

MASTER'S THESIS

Positive $C(K)$ -representations and positive spectral measures

On their one-to-one correspondence for reflexive Banach lattices

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Abstract

Is there a natural one-to-one correspondence between positive representations of spaces of continuous functions and positive spectral measures on Banach lattices? That is the question that we have investigated in this thesis. Representations and spectral measures are key notions throughout all sections.

After having given precise definitions of a positive spectral measure, a unital positive spectral measure, a positive representation, and a unital positive representation, we prove that there is a one-to-one relationship between the positive spectral measures and positive representations, and the unital positive spectral measures and unital positive representations, respectively. For example, between a unital positive representation $\rho : C(K) \rightarrow \mathcal{L}_r(X)$ and a unital positive spectral measure $E : \Omega \rightarrow \mathcal{L}_r(X)$, where K is a compact Hausdorff space, $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra, X a reflexive Banach lattice, and $\mathcal{L}_r(X)$ the space of all regular operators on X .

The map $\rho : C(K) \rightarrow \mathcal{L}_r(C(K))$, where K is a compact Hausdorff space, defined as pointwise multiplication on $C(K)$, is a unital positive representation. The subspace $\rho[f] := \overline{\rho(C(K))f} \subseteq C(K)$, for an $f \in C(K)$, has two partial orderings. One is inherited from $C(K)$, the other is newly defined. Using the defining property of elements of $\rho[f]$, we prove that they are equal.

A similar correspondence between representations and spectral measures on a general Banach space only exists under the assumption of R -boundedness ([PR07]). We show that the spectral measure that is generated via that correspondence is equal to the unital positive spectral measure that is generated from a unital positive representation in our setting.

Finally, we show that the correspondence between our unital positive spectral measures and unital positive representations is relevant in the context of covariance, belonging to the theory of crossed products, as well. We look at an example in which there is a one-to-one relationship between covariant representations and covariant spectral measures.

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1 Introduction

1.1 MOTIVATION AND QUESTIONS

Normal operators on a (possibly infinite dimensional) Hilbert space H are bounded linear maps $N : H \rightarrow H$, for which $NN^* = N^*N$. Here N^* denotes the Hermitian adjoint of N . In a finite dimensional setting, for example, the Hermitian adjoint of a square matrix with complex-valued entries is equal to the conjugate transpose of the matrix. A normal operator on a finite dimensional Hilbert space can be diagonalized. This is a well-known result, that is formally stated in the Spectral Theorem. There are two ways in which this diagonalization can be interpreted, one of which is generalizable to an infinite dimensional setting. The other relies heavily on the eigenvalues of the normal operator and the corresponding eigenvectors. Since eigenvalues may not exist for a normal operator on an infinite dimensional Hilbert space, we have to look at diagonalizability from a different angle. Suppose that N is a normal operator on a Hilbert space H with $\dim H = d < \infty$. Let $\lambda_1, \dots, \lambda_n$ be the distinct eigenvalues of N , and E_k the orthogonal projection of H onto $\ker(N - \lambda_k)$ for every $k \in \{1, \dots, n\}$. Then the Spectral Theorem says that

$$N = \sum_{k=1}^n \lambda_k E_k. \tag{1.1}$$

To generalize this way of expressing a normal operator to the case where $d = \infty$, we need the concept of a spectral measure, which replaces the orthogonal projections. The sum in (1.1) can then be replaced by an integral over the spectrum of N , denoted by $\sigma(N)$. The spectrum consists precisely of the eigenvalues of N whenever N is defined in a finite dimensional setting. The Spectral Theorem tells us that a normal operator N on an infinite dimensional Hilbert space can be expressed as

$$N = \int_{\sigma(N)} \lambda dE(\lambda), \tag{1.2}$$

referred to as the *spectral decomposition* of N .

To prove the Spectral Theorem (Theorem IX.2.2 of [CO07]), which is considered to be one of the landmarks of the theory of operators on a Hilbert space, one starts by looking at representations of commutative C^* -algebras. This makes sense, because the wanted result is a special case of such a theory. It can be shown that these representations correspond to certain operator-valued measures.

The spectral theory of linear operators on Hilbert spaces is well-developed, whereas there are very few general results on Banach spaces. There is a clear reason why there are more results on Hilbert spaces. Their geometry is relatively simple and well understood, in contrast to the geometry of general Banach spaces. The geometry is key in many proofs.

Banach lattices are equipped with a partial ordering. This makes them more interesting as a subject of research than general Banach spaces. More importantly, one has many more properties to hold on to while trying to prove statements. Arguments that can be obtained from this ordering property are sometimes analogous to the arguments that come from the inner product that is defined on the well-known Hilbert spaces.

That last observation is precisely what gave rise to questions that have been investigated in this thesis. Representations of spaces of continuous functions on Banach spaces have been studied in [PR07]. It is shown that they are related to spectral measures using the notion of R -boundedness. The most important question we were able to answer has to do with the Spectral Theorem, introduced above.

Is there a unique one-to-one correspondence between positive representations of spaces of continuous functions and positive spectral measures on Banach lattices?

The correspondence in the Hilbert space setting is a stepping stone for the Spectral Theorem on Hilbert spaces. We have found a similar result in the Banach lattice setting.

We relate our work to work that has already been done, in various ways. A specific subspace of our Banach lattice inherits a partial ordering of the original Banach lattice. For a certain unital positive representation ρ of $C(K)$ on $C(K)$, where K is a compact Hausdorff space, this subspace is $\rho[f] := \overline{\rho(C(K))}f \subseteq C(K)$ for an $f \in C(K)$. Since this subspace is a Banach lattice with a partial ordering itself, we can compare the two orderings.

To relate our work to the study of representations of the spaces of continuous functions on Banach spaces, we look at R -boundedness. This is the key ingredient for the correspondence between representations and spectral measures on Banach spaces.

The correspondence on Banach lattices we have proven to exist under certain circumstances, represents an example of covariant representations, an ingredient of the theory of crossed products. They are studied in [DJ11], and we show how our results fit into that theory.

1.2 HISTORICAL BACKGROUND

In the nineteen thirties the fundamentals of the theory of vector lattices, also known today as Riesz spaces, were founded by F. Riesz, L. Kantorovic, and H. Freudenthal ([SC74]). This happened only shortly after the start of the research on Banach spaces. Kantorovic and his school spent time investigating vector lattices in combination with normed vector spaces, a concept from Banach space theory. The investigation of normed vector lattices as well as the order-related linear maps on these vector lattices did however not keep pace with the fast developments in general functional analysis. This resulted in the two fields of research drifting more and more apart. The gap has become smaller and smaller from the nineteen sixties on, up until the point where the importance of studying the ties between these fields is now widely recognized, and the nice relationship between general Banach space theory and Riesz space theory is once more an object of study in functional analysis.

1.3 RELATED WORK

The approach for the research that resulted in this thesis used the way of thinking described above. We wanted to get to know more about Riesz spaces, in particular Banach lattices, through looking at the similarities between (certain) Banach spaces and Riesz spaces. More precisely, our approach towards the questions and proofs will follow Section IX.1 of [CO07]. We will discuss shortly what has been done there. This will help us understand what tools we need in order to get the wanted results in a Banach lattice setting.

Let K be a compact Hausdorff set, $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra, H a Hilbert space and $\mathcal{B}(H)$ the space of all bounded linear transformations $H \rightarrow H$. A representation $\rho : C(K) \rightarrow \mathcal{B}(H)$ is a $*$ -homomorphism with $\rho(\mathbf{1}) = I$. Also $\|\rho\| = 1$, and ρ is a positive map in the sense that $\rho(f) \geq 0$ whenever $f \geq 0$. By analogy with the Riesz Representation Theorem it is expected that $\rho(f) = \int f dE$ for some type of measure E , operator-valued rather than scalar-valued. This is indeed the case and those measures turn out to be of the following form.

Definition 1.1. Let K be a set, $\Omega \subseteq \mathcal{P}(K)$ a σ -algebra, and H a Hilbert space. A map $E : \Omega \rightarrow \mathcal{B}(H)$ is called a *spectral measure* whenever it has the following properties:

1. for each Δ in Ω , $E(\Delta)$ is an orthogonal projection of H onto $\text{ran}(E(\Delta))$;
2. $E(\emptyset) = 0$ and $E(K) = 1$;
3. $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$ for Δ_1 and Δ_2 in Ω ;
4. E is SOT-countably additive, that is, if $\{\Delta_n\}_{n=1}^\infty$ are pairwise disjoint sets from Ω , then $E(\cup_{n=1}^\infty \Delta_n) = \sum_{n=1}^\infty E(\Delta_n)$ (SOT).

Each spectral measure for (K, Ω, H) defines a family of countably additive measures on Ω via

$$E_{g,h}(\Delta) \equiv \langle E(\Delta)g, h \rangle,$$

where $f, g \in H$ and $\Delta \subset \Omega$, with total variation $\leq \|g\|\|h\|$. It is shown that spectral measures can be used to define representations. In order to be able to do this, we need to know how to integrate a bounded Ω -measurable function ϕ with respect to a spectral measure, which is meaningful since $\|E_{g,h}\| \leq \|g\|\|h\| < \infty$. It is done through a bounded sesquilinear form $H \times H \rightarrow \mathbb{C}$, $(g, h) \mapsto \int_K \phi dE_{g,h}$. This form is represented by a unique operator on H denoted by $\rho(\phi)$ such that

$$\langle \rho(\phi)g, h \rangle = \int_K \phi dE_{g,h}.$$

Proving that this ρ is a representation of $B(K)$, the space of all bounded Ω -measurable functions $K \rightarrow \mathbb{C}$, is fairly easy using the dense subspace $S(K)$, generated by the characteristic functions. It is also shown that $\rho(\phi)$ is a normal operator for every $\phi \in B(K)$. Proving the reverse statement, i.e., that representations of $C(K)$ generate spectral measures, is harder. An important ingredient is the Riesz Representation Theorem, which tells us that there is an isometric isomorphism of the regular Borel measures μ on K with the total variation norm onto the continuous linear functionals of $C_0(K)$, where K is locally compact, mapping $f \in C_0(K)$ to $\int f d\mu$. The proof starts with the extension of ρ to $\tilde{\rho}$ on $B(K)$, and then we

can define $E(\Delta)$ as $\tilde{\rho}(\chi_\Delta)$. Then $\tilde{\rho}$ is proven to be a representation as well. Proving that E has the wanted properties of a spectral measure is then the last – and nontrivial – step of the proof.

1.4 OUTLINE

We start with preliminaries in Section 2 on general functional analysis and more specific Riesz space theory, including Banach lattices. We end this section with Pettis' Theorem, which is an important ingredient for one of our main proofs.

We then define our most important concepts in Section 3, namely positive spectral measures, unital positive spectral measures, positive representations, and unital positive representations. We will see the relationship between positive spectral measures and unital positive spectral measures, and countably additive measures, which gives us a nice result on the norm bound of this measure.

The core of this thesis can be found in Sections 4, 5 and 6. We start by showing how representations can be obtained from spectral measures. Then we show the reverse, namely how to obtain spectral measures from representations. This will prove to be the more difficult direction of working with representations and spectral measures. There are certain conditions under which we find ourselves with a one-to-one correspondence between either positive spectral measures and positive representations, or unital positive spectral measures and unital positive representations. We distinguish between different settings in these sections as much as is interesting.

In Sections 7 and 8 we compare our results to the results in [PR07] on $C(K)$ -representations and R -boundedness. The road we have taken in the previous sections is less general than the one presented there. It has allowed us to use the structure of proofs used in Hilbert space situations, and it has brought us results similar to those in [PR07], as we will see.

The study of covariant representations of Banach algebra dynamical systems is part of the theory of crossed products. In Section 9 we see that the representations with which we have been dealing are part of this theory. We do this by showing, among other things, that an example of a representation of our kind fits into the definition of a covariant representation, and corresponds to a covariant spectral measure.

We conclude this thesis by summing up our most important results in Section 10.

2 Preliminaries

In this section we discuss basic functional analysis and the notation we use, the fundamentals of Riesz space theory, including order projections, we look at the definition of a Banach lattice and the regular norm, and state Pettis' Theorem (Theorem IV.10.1 of [DS58]). It certainly does not cover all theory available on the specific subjects, but is put together to present an overview of the necessary basic knowledge of the theory we use throughout this thesis. More on Riesz spaces can be found in for example [AB06].

2.1 GENERAL FUNCTIONAL ANALYSIS

We look at specific function spaces that are used throughout this thesis, some of their properties and possible inclusion relationships. We finish this subsection by looking at the simple functions.

Notationwise we start by saying that for a Banach space X , we denote its *dual* by X^* . That is, $X^* = \{f : X \rightarrow \mathbb{R} : f \text{ is a bounded linear map}\}$. We assume that all scalar-valued functions are real-valued functions. We say that a Banach space X is *reflexive* whenever $X^{**} = X$. Recall that linear transformations between normed linear spaces are bounded if and only if they are continuous (see Lemma 4.1 of [RY07]).

Details of the following can be found in Section III.1 of [CO07]. Whenever K is any topological space, we may define the set $C_b(K)$ as the Banach algebra of all continuous functions for which $\|f\|_\infty < \infty$. Its unit is the constant function $\mathbf{1}$, which takes the value 1 at each point of K . The set $C_0(K)$ is the space of all continuous functions $f : K \rightarrow \mathbb{F}$ such that for all $\epsilon > 0$, $\{x \in K : |f(x)| \geq \epsilon\}$ is compact. Such a function is said to *vanish at infinity*. The function space $C_0(K)$ is a Banach algebra when it is equipped with the supremum norm. Whenever K is not compact, this space does not have a unit function. In case K is a compact space, $C_0(K) = C_b(K) = C(K)$, the space of \mathbb{R} -valued continuous functions on K equipped with the usual sup-norm $\|\cdot\|_\infty$; its unit is the constant function $\mathbf{1}$. In this case, $C(K)$ is a commutative Banach algebra relative to pointwise operations. The set $B(K) = B(K, \Omega)$, where $\Omega \subseteq \mathcal{P}(K)$ is a σ -algebra, is the Banach algebra with identity consisting of all bounded Ω -measurable functions $\phi : K \rightarrow \mathbb{R}$ with the usual supremum norm $\|\cdot\|_\infty$; its unit is the constant function $\mathbf{1}$ as well.

We will make use of (inclusion) relationships between the spaces introduced above. This is why we have created a list describing these relations in the settings that will be considered (Table 2.1). Recall that continuity implies Borel measurability.

Let us now consider the situation in which K is an arbitrary set and $\Omega \subseteq \mathcal{P}(K)$ a σ -algebra. The function $\chi_\Delta \in B(K)$ is the characteristic function of Δ for every $\Delta \in \Omega$. A function

The space K is	$\Omega \subset \mathcal{P}(K)$ is a	Relevant relation is
Hausdorff	Borel σ -algebra	$C_b(K) \subseteq B(K)$
locally compact Hausdorff	σ -algebra	$C_0(K) \subseteq C_b(K)$
compact Hausdorff	σ -algebra	$C_0(K) = C_b(K) = C(K)$
	Borel σ -algebra	$C(K) \subseteq B(K)$

Table 2.1: Relevant relationships between used function spaces.

$f \in B(K)$ is called *simple* if there is a finite $n \in \mathbb{N}$ and a finite family of pairwise disjoint members $\{\Delta_i\}_{i=1}^n$ of Ω satisfying $\cup_{i=1}^n \Delta_i = K$ and $\{\alpha_i\}_{i=1}^n \in \mathbb{R}$ so that f has a representation

$$f = \sum_{i=1}^n \alpha_i \chi_{\Delta_i}.$$

Denote by $S(K) \subseteq B(K)$ the class of simple functions. This is a dense subspace of $B(K)$ in the sup-norm topology. This means, for example, that multiplicativity of a function on $S(K)$ implies multiplicativity on $B(K)$.

2.2 RIESZ SPACES

This subsection treats the definition of an operator, discusses the notion of positivity and defines a Riesz space. We illustrate the theory with a few examples. We state important properties of elements of and operators on Riesz spaces, which we use in the upcoming sections. Recall that a binary relation \geq on a vector space is called a *partial ordering* if it is reflexive, antisymmetric and transitive.

Definition 2.1. A real vector space V together with a partial ordering \geq on V is called an *ordered vector space* if it satisfies the following two axioms (and hence is compatible with the algebraic structure of V) for all $x, y, z \in V$ and for every $\alpha \geq 0$

1. $x \geq y \Rightarrow x + z \geq y + z$
2. $x \geq y \Rightarrow \alpha x \geq \alpha y$.

We also use the notation $y \leq x$ for $x \geq y$. A vector x in an ordered vector space V is *positive* whenever $x \geq 0$. The set $V^+ := \{x \in V : x \geq 0\}$ is called the *positive cone* of V .

Definition 2.2. An *operator* is a linear map between two vector spaces.

If V and W are vector spaces, then the map $T : V \rightarrow W$ is an operator if and only if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $x, y \in V$ and $\alpha, \beta \in \mathbb{R}$.

Definition 2.3. An operator $T : V \rightarrow W$ where V and W are ordered vector spaces is a *positive operator* if $T(x) \geq 0$ for all $x \geq 0$. We write $T \geq 0$ or $0 \leq T$.

Note that whenever $T : V \rightarrow W$ is an operator and V and W are ordered vector spaces, T is positive if and only if $T(V^+) \subset W^+$, which is equivalent to $x \leq y$ implying $T(x) \leq T(y)$.

Definition 2.4. An ordered vector space V is called a *Riesz space* if for all $x, y \in V$ both $\sup\{x, y\}$ and $\inf\{x, y\}$ exist in V .

We use the following notation:

$$x \vee y := \sup\{x, y\} \quad x \wedge y := \inf\{x, y\}.$$

A lot of examples of Riesz spaces are function spaces. A *function space* is a vector space V of real-valued functions on a set Ω such that for all $f, g \in V$ and $\omega \in \Omega$

$$[f \vee g](\omega) := \max\{f(\omega), g(\omega)\} \quad \text{and} \quad [f \wedge g](\omega) := \min\{f(\omega), g(\omega)\}$$

exist in V . It is clear that every function space V with the pointwise ordering, i.e., $f \leq g$ in $V \Leftrightarrow f(\omega) \leq g(\omega)$ for all $\omega \in \Omega$, is a Riesz space.

Example 2.5. The function space $C(\Omega)$ of all continuous real-valued functions on a topological space Ω with the pointwise ordering is a Riesz space. \triangleleft

Example 2.6. Let (K, Ω, μ) be a measure space, where K is an arbitrary set, Ω is a σ -algebra of subsets of K and μ an arbitrary measure, and $0 < p < \infty$. Then $L^p(K, \Omega, \mu)$ is the vector space consisting of all real-valued μ -measurable functions f on K for which $\int_K |f|^p d\mu < \infty$. Two functions having different values solely on sets of measure zero are considered to be equal, i.e., $f = g$ in $L^p(K, \Omega, \mu)$ whenever $f(x) = g(x)$ for μ -almost all $x \in K$. This means that $L^p(K, \Omega, \mu)$ consists of equivalence classes of functions rather than functions. The vector space $L^\infty(K, \Omega, \mu)$ consists of all real-valued μ -measurable functions f on K that are essentially bounded, i.e., for which $\text{ess sup}|f| < \infty$.

The space $L^p(K, \Omega, \mu)$ with $0 < p \leq \infty$ under the ordering $f \leq g$ if and only if $f(x) \leq g(x)$ for μ -almost all $x \in K$ is a Riesz space.

We write $L^p(K)$ whenever Ω and μ are clear. \triangleleft

Every element of a Riesz space has a unique positive and negative part, which we define below. This allows us to decompose such an element into these unique parts.

Definition 2.7. Let x be a vector in a Riesz space. The *positive part* is $x^+ := x \vee 0$. The *negative part* is $x^- := (-x) \vee 0$. The *absolute value* is $|x| := x \vee (-x)$.

Theorem 2.8. Let x be a vector in a Riesz space V . Then

1. $x = x^+ - x^-$
2. $|x| = x^+ + x^-$
3. $x^+ \wedge x^- = 0$.

Proof. See the proof of Theorem 1.5 in [AB06]. \square

Lemma 2.9. Let V and W be Riesz spaces and $T : V \rightarrow W$ a positive operator. Then for every $x \in V$

$$|T(x)| \leq T(|x|).$$

Proof. See the proof of Lemma 1.6 in [AB06]. □

Definition 2.10. Two vectors x and y in a Riesz space are called *disjoint*, notation $x \perp y$, if $|x| \wedge |y| = 0$. Arbitrary subsets A and B of a Riesz space are *disjoint subsets* if $a \perp b$ for every $a \in A$ and $b \in B$.

From Theorem 1.7 (5) of [AB06] we know that $x \perp y \Leftrightarrow |x + y| = |x - y|$.

Remark 2.11. Note that the notion of orthogonality in Hilbert spaces is similar to disjointness in Riesz spaces: if \mathcal{H} is a Hilbert space and $f, g \in \mathcal{H}$, then f and g are *orthogonal* if $\langle f, g \rangle = 0$, which is denoted by $f \perp g$. (see Definition I.2.1 of [CO07]).

Definition 2.12. Let A be a nonempty subset of a Riesz space V . Then

$$A^d := \{x \in V : x \perp y \text{ for all } y \in A\}$$

is called the *disjoint complement* of A .

Note that $A \cap A^d = \{0\}$. The notion of disjointness is crucial in the definition of order projections. Thus we come to talk of disjointness a little more in the next subsection, that is dedicated to order projections on Riesz spaces.

Theorem 2.13. *Let x, y and z be elements of a Riesz space. Then*

$$||x| - |y|| \leq |x + y| \leq |x| + |y| \quad (\text{Triangle Inequality})$$

and

$$|x \vee z - y \vee z| \leq |x - y| \quad \text{and} \quad |x \wedge z - y \wedge z| \leq |x - y|. \quad (\text{Birkhoff's Inequalities})$$

Proof. See the proof of Theorem 1.9 (1) in [AB06]. □

In any Riesz space we also have

$$|x^+ - y^+| \leq |x - y| \quad \text{and} \quad |x^- - y^-| \leq |x - y|$$

for arbitrary x and y . This is an immediate result of Birkhoff's Inequalities (Theorem 2.13), since $x^+ = x \vee 0$ and $x^- = x \wedge 0$ for an arbitrary x in a Riesz space.

A net $\{x_\alpha\}$ in a Riesz space is called *decreasing* if $\alpha \succeq \beta \Rightarrow x_\alpha \leq x_\beta$. We denote this by $x_\alpha \downarrow$. Analogously, $\{x_\alpha\}$ is called *increasing* if $\alpha \succeq \beta \Rightarrow x_\alpha \geq x_\beta$. We denote this by $x_\alpha \uparrow$. By $x_\alpha \downarrow x$ we mean $x_\alpha \downarrow$ and $\inf\{x_\alpha\} = x$. Analogously, $x_\alpha \uparrow x$ means $x_\alpha \uparrow$ and $\sup\{x_\alpha\} = x$.

Definition 2.14. An ordered vector space V is called *Archimedean* whenever $\frac{1}{n}x \downarrow 0$ in V for all $x \in V^+$.

All classical spaces used in functional analysis, including the function spaces and L^p -spaces, $0 < p \leq \infty$, are Archimedean. This is why we focus on Archimedean spaces in this thesis.

Assumption 2.15. From now on all Riesz spaces in this section are assumed to be Archimedean.

Definition 2.16. Let V and W be ordered vector spaces. The real vector space of all operators from V to W will be denoted by $\mathcal{L}(V, W)$. This space is an ordered vector space with the ordering defined by $T \geq S$ whenever $T - S$ is a positive operator.

Definition 2.17. The *modulus* for an operator $T : V \rightarrow W$, where V and W are both Riesz spaces, exists if

$$|T| := T \vee (-T)$$

exists, that is, $|T|$ is the supremum of $\{T, -T\}$ in $\mathcal{L}_b(V, W)$.

Let V be a Riesz space. Then the subset of V defined by

$$[x, y] := \{z \in V : x \leq z \leq y\}$$

with $x, y \in V$ and $x \leq y$ is called an *order interval*. Let A be a subset of a Riesz space V . Then A is called *bounded above* if there exists an $x \in V$ such that $y \leq x$ for every $y \in A$. Analogously, A is called *bounded below* if there exists an $x \in V$ such that $y \geq x$ for every $y \in A$. A subset of a Riesz space is called *order bounded* if it is both bounded above and below. Or, equivalently, if it is included in an order interval.

Definition 2.18. An operator $T : V \rightarrow W$ between Riesz spaces is called *order bounded* if it maps order bounded subsets to order bounded subsets. The space of all order bounded operators between Riesz spaces V and W is denoted by $\mathcal{L}_b(V, W)$.

Definition 2.19. An operator $T : V \rightarrow W$ between Riesz spaces is called *regular* if it can be written as the difference of two positive operators. The space of all regular operators between Riesz spaces V and W is denoted by $\mathcal{L}_r(V, W)$.

Equivalently, an operator is regular if there exists a positive operator $S : V \rightarrow W$ satisfying $T \leq S$. The space $\mathcal{L}_r(V, W)$ is the vector space generated by the positive operators. Every positive operator is order bounded, and so every regular operator is order bounded as well. The following inclusions hold for Riesz spaces V and W

$$\mathcal{L}_r(V, W) \subseteq \mathcal{L}_b(V, W) \subseteq \mathcal{L}(V, W).$$

The ordering inherited from $\mathcal{L}(V, W)$ turns $\mathcal{L}_r(V, W)$ and $\mathcal{L}_b(V, W)$ into ordered vector spaces.

Definition 2.20. A Riesz space is called *Dedekind complete* whenever every nonempty bounded above subset has a supremum.

Or, equivalently, whenever every nonempty bounded below subset has an infimum. The following statement is equivalent as well:

$$0 \leq x_\alpha \uparrow \leq x \Rightarrow \sup\{x_\alpha\} \text{ exists.}$$

Example 2.21. Let (K, Ω, μ) be a measure space, where K is an arbitrary set, Ω is a σ -algebra of subsets of K and μ an arbitrary measure, and $1 \leq p < \infty$. Then $L^p(K, \Omega, \mu)$ as defined in Example 2.6 is a Dedekind complete Riesz space. When μ is σ -finite, then $L^\infty(K, \Omega, \mu)$ is a Dedekind complete Riesz space as well. (See Example v on page 9 of [ME91].) ◁

Theorem 1.18 of [AB06] states the following: if V is a Riesz space and W is a Dedekind complete Riesz space, then $\mathcal{L}_b(V, W)$ is a Dedekind complete Riesz space. As can be seen on page 15 of [AB06], in this case $\mathcal{L}_b(V, W)$ coincides with the vector subspace generated by the positive operators in $\mathcal{L}(V, W)$. Hence $\mathcal{L}_r(V, W) = \mathcal{L}_b(V, W)$ whenever W is Dedekind complete. The space $\mathcal{L}_b(V, \mathbb{R})$ consists of order bounded linear functionals. It is called the *order dual* and denoted by V^\sim . Since \mathbb{R} is a Dedekind complete Riesz space, the reasoning above tells us that V^\sim is generated by the positive linear functionals.

2.3 ORDER PROJECTIONS ON RIESZ SPACES

In Hilbert spaces orthogonality is a property that is used in many proofs. We will see that the disjointness defined for Riesz spaces can be used as an analogue of orthogonality in Hilbert spaces. The definition of an order projection relies on disjointness and as we shall see, they have the right properties to replace certain notions from the proof in the Hilbert space setting. This explains our interest in order projections.

A Riesz subspace A of a Riesz space V is called *order dense* if for every $x \in V$ with $x > 0$ there exists a $y \in A$ such that $0 < y \leq x$. A subset A of a Riesz space is called *solid* if $|x| \leq |y|$ and $y \in A$ implies $x \in A$. A vector subspace is called an *ideal* whenever it is solid.

Definition 2.22. A net $\{x_\alpha\}$ in a Riesz space is called *order convergent* to a vector x (in symbols $x_\alpha \xrightarrow{o} x$) if there exists a net $\{y_\alpha\}$ with the same index set satisfying $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for all α .

This is abbreviated as follows: $|x_\alpha - x| \leq y_\alpha \downarrow 0$.

Definition 2.23. A subset A of a Riesz space is called *order closed* whenever $\{x_\alpha\} \subseteq A$ and $x_\alpha \xrightarrow{o} x$ imply $x \in A$.

Definition 2.24. An order closed ideal of a Riesz space is called a *band*.

Note that for a subset A of a Riesz space we have that A^d is always a band (Theorem 1.8 of [AB06]). The *ideal generated* by a nonempty subset A of a Riesz space V is the smallest ideal that includes A . It is denoted by E_A . The *band generated* by the set A is the smallest band that includes A and is denoted by B_A .

Theorem 2.25. *The band generated by a nonempty subset A of an Archimedean Riesz space is A^{dd} .*

Proof. See the proof of Theorem 1.39 in [AB06]. □

Hence if A is a band, then $A = A^{dd}$.

Definition 2.26. A band B in a Riesz space V is called a *projection band* whenever

$$V = B \oplus B^d.$$

Theorem 2.27. *If B is a band in a Dedekind complete Riesz space V , then $V = B \oplus B^d$.*

Proof. See the proof of Theorem 1.42 in [AB06]. □

Thus in a Dedekind complete Riesz space every band is a projection band.

Definition 2.28. An operator $P : V \rightarrow V$, where V is a vector space, is called a *projection* if $P^2 = P$.

A projection P on a Riesz space is called a *positive projection* whenever P is a positive operator. Let B be a projection band in a Riesz space V . Then every $x \in V$ has a unique decomposition $x = x_1 + x_2$ with $x_1 \in B$ and $x_2 \in B^d$. The map $P_B : V \rightarrow V$ defined via $x \mapsto x_1$ is clearly a positive projection. A map of this form is called an *order projection*.

Theorem 2.29. *For an operator $T : V \rightarrow V$, where V is a Riesz space, the following are equivalent:*

1. T is an order projection,
2. T is a projection satisfying $0 \leq T \leq I$, where I is the identity on V ,
3. T and $I - T$ have disjoint ranges, i.e., $Tx \perp y - Ty$ for all $x, y \in V$.

Proof. See the proof of Theorem 1.44 in [AB06]. □

The second statement in the previous theorem will prove to be a very useful characterization of an order projection. The third shows that order projections are similar to projections on Hilbert spaces – as was to be expected. From Proposition II.3.3 of [CO07] we know that an idempotent E on H (i.e., a bounded linear map E on H such that $E^2 = E$) is a projection (i.e., an idempotent E such that $\ker(E) = (\text{ran}(E))^\perp$) if and only if it is the orthogonal projection of H onto $\text{ran}(E)$. The orthogonal projection is the map such that Ef is the unique point such that $\text{ran}(E) \perp f - Ef$ for $f \in H$.

2.4 BANACH LATTICES

Banach lattices are spaces that have both a partial ordering and a norm with respect to which they are complete. This combination of the concepts of ordering between, size of, and distance between vectors makes Banach lattices interesting objects of study in the class of Banach spaces.

Definition 2.30. A norm $\|\cdot\|$ on a Riesz space V is called a *lattice norm* whenever $|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$ for $x, y \in V$.

Definition 2.31. A Riesz space equipped with a lattice norm is called a *normed Riesz space*.

In a general normed Riesz space V we have for arbitrary $x, y \in V$,

$$\begin{aligned} \|x\| &= \||x|\| \\ \|x^+ - y^+\| &\leq \|x - y\| \\ \||x| - |y|\| &\leq \|x - y\|. \end{aligned}$$

All normed Riesz spaces are Archimedean (see page 307 of [ZA83]). Since we deal in the sections to come with *normed* Riesz spaces only, we need no assumption like Assumption 2.15 in order to be able to use all theorems and properties stated in this section.

Definition 2.32. A normed Riesz space that is norm complete is called a *Banach lattice*.

We give a few examples of Banach lattices below.

Example 2.33. The space $C(\Omega)$ for a topological space Ω introduced in Example 2.5 is a Banach space with the usual sup-norm $\|\cdot\|_\infty$. It is easy to see that $\|\cdot\|_\infty$ is a lattice norm. Whenever $|f| \leq |g|$, we know that $|f(\omega)| \leq |g(\omega)|$ for all $\omega \in \Omega$. Hence $\sup\{|f(\omega)| : \omega \in \Omega\} \leq \sup\{|g(\omega)| : \omega \in \Omega\}$. Hence it is a Banach lattice. \triangleleft

Example 2.34. Let (K, Ω, μ) be a measure space, where K is an arbitrary set, Ω a σ -algebra of subsets of K and μ an arbitrary measure. The spaces $L^p(K, \Omega, \mu)$, where $1 \leq p \leq \infty$, introduced in Example 2.6 are Banach spaces with norm $\|\cdot\|_p$ (see Theorem 15.7 and Remark 2 in §15 of [BA01]). Whenever $|f| \leq |g|$, we know that $|f(\omega)| \leq |g(\omega)|$ for almost all $\omega \in \Omega$. Hence $\int_K |f|^p d\mu \leq \int_K |g|^p d\mu$. This implies that $\|\cdot\|_p$ is a lattice norm and so $L^p(K)$ is a Banach lattice. \triangleleft

Example 2.35. Let K be a locally compact, non-compact space. Then $C_0(K)$ is the Banach lattice of all real-valued continuous functions on K that vanish at infinity (see Example 3 on page 99 of [SC74]). \triangleleft

The two theorems below will help us understand the way certain maps are constructed in Section 7.

Theorem 2.36. *The norm dual of a normed Riesz space is a Banach lattice.*

Proof. See the proof of Theorem 4.1 in [AB06]. \square

Theorem 2.37. *The norm completion of a normed Riesz space V is a Banach lattice including V as a Riesz subspace.*

Proof. See the proof of Theorem 4.2 in [AB06]. \square

Positive operators between Banach lattices are continuous. An even more general result was proven, stated below.

Theorem 2.38. *Every positive operator from a Banach lattice to a normed Riesz space is continuous.*

Proof. See the proof of Theorem 4.3 in [AB06]. \square

We deal with reflexive Banach lattices frequently, and in this context the following result is essential. It is presented as Corollary 4.5 in [AB06].

Proposition 2.39. *The dual of a Banach lattice V coincides with its order dual, i.e., $V^* = V^\sim$.*

The regular norm, that will be defined below, has pleasant properties with respect to the natural operator norm on $\mathcal{L}_r(\mathcal{V}, \mathcal{L}_r(X))$ as we will see in Subsection 3.3.

Definition 2.40. Let Y_1 and Y_2 be two Banach lattices. If $T : Y_1 \rightarrow Y_2$ is an operator with modulus, then the *regular norm* is defined by

$$\|T\|_r := \||T|\| := \sup \{ \||T|x|\| : \|x\| \leq 1 \}.$$

Clearly, $\|T\| \leq \|T\|_r$ holds. Furthermore, if $T, S \in \mathcal{L}_r(Y_1, Y_2)$, then $|T| \leq |S|$ implies $\||T|\| \leq \||S|\|$. If we assume that Y_2 is Dedekind complete, then Proposition 1.3.6 of [ME91] tells us that $\|T\|_r = \||T|\|$ for arbitrary $T \in \mathcal{L}_r(Y_1, Y_2)$. Hence in this case $|T| \leq |S|$ implies $\|T\|_r \leq \|S\|_r$. This means that $\|\cdot\|_r$ is a lattice norm for $\mathcal{L}_r(Y_1, Y_2) = \mathcal{L}_b(Y_1, Y_2)$ (equality follows from Theorem 1.3.2 of [ME91]).

Theorem 2.41. *If Y_1 and Y_2 are Banach lattices, and Y_2 is Dedekind complete, then the space $\mathcal{L}_b(Y_1, Y_2)$ under the regular norm is a Dedekind complete Banach lattice.*

Proof. See the proof of Theorem 4.74 in [AB06]. \square

If X is a Dedekind complete Banach lattice, then $\mathcal{L}_b(X) = \mathcal{L}_r(X)$. Theorem 2.41 tells us that $\mathcal{L}_b(X)$ is a Banach lattice with lattice norm $\|\cdot\|_r$, hence $\mathcal{L}_r(X)$ is a Banach lattice with the regular norm as its lattice norm in this situation as well.

The next definition tells us about the strong operator topology, which we need to define positive spectral measures and unital positive spectral measures.

Definition 2.42. The *strong operator topology* is the topology defined on $\mathcal{L}(X)$, where X is a Banach space with norm $\|\cdot\|$, by the family of seminorms $\{p_x\}_{x \in X}$, where $p_x(T) = \|T(x)\|$ for each $T \in \mathcal{L}(X)$.

Note that for a net $\{T_\lambda\} \subset \mathcal{L}(X)$ we have

$$T_\lambda \rightarrow T \text{ (SOT)} \iff \|T_\lambda(x) - T(x)\| \rightarrow 0 \text{ for each } x \in X.$$

2.5 PETTIS' THEOREM

The theorem presented in this subsection is used to prove that positive representations generate positive spectral measures.

Let K be a fixed set and Ω a σ -algebra of subsets of K . Let μ be an additive set function defined on Ω with values in a Banach space X . We suppose in addition that μ is weakly countably additive, that is,

$$\sum_{n=1}^{\infty} x^*(\mu(E_n)) = x^* \left(\mu \left(\bigcup_{n=1}^{\infty} E_n \right) \right)$$

for each $x^* \in X^*$ and each sequence of disjoint sets E_n in Ω .

Theorem 2.43 (Pettis). *A weakly countably additive vector-valued set function μ with values in a Banach space X defined on a σ -algebra $\Omega \subseteq \mathcal{P}(K)$, where K is an arbitrary set, is countably additive.*

Proof. See the proof of Theorem IV.10.1 in [DS58]. □

3 Positive spectral measures and representations

We already know of spectral measures and representations on various function spaces. See for example Definitions VIII.5.1 and IX.1.1 of [CO07], Definition XV.2.1 of [DS71] and page 499 of [PR07]. We will define positive spectral measures and positive representations on a Banach lattice X . We use different versions of spectral measures and representations in order to be able to generalize as much as possible, and go into detail as much as turns out to be interesting. In this section we focus on the definitions themselves and look at some immediate results. We finish this section with a subsection that summarizes a few results regarding the positive representations and unital positive representations, and the regular norm.

3.1 POSITIVE SPECTRAL MEASURES

Let K be an arbitrary set and $\Omega \subseteq \mathcal{P}(K)$ a σ -algebra. Let X be a Banach lattice with norm $\|\cdot\|$ and $\mathcal{L}_r(X)$ the ordered vector space space of all regular operators of X into itself; its unit is the identity operator I on X . The notion of a spectral measure in a Banach lattice context is defined below.

Definition 3.1. The map $E : \Omega \rightarrow \mathcal{L}_r(X)$, where K is a set, $\Omega \subseteq \mathcal{P}(K)$ a σ -algebra, and X a Banach lattice, is called a *positive spectral measure* when

1. $E(\Delta)$ is a positive projection for all $\Delta \in \Omega$;
2. $E(\emptyset) = 0$;
3. $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$ for all $\Delta_1, \Delta_2 \in \Omega$, and
4. E is SOT-countably additive, that is, if $\{\Delta_n\}_{n=1}^\infty$ are pairwise disjoint sets from Ω , then $E(\cup_{n=1}^\infty \Delta_n) = \sum_{n=1}^\infty E(\Delta_n)$ (SOT).

We will use this definition immediately to prove a useful property of these spectral measures.

Lemma 3.2. *Let $E : \Omega \rightarrow \mathcal{L}_r(X)$ be a positive spectral measure, where K is an arbitrary set, $\Omega \subseteq \mathcal{P}(K)$ a σ -algebra, and X a Banach lattice. Let $x \in X$ and $x^* \in X^*$. We define $\mu_{x,x^*}(\Delta) = \langle E(\Delta)x, x^* \rangle$. Then μ_{x,x^*} is a countably additive measure and $\|\mu_{x,x^*}\| \leq \langle E(K)|x|, |x^*| \rangle$. Equality holds when either $x \geq 0$ and $x^* \geq 0$ or $x \leq 0$ and $x^* \leq 0$.*

Proof. We start by proving that μ_{x,x^*} is a countably additive measure for arbitrary $x \in X$ and $x^* \in X^*$. It is clear that

$$\mu_{x,x^*}(\emptyset) = \langle E(\emptyset)x, x^* \rangle = \langle 0, x^* \rangle = 0.$$

Let $\{\Delta_n\}_{n=1}^\infty \subset \Omega$ consist of pairwise disjoint subsets of K that have unions in Ω . Then we find

$$\mu_{x,x^*} \left(\bigcup_{n=1}^\infty \Delta_n \right) = \left\langle E \left(\bigcup_{n=1}^\infty \Delta_n \right) x, x^* \right\rangle = \sum_{n=1}^\infty \langle E(\Delta_n)x, x^* \rangle = \sum_{n=1}^\infty \mu_{x,x^*}(\Delta_n).$$

Hence μ_{x,x^*} is a countably additive measure.

We will now look at its *total variation*. It is defined for a measure μ as

$$\|\mu\| = |\mu|(K),$$

where $|\mu|$ is defined to be the *variation* of μ ,

$$|\mu|(\Delta) = \sup \left\{ \sum_{i=1}^n |\mu(\Delta_i)| : \{\Delta_i\}_{i=1}^n \text{ is a measurable partition of } \Delta \right\}$$

for every $\Delta \in \Omega$. Let $x \in X$ and $x^* \in X^*$ such that $x \geq 0$ and $x^* \geq 0$ and $E : \Omega \rightarrow \mathcal{L}_r(X)$ a positive spectral measure. Let μ_{x,x^*} be as defined above. Then

$$\begin{aligned} \|\mu_{x,x^*}\| &= |\mu_{x,x^*}|(K) \\ &= \sup \left\{ \sum_{j=1}^n |\mu_{x,x^*}(\Delta_j)| : \{\Delta_j\}_{j=1}^n \subset \Omega \text{ pairwise disjoint, measurable and } K = \bigcup_{j=1}^n \Delta_j \right\}, \end{aligned}$$

and using the positivity of x and x^* we find for arbitrary pairwise disjoint and measurable $\{\Delta_j\}_{j=1}^n \subset \Omega$ with the property that $K = \cup_{j=1}^n \Delta_j$,

$$\begin{aligned} \sum_{j=1}^n |\mu_{x,x^*}(\Delta_j)| &= \sum_{j=1}^n |\langle E(\Delta_j)x, x^* \rangle| \\ &= \sum_{j=1}^n \langle E(\Delta_j)x, x^* \rangle && \text{since } x \geq 0, x^* \geq 0, \text{ and } E(\Delta_j) \geq 0 \\ &= \left\langle E \left(\bigcup_{j=1}^n \Delta_j \right) x, x^* \right\rangle && \text{by Definition 3.5.4} \\ &= \langle E(K)x, x^* \rangle. \end{aligned}$$

Since our partition was arbitrary, we may conclude that

$$\|\mu_{x,x^*}\| = \langle E(K)x, x^* \rangle. \tag{3.1}$$

In case $x \leq 0$ and $x^* \leq 0$, we can use the same argument since then

$$|\langle E(\Delta)x, x^* \rangle| = \langle E(\Delta)x, x^* \rangle$$

holds for every $\Delta \in \Omega$ as well.

Now let $x \in X$ and $x^* \in X^*$ be arbitrary. Then Theorem 2.8.1 tells us that there are $x^+, x^- \in X$ positive such that $x = x^+ - x^-$ and $(x^*)^+, (x^*)^- \in X^*$ positive such that

$x^* = (x^*)^+ - (x^*)^-$. So we have for a measurable partition $\{\Delta_j\}_{j=1}^n \subset \Omega$ of K that for each $j = 1, \dots, n$

$$\begin{aligned} |\mu_{x,x^*}(\Delta_j)| &= |\mu_{x^+ - x^-, (x^*)^+ - (x^*)^-}(\Delta_j)| \\ &= |\langle E(\Delta_j)(x^+ - x^-), ((x^*)^+ - (x^*)^-) \rangle| \\ &\leq |\langle E(\Delta_j)x^+, (x^*)^+ \rangle| + |\langle E(\Delta_j)x^+, (x^*)^- \rangle| \\ &\quad + |\langle E(\Delta_j)x^-, (x^*)^+ \rangle| + |\langle E(\Delta_j)x^-, (x^*)^- \rangle|, \end{aligned}$$

and so

$$\begin{aligned} \sum_{j=1}^n |\mu_{x,x^*}(\Delta_j)| &\leq \sum_{j=1}^n |\mu_{x^+, (x^*)^+}(\Delta_j)| + \sum_{j=1}^n |\mu_{x^+, (x^*)^-}(\Delta_j)| \\ &\quad + \sum_{j=1}^n |\mu_{x^-, (x^*)^+}(\Delta_j)| + \sum_{j=1}^n |\mu_{x^-, (x^*)^-}(\Delta_j)|. \end{aligned} \quad (3.2)$$

Looking at the total variation of μ we find that

$$\begin{aligned} \|\mu_{x,x^*}\| &= |\mu_{x,x^*}|(K) \\ &= \sup \left\{ \sum_{j=1}^n |\mu_{x,x^*}(\Delta_j)| : \{\Delta_j\}_{j=1}^n \subset \Omega \text{ measurable partition of } K \right\} \\ &\leq |\mu_{x^+, (x^*)^+}|(K) + |\mu_{x^+, (x^*)^-}|(K) + |\mu_{x^-, (x^*)^+}|(K) + |\mu_{x^-, (x^*)^-}|(K) \quad \text{using (3.2)} \\ &= \|\mu_{x^+, (x^*)^+}\| + \|\mu_{x^+, (x^*)^-}\| + \|\mu_{x^-, (x^*)^+}\| + \|\mu_{x^-, (x^*)^-}\|. \end{aligned}$$

For positive elements of X and X^* we can use (3.1), and so we have

$$\begin{aligned} \|\mu_{x,x^*}\| &\leq \langle E(K)x^+, (x^*)^+ \rangle + \langle E(K)x^+, (x^*)^- \rangle + \langle E(K)x^-, (x^*)^+ \rangle + \langle E(K)x^-, (x^*)^- \rangle \\ &= \langle E(K)x^+ + E(K)x^-, (x^*)^+ + (x^*)^- \rangle \\ &= \langle E(K)|x|, |x^*| \rangle. \end{aligned}$$

This finishes the proof. \square

Both the regular norm and the operator norm are defined on elements of $\mathcal{L}_r(X)$, and they are equal for positive elements. Since for all $\Delta \in \Omega$, $E(\Delta)$ is positive, we only deal with positive elements of $\mathcal{L}_r(X)$ in this subsection. Therefore, we can write $\|\cdot\|$ for the norm on $\mathcal{L}_r(X)$. In Subsection 3.3 we look at these norms in more detail.

Remark 3.3. Defining a positive spectral measure analogous to a spectral measure defined on Hilbert spaces (see Definition 1.1) would imply mapping E from Ω to $\mathcal{B}(X)$. The space $\mathcal{B}(X)$ is the Banach algebra of all bounded operators of X into itself with the operator norm topology; its unit is the identity operator I on X . Recall that an arbitrary $T \in \mathcal{B}(X) \subseteq \mathcal{L}(X)$ is contained in $\mathcal{L}_r(X)$ whenever it can be written as the difference of two positive elements. We know that $E(\Delta) \geq 0$ for all $\Delta \in \Omega$ for a positive spectral measure E . Hence the image of E is in $\mathcal{L}_r(X)$.

It is clear that $\langle E(K)|x|, |x^*| \rangle \leq \|E(K)\| \|x\| \|x^*\|$. The following corollary is therefore an immediate result of the previous lemma.

Corollary 3.4. *Let $E : \Omega \rightarrow \mathcal{L}_r(X)$ be a positive spectral measure, where K is an arbitrary set, $\Omega \subseteq \mathcal{P}(K)$ a σ -algebra, and X a Banach lattice. Let $x \in X$ and $x^* \in X^*$. Let μ_{x,x^*} be the countably additive measure on Ω defined in Lemma 3.2. Then we have $\|\mu_{x,x^*}\| \leq \|E(K)\| \|x\| \|x^*\|$.*

We will see that certain assumptions in the lemmas yet to come will give us a stronger version of a positive spectral measure. We will now look at its definition and what will happen with our previous results when we are dealing with that so-called unital positive spectral measure. Let K be an arbitrary set and $\Omega \subseteq \mathcal{P}(K)$ a σ -algebra. Let X be a Banach lattice with norm $\|\cdot\|$.

Definition 3.5. The map $E : \Omega \rightarrow \mathcal{L}_r(X)$, where K is a set, $\Omega \subseteq \mathcal{P}(K)$ a σ -algebra, and X a Banach lattice, is called a *unital positive spectral measure* when

1. $E(\Delta)$ is an order projection for all $\Delta \in \Omega$;
2. $E(\emptyset) = 0$ and $E(K) = I$;
3. $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$ for all $\Delta_1, \Delta_2 \in \Omega$, and
4. E is SOT-countably additive, that is, if $\{\Delta_n\}_{n=1}^\infty$ are pairwise disjoint sets from Ω , then $E(\cup_{n=1}^\infty \Delta_n) = \sum_{n=1}^\infty E(\Delta_n)$ (SOT).

Looking at the proof of the previous lemma, we see that there are a few equations to which the new definition specifically applies. We start with equation (3.1), where both $x \in X$ and $x^* \in X^*$ are positive. Since unital positive spectral measures have the property that $E(K) = I$, we see that

$$\|\mu_{x,x^*}\| = \langle x, x^* \rangle. \tag{3.3}$$

When we try bounding the total variation of μ , we use this result and will find

$$\|\mu_{x,x^*}\| = \langle |x|, |x^*| \rangle.$$

The result in the previous corollary can be upgraded as well, using that $\langle |x|, |x^*| \rangle \leq \|x\| \|x^*\|$. This leads to the following corollary.

Corollary 3.6. *Let $E : \Omega \rightarrow \mathcal{L}_r(X)$ be a unital positive spectral measure, where K is an arbitrary set, $\Omega \subseteq \mathcal{P}(K)$ a σ -algebra, and X a Banach lattice. Let $x \in X$ and $x^* \in X^*$. We define $\mu_{x,x^*}(\Delta) = \langle E(\Delta)x, x^* \rangle$. Then μ_{x,x^*} is a countably additive measure and $\|\mu_{x,x^*}\| \leq \langle |x|, |x^*| \rangle$. Equality holds when either $x \geq 0$ and $x^* \geq 0$ or $x \leq 0$ and $x^* \leq 0$. Moreover, $\|\mu_{x,x^*}\| \leq \|x\| \|x^*\|$.*

Remark 3.7. In a setting where the Banach lattice X is replaced by a Hilbert space H , we have a spectral measure as defined in Definition 1.1. Clearly, these properties resemble the properties of our unital positive spectral measure most. This might suggest that we will need that version of a spectral measure in order to get a two-sided correspondence just as was possible in §1 of Chapter IX of [CO07]. And indeed, this makes it possible to use arguments in the proofs yet to come that are very similar to the ones used in the Hilbert space setting as has been summarized in Subsection 1.3. However, the positive spectral measure will provide us with interesting results as well.

3.2 POSITIVE REPRESENTATIONS

Let X be a Banach lattice with norm $\|\cdot\|$. The main results in this thesis make use of representations of $C_0(K)$, where K is a locally compact Hausdorff space, and $C(K)$, where K is a compact Hausdorff space. We use representations of $C_b(K)$, where K is a Hausdorff space, as well. In certain proofs we need to extend our representation to a representation on $B(K) = B(K, \Omega)$, where Ω is a σ -algebra of subsets of K . This can only be done if $B(K)$ contains the original function space. Since continuity of a function implies Borel measurability, Ω should consist of Borel subsets in any case. If K is a Hausdorff space, then $C_b(K) \subseteq B(K)$. If K is locally compact Hausdorff, then $C_0(K) \subseteq B(K)$. If K is compact Hausdorff, then $C(K) \subseteq B(K)$. See Table 2.1 for more details. Recall that the function spaces $B(K)$, $C_b(K)$, $C_0(K)$, and $C(K)$, with K as described above in each case, are all Banach lattices. In the following definition \mathcal{V} can be either $C(K)$, $C_b(K)$, $C_0(K)$, or $B(K)$, where K is appropriately chosen.

Definition 3.8. Let X be a Banach lattice, and \mathcal{V} any of the spaces described above. A map $\rho : \mathcal{V} \rightarrow \mathcal{L}_r(X)$ is called a *positive representation* of \mathcal{V} in X if it is a continuous, linear, multiplicative, and positive map.

Remark 3.9. Defining a positive representation analogous to a representation defined on Hilbert spaces (see Subsection 1.3) would imply mapping ρ to $\mathcal{B}(X)$. However, we know that $\rho \geq 0$ for a positive representation E . Recall that an arbitrary $T \in \mathcal{B}(X) \subseteq \mathcal{L}(X)$ is contained in $\mathcal{L}_r(X)$ whenever it can be written as the difference of two positive elements. Hence indeed, for arbitrary $\phi \in \mathcal{V}$, where \mathcal{V} is one of the function spaces above, there exist positive $\phi^+, \phi^- \in \mathcal{V}$ such that $\phi = \phi^+ - \phi^-$, since \mathcal{V} is a Banach lattice. Because of the linearity of ρ we find $\rho(\phi) = \rho(\phi^+) - \rho(\phi^-)$ and because ρ is positive, both $\rho(\phi^+)$ and $\rho(\phi^-)$ are positive, which means that $\rho(\phi) \in \mathcal{L}_r(X)$.

For the following definition we have the same arguments for choosing \mathcal{W} as we did before for \mathcal{V} . Recall that if K is locally compact Hausdorff and not compact, $\mathbf{1} \notin C_0(K)$, and so this definition is not applicable to $C_0(K)$ in that case. If K is a compact Hausdorff space, then $C_0(K) = C(K)$ and $\mathbf{1} \in C_0(K) = C(K)$. So \mathcal{W} is either $C(K)$, $C_b(K)$, or $B(K)$, where K is a compact Hausdorff space, a Hausdorff space, or an arbitrary set, respectively.

Definition 3.10. Let X be a Banach lattice, and \mathcal{V} any of the spaces described above. A map $\rho : \mathcal{W} \rightarrow \mathcal{L}_r(X)$ is called a *unital positive representation* of \mathcal{W} in X if it is a continuous, linear, multiplicative, and positive map for which we have $\rho(\mathbf{1}) = I$.

Whenever X is a Dedekind complete Banach lattice, $\mathcal{L}_r(X) = \mathcal{L}_b(X)$. Theorem 2.41 tells us that $\mathcal{L}_b(X)$ is a Dedekind complete Banach lattice with lattice norm $\|\cdot\|_r$, hence $\mathcal{L}_r(X)$ is a Dedekind complete Banach lattice with respect to the regular norm as well.

Remark 3.11. Using the fact that all function spaces we use here are Banach lattices, we see that Theorem 2.38 is applicable to the maps of Definition 3.8 and 3.10, whenever X is a Dedekind complete Banach lattice, which turns $\mathcal{L}_r(X)$ into a Banach lattice. It tells us that a positive operator from a Banach lattice to a normed Riesz space is automatically continuous.

Hence, whenever X is a Dedekind complete Banach lattice, the continuity of the map ρ with respect to the regular norm, as described in Definition 3.8 and Definition 3.10, is immediate after its positivity has been established.

We look at two examples of maps that have all the properties of a unital positive representation.

Example 3.12. Let K be a compact Hausdorff space. Then $C(K)$ is a Banach lattice with the sup-norm $\|\cdot\|_\infty$ and the constant function $\mathbf{1}$ as its unit. The map $\rho : C(K) \rightarrow \mathcal{L}_r(C(K))$ defined as $\rho(g)f = g \cdot f$ for all $f, g \in C(K)$, i.e., through pointwise multiplication, is a unital positive representation which we prove below.

Let $f \in C(K)$ arbitrary. Then, first of all, $\rho(\mathbf{1})f = \mathbf{1} \cdot f = f$. Hence $\rho(\mathbf{1}) = I$. We also have for $\alpha, \beta \in \mathbb{R}$ and $g, h \in C(K)$,

$$\begin{aligned} \rho(\alpha g + \beta h)f &= (\alpha g + \beta h) \cdot f \\ &= \alpha g \cdot f + \beta h \cdot f \\ &= \alpha \rho(g)f + \beta \rho(h)f \\ &= (\alpha \rho(g) + \beta \rho(h))f, \end{aligned}$$

which proves that ρ is linear. Furthermore, for arbitrary $\phi, \psi \in C(K)$ that

$$\begin{aligned} \rho(g \cdot h)f &= g \cdot h \cdot f = \rho(g)(h \cdot f) \\ &= \rho(g)\rho(h)f \end{aligned}$$

and so ρ is multiplicative. For arbitrary $f, g \in C(K)$ we have

$$\|\rho(g)f\|_\infty = \|g \cdot f\|_\infty \leq \|g\|_\infty \|f\|_\infty,$$

from which we conclude that $\|\rho\|_r := \sup\{\|\rho(g)\|_r : g \in C(K) \text{ such that } \|g\|_\infty \leq 1\} \leq 1$. Details on this norm can be found in Subsection 3.3. This implies that ρ is continuous as well. For an element $g \in C(K)$ that is positive we know that $\rho(g)f = g \cdot f$ is positive if f is positive. Hence ρ is a positive map. Hence the map ρ as defined above is linear, continuous, multiplicative, positive, and has the property that $\rho(\mathbf{1}) = I$. Hence ρ is a unital positive representation. \triangleleft

Example 3.13. When we turn the map of Example 3.12 into a map to $\mathcal{L}_r(L^p(K))$, where $1 < p < \infty$ and K a compact Hausdorff space, we get a similar result. We can use similar arguments to find that ρ is linear, multiplicative, positive, and has the property that $\rho(\mathbf{1}) = I$. Since $\mathcal{L}_r(L^p(K))$ is a Dedekind complete Banach lattice, the remark above tells us that continuity follows from the positivity of ρ . Hence the map $\rho : C(K) \rightarrow \mathcal{L}_r(L^p(K))$ defined as

$$\rho(g)f = g \cdot f$$

for all $g \in C(K)$ and $f \in L^p(K)$ is linear, continuous, multiplicative, positive, and has the property that $\rho(\mathbf{1}) = I$. Thus it is a unital positive representation. \triangleleft

Remark 3.14. When we apply Theorem 2.41 to $(\mathcal{L}_b(X, \mathbb{R}), \|\cdot\|_r)$, where X is a reflexive Banach lattice, we find that $(\mathcal{L}_b(X, \mathbb{R}), \|\cdot\|_r)$ is a Dedekind complete Banach lattice. Recall

that $\mathcal{L}_b(X, \mathbb{R})$ is the order dual of X , X^\sim . Since X is a Banach lattice, its order dual is equal to its dual X^* (see Proposition 2.39). Hence X^* is a Dedekind complete Banach lattice. When we apply the theorem to $(\mathcal{L}_b(X^*, \mathbb{R}), \|\cdot\|_r)$, we see that X^{**} is a Dedekind complete Banach lattice as well. Since X is reflexive, this implies that X is a Dedekind complete reflexive Banach lattice. Thus, if a Banach lattice is reflexive, it is automatically Dedekind complete as well.

3.3 REGULAR NORM

We will be looking at various norms of a positive representation or unital positive representation ρ in this thesis. This is why we have gathered key equalities in this subsection.

The space \mathcal{V} can be either of the function spaces mentioned earlier in the previous subsection, and let X be a Banach lattice. Let $\rho : \mathcal{V} \rightarrow \mathcal{L}_r(X)$ be a positive representation, hence $\rho \in \mathcal{L}_r(\mathcal{V}, \mathcal{L}_r(X))$. Recall that there are two norms on the space $\mathcal{L}_r(X)$. The two norms are the operator norm and the regular norm. Whenever X is a Dedekind complete Banach lattice, $(\mathcal{L}_r(X), \|\cdot\|_r)$ is a Dedekind complete Banach lattice as well. By definition of the regular norm

$$\|\rho(\phi)\|_r = \sup\{\|\rho(\phi)x\| : x \in X \text{ such that } \|x\| \leq 1\}$$

for all $\phi \in \mathcal{V}$. This implies that $\|\rho(\phi)\|_r = \|\rho(\phi)\|$ for a positive $\phi \in \mathcal{V}$. Because of these two norms, we get two operator norms on $\mathcal{L}_r(\mathcal{V}, \mathcal{L}_r(X))$. We denote them by

$$\|\rho\| := \sup\{\|\rho(\phi)\| : \phi \in \mathcal{V} \text{ such that } \|\phi\|_\infty \leq 1\}, \quad (3.4)$$

and

$$\|\rho\|_r := \sup\{\|\rho(\phi)\|_r : \phi \in \mathcal{V} \text{ such that } \|\phi\|_\infty \leq 1\}. \quad (3.5)$$

We will show how these two norms are related to each other.

Lemma 3.15. *Let X be a Dedekind complete Banach lattice, K an arbitrary set, and $\Omega \subseteq \mathcal{P}(K)$ a σ -algebra. If $\rho : (B(K), \|\cdot\|_\infty) \rightarrow (\mathcal{L}_r(X), \|\cdot\|_r)$ is a positive representation, then $\|\rho\| = \|\rho\|_r = \|\rho(\mathbf{1})\|$.*

Proof. By the definition of the regular norm we have for each $\phi \in B(K)$,

$$\|\rho(\phi)\|_r = \sup\{\|\rho(\phi)x\| : \|x\| \leq 1\}.$$

Now take an arbitrary $\phi \in B(K)$, then

$$-\mathbf{1} \leq \frac{\phi}{\|\phi\|_\infty} \leq \mathbf{1}$$

and so, using the positivity of ρ for the upper and linearity for the lower bound,

$$-\rho(\mathbf{1}) \leq \rho\left(\frac{\phi}{\|\phi\|_\infty}\right) \leq \rho(\mathbf{1}).$$

This implies that

$$\left| \frac{\rho(\phi)}{\|\phi\|_\infty} \right| \leq \rho(\mathbf{1}).$$

Using that $\rho(\phi) \in \mathcal{L}_r(X)$ for every $\phi \in B(K)$, and $\|\cdot\|_r$ is a lattice norm on $\mathcal{L}_r(X)$, we find

$$\left\| \frac{\rho(\phi)}{\|\phi\|_\infty} \right\|_r \leq \|\rho(\mathbf{1})\|$$

and so $\|\rho(\phi)\|_r \leq \|\rho(\mathbf{1})\| \|\phi\|_\infty$. Hence $\|\rho\|_r \leq \|\rho(\mathbf{1})\|$. Since $\|\rho\| \leq \|\rho\|_r$, which follows immediately from the definition of $\|\cdot\|_r$ in (3.5), and the fact that

$$\|\rho\| = \sup\{\|\rho(\phi)\| : \|\phi\|_\infty \leq 1\} \geq \|\rho(\mathbf{1})\|,$$

we find that $\|\rho\| = \|\rho\|_r = \|\rho(\mathbf{1})\|$. □

For unital positive representations we know that $\rho(\mathbf{1}) = I$ and so $\|\rho(\mathbf{1})\| = 1$. We will see that the result in this case is useful later on, which is why we state it here.

Corollary 3.16. *Let X be a Dedekind complete Banach lattice, K an arbitrary set, and $\Omega \subseteq \mathcal{P}(K)$ a σ -algebra. If $\rho : (B(K), \|\cdot\|_\infty) \rightarrow (\mathcal{L}_r(X), \|\cdot\|_r)$ is a unital positive representation, then $\|\rho\| = \|\rho\|_r = 1$.*

The lemma and corollary stated above yield results for other function spaces as well. For $C_b(K)$ with K Hausdorff we may simply copy the proofs, since no properties of $B(K)$ have been used that do not hold for $C_b(K)$. For $C(K)$ with K compact Hausdorff, the same arguments hold.

Corollary 3.17. *Let X be a Dedekind complete Banach lattice, K a Hausdorff space, and $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra. If $\rho : (C_b(K), \|\cdot\|_\infty) \rightarrow (\mathcal{L}_r(X), \|\cdot\|_r)$ is a positive representation, then $\|\rho\| = \|\rho\|_r = \|\rho(\mathbf{1})\|$. If ρ is a unital positive representation, then $\|\rho\| = \|\rho\|_r = 1$. Whenever K is a compact Hausdorff space, we get the same results for $C(K) = C_b(K)$.*

In Subsection 5.2 we discuss what bounds hold for a positive representation of $C_0(K)$, where K is locally compact Hausdorff.

Remark 3.18. As we mentioned earlier, whenever X is a Dedekind complete Banach lattice, $\mathcal{L}_r(X)$ is a Dedekind complete Banach lattice under the regular norm. This implies that $\mathcal{L}_r(\mathcal{V}, \mathcal{L}_r(X))$ is a Dedekind complete Banach lattice under the regular norm as well. Since we look at positive operators ρ , we know that $|\rho| = \rho$. Therefore $\| |\rho|(\phi) \| = \|\rho(\phi)\|$ and $\| |\rho|(\phi) \|_r = \|\rho(\phi)\|_r$. This implies that the regular norm of ρ in this case provides us with the same two norms as the operator norm of ρ , defined in (3.4) and (3.5). Hence there is no need to treat this norm separately.

Remark 3.19. Using Remark 3.14, we find that the results of this subsection all hold if we assume that X is a reflexive Banach lattice as well.

4 Generating positive representations

As we have mentioned before, generating representations from spectral measures has been done in Hilbert space setting. Using certain assumptions on the representations and spectral measures there are results concerning this correspondence in a Banach space setting as well, as can be found in [PR07]. We will look at this in more detail in Section 8. In this section we look at the Banach lattice setting and see that we can find results analogous to the Hilbert space situation, not needing any restrictions on our representations and spectral measures. We start with positive spectral measures, from which we continue onwards to unital positive spectral measures. Both generate representations; in the first case positive representations, in the second case unital positive representations.

4.1 THE GENERAL CASE

Let K be a set, $\Omega \subset \mathcal{P}(K)$ a σ -algebra, X a Banach lattice with norm $\|\cdot\|$. We let $E : \Omega \rightarrow \mathcal{L}_r(X)$ be a positive spectral measure and μ_{x,x^*} be the corresponding countably additive measure (see Lemma 3.2). Let $\phi \in B(K)$. We define the map $A_\phi : X \times X^* \rightarrow \mathbb{R}$ through

$$A_\phi(x, x^*) = \int_K \phi \, d\mu_{x,x^*}.$$

Hence for arbitrary $x \in X$ and $x^* \in X^*$

$$\begin{aligned} \|A_\phi(x, x^*)\| &= \left\| \int_K \phi \, d\mu_{x,x^*} \right\| \\ &\leq \|\phi\|_\infty \|\mu_{x,x^*}\| \end{aligned}$$

and using the upper bound that was stated in Corollary 3.4 we find

$$\|A_\phi(x, x^*)\| \leq \|\phi\|_\infty \|E(K)\| \|x\| \|x^*\|. \quad (4.1)$$

Hence A_ϕ is bounded for all $\phi \in B(K)$. We also know that $\langle \cdot, \cdot \rangle : X \times X^* \rightarrow \mathbb{R}$ as used in the definition of μ is a bilinear map and so for every $\Delta \in \Omega$ and arbitrary $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$, $x, y \in X$ and $x^*, y^* \in X^*$ we find

$$\begin{aligned} \mu_{\alpha x + \beta y, \alpha' x^* + \beta' y^*}(\Delta) &= \langle E(\Delta)\alpha x + \beta y, \alpha' x^* + \beta' y^* \rangle \\ &= \alpha\alpha' \langle E(\Delta)x, x^* \rangle + \alpha\beta' \langle E(\Delta)x, y^* \rangle + \beta\alpha' \langle E(\Delta)y, x^* \rangle + \beta\beta' \langle E(\Delta)y, y^* \rangle \\ &= \alpha\alpha' \mu_{x,x^*}(\Delta) + \alpha\beta' \mu_{x,y^*}(\Delta) + \beta\alpha' \mu_{y,x^*}(\Delta) + \beta\beta' \mu_{y,y^*}(\Delta) \end{aligned}$$

from which we may conclude that A_ϕ is bilinear, hence a bounded bilinear map for every $\phi \in B(K)$.

The following lemma is a general result on the uniqueness of bounded operators generated by bounded bilinear maps. This result is needed to say more about the map A_ϕ we have defined above.

Lemma 4.1. *If X is a Banach space, then all bounded bilinear forms on $X \times X^*$ are of the form $(x, x^*) \mapsto \langle x^*, A(x) \rangle$, where $A : X \rightarrow X^{**}$ is a unique and bounded operator. Conversely, if $A : X \rightarrow X^{**}$ is a bounded operator, then $(x, x^*) \mapsto \langle x^*, A(x) \rangle$ is a bounded bilinear form on $X \times X^*$.*

Proof. Let X be a Banach space. Let $U : X \times X^* \rightarrow \mathbb{R}$ be a bounded bilinear map. Let $x \in X$ arbitrary. Then $L_x : X^* \rightarrow \mathbb{R}$ sending $x^* \mapsto U(x, x^*)$ is a well-defined bounded and linear map.

It is clear that $L_x \in X^{**}$ for every $x \in X$, hence there exists an $A : X \rightarrow X^{**}$ such that $x \mapsto L_x$. Since U is bounded, we know that the following holds for a certain $M \in \mathbb{R}^+$

$$\|A(x)(x^*)\| = \|L_x(x^*)\| = \|U(x, x^*)\| \leq M\|x\|\|x^*\|$$

and so we find that A is bounded. Thus U is of the form $(x, x^*) \mapsto \langle x^*, Ax \rangle$ for every $x \in X$ and $x^* \in X^*$.

We look at the uniqueness of such a bounded operator. Suppose A and \tilde{A} are both bounded maps mapping $x \mapsto L_x$. Since L_x is a well-defined map, we may conclude that then $A(x) = \tilde{A}(x)$ for all $x \in X$. This implies that $A = \tilde{A}$.

Let $x, y \in X$ and $x^* \in X^*$ arbitrary. Let $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned} A(\alpha x + \beta y)(x^*) &= L_{\alpha x + \beta y}(x^*) \\ &= U(\alpha x + \beta y, x^*) \\ &= \alpha U(x, x^*) + \beta U(y, x^*) && \text{since } U \text{ is bilinear} \\ &= \alpha L_x(x^*) + \beta L_y(x^*) \\ &= \alpha A(x)(x^*) + \beta A(y)(x^*) \\ &= (\alpha A(x) + \beta A(y))(x^*). \end{aligned}$$

So indeed, A is a linear map.

Hence every bounded linear form on $X \times X^*$ is of the form $(x, x^*) \mapsto \langle x^*, Ax \rangle$, where $A : X \rightarrow X^{**}$ is a unique and bounded operator.

For the converse statement, we start by stating that there exists an $M > 0$ such that

$$\|A(x)(x^*)\| \leq M\|x\|\|x^*\|, \tag{4.2}$$

because A is a bounded operator. Define $U : X \times X^* \rightarrow \mathbb{R}$ via $U(x, x^*) = A(x)(x^*) = \langle x^*, A(x) \rangle$ for all $x \in X$ and $x^* \in X^*$. Firstly, we prove its bilinearity. Let $\alpha, \beta \in \mathbb{R}$, $x, y \in X$, and $x^*, y^* \in X^*$. Then

$$\begin{aligned} U(\alpha x + \beta y, x^*) &= A(\alpha x + \beta y)(x^*) \\ &= \alpha A(x)(x^*) + \beta A(y)(x^*) \\ &= \alpha U(x, x^*) + \beta U(y, x^*), \end{aligned}$$

and

$$\begin{aligned}
 U(x, \alpha x^* + \beta y^*) &= A(x)(\alpha x^* + \beta y^*) \\
 &= \alpha A(x)(x^*) + \beta A(x)(y^*) && \text{since } A(x) \in X^{**} \\
 &= \alpha U(x, x^*) + \beta U(x, y^*).
 \end{aligned}$$

From this we may conclude that U is a bilinear form on $X \times X^*$. Let $x \in X$ and $x^* \in X^*$. Then

$$\|U(x, x^*)\| = \|A(x)(x^*)\| \leq M\|x\|\|x^*\|,$$

where we used (4.2) for the last inequality. This implies that $\|U\| \leq M$, and so that U is a bounded bilinear form.

We look at the uniqueness of such a bounded bilinear form. Suppose U and \tilde{U} are both bounded bilinear forms mapping $(x, x^*) \mapsto A(x)(x^*)$. Uniqueness of A then tells us that $U(x, x^*) = \tilde{U}(x, x^*)$ for all $x \in X$ and $x^* \in X^*$, which implies that $U = \tilde{U}$.

Hence U is a well-defined bounded bilinear form on $X \times X^*$. □

From now on we will assume that X is a reflexive Banach lattice. Lemma 4.1 tells us then that for each $\phi \in B(K)$, A_ϕ is of the form $\langle \rho(\phi)x, x^* \rangle = \int_K \phi \, d\mu_{x, x^*}$ for all $x \in X$ and $x^* \in X^*$, where $\rho(\phi) : X \rightarrow X^{**} = X$ is a unique and bounded operator. We use the upper bound of $A_\phi(x, x^*)$ for arbitrary $x \in X$ and $x^* \in X^*$ from equation (4.1) to find that for all $\phi \in B(K)$,

$$\|\rho(\phi)\|_r = \|A_\phi\| \leq \|E(K)\|\|\phi\|_\infty. \tag{4.3}$$

We will write $\rho(\phi) = \int_K \phi \, dE$. We will show that ρ is a positive representation.

Lemma 4.2. *Let X be a reflexive Banach lattice, K an arbitrary set, and $\Omega \subseteq \mathcal{P}(K)$ a σ -algebra. If $E : \Omega \rightarrow \mathcal{L}_r(X)$ is a positive spectral measure on Ω and $\rho : B(K) \rightarrow \mathcal{L}_r(X)$ is defined via $\rho(\phi) = \int_K \phi \, dE$, then ρ is a positive representation of $B(K)$. Moreover, $\rho(\chi_\Delta) = E(\Delta)$ for all $\Delta \in \Omega$.*

Proof. We have to show that the map ρ is linear, multiplicative, and positive. Its continuity then follows from its positivity as we have remarked in the previous section.

Let $\alpha, \beta \in \mathbb{R}$, $\phi, \psi \in B(K)$, $x \in X$ and $x^* \in X^*$. Then

$$\begin{aligned}
 \langle \rho(\alpha\phi + \beta\psi)x, x^* \rangle &= \int_K \alpha\phi + \beta\psi \, d\mu_{x, x^*} \\
 &= \alpha \int_K \phi \, d\mu_{x, x^*} + \beta \int_K \psi \, d\mu_{x, x^*} \\
 &= \alpha \langle \rho(\phi)x, x^* \rangle + \beta \langle \rho(\psi)x, x^* \rangle \\
 &= \langle (\alpha\rho(\phi) + \beta\rho(\psi))x, x^* \rangle,
 \end{aligned}$$

and so ρ is linear.

Let $x \in X$, $x^* \in X^*$ and $\Delta \in \Omega$. As an immediate result of the definition of μ_{x,x^*} we find

$$\begin{aligned} \langle \rho(\chi_\Delta)x, x^* \rangle &= \int_K \chi_\Delta d\mu_{x,x^*} \\ &= \mu_{x,x^*}(\Delta) \\ &= \langle E(\Delta)x, x^* \rangle. \end{aligned}$$

This, together with the uniqueness of the operator $\rho(\phi)$ that was proven in Lemma 4.1, implies that $\rho(\chi_\Delta) = E(\Delta)$.

For $x \geq 0$ and $x^* \geq 0$ we have that $\mu_{x,x^*}(\Delta) \geq 0$, because $E(\Delta) \geq 0$ for all $\Delta \in \Omega$. Hence $\mu_{x,x^*} \geq 0$. Thus for a positive $\phi \in B(K)$, and $x \in X$ and $x^* \in X^*$ both positive as well, we have that

$$\langle \rho(\phi)x, x^* \rangle = \int_K \phi d\mu_{x,x^*}$$

is positive as well. Hence $\rho \geq 0$, which also implies that ρ is continuous.

Now let $\Delta_1, \Delta_2 \in \Omega$. Then

$$\begin{aligned} \rho(\chi_{\Delta_1}\chi_{\Delta_2}) &= \rho(\chi_{\Delta_1 \cap \Delta_2}) \\ &= E(\Delta_1 \cap \Delta_2) \\ &= E(\Delta_1)E(\Delta_2) && \text{by Definition 3.5.3} \\ &= \rho(\chi_{\Delta_1})\rho(\chi_{\Delta_2}). \end{aligned}$$

From this, the fact that ρ is continuous on $B(K)$ and that the subspace of simple functions is dense in $B(K)$, we may conclude that ρ is multiplicative. This proves indeed that ρ is a positive representation. \square

Now that we have proven that $\rho : B(K) \rightarrow \mathcal{L}_r(X)$ is a positive representation, we look at what the consequences are for maps defined on subspaces of $B(K)$. If K is Hausdorff and Ω the Borel σ -algebra, then $C_b(K) \subseteq B(K)$, hence $\rho_1 : C_b(K) \rightarrow \mathcal{L}_r(X)$ defined analogously to the representation of $B(K)$ in Lemma 4.2, is a positive representation as well. If K is locally compact Hausdorff, and not compact, and Ω the Borel σ -algebra, then $C_0(K) \subseteq C_b(K)$. Hence we can define a map on $C_0(K)$ similar to ρ_1 that is a positive representation. If K is compact Hausdorff and Ω the Borel σ -algebra, then $C(K) = C_b(K)$. Therefore, we can define a map on $C(K)$ similar to ρ_1 that is a positive representation as well. This is summarized in the following corollary.

Corollary 4.3. *Let X be a reflexive Banach lattice, K a Hausdorff space, and $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra. If $E : \Omega \rightarrow \mathcal{L}_r(X)$ is a positive spectral measure on Ω and $\rho : C_b(K) \rightarrow \mathcal{L}_r(X)$ is defined via $\rho(\phi) = \int_K \phi dE$, then ρ is a positive representation of $C_b(K)$. The same holds for $C_0(K)$ whenever K is locally compact Hausdorff, and for $C(K) = C_b(K)$ whenever K is a compact Hausdorff space.*

4.2 THE UNITAL CASE

When we change the assumptions used for Lemma 4.2, taking a unital positive spectral measure this time, we see that the representation that it generates is unital.

Lemma 4.4. *Let X be a reflexive Banach lattice, K an arbitrary set, and $\Omega \subseteq \mathcal{P}(K)$ a σ -algebra. If $E : \Omega \rightarrow \mathcal{L}_r(X)$ is a unital positive spectral measure on Ω and $\rho : B(K) \rightarrow \mathcal{L}_r(X)$ is defined via $\rho(\phi) = \int_K \phi dE$, then ρ is a unital positive representation of $B(K)$. Moreover, $\rho(\chi_\Delta) = E(\Delta)$ for all $\Delta \in \Omega$.*

Proof. Next to what was proven for Lemma 4.2, we have to show that $\rho(\mathbf{1}) = I$. We already know that $\rho(\mathbf{1}) = E(K)$. Since we assumed that E is a unital positive spectral measure, we know that $E(K) = I$. This shows immediately that indeed $\rho(\mathbf{1}) = I$, which finishes the proof. \square

Now that we have proven that $\rho : B(K) \rightarrow \mathcal{L}_r(X)$ is a unital positive representation, we look at what the consequences are for maps defined on subspaces of $B(K)$. If K is Hausdorff and Ω the Borel σ -algebra, then $C_b(K) \subseteq B(K)$, hence $\rho_1 : C_b(K) \rightarrow \mathcal{L}_r(X)$ defined analogously to the representation of $B(K)$ in Lemma 4.4, is a unital positive representation as well. If K is compact Hausdorff and Ω the Borel σ -algebra, then $C(K) = C_b(K)$. Therefore, all that has been said about ρ_1 still holds. Therefore, we can define a map on $C(K)$ similar to ρ_1 that is a unital positive representation as well. This is summarized in the following corollary.

Corollary 4.5. *Let X be a reflexive Banach lattice, K a Hausdorff space, and $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra. If $E : \Omega \rightarrow \mathcal{L}_r(X)$ is a unital positive spectral measure on Ω and $\rho : C_b(K) \rightarrow \mathcal{L}_r(X)$ is defined via $\rho(\phi) = \int_K \phi dE$, then ρ is a unital positive representation of $C_b(K)$. The same holds whenever K is a compact Hausdorff space.*

5 Generating positive spectral measures

In this section we look at the converse of the statement that was treated in Section 4, in which we showed that positive spectral measures generate positive representations. An important ingredient for the proof is the Riesz Representation Theorem (Theorem III.5.7 of [CO07]). Note that the Riesz Representation Theorem holds for representations of $C_0(K)$, where K is a locally compact space, which implies that we cannot prove the statement in this direction for $C_b(K)$, where K is a Hausdorff space, or $B(K)$, where K is an arbitrary set.

5.1 THE GENERAL CASE

This subsection is dedicated to the proof of the lemma stated below. It is a converse of Corollary 4.5, which stated that a map $\rho : C_0(K) \rightarrow \mathcal{L}_r(X)$ defined via $\rho(\phi) = \int_K \phi dE$, where K is locally compact Hausdorff, $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra, X a reflexive Banach lattice, and E a positive spectral measure, is a positive representation. This proof of this lemma is more intricate than that of its counterpart.

Lemma 5.1. *Let K be a locally compact Hausdorff space, $\Omega \subset \mathcal{P}(K)$ the Borel σ -algebra, and X a reflexive Banach lattice. Let $\rho : C_0(K) \rightarrow \mathcal{L}_r(X)$ be a positive representation. Then there exists a unique positive spectral measure E defined on the Borel subsets of K such that μ_{x,x^*} is a regular countably additive Borel measure for all $x \in X$ and $x^* \in X^*$, and $\rho(\phi) = \int_K \phi dE$ for all $\phi \in C_0(K)$.*

Proof. We start with the uniqueness of such a positive spectral measure. Suppose we have two; E and \tilde{E} , both corresponding to a countably additive measure, respectively μ and $\tilde{\mu}$ (see Lemma 3.2 and Corollary 3.4 for the properties of these countably additive measures). Let $x \in X$ and $x^* \in X^*$ arbitrary. Then we know that

$$\langle \rho(\phi)x, x^* \rangle = \int_K \phi d\mu_{x,x^*}$$

and

$$\langle \rho(\phi)x, x^* \rangle = \int_K \phi d\tilde{\mu}_{x,x^*}.$$

Then, because of the uniqueness in the Riesz Representation Theorem (Theorem III.5.7 of [CO07]), we know that $\mu_{x,x^*} = \tilde{\mu}_{x,x^*}$ for every $x \in X$, $x^* \in X^*$. Hence for every $\Delta \in \Omega$ we have that $\langle E(\Delta)x, x^* \rangle = \langle \tilde{E}(\Delta)x, x^* \rangle$ for every $x \in X$, $x^* \in X^*$. We may thus conclude that $E(\Delta) = \tilde{E}(\Delta)$ for all $\Delta \in \Omega$, i.e., $E = \tilde{E}$.

We will proceed with its existence. Let $x \in X$ and $x^* \in X^*$. Then for $\phi \in C_0(K)$ the map $\phi \mapsto \langle \rho(\phi)x, x^* \rangle$ is a linear functional on $C_0(K)$ with norm

$$|\langle \rho(\phi)x, x^* \rangle| \leq \|\rho\|_r \|\phi\|_\infty \|x\| \|x^*\| \leq M \|\phi\|_\infty \|x\| \|x^*\|. \quad (5.1)$$

Here we used that $\|\rho\|_r \leq M$ for a certain $M < \infty$, which holds because a map from $C_0(K)$ to $\mathcal{L}_r(X)$ is bounded. Furthermore we used that x^* is a bounded linear functional on X , and that $\rho(\phi)$ is a bounded operator on X . By the Riesz Representation Theorem (Theorem III.5.7 of [CO07]) there exists a unique regular Borel measure μ_{x,x^*} such that $\langle \rho(\phi)x, x^* \rangle = \int_K \phi d\mu_{x,x^*}$ for all $\phi \in C_0(K)$. Furthermore, we know that $(x, x^*) \mapsto \mu_{x,x^*}$ is bilinear, and from the last inequality in (5.1)

$$\|\mu_{x,x^*}\| \leq M \|x\| \|x^*\|.$$

For all $\phi \in B(K)$ we define $[x, x^*]_\phi = \int_K \phi d\mu_{x,x^*}$. Then $[\cdot, \cdot]_\phi$ is bilinear and $|[x, x^*]_\phi| \leq M \|\phi\|_\infty \|x\| \|x^*\|$. According to Lemma 4.1 we now have a unique operator $\tilde{\rho}(\phi) \in \mathcal{L}_r(X)$ such that $\langle \tilde{\rho}(\phi)x, x^* \rangle = \int_K \phi d\mu_{x,x^*}$ for all $\phi \in B(K)$. Then for all $\phi \in B(K)$, $\|\tilde{\rho}(\phi)\|_r \leq M \|\phi\|_\infty$.

Now we will prove that $\tilde{\rho}$ is a positive representation, and that it is equal to ρ on $C_0(K)$. Let $x \in X$ and $x^* \in X^*$. By definition of μ_{x,x^*} , it is clear that $\tilde{\rho}(\phi) = \rho(\phi)$ for all $\phi \in C_0(K)$. Now let $\phi, \psi \in B(K)$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned} \langle \tilde{\rho}(\alpha\phi + \beta\psi)x, x^* \rangle &= \int_K \alpha\phi + \beta\psi d\mu_{x,x^*} \\ &= \alpha \int_K \phi d\mu_{x,x^*} + \beta \int_K \psi d\mu_{x,x^*} \\ &= \alpha \langle \tilde{\rho}(\phi)x, x^* \rangle + \beta \langle \tilde{\rho}(\psi)x, x^* \rangle \\ &= \langle (\alpha\tilde{\rho}(\phi) + \beta\tilde{\rho}(\psi))x, x^* \rangle, \end{aligned}$$

and so $\tilde{\rho}$ is linear. From the definition of $\tilde{\rho}$ we know that it is bounded: $\|\tilde{\rho}(\phi)\|_r \leq M \|\phi\|_\infty$. For multiplicativity let $\phi, \psi \in C_0(K)$, then

$$\begin{aligned} \int_K \phi\psi d\mu_{x,x^*} &= \langle \rho(\phi\psi)x, x^* \rangle \\ &= \langle \rho(\phi)\rho(\psi)x, x^* \rangle \\ &= \int_K \phi d\mu_{\rho(\psi)x, x^*}. \end{aligned}$$

By uniqueness of μ_{x,x^*} we have

$$\psi d\mu_{x,x^*} = d\mu_{\rho(\psi)x, x^*}.$$

Thus for every $\tilde{\phi} \in B(K)$: $\int_K \tilde{\phi}\psi d\mu_{x,x^*} = \int_K \tilde{\phi} d\mu_{\rho(\psi)x, x^*}$. Hence

$$\begin{aligned} \int_K \tilde{\phi}\psi d\mu_{x,x^*} &= \int_K \tilde{\phi} d\mu_{\rho(\psi)x, x^*} \\ &= \langle \tilde{\rho}(\tilde{\phi})\rho(\psi)x, x^* \rangle \\ &= \langle \rho(\psi)x, (\tilde{\rho}(\tilde{\phi}))^* x^* \rangle \\ &= \int_K \psi d\mu_{x, (\tilde{\rho}(\tilde{\phi}))^* x^*}. \end{aligned}$$

So now for $\tilde{\psi} \in B(K)$ we have

$$\begin{aligned} \int_K \tilde{\phi} \tilde{\psi} d\mu_{x,x^*} &= \int_K \tilde{\psi} d\mu_{x,(\tilde{\rho}(\tilde{\phi}))^*x^*} \\ &= \langle \tilde{\rho}(\tilde{\psi})x, (\tilde{\rho}(\tilde{\phi}))^*x^* \rangle \\ &= \langle \tilde{\rho}(\tilde{\phi})\tilde{\rho}(\tilde{\psi})x, x^* \rangle \end{aligned}$$

and also

$$\int_K \tilde{\phi} \tilde{\psi} d\mu_{x,x^*} = \langle \tilde{\rho}(\tilde{\phi}\tilde{\psi})x, x^* \rangle,$$

which proves the multiplicativity. For $x \geq 0$, $x^* \geq 0$ we have that $\mu_{x,x^*} \geq 0$, because $\rho \geq 0$. Thus for a positive $\phi \in B(K)$, and $x \in X$ and $x^* \in X^*$ both positive as well, we have that

$$\langle \tilde{\rho}(\phi)x, x^* \rangle = \int_K \phi d\mu_{x,x^*}$$

is positive as well. Hence $\tilde{\rho} \geq 0$. Hence $\tilde{\rho}$ has all wanted properties and is a positive representation.

Now we define $E : \Omega \rightarrow \mathcal{L}_r(X)$ through $E(\Delta) = \tilde{\rho}(\chi_\Delta)$ for every $\Delta \in \Omega$. We will prove that E is a positive spectral measure. Let $\Delta \in \Omega$. Note that $\chi_\Delta^2 = \chi_{\Delta \cap \Delta} = \chi_\Delta$. Using the multiplicativity of $\tilde{\rho}$ we find

$$E(\Delta)^2 = \tilde{\rho}(\chi_\Delta)^2 = \tilde{\rho}(\chi_\Delta^2) = \tilde{\rho}(\chi_\Delta),$$

and so $E(\Delta)^2 = E(\Delta)$, which implies that $E(\Delta)$ is a projection. Furthermore we have $E(\Delta) = \tilde{\rho}(\chi_\Delta) \geq 0$, since $\chi_\Delta \geq 0$ and $\tilde{\rho}$ is positive, so $E(\Delta)$ is a positive projection. It is also clear that $E(\emptyset) = \tilde{\rho}(\chi_\emptyset) = \tilde{\rho}(0) = 0$.

For $\Delta_1, \Delta_2 \in \Omega$ we have

$$\begin{aligned} E(\Delta_1 \cap \Delta_2) &= \tilde{\rho}(\chi_{\Delta_1 \cap \Delta_2}) = \tilde{\rho}(\chi_{\Delta_1} \chi_{\Delta_2}) \\ &= \tilde{\rho}(\chi_{\Delta_1}) \tilde{\rho}(\chi_{\Delta_2}) \\ &= E(\Delta_1) E(\Delta_2). \end{aligned}$$

Let $\{\Delta_n\}_{n=1}^\infty \subset \Omega$ be pairwise disjoint elements of Ω , $x \in X$ and $x^* \in X^*$ both positive and $\Delta = \cup_{n=1}^\infty \Delta_n$. Then for an arbitrary $N \in \mathbb{N}$

$$\begin{aligned} \left| \langle E(\Delta)x, x^* \rangle - \sum_{n=1}^N \langle E(\Delta_n)x, x^* \rangle \right| &= \left| \int_\Delta d\mu_{x,x^*} - \sum_{n=1}^N \int_{\Delta_n} d\mu_{x,x^*} \right| \\ &= \left| \int_\Delta d\mu_{x,x^*} - \int_{\cup_{n=1}^N \Delta_n} d\mu_{x,x^*} \right| \\ &= \left| \int_K \chi_\Delta - \sum_{n=1}^N \chi_{\Delta_n} d\mu_{x,x^*} \right|, \end{aligned}$$

and for $N \rightarrow \infty$ it is immediate that $\chi_\Delta - \sum_{n=1}^N \chi_{\Delta_n} \rightarrow \chi_\emptyset = 0$, and so

$$\left| \langle E(\Delta)x, x^* \rangle - \sum_{n=1}^N \langle E(\Delta_n)x, x^* \rangle \right| \rightarrow 0.$$

Hence $\sum_{n=1}^{\infty} \langle E(\Delta_n)x, x^* \rangle = \langle E(\cup_{n=1}^{\infty} \Delta_n)x, x^* \rangle$ for $x \geq 0$ and $x^* \geq 0$. Now let $x \in X$ and $x^* \in X^*$ both be arbitrary. We know that there are positive $x^+, x^- \in X$ and positive $(x^*)^+, (x^*)^- \in X^*$ such that $x = x^+ - x^-$ and $x^* = (x^*)^+ - (x^*)^-$. Then for an arbitrary $N \in \mathbb{N}$

$$\begin{aligned} \left| \langle E(\Delta)x, x^* \rangle - \sum_{n=1}^N \langle E(\Delta_n)x, x^* \rangle \right| &\leq \left| \langle E(\Delta)x^+, (x^*)^+ \rangle - \sum_{n=1}^N \langle E(\Delta_n)x^+, (x^*)^+ \rangle \right| \\ &+ \left| -\langle E(\Delta)x^+, (x^*)^- \rangle + \sum_{n=1}^N \langle E(\Delta_n)x^+, (x^*)^- \rangle \right| \\ &+ \left| -\langle E(\Delta)x^-, (x^*)^+ \rangle + \sum_{n=1}^N \langle E(\Delta_n)x^-, (x^*)^+ \rangle \right| \\ &+ \left| \langle E(\Delta)x^-, (x^*)^- \rangle - \sum_{n=1}^N \langle E(\Delta_n)x^-, (x^*)^- \rangle \right|, \end{aligned}$$

hence

$$\begin{aligned} \left| \langle E(\Delta)x, x^* \rangle - \sum_{n=1}^N \langle E(\Delta_n)x, x^* \rangle \right| &\leq \left| \int_K \chi_{\Delta} - \sum_{n=1}^N \chi_{\Delta_n} d\mu_{x^+, (x^*)^+} \right| \\ &+ \left| -\int_K \chi_{\Delta} - \sum_{n=1}^N \chi_{\Delta_n} d\mu_{x^+, (x^*)^-} \right| \\ &+ \left| -\int_K \chi_{\Delta} - \sum_{n=1}^N \chi_{\Delta_n} d\mu_{x^-, (x^*)^+} \right| \\ &+ \left| \int_K \chi_{\Delta} - \sum_{n=1}^N \chi_{\Delta_n} d\mu_{x^-, (x^*)^-} \right|. \end{aligned}$$

Using the same argument as before each term will go to zero when $N \rightarrow \infty$. So now we have proven that for all $x \in X, x^* \in X^*$

$$\sum_{n=1}^{\infty} \langle E(\Delta_n)x, x^* \rangle = \left\langle E \left(\bigcup_{n=1}^{\infty} \Delta_n \right) x, x^* \right\rangle.$$

Theorem 2.43 by Pettis tells us now that for arbitrary $x \in X, x^* \in X^*$ we have

$$\sum_{n=1}^{\infty} E(\Delta_n) = E \left(\bigcup_{n=1}^{\infty} \Delta_n \right) \quad (\text{SOT}),$$

i.e., E is SOT-countably additive. The previous proves that E is indeed a positive spectral measure. By construction

$$\langle \tilde{\rho}(\phi)x, x^* \rangle = \int_K \phi d\mu_{x, x^*}$$

and

$$\begin{aligned} \mu_{x, x^*}(\Delta) &= \int_K \chi_{\Delta} d\mu_{x, x^*} \\ &= \langle \tilde{\rho}(\chi_{\Delta})x, x^* \rangle. \end{aligned}$$

Hence by definition of E

$$\mu_{x,x^*}(\Delta) = \langle E(\Delta)x, x^* \rangle,$$

so indeed E is a positive spectral measure defined on the Borel subsets of K . Moreover, we may use Lemma 3.2 to argue that μ_{x,x^*} is a countably additive measure on the Borel subsets of K as well as being a regular Borel measure. \square

So now we have proven that every positive representation of $C_0(K)$, where K is a locally compact Hausdorff space, on a reflexive Banach lattice generates a positive spectral measure on the Borel σ -algebra of subsets of K .

The following is an immediate result of the previous lemma.

Corollary 5.2. *Let $\rho : C_0(K) \rightarrow \mathcal{L}_r(X)$, where K is locally compact Hausdorff, $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra, and X a reflexive Banach lattice, be a positive representation. Then there exists a positive projection P such that*

$$0 \leq \rho(\phi) \leq \|\phi\|_\infty P \tag{5.2}$$

for all positive $\phi \in C_0(K)$.

Proof. Define $P := \tilde{\rho}(\mathbf{1}) = E(K)$, where $\tilde{\rho}$ as defined in the proof of Lemma 5.1. Then P is a positive projection, since E is a positive spectral measure. It is trivial that for an arbitrary $\phi \in C_0(K)$,

$$-\|\phi\|_\infty \mathbf{1} \leq \phi \leq \|\phi\|_\infty \mathbf{1}.$$

Let $\phi \in C_0(K)$ be an arbitrary positive element. Then by definition of a positive representation, $\rho(\phi) \geq 0$. The positivity of ϕ in our case implies that

$$0 \leq \phi \leq \|\phi\|_\infty \mathbf{1},$$

from which we find

$$0 \leq \rho(\phi) \leq \|\phi\|_\infty \tilde{\rho}(\mathbf{1}) = \|\phi\|_\infty P,$$

which finishes the proof. \square

5.2 REGULAR NORM

We extend the results of Subsection 3.3 to positive representations of $C_0(K)$, where K is a locally compact Hausdorff space.

Let K be a locally compact Hausdorff that is not compact, Ω the Borel σ -algebra, and X a reflexive Banach lattice. Then $C_0(K) \subseteq C_b(K)$. In this case $C_0(K)$ does not have a unit. This implies that the norm cannot be bounded by the norm of $\rho(\mathbf{1})$ as was done in Corollary

3.17. Note, however, that we can extend ρ to $B(K)$. Let us define $\tilde{\rho} : B(K) \rightarrow \mathcal{L}_r(X)$, such that $\tilde{\rho}(\phi) = \rho(\phi)$ for all $\phi \in C_0(K)$, which is a positive representation as well, as proven in Lemma 5.1. For all $\phi \in C_0(K)$ it is clear that

$$-\|\phi\|_\infty \mathbf{1} \leq \phi \leq \|\phi\|_\infty \mathbf{1}.$$

This implies that

$$-\|\phi\|_\infty \tilde{\rho}(\mathbf{1}) \leq \rho(\phi) \leq \|\phi\|_\infty \tilde{\rho}(\mathbf{1}),$$

from which we conclude

$$|\rho(\phi)| \leq \|\phi\|_\infty \tilde{\rho}(\mathbf{1}),$$

where we used the positivity of $\tilde{\rho}$. Since the regular norm is a lattice norm on $\mathcal{L}_r(X)$, and $\phi \in C_0(K)$ was arbitrary, we find that

$$\|\rho\|_r \leq \|\phi\|_\infty \|\tilde{\rho}(\mathbf{1})\|.$$

Hence $\|\rho\|_r \leq \|\tilde{\rho}(\mathbf{1})\|$. Using equations (3.4) and (3.5), we then find $\|\rho\| \leq \|\rho\|_r \leq \|\tilde{\rho}(\mathbf{1})\|$. This gives us the following result.

Corollary 5.3. *Let X be a reflexive Banach lattice, K a locally compact Hausdorff space, and $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra. If $\rho : (C_0(K), \|\cdot\|_\infty) \rightarrow (\mathcal{L}_r(X), \|\cdot\|_r)$ is a positive representation, then $\|\rho\| \leq \|\rho\|_r \leq \|\tilde{\rho}(\mathbf{1})\|$.*

5.3 COMMUTING OF $E(\Delta)$ WITH REGULAR OPERATORS ON X

Suppose X is a reflexive Banach lattice and K a locally compact Hausdorff space. Let $\Omega \subseteq \mathcal{P}(K)$ be the Borel σ -algebra.

Lemma 5.4. *Let $\rho : C_0(K) \rightarrow \mathcal{L}_r(X)$ be a positive representation and $\phi \in C_0(K)$. Let $E : \Omega \rightarrow \mathcal{L}_r(X)$ be the corresponding positive spectral measure and $\Delta \in \Omega$. Let $T \in \mathcal{L}_r(X)$. The map T commutes with all $\rho(\phi)$ if and only if T commutes with $E(\Delta)$, for all $\Delta \in \Omega$.*

Proof. “ \Rightarrow ” Suppose $T \in \mathcal{L}_r(X)$ commutes with $\rho(\phi)$, where $\phi \in C_0(K)$ and ρ is a positive representation of $C_0(K)$. Hence $\langle T\rho(\phi)x, x^* \rangle = \langle \rho(\phi)Tx, x^* \rangle$ for all $x \in X$, $x^* \in X^*$. Let $\Delta \in \Omega$. Then

$$\langle TE(\Delta)x, x^* \rangle = \langle E(\Delta)x, T^*x^* \rangle = \mu_{x, T^*x^*}(\Delta)$$

and

$$\langle E(\Delta)Tx, x^* \rangle = \mu_{Tx, x^*}(\Delta).$$

By assumption we know that T and $\rho(\phi)$ commute. Hence

$$\langle T\rho(\phi)x, x^* \rangle = \langle \rho(\phi)x, T^*x^* \rangle = \int_K \phi \, d\mu_{x, T^*x^*}$$

and

$$\langle \rho(\phi)Tx, x^* \rangle = \int_K \phi \, d\mu_{Tx, x^*}$$

imply via the Riesz Representation Theorem (Theorem III.5.7 of [CO07]) that $\mu_{x, T^*x^*} = \mu_{Tx, x^*}$ and so we may conclude that $\langle TE(\Delta)x, x^* \rangle = \langle E(\Delta)Tx, x^* \rangle$. Thus T commutes with $E(\Delta)$.

“ \Leftarrow ” Suppose T commutes with $E(\Delta)$, where $\Delta \in \Omega$ and E is a positive spectral measure on Ω . Then

$$\langle TE(\Delta)x, x^* \rangle = \langle E(\Delta)Tx, x^* \rangle = \mu_{Tx, x^*}(\Delta),$$

but also

$$\langle TE(\Delta)x, x^* \rangle = \langle E(\Delta)x, T^*x^* \rangle = \mu_{x, T^*x^*}(\Delta)$$

and so

$$\mu_{Tx, x^*}(\Delta) = \mu_{x, T^*x^*}(\Delta). \tag{5.3}$$

Now let $\phi \in C_0(K)$ and ρ a positive representation of $C_0(K)$, then

$$\langle T\rho(\phi)x, x^* \rangle = \langle \rho(\phi)x, T^*x^* \rangle = \int_K \phi \, d\mu_{x, T^*x^*}$$

and

$$\langle \rho(\phi)Tx, x^* \rangle = \int_K \phi \, d\mu_{Tx, x^*}.$$

Because of equation (5.3) we may now conclude that those two integrals are equal, and hence that $\langle T\rho(\phi)x, x^* \rangle = \langle \rho(\phi)Tx, x^* \rangle$. Thus T commutes with $\rho(\phi)$. \square

5.4 THE UNITAL CASE

We start out with unital positive representations here and let K be a compact Hausdorff space. Then $C_0(K) = C(K)$ and the unital positive spectral measure that is generated maps Borel subsets of K to order projections. The following is a corollary of Lemma 5.1.

Corollary 5.5. *Let K be a compact Hausdorff space, $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra, and X a reflexive Banach lattice. Let $\rho : C(K) \rightarrow \mathcal{L}_r(X)$ be a unital positive representation. Then there exists a unique unital positive spectral measure E defined on the Borel subsets of K such that μ_{x, x^*} is a regular countably additive Borel measure for all $x \in X$ and $x^* \in X^*$, and $\rho(\phi) = \int_K \phi \, dE$ for all $\phi \in C(K)$.*

Proof. We now deal with a unital positive representation ρ . We know that $\mathbf{1} \in C(K)$, and $\tilde{\rho}(\phi) = \rho(\phi)$ for all $\phi \in C(K)$, where $\tilde{\rho}$ as defined in the proof of Lemma 5.1. Thus we may

conclude that $\tilde{\rho}(\mathbf{1}) = \rho(\mathbf{1}) = I$. Hence $\tilde{\rho} : B(K) \rightarrow \mathcal{L}_r(X)$ is a unital positive representation as well. After having defined $E(\Delta) = \tilde{\rho}(\chi_\Delta)$ for each $\Delta \in \Omega$ we find that $E(\Delta)$ is not only a positive projection. We have $E(\Delta) \leq I$ for each $\Delta \in \Omega$, because $\chi_\Delta \leq \chi_K = \mathbf{1}$. Now Theorem 2.29 tells us that E is an order projection. Moreover, we have $E(K) = \tilde{\rho}(\chi_K) = \tilde{\rho}(\mathbf{1}) = I$, and so E is a unital positive spectral measure in this case. The rest of the statement is similar to Lemma 5.1, in which we dealt with positive representations. Hence this finishes the proof. \square

We conclude that in the setting of a unital positive representation of $C(K)$, where K is a compact Hausdorff space, on a reflexive Banach lattice, the representation generates a unital positive spectral measure, i.e., $E(\Delta)$ is an order projection and $E(K) = I$. Below we find an example, in which we look at the unital positive spectral measure that is generated by the unital positive representation of Example 3.13.

Example 5.6. Let (K, Ω, ν) be a measure space, where K is a compact Hausdorff space, $\Omega \subset \mathcal{P}(K)$ the Borel σ -algebra, ν a measure on Ω , and $1 < p < \infty$. Let $\rho : C(K) \rightarrow \mathcal{L}_r(L^p(K, \Omega, \nu))$ be the map defined as the multiplication on $L^p(K, \Omega, \nu)$, which is a Banach lattice (see Example 2.34). We show below that it corresponds to a unital positive spectral measure $E : \Omega \rightarrow \mathcal{L}_r(L^p(K, \Omega, \nu))$ having the following property: $E(\Delta) = M_{\chi_\Delta}$, i.e., it is the multiplication by χ_Δ . By Example 3.13 we know that ρ is a unital positive representation. Furthermore we know that $L^p(K, \Omega, \nu)$ is a reflexive Banach lattice for $1 < p < \infty$. Thus by Corollary 5.5 we can define $E : \Omega \rightarrow \mathcal{L}_r(L^p(K, \Omega, \nu))$, $E(\Delta) = \tilde{\rho}(\chi_\Delta)$, to be the corresponding unital positive spectral measure. Hence $\tilde{\rho}(\chi_\Delta)f = \chi_\Delta \cdot f$ and $\tilde{\rho}(\chi_\Delta)f = E(\Delta)f$. From this we conclude that indeed $E(\Delta) = M_{\chi_\Delta}$. \triangleleft

The following corollary, that builds on the result in Corollary 5.2, uses the notion of the center of a Banach lattice. The *center* of X is defined as

$$Z(X) := \{\pi \in \mathcal{L}_r(X) : \text{there exists an } M \in [0, \infty) \text{ with } |\pi(x)| \leq M|x| \text{ for all } x \in X\}.$$

See Definition 3.28 of [AA02] for more details.

Corollary 5.7. *Let K be compact Hausdorff, $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra, and X a reflexive Banach lattice. Suppose $\rho : C(K) \rightarrow \mathcal{L}_r(X)$ is a unital positive representation. Then ρ maps on the center of X , i.e., $\rho(C(K)) \subseteq Z(X)$.*

Proof. Since ρ is a unital positive representation, we have $\rho(\mathbf{1}) = \tilde{\rho}(\mathbf{1}) = I$. Let $\phi \in C(K)$ arbitrary. Then we find $-\|\phi\|_\infty \mathbf{1} \leq \phi \leq \|\phi\|_\infty \mathbf{1}$, which implies

$$-\|\phi\|_\infty \rho(\mathbf{1}) \leq \rho(\phi) \leq \|\phi\|_\infty \rho(\mathbf{1}).$$

The positivity of ρ then tells us that

$$|\rho(\phi)| \leq \|\phi\|_\infty \rho(\mathbf{1}),$$

and using that $\rho(\mathbf{1}) = I$, we find that $|\rho(\phi)| \leq \|\phi\|_\infty I$. Then

$$|\rho(\phi)(x)| \leq |\rho(\phi)|(|x|) \leq \|\phi\|_\infty I(|x|) = \|\phi\|_\infty |x|.$$

This implies that $\rho(\phi) \in Z(X)$ for all $\phi \in C(K)$. Hence $\rho(C(K)) \subseteq Z(X)$. \square

6 One-to-one correspondence

Under certain circumstances, positive representations generate positive spectral measures and vice versa. We investigate the assumptions that are necessary to let this happen. We start combining the results of Corollary 4.3 and Lemma 5.1, which lead to a correspondence between positive representations of $C_0(K)$, where K is locally compact Hausdorff, and positive spectral measures. We then move on and turn to the results of Corollary 4.3 and Corollary 5.5 in which we look at unital positive spectral measures and unital positive representations of $C(K)$, where K is a compact Hausdorff space.

Note that there is no one-to-one correspondence whenever K is merely a (Hausdorff) set. It is not possible to generate positive spectral measures from positive representations in that case, so the results in Section 4 for function spaces on such a K do not have a reverse statement analogous to Lemma 5.1 and Corollary 5.5.

6.1 LOCALLY COMPACT HAUSDORFF K

Let K be a locally compact Hausdorff space, $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra, and X a reflexive Banach lattice. Then we can combine the results of Corollary 4.3 and Lemma 5.1 to obtain the following theorem.

Theorem 6.1. *Let K be a locally compact Hausdorff space, $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra and X a reflexive Banach lattice. If $E : \Omega \rightarrow \mathcal{L}_r(X)$ is a positive spectral measure and $\rho : C_0(K) \rightarrow \mathcal{L}_r(X)$ is defined via $\rho(\phi) = \int_K \phi dE$, then ρ is a positive representation. If $\rho : C_0(K) \rightarrow \mathcal{L}_r(X)$ is a positive representation, then there exists a unique positive spectral measure $E : \Omega \rightarrow \mathcal{L}_r(X)$ such that $\rho(\phi) = \int_K \phi dE$.*

Other results in Sections 3, 4, and 5 tell us that in this setting, the following holds as well. According to Lemma 3.2, the positive spectral measure E generates a regular countably additive Borel measure μ_{x,x^*} for all $x \in X$ and $x^* \in X^*$ defined by $\mu_{x,x^*}(\Delta) = \langle E(\Delta)x, x^* \rangle$, and such that

$$\langle \rho(\phi)x, x^* \rangle = \int_K \phi d\mu_{x,x^*}.$$

From Corollary 3.4 we find that $\|\mu_{x,x^*}\| \leq \|E(K)\|\|x\|\|x^*\|$. We have also seen in Corollary 5.2 that there exists a positive projection P such that $0 \leq \rho(\phi) \leq \|\phi\|_\infty P$ for all positive $\phi \in C_0(K)$. Moreover, Corollary 5.3 tells us that positive representations on $C_0(K)$ have the property that $\|\rho\| \leq \|\rho\|_r \leq \|\tilde{\rho}(\mathbf{1})\|$, where $\tilde{\rho}$ is the positive representation of $B(K)$ such that $\tilde{\rho}(\phi) = \rho(\phi)$ for all $\phi \in C_0(K)$.

6.2 COMPACT HAUSDORFF K

Let K be a compact Hausdorff space, $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra, and X a reflexive Banach lattice. Then we can combine the results of Corollary 4.3 and Corollary 5.5 to obtain the following theorem.

Theorem 6.2. *Let K be a compact Hausdorff space, $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra and X a reflexive Banach lattice. If $E : \Omega \rightarrow \mathcal{L}_r(X)$ is a unital positive spectral measure and $\rho : C(K) \rightarrow \mathcal{L}_r(X)$ is defined via $\rho(\phi) = \int_K \phi dE$, then ρ is a unital positive representation. If $\rho : C(K) \rightarrow \mathcal{L}_r(X)$ is a unital positive representation, then there exists a unique regular unital positive spectral measure $E : \Omega \rightarrow \mathcal{L}_r(X)$ such that $\rho(\phi) = \int_K \phi dE$.*

Again, other results in Sections 3, 4, and 5 tell us that the following holds as well in this setting. The unital positive spectral measure E generates a regular countably additive Borel measure μ_{x,x^*} for all $x \in X$ and $x^* \in X^*$ defined by $\mu_{x,x^*}(\Delta) = \langle E(\Delta)x, x^* \rangle$, and such that

$$\langle \rho(\phi)x, x^* \rangle = \int_K \phi d\mu_{x,x^*}.$$

Moreover, $\|\mu_{x,x^*}\| \leq \|x\|\|x^*\|$ holds for all $x \in X$ and $x^* \in X^*$. See Corollary 3.6 for more details. We have also seen in Corollary 3.17 that unital positive representations on $C(K)$ have the property that $\|\rho\| = \|\rho\|_r = 1$. Another result, stated in Corollary 5.7, tells us that ρ maps on the center of X , i.e., $\rho(C(K)) \subseteq Z(X)$.

7 Lattice structures

In this section we look at the partial ordering on the closure of orbits in X of our representations on $C(K)$. There is an ordering that is inherited from the Banach lattice X . On the other hand, there is an ordering that comes from an isomorphism onto this subspace of X as well. That second structure is defined in [PR07], where they are dealing with a similar situation, with one important difference: they start with a Banach space X instead of a lattice. Since we have added the property of X being a lattice, we can speak of the inherited ordering on the closure of orbits in X of our representations. We look at these orderings to understand what properties elements of the closure of the image of our representations have. More specifically, we will look at the representation of $C(K)$ that was introduced in Example 3.12.

7.1 GENERAL SETTING

Let K be a compact Hausdorff space, $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra and X a Banach lattice. Suppose $\rho : C(K) \rightarrow \mathcal{L}_r(X)$ is a unital positive representation. Then there are two partial orderings defined on

$$\rho[f] := \overline{\rho(C(K))f} \subseteq X,$$

induced by $f \in X$, $f \geq 0$, namely the one inherited from X and one coming from an isomorphism onto $\rho[f]$ defined along the lines of [PR07].

We start by following pages 499, 500 and 501 of [PR07] and the definition of positivity in $\rho[f]$ on page 501.

Every $f \in X$ defines a linear map $\rho_f : C(K) \rightarrow X$ that maps $\phi \in C(K)$ to $\rho(\phi)f$. So $\rho[f] = \overline{\text{Ran}(\rho_f)}$. Lemma 2.4 of [PR07] tells us that the map $p_f : C(K) \rightarrow [0, \infty)$ defined by

$$\begin{aligned} p_f(\phi) &= \sup\{\|\rho_f(\psi)\| : \psi \in C(K), |\psi| \leq |\phi|\} \\ &= \sup\{\|\rho(\psi)f\| : \psi \in C(K), |\psi| \leq |\phi|\} \end{aligned}$$

is a continuous lattice seminorm on $C(K)$. We define

$$\mathcal{N}_f = \text{Ker}(\rho_f) = \{\phi \in C(K) : \rho(\phi)f = 0\},$$

which is a closed order ideal (i.e., a closed subspace which is solid) of $C(K)$. It is closed because it is equal to the kernel of ρ_f , and solid because it is equal to the kernel of p_f . Then we define the map $q_f : C(K) \rightarrow C(K) / \mathcal{N}_f$ via $\phi \mapsto \tilde{\phi}$, and define

$$\|\tilde{\phi}\|_f := p_f(\phi).$$

Then $\|\cdot\|_f$ is a lattice norm on $C(K) / \mathcal{N}_f$. Now the following map is an isomorphism of normed spaces onto $\text{Ran}(\rho_f)$:

$$\begin{aligned} \tilde{\rho}_f : C(K) / \mathcal{N}_f &\rightarrow \rho[f] \subseteq X \\ \tilde{\phi} &\mapsto \rho(\phi)f. \end{aligned}$$

It is clear that $\text{Ran}(\tilde{\rho}_f) = \text{Ran}(\rho_f)$. We denote the norm completion of

$$\left(C(K) / \mathcal{N}_f, \|\cdot\|_f \right)$$

by $L^1(\rho_f)$. This is a real Banach lattice and there is a unique linear extension $\tilde{\rho}_f : L^1(\rho_f) \rightarrow X$, which is a linear isomorphism onto $\rho[f]$. Hence for a given $g \in \rho[f]$ there is a unique $z \in L^1(\rho_f)$ such that $g = \tilde{\rho}_f(z)$. For any $g \in \text{Ran}(\rho_f)$ there is a unique $\tilde{\phi} \in C(K) / \mathcal{N}_f$ such that $\tilde{\rho}_f(\tilde{\phi}) = \rho(\phi)f = g$. Via the isomorphism $\tilde{\rho}_f$ we equip $\rho[f]$ with a real Riesz space structure in the following way, where $g \in \rho[f]$ and $z \in L^1(\rho_f)$ is the unique element such that $g = \tilde{\rho}_f(z)$,

$$g \geq 0 \Leftrightarrow z \geq 0. \tag{7.1}$$

This is how the ordering coming from the isomorphism $\tilde{\rho}_f$ is defined whenever $f \in X$. For an arbitrary $f \in X$ we find that (7.1) is equivalent to the following statement for an arbitrary $g \in \rho[f]$: there exists a sequence $\{[\phi_n]\}_{n=1}^\infty \subset \left(C(K) / \mathcal{N}_f \right)^+$ with

$$\lim_{n \rightarrow \infty} \|[\phi_n] - z\|_f = 0. \tag{7.2}$$

The meaning of this sequence $\{[\phi_n]\}_{n=1}^\infty$ of equivalence classes being positive is as follows: for every $n \in \mathbb{N}$ there exists a $\psi_n \in [\phi_n]$ for which $\psi_n \geq 0$. Hence the defining equivalence (7.1) can also be described as

$$g \geq 0 \Leftrightarrow \text{there exists a sequence } \{\psi_n\}_{n=1}^\infty \in C(K)^+ \text{ for which } \lim_{n \rightarrow \infty} \rho(\psi_n)f = g. \tag{7.3}$$

We use this defining relation in the next subsection.

The ordering that is inherited from X in the case below is the expected, or ‘natural’, ordering, as we will see.

7.2 MULTIPLICATION ON $C(K)$

Let K be compact Hausdorff. The map $\rho : C(K) \rightarrow \mathcal{L}_r(C(K))$ defined for arbitrary $f, g \in C(K)$ through

$$\rho(g)f = g \cdot f$$

is a unital positive representation according to Example 3.12. For this map we have

$$\rho[f] = \overline{C(K) \cdot f}$$

for an $f \in C(K)$. Our goal is to understand the ordering structure on this subspace of $C(K)$ defined via the isomorphism as described in the previous subsection. Furthermore, we are interested in its relationship with the ordering structure it inherits from $C(K)$.

We start by defining the notion of an annihilator, which will be useful later on. Let $f \in C(K)$ be arbitrary. The *annihilator* of the set $\rho[f] = \overline{C(K) \cdot f} \subseteq C(K)$ in $C(K)^*$ is defined as

$$\rho[f]^\perp = \{\phi \in C(K)^* : \phi(g) = 0 \text{ for all } g \in \rho[f]\}$$

and the *annihilator* of the set $\rho[f]^\perp \subseteq C(K)^*$ in $C(K)$ is defined as

$${}^\perp(\rho[f]^\perp) = \left\{g \in C(K) : \phi(g) = 0 \text{ for all } \phi \in \rho[f]^\perp\right\}.$$

Note that since the annihilator is a closed subset,

$${}^\perp(\rho[f]^\perp) = {}^\perp((C(K) \cdot f)^\perp)$$

Recall that elements ϕ of the dual of $C(K)$ correspond to a measure μ on the Borel σ -algebra of K such that

$$\phi_\mu(g) = \int_K g \, d\mu.$$

Thus $\phi_\mu \in (C(K) \cdot f)^\perp$ implies that

$$\int_K g \cdot f \, d\mu = 0 \tag{7.4}$$

for all $g \in C(K)$. Since this holds for all g , equation (7.4) implies that

$$\int_K f \, d\mu = 0.$$

Hence, by Theorem 13.2 of [BA01], $f = 0$ μ -almost everywhere. This is the case if and only if for an arbitrary $f \in C(K)$ we have $\text{supp}(\mu) \subseteq N_f$, where μ is the measure corresponding to an arbitrary element of $(C(K) \cdot f)^\perp$, and where N_f is the kernel of f .

We now move on to finding defining properties of elements of $\rho[f]$. Let $g \in C(K)$ be arbitrary and suppose that $N_f \subseteq N_g$. We know now that $\text{supp}(\mu) \subseteq N_f$. Then $\text{supp}(\mu) \subseteq N_g$, hence

$$\int_K g \, d\mu = 0$$

as well, i.e., $g = 0$ μ -almost everywhere (see Theorem 13.2 of [BA01]). Since μ corresponded to an arbitrary element of $(C(K) \cdot f)^\perp$, this implies that

$$g \in {}^\perp((C(K) \cdot f)^\perp) = {}^\perp(\rho[f]^\perp).$$

Since $\rho[f]$ is a subspace of $C(K)$, it is closed under scalar multiplication and addition. It is therefore clear that it is a closed convex balanced hull. Thus Theorem V.1.8 of [CO07] (the Bipolar Theorem) tells us that

$${}^\perp(\rho[f]^\perp) = \rho[f].$$

Hence $N_g \subseteq N_f$ for an arbitrary $g \in C(K)$ and $\text{supp}(\mu) \subseteq N_f$ together imply that $g \in \rho[f]$. The reverse argument holds as well. If $g \in \rho[f]$, then g is the limit of a product of a sequence of elements of $C(K)$ and f , or a product of a single element of $C(K)$ and f . Thus g certainly has the same zeros as f . Hence clearly $N_f \subseteq N_g$. So the elements of $\rho[f]$ are precisely the elements $g \in C(K)$ for which $N_f \subseteq N_g$.

Let $f \in C(K)$ be a positive element, i.e., $f(x) \geq 0$ for all $x \in K$. Now suppose that $g \in C(K)$ is positive in that ‘natural’ sense as well, and $g \in \rho[f]$, i.e., $N_f \subseteq N_g$. Then there exists a sequence $\{g_n\}_{n=1}^{\infty} \subset C(K)$ for which $\lim_{n \rightarrow \infty} g_n \cdot f = g$. Then also $\lim_{n \rightarrow \infty} (g_n \cdot f)^+ = g^+$, which implies that $\lim_{n \rightarrow \infty} g_n^+ \cdot f = g$, since both f and g are positive. This is precisely how positivity was defined in (7.1).

Note that this argument holds whenever f and g are both negative in the ‘natural’ sense as well. This is stated in the lemma below.

Lemma 7.1. *Let K be compact Hausdorff. The map $\rho : C(K) \rightarrow \mathcal{L}_r(C(K))$ defined for arbitrary $f, g \in C(K)$ through*

$$\rho(g)f = g \cdot f$$

is a unital positive representation. Let $f \in C(K)$. An element $g \in C(K)$ that has the same sign as f , and that is contained in $\rho[f]$, i.e., $N_f \subseteq N_g$, is positive in the sense of (7.1).

We return to the setting where $f \in C(K)$ is a positive element. We now start with an element $g \in C(K)$ that is positive in the sense of (7.1), and that is contained in $\rho[f]$, i.e., $N_f \subseteq N_g$. Then we know from (7.3) that there exists a positive sequence (in the ‘natural’ sense) $\{g_n\}_{n=1}^{\infty} \in C(K)$ for which $\lim_{n \rightarrow \infty} g_n \cdot f = g$. Since f was positive by assumption as well, $g(x) \geq 0$ for all $x \in K$. Hence the two orderings on $\rho[f]$ coincide.

8 Generalizing to Banach spaces

In [PR07] one can find a detailed study of representations of $C(K)$ on $\mathcal{B}(X)$, where X is a Banach space. The notion of R -boundedness for families of operators is used for this purpose. Among the results is a result similar to Theorem 6.2, which stated a one-to-one correspondence between unital positive spectral measures and unital positive representations. This result uses the R -boundedness to make up for the lack of certain specific lattice properties. First we define a spectral measure on a Banach space. Then we look at the mentioned result in detail and at how it fits into our setting. We finish by comparing our generated spectral measure with the one generated in [PR07]. We will see that indeed our setting is a special version of the setting in [PR07], but with more direct proofs.

8.1 SPECTRAL MEASURES IN A BANACH SPACE SETTING

Suppose K is a set, $\Omega \subseteq \mathcal{P}(K)$ a σ -algebra, X is a Banach space, and $\mathcal{B}(X)$ the Banach algebra of all bounded operators of X into itself with the operator norm topology; its unit is the identity operator I on X . Earlier on we were able to define spectral measures $E : \Omega \rightarrow \mathcal{L}_r(X)$ with properties depending on the ordering of X . Since the ordering is left out in a general Banach space setting, we have to change the definition of a spectral measure. The following definition is based on Definition XV.2.1 of [DS71].

Definition 8.1. The map $E : \Omega \rightarrow \mathcal{B}(X)$, where K is a set, $\Omega \subseteq \mathcal{P}(K)$ a σ -algebra, and X a Banach space, is called a *spectral measure* when

1. $E(K) = 1$;
2. $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$ for all $\Delta_1, \Delta_2 \in \Omega$;
3. $E(\Delta_1 \cup \Delta_2) = E(\Delta_1) + E(\Delta_2) - E(\Delta_1)E(\Delta_2)$ for all $\Delta_1, \Delta_2 \in \Omega$, and
4. $E(\Delta)$ is a projection, i.e., $E(\Delta)^2 = E(\Delta)$.

A spectral measure is called *bounded* if

$$\sup \{ \|E(\Delta)\| : \Delta \in \Omega \} < \infty.$$

Compared to the properties of positive spectral measures and unital positive spectral measures, we miss specific knowledge of an arbitrary countable partition of K or of the relationship between arbitrary x and x^* . This is what is needed to find results analogous to Lemma 3.2, Corollary 3.4 and Corollary 3.6, in which we found a countably additive measure corresponding to a positive spectral measure and a unital positive spectral measure, and were able to bound the norm of this measure. This is needed in the various proofs of statements about generating representations. Thus, adding a property that bounds the norm in some way seems natural. This brings us to the next subsection.

8.2 INTEGRATION WITH RESPECT TO AN OPERATOR-VALUED SET FUNCTION

On page 890 of [DS63] the operator calculus for a normal operator T on a finite dimensional Hilbert space H is given by the formula

$$f(T) = \int_{\sigma(T)} f dE,$$

where $\sigma(T) = \{\lambda_1, \dots, \lambda_k\}$ denotes the spectrum of T , E is a resolution of the identity for T , and the integral is defined as the finite sum $\sum_{i=1}^k f(\lambda_i)E(\lambda_i)$. It is argued that in the study of normal operators a general form of this formula is necessary. On pages 891 and 892 of [DS63] the integral of a bounded scalar function with respect to an operator-valued set function is defined. We have summarized this construction below, because this construction of an integral is used in generating a representation in [PR07]. We will look into the results of [PR07] in the next subsection, and see that the construction presented here is a crucial part of those results.

Let K be a set, $\Omega \subseteq \mathcal{P}(K)$ a σ -algebra, and X a Banach space. Let $E : \Omega \rightarrow \mathcal{B}(X)$ be an additive and bounded map. The functions on which this integral will be defined are the bounded Ω -measurable functions, denoted by $B(K)$. The term Ω -simple function is used for functions $f \in B(K)$ of the form

$$f = \sum_{i=1}^n \alpha_i \chi_{\Delta_i}, \tag{8.1}$$

where $\alpha_i \in \mathbb{R}$ and $\Delta_i \in \Omega$ for all $i \in \{1, \dots, n\}$. For $\alpha_i, \beta_i \in \mathbb{R}$ and $\Delta_i, \Sigma_i \in \Omega$ for $i \in \{1, \dots, n\}$ the following implication holds:

$$\sum_{i=1}^n \alpha_i \chi_{\Delta_i} = \sum_{i=1}^n \beta_i \chi_{\Sigma_i} \Rightarrow \sum_{i=1}^n \alpha_i E(\Delta_i) = \sum_{i=1}^n \beta_i E(\Sigma_i).$$

Hence we may define the *integral* of the Ω -simple function f defined in (8.1) by

$$\int_K f dE = \sum_{i=1}^n \alpha_i E(\Delta_i).$$

The absolute value of this integral can be bounded by a constant times the sup-norm of f as can be seen on page 892 of [DS63]. This shows that if a sequence of Ω -simple functions converges in $B(K)$ to a function f , then the sequence of their integrals converges as well. Furthermore, the limit only depends on f and does not depend on the sequence. So if the sequence $\{f_n\}_{n=1}^\infty$ of Ω -simple functions converges to $f \in B(K)$, then we may define the *integral* of f by

$$\int_K f dE = \lim_{n \rightarrow \infty} \int_K f_n dE.$$

The following lemma summarizes the properties of the map sending f to this integral. Details can be found on page 892 of [DS63].

Lemma 8.2. *Let K is an arbitrary set, $\Omega \subseteq \mathcal{P}(K)$ a σ -algebra, $f \in B(K)$, and X a Banach space. Suppose $E : \Omega \rightarrow \mathcal{B}(X)$ is an additive and bounded map. Then the map $f \mapsto \int_K f dE$ is a continuous, linear map from $B(K)$ to $\mathcal{B}(X)$. If E is a spectral measure, then the map $f \mapsto \int_K f dE$ is a homomorphism.*

8.3 R -BOUNDEDNESS

Indeed, using R -boundedness we can find a one-to-one correspondence between representations and spectral measures. It is stated and proven in [PR07], but we summarize it here to show its relevance and get a grip on the analogy to what has been proven in the previous sections of this thesis.

We assume in this subsection that X is a Banach space, K a compact Hausdorff space, and $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra. Denote by $\{r_j\}_{j=1}^\infty$ the sequence of Rademacher functions on the interval $[0, 1]$. We will start with the definition of R -boundedness.

Definition 8.3. A non-empty collection $\mathcal{T} \subseteq \mathcal{B}(X)$ is called R -bounded if there exists an $M \geq 0$ such that

$$\left(\int_0^1 \left\| \sum_{j=1}^n r_j(t) T_j x_j \right\|^2 dt \right)^{\frac{1}{2}} \leq M \left(\int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\|^2 dt \right)^{\frac{1}{2}} \quad (8.2)$$

for all $\{T_j\}_{j=1}^n \subseteq \mathcal{T}$, all $\{x_j\}_{j=1}^n$ and all $n \in \mathbb{N} \setminus \{0\}$.

If $\mathcal{T} \subseteq L(X)$ is R -bounded, then the smallest constant $M \geq 0$ for which (8.2) holds is denoted by $M_{\mathcal{T}}$ and is called the R -bound of \mathcal{T} . An R -bounded representation $\rho : C(K) \rightarrow \mathcal{B}(X)$ is a representation such that the set

$$\{\rho(\phi) : \phi \in C(K), \|\phi\|_\infty \leq 1\}$$

is R -bounded in $\mathcal{B}(X)$. An R -bounded spectral measure $E : \Omega \rightarrow \mathcal{B}(X)$ is a spectral measure with the property that the set $\text{Ran}(E) = \{E(A) : A \in \Omega\}$ is R -bounded in $\mathcal{B}(X)$. The following result is stated and proven in Proposition 2.17 of [PR07].

Lemma 8.4. *If $\rho : C(K) \rightarrow \mathcal{B}(X)$ is an R -bounded representation, then there exists a regular, strong operator countably additive and R -bounded spectral measure $E : \Omega \rightarrow \mathcal{B}(X)$ satisfying $\rho(\phi) = \int_K \phi dE$ for all $\phi \in C(K)$.*

The converse of this lemma is then stated and a proof is sketched in Remark 2.18 of [PR07]. For this proof the following lemma is essential, which follows from the reasoning along the lines of pages 891 and 892 of [DS63], summarized in Subsection 8.2.

Lemma 8.5. *Let $E : \Omega \rightarrow \mathcal{B}(X)$ be a spectral measure. Then the map $\rho : C(K) \rightarrow \mathcal{B}(X)$ defined as $\phi \mapsto \int_K \phi dE$ is a representation.*

This leads us to the following lemma.

Lemma 8.6. *If $E : \Omega \rightarrow \mathcal{B}(X)$ is any regular, strong operator countably additive and R -bounded spectral measure, and a representation $\rho_E : C(K) \rightarrow \mathcal{B}(X)$ is defined by $\phi \mapsto \int_K \phi dE$, then ρ_E is necessarily R -bounded.*

So indeed there is a two-sided correspondence in the Banach space setting as well, though solely if one assumes that the strong R -boundedness property holds for either the representation or the spectral measure.

8.4 COMPARISON OF SPECTRAL MEASURES

A unital positive representation of $C(K)$ on $\mathcal{L}_r(X)$ generates a unital positive spectral measure. On page 502 of [PR07] a slightly different approach generates a spectral measure as well, starting out with a similar unital positive representation.

Let X be a reflexive Banach lattice, K a compact Hausdorff space, $\Omega \subset \mathcal{P}(K)$ the Borel σ -algebra. Every unital positive representation $\rho : C(K) \rightarrow \mathcal{L}_r(X)$ generates a unique unital positive spectral measure $E : \Omega \rightarrow \mathcal{L}_r(X)$ such that $\rho(\phi) = \int_K \phi dE$ for all $\phi \in C(K)$. That E corresponds to a regular countably additive Borel measure μ_{x,x^*} , $x \in X$ and $x^* \in X^*$, in the following way for $\phi \in C(K)$,

$$\langle \rho(\phi)x, x^* \rangle = \int_K \phi d\mu_{x,x^*},$$

i.e., $\mu_{x,x^*}(\Delta) := \langle E(\Delta)x, x^* \rangle$ for all $\Delta \in \Omega$. According to Proposition 2.13 of [PR07] there is a spectral measure $P : \Omega \rightarrow \mathcal{B}(X)$, under certain circumstances, such that $\rho(\phi) = \int_K \phi dP$ for all $\phi \in C(K)$. As we can see there, it holds by definition that $\langle P(\Delta)x, x^* \rangle = \mu_{x,x^*}(\Delta)$. Since μ_{x,x^*} is a unique measure by the Riesz Representation Theorem (Theorem III.5.7 of [CO07]), we may conclude that $P(\Delta) = E(\Delta)$ for all $\Delta \in \Omega$.

Hence the spectral measure corresponding to the unital positive representation in [PR07] and in this theses are equal.

9 Covariant representations and spectral measures

The crossed product construction from crossed product theory for C^* -algebras provides a way to construct C^* -algebras from more elementary building blocks. The crossed product theory started to develop in the nineteen fifties and has extended ever since. The basics for a generalization to Banach algebras of this theory are contained in [DJ11]. The motivation for this generalization is twofold. Firstly, it seems natural to have the possibility to generate Banach algebras from elementary building blocks, and secondly there are possible applications for these algebras in the theory of Banach representations of locally compact groups. The notion of covariant representations comes to table when talking of the detailed construction of the crossed products on Banach algebras. In this section we show that our two-sided correspondence between positive representations and positive spectral measures belongs to this theory. We start with the necessary definitions and notations, and continue by proving a one-to-one correspondence between covariant representations and covariant spectral measures for a specific example. For more details on this theory, see [DJ11].

9.1 DEFINITIONS AND NOTATION

If G is a group, then we denote its unit element by e . If X is a normed space, then let $B(X)$ denote the normed algebra of bounded maps on X and $\text{Inv}(X)$ the group of invertible elements of $B(X)$. If A is a normed algebra, then we write $\text{Aut}(A)$ for its group of bounded automorphisms. Next to the representations of function spaces we have already encountered, we need the definition of a representation of a group.

Definition 9.1. A *representation* U of a group G on a normed space X is a group homomorphism $U : G \rightarrow \text{Inv}(X)$.

For clarity, we sometimes use the shorthand notation U_g , where $g \in G$, instead of $U(g)$. Not only for this specific representation, but for all maps we come across. Representations of algebras are defined below, as well as the triple that we define to be the normed or Banach algebra dynamical system.

Definition 9.2. A *representation* π of an algebra A on a normed space X is an algebra homomorphism $\pi : A \rightarrow B(X)$.

Definition 9.3. Let A be a normed algebra, G a locally compact group, and $\alpha : G \rightarrow \text{Aut}(A)$ a map such that $g \mapsto \alpha_g(a)$ is continuous for all $a \in A$. Then the triple (A, G, α) is called a *normed algebra dynamical system*. If A is a Banach algebra, we speak of a *Banach algebra*

dynamical system.

We have all tools necessary to define the main object of interest in crossed product theory.

Definition 9.4. Let (A, G, α) be a normed algebra dynamical system and X a normed space. A *covariant representation* of (A, G, α) on X is a pair (ρ, U) , where ρ is a representation of A on X , and U a representation of G on X , such that for all $a \in A$ and $g \in G$,

$$U_g \rho(a) U_g^{-1} = \rho(\alpha_g(a)). \quad (9.1)$$

The definition below is useful in the context of our representations and spectral measures, as we will see in the next subsection.

Definition 9.5. Let K be a compact Hausdorff space, $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra, and X a reflexive Banach lattice. Let G be a locally compact group. A *covariant spectral measure* on X is a pair (E, U) , where $E : \Omega \rightarrow \mathcal{B}(X)$ is a positive spectral measure, and U a representation of G on X , such that for all $\Delta \in \Omega$ and $g \in G$,

$$U_g E(\Delta) U_g^{-1} = E(g(\Delta)). \quad (9.2)$$

9.2 AN EXAMPLE

In this subsection we look at an example of a Banach algebra dynamical system. The unital positive representation $\rho : C(K) \rightarrow \mathcal{B}(L^p(K))$, $1 < p < \infty$, for a compact Hausdorff space K from Example 3.13 will fit into the definition of a covariant representation and prove to correspond to a covariant spectral measure using the correspondence we have stated formally in Theorem 6.2.

Let K be a compact Hausdorff space. The space $C(K)$ denotes the continuous \mathbb{R} -valued functions (which is a Banach algebra according to Example VII.1.4 of [CO07]), $\mathcal{B}(C(K))$ all bounded operators on $C(K)$ and $\text{Aut}(C(K))$ all isomorphisms on $C(K)$. Let G be an arbitrary locally compact group with unit element e .

Let $(g, x) \mapsto g(x)$, from $G \times K$ to K , be a continuous map such that $K \rightarrow K$, $x \mapsto g(x)$ is a homeomorphism. For a fixed $g \in G$ we have the continuous map $\alpha_g : C(K) \rightarrow C(K)$, $f \mapsto \{x \mapsto f \circ g^{-1}(x)\}$. The map $\alpha : G \rightarrow \text{Aut}(C(K))$ is a homomorphism. The map $g \mapsto \alpha_g(f)$ for a fixed $f \in C(K)$, from G to $C(K)$ is continuous as well. Hence $(C(K), G, \alpha)$ is a Banach algebra dynamical system.

Let $1 < p < \infty$, and let μ be an G -invariant measure on Ω . Then $L^p(K) = L^p(K, \mu)$ is a reflexive Banach lattice (Example III.11.3 of [CO07] and Example 2.34).

Now define the map $\rho : C(K) \rightarrow \mathcal{B}(L^p(K))$ as

$$\rho(f)\phi = f \cdot \phi$$

for all $f \in C(K)$ and $\phi \in L^p(K)$. According to Remark 3.9 and Example 3.13, the map ρ is a unital positive representation. Let $\Omega \subseteq \mathcal{P}(K)$ be the Borel σ -algebra. Theorem 6.2

states that we may define the regular unital positive spectral measure $E : \Omega \rightarrow \mathcal{B}(L^p(K))$ such that $\rho(f) = \int_K f dE$. Remark 3.3 tells us that we may define E like this, because the image of E is still in $\mathcal{L}_r(L^p(K))$. The unital positive spectral measure E generates a regular countably additive Borel measure μ_{x,x^*} for all $x \in L^p(K)$ and $x^* \in L^p(K)^*$ defined by $\mu_{x,x^*}(\Delta) = \langle E(\Delta)x, x^* \rangle$, and such that

$$\langle \rho(\phi)x, x^* \rangle = \int_K \phi d\mu_{x,x^*}.$$

Note that ρ being a unital positive representation makes it a representation in the sense of Definition 9.2. Now we define the group representation

$$U : G \rightarrow \mathcal{B}(L^p(K)) \\ g \mapsto \{f \mapsto \{x \mapsto f(g^{-1}(x))\}\}.$$

The G -invariance of μ ensures that for an arbitrary $g \in G$, $f \mapsto \{x \mapsto f(g^{-1}(x))\}$ is indeed a map in $\mathcal{B}(L^p(K))$. We can now speak of (ρ, U) being a covariant representation of $(C(K), G, \alpha)$, since both ρ and U satisfy all assumed properties of Definition 9.4. Since E and U both satisfy the assumptions in Definition 9.5, we can speak of (E, U) being a covariant spectral measure.

The results below hold not only for the example described above, as we will see. We need a result from measure theory before moving on to these main results. Suppose V and W are two spaces, both having a σ -algebra, Ω and Σ , respectively. Let μ be a measure on V . Then we can define a measure on W via $\mu^\phi(\Delta) = \mu(\phi^{-1}(\Delta))$ for all $\Delta \in \Sigma$. Then

$$\int_X f \circ \phi d\mu = \int_Y f d\mu^\phi \tag{9.3}$$

for an integrable $f : W \rightarrow \mathbb{R}$ (Theorem 3.6.1 of [BO07]).

Lemma 9.6. *Let K be a compact Hausdorff space, $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra, X a reflexive Banach lattice, and $\rho : C(K) \rightarrow \mathcal{B}(X)$ a unital positive representation. Let E be the unital positive spectral measure corresponding to ρ as described in Theorem 6.2. Suppose (ρ, U) is a covariant representation of a Banach algebra dynamical system $(C(K), G, \alpha)$ on X . Then (E, U) is a covariant spectral measure.*

Proof. We have to show that the pair (E, U) satisfies equation (9.2). Fix $g \in G$. Then define

$$\rho_1(f) = U_g \rho(f) U_g^{-1} \\ \rho_2(f) = \rho(\alpha_g(f)),$$

for all $f \in C(K)$. By covariance of the pair (ρ, U) we know $\rho_1(f) = \rho_2(f)$ for all $f \in C(K)$. For all $x \in X$ and $x^* \in X^*$ we find, using that $\langle \rho(f)x, x^* \rangle = \int_K f d\mu_{x,x^*}$,

$$\begin{aligned} \langle \rho_1(f)x, x^* \rangle &= \langle U_g \rho(f) U_g^{-1} x, x^* \rangle \\ &= \langle \rho(f) U_g^{-1} x, (U_g)^* x^* \rangle \\ &= \int_K f d\mu_{U_g^{-1} x, (U_g)^* x^*}. \end{aligned}$$

We also have

$$\begin{aligned} \langle \rho_2(f)x, x^* \rangle &= \langle \rho(\alpha_g(f))x, x^* \rangle \\ &= \int_K f \circ g^{-1} d\mu_{x,x^*} \\ &= \int_K f d\mu_{x,x^*}^{\phi_g}, \end{aligned}$$

where $\phi_g(k) := g^{-1}(k)$ for all $k \in K$. The equality in the last step follows from (9.3). Uniqueness in the Riesz Representation Theorem implies that for all $\Delta \in \Omega$,

$$\mu_{U_g^{-1}x, (U_g)^*x^*}(\Delta) = \mu_{x,x^*}^{\phi_g}(\Delta). \quad (9.4)$$

We look at both measures separately to see how they can be expressed in terms of the unital positive spectral measure to which they correspond. The lefthandside can be expressed as follows

$$\begin{aligned} \mu_{U_g^{-1}x, (U_g)^*x^*}(\Delta) &= \langle E(\Delta)U_g^{-1}x, (U_g)^*x^* \rangle \\ &= \langle U_g E(\Delta)U_g^{-1}x, x^* \rangle \end{aligned}$$

and the righthandside as

$$\begin{aligned} \mu_{x,x^*}^{\phi_g}(\Delta) &= \mu_{x,x^*}(\phi_g^{-1}(\Delta)) \\ &= \mu_{x,x^*}(g(\Delta)) \\ &= \langle E(g(\Delta))x, x^* \rangle. \end{aligned}$$

So from the equality in equation (9.4) we may conclude that

$$U_g E(\Delta)U_g^{-1} = E(g(\Delta))$$

which finishes the proof. \square

The converse statement can be found below.

Lemma 9.7. *Let K be a compact Hausdorff space, $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra, X a reflexive Banach lattice, and $\rho : C(K) \rightarrow \mathcal{B}(X)$ a unital positive representation. Let E be the unital positive spectral measure corresponding to ρ as described in Theorem 6.2. Suppose (E, U) is a covariant spectral measure. Then (ρ, U) is a covariant representation of $(C(K), G, \alpha)$ on X .*

Proof. We have to show that the pair (ρ, U) satisfies (9.1). Fix $g \in G$. For arbitrary $x \in X$ and $x^* \in X^*$, let μ_{x,x^*} be the regular countably additive Borel measure that corresponds to the unital positive spectral measure E . Firstly we have for an arbitrary $\Delta \in \Omega$,

$$\begin{aligned} \langle U_g E(\Delta)U_g^{-1}x, x^* \rangle &= \langle E(\Delta)U_g^{-1}x, (U_g)^*x^* \rangle \\ &= \mu_{U_g^{-1}x, (U_g)^*x^*}(\Delta). \end{aligned}$$

For every $f \in C(K)$ this means that

$$\begin{aligned} \int_K f \, d\mu_{U_g^{-1}x, (U_g)^*x^*} &= \langle \rho(f)U_g^{-1}x, (U_g)^*x^* \rangle \\ &= \langle U_g\rho(f)U_g^{-1}x, x^* \rangle. \end{aligned} \tag{9.5}$$

On the other hand we have

$$\begin{aligned} \langle E(g(\Delta))x, x^* \rangle &= \mu_{x, x^*}(g(\Delta)) \\ &= \mu_{x, x^*}(\phi_g^{-1}(\Delta)) \\ &= \mu_{x, x^*}^{\phi_g}(\Delta). \end{aligned}$$

Using (9.3) we then find that for every $f \in C(K)$,

$$\begin{aligned} \int_K f \, d\mu_{x, x^*}^{\phi_g} &= \int_K f \circ g^{-1} \, d\mu_{x, x^*} \\ &= \langle \rho(f \circ g^{-1})x, x^* \rangle \\ &= \langle \rho(\alpha_g(f))x, x^* \rangle. \end{aligned} \tag{9.6}$$

Since by assumption equality (9.2) holds, expressions (9.5) and (9.6) are equal, and so for all $\Delta \in \Omega$

$$U_g\rho(f)U_g^{-1} = \rho(\alpha_g(f)).$$

This identity tells us that (ρ, U) is a covariant representation of $(C(K), G, \alpha)$ on X . □

The result of the two lemmas belonging is summarized in the following corollary.

Corollary 9.8. *Let K be a compact Hausdorff space, $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra, X a reflexive Banach lattice, and $\rho : C(K) \rightarrow \mathcal{B}(X)$ a unital positive representation. Let E be the unital positive spectral measure corresponding to ρ as described in Theorem 6.2. Then (ρ, U) is a covariant representation of $(C(K), G, \alpha)$ on X if and only if (E, U) is a covariant spectral measure.*

10 Conclusions

In this thesis we have showed that there are strong relationships between positive spectral measures and positive representations, and unital positive spectral measures and unital positive representations. We present our main results below.

We started with an important property of a positive spectral measure E (Lemma 3.2). It corresponds to a countably additive measure μ_{x,x^*} defined as $\mu_{x,x^*}(\Delta) = \langle E(\Delta)x, x^* \rangle$. This measure has the property that $\|\mu_{x,x^*}\| \leq \langle E(K)|x, |x^* \rangle$. Moreover, we have that $\|\mu_{x,x^*}\| \leq \|E(K)\| \|x\| \|x^*\|$ (Corollary 3.4). Whenever E is a unital positive spectral measure, we know that $E(K) = I$. This implies that in that case $\|\mu_{x,x^*}\| \leq \langle |x, |x^* \rangle \leq \|x\| \|x^*\|$ (Corollary 3.6).

The one-to-one correspondence between positive spectral measures and positive representations we have proven through looking at both ways separately can be stated as follows (Theorem 6.1).

Let K be a locally compact Hausdorff space, $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra, and X a reflexive Banach lattice. If $E : \Omega \rightarrow \mathcal{L}_r(X)$ is a positive spectral measure and $\rho : C_0(K) \rightarrow \mathcal{L}_r(X)$ is defined via $\rho(\phi) = \int_K \phi dE$, then ρ is a positive representation. If $\rho : C_0(K) \rightarrow \mathcal{L}_r(X)$ is a positive representation, then there exists a unique regular positive spectral measure $E : \Omega \rightarrow \mathcal{L}_r(X)$ such that $\rho(\phi) = \int_K \phi dE$.

We found similar results when looking at unital positive spectral measures and unital positive representations, leading to the following correspondence (Theorem 6.2).

Let K be a compact Hausdorff space, $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra, and X a reflexive Banach lattice. If $E : \Omega \rightarrow \mathcal{L}_r(X)$ is a unital positive spectral measure and $\rho : C(K) \rightarrow \mathcal{L}_r(X)$ is defined via $\rho(\phi) = \int_K \phi dE$, then ρ is a unital positive representation. If $\rho : C(K) \rightarrow \mathcal{L}_r(X)$ is a unital positive representation, then there exists a unique regular unital positive spectral measure $E : \Omega \rightarrow \mathcal{L}_r(X)$ such that $\rho(\phi) = \int_K \phi dE$.

More results concerning these positive spectral measures, positive representations, unital positive spectral measures, and unital positive representations are summarized in Section 6. Altering $C_0(K)$ or $C(K)$ together with K into certain other function spaces can be done using certain inclusion relationships only for one direction of the statement, namely where we generate representations from spectral measures. This also shows that the part where we show that spectral measures can be generated from representations is certainly the most intricate of the two parts.

Our representations map on $\mathcal{L}_r(X)$, the collection of regular operators on the reflexive Banach lattice X . Thus the closure of orbits in X , $\rho[f]$ for an $f \in X$, of such a representation ρ inherits a partial ordering from X . In [PR07] a partial ordering has been defined on this subspace. We

have seen in Section 7 that in case our unital positive representation $\rho : C(K) \rightarrow \mathcal{L}_r(C(K))$ is defined as the pointwise multiplication on $C(K)$, $\rho[f]$ consists precisely of elements $g \in C(K_0)$ for which $N_f \subseteq N_g$. This precise description of the space $\rho[f]$ led to the conclusion that both orderings are equal.

When looking at what has been found in [PR07] in Section 8, in which a more general approach to the statements has been taken, we can see that our spectral measure is defined in the same way as the one in that article. We see that if we would change X into a Banach space, we would need R -boundedness to find a similar correspondence result.

If we place our representations ρ in the context of a Banach algebra dynamical system, and look at covariant representations (ρ, U) and covariant spectral measures (E, U) , where U is a group representation on a group G , we find a one-to-one correspondence as well (Corollary 9.8).

Let K be a compact Hausdorff space, $\Omega \subseteq \mathcal{P}(K)$ the Borel σ -algebra, X a reflexive Banach lattice, and $\rho : C(K) \rightarrow \mathcal{B}(X)$ a unital positive representation. Let E be the unital positive spectral measure corresponding to ρ as described in Theorem 6.2. Then (ρ, U) is a covariant representation of $(C(K), G, \alpha)$ on X if and only if (E, U) is a covariant spectral measure.

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