interpret equation 4.1 as follows:
Consider the flow $\hat{i}^\phi(u, v)$, which goes from u to v. The dropped potential

is then given by the multiplication of the flow $\hat{\imath}^{\phi}(u,v)$ with the resistor $\frac{r^s(u,v)}{2}$. The total drop in potential before the voltage amplifier $\lambda(v,u)$ is then given by $\phi(u) - \hat{\imath}^{\phi}(u,v)\frac{r^s(u,v)}{2}$, which then gets multiplied with the voltage amplifier $\lambda(v,u)$.
The dropped potential after the voltage amplifier is then given by the multiplication of $\hat{\imath}^{\phi}(u,v)$ times the resistor $\frac{r^s(u,v)}{2}$. That matches then the voltage at v, $\phi(v)$.
Hence, we end up with equation 4.1.

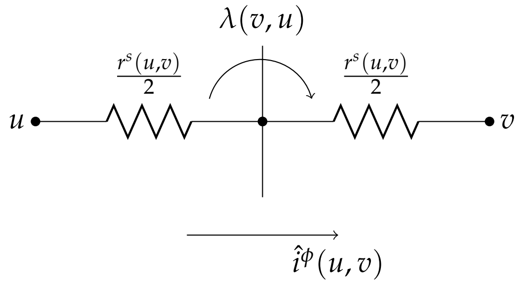

Figure 4.1: Electrical component

Now consider two alternative components, where there is either only a resistor before, respectively after, the voltage amplifier. Note that that directly means that there is only a contribution coming from $\hat{\imath}^{\phi}(u,v)$ before, respectively after, the voltage amplifier. We refer to these two alternative components as the **primer**, respectively **secunder**.
These components are illustrated by the following figures:

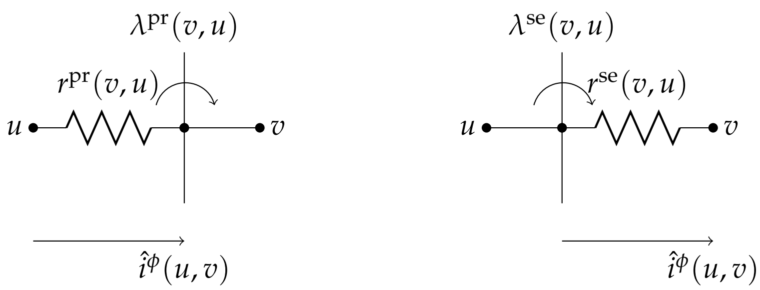

Figure 4.2: Primer and secunder unit

Following the interpretation of the electrical component $\big(r^s(u,v), \lambda(v,u)\big)$ does lead to the following interpretation for the primer:
The dropped potential is given by the multiplication of $\hat{i}^\phi(u,v)$ and $r^{\text{pr}}(v,u)$. Note that there is no drop in potential after the voltage amplifier $\lambda^{\text{pr}}(v,u)$. So the total drop in potential is given by $\big(\phi(u) - \hat{i}^\phi(u,v)r^{\text{pr}}(v,u)\big)\lambda^{\text{pr}}(v,u)$. So we have for the primer that

$$\phi(v) = \Big(\phi(u) - \hat{i}^\phi(u,v)r^{\text{pr}}(v,u)\Big)\lambda(v,u)^{\text{pr}}.$$

Let us interpret the secunder is as follows:
The drop in potential before the voltage amplifier is directly $\phi(u)\lambda(v,u)^{\text{se}}$. The drop after the voltage amplifier is then given by the multiplication of $\hat{i}^\phi(u,v)$ and $r^{\text{se}}(v,u)$. So the total drop in potential is given by $\phi(u)\lambda(v,u)^{\text{se}} - \hat{i}^\phi(u,v)r^{\text{se}}(v,u)$.
So we have for the secunder that

$$\phi(v) = \phi(u)\lambda(v,u)^{\text{se}} - \hat{i}^\phi(u,v)r^{\text{se}}(v,u).$$

They have to be equivalent to (1), hence we see that

$$\lambda(v,u)^{\text{pr}} = \lambda(v,u)^{\text{se}} = \lambda(v,u). \tag{4.3}$$

$$r(v,u)^{\text{pr}} = r^s(u,v)\frac{\lambda(v,u)+1}{2\lambda(v,u)} \text{ and } r(v,u)^{\text{se}} = r^s(u,v)\frac{\lambda(v,u)+1}{2}. \tag{4.4}$$

4.2.4 Series, parallel and Delta-Star transformation law

The following section will hold results on transformation laws for electrical networks. These results can be used for an underlying graph structure which allows multiple edges between two vertices. For more information on this, we refer to section 3.1.3.

Consider an electrical network (G, c, A, Z) with $G = (V, E, -, +)$. We present and prove in this section the the series and parallel laws. To prove these transformation laws, we will use the primer and secunder component, as introduced in the previous section. We will also present the Delta-Star transformation law, where proving this relies upon the use of the series and parallel transformation laws.

We start by presenting the **series law**. This law is used to reduce electrical components which are in series.

Lemma 4.3 (Series law). *Let $w \in V \setminus (A \cup Z)$ be a node of degree 2 with neighbors u_1 and u_2.*
Suppose that the edges $\langle u_1, w \rangle$ and $\langle w, u_2 \rangle$ have electrical components (R, λ) and (Q, μ), respectively.
These edges can be replaced by a single edge $\langle u_1, u_2 \rangle$ with electrical component $(R\frac{(\lambda+1)\mu}{\lambda\mu+1} + Q\frac{\mu+1}{\lambda\mu+1}, \lambda\mu)$.

Proof The pictures sketch the steps which are used.

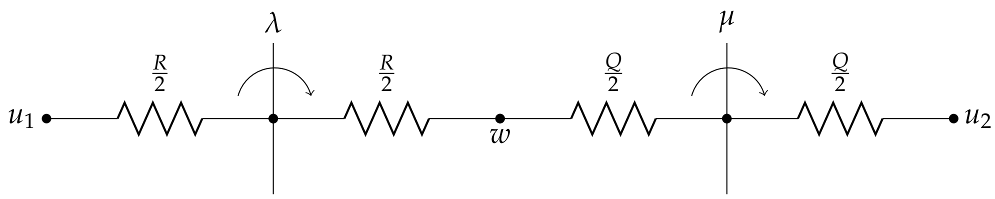

Figure 4.3: Electrical components (R, λ) and (Q, μ)

Replace the electrical components (R, λ) and (Q, μ) by their associated primer and secunder component, respectively, to obtain the electrical components (R^{pr}, λ) and (Q^{se}, μ).
This can be directly transformed to the electrical components $(R^{\text{pr}}, 1)$ and $(Q^{\text{se}}, \lambda\mu)$.

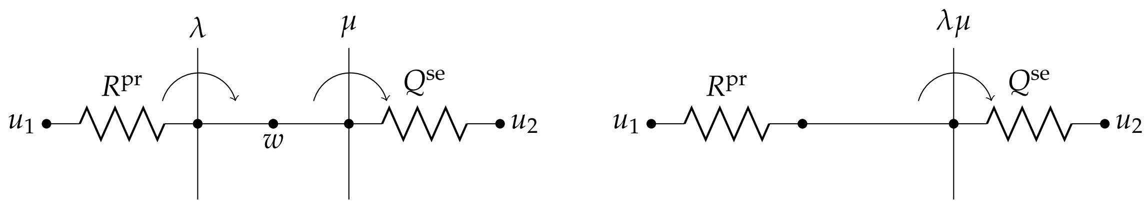

Figure 4.4: Electrical components (R^{pr}, λ) and (Q^{se}, μ); Electrical components $(R^{\text{pr}}, 1)$ and $(Q^{\text{se}}, \lambda\mu)$

Transform the secunder component $(Q^{\text{se}}, \lambda\mu)$ into the primer component $(Q'^{\text{pr}}, \lambda\mu)$ and then transform this into one electrical component $(S^{\text{pr}}, \lambda\mu)$.

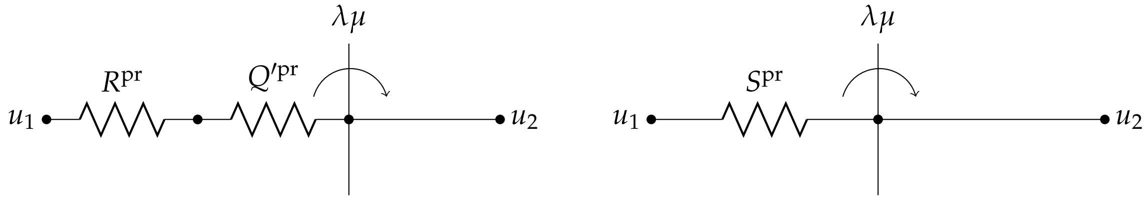

Figure 4.5: Electrical components $(R^{\text{pr}}, 1)$ and $(Q'^{\text{pr}}, \lambda\mu)$; Electrical component $(S^{\text{pr}}, \lambda\mu)$

Convert the component $(S^{\text{pr}}, \lambda\mu)$ back to $(S, \lambda\mu)$.

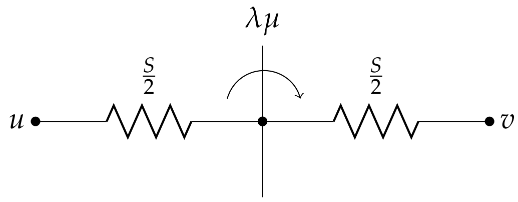

Figure 4.6: Electrical component $(S, \lambda\mu)$

Use the replacements for the voltage amplifier and resistance following from equations 4.3 and 4.4 given in the previous section. This leads to the following set of equations for the steps given in reversed order, starting at the end:

$$S = S^{\text{pr}} \frac{2\lambda\mu}{\lambda\mu + 1} = (R^{\text{pr}} + Q'^{\text{pr}}) \frac{2\lambda\mu}{\lambda\mu + 1}$$

$$= R^{\text{pr}} \frac{2\lambda\mu}{\lambda\mu + 1} + Q^{\text{se}} \frac{2}{\lambda\mu + 1} = R \frac{\lambda + 1}{2\lambda} \frac{2\lambda\mu}{\lambda\mu + 1} + Q \frac{\mu + 1}{2} \frac{2}{\lambda\mu + 1}$$

$$= R \frac{(\lambda + 1)\mu}{\lambda\mu + 1} + Q \frac{\mu + 1}{\lambda\mu + 1}. \qquad \square$$

The second transformation law which we present and prove is the **parallel law**. This law is used to reduce electrical component which are in parallel.

Lemma 4.4 (Parallel law). *Let $u, v \in V, u \neq v$ and let $e_1, e_2 \in E$ be distinct edges such that $e_1 = e_2 = \langle u, v \rangle$.*
Suppose that the edges e_1 and e_2 have electrical components (R, λ) and (Q, μ), respectively.

Replace these edges by a single edge $e = \langle u, v\rangle$ *with electrical component* $(\frac{RQ}{R+Q}, \frac{Q(\mu+1)}{Q(\mu+1)+R(\lambda+1)}\lambda + \frac{R(\lambda+1)}{Q(\mu+1)+R(\lambda+1)}\mu)$.

Proof The pictures sketch the steps which are used.
The goal is to transform the electrical components $(R,\lambda),(Q,\mu)$ into one component (S,ν).

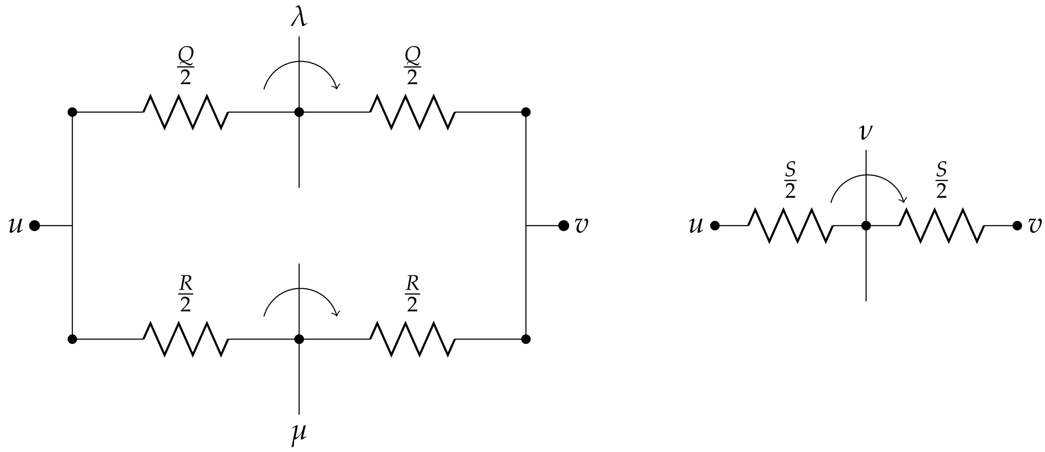

Figure 4.7: Electrical component parallel with (R,λ) and (Q,μ) and equivalent electrical component (S,ν)

We want to replace these components such that the voltage ϕ does not change and that the flow $\hat{i}^\phi$ stays the same. Let e_1, e_2, e be the edge from u to v with the electrical component $(Q,\lambda),(R,\mu),(S,\nu)$, respectively. Hence, we have the following equality for the flow $\hat{i}^\phi$:

$$\hat{i}^\phi(e) = \hat{i}^\phi(e_1) + \hat{i}^\phi(e_2).$$

Note that

$$\hat{i}^\phi(e_1) = \frac{2}{R(1+\lambda)}(\lambda\phi(u) - \phi(v)).$$

$$\hat{i}^\phi(e_2) = \frac{2}{Q(1+\mu)}(\mu\phi(u) - \phi(v)).$$

$$\hat{i}^\phi(e) = \hat{i}^\phi(e_1) + \hat{i}^\phi(e_2) = 2\Big((\frac{\lambda}{R(1+\lambda)} + \frac{\mu}{Q(1+\mu)})\phi(u) - (\frac{1}{R(1+\lambda)} + \frac{1}{Q(1+\mu)})\phi(v)\Big).$$

Note that

$$\hat{i}^\phi(e) = \frac{2}{S(1+\nu)}(\nu\phi(u) - \phi(v)).$$

Hence

$$\frac{\nu}{S(1+\nu)} = (\frac{\lambda}{R(1+\lambda)} + \frac{\mu}{Q(1+\mu)}) = \frac{\lambda Q(1+\mu) + \mu R(1+\lambda)}{R(1+\lambda)Q(1+\mu)}.$$

$$\frac{1}{S(1+\nu)} = (\frac{1}{R(1+\lambda)} + \frac{1}{Q(1+\mu)}) = \frac{Q(1+\mu) + R(1+\lambda)}{R(1+\lambda)Q(1+\mu)}.$$

That means that

$$\nu = \frac{\frac{\nu}{S(1+\nu)}}{\frac{1}{S(1+\nu)}} = \frac{\lambda Q(1+\mu) + \mu R(1+\lambda)}{Q(1+\mu) + R(1+\lambda)} = \frac{Q(\mu+1)}{Q(\mu+1) + R(\lambda+1)}\lambda + \frac{R(\lambda+1)}{Q(\mu+1) + R(\lambda+1)}\mu.$$

So that gives

$$1+\nu = \frac{\lambda Q(1+\mu) + \mu R(1+\lambda) + Q(1+\mu) + R(1+\lambda)}{Q(1+\mu) + R(1+\lambda)}$$

$$= \frac{Q(1+\mu)(1+\lambda) + R(1+\mu)(1+\lambda)}{Q(1+\mu) + R(1+\lambda)} = \frac{(Q+R)(1+\mu)(1+\lambda)}{Q(1+\mu) + R(1+\lambda)}.$$

Hence, we end up with

$$S = \frac{R(1+\lambda)Q(1+\mu)}{Q(1+\mu) + R(1+\lambda)} \cdot \frac{1}{1+\nu} = \frac{R(1+\lambda)Q(1+\mu)}{Q(1+\mu) + R(1+\lambda)} \cdot \frac{Q(1+\mu) + R(1+\lambda)}{(Q+R)(1+\mu)(1+\lambda)}.$$

$$= \frac{RQ}{R+Q}. \qquad \square$$

The third transformation law is the following, the **Delta-Star transformation**:

Lemma 4.5 (Delta-Star transformation). *Let $x, y, z, w \in V$ be distinct vertices with $x \sim w, y \sim w, z \sim w$ in the star $x \sim y \sim z \sim x$ forming a cycle in the delta.*
Let $(Q, \mu), (R, \lambda), (S, \nu)$ be the electrical components in the star configuration for the edges $\langle x, w \rangle, \langle y, w \rangle, \langle z, w \rangle$, respectively.
Let $(S', \nu'), (Q', \mu'), (R', \lambda')$ be the electrical components in the delta configuration for the edges $\langle x, y \rangle, \langle y, z \rangle, \langle z, x \rangle$, respectively.
For simplicity of writing, we give the equalities for the secunder components.

Any star configuration can be transformed into an equivalent delta configuration, where the resistances and voltage amplifiers are given by

$$\lambda S^{se} S'^{se} = K$$

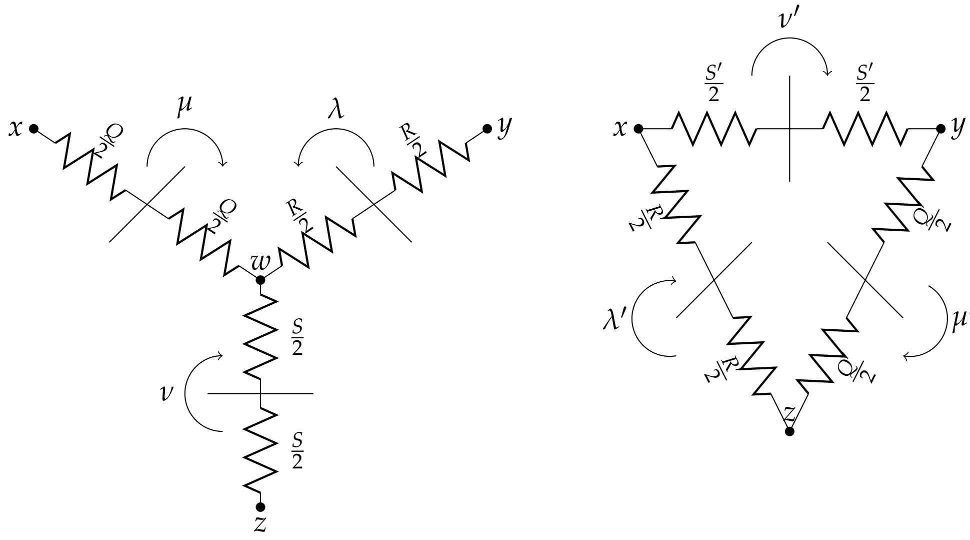

Figure 4.8: Star and Delta

$$\nu Q^{se} Q'^{se} = K$$
$$\mu R^{se} R'^{se} = K$$

and $\lambda' = \frac{\nu}{\mu}, \nu' = \frac{\mu}{\lambda}, \mu' = \frac{\lambda}{\nu}$ *with*

$$K := R^{se} S^{se} + Q^{se} S^{se} + Q^{se} R^{se}$$

Not any delta configuration can be transformed into a star configuration.
A delta configuration can only be transformed into an equivalent star configuration if and only if $\lambda' \cdot \nu' \cdot \mu' = 1$.
Then our resistances and voltage amplifiers are not uniquely determined, however we still have for any $\alpha > 0$ *that*

$$\lambda S^{se} S'^{se} = K$$
$$\nu Q^{se} Q'^{se} = K$$
$$\mu R^{se} R'^{se} = K$$

with $\lambda = \lambda'^{\frac{1}{3}} \mu'^{\frac{2}{3}} \alpha, \nu = \lambda'^{\frac{2}{3}} \nu'^{\frac{1}{3}} \alpha, \mu = \nu'^{\frac{2}{3}} \mu'^{\frac{1}{3}} \alpha$ *and*

$$K := \frac{\alpha^2 R'^{se} Q'^{se} S'^{se}}{\lambda'^{\frac{2}{3}} \nu'^{\frac{1}{3}} Q'^{se} + \nu'^{\frac{2}{3}} \mu'^{\frac{1}{3}} R'^{se} + \lambda'^{\frac{1}{3}} \mu'^{\frac{2}{3}} S'^{se}}$$

Proof We refer for the proof of this to [3].

4.2.5 Energy

Interesting for electrical networks is the **energy** of flows and potentials. Results on this will be given in this section.

Consider the electrical networks $(G, c, A, Z), (G, \hat{c}, A, Z), (G, c^s, A, Z)$ with $G = (V, E)$ with associated Markov chains $X, \hat{X}, X^s$ with pairs $(\pi, P), (\pi, \hat{P}), (\pi, P^s)$, respectively.
We refer to definitions 3.13 and 3.18, where we defined the energy of a flow, respectively potential. We define energy with respect to the symmetrized resistance, respectively symmetrized conductance, as follows:

Definition 4.10 (Energy). *Define the energy of a flow $\theta \in \ell^2_-(E)$ by*

$$\mathscr{E}(\theta) := \|\theta\|^2_{r_s}.$$

Define the energy of a potential $F \in \ell^2(V)$ by

$$\mathscr{E}(F) = \|dF\|^2_{c_s}.$$

We have an equivalence between the energy of voltage and associated flow:

Proposition 4.4. *Suppose that $\phi, \hat{\phi}$ are voltages with respect to P and $\hat{P}$ and consider the associated system of flows $i^{\phi}, \hat{i}^{\phi}, i^{\hat{\phi}}, \hat{i}^{\hat{\phi}}$ and the defined symmetrized flows $i_s^{\phi}, i_s^{\hat{\phi}}$ from definition 4.5. It follows then that*

$$\mathscr{E}(\phi) = \mathscr{E}(i_s^{\phi})$$

$$\mathscr{E}(\hat{\phi}) = \mathscr{E}(i_s^{\hat{\phi}}).$$

Proof We get the following:

$$\mathscr{E}(\phi) = \langle d\phi, d\phi \rangle_{c^s} = \langle d\phi, i_s^{\phi} \rangle = \langle i_s^{\phi}, i_s^{\phi} \rangle_{r_s} = \mathscr{E}(i_s^{\phi}).$$

$$\mathscr{E}(\hat{\phi}) = \langle d\hat{\phi}, d\hat{\phi} \rangle_{c^s} = \langle d\hat{\phi}, i_s^{\hat{\phi}} \rangle = \langle i_s^{\hat{\phi}}, i_s^{\hat{\phi}} \rangle_{r_s} = \mathscr{E}(i_s^{\hat{\phi}}) \qquad \square.$$

Let us define the symmetric current flow as follows:

Definition 4.11 (Symmetric current flow). *Define $i_s^{\phi^s}$ to be the current flow with respect to ϕ^s, where ϕ^s is the voltage with respect to P^s. For all $e \in E$:*

$$i_s^{\phi^s}(e) = c^s(e) d\phi^s(e).$$

We see that $i_s^{\phi^s}$ satisfies Kirchoff's node law with boundary $A \cup Z$, since for all $u \in V \setminus (A \cup Z)$:

$$d^* i_s^{\phi^s}(u) = \sum_{v:v\sim u} c^s(u,v)(\phi^s(u) - \phi^s(v)) = \sum_{v:v\sim u} \pi(u) p_{uv}^s(\phi^s(u) - \phi^s(v))$$

$$= \pi(u)\Big(\phi^s(u) \sum_{v:v\sim u} p_{uv}^s - \sum_{v:v\sim u} p_{uv}^s \phi^s(v)\Big) = \pi(u)(\phi^s(u) - \phi^s(u)) = 0.$$

Also note that $i_s^{\phi^s}$ satisfies Kirchoff's cycle law, since for every oriented cycle $e_1, ..., e_n$ of G:

$$\sum_{i=1}^{n} i_s^{\phi^s}(e_i) r_s(e_i) = \sum_{i=1}^{n} d\phi^s(e_i) = 0.$$

We see that $i_s^{\phi^s}$ is actually the **current flow** from the previous chapter.

Recall lemma 3.3. Suppose that ϕ^s are voltages with respect to P^s, with boundary conditions $\phi^s \restriction A = \phi_A^s, \phi^s \restriction Z = \phi_Z^s$, where ϕ_A^s, ϕ_Z^s are constants. Let $i_s^{\phi^s}$ be the associated flow. We see that

$$\mathscr{E}(\phi^s) = \text{Strength}(i_s^{\phi^s})(\phi_A^s - \phi_Z^s) = \mathscr{E}(i_s^{\phi^s}).$$

Proving this is similar to the proof of lemma 3.3.

A direct consequence of the symmetric current flow is the following:

Lemma 4.6. *Consider the electrical network (G, c^s, A, Z) with $G = (V, E)$, with $A = \{a\}$ and with c^s defined as in definition 4.3. Consider the corresponding Markov chain X^s with associated pair (π, P^s).*
Define $\mathbf{P}^s[a \to Z] := \mathbf{P}_a^s(\tau_Z^s < \tau_a^{s+})$ to be the probability that a random walk starting at a will hit Z before it returns to a with respect to P^s.
Suppose now that ϕ^s is a voltage with respect to P^s with boundary conditions $\phi^s \restriction A = \phi_A^s, \phi^s \restriction Z = \phi_Z^s$, with ϕ_A^s, ϕ_Z^s being constants and let $i_s^{\phi^s}$ be define as in definition 4.11. Then,

$$\mathbf{P}^s[a \to Z] = \frac{d^* i_s^{\phi^s}(a)}{\pi(a)(\phi_A^s - \phi_Z^s)}.$$

Proof Proving this is similar to the proof given in section 3.1.1. □

Let us define the effective conductance with respect to the symmetrized Markov chain X^s with pair (π, P^s) as follows:

Definition 4.12. *Define the effective conductance between the sets A and Z with respect to P^s as follows:*

$$\mathscr{C}^s(A \leftrightarrow Z) := \sum_{a \in A} \mathscr{C}^s(a \leftrightarrow Z) := \sum_{a \in A} \pi(a)\mathbf{P}^s[a \to Z].$$

Define the effective resistance between the sets A and Z with respect to P^s as follows:

$$\mathscr{R}^s(a \leftrightarrow Z) := \mathscr{C}^s(A \leftrightarrow Z)^{-1}.$$

We get the following lemma by combining definition 4.12 with lemma 4.6:

Lemma 4.7. *Let (G, c^s, A, Z) be the electrical network with $G = (V, E)$. Let ϕ^s be a voltage with boundary conditions $\phi^s \upharpoonright A = \phi^s_A, \phi^s \upharpoonright Z = \phi^s_Z$, with ϕ^s_A, ϕ^s_Z being constants and let $i_s^{\phi^s}$ be defined as in definition 4.11. Then*

$$\phi^s_A - \phi^s_Z = \frac{Strength(i_s^{\phi^s})}{\mathscr{C}^s(A \leftrightarrow Z)}.$$

Proof This proof is similar to the proof of lemma 3.4. □

4.2.6 Thomson's and Dirichlet's principles

Just as in chapter 3, we present and prove Thomson's and Dirichlet's principles.

Consider the electrical networks $(G, c, A, Z), (G, \hat{c}, A, Z), (G, c^s, A, Z)$ with $G = (V, E)$ with associated Markov chains $X, \hat{X}, X^s$ with pairs $(\pi, P), (\pi, \hat{P}), (\pi, P^s)$, respectively.
Also consider the voltages $\phi, \hat{\phi}, \phi^s$ with respect to $P, \hat{P}, P^s$, respectively. Suppose that they admit the same boundary conditions $\phi \upharpoonright A = \hat{\phi} \upharpoonright A = \phi^s \upharpoonright A = \phi_A, \phi \upharpoonright Z = \hat{\phi} \upharpoonright Z = \phi^s \upharpoonright Z = \phi_Z$, with ϕ_A, ϕ_Z being constants.
Consider the associated flows $i_s^{\phi}, i_s^{\hat{\phi}}, i_s^{\phi^s}$.
Recall the spaces $\mathcal{I}^1_{A,Z}$ and $\mathcal{U}^1_{A,Z}$ from definition 3.17.

Let us now define a whole class of transition matrices with respect to a probability measure as follows:

Definition 4.13. *Suppose that π is a probability measure. Define $\mathcal{P}$ to be the class of transition matrices P such that π is stationary with respect to P and such* *that ϕ is harmonic with respect to P on $V \setminus (A \cup Z)$.*

Note that $P, \hat{P}, P^s \in \mathcal{P}$ and that we find that $\phi, \hat{\phi}, \phi^s$ are the associated harmonic functions with respect to $P, \hat{P}, P^s$ on $V \setminus (A \cup Z)$, respectively.
We see that our symmetrized voltage with respect to the symmetrized Markov chain minimizes the energy:

Proposition 4.5.

$$\mathscr{E}(\phi^s) = \min_{\overline{P} \in \mathcal{P}} \mathscr{E}(\phi).$$

Its minimum is attained at $\overline{P} = P^s$.

Proof We state that the classical Dirichlet principle tells us that symmetrizing a Markov chain never increases the energy of the voltage. So that means that

$$\mathscr{E}(\phi^s) \leq \mathscr{E}(\phi).$$

Hence, we see that

$$\mathscr{E}(\phi^s) = \min_{\overline{P} \in \mathcal{P}} \mathscr{E}(\phi)$$

with minimum attained at $\overline{P} = P^s$. □

Suppose that $\phi \in \mathcal{U}^1_{A,Z}$ satisfy Kirchoff laws from chapter 3. We have then by the classical Dirichlet principle that

$$\mathscr{E}(\phi) = \min_{F \in \mathcal{U}^1_{A,Z}} \mathscr{E}(F)$$

with the minimum uniquely attained at $F = \phi$.

This leads to the following version of Dirichlet's principle:

Theorem 4.1 (Dirichlet's principle). *Suppose that $\phi \in \mathcal{U}^1_{A,Z}$. By the previous statement and proposition 4.5, we see that*

$$\mathscr{E}(\phi^s) = \min_{\overline{P} \in \mathcal{P}} \mathscr{E}(\phi')$$

where ϕ' is the harmonic function with respect to $\overline{P}$ on $V \setminus (A \cup Z)$.
The minimum is attained at $\overline{P} = P^s$.

$$\mathscr{E}(\phi) = \min_{F \in \mathcal{U}^1_{A,Z}} \mathscr{E}(F)$$

with minimum attained at $F = \phi$.
That means that the minimal energy is attained for a harmonic function ϕ^s with respect to P^s at $V \setminus (A \cup Z)$ with boundary conditions $\phi^s \upharpoonright A = 1, \phi^s \upharpoonright Z = 0$.

Proof This follows directly by the classical Dirichlet principle and by using proposition 4.5. □

We do see that our symmetrized current flow with respect to the symmetrized Markov chain minimizes the energy:

Proposition 4.6.

$$\mathscr{E}(i_s^{\phi^s}) = \min_{\overline{P} \in \mathcal{P}} \mathscr{E}(i_s^{\phi})$$

where i_s^{ϕ} is the associated flow with respect to ϕ, i.e. $i_s^{\phi} = d\phi \cdot c^s$.
Its minimum is attained at $\overline{P} = P^s$.

Proof We state that symmetrizing a Markov chain never increases the energy of a current flow. So that means that

$$\mathscr{E}(i_s^{\phi^s}) \leq \mathscr{E}(i_s^{\phi}).$$

So we get that

$$\mathscr{E}(i_s^{\phi^s}) = \min_{\overline{P} \in \mathcal{P}} \mathscr{E}(i_s^{\phi}).$$

Its minimum is attained at $\overline{P} = P^s$. □

Suppose that $i_s^{\phi} \in \mathcal{I}_{A,Z}^1$ satisfy Kirchoff laws from chapter 3. We have then by the classical Thomson principle that

$$\mathscr{E}(i_s^{\phi}) = \min_{\theta \in \mathcal{I}_{A,Z}^1} \mathscr{E}(\theta)$$

with the minimum uniquely attained at $\theta = i_s^{\phi}$.

This leads to the following version of Thomson's principle:

Theorem 4.2 (Thomson's principle). *Suppose that $i_s^{\phi} \in \mathcal{I}_{A,Z}^1$. By the previous statement and proposition 4.6, we see that*

$$\mathscr{E}(i_s^{\phi^s}) = \min_{\overline{P} \in \mathcal{P}} \mathscr{E}(i_s^{\prime\phi})$$

where $i_s^{'\phi}$ is the associated flow with respect to ϕ, i.e. $i_s^{'\phi} = d\phi \cdot c^s$.
Its minimum is attained at $\overline{P} = P^s$.

$$\mathscr{E}(i_s^{\phi}) = \min_{\theta \in \mathcal{I}^1_{A,Z}} \mathscr{E}(\theta)$$

with the minimum uniquely attained at $\theta = i_s^{\phi}$.
That means that the minimal energy is attained for a flow $i_s^{\phi^s}$ associated to the harmonic function ϕ^s with respect to P^s at $V \setminus (A \cup Z)$ such that $i_s^{\phi^s} \in \mathcal{U}^1_{A,Z}$.

Proof This follows directly by the classical Thomson's principle and by using proposition 4.6. □

Let us state the following results on energy and effective conductance:
Consider the electrical network (G, c^s, A, Z) with $G = (V, E)$.
Let ϕ^s be the voltage with boundary conditions $\phi^s \restriction A = \phi^s_A, \phi^s \restriction Z = \phi^s_Z$, with ϕ^s_A, ϕ^s_Z being constants and let $i_s^{\phi^s}$ be defined as in definition 4.11. We get then that

$$\mathscr{E}(i_s^{\phi^s}) = \langle i_s^{\phi^s}, i_s^{\phi^s} \rangle_{r_s} = \langle i_s^{\phi^s}, d\phi^s \rangle = \langle d^* i_s^{\phi^s}, \phi^s \rangle = \text{Strength}(i_s^{\phi^s})(\phi^s_A - \phi^s_Z).$$

$$\mathscr{E}(\phi^s) = \langle d\phi^s, d\phi^s \rangle_{c^s} = \langle i_s^{\phi^s}, d\phi^s \rangle = \langle d^* i_s^{\phi^s}, \phi^s \rangle = \text{Strength}(i_s^{\phi^s})(\phi^s_A - \phi^s_Z).$$

From lemma 4.7 we know that

$$\mathscr{C}^s(A \leftrightarrow Z) = \frac{\text{Strength}(i_s^{\phi^s})}{\phi^s_A - \phi^s_Z} = \frac{1}{\mathscr{R}^s(A \leftrightarrow Z)}.$$

So we see then that if $\phi^s \in \mathcal{U}^1_{A,Z}$, that $\mathscr{E}(\phi^s) = \mathscr{C}^s(A \leftrightarrow Z)$.
We also see that if $i_s^{\phi^s} \in \mathcal{I}^1_{A,Z}$, that $\mathscr{E}(i_s^{\phi^s}) = \mathscr{R}^s(A \leftrightarrow Z)$.

Chapter 5

Random spanning trees and Transfer current theorem

This chapter links reversible Markov chains with random spanning trees and it links general Markov chains with random rooted spanning trees. We will introduce Wilson's algorithm and determine the probability distribution of picking a random (rooted) spanning tree produced by Wilson's algorithm.
This chapter concludes with the Transfer current theorem, which will be proven by using the connection between electrical networks with random spanning trees. The Transfer current theorem tells us that the probability distribution of seeing certain edges in a random spanning tree is a determinantal point process. Recent results on the Transfer current theorem related to general Markov chains can be found in [2] and [6].

This chapter starts with some basic theory regarding spanning trees and rooted spanning trees. In section 5.1 definitions and notation with regard to spanning trees and rooted spanning trees, will be introduced.
Section 5.2 introduces Wilson's algorithm, more on this can be found in [19] and [20]. The algorithm presented uses the notion of loop-erasure. Section 5.2 also shows the probability law for loop-erasure, where more on this can be found in [16]. By using this result, we obtain the probability law of producing a random rooted spanning tree and random spanning tree by using Wilson's algorithm.
Section 5.3 is devoted to the connection between unit current flows and

observing a certain edge in a random spanning tree.
Section 5.4 concludes this section with the Transfer current theorem from Burton and Pemantle [5]. More background on this can be found in [12] and [14].
More background on this chapter can be found in [15].

Suppose that $G = (V, E)$ is a (strongly) connected graph. We recall from section 2.1, that a graph $H = (W, F)$ is a spanning subgraph if $W = V$ and $F \subseteq E$.
Suppose that G is undirected. If H is connected and has no cycles, we call H a **tree**. If H is a tree and a spanning subgraph of G, we call H a **spanning tree**.
Suppose that G is directed. If H has no cycles and there exists one vertex, such that there is exactly one path from this vertex to any other vertex, then we call this vertex the **root**. We refer to H as a **rooted spanning tree**.

5.1 Spanning trees

This section introduces the basic definitions and notation with regard to spanning trees and rooted spanning trees. These definitions will be used throughout this chapter.

Whenever we refer to the weighted (un)directed graph (G, c) with $G = (V, E)$, we assume that $G = (V, E)$ is an (un)directed (strongly) connected graph.

Let us define the set of spanning trees and the weight of a spanning tree:

Definition 5.1. *Consider a weighted undirected graph (G, c) with $G = (V, E)$. Let $\mathcal{T}^G$ be the set of all spanning trees of G.*

We note that a spanning tree is uniquely determined by its edge-set.

Definition 5.2. *Consider a weighted undirected graph (G, c) with $G = (V, E)$. Define the weight of a spanning tree as*

$$c(T) := \prod_{e \in T} c(e) \text{ for all } T \in \mathcal{T}^G.$$

Let us define the probability to pick a random spanning tree from $\mathcal{T}^G$:

Definition 5.3. *Consider a weighted undirected graph (G, c) with $G = (V, E)$. Let τ be picked proportionally to its weight at random from $\mathcal{T}^G$:*

$$\mathbf{P}(\tau = T) := \frac{c(T)}{Z^G} \text{ for all } T \in \mathcal{T}^G$$

with $Z^G := \sum_{T \in \mathcal{T}^G} c(T)$.

Define the set of rooted spanning trees as follows:

Definition 5.4. *Consider a weighted directed graph (G, c) with $G = (V, E)$. Let $r \in V$. Define Σ_r^G to be the set of rooted spanning trees of G with root r.*
Define

$$\Sigma^G := \bigcup_{r \in V} \Sigma_r^G.$$

We note that a rooted spanning tree is uniquely determined by its edge-set.

Recall from section 2.2.1 that we have a (one-to-one) correspondence between (irreducible) general Markov chains and directed (strongly connected) weighted graphs. Consider the associated Markov chain X with the pair (π, P).
Let $p_e := p_{e^- e^+}$ with $e = \langle e^-, e^+ \rangle$ for all $e \in E$.

Define the weight of a rooted spanning tree and the probability to pick a random rooted spanning tree with given root r from Σ_r^G as follows:

Definition 5.5. *Consider a weighted directed graph (G, c) with $G = (V, E)$ and let $r \in V$ be a fixed vertex.*
Define the weight of a rooted spanning tree as

$$\alpha(A) := \prod_{e \in A} p_e \text{ for all } A \in \Sigma_r^G.$$

Definition 5.6. *Consider a weighted directed graph (G, c) with $G = (V, E)$ and let $r \in V$ be a fixed vertex.*
Let σ be picked proportionally to its weight at random from Σ_r^G:

$$\mathbf{P}(\sigma = A) := \frac{\alpha(A)}{Z_r^G} \text{ for all } A \in \Sigma_r^G$$

with $Z_r^G := \sum_{A \in \Sigma_r^G} \alpha(A)$.

5.2 Wilson's algorithm

This section is devoted to Wilson's algorithm, more on this can be found in [20] and [19]. An important concept used for the presented Wilson's algorithm, is loop-erasure. We show that the probability distribution to pick a random spanning tree and random rooted spanning tree by using Wilson's algorithm corresponds with the probability distribution from definitions 5.3 and 5.6, respectively. This relies upon a theorem with regard to the probability distribution for loop-erasure. More on this can be found in [16].

Consider a weighted directed graph (G, c) with $G = (V, E)$ and consider the associated Markov chain X with the pair (π, P).

Define **loop-erasure** as follows:
Consider two vertices $u, v \in V, u \neq v$. Sample a random walk $\mathcal{W}$ from our Markov chain X starting at u until it hits v, where we write $\mathcal{W} = (u_0 = u, u_1, ..., u_k = v), u_i \in V, 0 \leq i \leq k$, where $k \geq 1$, with $u_i \sim u_{i+1}$ for all $0 \leq i \leq k-1$.
This walk can have self-intersections. We now construct a non-intersecting sub-walk, which we will denote by $LE(\mathcal{W})$, where we have removed all the loops from $\mathcal{W}$.
Define

$$J := \min\Big\{ j \geq 1 : u_j = u_i \text{ for some } 0 \leq i < j \Big\}.$$

If such J exists, then fix I to be the value of i satisfying $0 \leq I < J$ and define $u_I := u_J$.
We now have the loop $(u_I, u_{I+1}, ..., u_J)$. Remove this loop from $\mathcal{W}$ to obtain

$$\mathcal{W}' = (u_0 = u, u_1, ..., u_I, u_{J+1}, ..., u_k = v).$$

We keep repeating this procedure, until such J does not exists. That means that we have removed all the loops. Define this outcome to be $LE(\mathcal{W})$, which then is a non-intersecting sub-walk from u to v.
We can interpret $LE(\mathcal{W})$ as a graph with a vertex-set and edge-set.

Let $r \in V$ be a fixed vertex. We generate random rooted spanning trees with root r from Σ_r^G by using the following approach for Wilson's algorithm.

Wilson's algorithm:
Let $v_1, v_2, ..v_{|V|-1}$ be an arbitrary, but fixed ordering of $V \setminus \{r\}$.
(1) : Set $B_0 = \{r\}, E_0 = \emptyset$.
(2) : Let $i_1 = \min\{k \geq 1 : v_k \in B_0^c\}$, here $i_1 = 1$. Sample a random walk $\mathcal{W}_1$ from our Markov chain X starting at v_{i_1} until it hits B_0. Let $\hat{B}_1, \hat{E}_1$ be the vertex-set, respectively edge-set coming from the loop-erased walk $LE(\mathcal{W}_1)$. Take $B_1 = B_0 \cup \hat{B}_1, E_1 = E_0 \cup \hat{E}_1$.
(3) : Iterate the process, by taking $i_j = \min\{k \geq 1 : v_k \in B_{j-1}^c\}$. Sample a random walk $\mathcal{W}_j$ from our Markov chain X starting at v_{i_j} until it hits B_{j-1}. Let $\hat{B}_j, \hat{E}_j$ be the vertex-set, respectively edge-set coming from the loop-erased walk $LE(\mathcal{W}_j)$. Take $B_j = B_{j-1} \cup \hat{B}_j, E_j = E_{j-1} \cup \hat{E}_j$.
(4) : Stop when $B_N = V$ for some $N \geq 0$ and set $\sigma = E_N$.
σ is then a random rooted spanning tree with root r.
Note that whenever we forget about the root and orientation of the edges, we have a random spanning tree.

Recall that we consider (strongly) connected finite graphs, which means that the associated Markov chains are irreducible. That means that Wilson's algorithm stops with probability 1, so the number of steps of the algorithm, denoted by N, is finite.

Theorem 5.1 (Wilson). *σ is a random rooted spanning tree with root r and*

$$\mathbf{P}^W(\sigma = A) = \frac{\alpha(A)}{Z_r^G}, A \in \Sigma_r^G$$

with $\mathbf{P}^W(\cdot)$ referring to the probability law coming from Wilson's algorithm.

To prove this theorem, we will first prove the following theorem regarding loop-erasure. More can be found in [16].

Theorem 5.2. *Consider a weighted directed graph (G, c) with $G = (V, E)$ and consider the associated Markov chain $X = (X_n)_{n \geq 0}$ with the pair (π, P).*
Sample a random walk $\mathcal{W}$ from our Markov chain X and let $LE(\mathcal{W})$ be the loop-erased walk $\mathcal{W}$ on V.

Let $\emptyset \neq B \subseteq V$. Now let $\gamma_l(B) = (u_0, ..., u_{l-1}, u_l), u_i \in V \setminus B, 0 \leq i \leq l-1, u_l \in B$, such that $\gamma_l(B)$ is a walk with l edges which is stopped when it hits B. We have that

$$\mathbf{P}_{u_0}\Big(LE(\mathcal{W}) = \gamma_l(B)\Big) = \frac{\det[\mathbb{1} - P]_{B^c \setminus \{u_0, ..., u_{l-1}\}}}{\det[\mathbb{1} - P]_{B^c}} \prod_{i=0}^{l-1} p_{u_i u_{i+1}}$$

where $\mathbf{P}_{u_0}\Big(LE(\mathcal{W}) = \cdot\Big)$ refers to the probability law of the loop-erased walk started at u_0.

Proof Define $\gamma_{l-i}(B) := (u_i, ..., u_l), 0 \leq i \leq l$.
Define the events $A_i := \Big\{LE(\mathcal{W}) = \gamma_{l-i}(B)\Big\}$ and $H_i := \Big\{\tau_B < \tau_{u_i}^+\Big\}$.
Note that for all $0 \leq i \leq l, A_i$ and H_i^c are independent when both are started from u_i, cause of the definition of loop-erasure. That means that we can write for all $0 \leq i \leq l$:

$$\mathbf{P}_{u_i}(A_i, H_i^c) = \mathbf{P}_{u_i}(A_i)\mathbf{P}_{u_i}(H_i^c).$$

Hence, we see that for all $0 \leq i \leq l$:

$$\mathbf{P}_{u_i}(A_i) = \mathbf{P}_{u_i}(A_i, H_i^c) + \mathbf{P}_{u_i}(A_i, H_i) = \mathbf{P}_{u_i}(A_i)\mathbf{P}_{u_i}(H_i^c) + \mathbf{P}_{u_i}(A_i, H_i).$$

That means that for all $0 \leq i \leq l$:

$$\mathbf{P}_{u_i}(A_i) = \frac{\mathbf{P}_{u_i}(A_i, H_i)}{1 - \mathbf{P}_{u_i}(H_i^c)} = \frac{\mathbf{P}_{u_i}(A_i, H_i)}{\mathbf{P}_{u_i}(H_i)}. \tag{5.1}$$

Recall that we consider a weighted directed graph (G, c) with $G = (V, E)$ with the associated Markov chain X with the pair (π, P).
We also have for all $0 \leq i \leq l-1$:

$$\mathbf{P}_{u_i}(A_i, H_i) = \mathbf{P}_{u_{i+1}}(A_{i+1}) p_{u_i u_{i+1}} \tag{5.2}$$

which directly follows.
Recall from section 4.2.2 that we have for all $u \in V : \mathbf{P}[u \to B] = \mathbf{P}_u(\tau_B < \tau_u^+)$.
Recall from section 3.1.2 and definition 3.9, that we have for all $u \in V$: $\mathcal{G}^{\tau_B}(u, u) = \mathbf{P}[u \to B]^{-1}$ and that $\mathcal{G}^{\tau_B}(u, u) = \mathbf{E}_u\Big(\mathcal{C}_u(\tau_B)\Big)$ for all $u \in V$.
Use that for all $u \in V$:

$$\mathbf{E}_u\Big(\mathcal{C}_u(\tau_B)\Big) = \sum_{k \geq 0} [P]_{B^c}^k(u, u) = \Big[\mathbb{1} - P\Big]_{B^c}^{-1}(u, u).$$

For an inverse matrix, we get then by Cramer's formula [7] for all $u \in V$:

$$\Big[\mathbb{1} - P\Big]_{B^c}^{-1}(u,u) = \frac{\det[\mathbb{1} - P]_{B^c \setminus \{u\}}}{\det[\mathbb{1} - P]_{B^c}}.$$

That means that for all $0 \leq i \leq l$:

$$\mathbf{P}_{u_i}(H_i) = \frac{\det[\mathbb{1} - P]_{B^c \setminus \bigcup_{j=0}^{i}\{u_j\}}}{\det[\mathbb{1} - P]_{B^c \setminus \bigcup_{j=0}^{i-1}\{u_j\}}} \tag{5.3}$$

with the convention that $\bigcup_{j=0}^{i-1}\{u_j\} = \emptyset$ for $i = 0$.
We are now capable of proving theorem 5.2 by using equations 5.1, 5.2 and 5.3.

$$\mathbf{P}_{u_0}(A_0) = \frac{\mathbf{P}_{u_0}(A_0, H_0)}{\mathbf{P}_{u_0}(H_0)} = \frac{\mathbf{P}_{u_1}(A_1)}{\mathbf{P}_{u_0}(H_0)} p_{u_0 u_1}$$

where we used equation 5.1 for the first equality and equation 5.2 for the second equality.
Hence, by iteration it then follows that

$$\mathbf{P}_{u_0}(A_0) = \frac{\mathbf{P}_{u_l}(A_l)}{\prod_{i=0}^{l-1} \mathbf{P}_{u_i}(H_i)} \prod_{i=0}^{l-1} p_{u_i u_{i+1}}.$$

Now use equation 5.3, so that we can write:

$$\prod_{i=0}^{l-1} \mathbf{P}_{u_i}(H_i) = \prod_{i=0}^{l-1} \frac{\det[\mathbb{1} - P]_{B^c \setminus \bigcup_{j=0}^{i}\{u_j\}}}{\det[\mathbb{1} - P]_{B^c \setminus \bigcup_{j=0}^{i-1}\{u_j\}}} = \frac{\det[\mathbb{1} - P]_{B^c \setminus \bigcup_{j=0}^{l-1}\{u_j\}}}{\det[\mathbb{1} - P]_{B^c}}.$$

Note that $\mathbf{P}_{u_l}(A_l) = \mathbf{P}_{u_l}\Big(LE(\mathcal{W}) = \gamma_0(B)\Big) = \mathbf{P}_{u_l}\Big(LE(\mathcal{W}) = (u_l)\Big) = 1$.
That means that

$$\mathbf{P}_{u_0}(A_0) = \frac{\det[\mathbb{1} - P]_{B^c \setminus \bigcup_{j=0}^{l-1}\{u_j\}}}{\det[\mathbb{1} - P]_{B^c}} \prod_{i=0}^{l-1} p_{u_i u_{i+1}}. \qquad \square$$

Remark 5.1. *By convention, let* $\det[\mathbb{1} - P]_{\emptyset} = 1$.

We have all the tools needed to prove theorem 5.1.

Proof
Consider a weighted directed graph (G, c) with $G = (V, E)$ and consider the associated Markov chain X with the pair (π, P).
Fix a vertex r to be the root and fix a rooted spanning tree $A \in \Sigma_r^G$.

Recall from theorem 5.1, that $\mathbf{P}^W(\cdot)$ denote the probability law coming from Wilson's algorithm.
Consider Wilson's algorithm. First let $v_1, ..v_{|V|-1}$ be an arbitrary, but fixed ordering of $V \setminus \{r\}$.
We follow the steps from Wilson's algorithm:
(1) : Set $B_0 = \{r\}, E_0 = \emptyset$.
(2) : We have $i_1 = 1$. From v_1, we have a unique path to hit B_0 going by $l_1 \geq 1$ edges, we denote this path by $\gamma_{l_1}(B_0)$. Consider a random walk $\mathcal{W}_1$ from our Markov chain X starting at v_1 until it hits B_0. We let $\hat{B}_1, \hat{E}_1$ be the vertex-set, respectively edge-set coming from the loop-erased path $LE(\mathcal{W}_1)$. Take $B_1 = B_0 \cup \hat{B}_1, E_1 = E_0 \cup \hat{E}_1$. By theorem 5.2, we see that

$$\mathbf{P}_{u_1}\Big(LE(\mathcal{W}_1) = \gamma_{l_1}(B_0)\Big) = \frac{\det[\mathbb{1} - P]_{B_1^c}}{\det[\mathbb{1} - P]_{B_0^c}} \prod_{e \in \hat{E}_1} p_e.$$

(3) : We have $i_j = \min\{k \geq 1 : v_k \in B_{j-1}^c\}$. From v_{i_j}, we have a unique path to hit B_{j-1} going by $l_j \geq 1$ edges, we denote this path by $\gamma_{l_j}(B_{j-1})$. Consider a random walk $\mathcal{W}_j$ from our Markov chain X starting at v_{i_j} until it hits B_{j-1}. We let $\hat{B}_j, \hat{E}_j$ be the vertex-set, respectively edge-set coming from the loop-erased path $LE(\mathcal{W}_j)$. By theorem 5.2, we see that

$$\mathbf{P}_{u_{i_j}}\Big(LE(\mathcal{W}_j) = \gamma_{l_j}(B_{j-1})\Big) = \frac{\det[\mathbb{1} - P]_{B_j^c}}{\det[\mathbb{1} - P]_{B_{j-1}^c}} \prod_{e \in \hat{E}_j} p_e.$$

(4) : Stop when $B_N = V$ for some $N \geq 0$ and set $\sigma = E_N$.

That means that we obtain by iteration:

$$\mathbf{P}^W(\sigma = A) = \prod_{j=1}^{N} \mathbf{P}_{v_{i_j}}\big((LE(\mathcal{W}_j) = \gamma_{l_j}(B_{j-1})\big) = \prod_{j=1}^{N}\Big(\frac{\det[\mathbb{1} - P]_{B_j^c}}{\det[\mathbb{1} - P]_{B_{j-1}^c}} \prod_{e \in \hat{E}_j} p_e\Big).$$

Note that $\bigcup_{j=1}^N \hat{E}_j = A$ and that $\hat{E}_j \cap \hat{E}_i = \emptyset, 1 \leq i,j \leq N, i \neq j$.
That means that we can write

$$\prod_{j=1}^{N}\Big(\prod_{e \in \hat{E}_j} p_e\Big) = \prod_{e \in A} p_e = \alpha(A).$$

Also note that

$$\prod_{j=1}^{N} \frac{\det[\mathbb{1} - P]_{B_j^c}}{\det[\mathbb{1} - P]_{B_{j-1}^c}} = \frac{\det[\mathbb{1} - P]_{B_N^c}}{\det[\mathbb{1} - P]_{B_0^c}} = \frac{1}{\det[\mathbb{1} - P]_{V \setminus \{r\}}}.$$

We used here that $B_N^c = V^c = \emptyset$ and that $\det[\mathbb{1} - P]_\emptyset = 1$, as stated in remark 5.1.
Also note that $B_0^c = V \setminus \{r\}$.
That means that

$$\mathbf{P}^W(\sigma = A) = \frac{\alpha(A)}{\det[\mathbb{1} - P]_{V\setminus\{r\}}}.$$

Now note that

$$\sum_{A \in \Sigma_r^G} \mathbf{P}^W(\sigma = A) = \frac{1}{\det[\mathbb{1} - P]_{V\setminus\{r\}}} \sum_{A \in \Sigma_r^G} \alpha(A) = 1.$$

Hence, we see that

$$\det[\mathbb{1} - P]_{V\setminus\{r\}} = \sum_{A \in \Sigma_r^G} \alpha(A) = Z_r^G.$$

So that means that

$$\mathbf{P}^W(\sigma = A) = \frac{\alpha(A)}{Z_r^G} = \mathbf{P}(\sigma = A),$$

where we refer to definition 5.6.
We see from this proof that the produced random rooted spanning tree is independent on the ordering of $V \setminus \{r\}$. □

We will now prove that when we generate a random rooted spanning tree by using Wilson's algorithm and when we forget about the orientation of the edges and the root, we have probability corresponding to definition 5.3 to pick a random spanning tree.
Consider a weighted directed graph (G, c) with $G = (V, E)$ and consider the associated Markov chain X with the pair (π, P).
Fix a vertex r to be the root and take Wilson's algorithm to generate a random rooted spanning tree σ with root r.
We get then by theorem 5.1, that $\mathbf{P}^W(\sigma = A) = \frac{\alpha(A)}{Z_r^G}$.
Note that r is then not the tail of any edge in A, which means that we can write

$$\alpha(A) = \prod_{e \in A} p_e = \frac{\prod_{e\in A} \pi(e^-) p_e}{\prod_{u \in V\setminus\{r\}} \pi(u)} = \pi(r) \frac{\prod_{e\in A} \pi(e^-) p_e}{\prod_{u\in V} \pi(u)}.$$

If we forget about the orientation of the edges of A and about the root r, we obtain a spanning tree T of G. This means that we can write

$$c(T) = \prod_{e \in A} \pi(e^-) p_e.$$

So that means that have the following equality:

$$\alpha(A) = \frac{\pi(r)}{\prod_{u \in V} \pi(u)} c(T).$$

Note that this means that τ is picked with probability proportional to its weight at random from $\mathcal{T}^G$, as in definition 5.3, where we used that T is independent on the root r. The constant Z^G follows by using that $1 = \sum_{T \in \mathcal{T}^G} \mathbf{P}(\tau = T)$.

5.3 Current flows and spanning trees

This section connects the probability of seeing a certain edge in a random spanning tree with electrical networks as presented in chapter 3, and in specific with the unit current flow.

Let (G, c, A, Z) be an electrical network with $G = (V, E)$ being an undirected graph, where $A = \{e^-\}, Z = \{e^+\}$. We define $i^e(\cdot)$ with $e = \langle e^-, e^+ \rangle$, to denote the unit current flow from e^- to e^+.

The following proposition connects the probability of seeing a certain edge in a random spanning tree with the unit current flow.

Proposition 5.1 (Unit current flow interpretation). *Consider the electrical network (G, c, A, Z) with $G = (V, E)$ being an undirected graph and consider the associated Markov chain X with the pair (π, P).*
Let T be a weighted spanning tree picked with probability proportionally to its weights at random from $\mathcal{T}^G$. For all $e \in E$, we set $A = \{e^-\}$ and $Z = \{e^+\}$. The following equality then holds:

$$\mathbf{P}(e \in T) = \mathbf{P}_{e^-}[\textit{1 st hit } e^+ \textit{ via } e] = i^e(e) = c(e)\mathscr{R}(e^- \leftrightarrow e^+)$$

with $\mathbf{P}_{e^-}[\cdot]$ referring to the probability law coming from Wilson's algorithm with starting point e^-.

Proof Let us start by proving the first equality:
Fix e^+ to be the root of T and start Wilson's algorithm by taking e^- as the starting vertex. We either hit e^+ via traveling along e when we start from

e^- or we hit e^+ via traveling along another edge. The probability of seeing exactly the edge e is then the desired one.
We now prove the second equality:
Sample a random walk starting at e^- until it hits e^+. That means that

$$\mathbf{P}_{e^-}[\text{1st hit } e^+ \text{ via } e] = \mathbf{E}_{e^-}\left(S_e^{\tau_{e^+}}\right).$$

where we write $\mathbf{E}_{e^-}\left(\cdot\right)$ to refer to the random walk starting from e^-. This equality holds cause we are looking at the expected number of crossings of e. Note that

$$\mathbf{E}_{e^-}\left(S_{-e}^{\tau_{e^+}}\right) = 0.$$

So we get by using proposition 3.4 that

$$i^e(e) = \mathbf{E}_{e^-}\left(S_e^{\tau_{e^+}} - S_{-e}^{\tau_{e^+}}\right) = \mathbf{E}_{e^-}\left(S_e^{\tau_{e^+}}\right) = \mathcal{G}^{\tau_{e^+}}(e^-, e^-)p_e.$$

Now note that we saw in section 3.1.2 that

$$\mathcal{G}^{\tau_{e^+}}(e^-, e^-) = \mathbf{P}_{e^-}[e^- \to e^+]^{-1} = \pi(e^-)\mathscr{R}(e^- \leftrightarrow e^+).$$

Hence, we get that

$$\mathcal{G}^{\tau_{e^+}}(e^-, e^-)p_e = \mathbf{P}[e^- \to e^+]^{-1}p_e = \pi(e^-)\mathscr{R}(e^- \leftrightarrow e^+)p_e = c(e)\mathscr{R}(e^- \leftrightarrow e^+).$$

□

5.4 Transfer current theorem

This section is devoted to the Transfer current theorem, which is theorem 5.3. This theorem and the proof of it uses results from this chapter and from chapter 3.
Let (G, c, A, Z) be an electrical network with $G = (V, E)$ being an undirected graph.

We start by defining the so-called Transfer current matrix.

Definition 5.7 (Transfer current matrix).

$$Y(e, e') := i^e(e').$$

So $Y(e, e')$ is the current flow that goes across e' when a unit current flow is imposed between the endpoints of e.

We have the following equality:

Lemma 5.1.

$$\frac{\langle P_\bigstar \chi^e, \chi^{e'} \rangle_r}{\langle \chi^{e'}, \chi^{e'} \rangle_r} = i^e(e').$$

Proof First notice that $P_\bigstar \chi^e = i^e$. That means that

$$\langle P_\bigstar \chi^e, \chi^{e'} \rangle_r = \langle i^e, \chi^{e'} \rangle_r = i^e(e')r(e')$$

where we used lemma 3.10. By this same lemma, it follows that

$$\langle \chi^{e'}, \chi^{e'} \rangle_r = r(e').$$

This proves the lemma. □

And the following so-called reciprocity law follows:

Lemma 5.2 (Reciprocity law).

$$Y(e, e')r(e') = Y(e', e)r(e).$$

Proof Notice that $P_\bigstar$ is an orthogonal projection and thereby self-adjoint. That means that

$$\langle P_\bigstar \chi^e, \chi^{e'} \rangle_r = \langle \chi^e, P_\bigstar \chi^{e'} \rangle_r.$$

$$\langle P_\bigstar \chi^e, \chi^{e'} \rangle_r = i^e(e')r(e') = Y(e, e')r(e').$$

$$\langle \chi^e, P_\bigstar \chi^{e'} \rangle_r = i^{e'}(e)r(e) = Y(e', e)r(e).$$

This proves the lemma. □

Suppose that we have an undirected graph $G = (V, E)$ and that we contract an edge $e \in E$ with $e = \langle e^-, e^+ \rangle, e^-, e^+ \in V, e^- \neq e^+$.
That means that we remove the edge e, but we combine its endpoints into one vertex. Let us denote the graph obtained by contracting the edge e by $G.\{e\}$. We will use the following notation for contracting subgraphs:

Notation 5.1. *Let F be a subgraph of G and let $G.F$ denote the graph where edges $f \in F$ are contracted.*

We now give an example of contraction:
Example of contraction Let $G = (V, E)$ be the graph in the following figure, where F is the edge-set with the edges $f_i, 1 \le i \le 6$.
The contraction of the edges in F leads to the contracted graph $G.F$, as shown in the following figure:

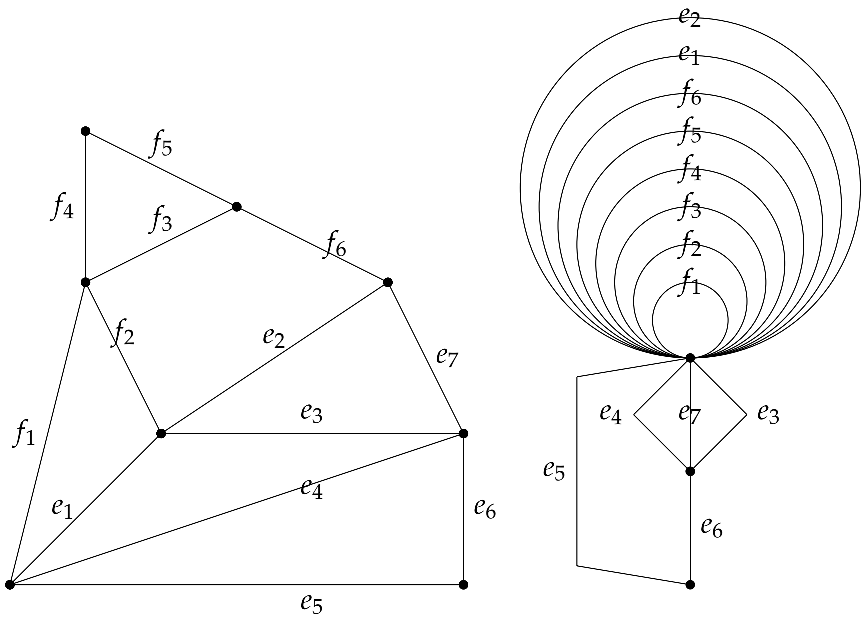

Figure 5.1: Graph G before contraction and graph $G.F$ after contraction

We see that the edges e_1, e_2 form an undirected cycle with edges of F, hence they become a loop when contracted.

We introduce the following notation for the current flow in a contracted graph:

Notation 5.2. *Let F be a subgraph of G. We then let $\hat{i}^e$ be the unit current flow in $G.F$ with $e = \langle e^-, e^+ \rangle \in E$ such that $e \notin F$ and e does not form any undirected cycle with edges of F, otherwise we will get a loop when the edges of F are contracted.*

Let us define the following:

Definition 5.8. *Define $W := span\{i^f : f \in F\}$.*

A proposition which is essential to prove the Transfer current theorem, is the following:

Proposition 5.2.

$$(1) : \hat{i}^e = (P_{\underset{W}{\perp}} i^e) \restriction (E \setminus F)$$

$$(2) : (P_{\underset{W}{\perp}} i^e) \restriction F = 0$$

with $P_{\underset{W}{\perp}}$ denoting the orthogonal projection onto the orthocomplement of W.

Proof Note that $W \subseteq \bigstar$ and that $i^e \in \bigstar$. As a consequence, we get that

$$i^e = P_W i^e + P_{\underset{W}{\perp}} i^e$$

and recall that we have that $P_\bigstar \chi^f = i^f$.
Now let $\hat{\bigstar}$ be the star space in $G.F$ and let $\hat{\Diamond}$ be the cycle space in $G.F$.
So we have that $\hat{i}^e = P_{\hat{\bigstar}} \chi^e$.
We know that cycles in G remain cycles when F is contracted, hence, we can decompose $\Diamond$ and $\hat{\Diamond}$ such that every function of $\Diamond$ is also a function of $\hat{\Diamond}$.
Also note that contraction of the edges in F makes these edges loops in $G.F$, hence we get that all the functions of W are also functions of $\hat{\Diamond}$. So we know for sure that all the functions of $(\Diamond \cup W)$ are at least functions of $\hat{\Diamond}$.
Now suppose that we have a cycle $e_1, ..., e_k$ in $G.F$. If now $e_{i+1}^- = e_i^+$ in G for all $1 \leq i \leq k-1$, then no edge will get in between the endpoints when F is contracted. So suppose that there exists some $1 \leq i \leq k-1$ such that $e_{i+1}^- = v_2 \neq v_1 = e_i^+$ in G. That means that v_1 and v_2 belong to the same connected component of F, so they are contracted to one vertex in $G.F$. So we can then insert edges from F to obtain a path from v_1 to v_2, which means that $\hat{\Diamond} \subseteq (\Diamond \cup W)$.
Therefore, we get that

$$\hat{\Diamond} = \Diamond \oplus W.$$

Now recall that $\ell^2_-(E) = \bigstar \oplus \Diamond = \hat{\bigstar} \oplus \hat{\Diamond}$ with $\bigstar$ and $\Diamond$ being orthogonal to each other and $\hat{\bigstar}$ and $\hat{\Diamond}$ being orthogonal to each other. Cause $\Diamond \subseteq \hat{\Diamond}$, we get that $\hat{\bigstar} \subseteq \bigstar$.
So we can write the following:

$$\bigstar = \hat{\bigstar} \oplus (\bigstar \cap \hat{\Diamond}).$$

$$\hat{\lozenge} = (\bigstar \cap \hat{\lozenge}) \oplus \lozenge.$$

Therefore:

$$\ell^2_-(E) = \hat{\bigstar} \oplus (\bigstar \cap \hat{\lozenge}) \oplus \lozenge.$$

Now note that

$$\bigstar \cap \hat{\lozenge} = P_{\bigstar}\hat{\lozenge} = P_{\bigstar}\lozenge + P_{\bigstar}W = W$$

cause $\bigstar$ and $\lozenge$ are orthogonal to each other.
So that gives then for $\hat{i}^e \in G.F$:

$$\hat{i}^e = P_{\hat{\bigstar}}\chi^e = P_{\underset{W}{\perp}} P_{\bigstar}\chi^e = P_{\underset{W}{\perp}} i^e.$$

That means that for $f \in F$:

$$\langle \hat{i}^e, \chi^f \rangle_r = \langle P_{\underset{W}{\perp}} i^e, \chi^f \rangle_r = \langle P_{\bigstar} P_{\underset{W}{\perp}} i^e, \chi^f \rangle_r$$

$$= \langle P_{\underset{W}{\perp}} i^e, P_{\bigstar}\chi^f \rangle_r = \langle P_{\underset{W}{\perp}} i^e, i^f \rangle_r = 0$$

cause $e \notin F$ and e does not form any undirected cycle with edges in F. So that proves (2).
Since $P_{\underset{W}{\perp}} i^e \in \bigstar$, it satisfies the cycle law in G, so there is no flow across the edges in F, this also satisfies the cycle law in $G.F$ ($\bigstar = \hat{\bigstar} \oplus W$).
Since i^e is orthogonal to all stars in G except at the endpoints of e, we get that i^e is orthogonal to all stars in $G.F$ other than the endpoints of e.
So that means that we can write:

$$P_{\underset{W}{\perp}} i^e = i^e - P_W i^e = i^e - \sum_{f \in F} \alpha_f i^f$$

for some constants α_f. So $i^e - \sum_{f \in F} \alpha_f i^f$ is orthogonal to all stars in $G.F$ except at the endpoints of e, there the inner products are ± 1. This proves (1). □

We are now ready to state the Transfer current theorem, which determines the edge process in a random spanning tree.

Theorem 5.3 (Transfer current theorem). *Consider an electrical network (G, c, A, Z) with $G = (V, E)$ being an undirected graph. Let T be a weighted spanning tree picked with probability proportionally to its weights at random from $\mathcal{T}^G$. For any distinct edges $e_1, ...e_k \in E$:*

$$\mathbf{P}\Big(e_1, ..., e_k \in T\Big) = det\ [Y(e_i, e_j)]_{1 \le i,j \le k}.$$

This process belongs to the class of determinantal processes. For more on these processes, the reader is referred to [11].

Proof Suppose that we can form a cycle of the edges $e_1, ...e_k$. Note that we can write this cycle as $\sum_j a_j \chi^{e_j}$ with $a_j \in \{-1, 0, 1\}$. That means that for $1 \leq m \leq k$:

$$\sum_j a_j r(e_j) Y(e_m, e_j) = \sum_j a_j r(e_j) i^{e_m}(e_j) = 0$$

by applying Kirchoff's cycle law to the current flow i^{e_m}. But that means that we have a linear combination of the corresponding columns of $[Y(e_i, e_j)]$ which equals zero.
Hence, we see that the determinant is zero. Now recall that cycles cannot appear in a spanning tree, therefore both sides of our theorem are zero. So we assume that we have no such cycles for the remainder of the proof.

First recall that when $k = 1$ we get for any edge $e \in E$:

$$\mathbf{P}\Big(e \in T\Big) = Y(e, e) = i^e(e)$$

which has been proven already, recall proposition 5.1.
Define for simplicity $Y_m := [Y(e_i, e_j)]_{1 \leq i,j \leq m}$ for $1 \leq m \leq k$. We want to carry the induction from $m = k - 1$ to $m = k$. Recall from conditional probability that for any distinct edges $e_1, ..., e_k \in G$:

$$\mathbf{P}\Big(e_1, ..., e_k \in T\Big) = \mathbf{P}\Big(e_k \in T | e_1, ..., e_{k-1} \in T\Big) \mathbf{P}\Big(e_1, ..., e_{k-1} \in T\Big).$$

So we want to show that

$$\det Y_k = \mathbf{P}\Big(e_k \in T | e_1, ..., e_{k-1} \in T\Big) \det Y_{k-1}.$$

First contract the edges $e_1, ..., e_{k-1}$ to obtain the graph $G.\{e_1, ..., e_{k-1}\}$. Let $\hat{i}^{e_k}$ be the associated unit current flow of the electrical network $(G.\{e_1, ..., e_{k-1}\}, c, A, Z)$. We know then from proposition 5.2 that

$$\mathbf{P}\Big(e_k \in T | e_1, ...e_{k-1} \in T\Big) = \hat{i}^{e_k}(e_k).$$

Let $W := \text{span}\{i^e : e \in E\}$. We have that

$$P_{\underset{W}{\perp}} i^{e_k} = i^{e_k} - \sum_{m=1}^{k-1} a_m i^{e_m}.$$

Now define a matrix $\hat{Y}$ with (m, j)-entry being the same as that of Y_{k-1} for $m, j < k$ and as (k, j)-entry:

$$i^{e_k}(e_j) - \sum_{m=1}^{k-1} a_m i^{e_m}(e_j) = (P_{\underset{W}{\perp}} i^{e_k})(e_j) = \begin{cases} 0 \text{ if } j < k \\ \hat{i}^{e_k}(e_k) \text{ if } j = k \end{cases}$$

by proposition 5.2.

Note then that det $\hat{Y}$ = det Y_k, by the property of the determinant that adding one row to a linear combination of the other rows, the determinant doesn't change.

So what we use here is that we can write for $1 \leq j \leq k$:

$$Y(e_k, e_j) = i^{e_k}(e_j) = (P_{\underset{W}{\perp}} i^{e_k})(e_j) + \sum_{m=1}^{k-1} a_m i^{e_m}(e_j)$$

$$= (P_{\underset{W}{\perp}} i^{e_k})(e_j) + \sum_{m=1}^{k-1} a_m Y(e_m, e_j) = \begin{cases} \sum_{m=1}^{k-1} a_m Y(e_m, e_j) \text{ if } j < k \\ \hat{i}^{e_k}(e_k) + \sum_{m=1}^{k-1} a_m Y(e_m, e_k) \text{ if } j = k \end{cases}.$$

That gives then that

$$\det Y_k = \det \hat{Y} = \hat{i}^{e_k}(e_k) \det Y_{k-1} = \mathbf{P}\Big(e_k \in T | e_1, ..., e_{k-1} \in T\Big) \det Y_{k-1}. \quad \square$$

A direct consequence of the Transfer current theorem is the following corollary:

Corollary 5.1 (Negative association). *Recall from definition 5.7 that $Y(e, f) = i^e(f)$. By using theorem 5.3, we get for distinct edges $e, f \in G$ that:*

$$\mathbf{P}\Big(e, f \in T\Big) - \mathbf{P}\Big(e \in T\Big)\mathbf{P}\Big(f \in T\Big) = det \begin{pmatrix} Y(e,e) & Y(e,f) \\ Y(f,e) & Y(f,f) \end{pmatrix} - Y(e,e) \cdot Y(f,f)$$

$$= -Y(e, f)Y(f, e).$$

Recall from lemma 5.2 that $Y(f, e) = Y(e, f)r(f)c(e)$. Hence:

$$\mathbf{P}\Big(e, f \in T\Big) - \mathbf{P}\Big(e \in T\Big)\mathbf{P}\Big(f \in T\Big) = -c(e)r(f)Y(e, f)^2 \leq 0.$$

This is referred to as negative association, which actually is a ground fact about determinantal processes.

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